

# CODES AND DESIGNS IN GRASSMANNIAN SPACES.

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ABSTRACT. The notion of  $t$ -design in a Grassmannian space  $\mathcal{G}_{m,n}$  was introduced by the first two authors and G. Nebe in a previous paper. In the present work, we give a general lower bound for the size of such designs. The method is inspired by Delsarte, Goethals and Seidel work in the case of spherical designs. This leads us to introduce a notion of  $f$ -code in Grassmannian spaces, for which we obtain upper bounds, as well as a kind of duality tight-designs/tight-codes. The bounds are in terms of the dimensions of the irreducible representations of the orthogonal group  $O(n)$  occurring in the decomposition of the space  $L^2(\mathcal{G}_{m,n}^\circ)$  of square integrable functions on  $\mathcal{G}_{m,n}^\circ$ , the set of oriented grassmannians.

## 1. INTRODUCTION.

There are various combinatorial problems related to finite sets of Euclidean spheres. Among those, two, in a sense dual to each other, have received much attention, namely the notions of spherical  $t$ -design ( $t$  an integer), and spherical  $A$ -codes ( $A$  a finite set in  $[-1, 1]$ ). The notion of spherical design was motivated by numerical integration: a spherical  $t$ -design is a finite subset  $X$  of a sphere  $S^{d-1}$ , such that the integral over  $S^{d-1}$  of a polynomial function up to degree  $t$  coincides with its average value at the points of  $X$ . It is thus important, for instance for applications, to find designs with smallest possible cardinality. So the question of finding a *lower* bound for the size of a spherical design is central. As for  $A$ -codes, it is natural conversely to ask for an *upper* bound of their size : an  $A$ -code is a finite set in a sphere  $S^{d-1}$  such that the scalar products of pairwise distinct elements belong to a fixed set  $A \subset [-1, 1]$ . When  $A = [-1, 1/2]$ , finding an upper bound is equivalent to the kissing number problem, known as the problem of the thirteen spheres when  $n = 3$ . In their landmark paper [5], Delsarte, Goethals and Seidel proposed a general method, based on harmonic analysis on the orthogonal group, to study both questions.

The problem of packings, and related combinatorial questions, in the Grassmannian spaces  $\mathcal{G}_{m,n}$  of  $m$ -dimensional subspaces of  $\mathbb{R}^n$  have been investigated in a series of recent papers (see [3], [4]). In [1], a theory of designs was developed in that framework. One task of the present paper is to define a notion of  $f$ -code in Grassmannian spaces, which reduces to  $A$ -code

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when  $m = 1$  (the codes in the first Grassmannian  $\mathcal{G}_{1,n}$  are in one-to-one correspondance with the antipodal codes of the unit sphere). Then, inspired by Delsarte, Goethals and Seidel's works, we establish lower/upper bounds for the size of such designs/codes, which involve the dimensions of some irreducible representations of  $O(n)$ .

## 2. ZONAL FUNCTIONS ON GRASSMANNIAN SPACES.

Let  $\mathcal{G}_{m,n} \simeq O(n)/O(m) \times O(n-m)$  be the Grassmannian space of  $m$ -dimensional subspaces of  $\mathbb{R}^n$ . Recall (see [1], [11]), that the orbits under  $O(n)$  of pairs  $(p, q) \in \mathcal{G}_{m,n} \times \mathcal{G}_{m,n}$  are parametrized by the  $m$ -tuples

$$1 \geq t_1 \geq t_2 \geq \dots \geq t_m \geq 0.$$

Namely, to a couple  $(p, q)$  of  $m$ -dimensional subspaces, one associates the  $m$ -tuple  $t_1 = \cos \theta_1, \dots, t_m = \cos \theta_m$ , where  $0 \leq \theta_1 \leq \dots \leq \theta_m \leq \frac{\pi}{2}$  are the *principal angles* between  $p$  and  $q$ . One way to compute the  $t_i$ , is as follows : denoting by  $p_0$  the subspace generated by the first  $m$  vectors of the canonical basis of  $\mathbb{R}^n$ , and writing  $p = g \cdot p_0$ ,  $q = h \cdot p_0$ , with suitable  $g, h$  in  $O(n)$ , then the  $y_i := t_i^2$  are the eigenvalues of the  $m \times m$  symmetric matrix  $AA^t$ , where  $A$  is the  $m$ -size block appearing in the block-decomposition

$$(1) \quad g^{-1}h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Moreover,  $g$  and  $h$  are defined up to multiplication by an element in  $\text{Stab}(p_0)$ , and may be chosen so that  $B = D = 0$  and

$$(2) \quad A = \begin{pmatrix} \cos \theta_1 & 0 & \dots & 0 \\ 0 & \cos \theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \cos \theta_m \end{pmatrix}, C = \begin{pmatrix} \sin \theta_1 & 0 & \dots & 0 \\ 0 & \sin \theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sin \theta_m \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Besides  $\mathcal{G}_{m,n}$ , we have to consider the set  $\mathcal{G}_{m,n}^\circ$  of *oriented*  $m$ -dimensional subspaces of  $\mathbb{R}^n$ . We may view the elements of  $\mathcal{G}_{m,n}^\circ$  as couples  $\tilde{p} = (p, s)$ , with  $p$  an  $m$ -dimensional subspace, and  $s$  an element in  $\bigwedge^m p$ . The action of  $O(n)$  on these couples is given by

$$g \cdot (p, s) := (gp, gs),$$

so that if we fix an orientation  $s_0$  on  $p_0$ , the stabilizer of  $(p_0, s_0)$  identifies with  $SO(m) \times O(n-m)$ . Consequently

