Improvement of Reduced Order Modeling based on Proper Orthogonal Decomposition

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Summary

Context and flow configuration

I - A pressure extended Reduced Order Model based on POD

- ▶ Proper Orthogonal Decomposition (POD)
- ▶ Reduced Order Model (ROM)

II - Stabilization of reduced order models

- ▶ Residuals based stabilization method
- ▶ Classical SUPG and VMS methods

III - Improvement of the functional subspace

- ▶ Krylov like method
- ▶ An hybrid DNS/POD ROM method (Database modification)

Conclusions

Context and flow configuration

▷ Context

- Need of reduced order model for flow control purpose

  ↪ To reduce the CPU time
  ↪ To reduce the memory storage during adjoint-based minimization process

- Optimization + POD ROM methods

  ↪ Generalized basis, no POD basis actualization : fast but no "convergence" proof
  ↪ Trust Region POD (TRPOD), POD basis actualization : proof of convergence !

- Drawbacks

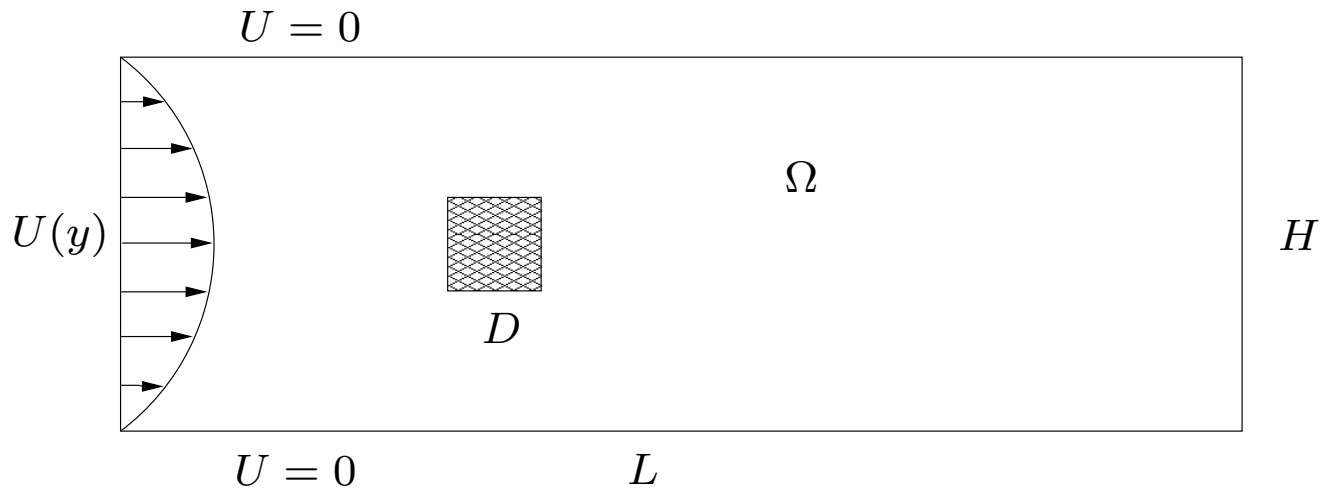
  ↪ Need to stabilize POD ROM (lack of dissipation, numerical issues, pressure term)
  ↪ Basis actualization : DNS $\longrightarrow$ high numerical costs !

- Solutions

  ↪ Efficient ROM & stabilization
  ↪ Low costs functional subspace adaptation during optimization process

Context and flow configuration

- **Flow Configuration**
  - 2-D Confined flow past a square cylinder in laminar regime
  - Viscous fluid, incompressible and newtonian
  - No control



- **Numerical methods**
  - Penalization method for the square cylinder
  - Multigrids V-cycles method in space *C.-H. Bruneau solver*
  - Gear method in time

I - A pressure extended Reduced Order Model

Proper Orthogonal Decomposition (POD), Lumley (1967)

▷ Look for the flow realization $\boldsymbol{\Phi}(\boldsymbol{X})$ that is "the closest" in an average sense to realizations $\boldsymbol{U}(\boldsymbol{X})$.

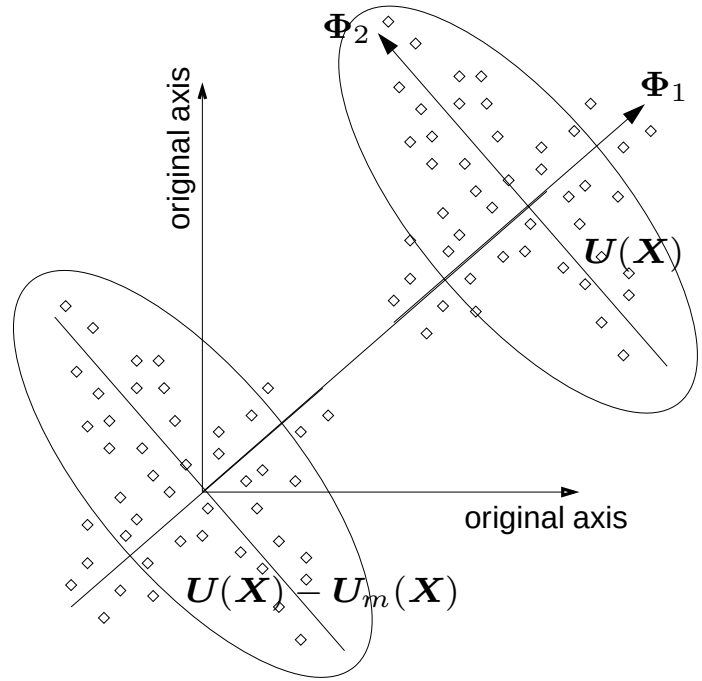
$$(\boldsymbol{X} = (\boldsymbol{x}, t) \in \mathcal{D} = \Omega \times \mathbb{R}^+)$$

▷ $\boldsymbol{\Phi}(\boldsymbol{X})$ solution of problem :

$$\max_{\boldsymbol{\Phi}} \langle |(\boldsymbol{U}, \boldsymbol{\Phi})|^2 \rangle, \quad \|\boldsymbol{\Phi}\|^2 = 1.$$

▷ Optimal convergence *in* L^2 *norm* de $\boldsymbol{\Phi}(\boldsymbol{X})$

$\Rightarrow$ Dynamical reduction possible.



Lumley J.L. (1967) : The structure of inhomogeneous turbulence. *Atmospheric Turbulence and Wave Propagation*, ed. A.M. Yaglom & V.I. Tatarski, pp. 166-178.

I - A pressure extended Reduced Order Model

▷ Equivalent with Fredholm equation :

$$\int_{\mathcal{D}} R_{ij}(\boldsymbol{X}, \boldsymbol{X'})\Phi_n^{(j)}(\boldsymbol{X'})\, d\boldsymbol{X'} = \lambda_n \Phi_n^{(i)}(\boldsymbol{X}) \qquad n = 1, .., N_s$$

$$\hookrightarrow R(\boldsymbol{X}, \boldsymbol{X'}) : \textit{Space-time correlation tensor}$$

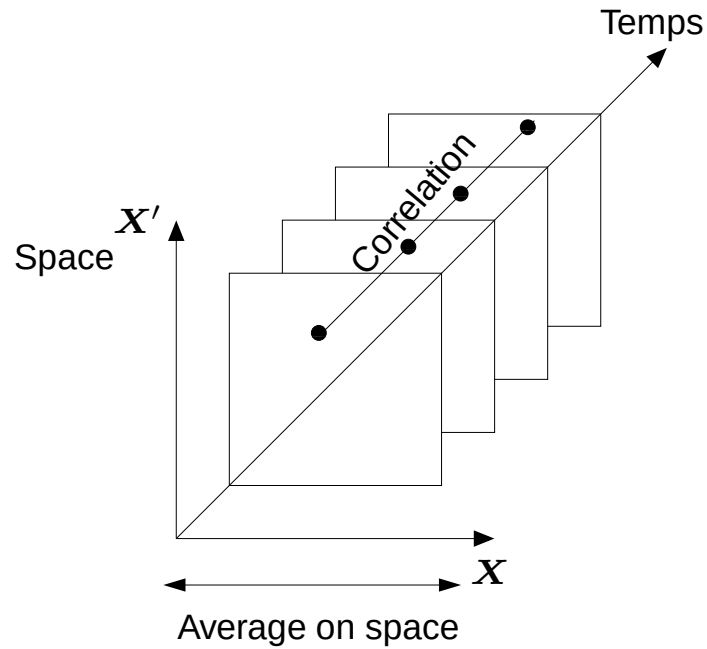
▷ Snapshots method, Sirovich (1987) :

$$\int_T C(t, t')a_n(t')\, dt' = \lambda_n a_n(t)$$

$$\hookrightarrow C(t, t') : \textit{Temporal correlations}$$

▷ $\boldsymbol{\Phi}(\boldsymbol{X})$ flow basis :

$$\boldsymbol{U}(\boldsymbol{x}, t) = \sum_{n=1}^{N_s} a_n(t)\boldsymbol{\Phi}_n(\boldsymbol{x}).$$



Sirovich L. (1987) : Turbulence and the dynamics of coherent structures. Part 1,2,3 *Quarterly of Applied Mathematics*, **XLV** N° 3, pp. 561–571.

I - A pressure extended Reduced Order Model

Truncation of the POD basis to keep 99% of the Relative Information Content

$$RIC(M) = \sum_{k=1}^{M} \lambda_k \Big/ \sum_{k=1}^{N_s} \lambda_k$$

Example at $Re = 200$ for $\boldsymbol{U} = (\boldsymbol{u},\, p)^T$ with $N_s = 200$

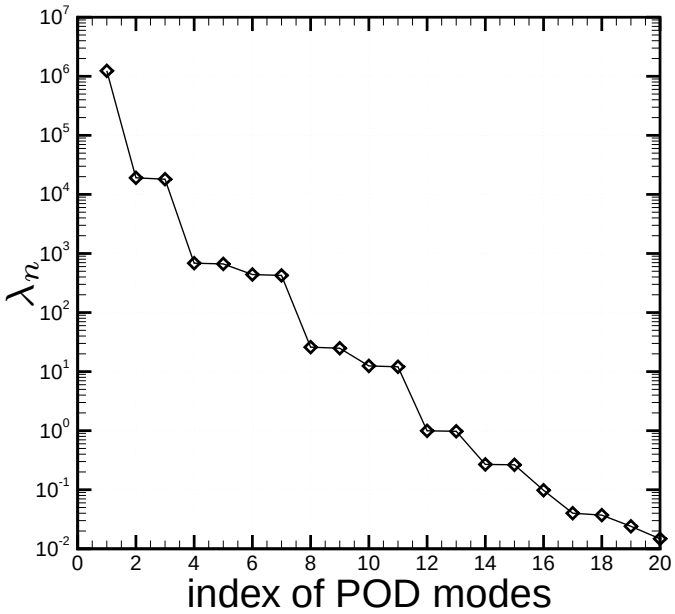


Fig. : *POD spectrum.*

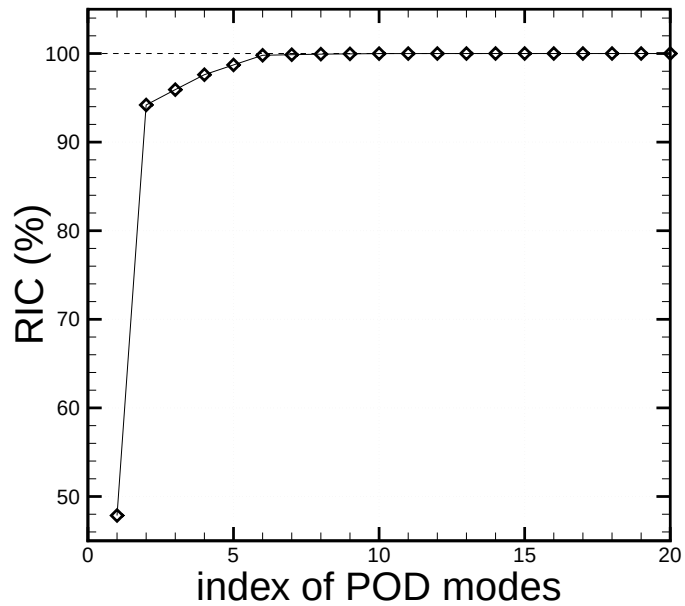


Fig. : *RIC(M), M nb modes POD retenus.*

$$N_r = \arg\min_{M} RIC(M) \text{ s.t. } RIC(N_r) > 99\% \quad \Rightarrow \quad \boldsymbol{N_r = 5}\,!$$

I - A pressure extended Reduced Order Model

The POD basis is $\mathbf{\Phi} = (\boldsymbol{\phi}, \psi)^T$

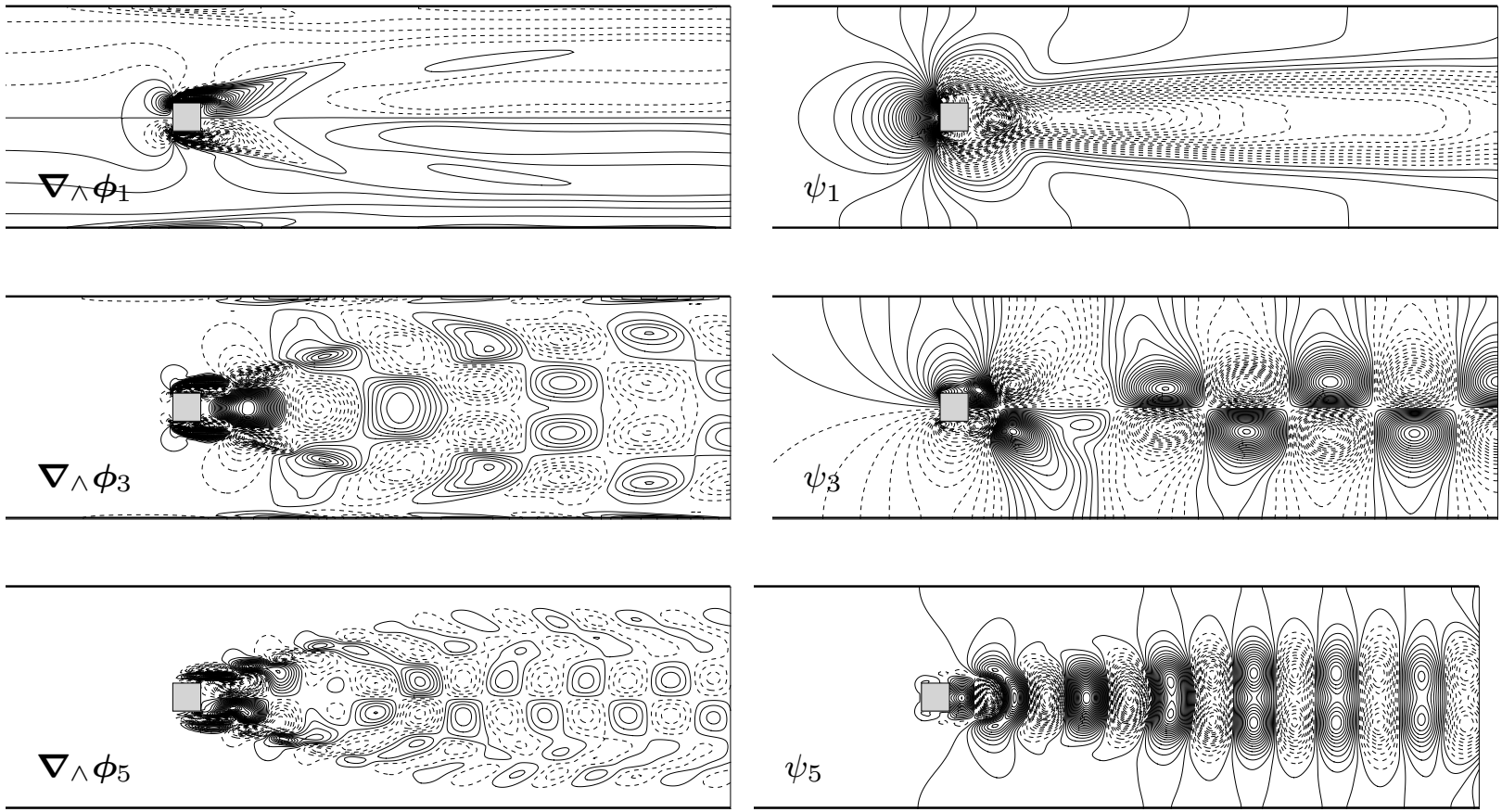


Fig. : *Representation of some POD modes. Iso-vorticity (left) and isobars (right). Dashed lines represent negative values (the pressure reference is arbitrarily chosen to be zero*

I - A pressure extended Reduced Order Model

▶ **Momentum conservation**

Detailled model (exact)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \frac{1}{Re}\Delta \boldsymbol{u}$$

Galerkin projection using $\widetilde{\boldsymbol{u}}(\boldsymbol{x},\,t) = \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})$ and $\widetilde{p}(\boldsymbol{x},\,t) = \sum_{i=1}^{N_r} a_i(t)\psi_i(\boldsymbol{x})$:

$$\left(\boldsymbol{\phi}_i,\, \sum_{j=1}^{N_r} \boldsymbol{\phi}_j \,\frac{\mathrm{d}a_j}{\mathrm{d}t} + \sum_{j=1}^{N_r}\sum_{k=1}^{N_r} (\boldsymbol{\phi}_j\cdot\boldsymbol{\nabla})\,\boldsymbol{\phi}_k\, a_j a_k + \sum_{j=1}^{N_r} \boldsymbol{\nabla}\psi_j\, a_j - \frac{1}{Re}\sum_{j=1}^{N_r}\Delta\boldsymbol{\phi}_j\, a_j\right)_{\Omega} = 0.$$

The Reduced Order Model is then :

$$\sum_{j=1}^{N_r} L_{ij}^{(m)} \frac{\mathrm{d}a_j}{\mathrm{d}t} = \sum_{j=1}^{N_r} B_{ij}^{(m)} a_j + \sum_{j=1}^{N_r}\sum_{k=1}^{N_r} C_{ijk}^{(m)} a_j a_k$$

↪ The ROM does not satisfy *a priori* the mass conservation
(for non divergence free modes, as NSE-Residual modes)

I - A pressure extended Reduced Order Model

▶ **Mass conservation**

Detailled model

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

Projection onto the POD basis

$$\sum_{j=1}^{N_r} a_j \boldsymbol{\nabla} \cdot \boldsymbol{\phi}_j = 0$$

Minimizing residuals in a least squares sense, we obtain :

$$\sum_{j=1}^{N_r} B_{ij}^{(c)} a_j = 0, \quad \text{where} \quad B_{ij}^{(c)} = (\boldsymbol{\nabla} \cdot \boldsymbol{\phi}_j)^T \boldsymbol{\nabla} \cdot \boldsymbol{\phi}_j$$

The modified ROM that satisfies both momentum and continuity equation writes :

$$\sum_{j=1}^{N_r} L_{ij}^{(m)} \frac{\mathrm{d}a_j}{\mathrm{d}t} = \sum_{j=1}^{N_r} \left(B_{ij}^{(m)} + \alpha B_{ij}^{(c)} \right) a_j + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} C_{ijk}^{(m)} a_j a_k$$

↪ The ROM has moreover to satisfy the flow rate conservation.

