TERMINAL EXAM
SUCCINCT CORRECTION

Exercise 1

1) We denote $M = \min(R, S)$ with $R \sim \mathcal{E}(\alpha)$, $S \sim \mathcal{E}(\beta)$, $R \perp\!\!\!\perp S$

~~then~~ $t \in \mathbb{R}$

$$P(M > t) = P(R > t, S > t) = P(R > t)P(S > t) \quad \text{since } R \perp\!\!\!\perp S$$

$$= \begin{cases} e^{-\alpha t} e^{-\beta t} = e^{-(\alpha+\beta)t} & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases}$$

So $M \sim \mathcal{E}(\alpha+\beta)$.

$$P(M = R) = P(R \leq S) = \iint_{\mathbb{R}^2} \mathbb{1}_{r \leq s}\, f_{(R,S)}(r,s)\, dr\, ds$$

$$= \int_0^{+\infty}\int_0^{+\infty} \alpha\beta\, \mathbb{1}_{r \leq s}\, e^{-\alpha r} e^{-\beta s}\, dr\, ds = \alpha \int_0^{+\infty} \left(\int_r^{+\infty} \beta e^{-\beta s}\, ds \right) e^{-\alpha r}\, dr$$

$$= \alpha \int_0^{+\infty} e^{-\beta r} e^{-\alpha r}\, dr = \frac{\alpha}{\alpha+\beta} \qquad a/$$

2) $(Z_n)_{n \in \mathbb{N}}$ embedded Markov chain.

$P(Z_{n+1} = 0 \mid Z_n = 2) = c$ (when one component fails, the other one can fail instantly with probability c.)

So $P(Z_{n+1} = 1 \mid Z_n = 2) = 1 - c$.

②

$P(Z_{n+1}=0 \mid Z_n=0) = 1$ absorbing state (when the two electronic components fail, the machine is out of order)

When $Z_n = 1$, one component has failed and the other one works.

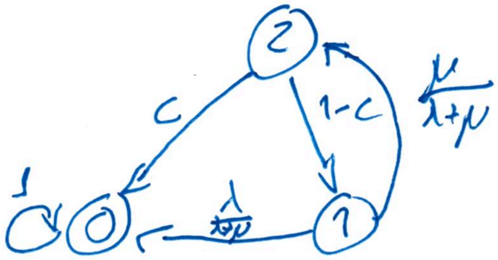
R: repair time of the component that has failed. $R \sim \mathcal{E}(\mu)$

S: failure time of the working component. $S \sim \mathcal{E}(\lambda)$.

$R \perp\!\!\!\perp S$.

$P(Z_{n+1}=2 \mid Z_n=1) = P(R<S) = \frac{\mu}{\lambda+\mu}$ (question 1)

and $P(Z_{n+1}=0 \mid Z_n=1) = \frac{\lambda}{\lambda+\mu}$



$E = \{0,1,2\}$

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\lambda}{\lambda+\mu} & 0 & \frac{\mu}{\lambda+\mu} \\ c & 1-c & 0 \end{pmatrix}$$

3) $\{0\}$ is an absorbing state so $q_{\{0\}} = +\infty$

By using question 1) ~~and~~ $q_{\{1\}} = \lambda+\mu$ and $q_{\{2\}} = 2\lambda$

so, $$Q = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 2\lambda c & 2\lambda(1-c) & -2\lambda \end{pmatrix}$$

4) $\{0\}$ absorbing state.

Since we have $2 \to 0$ and $1 \to 0$ ~~[illegible]~~
and $1 \leftrightarrow 2$

$\{1,2\}$ is a transient classe.

5) We denote $d_x^{\{0\}}$ the expected hitting time of $\{0\}$ starting from x.

We are looking for $d_2^{\{0\}}$.