$$\mathcal{G}_{m,n}^\circ \simeq SO(n)/SO(m) \times SO(n-m) \simeq O(n)/SO(m) \times O(n-m),$$

which is a 2 to 1 covering of  $\mathcal{G}_{m,n}$ . The orbits under  $O(n)$  of pairs  $(\tilde{p}, \tilde{q}) \in \mathcal{G}_{m,n}^\circ \times \mathcal{G}_{m,n}^\circ$  can be likewise parametrized by  $(m+1)$ -tuples  $(\epsilon, t_1, \dots, t_m)$ ,

where  $t_1, \dots, t_m$  are defined as above, and  $\epsilon = \frac{\det A}{|\det A|} \in \{\pm 1\}$ , where  $A$  is as in (1). We still have a canonical block-decomposition like (2), but with top-left block

$$(2') \quad A = \begin{pmatrix} \epsilon \cos \theta_1 & 0 & \dots & 0 \\ 0 & \cos \theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \cos \theta_m \end{pmatrix}.$$

Both  $\mathcal{G}_{m,n}$  and  $\mathcal{G}_{m,n}^\circ$  inherit from the Haar measure of  $O(n)$ , a measure denoted  $dp$  and  $d\tilde{p}$  respectively. Since we will be mainly interested in non-oriented grassmanians, we normalize it so as  $\int_{\mathcal{G}_{m,n}} dp = 1$  (whence  $\int_{\mathcal{G}_{m,n}^\circ} d\tilde{p} = 2$ ). Accordingly, the space  $L^2(\mathcal{G}_{m,n}^\circ)$  of square integrable functions on  $\mathcal{G}_{m,n}^\circ$  is endowed with the scalar product

$$\langle f, g \rangle = \frac{1}{2} \int_{\mathcal{G}_{m,n}^\circ} f(\tilde{p}) \overline{g(\tilde{p})} d\tilde{p},$$

so that its restriction to  $L^2(\mathcal{G}_{m,n})$  satisfies

$$\langle f, g \rangle = \int_{\mathcal{G}_{m,n}} f(p) \overline{g(p)} dp, \quad f \in L^2(\mathcal{G}_{m,n}), g \in L^2(\mathcal{G}_{m,n}).$$

The group  $O(n)$  acts isometrically on  $L^2(\mathcal{G}_{m,n}^\circ)$  by

$$\sigma \cdot f(\tilde{p}) = f(\sigma^{-1}\tilde{p}).$$

The structure of  $L^2(\mathcal{G}_{m,n}^\circ)$  as an  $O(n)$ -module is well-known, and is given for instance in [9], p. 546. To be precise, if we consider the subset  $\mathcal{R}(\mathcal{G}_{m,n}^\circ)$  of regular functions on  $\mathcal{G}_{m,n}^\circ$  (i.e. the set of functions induced by regular functions on  $O(n)$ ) which is a dense subset of  $L^2(\mathcal{G}_{m,n}^\circ)$ , we have the following decomposition

$$\mathcal{R}(\mathcal{G}_{m,n}^\circ) = \bigoplus H_{m,n}^\mu$$

in pairwise orthogonal non isomorphic irreducible  $O(n)$ -submodules  $H_{m,n}^\mu$ , the sum being over partitions  $\mu = \mu_1 \geq \mu_2 \geq \dots \mu_m \geq 0$ , of depth at most  $m$ , with  $\mu_i \equiv \mu_j \pmod{2}$  for all  $(i, j)$ . We call these partitions  $m$ -admissible, or *admissible* for short. They split into *odd* and *even*, according to the parity of the  $\mu_i$ .

**Remark 2.1.** *For a given even partition  $\mu$ , the admissibility does not depend on  $m$ , as long as  $\text{depth}(\mu) \leq m \leq \frac{n}{2}$ , whereas for  $\mu$  odd it does, since in that case the  $\mu_i$  have to be nonzero for all  $1 \leq i \leq m$ .*

It turns out that the  $O(n)$ -isomorphism class of  $H_{m,n}^\mu$  is independent of  $m$ , provided that  $\text{depth}(\mu) \leq m \leq \frac{n}{2}$  and  $\mu$  is  $m$ -admissible (see [1] for a more detailed description of  $H_{m,n}^\mu$ ). The space  $H_{m,n}^\mu$  is isomorphic to the irreducible representation of  $O(n)$  canonically associated to the partition  $\mu$ , and denoted  $V_n^\mu$  in [9]. We shall denote the dimension of this space  $d_\mu$ .

The sum over *even*, resp. *odd*, partitions corresponds respectively to  $\mathcal{R}(\mathcal{G}_{m,n})$  and its orthogonal complement  $\mathcal{R}(\mathcal{G}_{m,n})^\perp$ . This also corresponds to the eigenspace decomposition of  $\mathcal{R}(\mathcal{G}_{m,n}^\circ)$  with respect to the canonical involution  $\sigma^*$  induced by orientation changing, namely

$$\mathcal{R}(\mathcal{G}_{m,n}) = \mathcal{R}(\mathcal{G}_{m,n}^\circ)^+ := \{f \in \mathcal{R}(\mathcal{G}_{m,n}^\circ) \mid \sigma^*(f) = f\},$$

and

$$\mathcal{R}(\mathcal{G}_{m,n})^\perp = \mathcal{R}(\mathcal{G}_{m,n}^\circ)^- := \{f \in \mathcal{R}(\mathcal{G}_{m,n}^\circ) \mid \sigma^*(f) = -f\}.$$