I - A pressure extended Reduced Order Model

▶ **Flow rate conservation**

For the 2-D confined flow :

$$\int_{\mathcal{S}} u \, \mathrm{d}s = c,$$

For each slides $\mathcal{S}_l$:

$$\sum_{i=1}^{N_r} a_j(t) \int_{\mathcal{S}_l} \boldsymbol{\phi}_j^u \, \mathrm{d}s = c,$$

$$\sum_{j=1}^{N_r} \frac{\mathrm{d}a_j}{\mathrm{d}t} \int_{\mathcal{S}_l} \boldsymbol{\phi}_j^u \, \mathrm{d}s = 0.$$

In a least square sense :

$$\sum_{j=1}^{N_r} L_{ij}^{(r)} \frac{\mathrm{d}a_j}{\mathrm{d}t} = 0.$$

Finally, the ROM writes

$$\sum_{j=1}^{N_r} \left(L_{ij}^{(m)} + \beta L_{ij}^{(r)} \right) \frac{\mathrm{d}a_j}{\mathrm{d}t} = \sum_{j=1}^{N_r} \left(B_{ij}^{(m)} + \alpha B_{ij}^{(c)} \right) a_j + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} C_{ijk}^{(m)} a_j a_k,$$

with initial conditions $a_i(0) = (\boldsymbol{U}(\boldsymbol{x}, 0), \boldsymbol{\Phi}_i(\boldsymbol{x}))_\Omega \quad i = 1, \cdots, N_r.$

I - A pressure extended Reduced Order Model

▶ **Advantage** no modelisation of the pressure term

$Re = 200$, 11 modes $\Rightarrow$ convergence towards the exact limit cycles (= DNS)

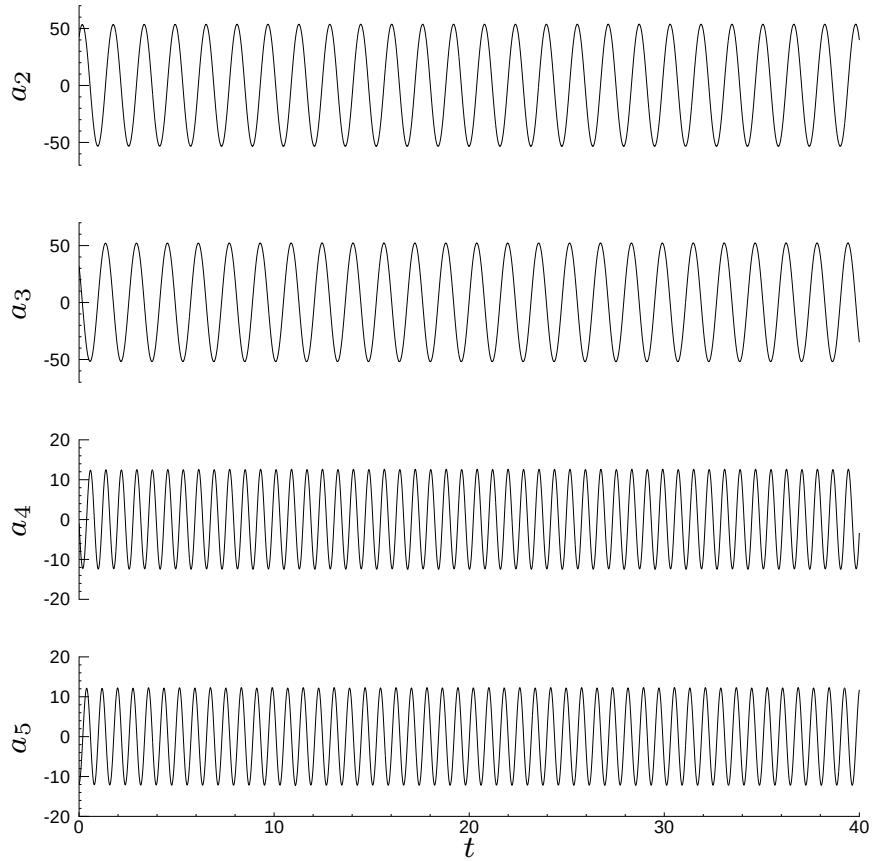


Fig. : *Temporal evolution of the POD ROM coefficients over* 25 *vortex shedding periods*

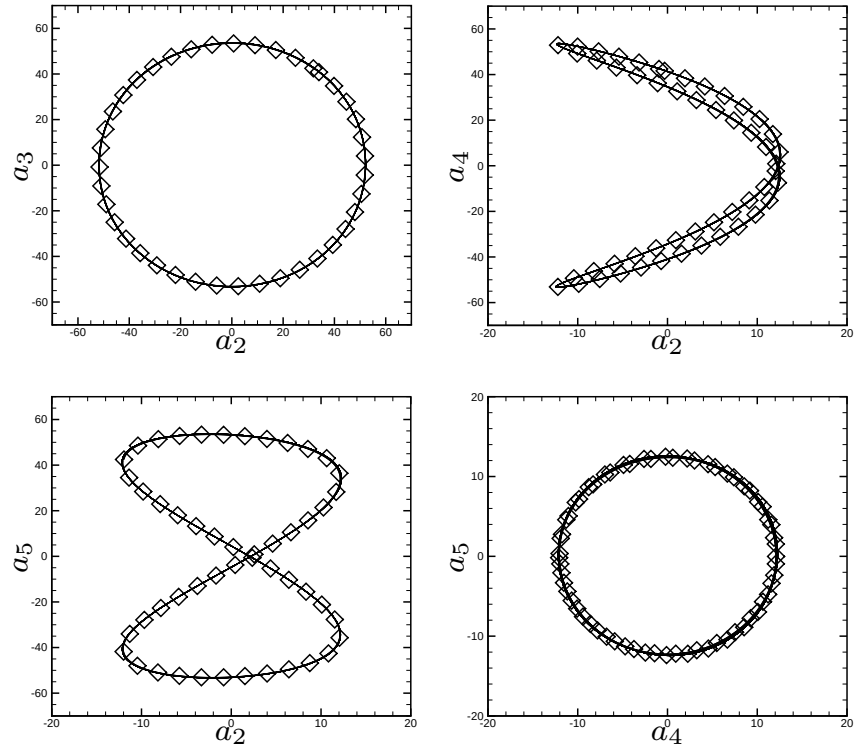
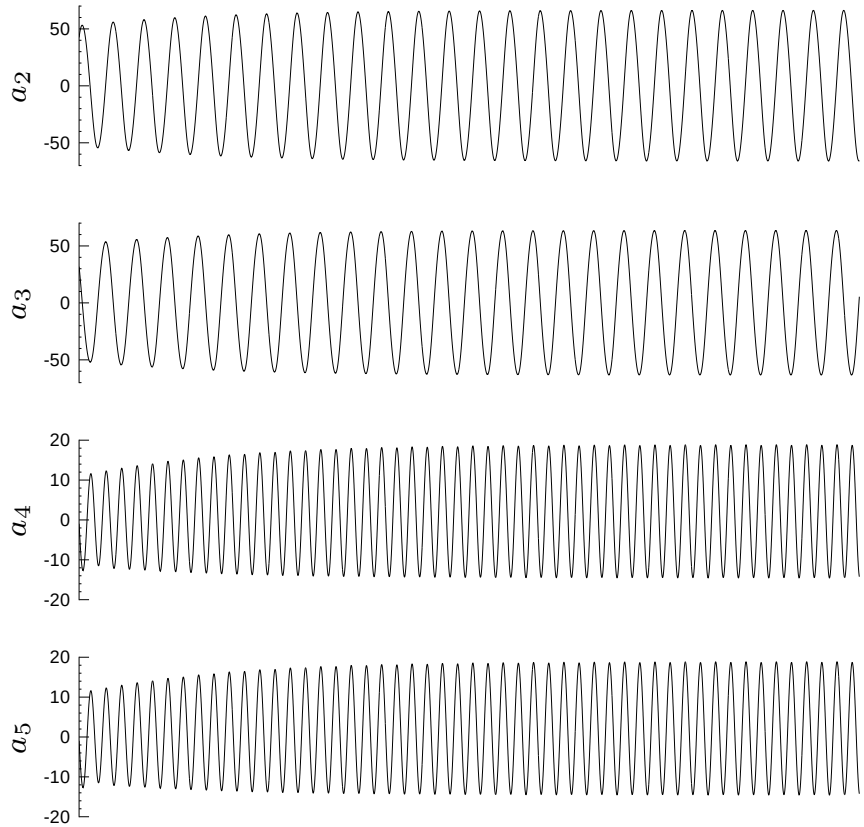


Fig. : *Limit cycles of the POD ROM coefficients over* 25 *vortex shedding periods*

I - A pressure extended Reduced Order Model

▶ **Drawbaks** same as usual, *i.e.* lack of dissipation...

$Re = 200$, 5 modes $\Rightarrow$ convergence towards an erroneous limit cycles ($\neq$ DNS)



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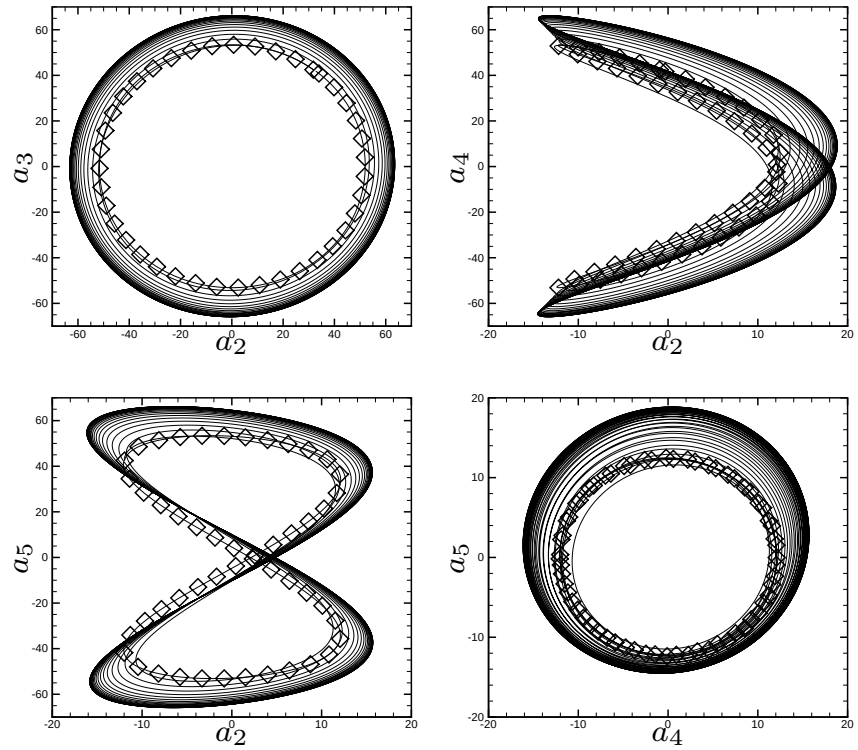

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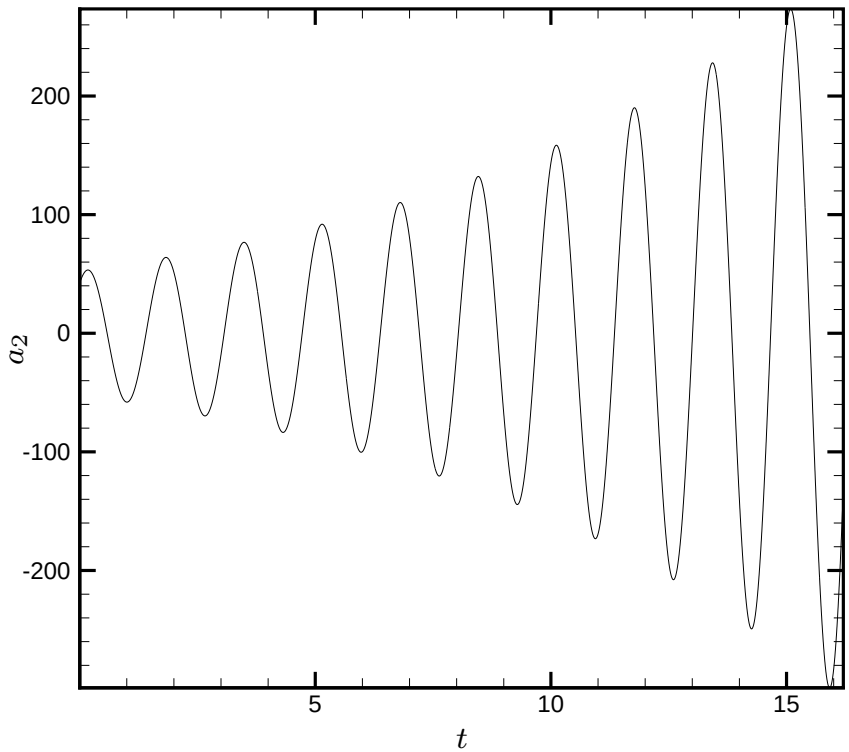
$Re = 200$, 3 modes $\Rightarrow$ exponential divergence



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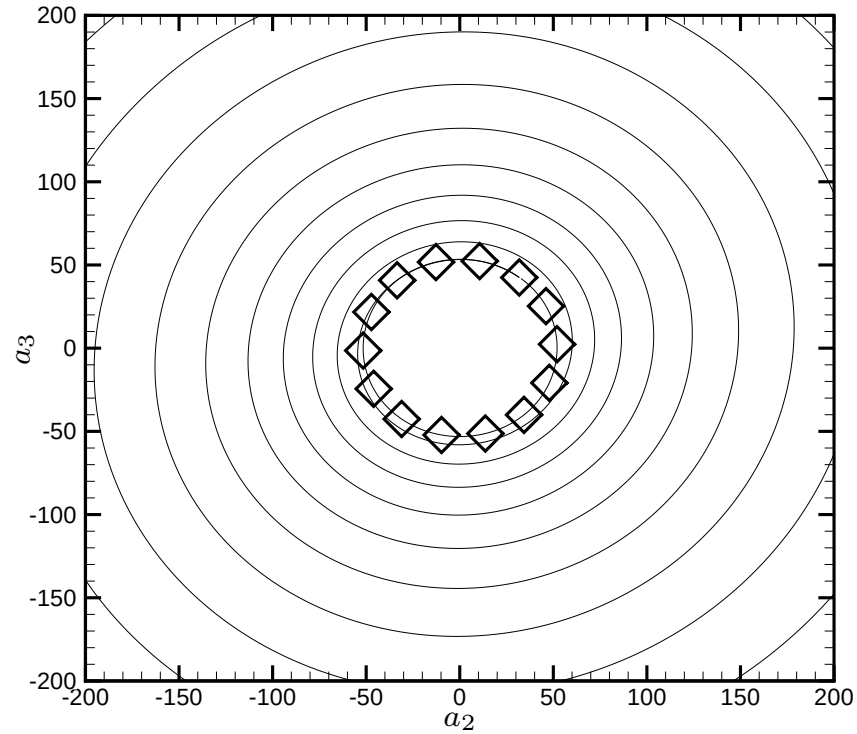


Fig. : *Limit cycles of the POD ROM coefficients over* 25 *vortex shedding periods*

II - POD ROM stabilization

▶ **Overview of stabilization methods (non-exhaustive)**

- Eddy viscosity
  - ↪ Heisenberg viscosity
  - ↪ Spectral vanishing viscosity
  - ↪ Optimal viscosity
- Penalty method
- Calibration of POD ROM coefficients

▶ **"New" stabilization methods in POD ROM context**

- *Residuals based stabilization method*
- *Streamline Upwind Petrov-Galerkin (SUPG) and Variational Multi-scale (VMS) methods*

II - POD ROM stabilization

▶ **Residuals based stabilization method**

$\Rightarrow$ **Idea** add dominant POD-NSE residual modes to the existing basis

$\hookrightarrow$ The POD-NSE residuals are $\mathcal{L}(\widetilde{\boldsymbol{u}}(\boldsymbol{x},\, t),\, \widetilde{p}(\boldsymbol{x},\, t)) = \boldsymbol{R}(\boldsymbol{x},\, t)$,
where $\widetilde{\boldsymbol{u}}$ and $\widetilde{p}$ obtained using POD and $\mathcal{L}$ is the NSE operator

Model $A^{[N_r]}$, **unstable POD ROM** built with N_r basis functions $\boldsymbol{\Phi}_i(\boldsymbol{x})$.