$\left(d_x^{\{0\}}\right)_{x\in E}$ is the smallest positive solution of the system

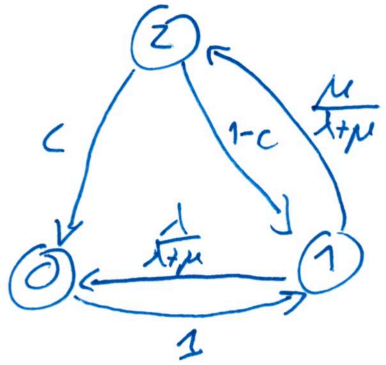
$$\begin{cases} d_0^{\{0\}} = 0 \\ \sum_{y\in E} Q_{xy} d_y^{\{0\}} = -1 \quad \forall x \neq 0 \end{cases} \iff \begin{cases} d_0^{\{0\}} = 0 \\ -(\lambda+\mu) d_1^{\{0\}} + \mu d_2 = -1 \\ 2\lambda(1-c) d_1^{\{0\}} - 2\lambda d_2^{\{0\}} = -1 \end{cases}$$

$$\iff \begin{cases} d_0^{\{0\}} = 0 \\ d_1^{\{0\}} = \dfrac{2\lambda+\mu}{2\lambda^2+2\lambda\mu c} \\ d_2^{\{0\}} = \dfrac{3\lambda+\mu-2\lambda c}{2\lambda^2+2\lambda\mu c} \end{cases}$$

and so $$d_2^{\{0\}} = \frac{3\lambda+\mu-2\lambda c}{2\lambda^2+2\lambda\mu c}$$

Part II

1) $$\tilde{\Pi} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\lambda}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} \\ c & (1-c) & 0 \end{pmatrix}$$



2) Now $q_{\{0\}} = 2\mu \qquad q_{\{1\}} = \lambda+\mu \qquad q_{\{2\}} = 2\lambda$

So $$Q = \begin{pmatrix} -2\mu & 2\mu & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 2\lambda c & 2\lambda(1-c) & -2\lambda \end{pmatrix}$$

3) $2 \to 0 \to 1 \to 2$ So $\{0,1,2\}$ is a recurrent class

The Markov process is irreducible.

4) E is finite and the process is irreducible so we are in the positive recurrent framework: there exists a unique invariant probability measure ν and ν is solution of $\begin{cases} \nu Q = 0 \\ \nu_0+\nu_1+\nu_2 = 1 \end{cases}$

$$\iff \begin{cases} -2\mu\nu_0 + \lambda\nu_1 + 2\lambda c\,\nu_2 = 0 \\ 2\mu\nu_0 - (\lambda+\mu)\nu_1 + 2\lambda(1-c)\nu_2 = 0 \\ \mu\nu_1 - 2\lambda\nu_2 = 0 \\ \nu_0+\nu_1+\nu_2 = 1 \end{cases}$$

$$\iff \begin{cases} \nu_0+\nu_1+\nu_2 = 1 \\ \nu_2 = \frac{\mu}{2\lambda}\nu_1 \\ \nu_0 = \frac{\lambda+\mu c}{2\mu}\nu_1 \end{cases} \iff \begin{cases} \nu_1 = \dfrac{1}{\frac{\mu}{2\lambda} + \frac{\lambda+\mu c}{2\mu} + 1} \\ \nu_0 = \frac{\lambda+\mu c}{2\mu}\left(\frac{\mu}{2\lambda} + \frac{\lambda+\mu c}{2\mu} + 1\right)^{-1} \\ \nu_2 = \frac{\mu}{2\lambda}\left(\frac{\mu}{2\lambda} + \frac{\lambda+\mu c}{2\mu} + 1\right)^{-1} \end{cases}$$

5) The machine works when we are in states $\{1,2\}$ and does not work in state $\{0\}$.

So the gain per unit of time G_t is given by

$$G_t = \frac{1}{t}\int_0^t 100 \times \mathbb{1}_{\{1,2\}}(X_t) - 50 \times \mathbb{1}_{\{0\}}(X_t)\,dt \xrightarrow{t\to+\infty} 100\times(\nu_1+\nu_2) - 50\nu_0$$

thanks to the ergodic theorem.

$$100\times(\nu_1+\nu_2) - 50\nu_0 = \left(100\left(1+\frac{\mu}{2\lambda}\right) - 50\left(\frac{\lambda+\mu c}{2\mu}\right)\right)\left(\frac{\mu}{2\lambda} + \frac{\lambda+\mu c}{2\mu} + 1\right)^{-1}$$

In particular, this quantity is positive iff $2\mu^2 - \lambda^2 + \lambda\mu(4-c) > 0$

i.e. $\frac{\mu}{\lambda}$ large enough!