Let  $\mathcal{Z}^\circ$  (resp.  $\mathcal{Z}$ ) be the set of  $O(n)$ -invariant functions  $F$  on  $\mathcal{G}_{m,n}^\circ \times \mathcal{G}_{m,n}^\circ$  (resp.  $\mathcal{G}_{m,n} \times \mathcal{G}_{m,n}$ ), such that

$$(3) \quad F(\tilde{p}, \cdot) \in \mathcal{R}(\mathcal{G}_{m,n}^\circ), F(\cdot, \tilde{q}) \in \mathcal{R}(\mathcal{G}_{m,n}^\circ) \text{ for all } (\tilde{p}, \tilde{q}) \in \mathcal{G}_{m,n}^\circ \times \mathcal{G}_{m,n}^\circ,$$

resp.

$$(3') \quad F(p, \cdot) \in \mathcal{R}(\mathcal{G}_{m,n}), F(\cdot, q) \in \mathcal{R}(\mathcal{G}_{m,n}) \text{ for all } (p, q) \in \mathcal{G}_{m,n} \times \mathcal{G}_{m,n}.$$

As usual, we call such functions *zonal*. Alternatively, if a base point  $\tilde{p}$  is fixed, one can identify  $\mathcal{Z}^\circ$  with  $\mathcal{R}(\mathcal{G}_{m,n}^\circ)^{\text{Stab}(\tilde{p})}$ , mapping  $F \in \mathcal{Z}^\circ$  on  $F(\tilde{p}, \cdot) \in \mathcal{R}(\mathcal{G}_{m,n}^\circ)^{\text{Stab}(\tilde{p})}$ , and similarly  $\mathcal{Z}$  identifies with  $\mathcal{R}(\mathcal{G}_{m,n})^{\text{Stab}(p)}$

As explained in [1], [11], it follows from the Frobenius reciprocity theorem that  $H_{m,n}^{\mu, \text{Stab}(\tilde{p})}$  is one-dimensional for any  $\mu$  (if  $\mu$  is even, then  $H_{m,n}^\mu \subset \mathcal{R}(\mathcal{G}_{m,n}) = \mathcal{R}(\mathcal{G}_{m,n}^\circ)^+$ , so that  $H_{m,n}^{\mu, \text{Stab}(\tilde{p})} = H_{m,n}^{\mu, \text{Stab}(p)}$ ). Consequently, to each summand  $H_{m,n}^\mu$  is attached a unique (up to scaling) *zonal function*  $P_\mu$ , which can be computed in the following way: denoting by  $d_\mu$  the dimension of  $H_{m,n}^\mu$ , and  $\{e_{\mu,i}\}_{1 \leq i \leq d_\mu}$  an orthonormal basis of it, one has

$$(4) \quad P_\mu(\tilde{p}, \tilde{p}') := \frac{1}{d_\mu} \sum_{i=1}^{d_\mu} e_{\mu,i}(\tilde{p}) \overline{e_{\mu,i}(\tilde{p}')}$$

The results of the next section rely on the following properties of the  $P_\mu$ :

**Lemma 2.2.**

- (i)  $P_\mu(\tilde{p}, \tilde{p}) = 1$ , for any  $\mu$  and  $\tilde{p}$ .
- (ii) For any  $\lambda, \mu$  and  $\tilde{p}, \tilde{p}'$ , one has

$$(5) \quad \langle P_\mu(\tilde{p}, \cdot), P_\lambda(\tilde{p}', \cdot) \rangle = \frac{\delta_{\lambda,\mu}}{d_\mu} P_\mu(\tilde{p}, \tilde{p}'),$$

if, for fixed  $\tilde{p}$ , we view the map  $\tilde{q} \mapsto P_\mu(\tilde{p}, \tilde{q})$  as a function in  $L^2(\mathcal{G}_{m,n}^\circ)$ . In particular, for any finite set  $X \subset \mathcal{G}_{m,n}^\circ$ , the matrix  $(P_\mu(\tilde{p}, \tilde{p}'))_{\tilde{p}, \tilde{p}' \in X^2}$  is positive semidefinite.

- (iii) For any  $\lambda, \mu$ , one has

$$(6) \quad P_\lambda P_\mu = \sum_{\tau} c_{\lambda,\mu}(\tau) P_\tau,$$

with non-negative coefficients  $c_{\lambda,\mu}(\tau)$ . In particular,  $c_{\lambda,\mu}(0) = \frac{\delta_{\lambda,\mu}}{d_\mu}$ .

*Proof.* Since  $P_\mu(\tilde{p}, \tilde{p})$  does not depend on  $\tilde{p}$ , one has

$$\begin{aligned} P_\mu(\tilde{p}, \tilde{p}) &= \frac{1}{2} \int_{\mathcal{G}_{m,n}^\circ} P_\mu(\tilde{p}, \tilde{p}) d\tilde{p} = \frac{1}{2d_\mu} \sum_{i=1}^{d_\mu} \int_{\mathcal{G}_{m,n}^\circ} |e_{\mu,i}(\tilde{p})|^2 d\tilde{p} \\ &= \frac{1}{d_\mu} \sum_{i=1}^{d_\mu} \langle e_{\mu,i}, e_{\mu,i} \rangle = 1, \end{aligned}$$

which proves (i). As for (ii), this is clear using (4) and the orthogonality relations between the  $e_{\mu,i}$ . Finally, assertion (iii) is classical, see [10] Theorem 3.1. □

The algebraic structure of  $\mathcal{Z}$  and  $\mathcal{Z}^o$  can be easily deduced from [9]. For lack of reference, we state it in the next proposition