Algorithm

1. Integrate the ROM to obtain $a_i(t)$ and extract N_s snapshots $a_i(t_k)$, $k = 1, \ldots, N_s$.
2. Compute $\widetilde{\boldsymbol{u}}(\boldsymbol{x},\, t_k) = \sum_{i=1}^{N_r} a_i(t_k)\boldsymbol{\phi}_i(\boldsymbol{x})$, $\widetilde{p}(\boldsymbol{x},\, t_k) = \sum_{i=1}^{N_r} a_i(t_k)\psi_i(\boldsymbol{x})$, and $\boldsymbol{R}(\boldsymbol{x},\, t_k)$.
3. Compute the POD modes $\boldsymbol{\Psi}(\boldsymbol{x})$ of the NSE residuals.
4. Add the K first residual modes $\boldsymbol{\Psi}(\boldsymbol{x})$ to the existing POD basis $\boldsymbol{\Phi}_i(\boldsymbol{x})$ and build a new ROM (here the mass and flow rate constraints are important).

Model $B^{[N_r;K]}$, **PODRES ROM** built with N_r POD basis functions $\boldsymbol{\Phi}_i(\boldsymbol{x})$
$+\ K$ RES basis functions $\boldsymbol{\Psi}_i(\boldsymbol{x})$

II - POD ROM stabilization

▶ **SUPG and VMS methods**

⇒ **Idea** approximate the fine scales using the NSE residuals $\boldsymbol{R} = (\boldsymbol{R}_M,\, R_C)^T$

$$\boldsymbol{u}'(\boldsymbol{x},\, t) = \tau_M\, \boldsymbol{R}_M(\boldsymbol{x},\, t) \quad \text{and} \quad p'(\boldsymbol{x},\, t) = \tau_C\, R_C(\boldsymbol{x},\, t)$$

↪ Class of penalty methods, *i.e.*

$$\sum_{j=1}^{N_r} L_{ij} \frac{da_j}{dt} = \sum_{j=1}^{N_r} B_{ij} a_j + \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} C_{ijk} a_j a_k + F_i(t)$$

- **Model $C^{[N_r]}$, SUPG method**

$$F_i^{SUPG}(t) = (\widetilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{\Phi}_i + \nabla \Psi_i,\, \tau_M\, \boldsymbol{R}_M(\boldsymbol{x},\, t))_\Omega + (\nabla \cdot \boldsymbol{\Phi}_i,\, \tau_C\, R_C(\boldsymbol{x},\, t))_\Omega$$

- **Model $D^{[N_r]}$, VMS method**

$$F_i^{VMS}(t) = F_i^{SUPG}(t) + (\widetilde{\boldsymbol{u}} \cdot (\nabla \boldsymbol{\Phi}_i)^T,\, \tau_M\, \boldsymbol{R}_M(\boldsymbol{x},\, t))_\Omega - (\nabla \boldsymbol{\Phi}_i,\, \tau_M\, \boldsymbol{R}_M(\boldsymbol{x},\, t) \otimes \tau_M\, \boldsymbol{R}_M(\boldsymbol{x},\, t))_\Omega$$

↪ Parameters τ_M and τ_C are determined using adjoint based minimization method

II - POD ROM stabilization

- $Re = 200$ and $N_r = 5$ POD basis function $\rightarrow$ erroneous limit cylcles

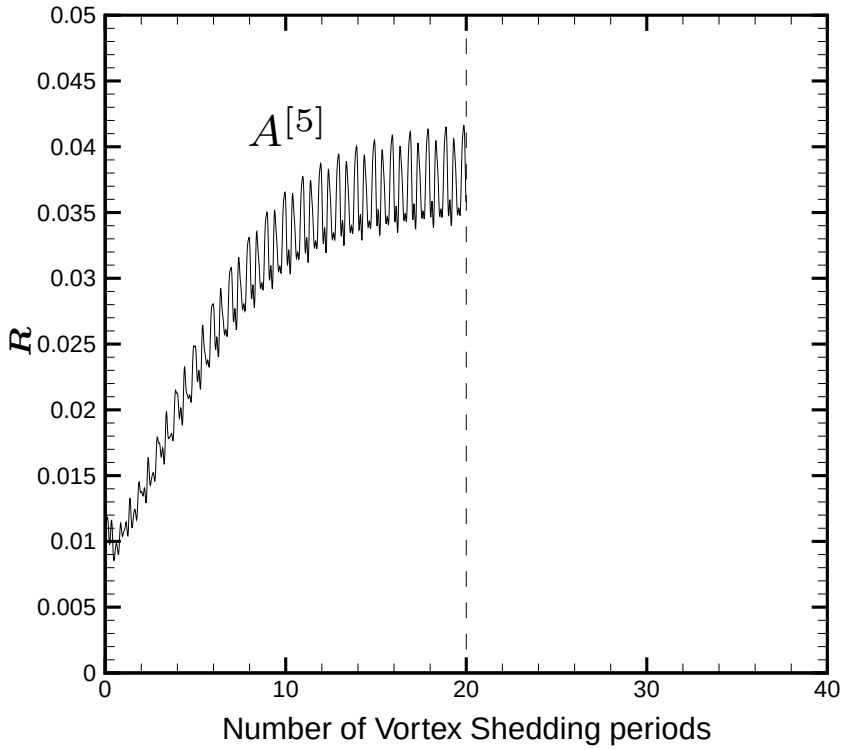
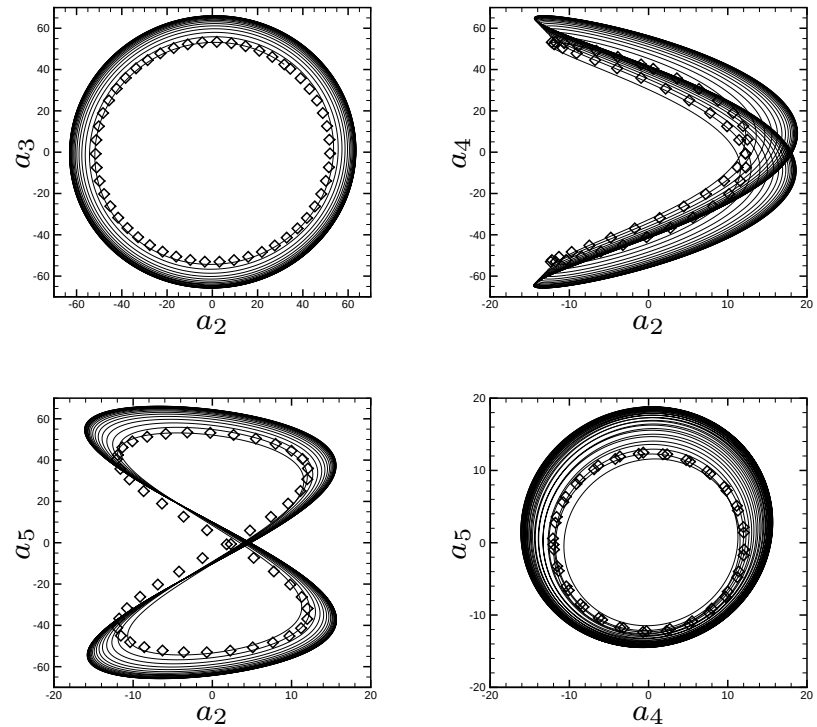


Fig. : *temporal evolution of the L_2 norm of the POD-NSE residuals*



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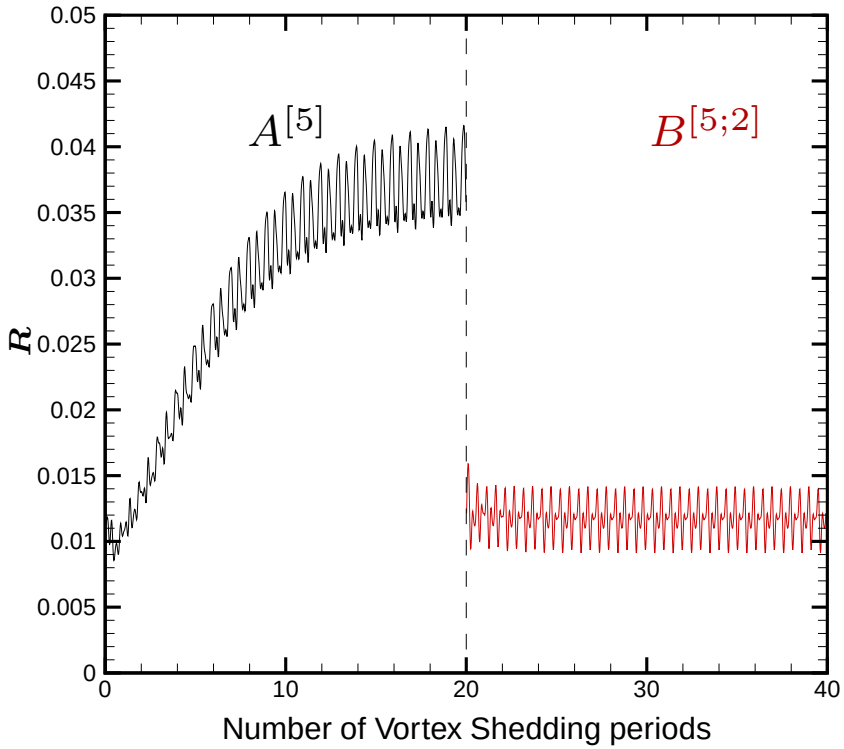

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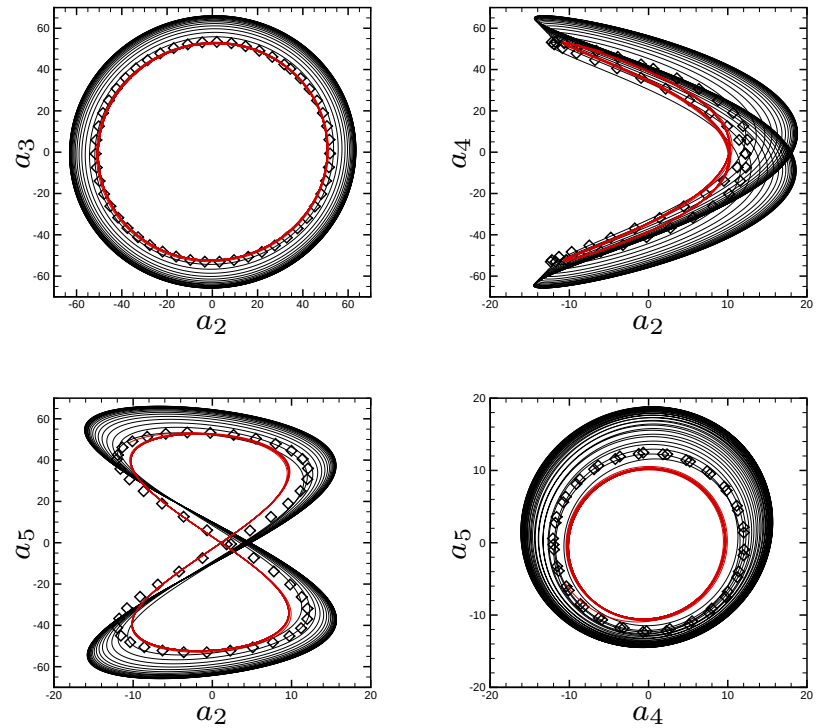

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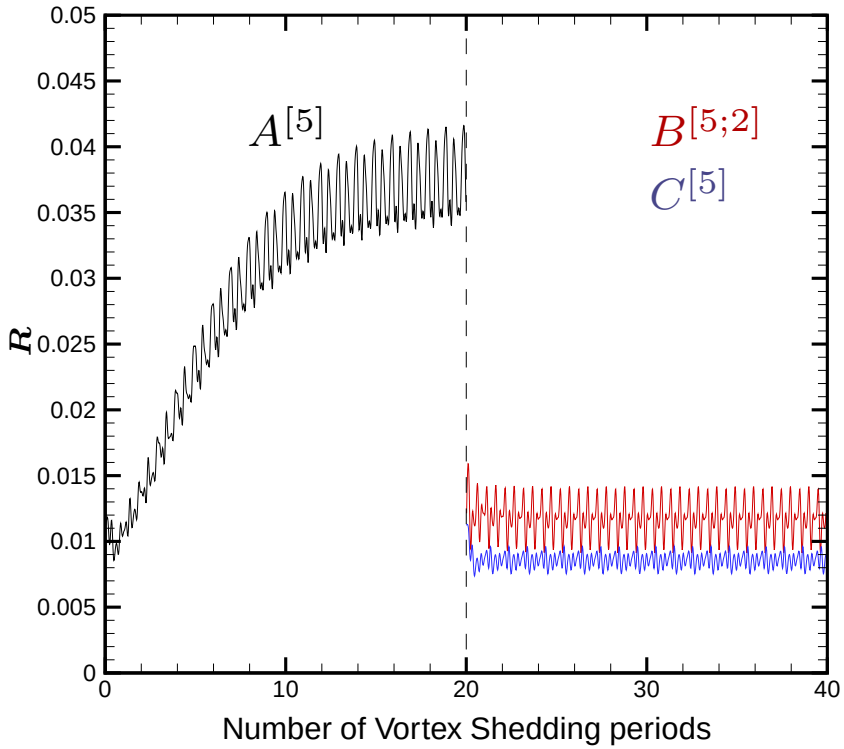

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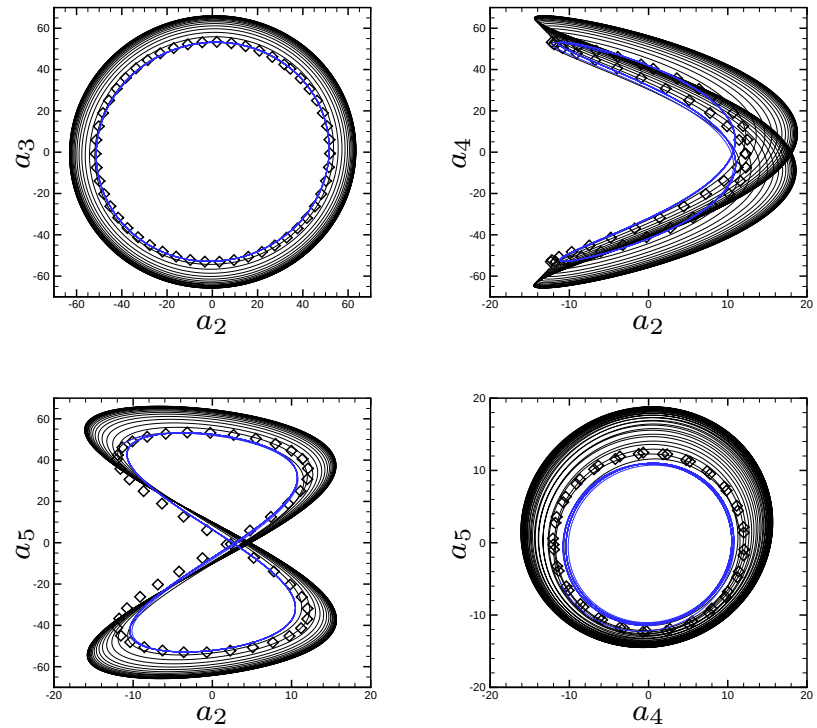

