Université de Bordeaux
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Complementary results on Poisson processes

1 Exponential distribution properties

Proposition 1.1 (Memorylessness) *A positive random variable S has an exponential distribution if and only if it satisfies the memoryless property*

$$\mathbb{P}(S > s + t | S > s) = \mathbb{P}(S > t)$$

for all $s, t \geqslant 0$.

Proof. Let us assume that S has an exponential distribution $\mathcal{E}(\lambda)$. We have

$$\mathbb{P}(S > s + t | S > s) = \frac{\mathbb{P}(S > s + t, S > s)}{\mathbb{P}(S > s)} = e^{-\lambda t} = \mathbb{P}(S > t).$$

On the contrary, Let us assume that S satisfies the memoryless property. We define $g(t) = \mathbb{P}(S > t)$ for all $t \geqslant 0$. We remark that g is a decreasing function on $\mathbb{R}^+$ such that $\lim_{t\to 0} g(t) = 1$ and $\lim_{t\to +\infty} g(t) = 0$. Moreover,

$$\begin{aligned} \mathbb{P}(S > s + t | S > s) &= \frac{\mathbb{P}(S > s + t, S > s)}{\mathbb{P}(S > s)} = \frac{g(s+t)}{g(s)}, \\ \mathbb{P}(S > t) &= g(t), \end{aligned}$$

and thus $g(s)g(t) = g(s + t)$ for all $s, t \geqslant 0$. By using the following lemma we conclude that $g(t) = e^{-\lambda t}$: S has an exponential distribution. □

Lemma 1.1 *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ a multiplicative non increasing function (i.e. $g(s)g(t) = g(s + t)$ for all $s, t \geqslant 0$), such that $\lim_{t\to 0} g(t) = 1$ and $\lim_{t\to +\infty} g(t) = 0$. Then there exists $\lambda > 0$ such that $g(t) = e^{-\lambda t}$ for all $t \geqslant 0$.*

Proof. Let n an integer. We have

$$g(n) = g(1 + ... + 1) = g(1)^n$$

using the multiplicativity property. Let us consider now $n \in \mathbb{N}^*$, then we get

$$g(1) = g\left(\frac{1}{n} + ... + \frac{1}{n}\right) = g\left(\frac{1}{n}\right)^n,$$

and thus $g(1/n) = g(1)^{1/n}$. We can deduce that for all $r = p/q \in \mathbb{Q}^+$ we have $g(r) = g(p/q) = g(1)^{p/q}$. Finally we consider $t \in \mathbb{R}$. There exists an increasing sequence $(r_n)_{n\geqslant 0}$

and a decreasing sequence $(s_n)_{n\geqslant 0}$ of rational numbers tending to t such that $r_n \leqslant t \leqslant s_n$ for all $n \in \mathbb{N}$. Then, we have for all n

$$g(r_n) \leqslant g(t) \leqslant g(s_n).$$

Since r_n and s_n rational numbers, we deduce

$$g(1)^{r_n} \leqslant g(t) \leqslant g(1)^{s_n}.$$

Finally, since g is non increasing, we pass to the limit in the previous inequality to obtain $g(t) = g(1)^t$ for all $t \geqslant 0$. Since $\lim_{t\to 0} g(t) = 1$ and $\lim_{t\to+\infty} g(t) = 0$, we get $0 < g(1) < 1$. By setting $\lambda = -\log(g(1)) > 0$ we obtain the result: $g(t) = e^{-\lambda t}$ for all $t \geqslant 0$. □

2 Equivalent definitions of Poisson process

Theorem 2.1 *Let $(T_n)_{n\geqslant 1}$ a point process on $\mathbb{R}^+$, $(N_t)_{t\geqslant 0}$ its random counting function and $\lambda > 0$. Then the three following propositions are equivalent:*

1. *(N_t) is a Poisson process with intensity λ.*
2. *Increments of $(N_t)_{t\geqslant 0}$ are independent and we have the following asymptotic expansions, uniform with respect to t, when h tends to 0*

$$\begin{aligned}\mathbb{P}(N_{t+h} - N_t = 0) &= 1 - \lambda h + o(h)\\ \mathbb{P}(N_{t+h} - N_t = 1) &= \lambda h + o(h).\end{aligned}$$

3. *Waiting times between jumps $(S_n)_{n\geqslant 1}$ are i.i.d. with law $\mathbb{E}(\lambda)$.*

Proof. We have already seen in the course that $1 \Rightarrow 2$ and $1 \Rightarrow 3$. So it is sufficient to show $3 \Rightarrow 2$ and $2 \Rightarrow 1$ to obtain all equivalences. Let us assume that 3 is fulfilled and let us try to prove 2. We start by proving that, under this hypothesis, for all time $s \geqslant 0$, the process $N_t^s = N_{t+s} - N_s$ is independent with $(N_r, 0 \leqslant r \leqslant s)$ and has waiting times (S_n^s) i.i.d. with distribution $\mathcal{E}(\lambda)$. Since N_s takes its values in $\mathbb{N}$, it is sufficient to Prove the result conditionally to $N_s = i$ for $i \in \mathbb{N}$.Then we have $S_1^s = S_{i+1} - (s - T_i)$ and $S_n^s = S_{n+1}$ for $n \geqslant 2$. So, for $n \geqslant 2$, (S_n^s) are i.i.d. with law $\mathcal{E}(\lambda)$ and independent to the past $(N_r, 0 \leqslant r \leqslant s)$. We calculate now the law of S_1^s. We have

$$\begin{aligned}\mathbb{P}(S_1^s > t | N_s = i) &= \mathbb{P}(S_{i+1} > t + s - T_i | T_i \leqslant s, S_{i+1} > s - T_i)\\ &= \frac{\mathbb{P}(S_{i+1} > t + s - T_i, T_i \leqslant s, S_{i+1} > s - T_i)}{\mathbb{P}(T_i \leqslant s, S_{i+1} > s - T_i)}\\ &= \frac{\mathbb{E}[\mathbb{E}[\mathbb{1}_{\{S_{i+1} > t+s-T_i, T_i \leqslant s, S_{i+1} > s - T_i\}} | T_i]]}{\mathbb{E}[\mathbb{E}[\mathbb{1}_{\{T_i \leqslant s, S_{i+1} > s - T_i\}} | T_i]]}\end{aligned}$$