**Proposition 2.3.**

(i) *There is an isomorphism*

$$\mathcal{Z} \simeq \mathbb{C}[Y_1, \dots, Y_m]^{S_m},$$

*the ring of symmetric polynomials in  $m$  variables, mapping  $Y_i$  to  $y_i = y_i(p, q)$ . Similarly, one has*

$$\mathcal{Z}^o \simeq \mathbb{C}[Y_1, \dots, Y_m]^{S_m}[\theta], \text{ with } \theta^2 = Y_1 \cdots Y_m,$$

*mapping  $\theta$  to  $\epsilon t_1 \cdots t_m$ , where  $\epsilon = \epsilon(\tilde{p}, \tilde{q})$ ,  $t_i = t_i(\tilde{p}, \tilde{q})$ . Moreover, the eigenspace decomposition of  $\mathcal{Z}^o$  with respect to the involution  $\sigma^*$  is given by*

$$\mathcal{Z}^{o+} = \mathcal{Z} \text{ and } \mathcal{Z}^{o-} = \epsilon t_1 \cdots t_m \mathcal{Z} \simeq \theta \mathbb{C}[Y_1, \dots, Y_m]^{S_m}$$

(ii) *The  $P_\mu$  corresponding to even partitions may be expressed as*

$$P_\mu(p, q) = p_\mu(y_1(p, q), \dots, y_m(p, q))$$

*with  $p_\mu(Y_1, \dots, Y_m)$  a symmetric polynomial of total degree  $\frac{|\mu|}{2}$ , and those corresponding to odd partitions as*

$$P_\mu(\tilde{p}, \tilde{q}) = (\epsilon t_1 \cdots t_m) p_\mu(y_1, \dots, y_m),$$

*with  $p_\mu(Y_1, \dots, Y_m)$  a symmetric polynomial of degree  $\frac{|\mu|-m}{2}$ .*

*Proof.* (i) As explained above, we can identify  $\mathcal{Z}$  with  $\mathcal{R}(\mathcal{G}_{m,n})^{\text{Stab}(p)}$  (resp.  $\mathcal{Z}^o$  with  $\mathcal{R}(\mathcal{G}_{m,n}^\circ)^{\text{Stab}(\tilde{p})}$ ),  $p$  (resp.  $\tilde{p}$ ) being any fixed base point. From the isomorphism  $\text{Stab}(p) \simeq O(m) \times O(n-m)$ , it is easily seen, using (2), that an element  $F = F(p, \cdot) \in \mathcal{R}(\mathcal{G}_{m,n})^{\text{Stab}(p)}$  is of the form

$$F = P(\cos \theta_1, \dots, \cos \theta_m, \sin \theta_1, \dots, \sin \theta_m),$$

where  $P(T_1, \dots, T_m, Z_1, \dots, Z_m)$  is a polynomial, symmetric in  $T_1, \dots, T_m$  and  $Z_1, \dots, Z_m$  respectively. Now the  $\text{Stab}(p)$ -invariance also implies that all the exponents are even, so that  $F$  is indeed a symmetric polynomial

in  $Y_1 = T_1^2, \dots, Y_m = T_m^2$ , which is the first part of assertion (i). As for the second part of the assertion, one first shows in the same way that a  $\text{Stab}(\tilde{p})$ -invariant element in  $\mathcal{R}(\mathcal{G}_{m,n}^\circ)$  is of the form

$$F = F(\tilde{p}, \cdot) = P(\epsilon \cos \theta_1, \dots, \cos \theta_m, \sin \theta_1, \dots, \sin \theta_m),$$

where  $P(T_1, \dots, T_m, Z_1, \dots, Z_m)$  is a polynomial, symmetric in  $T_1, \dots, T_m$  and  $Z_1, \dots, Z_m$  respectively. Since  $\text{Stab}(\tilde{p}) \simeq SO_m \times O_{n-m}$ , the  $\text{Stab}(\tilde{p})$ -invariance also implies that the exponents in the last  $m$  variables are even, whereas the exponents in the first  $m$  ones are only restricted to have the same parity. Consequently,  $P$  is the sum of a polynomial in  $Y_1 = T_1^2, \dots, Y_m = T_m^2$  plus  $T_1 \cdots T_m$  times a polynomial in  $Y_1, \dots, Y_m$ , as asserted. The eigenspace decomposition is clear.

As for assertion (ii), we only need to observe that the  $P_\mu$  belong to  $\mathcal{Z}^{o+}$  or  $\mathcal{Z}^{o-}$  according to as  $\mu$  is even or odd, and that the  $p_\mu$  have total degree  $|\mu|$  in  $T_1, \dots, T_m$ .  $\square$

### 3. BOUNDS ON CODES AND DESIGNS.

Among the various equivalent definitions of a  $t$ -design given in [1] we recall the following one (see [1] Proposition 4.2.)

**Definition 3.1.** *A finite subset  $\mathcal{D}$  of  $\mathcal{G}_{m,n}$  is a  $2k$ -design if*

$$(7) \quad \forall \varphi \in H_{2k}^+, \quad \langle \varphi, 1 \rangle = \frac{1}{|\mathcal{D}|} \sum_{p \in \mathcal{D}} \varphi(p).$$

As for spherical codes, the natural generalization to our context is as follows:

**Definition 3.2.** *Let  $f(Y_1, \dots, Y_m)$  be a symmetric polynomial such that  $f(1, \dots, 1) = 1$ . A finite subset  $\mathcal{D}$  of the Grassmannian space  $\mathcal{G}_{m,n}$  is a  $f$ -code, if for any pair  $(p, q)$  of distinct elements in  $\mathcal{D}$  one has*

$$f(y_1(p, q), \dots, y_m(p, q)) = 0.$$

On the other hand, one can associate canonically to a symmetric polynomial  $f(Y_1, \dots, Y_m)$  as above, an  $O(n)$ -invariant function  $F$  on  $\mathcal{G}_{m,n} \times \mathcal{G}_{m,n}$ , satisfying  $F(p, p) = 1$ , by the formula:

$$F(p, q) := f(y_1(p, q), \dots, y_m(p, q)),$$

and the definition of an  $f$ -code now reads

$$(8) \quad F(p, q) = \delta_{p,q}, \quad (p, q) \in \mathcal{D}^2.$$

The following notion of *type* is consistent with [6] Definition 5.4.:

**Definition 3.3.** *The type of an  $f$ -code is 1 if  $Y_1 \cdots Y_m$  divides the polynomial  $f(Y_1, \dots, Y_m)$ , and 0 otherwise.*

For any integer  $k$ , we define

$$H_k = \bigoplus_{\substack{|\mu| \leq k \\ \mu \text{ admissible}}} H_{m,n}^\mu.$$

It decomposes under  $\sigma^*$  as  $H_k = H_k^+ \oplus H_k^-$ , and we have, for the respective dimensions  $d_k^\pm$  of  $H_k^\pm$ ,

$$d_k^+ := \sum_{\substack{|\mu| \leq k \\ \mu \text{ even} \\ \text{depth}(\mu) \leq m}} d_\mu, \text{ resp. } d_k^- := \sum_{\substack{|\mu| \leq k \\ \mu \text{ odd} \\ \text{depth}(\mu) \leq m}} d_\mu.$$

It's worth noticing that, from remark 2.1,  $d_k^+$  does not depend on  $m$ , while  $d_k^-$  does. The next two theorems establish bounds for  $t$ -designs and  $f$ -codes in terms of these numbers. Some explicit values of  $d_k^+$  and  $d_k^-$  are collected in the Appendix (the  $d_\mu$  are computed from the formulas in [7] §24.2, pp 407-410):

**Remark 3.4.** In [1], we considered only non-oriented grassmannians, and what was denoted  $H_k$  there, corresponds to what is denoted  $H_k^+$  here.

**Theorem 3.5.** Let  $\mathcal{D} \subset \mathcal{G}_{m,n}$  be a  $2k$ -design. Then

$$(9) \quad |\mathcal{D}| \geq \max\{d_k^+, d_k^-\}.$$

If equality holds in (9), then  $\mathcal{D}$  is an  $f$ -code for  $f = \frac{1}{d_k^+} \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} d_\mu p_\mu$  or  $f = \frac{Y_1 \cdots Y_m}{d_k^-} \sum_{\substack{|\mu| \leq k \\ \mu \text{ odd}}} d_\mu p_\mu$ , depending on whether  $d_k^+ \geq d_k^-$  or not.

*Proof.* Let  $s$  be a section of the canonical surjection  $\mathcal{G}_{m,n}^\circ \rightarrow \mathcal{G}_{m,n}$ , and  $\tilde{\mathcal{D}} = s(\mathcal{D})$ . Fix  $\tilde{p}$  and  $\tilde{q}$  in  $\mathcal{G}_{m,n}^\circ$ . If  $\mu$  and  $\lambda$  are two partitions of degree  $\leq k$ , the formula

$$\varphi(\tilde{p}') = P_\mu(\tilde{p}, \tilde{p}') P_\lambda(\tilde{q}, \tilde{p}')$$

defines an element in  $H_{2k}$ . If moreover  $\mu$  and  $\lambda$  are both even (resp. odd), then  $\varphi$  is  $\sigma^*$ -invariant, so it belongs to  $H_{2k}^+$ . Consequently (7) applies in both cases and reads:

$$\begin{aligned} \frac{1}{|\mathcal{D}|} \sum_{\tilde{p}' \in \tilde{\mathcal{D}}} \varphi(\tilde{p}') &= \frac{1}{|\mathcal{D}|} \sum_{p' \in \mathcal{D}} \varphi(p') \\ &= \langle \varphi, 1 \rangle_{L^2(\mathcal{G}_{m,n})} \\ &= \langle P_\mu(\tilde{p}, \cdot) P_\lambda(\tilde{q}, \cdot), 1 \rangle_{L^2(\mathcal{G}_{m,n})} \\ &= \langle P_\mu(\tilde{p}, \cdot), P_\lambda(\tilde{q}, \cdot) \rangle_{L^2(\mathcal{G}_{m,n}^\circ)} \\ &= \frac{\delta_{\lambda,\mu}}{d_\mu} P_\mu(\tilde{p}, \tilde{q}). \end{aligned}$$

In other words, the matrices  $S_\mu := (d_\mu P_\mu(\tilde{p}, \tilde{p}'))_{\tilde{p}, \tilde{p}' \text{ in } \tilde{\mathcal{D}}, |\mu| \leq k}$ , satisfy the relations

$$S_\mu S_\lambda = \delta_{\lambda, \mu} |\tilde{\mathcal{D}}| S_\mu = \delta_{\lambda, \mu} |\mathcal{D}| S_\mu,$$

as long as  $\mu$  and  $\lambda$  are both even (resp. odd). Setting  $S^+ := \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} S_\mu$ , resp.  $S^- := \sum_{\substack{|\mu| \leq k \\ \mu \text{ odd}}} S_\mu$ , it follows that

$$(10) \quad S^{\pm 2} = |\mathcal{D}| S^\pm.$$

On the other hand, one has  $\text{Tr } S_\mu = d_\mu |\mathcal{D}|$ , from Lemma 2.2, so that  $\text{Tr } S^+ = \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} \text{Tr } S_\mu = d_k^+ |\mathcal{D}|$ , and likewise  $\text{Tr } S^- = d_k^- |\mathcal{D}|$ . Therefore,

$$d_k^\pm = \frac{\text{Tr } S^\pm}{|\mathcal{D}|} = \text{rank } S^\pm \leq |\mathcal{D}|,$$

whence the conclusion.