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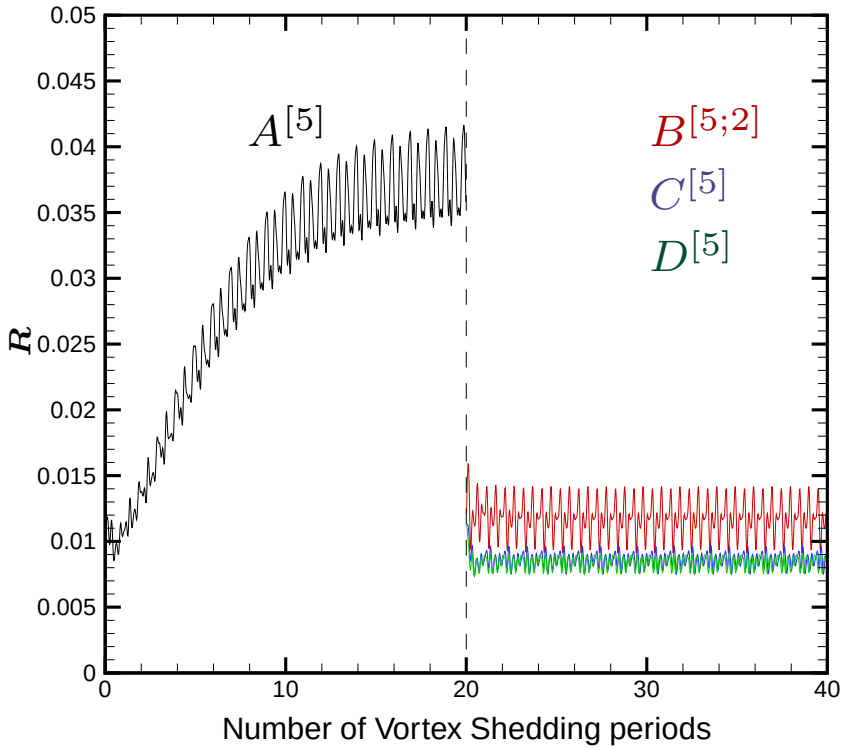

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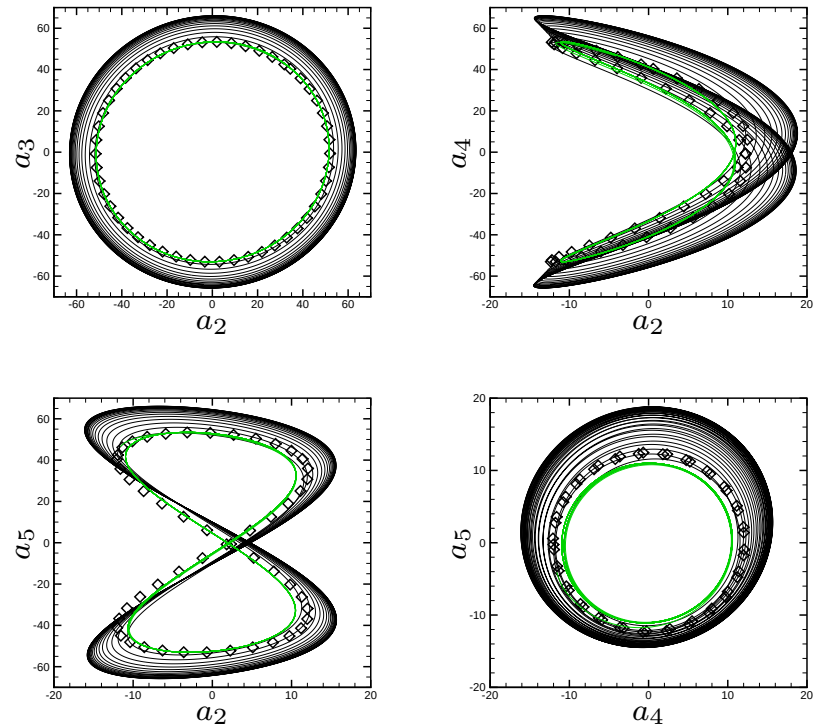

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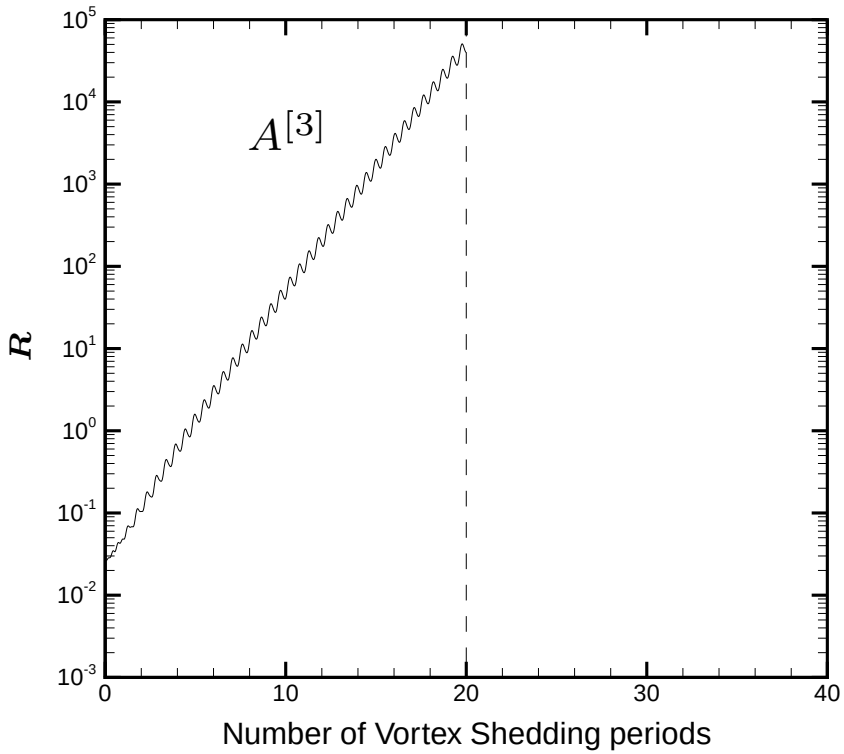
II - POD ROM stabilization

▶ $Re = 200$ and $N_r = 3$ POD basis function $\rightarrow$ divergence



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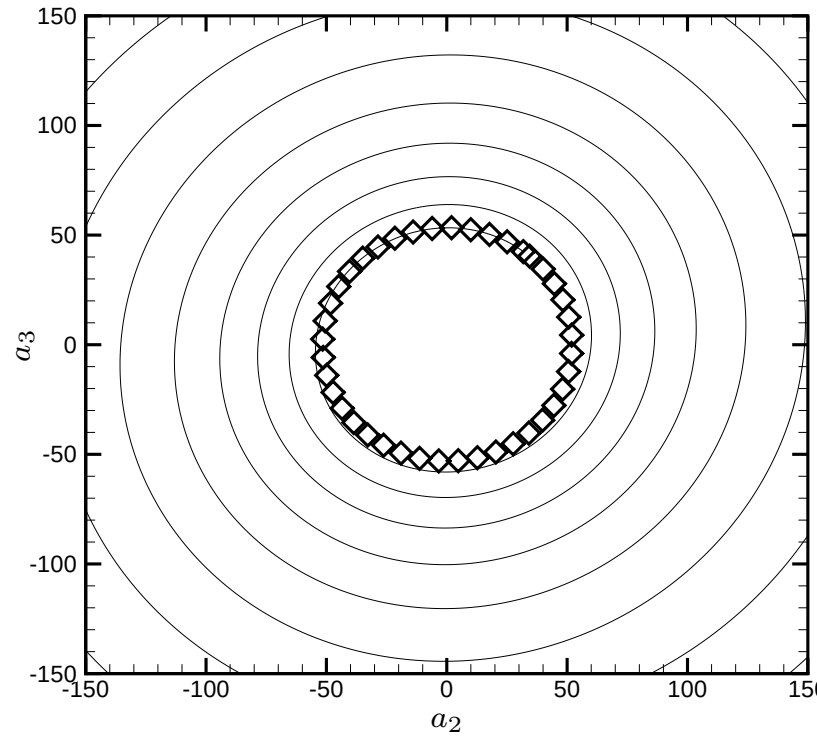

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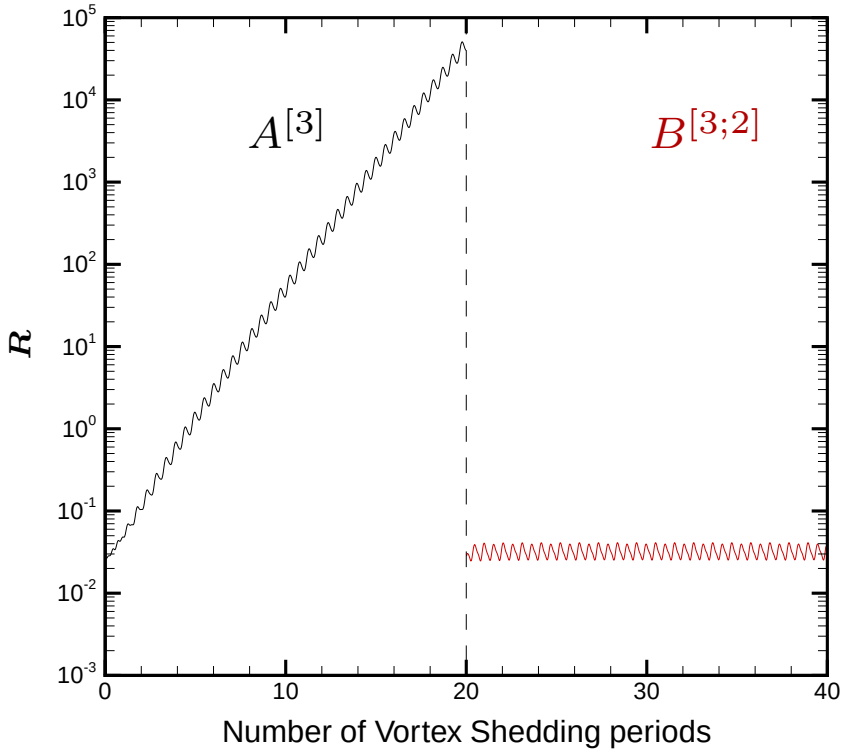

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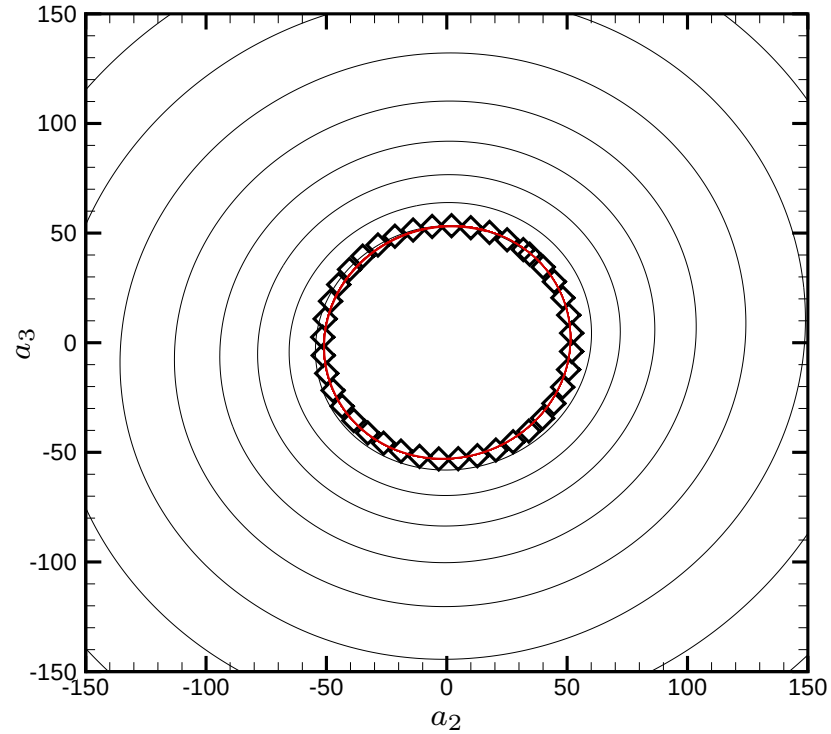

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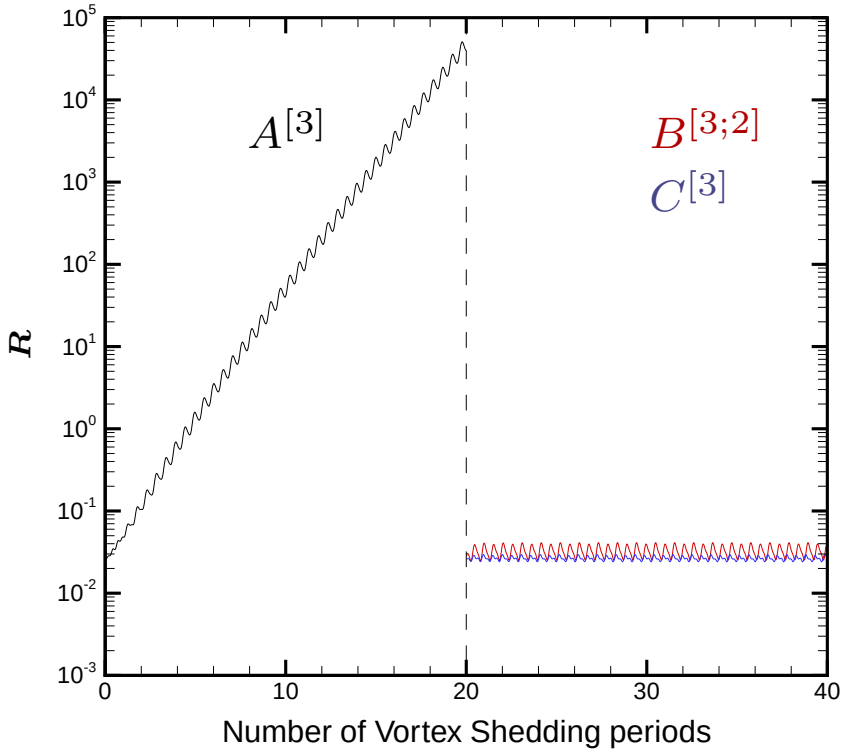

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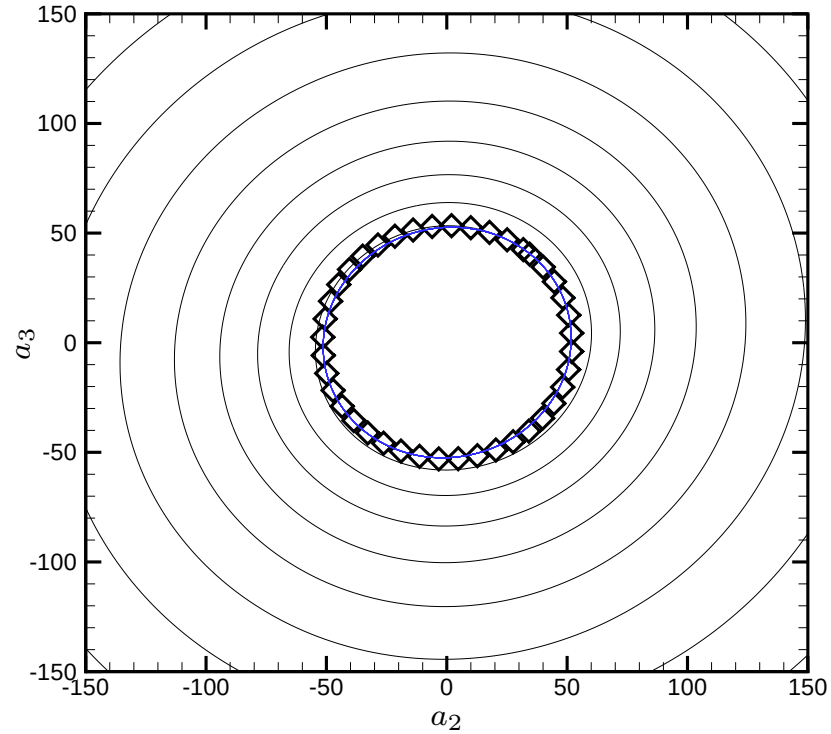

