Optimal partitions, regularized solutions, and application to image classification

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Abstract

This article is concerned with the problem of finding a regular and homogeneous partition of a Lipschitz open set Ω of $\mathbb{R}^2$. This type of problem occurs in image classification which consists in assigning a label (that is a class or a phase) to each pixel of an observed image. Such a problem can numerically be solved by using a level set formulation. But this approach generally relies on heuristic arguments. We intend here to give a theoretical justification in the two phases case. In the first part of the paper, we prove that the partition problem we consider admits a solution. And in the second part, we justify the numerical approach which is usually done to compute a solution.

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1. Introduction

This paper is concerned with the problem of finding a regular and homogeneous partition of a Lipschitz open set Ω of $\mathbb{R}^2$. A partitionning of Ω consists in searching for a family of sets $\{E_i\}_{i=1\dots I}$ such that $\Omega = \bigcup_{i=1}^I E_i \bigcup_i \Gamma_i$ with $E_i \bigcap E_j = \emptyset$ if $i \neq j$, and where $\Gamma_i = \partial E_i \bigcap \Omega$ is the intersection of the boundary of E_i with Ω. In fact, $\Gamma_i = \bigcup_{j\neq i} \Gamma_{ij}$, where $\Gamma_{ij} = \Gamma_{ji}$, $i \neq j$, is the interface between E_i and E_j. By regular, we mean that Γ_i's are of minimal length. We also want to get an homogeneous partition in the sense that each set E_i is homogeneous with respect to a given criterion. Typically this type of problem occurs in image classification which consists in assigning a label (that is a class or a phase) to each pixel of an observed image. Here the feature criterion is for example the spatial distribution of the intensity, e.g. each class is the set of pixels having the same distribution. Other discriminant features can be used such as texture for instance. An efficient way to get an optimal partition is to search for E_i's as minimizers of a partitionning functional of the form:

$$F(E_1,\dots,E_I) = \left\{ \sum_{i=1}^I |\Gamma_i| + \sum_{i=1}^I \int_\Omega B_i \ ; \ \Omega = \bigcup_{i=1}^I E_i \bigcup \Gamma_i \ , \ E_i \bigcap E_j = \emptyset \text{ if } i \neq j \right\} \tag{1.1}$$

where $|\Gamma_i|$ stands for the one-dimensional Hausdorff measure of Γ_i, and B_i is a given data term.

A difficulty in the above formulation comes from the fact that unknowns are sets and not functions. To overcome this difficulty which is a real problem for numerical computations, we often use a level set formulation. If we assume that we can write $E_i = \{\phi_i > 0\}$ for some Lipschitz function ϕ_i, with $\partial E_i = \{\phi_i = 0\}$, then we can rewrite (1.1) as:

$$F(E_1,\dots,E_I) = \left\{ \sum_{i=1}^I \int_{\phi_i=0} ds + \sum_{i=1}^I \int_\Omega B_i H(\phi_i) \ ; \ \sum_{i=1}^I H(\phi_i) = 1 \right\} \tag{1.2}$$

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where H stands for the Heaviside function. In fact, in real applications, we relax the partionning constraint $\sum_{i=1}^{I} H(\phi_i) = 1$ by considering the functional:

$$F(E_1, \ldots, E_I) = \sum_{i=1}^{I} \int_{\phi_i=0} ds + \sum_{i=1}^{I} \int_\Omega B_i H(\phi_i) + \int_\Omega \left(\sum_{i=1}^{I} H(\phi_i) - 1 \right)^2 \tag{1.3}$$

or an approximated version of (1.3):

$$F_\alpha(E_1, \ldots, E_I) = \sum_{i=1}^{I} \int_\Omega \delta_\alpha(\phi_i)|\nabla \phi_i| + \sum_{i=1}^{I} \int_\Omega B_i H_\alpha(\phi_i) + \int_\Omega \left(\sum_{i=1}^{I} H_\alpha(\phi_i) - 1 \right)^2 \tag{1.4}$$

where δ_α and H_α are respectively smooth approximations of the Dirac measure and the Heaviside function.