When equality holds, then (10) implies that  $S^+ = |\mathcal{D}| I_{|\mathcal{D}|} = d_k^+ I_{|\mathcal{D}|}$  resp.  $S^- = d_k^- I_{|\mathcal{D}|}$ , depending on whether  $d_k^+ \geq d_k^-$  or not, where  $I_{|\mathcal{D}|}$  stands for the identity matrix in dimension  $|\mathcal{D}|$ . This means that  $F(p, q) = \delta_{p, q}$ , for all  $(p, q) \in \mathcal{D}^2$ , where  $F = \frac{1}{d_k^+} \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} d_\mu P_\mu$ , resp.  $\frac{1}{d_k^-} \sum_{\substack{|\mu| \leq k \\ \mu \text{ odd}}} d_\mu P_\mu$ . In the first case, this is clearly equivalent to the assertion that  $\mathcal{D}$  be an  $f$ -code, according to (8) and the definition of  $f$ . This is also the case in the second case, since each  $P_\mu$ ,  $\mu$  odd, is divisible by the product  $t_1 \cdots t_m$ , so that

$$\frac{1}{d_k^-} \sum_{\substack{|\mu| \leq k \\ \mu \text{ odd}}} d_\mu P_\mu(p, q) = \delta_{p, q} \Leftrightarrow \frac{1}{d_k^-} \sum_{\substack{|\mu| \leq k \\ \mu \text{ odd}}} (t_1 \cdots t_m) d_\mu P_\mu(p, q) = \delta_{p, q}.$$

□

**Theorem 3.6.** *Any  $f$ -code  $\mathcal{D}$  in  $\mathcal{G}_{m, n}$  satisfies*

$$(11) \quad |\mathcal{D}| \leq d_k^+$$

where  $k = 2 \deg f$ . If moreover  $f$  is of type 1, then

$$(12) \quad |\mathcal{D}| \leq d_k^-$$

where  $k = 2 \deg f - m$ . Whenever equality holds in (11), resp. (12), then

$$f = \frac{1}{d_k^+} \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} d_\mu p_\mu,$$

resp.

$$f = \frac{Y_1 \cdots Y_m}{d_k^-} \sum_{\substack{|\mu| \leq k \\ \mu \text{ odd}}} d_\mu p_\mu,$$

and  $\mathcal{D}$  is a  $2k$ -design.



*Proof.* Setting  $k = 2 \deg f$ , we first see that the functions  $F(p, \cdot)$ ,  $p \in \mathcal{D}$  are in  $H_k^+$ . We claim that they form a linearly independent system. Indeed, if  $\sum_{p \in \mathcal{D}} \lambda_p F(p, \cdot) = 0$ , then evaluating the left-hand side successively on each  $p \in \mathcal{D}$ , and using (8), we see that  $\lambda_p = 0$  for all  $p \in \mathcal{D}$ . Hence  $|\mathcal{D}| \leq \dim H_k^+ = d_k^+$ , which is the first assertion. As for the second one, if  $f$  is divisible by  $Y_1 \cdots Y_m$ , we write it as  $f(Y_1, \dots, Y_m) = (Y_1 \cdots Y_m)g(Y_1, \dots, Y_m)$ . Then the functions  $t_1(p, \cdot) \cdots t_m(p, \cdot)G(p, \cdot)$ ,  $p \in \mathcal{D}$  are linearly independent elements in  $H_k^-$ , with  $k = 2 \deg f - m$ , and the inequality  $|\mathcal{D}| \leq d_k^-$  follows, as in the first case.

To see when equality is achieved, let us assume, for instance that  $|\mathcal{D}| = d_k^+$ ,  $k = 2 \deg f$  (the case  $|\mathcal{D}| = d_k^-$ ,  $k = 2 \deg f - m$  for  $\mathcal{D}$  of type 1 is dealt with similarly). Under this assumption, the family  $\{F(p, \cdot), p \in \mathcal{D}\}$  is now a basis of  $H_k^+$ . Moreover, it is readily checked that the following formula holds for any  $\varphi$  in  $H_k^+$ :

$$(13) \quad \varphi = \sum_{p \in \mathcal{D}} \varphi(p) F(p, \cdot).$$

On the other hand, we know from Proposition 2.3 that  $F$  (resp.  $f$ ) may be written as a linear combination of the  $P_\mu$  (resp.  $p_\mu$ ),

$$(14) \quad F = \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} f_\mu P_\mu \quad (\text{resp. } f = \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} f_\mu p_\mu).$$

Let  $P := \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} d_\mu P_\mu$ . What we want to show is that  $F = \frac{1}{d_k^+} P$ , or in other words that

$$(15) \quad f_\mu = \frac{d_\mu}{d_k^+}, \text{ for } |\mu| \leq k, \mu \text{ even.}$$