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II - POD ROM stabilization

▶ $Re = 200$ and $N_r = 3$ POD basis function $\rightarrow$ divergence

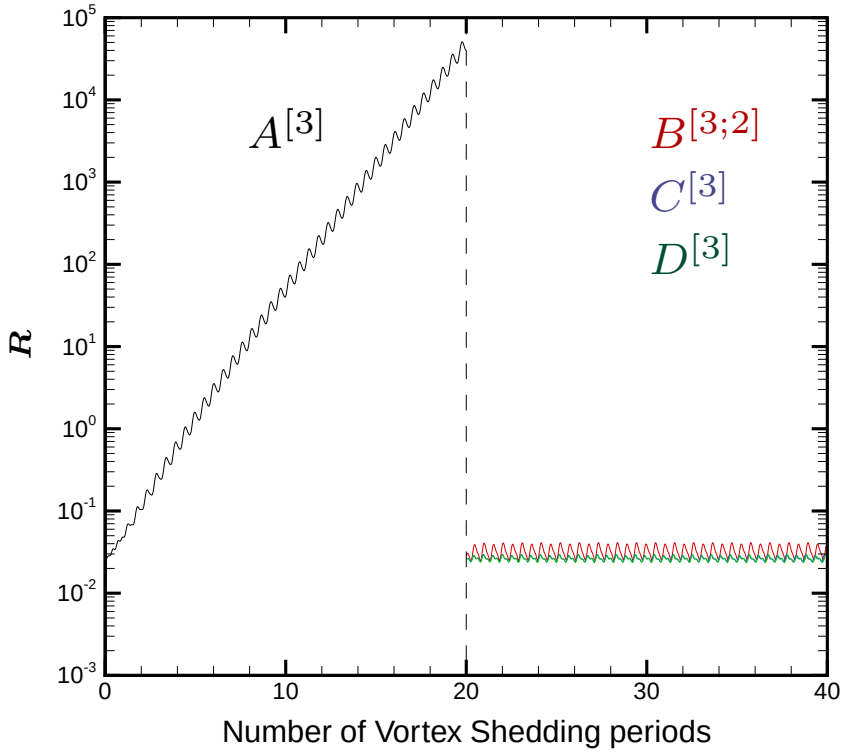


Fig. : *temporal evolution of the L_2 norm of the POD-NSE residuals*

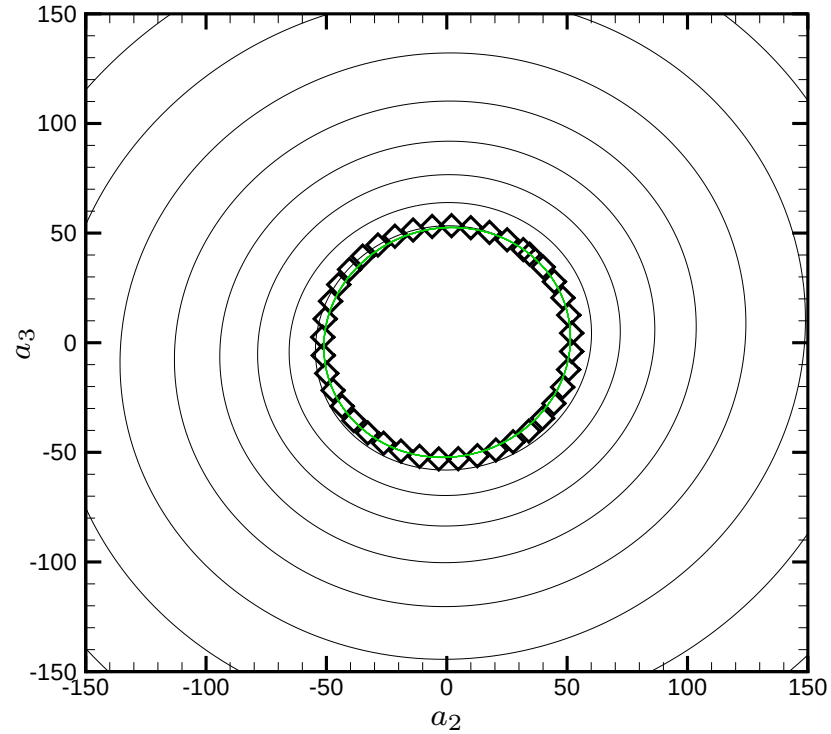
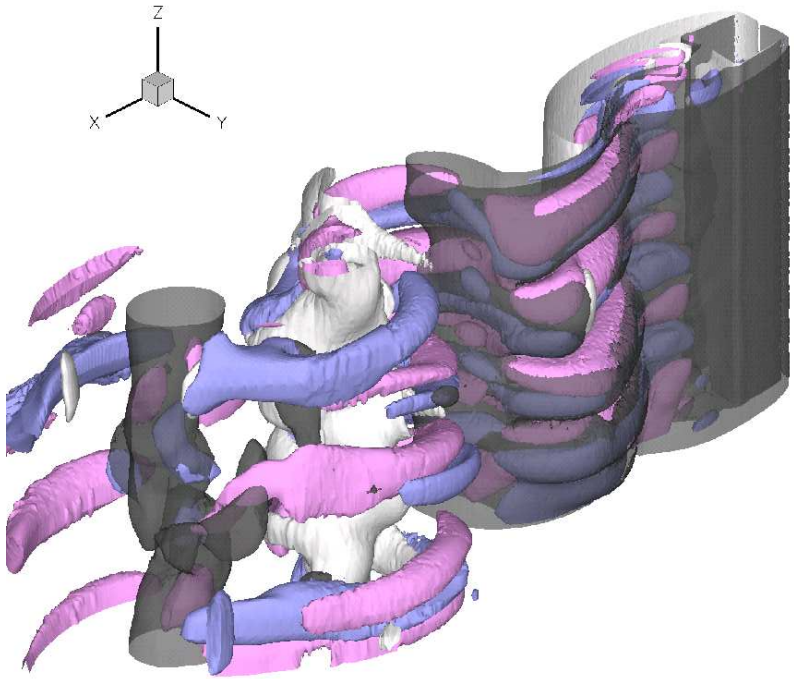
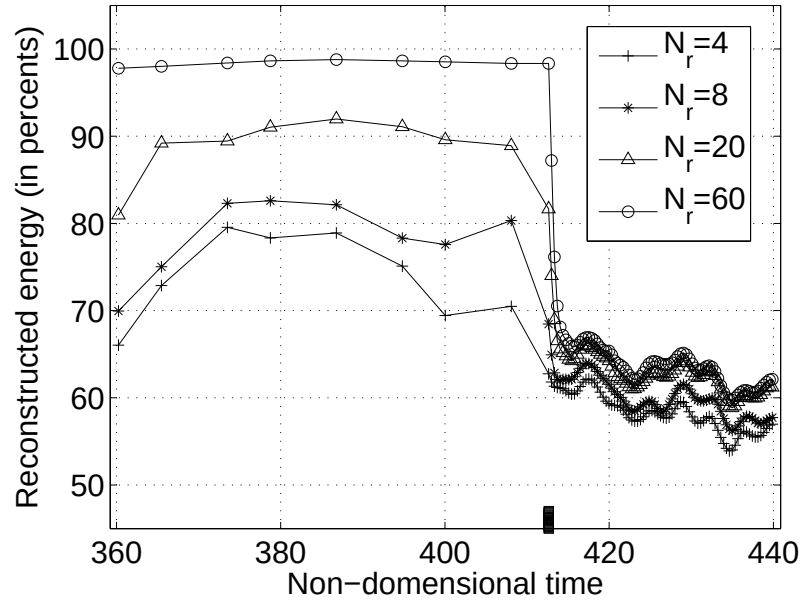


Fig. : *Limit cycle of the POD ROM coefficients over 20 vortex shedding periods*

III - Improvement of the functional subspace

▶ **Functional subspace drawbacks**, $\Phi_n(\boldsymbol{x})$: lack of representativity of 3D flows *outside the database*





Figures results from Buffoni *etal.* Journal of Fluid Mech. **569** (2006)

- Problems for 3D flow control
- Erroneous turbulence properties (spectrum, *etc*)

⇒ **Goal : determine $\Phi_n(\boldsymbol{x})$ at Re_2 starting from $\Phi_n(\boldsymbol{x})$ at Re_1 for low numerical costs.**

III - Improvement of the functional subspace

▷ **Method 1 : Krylov-like method to improve the functional subspace $\boldsymbol{\Phi}_n(\boldsymbol{x})$**

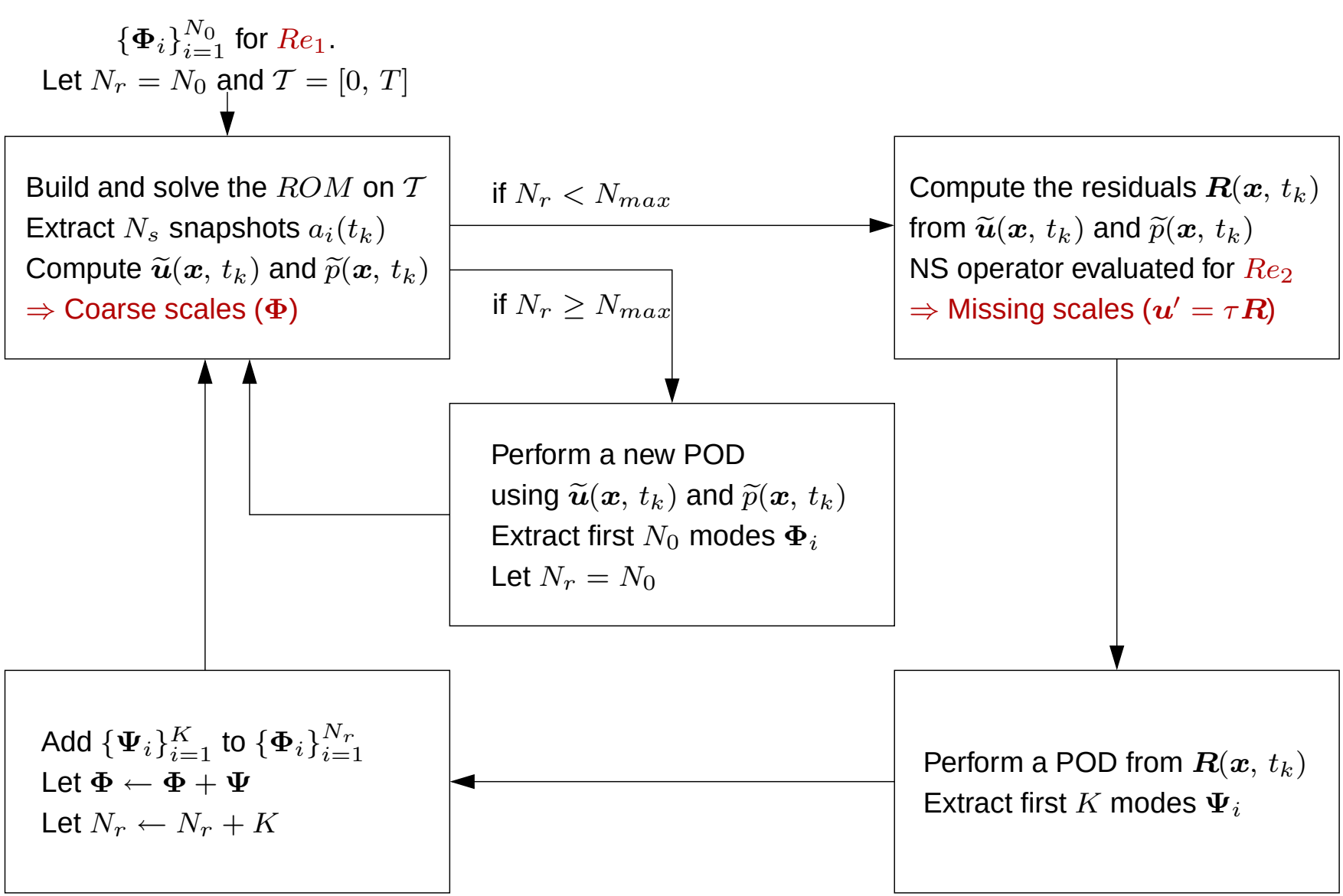
○ ***Use of the POD-NSE residuals :*** $\mathcal{L}(\widetilde{\boldsymbol{u}}(\boldsymbol{x},\, t),\, \widetilde{p}(\boldsymbol{x},\, t)) = \boldsymbol{R}(\boldsymbol{x},\, t)$,
$\widetilde{\boldsymbol{u}}$ and $\widetilde{p}$ are POD fields, $\mathcal{L}$ is the NSE operator

Algorithm

Start with the POD basis to be improved, $\boldsymbol{\Phi}_i$ with $i = 1, \ldots, N_r$. Let $N_0 = N_r$.

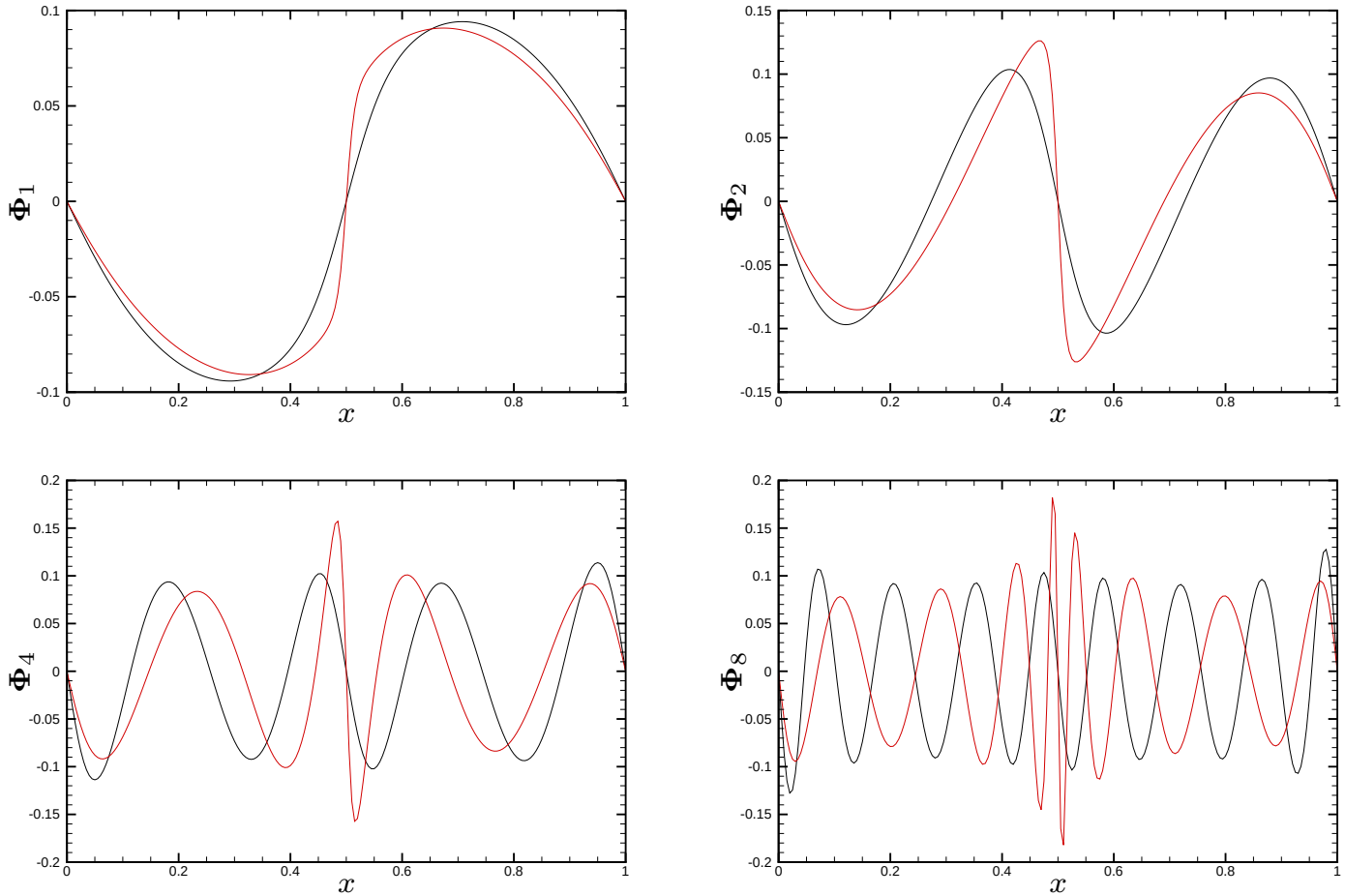
1. Build and solve the corresponding ROM to obtain $a_i(t)$ and extract N_s snapshots $a_i(t_k)$ with $i = 1, \ldots, N_r$ and $k = 1, \ldots, N_s$.
2. Compute $\widetilde{\boldsymbol{u}}(\boldsymbol{x},\, t_k) = \sum_{i=1}^{N_r} a_i(t_k)\boldsymbol{\phi}_i(\boldsymbol{x})$, $\widetilde{p}(\boldsymbol{x},\, t_k) = \sum_{i=1}^{N_r} a_i(t_k)\psi_i(\boldsymbol{x})$, and $\boldsymbol{R}(\boldsymbol{x},\, t_k)$.
3. Compute the POD modes $\boldsymbol{\Psi}(\boldsymbol{x})$ of the NSE residuals.
4. Add the K first residual modes $\boldsymbol{\Psi}(\boldsymbol{x})$ to the existing POD basis $\boldsymbol{\Phi}_i(\boldsymbol{x})$
   - $\boldsymbol{\Phi} \leftarrow \boldsymbol{\Phi} + \boldsymbol{\Psi}$
   - $N_r \leftarrow N_r + K$
   - If N_r is below than a threshold, return to 1. Else, go to 5.
5. Perform a new POD compression with $N_r = N_0$.
   - If convergence is satisfied, stop. Else, return to 1.

