Notes de correction   Examen 17/12/2013

I.1 f est continue sur $[a,b]$ donc uniformément continue ;

$\forall \varepsilon>0\ \exists \eta(\varepsilon)>0 \ / \ \forall t,t' \in [a,b] \quad |t-t'|\leq \eta \Rightarrow |f(t)-f(t')|\leq \varepsilon$

Soit $N(\varepsilon)$ / $\dfrac{b-a}{N(\varepsilon)} < \eta(\varepsilon)$ et $a_k = a + k\dfrac{b-a}{N(\varepsilon)}$, $0\leq k \leq N(\varepsilon)$

$\forall t \in [a,b] \ \exists k \in [\![0, N(\varepsilon)-1]\!] \ / \ t \in [a_k, a_{k+1}]$

Donc $(t, f(t)) \in [a_k, a_{k+1}] \times [f(a_k)-\varepsilon, f(a_k)+\varepsilon]$

Donc $\Gamma(a,b) \subset \displaystyle\bigcup_{k=0}^{N(\varepsilon)-1} [a_k, a_{k+1}] \times [f(a_k)-\varepsilon, f(a_k)+\varepsilon]$

Donc $\lambda(\Gamma(a,b)) \leq \displaystyle\sum_{k=0}^{N(\varepsilon)-1} \frac{b-a}{N(\varepsilon)} \times 2\varepsilon = 2(b-a)\varepsilon$

On en déduit que $\lambda(\Gamma(a,b)) = 0$ $\square$

I.2 Par le théorème des accroissements finis $|\varphi(t)-\varphi(t')| \leq \|\varphi'\|_{L^\infty([a,b])}|t-t'|$

donc si $t \in [a_k, a_{k+1}]$ $\quad |\varphi(t)-\varphi(a_k)| \leq \|\varphi'\|_{L^\infty(a,b)} \dfrac{b-a}{N(\varepsilon)}$

et donc $(\varphi(t), f(t)) \in \left[\varphi(a_k) - \dfrac{b-a}{N(\varepsilon)}\|\varphi'\|_{L^\infty}, \varphi(a_k) + \dfrac{b-a}{N(\varepsilon)}\|\varphi'\|_{L^\infty}\right]$

$\times [f(a_k)-\varepsilon, f(a_k)+\varepsilon] \equiv Q_k$

Donc $G(a,b) \subset \displaystyle\bigcup_{k=0}^{N(\varepsilon)-1} Q_k$, $\lambda(G(a,b)) \leq \displaystyle\sum_{0}^{N(\varepsilon)-1} \lambda(Q_k) = 4\varepsilon(b-a)\|\varphi'\|_{L^\infty}$

On en déduit que $\lambda(G(a,b)) = 0$ $\square$

I.3 $G = \displaystyle\bigcup_{n=0}^{\infty} G(-n,n)$ $\quad \lambda(G(-n,n)) = 0 \Rightarrow \lambda(G) = 0$ $\square$

I.4 Si $\varphi = f =$ fonction de Peano avec $[a,b] = [0,1]$

on a $G(a,b) = [0,1]\times[0,1]$ non négligeable ! $\square$

II.1 $f(x) = \dfrac{\cos x}{\log(1+\sqrt{x})} \in C^0(]0,\pi])$

Or $x \to 0$ $\log(1+\sqrt{x}) \sim \sqrt{x}$, $\cos x \sim 1$

donc $f(x) \sim \dfrac{1}{\sqrt{x}} \in L^1(]0,\pi[)$

Donc $f \in L^1(]0,\pi[)$.

II.2 $$\int_0^R f(x)\,dx = \int_{]0,\pi[} f(x)\,d\lambda(x) + \int_\pi^R f(x)\,dx$$

$$\int_\pi^R \frac{\cos x}{\log(1+\sqrt{x})}\,dx = \left[\frac{\sin x}{\log(1+\sqrt{x})}\right]_\pi^R + \frac{1}{2}\int_\pi^R \frac{\sin x}{\sqrt{x}(1+\sqrt{x})(\log(1+\sqrt{x}))^2}\,dx$$

$$= \frac{\sin R}{\log(1+\sqrt{R})} + \int_{\log(1+\sqrt{\pi})}^{\log(1+\sqrt{R})} \frac{\sin[(e^t-1)^2]}{t^2}\,dt \quad (t = \log(1+\sqrt{x}))$$

Or $\left|\dfrac{\sin((e^t-1)^2)}{t^2}\right| \leq \dfrac{1}{t^2} \in L^1(\log(1+\sqrt{\pi}), \infty)$

donc $$\int_\pi^R \frac{\cos x}{\log(1+\sqrt{x})}\,dx \xrightarrow[R\to\infty]{} \int_{\log(1+\sqrt{\pi})}^{\infty} \frac{\sin(e^t-1)^2}{t^2}\,dt$$ □.