Exercise 2

1) Q is an infinitesimal generator if $\sum_{y\in E} Q_{xy} = 0$

so, for $x>0$, $Q_{xx} = -(Q_{x,x+1} + Q_{x,x-1}) = -(\lambda+\nu)x$

2) We have $q_0 = \lambda$, $q_1 = \lambda+\mu$, ... $q_n = n(\lambda+\mu)$

So q_n is an increasing function (with respect to n)

Moreover $Z_0 = x_0$, $Z_1 \leq x_0+1$, $Z_2 \leq x_0+2$, ... $Z_n \leq x_0+n$

So $\sum_{n=0}^{+\infty} \frac{1}{q_{Z_n}} \geq \sum_{n=0}^{+\infty} \frac{1}{q_{x_0+n}} \geq \sum_{n=1}^{+\infty} \frac{1}{(\lambda+\mu)(x_0+n)} = +\infty$ (the serie $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges)

the non-explosion condition is satisfied.

3) For all $x \in \mathbb{N}^*$ we have $x-1 \leftrightarrow x \leftrightarrow x+1$

So the process is irreducible ($\mathbb{N}$ is a communicating class)

4) a) Π transition matrix of the embedded Markov chain $(Z_n)_{n\geq 0}$:

if $x \in \mathbb{N}^*$,

$$\Pi_{xy} = \begin{cases} \frac{\lambda}{\lambda+\nu} & \text{if } y = x+1 \\ \frac{\nu}{\lambda+\nu} & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Pi_{0y} = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

so the embedded Markov chain is a non-symmetric random walk (symetric iff $\lambda=\nu$) reflected in $\{0\}$.

Let us denote $(\tilde{Z}_n)_{n\geq 0}$ the non-symetric random walk not reflected in $\{0\}$ associated to

$(Z_n)_{n\geq 0}$.

i.e, $\tilde{Z}_{n+1}-\tilde{Z}_n = Z_{n+1}-Z_n$ when $Z_n \in \mathbb{N}^*$

and $\tilde{Z}_{n+1}-\tilde{Z}_n = \begin{cases} 1 & \text{with probability } \frac{\lambda}{\lambda+\nu} \\ -1 & \text{———} \quad \frac{\nu}{\lambda+\nu} \end{cases}$ when $Z_n = 0$.

So $Z_n \geq \tilde{Z}_n$. Moreover $\tilde{Z}_n \xrightarrow{n\to+\infty} +\infty$ since $\frac{\lambda}{\lambda+\nu} > \frac{\nu}{\lambda+\nu}$

So $Z_n \xrightarrow{n\to+\infty} +\infty$ a.s. ~~$(Z_n)_{n\geq 0}$ is~~

$(Z_n)_{n\geq 0}$ is transient, so $(X_t)_{t\geq 0}$ is transient.

b) We use same notation than in a). Let us assume that $x_0 = 0$.

So $Z_0 = 0$ and $Z_1 = 1$

We denote $(\tilde{Z}_n)_{n\geq 1}$ the non-symetric random walk not reflected in $\{0\}$ associated to

$(Z_n)_{n\geq 1}$.

~~Since $\tilde{Z}_n \xrightarrow{n\to+\infty} -\infty$,~~

First of all $\underline{Z_n = \tilde{Z}_n}$ for $n \underline{\leq R_0}$ with R_0 the time of the first return to $\{0\}$.

Moreover $\tilde{Z}_n \xrightarrow{n\to+\infty} -\infty$, so $P(\exists n\geq 1, \tilde{Z}_n = 0) = 1$.

So $R_0 < +\infty$: $\{0\}$ is a recurrent state for $(Z_n)_{n\geq 0}$.

$(Z_n)_{n\geq 0}$ is recurrent and so $(X_t)_{t\geq 0}$ is recurrent.

Exercise 3

1) We denote $Y_t = \sum_{n=1}^{N_t} Z_n$. Let us show that Y_t has stationnary and independant increments.