Functionals of the kind (1.1),(1.2), (1.3) or (1.4) have been abundantly used in the image processing literature [6, 8, 22, 21, 10, 24], but as far as we know, the theoretical study of such minimization problems has not been adressed yet. The goal of this paper is to fill this gap in the two phases case (the case $I > 2$ is more technical and will be adressed in a future work). Therefore, in this case, we only need one unknown function ϕ and (1.3), (1.4) simplified:

$$\inf_{\phi \in Lip(\Omega)} \left(\int_{\phi=0} ds + \int_\Omega BH(\phi) \right) \tag{1.5}$$

$$\inf_{\phi \in Lip(\Omega)} \left(\int_\Omega \delta_\alpha(\phi)|\nabla \phi| + \int_\Omega BH_\alpha(\phi) \right) \tag{1.6}$$

where $B \in L^\infty(\Omega)$. Such simplified functionals are used in [8, 22, 21, 10, 24]. But in all these works all the computations are formal and we intend here to develop a theoretical justification.

The plan of the paper is as follows. We first recall a few basic facts about sets with finite perimeter in Section 2. We are then in position to study (1.5): we prove the existence of a solution for problem (1.5) in Section 3. In Section 4, thanks to a change of function $\psi = H_\alpha(\phi)$, we get the existence of Lipschitz function for problem (1.6). This jusitifies the numerical approach used in [8, 22, 21, 10, 24]. We end this paper by showing some experimental classification results obtained by minimizing (1.6) in Section 5.

2. Sets with finite perimeter

For this section, we refer the reader to [4, 2, 14, 18, 13], our presentation being inspired essentially from [4]. We first recall the definition of $BV(\Omega)$ (we suppose that Ω, the domain of the image, is a bounded Lipschitz open set of $\mathbb{R}^2$):

Definition 2.1. $BV(\Omega)$ is the subspace of functions $u \in L^1(\Omega)$ such that the following quantity is finite:

$$J(u) = \sup \left\{ \int_\Omega u(x) \text{div}\,(\xi(x)) dx / \xi \in C_c^1(\Omega; \mathbb{R}^2), \|\xi\|_{L^\infty(\Omega)} \leq 1 \right\} \tag{2.1}$$

where $C_c^1(\Omega)$ stands for the set of functions in $C^1(\Omega)$ with compact support in Ω. $BV(\Omega)$ endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + J(u)$ is a Banach space. If $u \in BV(\Omega)$, the distributional derivative Du is a bounded Radon measure and (2.1) corresponds to the total variation $|Du|(\Omega)$.

We now come to the definition of a set with finite perimeter:

Definition 2.2. Let E be a measurable subset of $\mathbb{R}^2$. Then for any open set $\Omega \subset \mathbb{R}^2$, we call perimeter of E in Ω, denoted by $P(E, \Omega)$, the total variation of $\mathbf{1}_E$ in Ω, i.e.:

$$P(E, \Omega) = \sup \left\{ \int_E \text{div}\,(\xi(x)) dx / \xi \in C_c^1(\Omega; \mathbb{R}^2), \|\xi\|_{L^\infty(\Omega)} \leq 1 \right\} \tag{2.2}$$

We say that E has finite perimeter if $P(E, \Omega) < \infty$.

Remark: If E has a C^1-boundary, this definition of the perimeter corresponds to the classical one. We then have:

$$P(E,\Omega) = \mathcal{H}^1(\partial E \bigcap \Omega) \tag{2.3}$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure [4]. The result remains true when E has a Lipschitz boundary.

In the general case, if E is any open set in Ω, and if $\mathcal{H}^1(\partial E \bigcap \Omega) < +\infty$, then:

$$P(E,\Omega) \leq \mathcal{H}^1(\partial E \bigcap \Omega) \tag{2.4}$$

Definition 2.3. We denote by $\mathcal{F}E$ the reduced boundary of E.

$$\mathcal{F}E = \left\{ x \in \text{ support } \left(|D_{\mathbf{1}_E}| \bigcap \Omega \right) \;/\; \nu_E = \lim_{\rho \to 0} \frac{D_{\mathbf{1}_E(B_\rho(x))}}{\left|D_{\mathbf{1}_E(B_\rho(x))}\right|} \text{ exists and verifies } |\nu_E| = 1 \right\} \tag{2.5}$$

