Improvement of Reduced Order Modeling based on Proper Orthogonal Decomposition

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Abstract: This study focuses on stabilizing Reduced Order Model based on Proper Orthogonal Decomposition (POD) and on improving the POD functional subspace. A modified reduced order model (ROM) that incorporates directly the pressure term is proposed. The ROM is obtained by seeking a solution that lives in the POD subspace and at the same time minimizes the Navier-Stokes residuals. Both ROM stabilization and POD subspace adaptation make use of methods based on the fine scale equation that is approximated using the residuals of the Navier-Sokes equations. Results are shown for the 2D confined cylinder wake flow.

These last decades, the conception and the optimization of the aerodynamics of ground vehicles and airplanes has been considered using detailed numerical simulations. Now, and despite of the considerable progress made in the numerical field, it is still very difficult to solve this kind of problems for complex flows in real time, that is, *in fine*, a major stake for industrials. To undergo this difficulty, it is possible to approximate the detailed model of flow dynamics by a Reduced Order Model (ROM). In this study we use ROM based on the Proper Orthogonal Decomposition (POD, see Sirovich (1987) or Cordier and Bergmann (2002). for more details). The main drawbacks of POD ROM are:

1. Only the coarse scales are solved. Since the main part of dissipation takes place in the fine scales, which are not solved, the POD ROM is not able to dissipate a sufficient amount of energy. During the POD ROM integration, ***some spurious divergences can occur.***

2. A POD basis is only optimal to represent the dynamics included in the database used to build it. ***This same basis is not optimal to represent other dynamics*** (Prabhu et al., 2001).

The main objective of the present work is to report recent improvements on the two points raised just before, *i.e.* derive a low cost strategy (1) to stabilize the POD ROM and (2) to adapt the functional subspace $\mathcal{S}_N^{POD}$, spanned by the first N eigenfunctions $\boldsymbol{\Phi}_i$, to capture other dynamics than that used to build the basis. These strategies will be tested onto a paradigm of separated flow, the 2D confined square cylinder wake flow in the laminar regime.

Pressure extended ROM After having computed a basis $\{\boldsymbol{\Phi}\}_{k=1}^{N_s}$ extended to the pressure, a robust and precise ROM can be obtained by seeking a solution that both lives in the POD subspace and minimizes the residuals of the Navier-Stokes equations (NSE). The pressure term is directly evaluated from the pressure mode. A robust ROM has to satisfy the momentum equations, the continuity equation as well as the flow rate conservation. The continuity equation is always satisfied using divergence free modes (POD), but it is not the case using non-divergence free modes (residuals). This kind of ROM writes:

$$\sum_{j=1}^{N} L_{ij}\frac{d\,a_j}{dt} = \sum_{j=1}^{N} B_{ij}a_j + \sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}a_j a_k. \quad (1)$$

Coefficients L, B and C have momentum, continuity and flow rate contributions. The coarse scales are computed using the POD expansion $\overline{\boldsymbol{U}}(\boldsymbol{x},\,t) = \sum_{i=1}^{N} a_i(t)\boldsymbol{\Phi}_i(\boldsymbol{x})$, where N is a given cutoff threshold. For latter convenience, we note $\boldsymbol{U}(\boldsymbol{x},\,t)$ the numerical solution of the full Navier-Stokes equations, and $\boldsymbol{U}'(\boldsymbol{x},\,t)$ the correction term such that $\boldsymbol{U}(\boldsymbol{x},\,t) = \overline{\boldsymbol{U}}(\boldsymbol{x},\,t) + \boldsymbol{U}'(\boldsymbol{x},\,t)$.

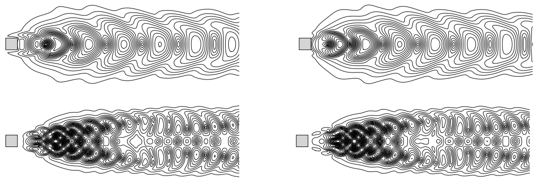

Figure 1: Norm of POD modes $\boldsymbol{\phi}_k$, $k = 2, \ldots, 5$.

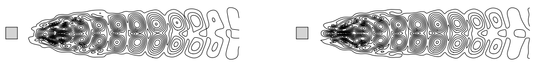

Figure 2: Norm of residual modes $\boldsymbol{\phi}_6$, and $\boldsymbol{\phi}_7$.