Let us consider $0 < t_1 < t_2$, $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$

$$\varphi(x_1, x_2) = \mathbb{E}\left[e^{ix_1 Y_{t_1} + ix_2(Y_{t_2} - Y_{t_1})}\right] = \mathbb{E}\Bigg[\underbrace{\mathbb{E}\left[e^{ix_1 \sum_{j=1}^{N_{t_1}} Z_j + ix_2 \sum_{j=N_{t_1}+1}^{N_{t_2}} Z_j} \,\Big|\, N_{t_1}, N_{t_2} - N_{t_1}\right]}_{:= f(N_{t_1}, N_{t_2} - N_{t_1})}\Bigg]$$

with $f(k_1, k_2) = \mathbb{E}\left[e^{ix_1 \sum_{j=1}^{k_1} Z_j + ix_2 \sum_{j=k_1+1}^{k_1+k_2} Z_j}\right]$ since $(Z_j) \perp\!\!\!\perp (N_{t_1}, N_{t_2} - N_{t_1})$

$= (\phi(x_1))^{k_1} (\phi(x_2))^{k_2}$ since $(Z_j)_j$ are independent

where ϕ is the caracteristic function of Z_1.

So $\varphi(x_1, x_2) = \mathbb{E}\left[\phi(x_1)^{N_{t_1}} \phi(x_2)^{N_{t_2} - N_{t_1}}\right]$

$= \mathbb{E}\left[\phi(x_1)^{N_{t_1}}\right] \mathbb{E}\left[\phi(x_2)^{N_{t_2} - N_{t_1}}\right]$ since $N_{t_1} \perp\!\!\!\perp N_{t_2} - N_{t_1}$

$= \mathbb{E}\left[e^{ix_1 Y_{t_1}}\right] \mathbb{E}\left[e^{ix_2(Y_{t_2} - Y_{t_1})}\right]$

$\Rightarrow Y_{t_1} \perp\!\!\!\perp Y_{t_2} - Y_{t_1}$ (OK with more increments...)

Moreover $\mathbb{E}\left[e^{ix_2(Y_{t_2} - Y_{t_1})}\right] = \mathbb{E}\left[\phi(x_2)^{N_{t_2} - N_{t_1}}\right] = \mathbb{E}\left[\phi(x_2)^{N_{t_2 - t_1}}\right]$ since $N_{t_2} - N_{t_1} \sim N_{t_2 - t_1}$

So $Y_{t_2} - Y_{t_1} \sim Y_{t_2 - t_1}$.

$\Rightarrow Y_t$ has stationnary and independent increments

since $t \mapsto at$, $t \mapsto bW_t$ and $t \mapsto Y_t$ are independent and have stationnary and independent increments, $(X_t)_t$ has stationnary and independent increments.

2) ~~$E = \{ x \in \mathbb{R} / \exists n \in \mathbb{N}^*, \exists x_1, \dots x_n$~~

We denote $\nu = \sum_{y \in F} \delta_y$,

Then $E = \{ x \in \mathbb{R} \mid \exists n \in \mathbb{N}, \exists x_1, \dots x_n \in F, x = \sum_{i=1}^{n} x_i \}$

Let us denote $T_1, \dots, T_n, \dots$ the jump times of the Poisson process. So

$$X_t = \sum_{n \geq 0} \left(\sum_{i=1}^{n} Z_i \right) \mathbb{1}_{[T_n, T_{n+1}[}(t) \rightarrow (X_t) \text{ is a random jump function.}$$

Its a jump Markov process since $(X_t)_t$ has independent increments.

Indeed, $\forall n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \dots < t_n < s < t$, $\forall x_0, x_1, \dots, x_n, x, y \in E$

$$\mathbb{P}(X_t = y \mid X_{t_0} = x_0, \dots X_{t_n} = x_n, X_s = x) = \frac{\mathbb{P}(X_t = y, \dots X_{t_n} = x_n, X_s = x)}{\mathbb{P}(X_{t_0} = x_0, \dots, X_s = x)}$$

$$= \frac{\mathbb{P}(X_t - X_s = y - x, X_s - X_{t_n} = x - x_n, \dots, X_{t_0} - X_0 = x_0)}{\mathbb{P}(X_s - X_{t_n} = x - x_n, \dots, X_{t_0} - X_0 = x_0)}$$

$$= \mathbb{P}(X_t - X_s = y - x) = \mathbb{P}(X_t = y \mid X_s = x)$$

since $\mathbb{P}(X_t - X_s = y - x)$ does not depend on $x_0, x_1, \dots x_n$.