II.3 $$\int_\pi^{n\pi} \left|\frac{\cos x}{\log(1+\sqrt{x})}\right| dx = \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\cos x|}{\log(1+\sqrt{x})}\,dx$$

$$= \sum_{k=1}^{n-1} \int_0^\pi \frac{|\cos(k\pi + x)|}{\log(1+\sqrt{k\pi + x})}\,dx$$

$$\geq \sum_{k=1}^{n-1} \frac{1}{\log(1+\sqrt{(k+1)\pi})} \int_0^\pi |\cos x|\,dx \geq \frac{2}{\sqrt{\pi}} \sum_1^{n-1} \frac{1}{\sqrt{k+1}} \xrightarrow{n\to\infty} \infty$$

□

III - 1 $\displaystyle\int_X |f|^p \, d\mu(x) \leq \|f\|_{L^\infty}^p \, \mu(X) < \infty \Rightarrow f \in L^p$ $\square$

III - 2 Soit $p_\infty \in [1, \infty[$ et $p_n \to p_\infty$, $p_n \in [1, \infty[$

- $|f(x)|^{p_n} \to |f(x)|^{p_\infty}$ $n \to \infty$

$(\sup p_n = p_* < \infty)$

- $|f(x)|^{p_n} \leq 1$ si $|f(x)| \leq 1$ et $|f(x)|^{p_n} \leq |f(x)|^{p_*}$ si $|f(x)| \geq 1$.

donc $|f(x)|^{p_n} \leq 1 + |f(x)|^{p_*} \leq 1 + \|f\|_{L^\infty}^{p_*} \in L^1(X)$

Théorème de Convergence dominée $\Rightarrow \displaystyle\int_X |f(x)|^{p_n} d\mu \to \int_X |f(x)|^{p_\infty} d\mu$ $\square$

III . 3 $\displaystyle\infty > \int_X |f(x)|^p d\mu \geq \int_{\{|f| \geq A\}} |f(x)|^p d\mu \geq A^p \mu(\{x / |f(x)| \geq A\})$

$\Rightarrow \quad A \left(\mu(\{x / |f(x)| \geq A\}) \right)^{1/p} \leq \|f\|_{L^p}$

$A < \|f\|_{L^\infty} \Rightarrow \mu(\{x \,/\, |f(x)| \geq A\}) > 0$

$\Rightarrow \left(\mu(\{x / |f(x)| \geq A\}) \right)^{1/p} \longrightarrow 1 \quad p \to \infty$

$\Rightarrow \quad A \leq \liminf_{p \to \infty} \|f\|_{L^p} \leq \limsup_p \|f\|_{L^p}$

$\leq \limsup \|f\|_{L^\infty} (\mu(X))^{1/p}$

$= \|f\|_{L^\infty}$

On a montré que $\forall A < \|f\|_{L^\infty}$

$A \leq \liminf \|f\|_{L^p} \leq \limsup \|f\|_{L^p} \leq \|f\|_{L^\infty}$

Donc $\liminf \|f\|_{L^p} = \limsup \|f\|_{L^\infty} = \|f\|_{L^\infty}$ $\square$.

$\underline{IV.1}$ $f(x,\theta) = \cos(x\cos\theta)$

- x fixé $(\theta \mapsto f(x,\theta)) \in C^0([0,\pi]) \subset L^1(0,\pi)$
- θ fixé $(x \to f(x,\theta)) \in C^1(\mathbb{R})$
- $\frac{\partial f}{\partial x}(x,\theta) = -\cos\theta \sin(x\cos\theta)$

$$\left|\frac{\partial f}{\partial x}(x,\theta)\right| \leq 1 \in L^1([0,\pi])$$

$\Rightarrow \left(x \mapsto \int_0^\pi f(x,\theta)\, d\theta\right) \in C^1$ et

$$J'(x) = -\int_0^\pi \cos\theta \sin(x\cos\theta)\, d\theta \,.$$

$\underline{IV.2}$ $g(x,\theta) = -\cos\theta \sin(x\cos\theta)$

- x fixé $(\theta \mapsto g(x,\theta)) \in C^0([0,\pi]) \subset L^1(0,\pi)$
- θ fixé $(x \mapsto g(x,\theta)) \in C^1(\mathbb{R})$
- $\frac{\partial g}{\partial x}(x,\theta) = -\cos^2\theta \cos(x\cos\theta)$

$$\left|\frac{\partial g}{\partial x}\right| \leq 1 \in L^1([0,\pi])$$

$\Rightarrow \left(x \to \int_0^\pi g(x,\theta)\, d\theta\right) \in C^1$ et

$$J''(x) = -\int_0^\pi \cos^2\theta \cos(x\cos\theta)\, d\theta \,.$$

$\underline{IV.3}$ $J'(x) = -\int_0^\pi \underbrace{\cos\theta}_{v'} \underbrace{\sin(x\cos\theta)}_{u}\, d\theta = \left[\sin\theta \sin(x\cos\theta)\right]_0^\pi - x\int_0^\pi \sin^2\theta \cos(x\cos\theta)\, d\theta$

$$= -x\int_0^\pi \sin^2\theta \cos(x\cos\theta)\, d\theta$$

$$xJ'' + J' + xJ = \int_0^\pi \left(-x\cos^2\theta - x\sin^2\theta + x\right)\cos(x\cos\theta)\, d\theta$$

$$= 0 \quad \square$$