Residual-based Stabilization method of the POD ROM The idea is to model the fine scale equation, that is equal to the NSE residuals obtained from the coarse scales. This method is inspired by the variational multiscale residual-based turbulence modeling (for instance, see Bazilevs et al., 2007). During the integration of the POD ROM we compute the coarse scales $\overline{\boldsymbol{U}}(\boldsymbol{x}, t)$ and evaluate the NSE residuals, denoted $\mathcal{L}(\overline{\boldsymbol{U}}(\boldsymbol{x}, t)) = \boldsymbol{R}(\boldsymbol{x}, t)$. Starting from a NSE residuals database, we compute POD residual basis. We then add a few POD residual modes to the original POD basis (Gram-Schmidt) to model the interaction with non-resolved modes. For our test case at $Re = 100$, two reduced order models are built: the first one, named A, uses only a 5 POD basis functions, and the second one, named B uses 2 additional residuals modes (*i.e.* 7 basis functions). These POD and residual modes are presented in Fig. 1 and 2. It is noticeable in Fig. 3 that while the solution of system A diverges from the exact limit cycle to reach an erroneous one after few vortex shedding periods, solution of system B stays on the exact limit cycle, *i.e.* the Navier-Stokes attractor. Modes 1 to 5 model the coarse scales, and residual modes 6 and 7 model the fine scale equation. Model B is then a better approximation of the full NSE (coarse and fine scales) than model A, which approximates an erroneous model (uncomplete NSE). As shown in Fig. 4, both the norms of reconstruction error, $\boldsymbol{U}(\boldsymbol{x}, t) - \overline{\boldsymbol{U}}(\boldsymbol{x}, t)$, and of the NSE residuals are lower for model B.

Functional subspace adaptation Let $\boldsymbol{U}(\boldsymbol{x}, t)$ be the solution of NSE for a given dynamics (for instance, at Reynolds number Re_{NS}), and $\overline{\boldsymbol{U}}(\boldsymbol{x}, t)$ be the solution obtained minimizing NSE residuals onto a POD subspace that corresponds to an other dynamics (at Reynolds number $Re_{POD} \neq Re_{NS}$). The goal is to give an approximation $\widetilde{\boldsymbol{U}}'(\boldsymbol{x}, t)$ of the error $\boldsymbol{U}(\boldsymbol{x}, t) - \overline{\boldsymbol{U}}(\boldsymbol{x}, t)$ in order to improve the POD subspace. A new POD is per-

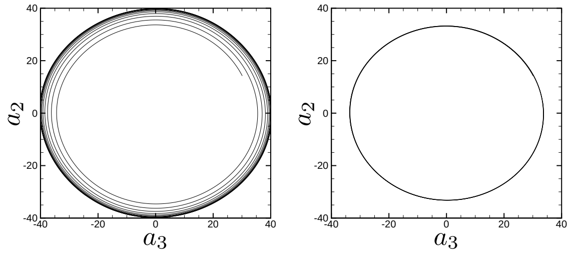

Figure 3: POD ROM limit cycles (a_2, a_3). Left, system A; right, system B.

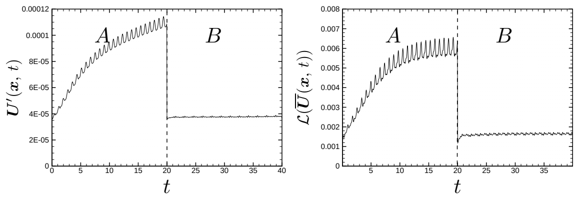

Figure 4: Temporal evolution of $\mathcal{L}_2$ norms: left, "error" $\boldsymbol{U}'(\boldsymbol{x}, t)$; right, NS residual $\mathcal{L}(\overline{\boldsymbol{U}}(\boldsymbol{x}, t))$.

formed from a database of the field $\widetilde{\boldsymbol{U}}(\boldsymbol{x}, t) = \overline{\boldsymbol{U}}(\boldsymbol{x}, t) + \widetilde{\boldsymbol{U}}'(\boldsymbol{x}, t)$. This process is iterated several times until convergence, for instance when $\widetilde{\boldsymbol{U}}'(\boldsymbol{x}, t)$ is minimized. Our method consits in expanding the error (that is the sum of all missing scales) onto the "coarse scales" residuals basis, *i.e.* $\widetilde{\boldsymbol{U}}'(\boldsymbol{x}, t) = M(t)\boldsymbol{R}(\boldsymbol{x}, t)$ with $M \in \mathbb{R}^{n_c \times n_c}$, where n_c denotes the number of Navier-Stokes components (here $n_c = 3$ for u, v and p). The unknown matrices $M(t)$ are determined using a Galerkin projection of the "fine scale" equations onto the NSE residual basis. First results, for $Re_{NS} = 200$ and $Re_{POD} = 100$, show that the functional subspace can be improved. Indeed, a 30% error reduction is obtained.

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