a) $E = \{ x \in \mathbb{R} \mid \exists (n_1, n_2) \in \mathbb{N}^2, x = n_1 - n_2 \pi \}$

we have $\{n_1 - n_2\pi\} \rightarrow \{n_1 + 1 - n_2\pi\}$ but ~~we do~~ $\{n_1 + 1 - n_2\pi\} \not\rightarrow \{n_1 - n_2\pi\}$

$\{n_1 - n_2\pi\} \rightarrow \{n_1 - (n_2+1)\pi\}$ $\quad \{n_1 - (n_2+1)\pi\} \not\rightarrow \{n_1 - n_2\pi\}$

(The reason is : ~~π is a transcendental numb~~ $\pi \notin \mathbb{Q}$, so $\mathbb{Q}[\pi]$ is a $\mathbb{Q}$-ev of dimension 2
$(1, \pi)$ is a basis)

So all the communicating classes are singletons and are transient.

$$Q_{n_1 - n_2\pi, n_1 + 1 - n_2\pi} = Q_{n_1 - n_2\pi, n_1 - (n_2+1)\pi} = \lambda/2$$

with λ the intensity of the Poisson process

$$Q_{n_1 - n_2\pi, n_1 - n_2\pi} = -\lambda$$

$Q_{xy} = 0$ otherwise.

c) $E = \mathbb{Z}$.

$\forall k \in \mathbb{Z}$ $\quad k \to k-1 \to k+1 \to k$

So E is irreducible.

$$\frac{X_t}{t} = \frac{N_t}{t} \times \frac{\sum_{i=1}^{N_t} Z_i}{N_t} \longrightarrow \lambda \times \mathbb{E}[Z_1] \text{ a.s.}$$ thanks to the law of large number (we recall that $N_t \xrightarrow{t\to+\infty} +\infty$ a.s)

$$\frac{X_t}{t} \longrightarrow \lambda \times \frac{2-1}{2} = \frac{\lambda}{2} > 0$$

So $X_t \xrightarrow{t\to+\infty} +\infty$ a.s. and E is transient.

3) a) $\mathbb{E}\left[\sum_{m=1}^{N_t} Z_m \mid N_t = k\right] = \mathbb{E}\left[\sum_{m=1}^{k} Z_m \mid N_t = k\right] = \mathbb{E}\left[\sum_{m=1}^{k} Z_m\right]$ since $(Z_i) \perp\!\!\!\perp N_t$

$= k\,\mathbb{E}[Z_1]$ and so $\mathbb{E}\left[\sum_{m=1}^{N_t} Z_m \mid N_t\right] = N_t\,\mathbb{E}[Z_1]$

$$\mathbb{E}\left[\sum_{m=1}^{N_t} Z_m\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{m=1}^{N_t} Z_m \mid N_t\right]\right] = \mathbb{E}\left[N_t\,\mathbb{E}[Z_1]\right] = \mathbb{E}[N_t]\,\mathbb{E}[Z_1] = \lambda t\,\mathbb{E}[Z_1].$$

thanks to the previous calculation

b) $\mathbb{E}[X_t \mid \mathcal{F}_s] = at + b\,\mathbb{E}[W_t \mid \mathcal{F}_s] + \mathbb{E}\left[\sum_{m=1}^{N_t} Z_m \mid \mathcal{F}_s\right]$

$$= at + bW_s + \underbrace{b\,\mathbb{E}[W_t - W_s \mid \mathcal{F}_s]}_{=0 \;\; W_t - W_s \perp\!\!\!\perp \mathcal{F}_s^W \text{ and so } W_t - W_s \perp\!\!\!\perp \mathcal{F}_s} + \sum_{m=1}^{N_s} Z_m + \underbrace{\mathbb{E}\left[\sum_{m=N_s+1}^{N_t - N_s + N_s} Z_m \mid \mathcal{F}_s\right]}_{=\phi(N_s)}$$

where $\phi(k) = \mathbb{E}\left[\sum_{m=k+1}^{N_t - N_s + k} Z_m \mid \mathcal{F}_s, N_s = k\right] = \mathbb{E}\left[\sum_{m=k+1}^{N_t - N_s + k} Z_m\right]$ since $N_t - N_s \perp\!\!\!\perp \mathcal{F}_s^N$ and so $N_t - N_s \perp\!\!\!\perp \mathcal{F}_s$, $(Z_i)_{i \geq k+1} \perp\!\!\!\perp \mathcal{F}_s$