Definition 2.4. For all $t \in [0,1]$, we denote by E^t the set

$$\left\{ x \in \mathbb{R}^2 \;/\; \lim_{\rho \to 0} \frac{|E \bigcap B_\rho(x)|}{|B_\rho(x)|} = t \right\} \tag{2.6}$$

of points where E is of density t, where $B_\rho(x) = \{y \;/\; \|x-y\| \leq \rho\}$. We set $\partial^* E = \mathbb{R}^2 \backslash \left(E^0 \bigcup E^1\right)$ the essential boundary of E.

Theorem 2.1. *[Federer [4]]. Let E a set with finite perimeter in Ω. Then:*

$$\mathcal{F}E \bigcap \Omega \subset E^{1/2} \subset \partial^* E \tag{2.7}$$

and

$$\mathcal{H}^1 \left(\Omega \backslash \left(E^0 \bigcup \mathcal{F}E \bigcup E^1 \right) \right) = 0 \tag{2.8}$$

Remark: If E is Lipschitz, then $\partial E \subset \partial^* E$. In particular, since we always have $\mathcal{F}E \subset \partial E$ (see [13]):

$$P(E,\Omega) = \mathcal{H}^1(\partial E \bigcap \Omega) = \mathcal{H}^1(\partial^* E \bigcap \Omega) = \mathcal{H}^1(\mathcal{F}E \bigcap \Omega) \tag{2.9}$$

Theorem 2.2. *[De Giorgi [4]]. Let E a Lebesgue measurable set of $\mathbb{R}^2$. Then $\mathcal{F}E$ is 1-rectifiable.*

We recall that E is 1-rectifiable if and only if there exist Lipschitz functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ such that $E \subset \bigcup_{i=0}^{+\infty} f_i(\mathbb{R})$.

3. The two phases problem

In this section, Ω will be a bounded Lipschitz open set of $\mathbb{R}^2$. We consider a functional involving two terms: a length penalization term, and a data term. We denote by E a set in Ω such that E and $\Omega \backslash E$ correspond respectively to the two phases. We call a partitionning functional, the functional:

$$G(E) = \int_\Omega |D\mathbf{1}_E| + \int_\Omega B\mathbf{1}_E \tag{3.1}$$

We assume that $B \in L^\infty(\Omega)$. Let us introduce the three following problems:

1.

$$\inf_{\phi \in Lip(\Omega)} \left(\int_{\phi=0} ds + \int_\Omega BH(\phi) \right) \tag{3.2}$$

2.

$$\inf_{E \in \mathcal{B}(\Omega)} \left(P(E,\Omega) + \int_E B \right) \tag{3.3}$$

3.

$$\inf_{\psi \in BV(\Omega),\ 0 \leq \psi \leq 1} \left(\int_\Omega |D\psi| + \int_\Omega B\psi \right) \tag{3.4}$$

where $\mathcal{B}(\Omega)$ stands for the set of Borel subsets in Ω, and $Lip(\Omega)$ stands for the set of functions which are Lipschitz in Ω.

Remark: The functional (1.6) can be seen as an approximated version of (3.2), in the sense that for all Lipschitz function ϕ, we have (see (4.5)):

$$\lim_{\alpha\to 0}\int_\Omega \delta_\alpha(\phi)|\nabla\phi| + \int_\Omega B_\alpha H(\phi) = \int_{\phi=0} ds + \int_\Omega BH(\phi) \tag{3.5}$$

In the sequel, we will show that these two problems are very close to each other. Problem (3.3) corresponds to the classification problem with two phases that we have introduced with (3.1). It is a classical problem in the theory of minimal sets with prescribed mean curvature [2, 17, 18]. The theory of sets with finite perimeter has already been used many times in image processing [3, 11, 19, 20]. In (3.4), if ψ is a characteristic function, then we exactly get (3.1). The main result of this section is that there exists a characteristic function which is a solution of (3.4) (Proposition 3.2). Then we will be in position to show that (3.2) admits a solution (Proposition 3.3).