But S_{i+1} and T_i are independent, thus properties of conditional expectation give us

$$\mathbb{P}(S_1^s > t | N_s = i) = \frac{\mathbb{E}[f(T_i)]}{\mathbb{E}[f(T_i)]},$$

with

$$\begin{aligned} f(u) &= \mathbb{E}[\mathbb{1}_{\{S_{i+1}>t+s-u, u\leqslant s, S_{i+1}>s-u\}}] \\ &= \mathbb{P}(S_{i+1}>t+s-u|S_{i+1}>s-u)\mathbb{P}(S_{i+1}>s-u)\mathbb{1}_{u\leqslant s} \\ &= \mathbb{P}(S_{i+1}>t)\mathbb{P}(S_{i+1}>s-u)\mathbb{1}_{u\leqslant s} \\ &= e^{-\lambda t}e^{-\lambda(s-u)}\mathbb{1}_{u\leqslant s} \end{aligned}$$

by using the memoryless property, and

$$\begin{aligned} g(u) &= \mathbb{E}[\mathbb{1}_{\{u\leqslant s, S_{i+1}>s-u\}}] \\ &= \mathbb{P}(S_{i+1}>s-u)\mathbb{1}_{u\leqslant s} \\ &= e^{-\lambda(s-u)}\mathbb{1}_{u\leqslant s}. \end{aligned}$$

Thus we obtain

$$\mathbb{P}(S_1^s>t|N_s=i)=\frac{\mathbb{E}[e^{-\lambda t}e^{-\lambda(s-T_i)}\mathbb{1}_{T_i\leqslant s}]}{\mathbb{E}[e^{-\lambda(s-T_i)}\mathbb{1}_{T_i\leqslant s}]}=e^{-\lambda t},$$

so $S_1^s\sim\mathbb{E}(\lambda)$. We also get the independence with the past by showing that

$$\mathbb{P}(S_1^s>t,S_1>s_1,...,S_i>s_i|N_s=i)=e^{-\lambda t}\mathbb{P}(S_1>s_1,...,S_i>s_i|N_s=i).$$

We deduce that increments of (N_t) are independent under assumption 3. Moreover, $(N_{t+h}-N_t)$ and (N_h) have the same law, so we have

$$\begin{aligned} \mathbb{P}(N_{t+h}-N_t\geqslant 1) &= \mathbb{P}(N_h\geqslant 1)=\mathbb{P}(T_1\leqslant h)=\mathbb{P}(S_1\leqslant h) \\ &= 1-e^{-\lambda h}=\lambda h+o(h) \end{aligned}$$

uniformly with respect to t, when h is small. By same arguments we get

$$\begin{aligned} 0\leqslant\mathbb{P}(N_{t+h}-N_t\geqslant 2) &= \mathbb{P}(N_h\geqslant 2)=\mathbb{P}(T_2\leqslant h) \\ &\leqslant \mathbb{P}(S_1\leqslant h,S_2\leqslant h)=(1-e^{-\lambda h})^2=o(h) \end{aligned}$$

uniformly with respect to t, when h is small. The difference gives us

$$\begin{aligned} \mathbb{P}(N_{t+h}-N_t=1) &= \mathbb{P}(N_{t+h}-N_t\geqslant 1)-\mathbb{P}(N_{t+h}-N_t\geqslant 2)=\lambda h+o(h) \\ \mathbb{P}(N_{t+h}-N_t=0) &= 1-\mathbb{P}(N_{t+h}-N_t\geqslant 1)=1-\lambda h+o(h), \end{aligned}$$

which prove 2.

Now we assume that 2 is fulfilled and we try to prove 1. We have independence of increments, we just have to show stationarity. We will calculate the characteristic function of $N_{t+s}-N_t$ and check that it does not depend on t. Using independence of increments, we have, for all $u\in\mathbb{R}$,

$$\begin{aligned} \mathbb{E}[e^{iu(N_{t+s}-N_t)}] &= \mathbb{E}[\prod_{j=1}^n e^{iu(N_{t+j\frac{s}{n}}-N_{t+(j-1)\frac{s}{n}})}]=\prod_{j=1}^n\mathbb{E}[e^{iu(N_{t+j\frac{s}{n}}-N_{t+(j-1)\frac{s}{n}})}] \\ &= \prod_{j=1}^n\Big(1-\lambda\frac{s}{n}+e^{iu}\lambda\frac{s}{n}+o(1/n)\Big) \end{aligned}$$

where $o(1/n)$ is uniform with respect to j. So we have

$$\mathbb{E}[e^{iu(N_{t+s}-N_t)}]=e^{\lambda s(e^{iu}-1)}+o(1).$$

Then, when n tends to the limit, we get $N_{t+s}-N_t\sim\mathcal{P}(\lambda s)$ which gives us 1. □