III - Improvement of the functional subspace



III - Improvement of the functional subspace

▶ **First test case :** 1D burgers equation $\underline{Re_1 = 50} \rightarrow \underline{Re_2 = 300}$ ($\mathbf{\Phi}^{Re_1} \cdot \mathbf{\Phi}^{Re_2} \approx 0.5$)



III - Improvement of the functional subspace

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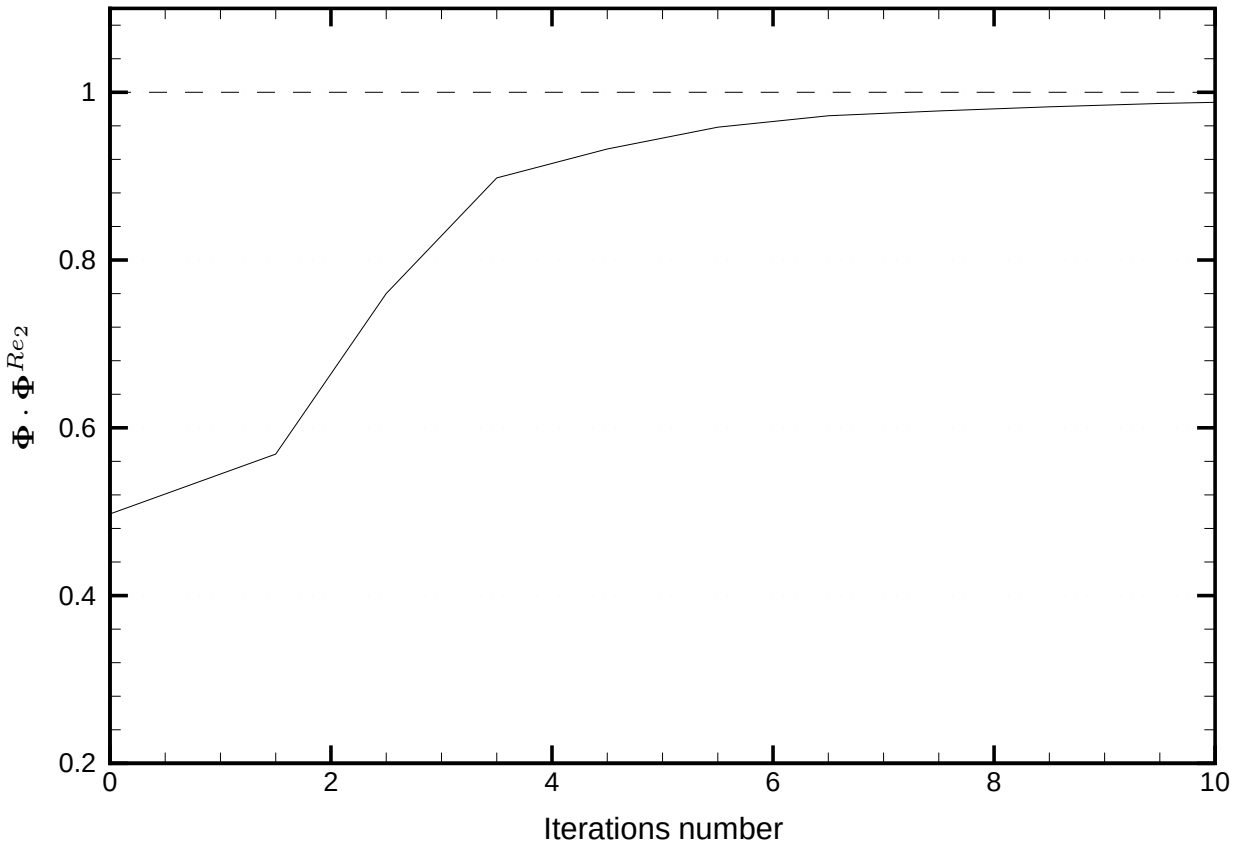
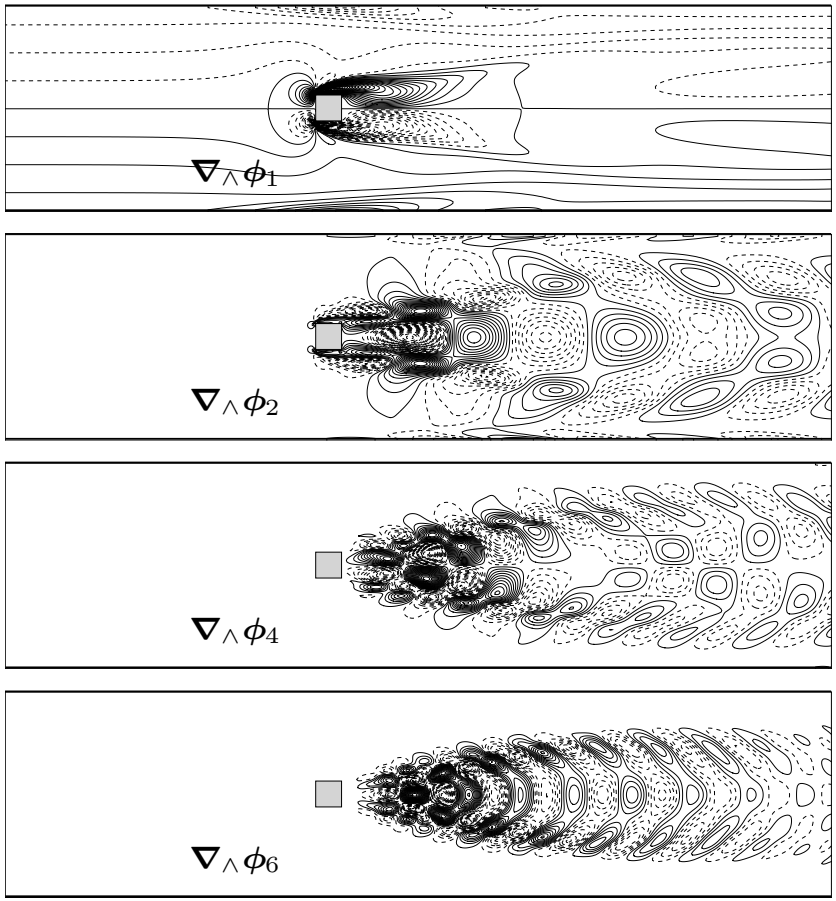


$\hookrightarrow$ Only 6 ROM integrations (on $T = 1$) are necessary to converge (no DNS ! !)

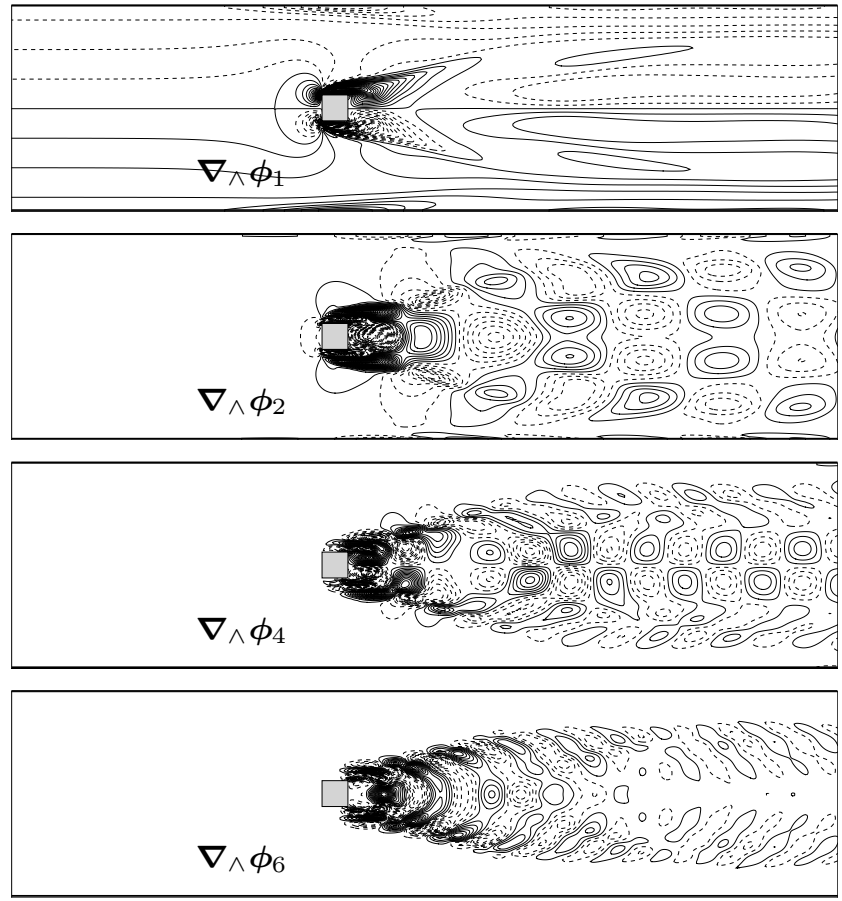
III - Improvement of the functional subspace

▶ **Second test case :** 2D NSE equations $\underline{Re_1 = 100} \rightarrow \underline{Re_2 = 200}$ ($\mathbf{\Phi}^{Re_1} \cdot \mathbf{\Phi}^{Re_2} \approx 0.5$)

Initial basis, $Re_1 = 100$

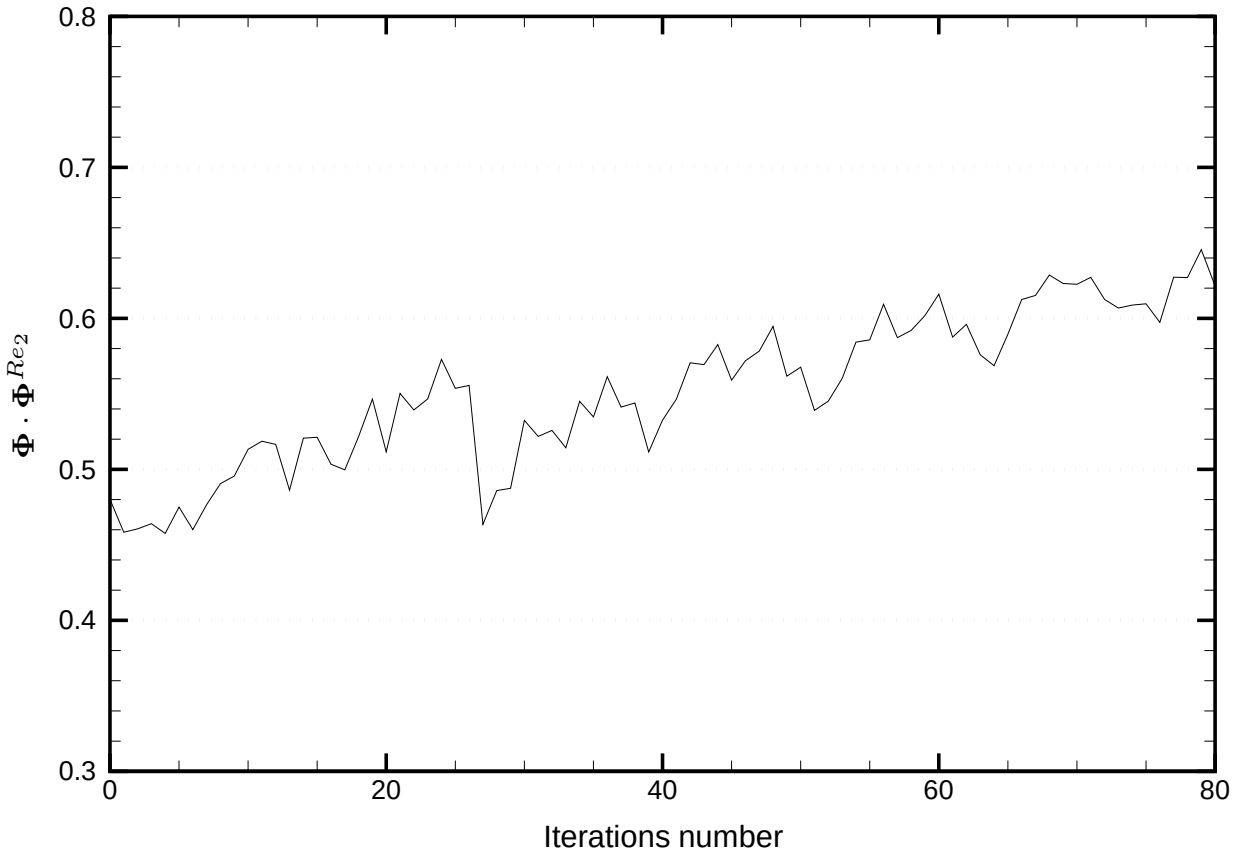


"Target" basis, $Re_2 = 200$



III - Improvement of the functional subspace

▶ **Second test case :** 2D NSE equations $\underline{Re_1 = 100} \rightarrow \underline{Re_2 = 200}$ ($\mathbf{\Phi}^{Re_1} \cdot \mathbf{\Phi}^{Re_2} \approx 0.5$)



↪ No convergence...

III - Improvement of the functional subspace

▶ **Observations**

- the decomposition $\boldsymbol{U}'(\boldsymbol{x},\, t) = \tau \boldsymbol{R}(\boldsymbol{x},\, t)$ is used to stabilize the ROM
  - ↪ Very good results for NSE
- the decomposition $\boldsymbol{U}'(\boldsymbol{x},\, t) = \tau \boldsymbol{R}(\boldsymbol{x},\, t)$ is used to improve POD basis
  - ↪ Very good results for Burgers, quite bad results for NSE

▶ **Possible explanation**

- the decomposition $\boldsymbol{U}'(\boldsymbol{x},\, t) = \tau \boldsymbol{R}(\boldsymbol{x},\, t)$ is only valid for
  - ↪ Small values of $\boldsymbol{U}'(\boldsymbol{x},\, t)$ (for instance, non resolved POD modes).
  - ↪ Can we find a good approximation τ of the elementary Green's function ? Not sure...

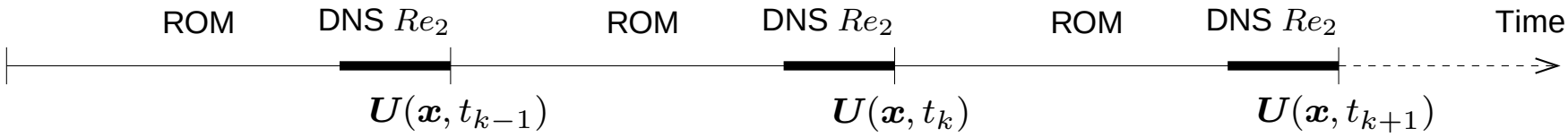
▶ **Future works**

- Look for an other decomposition for the missing scales $\boldsymbol{U}'(\boldsymbol{x},\, t)$
  - ↪ $\boldsymbol{U}'(\boldsymbol{x},\, t) = M(t)\boldsymbol{R}(\boldsymbol{x},\, t)$, where $M \in \mathbb{R}^{3\times 3}$

III - Improvement of the functional subspace

▷ **Method 2 : hybrid ROM-DNS method to adapt the functional subspace $\Phi_n(\boldsymbol{x})$**

○ ***Database modification :*** statistics evolution $\Rightarrow \varphi : \Phi^{(k)} \mapsto \Phi^{(k+1)}$



1. Database modification $[\boldsymbol{U}(\boldsymbol{x},\, t_1)\, \boldsymbol{U}(\boldsymbol{x},\, t_2)\, \ldots \boldsymbol{U}(\boldsymbol{x},\, t_{N_r})\,]$

$$\widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}(\boldsymbol{x}, t_k) = \sum_{n=1}^{N_r} a_n(t_k)\boldsymbol{\phi}_n(\boldsymbol{x}),$$

One snapshot modification using few DNS iterations

$$\boldsymbol{U}(\boldsymbol{x}, t_s) = \widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}(\boldsymbol{x}, t_s) + \boldsymbol{U}_s^{\perp}(\boldsymbol{x}, t_s).$$

In a general way

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}, t_k) = \widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}(\boldsymbol{x}, t_k) + \delta_{ks}\boldsymbol{U}^{\perp}(\boldsymbol{x}, t_s),$$

III - Improvement of the functional subspace

2 Modification temporal correlations tensor

$$C(t_k,t_l) = (\boldsymbol{U}(\boldsymbol{x},t_k),\boldsymbol{U}(\boldsymbol{x},t_l))_\Omega$$

$$= \left(\sum_{i=1}^{N_r} a_i(t_k)\boldsymbol{\phi}_i(\boldsymbol{x}) + \boldsymbol{U}^\perp(\boldsymbol{x},t_k), \sum_{j=1}^{N_r} a_j(t_l)\boldsymbol{\phi}_j(\boldsymbol{x}) + \boldsymbol{U}^\perp(\boldsymbol{x},t_l)\right)_\Omega$$

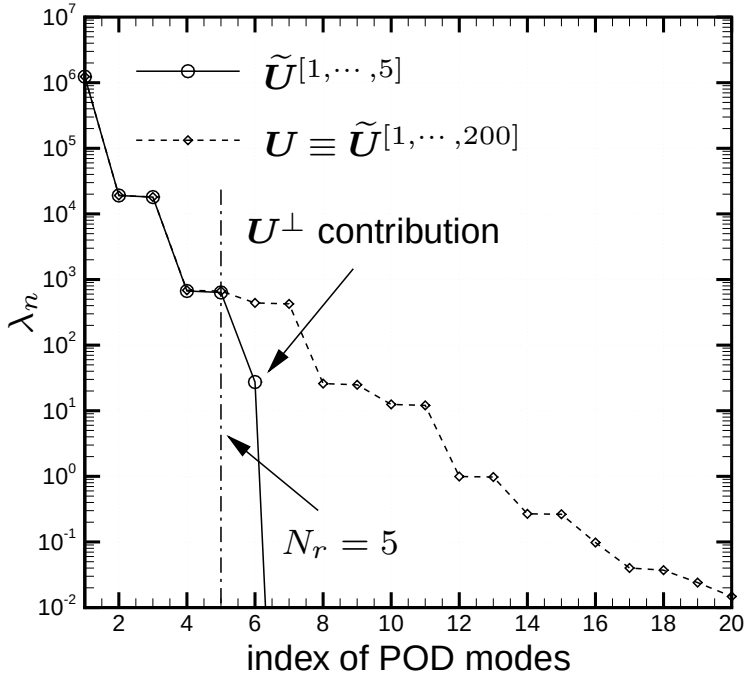
$$= \sum_{i=1}^{N_r}\sum_{j=1}^{N_r} a_i(t_k)a_j(t_l)\underbrace{(\boldsymbol{\phi}_i(\boldsymbol{x}),\boldsymbol{\phi}_j(\boldsymbol{x}))_\Omega}_{=\delta_{ij}} + \left(\boldsymbol{U}^\perp(\boldsymbol{x},t_k),\boldsymbol{U}^\perp(\boldsymbol{x},t_l)\right)_\Omega$$

$$+ \sum_{i=1}^{N_r} a_i(t_k)\underbrace{\left(\boldsymbol{\phi}_i(\boldsymbol{x}),\boldsymbol{U}^\perp(\boldsymbol{x},t_l)\right)_\Omega}_{=0} + \sum_{j=1}^{N_r} a_{lj}\underbrace{\left(\boldsymbol{U}^{\perp\,T}(\boldsymbol{x},t_k),\boldsymbol{\phi}_j(\boldsymbol{x})\right)_\Omega}_{=0}.$$