3.1 Study of problem (3.3)

Problem (3.3) has been completely solved by Massari in [17]. We use here the presentation of [2]. We have the three following results (whose proofs are given in [2, 18]):

Proposition 3.1. *Problem (3.3) admits at least one solution.*

Theorem 3.1. *Let E a solution of (3.3). Then $\mathcal{F}E = \partial^* E$, and $\mathcal{F}E$ is $C^{1,1}$.*

Thanks to these results, we know that problem (3.3) has at least one solution $\tilde{E}$, and that moreover $\tilde{E}$ has a $C^{1,1}$-boundary. In particular, we have: $P(\tilde{E},\Omega) = \mathcal{H}^1(\partial\tilde{E}\bigcap\Omega)$ (see (2.9)).

3.2 Study of problem (3.4)

Proposition 3.2. *There exists $\tilde{E}\in\mathcal{B}(\Omega)$ solution of (3.3) such that $\mathbf{1}_{\tilde{E}}$ is a solution of (3.4).*

Proof: We consider the following functional defined on $\{\psi\in BV(\Omega),\ 0\leq\psi\leq 1\}$:

$$G(\psi) = \int_\Omega B\psi + \int_\Omega |D\psi| \tag{3.6}$$

G is convex and l.s.c. on $BV-$weak *. Thanks to the condition $0\leq\psi\leq 1$ we can conclude that there exists a solution $\tilde{\psi}$ for (3.4).

Let us recall the co-area formula [4]:

$$\int_\Omega |D\psi| = \int_{-\infty}^{+\infty} \left|D\mathbf{1}_{\{\psi\geq s\}}\right| ds = \int_{-\infty}^{+\infty} P\left(\{\psi\geq s\},\ \Omega\right) ds \tag{3.7}$$

As we impose the condition $0\leq\psi\leq 1$, we can rewrite (3.7):

$$\int_\Omega |D\psi|) = \int_0^1 \left|D\mathbf{1}_{\{\psi\geq s\}}\right| ds = \int_0^1 P\left(\{\psi\geq s\},\ \Omega\right) ds \tag{3.8}$$

We concentrate now on the second term of the functional. We first remark that, since $0\leq\psi\leq 1$:

$$\psi(x) = \int_0^{\psi(x)} ds = \int_0^1 \mathbf{1}_{\psi(x)\geq s}\, ds \tag{3.9}$$

and thus:

$$\begin{aligned}\int_\Omega B\psi &= \int_\Omega B(x)\left(\int_0^1 \mathbf{1}_{\{\psi(x)\geq s\}} ds\right) dx \\ \text{(Fubini)} &= \int_0^1\int_\Omega B(x)\mathbf{1}_{\{\psi(x)\geq s\}} dx\, ds \\ &= \int_0^1\int_{\{\psi\geq s\}} B(x)\, dx\, ds\end{aligned}$$

Hence:

$$G(\psi) = \int_\Omega (|D\psi| + B\psi) = \int_0^1 \left(P(\{\psi \geq s\},\ \Omega) + \int_{\{\psi \geq s\}} B \right) ds \tag{3.10}$$

which implies:

$$\inf_{\psi \in BV(\Omega),\ 0 \leq \psi \leq 1} \int_\Omega (|D\psi| + B\psi) = \inf_{\psi \in BV(\Omega),\ 0 \leq \psi \leq 1} \int_0^1 \left(P(\{\psi \geq s\},\ \Omega) + \int_{\{\psi \geq s\}} B \right) ds \tag{3.11}$$

Let $\tilde{E}$ a solution of problem (3.3). We denote by $\tilde{\psi} = \mathbf{1}_{\tilde{E}}$. We want to show that $\tilde{\psi}$ is a solution of (3.4). Let $s \in (0,1)$.

- Let $x \in \tilde{E}$. Then $\tilde{\psi}(x) = \mathbf{1}_{\tilde{E}}(x) = 1$. Thus $\tilde{\psi}(x) \geq s$. Hence $\tilde{E} \subset \{\tilde{\psi} \geq s\}$.
- Let $x \in \{\tilde{\psi} \geq s\}$. Then $\tilde{\psi}(x) \geq s > 0$. Thus $\tilde{\psi}(x) = 1$ (since $\tilde{\psi} = 0$ or 1). Hence $\{\tilde{\psi} \geq s\} \subset \tilde{E}$.