$= \mathbb{E}[N_t - N_s]\,\mathbb{E}[Z_1] = \lambda(t-s)\,\mathbb{E}[Z_1]$.

So $\mathbb{E}[X_t \mid \mathcal{F}_s] = at + bW_s + \sum_{m=1}^{N_s} Z_m + \lambda(t-s)\,\mathbb{E}[Z_1] = X_s + a(t-s) + \lambda(t-s)\,\mathbb{E}[Z_1]$

And so (X_t) is a martingale iff $a+\lambda\mathbb{E}[Z_1]=0$

c) $\mathbb{E}[X_t^2-dt|\mathcal{F}_s] = \mathbb{E}\left[\left(at+bW_t+\sum_{n=1}^{N_t}Z_n\right)^2-dt|\mathcal{F}_s\right] = \mathbb{E}\left[(at+bW_t+cN_t)^2-dt|\mathcal{F}_s\right]$

$$\mathbb{E}[X_t^2|\mathcal{F}_s] = \mathbb{E}\left[\left(X_s+a(t-s)+b(W_t-W_s)+c(N_t-N_s)\right)^2|\mathcal{F}_s\right]$$

$$= X_s^2+2a(t-s)X_s+2bX_s\underbrace{\mathbb{E}[W_t-W_s|\mathcal{F}_s]}_{=0}+2cX_s\underbrace{\mathbb{E}[N_t-N_s|\mathcal{F}_s]}_{=\lambda(t-s)}$$

$$+2a(t-s)b\underbrace{\mathbb{E}[W_t-W_s|\mathcal{F}_s]}_{=0}+2a(t-s)c\underbrace{\mathbb{E}[N_t-N_s|\mathcal{F}_s]}_{=\lambda(t-s)}+2bc\underbrace{\mathbb{E}[(W_t-W_s)(N_t-N_s)|\mathcal{F}_s]}_{=0 \text{ since } W_t-W_s \perp\!\!\!\perp N_t-N_s \perp\!\!\!\perp \mathcal{F}_s}$$

$$+a^2(t-s)^2+b^2\underbrace{\mathbb{E}[(W_t-W_s)^2|\mathcal{F}_s]}_{=t-s}+c^2\underbrace{\mathbb{E}[(N_t-N_s)^2|\mathcal{F}_s]}_{\lambda(t-s)+\lambda^2(t-s)^2}$$

$$= X_s^2+2a(t-s)X_s+2c\lambda(t-s)X_s+2ac\lambda(t-s)^2+a^2(t-s)^2+b^2(t-s)$$

$$+c^2\lambda(t-s)+c^2\lambda^2(t-s)^2$$

$$= X_s^2+X_s[2a+2c\lambda](t-s)+(a+c\lambda)^2(t-s)^2+(b^2+c^2\lambda)(t-s)$$

We impose $\underline{a+c\lambda=0}$

Then $\mathbb{E}[X_t^2-dt|\mathcal{F}_s] = X_s^2-ds+(b^2+c^2\lambda-d)(t-s)$

We impose also $b^2+c^2\lambda-d=0$.

Then, $\mathbb{E}[X_t^2-dt|\mathcal{F}_s]=X_s^2-ds$: we have a Martingale.

d) $(N^1, N^2, \dots N^n)$ n independent Poisson processes with intensity $\lambda p_1, \lambda p_2, \dots \lambda p_n$

Then $X_t=\sum_{k=1}^{n}x_k N_t^k$ by adapting exercice 11 in the first sheet of exercises.

$N_t^k = \#\{n\le N_t \mid Z_n = x_k\}$