To that end, it is sufficient to show that

$$(16) \quad 0 \leq f_\mu \leq \frac{d_\mu}{d_k^+}, \text{ for } |\mu| \leq k, \mu \text{ even.}$$

since applying (14) to any  $p$  in  $\mathcal{D}$ , we see that  $1 = F(p, p) = \sum f_\mu P_\mu(p, p) = \sum f_\mu$ , so the right inequality in (16) is an equality. First we note, using (5) and the above decomposition, that

$$\langle F(q, \cdot), \sum_{|\mu| \leq k} d_\mu P_\mu(p, \cdot) \rangle = F(p, q),$$

so that the condition  $F(p, q) = \delta_{p,q}$ ,  $(p, q) \in \mathcal{D}^2$  implies that the family  $\{\sum_{|\mu| \leq k} d_\mu P_\mu(p, \cdot), p \in \mathcal{D}\}$  is a basis, dual to  $\{F(p, \cdot), p \in \mathcal{D}\}$  with respect to the scalar product  $\langle \cdot, \cdot \rangle$ . Consequently, the matrix  $S = (P(p, q))_{p, q \in \mathcal{D}^2} = (\langle P(p, \cdot), P(q, \cdot) \rangle)_{p, q \in \mathcal{D}^2}$  is invertible and its inverse is given by

$$S^{-1} = (\langle F(p, \cdot), F(q, \cdot) \rangle)_{p, q \in \mathcal{D}^2}.$$

One easily checks, using lemma 2.2 that

$$\langle F(p, \cdot), P_\mu(q, \cdot) \rangle = \frac{f_\mu}{d_\mu} P_\mu(q, p),$$

for  $p, q$  in  $\mathcal{D}^2$ , and  $|\mu| \leq k$ ,  $\mu$  even. But according to (13), this means that the functions  $P_\mu(q, \cdot)$ ,  $q \in \mathcal{D}$ ,  $|\mu| \leq k$ ,  $\mu$  even, are eigenfunctions of the matrix  $(\langle F(p, \cdot), F(q, \cdot) \rangle)_{p, q \in \mathcal{D}^2} = S^{-1}$ , with corresponding eigenvalue  $\frac{f_\mu}{d_\mu}$ . Thus the  $\frac{f_\mu}{d_\mu}$  are eigenvalues of the Gram matrix of a basis of  $H_k^+$ , hence positive. Now, writing (14) for all  $(p, q) \in \mathcal{D}^2$ , and adding up we obtain

$$|\mathcal{D}| = \sum_{(p, q) \in \mathcal{D}^2} F(p, q) = \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} f_\mu \sum_{(p, q) \in \mathcal{D}^2} P_\mu(p, q)$$

whence

$$(17) \quad |\mathcal{D}|(1 - f_0|\mathcal{D}|) = \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}, \mu \neq 0}} f_\mu \sum_{(p, q) \in \mathcal{D}^2} P_\mu(p, q) \geq 0,$$

because of the positivity of the matrix  $(P_\mu(p, p'))_{p, p' \in \mathcal{D}^2}$  (lemma (2.2)(ii)), so that  $f_0 \leq \frac{1}{|\mathcal{D}|} = \frac{1}{d_k^+}$ . If we now consider the annihilator polynomial

$$F_\lambda := P_\lambda F = \sum_{\substack{|\mu| \leq k \\ \mu \text{ even}}} g_{\lambda, \mu} P_\mu,$$

we contend that the  $g_{\lambda, \mu}$  are nonnegative and that  $g_{\lambda, 0} = \frac{f_\lambda}{d_\lambda}$ : this is an easy consequence of lemma 2.2 (6). Consequently, the argument used to get (17) still holds, and we obtain  $g_{\lambda, 0} = \frac{f_\lambda}{d_\lambda} \leq \frac{1}{d_k^+}$ , as desired.

It remains to prove that  $\mathcal{D}$  is a  $2k$ -design. From [1] Proposition 4.2., it amounts to prove that

$$\forall \varphi \in H_{2k}^+, \langle \varphi, 1 \rangle = \frac{1}{|\mathcal{D}|} \sum_{p \in \mathcal{D}} \varphi(p) = \frac{1}{d_k^+} \sum_{p \in \mathcal{D}} \varphi(p).$$

It's enough to check this for functions of the form  $gh$ , with  $g, h$  in  $H_k^+$ , since they generate  $H_{2k}^+$ . Using the expansion (13) of  $g$  and  $h$ , we see that

$$\begin{aligned} \langle 1, gh \rangle &= \langle g, \bar{h} \rangle \\ &= \sum_{p, q \in \mathcal{D}^2} g(p)h(q) \langle F(p, \cdot), F(q, \cdot) \rangle \\ &= \frac{1}{|\mathcal{D}|} \sum_{p \in \mathcal{D}} g(p)h(p), \end{aligned}$$

whence the conclusion.  $\square$

4. EXAMPLES.

4.1. **The case  $m = 1$ .** This is the case of the projective space over the real numbers, the codes of which are studied in [6]. The  $2k$ -designs in the real projective space can be viewed as *antipodal*  $(2k + 1)$ -designs on the unit sphere of the Euclidean space for which absolute bounds are given in [5]. We recover here these bounds, since for  $\mu = \mu_1 \geq 0$  the space  $H_{1,n}^\mu$  is isomorphic to the space of harmonic polynomials in  $n$  variables of degree  $\mu_1$ . One has  $d_k^+ = \binom{n+k-1}{n-1}$  and  $d_k^- = \binom{n+k-2}{n-1}$  if  $k$  is even, and *vice versa* if  $k$  is odd.

A  $t$ -design is called tight if its cardinality attains this lower bound. Tight  $t$ -designs are only known for  $(n, t) = (7, 4), (8, 6), (23, 4), (23, 6), (24, 10)$ . Moreover, it is known that tight  $t$ -designs cannot exist when  $t \geq 8$ , apart from the  $(24, 10)$  given by the lines supporting the minimal vectors of the Leech lattice (see [2]).