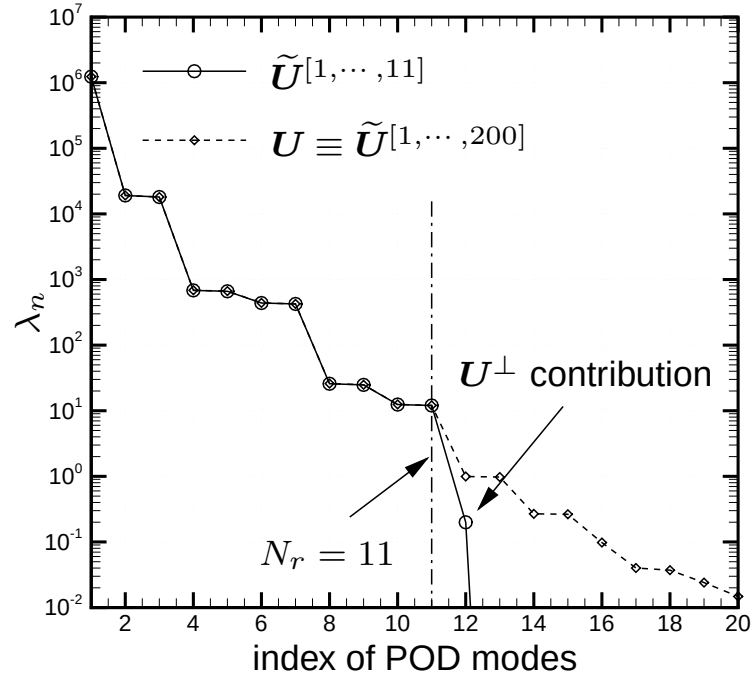
Final approximation

$$C(t_k,t_l) = \sum_{i=1}^{N_r} a_i(t_k)a_i(t_l) + \delta_{ks}\delta_{ls}\int_\Omega \sum_{i=1}^{n_c} U^{\perp\,i}(\boldsymbol{x},t_s)U^{\perp\,i}(\boldsymbol{x},t_s)\,d\boldsymbol{x}.$$

III - Improvement of the functional subspace



$N_r = 5$



$N_r = 11$

Fig. : *Comparison of the temporal correlation tensor eigenvalues evaluated from the exact field, $\boldsymbol{U}$, and from the N_r-modes approximated one, $\widetilde{\boldsymbol{U}}^{[1,\cdots,N_r]}$.*

$\hookrightarrow$ Very good approximation, and very low costs method !

III - Improvement of the functional subspace

3 Functional subspace adaptation

$$\boldsymbol{\phi}_k^{(n+1)}(\boldsymbol{x}) = \frac{1}{\lambda_k^{(n+1)}} \sum_{j=1}^{N_r} \widetilde{\boldsymbol{U}}^{(n)}(\boldsymbol{x}, t_j) a_k^{(n+1)}(t_j)$$

$$\boldsymbol{\phi}_k^{(n+1)}(\boldsymbol{x}) = \frac{1}{\lambda_k^{(n+1)}} \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} a_k^{(n+1)}(t_j) a_i^{(n)}(t_j)\, \boldsymbol{\phi}_i^{(n)}(\boldsymbol{x}) + \frac{1}{\lambda_k} \boldsymbol{U}^{\perp\,(n)}(\boldsymbol{x}, t_s) a_k^{(n+1)}(t_s).$$

$$\boldsymbol{\phi}_k^{(n+1)}(\boldsymbol{x}) = \sum_{i=1}^{N_r} K_{ki}^{(n+1)} \boldsymbol{\phi}_i^{(n)}(\boldsymbol{x}) + \boldsymbol{S}_k^{(n+1)}(\boldsymbol{x}).$$

Taken $S^{(n+1)}$ with elements $S_{ij}^{(n+1)} = S_i^{j\,(n+1)}$, the actualized basis is obtained using the linear application $\varphi : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n \times \mathbb{R}^n$ defined as

$$\boxed{\varphi : \phi^{(n)} \mapsto \phi^{(n+1)} = \phi^{(n)} K^{(n+1)} + S^{(n+1)}}$$

Incrementation $n = n + 1$.

III - Improvement of the functional subspace

▶ **Results for a dynamical evolution from** $Re_1 = 100$ **to** $Re_2 = 200$

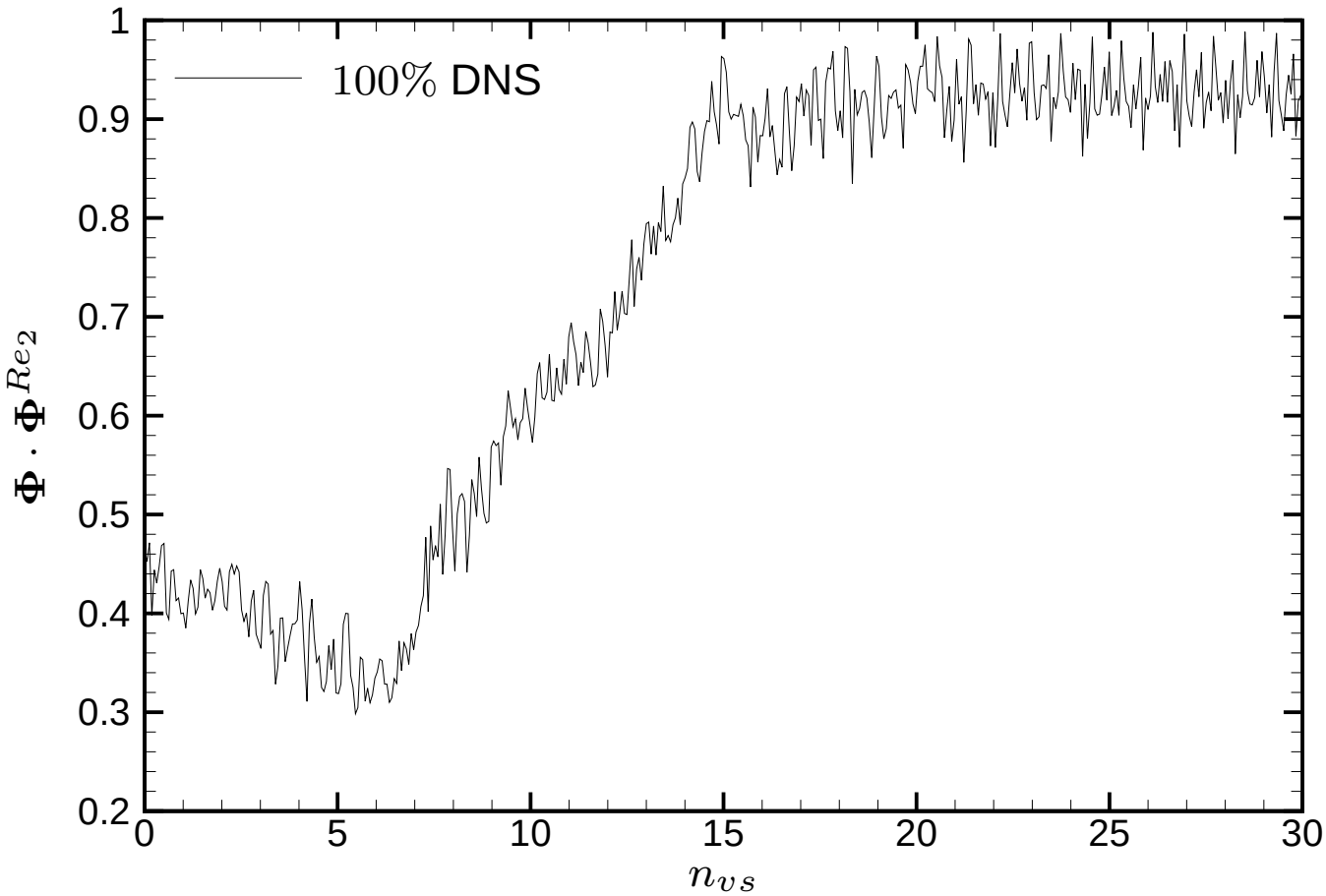


Fig. : *temporal evolution of the POD basis*

III - Improvement of the functional subspace

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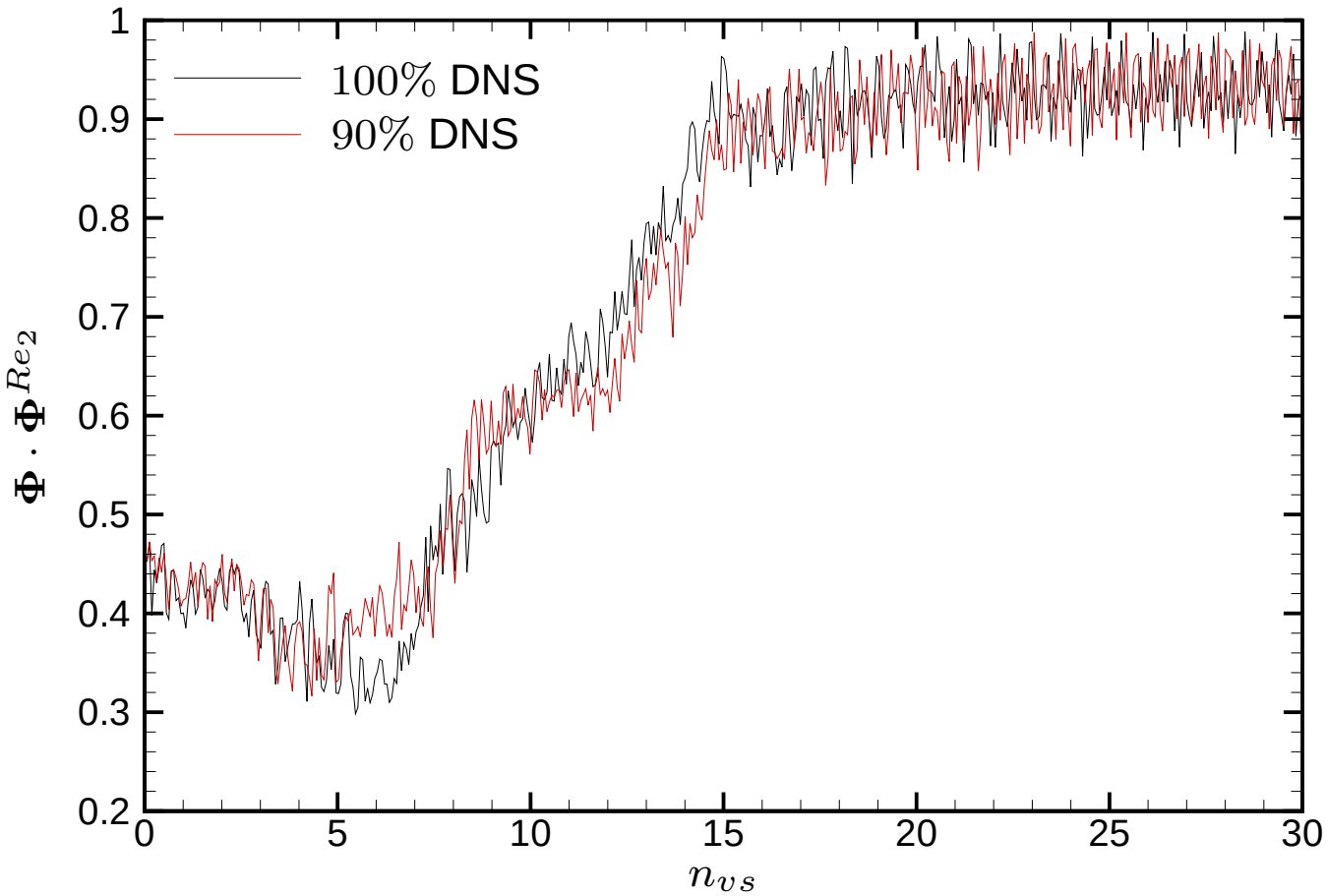

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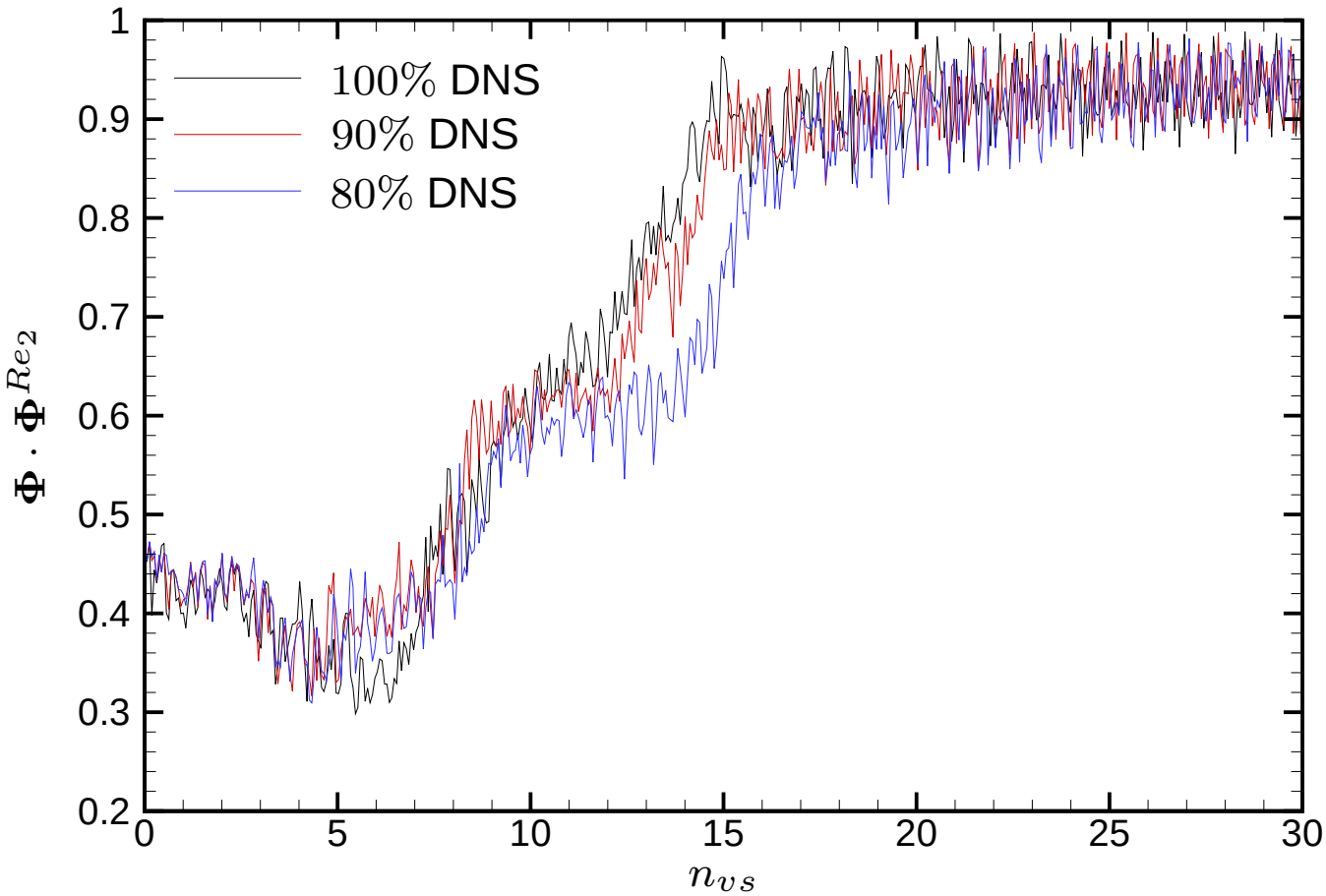