We eventually deduce that, $\forall s \in (0,1)$: $\tilde{E} = \{\tilde{\psi} \geq s\}$.

Now, we consider $\psi \in BV(\Omega)$ such that $0 \leq \psi \leq 1$. We have (thanks to (3.10)):

$$G(\psi) = \int_\Omega (|D\psi| + B\psi) = \int_0^1 \left(P(\{\psi \geq s\},\ \Omega) + \int_{\{\psi \geq s\}} B \right) ds \tag{3.12}$$

In the same way as above:

$$\begin{aligned} G(\tilde{\psi}) &= \int_0^1 \left(P\left(\{\tilde{\psi} \geq s\},\ \Omega\right) + \int_{\{\tilde{\psi} \geq s\}} B \right) ds \\ &= \int_0^1 \left(P\left(\tilde{E},\ \Omega\right) + \int_{\tilde{E}} B \right) ds \\ &= P\left(\tilde{E},\ \Omega\right) + \int_{\tilde{E}} B \end{aligned}$$

As $\tilde{E}$ is a solution of problem (3.3), we have:

$$P\left(\tilde{E},\ \Omega\right) + \int_{\tilde{E}} B \leq P(\{\psi \geq s\},\ \Omega) + \int_{\{\psi \geq s\}} B \tag{3.13}$$

And by integrating (3.13) with respect to s, we get for any $\psi \in BV(\Omega)$, $0 \leq \psi \leq 1$:

$$\underbrace{P\left(\tilde{E},\ \Omega\right) + \int_{\tilde{E}} B}_{=G(\tilde{\psi})} \leq \underbrace{\int_0^1 \left(P(\{\psi \geq s\},\ \Omega) + \int_{\{\psi \geq s\}} B \right) ds}_{=G(\psi)} \tag{3.14}$$

And $\tilde{\psi}$ is therefore a solution of problem (3.4).

■

We have thus shown that problem (3.4) has at least one solution which is a characteristic function. We are in position to treat problem (3.2).

3.3 Study of problem (3.2)

Proposition 3.3. *Let $\tilde{E}$ be a solution of problem (3.3), and let $\tilde{\phi}$ be the Euclidean signed distance function to $\partial\tilde{E}$ (see Definition 3.1). Then $\tilde{\phi}$ is a solution of problem (3.2).*

To show Proposition (3.3), we will need the two following lemmas:

Lemma 3.1. *Let ϕ Lipschitz. We set $\psi = H(\phi)$. Then: $\int_{\{\phi=0\}} ds \geq \int_\Omega |D\psi|$.*

Proof: Thanks to the co-area formula [4], we have:

$$\int_\Omega |D\psi| = \int_{-\infty}^{+\infty} P(\{\psi \geq s\}, \Omega)\, ds = \int_0^1 P(\{\psi \geq s\}, \Omega)\, ds \tag{3.15}$$

since $0 \leq \psi \leq 1$. Thus $\int_\Omega |D\psi| = \int_0^1 P(\{H(\phi) \geq s\}, \Omega)\, ds$. We set $E = \{\phi > 0\}$ (E is an open set since ϕ is continuous). We therefore have $E = \{H(\phi) > s\}$, $\forall s > 0$. We have:

$$\int_0^1 P(\{H(\phi) \geq s\}, \Omega)\, ds = \int_0^1 P(E, \Omega)\, ds = P(E, \Omega) \tag{3.16}$$

Hence $\int_\Omega |D\psi| = P(E, \Omega)$. Then we conclude by remarking that:

$$P(E, \Omega) \leq \mathcal{H}^1(\partial E \bigcap \Omega) \leq \int_{\{\phi=0\}} ds \tag{3.17}$$

■

The following lemma is straightforward.