4.2. **The case  $k = 2$ .** In [3] and [4], packings in the Grassmannian spaces are considered, with respect to the so-called *chordal distance*, given in our notations by

$$d^2(p, q) = m - \sum_{i=1}^m y_i.$$

In [3], a *simplex bound* is settled for the sets  $\mathcal{D}$  for which  $d(p, q) \geq d$  (using an isometric embedding into the Euclidean sphere of  $\mathbb{R}^{(n-1)(n+2)/2}$ ). Equality holds if and only if  $|\mathcal{D}| = \frac{n(n+1)}{2}$  and  $d(p, q)$  is constant.

In [4, §5], an infinite family of packings in  $\mathcal{G}_{\frac{p-1}{2}, p}$  meeting this bound, is constructed. Here  $p$  is a prime, which is either equal to 3 or congruent to  $-1$  modulo 8. Let us denote it by  $\mathcal{D}_p$ . Then one has:

**Proposition 4.1.**  $\mathcal{D}_p$  is a tight 4-design in  $\mathcal{G}_{\frac{p-1}{2}, p}$ .

*Proof.* According to [4] Theorem 3,  $\mathcal{D}_p$  consists on  $\frac{p(p+1)}{2} = d_{[0, \dots, 0]} + d_{[2, 0, \dots, 0]} = d_2^+$  subspaces with same pairwise chordal distance  $d^2 = \frac{(p+1)^2}{4(p+2)}$ . Since  $d^2 = \sum \sin^2 \theta_i = \frac{p-1}{2} - \sum y_i$ , the conclusion follows, applying Theorem 3.6 to the polynomial  $f = \frac{4(p+2)(\sum Y_i) - (p^2-5)}{p^2-5}$ .  $\square$

4.3. **The case  $k = 3$ .** From the definitions, one has  $d_3^+ = d_2^+ = \frac{n(n+1)}{2}$ , and  $d_3^-$  equals 0 unless  $m = 1$  in which case  $d_3^- = \binom{n+2}{3}$ , or  $m = 3$  in which case  $d_3^- = d_{(1,1,1)} = \binom{n}{3}$ . Therefore, it is very unlikely that tight 6-designs exist for  $m \neq 1, 3$ .

A family of packings in the Grassmannian  $\mathcal{G}_{2^k, 2^m}$  is constructed in [4, Theorem 1], each of them are orbits under the Clifford group  $\mathcal{C}_m$ . We have checked that, for  $m = 2, 3, 4$ , and for  $(m, k) = (5, 4)$ , these packings are 6-designs. For each  $m$ , the smallest of these sets corresponds to  $k = m - 1$  and its cardinality equals  $2^{2m} + 2^m - 2 = 2(d_2^+ - 1)$ .

**Remark 4.2.** *It is known that the orbits of the Clifford group on the first Grassmannian provide 6-designs, because the first non trivial invariant polynomial of this group has degree 8 (and corresponds to the Hamming code, see [12] and the earlier work of B. Runge). We conjecture that the orbits of the Clifford group on all the Grassmannians provide 6-designs. This, according to [1, Th 4.5, Rem 4.6], is equivalent to the fact that the  $C_m$ -invariants of the  $(n = 2^m)$   $Gl_n$ -irreducible modules canonically associated to the partitions  $(4, 2)$  and  $(2, 2, 2)$  (denoted  $F_n^\mu$  in [9]) have dimension 1.*

## 5. APPENDIX.

We list below some values of  $d_k^+$  and  $d_k^-$  for  $m = 2, 3$  and 4.

m=2				m=3				m=4			
$n$	$k$	$d_k^+$	$d_k^-$	$n$	$k$	$d_k^+$	$d_k^-$	$n$	$k$	$d_k^+$	$d_k^-$
4	1	1	0	6	1	1	0	8	1	1	0
4	2	10	3	6	2	21	0	8	2	36	0
4	3	10	3	6	3	21	10	8	3	36	0
4	4	40	18	6	4	210	10	8	4	630	35
4	5	40	18	6	5	210	136	8	5	630	35
5	1	1	0	7	1	1	0	9	1	1	0
5	2	15	10	7	2	28	0	9	2	45	0
5	3	15	10	7	3	28	35	9	3	45	0
5	4	105	91	7	4	378	35	9	4	990	126
5	5	105	91	7	5	378	651	9	5	990	126
6	1	1	0	8	1	1	0	10	1	1	0
6	2	21	15	8	2	36	0	10	2	55	0
6	3	21	15	8	3	36	56	10	3	55	0
6	4	210	190	8	4	630	56	10	4	1485	210
6	5	210	190	8	5	630	1352	10	5	1485	210
7	1	1	0	9	1	1	0	11	1	1	0
7	2	28	21	9	2	45	0	11	2	66	0
7	3	28	21	9	3	45	84	11	3	66	0
7	4	378	351	9	4	990	84	11	4	2145	330
7	5	378	351	9	5	990	2541	11	5	2145	330
8	1	1	0	10	1	1	0	12	1	1	0
8	2	36	28	10	2	55	0	12	2	78	0
8	3	36	28	10	3	55	120	12	3	78	0
8	4	630	595	10	4	1485	120	12	4	3003	495
8	5	630	595	10	5	1485	4432	12	5	3003	495
9	1	1	0	11	1	1	0	13	1	1	0
9	2	45	36	11	2	66	0	13	2	91	0
9	3	45	36	11	3	66	165	13	3	91	0
9	4	990	946	11	4	2145	165	13	4	4095	715
9	5	990	946	11	5	2145	7293	13	5	4095	715

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