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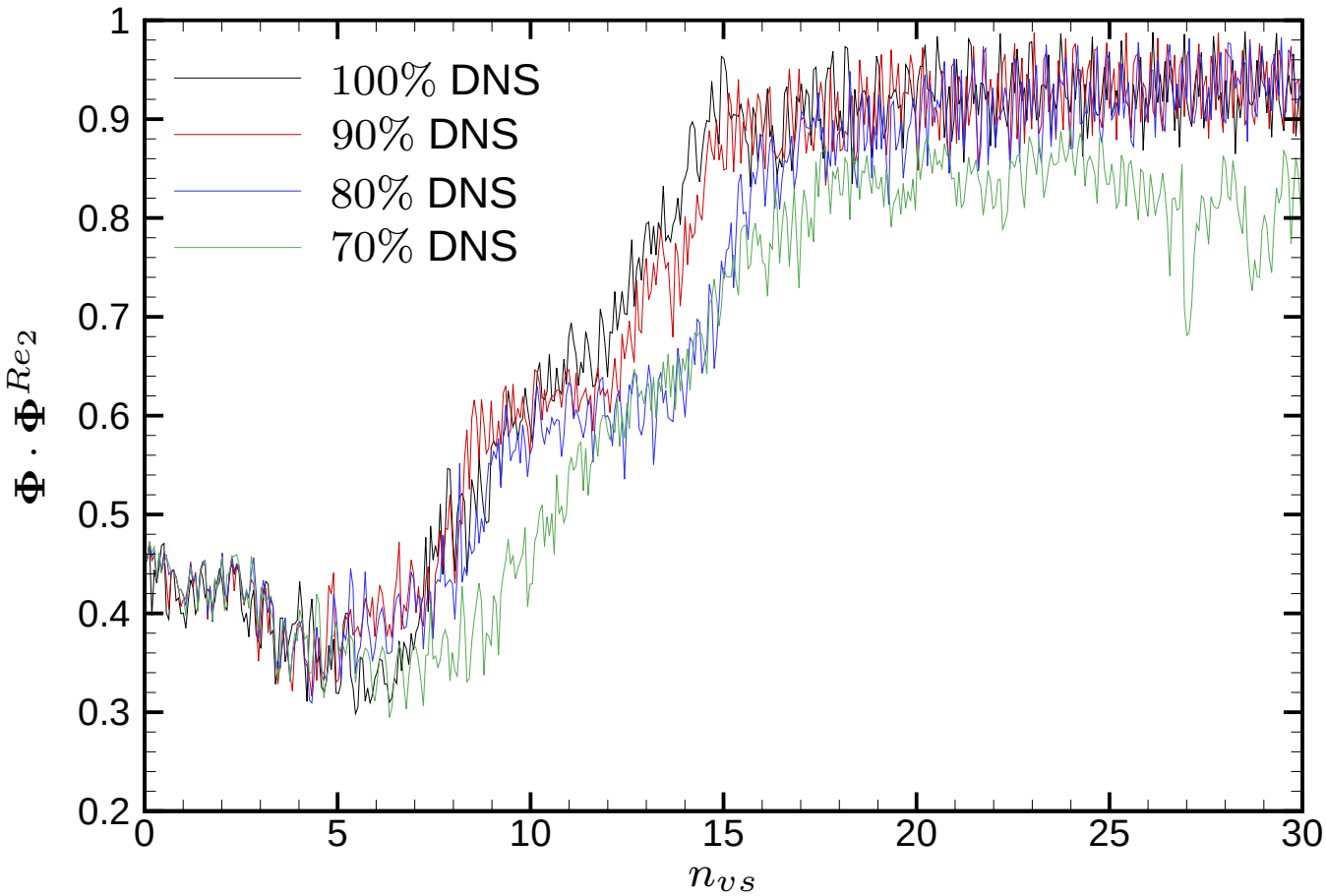

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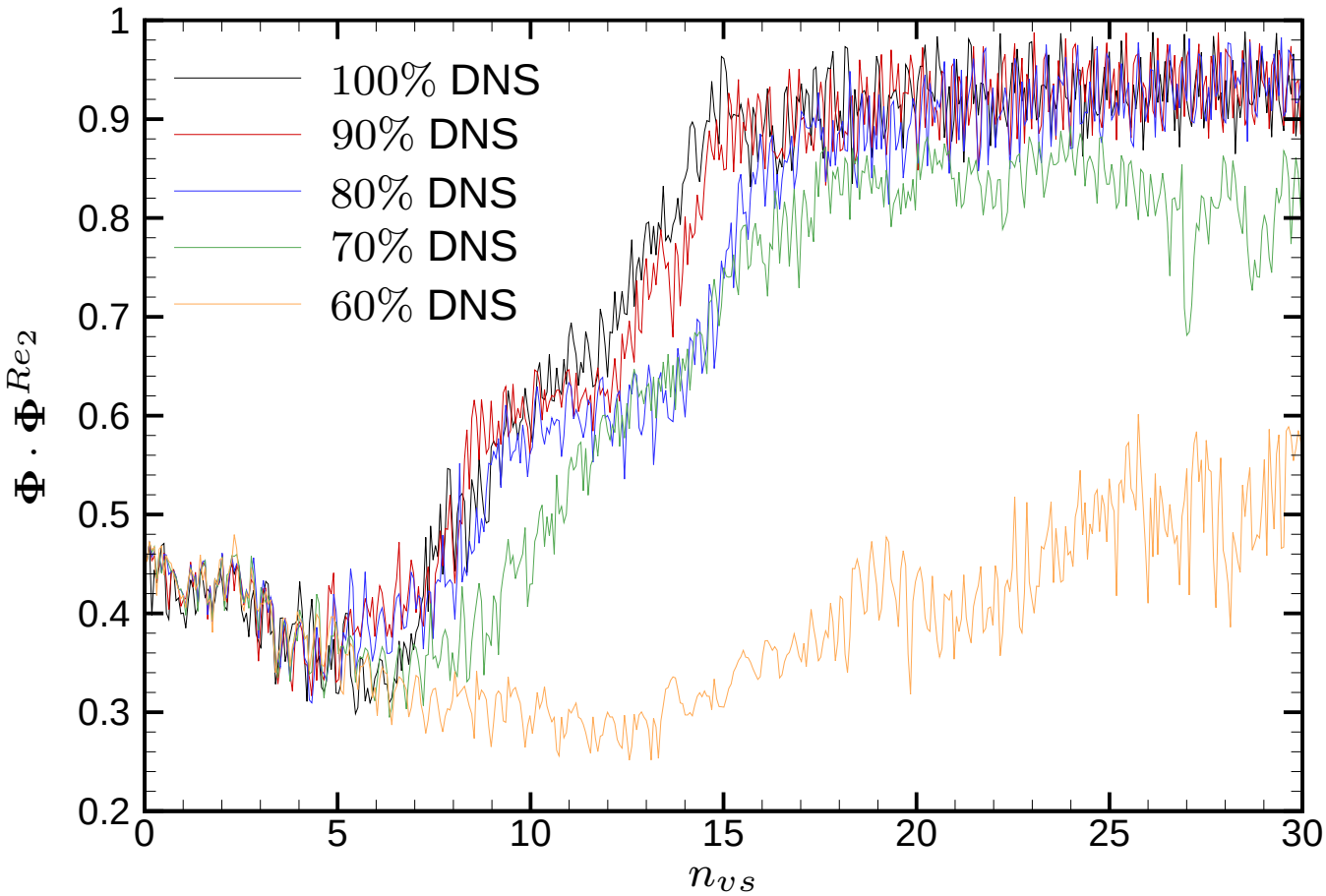


Fig. : *temporal evolution of the POD basis*

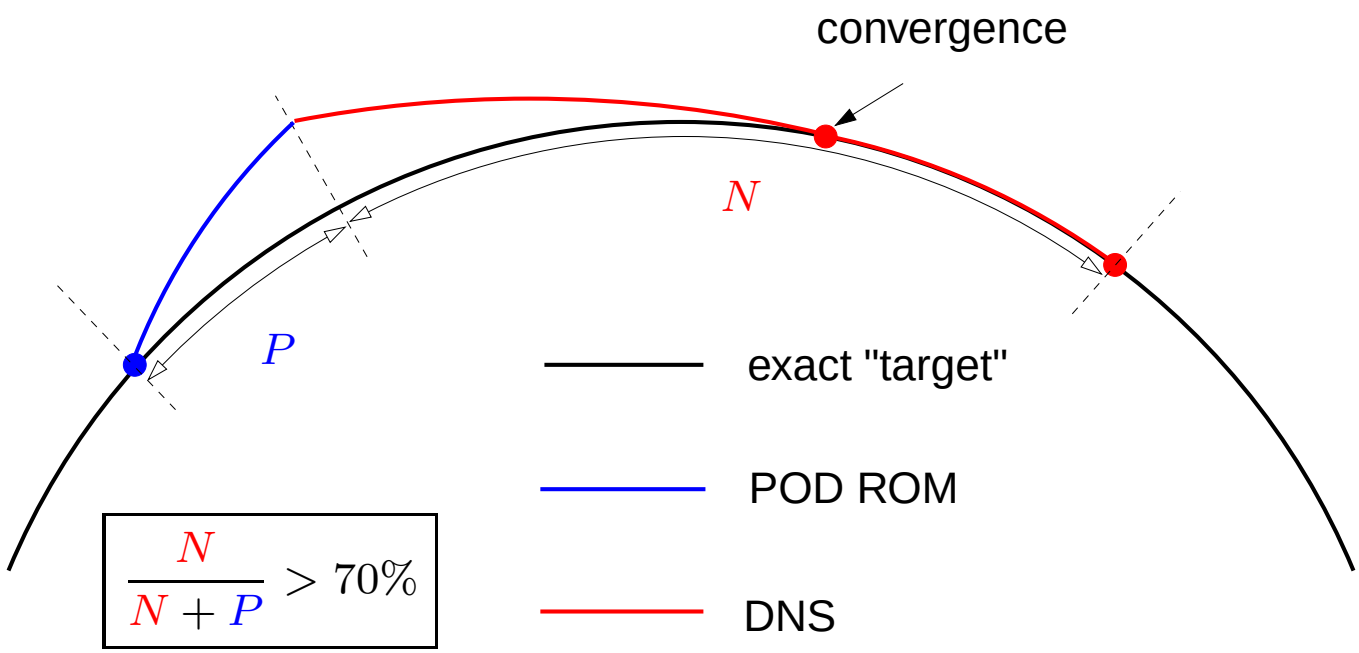
III - Improvement of the functional subspace

▶ **Observations**

Results are very good if a sufficient amount of DNS is performed

↪ Good for a percentage $\frac{DNS}{DNS+PODROM}$ greater than 70%

▶ **Possible explanation**



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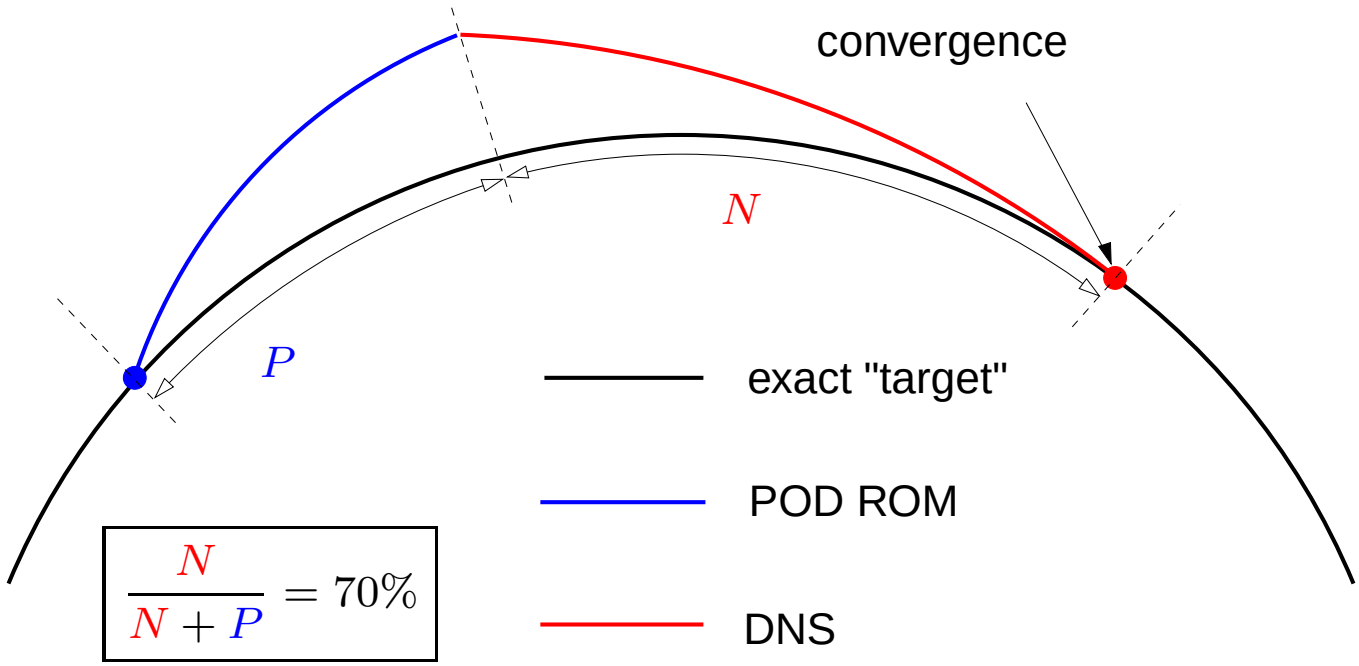

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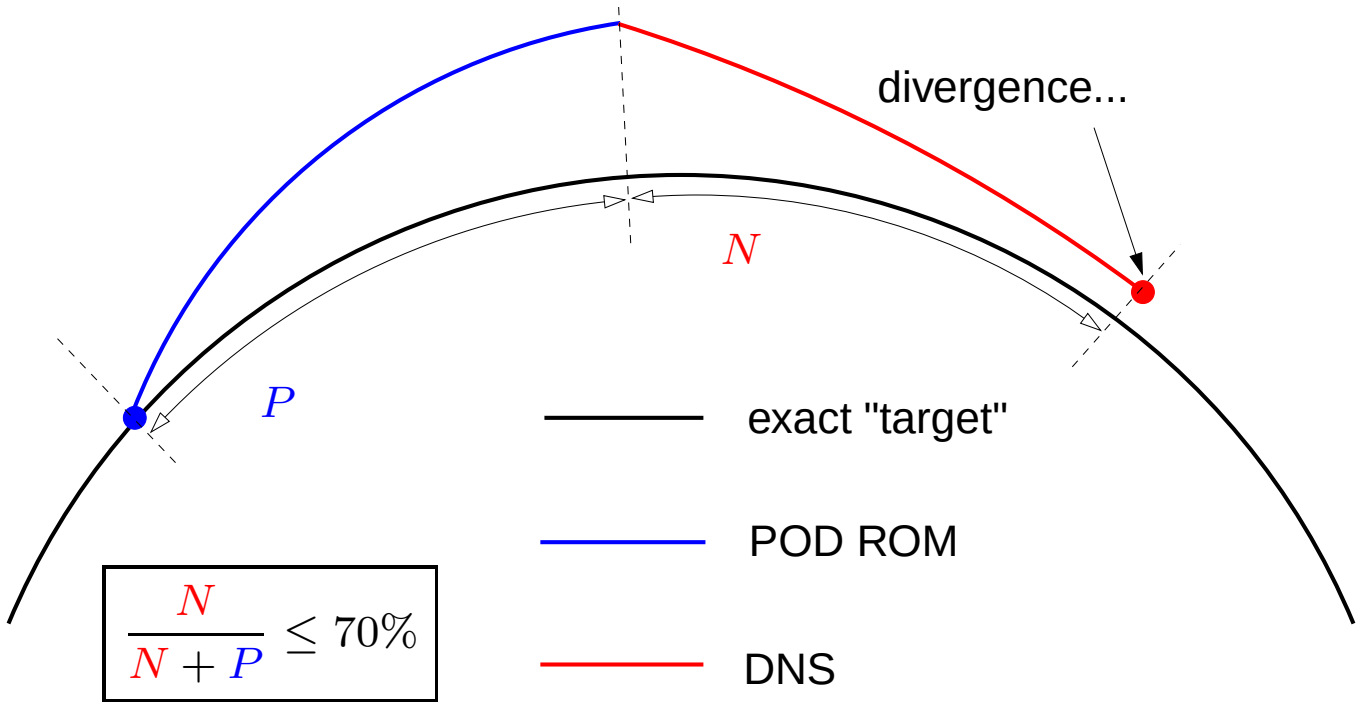

Conclusions

- **A pressure extended Reduced Order Model**
  - The pressure is naturally included in the ROM $\Rightarrow$ no modelisation of pressure term...
  - ... but need of modelisation interaction with non resolved modes (dissipation)
- **Stabilization of Reduced Order Models based on POD**
  - Add some residual modes $\Rightarrow$ Good results
  - SUPG and VMS methods $\Rightarrow$ Very good results
- **Try to improve the functional subspace**
  - Improvement using POD-NSE residuals (Krylov like method)
    - $\hookrightarrow$ Very good for 1D burgers equation but quite poor results for 2D NSE equations
    - $\hookrightarrow$ Problem with the continuity equation ?
    - $\hookrightarrow$ Missing scales $\neq$ "fine scales" $\Rightarrow$ approximation $\boldsymbol{U}'(\boldsymbol{x},\, t) = \tau \boldsymbol{R}(\boldsymbol{x},\, t)$ not good !
  - Database modification : an hybrid DNS/ROM method
    - $\hookrightarrow$ Fast evaluation of temporal correlations tensor
    - $\hookrightarrow$ Linear actualization of the POD basis
    - $\hookrightarrow$ DNS must correct ROM $\Rightarrow$ good results for amount of DNS greater than 70%