Lemma 3.2. *Let $A \in \mathcal{B}(\Omega)$. The Euclidean signed distance function u to A is 1-Lipschitz.*

Definition 3.1. If $A \in \mathcal{B}(\Omega)$, we call Euclidean signed distance function to A the function u defined by:

$$u(x) = \begin{cases} d(x, A) & \text{if } x \in \Omega \backslash A \\ d(x, \Omega \backslash A) & \text{if } x \in A \end{cases} \tag{3.18}$$

where $d(., A)$ stands for the Euclidean distance to A.

Proof of Proposition 3.3: Thanks to Lemma 3.2, we know that $\tilde{\phi}$ is 1-Lipschitz. Let $\phi \in Lip(\Omega)$. We denote by $E = \{\phi > 0\}$. Thanks to the proof of Lemma 3.1, we have: $\int_{\phi=0} ds \geq P(E, \Omega)$. Thus:

$$\int_{\phi=0} ds + \int_\Omega BH(\phi) \geq P(E, \Omega) + \int_\Omega B\mathbf{1}_E \geq P(\tilde{E}, \Omega) + \int_\Omega B\mathbf{1}_{\tilde{E}} \tag{3.19}$$

since $\tilde{E}$ is a solution of (3.3). Hence:

$$\int_{\phi=0} ds + \int_\Omega BH(\phi) \geq P(\tilde{E}, \Omega) + \int_\Omega BH(\tilde{\phi}) \tag{3.20}$$

But $\tilde{E}$ being a solution of (3.3), we know, thanks to Proposition 3.1, that $\tilde{E}$ has a $C^{1,1}$-boundary, and we therefore have: $P(\tilde{E}, \Omega) = \mathcal{H}^1(\partial \tilde{E} \bigcap \Omega) = \int_{\tilde{\phi}=0} ds$. We thus conclude that:

$$\int_{\phi=0} ds + \int_\Omega BH(\phi) \geq \int_{\tilde{\phi}=0} ds + \int_\Omega BH(\tilde{\phi}) \tag{3.21}$$

which shows that $\tilde{\phi}$ is a solution of problem (3.2).

■

Remark: In the proof of Proposition 3.3, we have shown the following result:

Lemma 3.3. *Let E an open set with C^1-boundary, and let ϕ the signed distance function to ∂E. Then: $\int_{\phi=0} ds = P(E, \Omega)$.*

3.4 Equivalence between problems (3.2), (3.3) and (3.4)

Now that we have studied problems (3.2), (3.3) and (3.4), and that we have seen that they present several connections, we want to state a kind of equivalence result between them. To do so, we will need the following result.

Proposition 3.4. *Let ϕ a solution of problem (3.2). Then $\psi = H(\phi)$ is a solution of problem (3.4).*

Proof: According to Lemma 3.1, we have: $\int_{\{\phi=0\}} ds \geq \int_\Omega |D\psi|$. Thus:

$$\int_{\phi=0} ds + \int_\Omega BH(\phi) \geq \int_\Omega |D\psi| + \int_\Omega B\psi \tag{3.22}$$

$$\geq \int_\Omega |D\tilde{\psi}| + \int_\Omega B\tilde{\psi} \tag{3.23}$$

where $\tilde{E}$ is a solution of problem (3.3), and $\tilde{\psi} = \mathbf{1}_{\tilde{E}}$ (thanks to Proposition 3.2, we know that $\tilde{\psi}$ is a solution of problem (3.4)). We denote by $\tilde{\phi}$ the signed distance function to $\partial\tilde{E}$. Thanks to Proposition 3.3, $\tilde{\phi}$ is a solution of problem (3.2), and we thus have:

$$\int_{\{\phi=0\}} ds + \int_\Omega BH(\phi) = \int_{\{\tilde{\phi}=0\}} ds + \int_\Omega BH(\tilde{\phi}) \tag{3.24}$$

But with Lemma 3.3, we have:

$$\int_{\phi=0} ds + \int_\Omega BH(\phi) = P(\tilde{E},\Omega) + \int_\Omega B\mathbf{1}_{\tilde{E}} = \int_{\tilde{\phi}=0} ds + \int_\Omega BH(\tilde{\phi}) \tag{3.25}$$

We conclude that (3.22) and (3.23) are in fact equalities, and therefore that ψ is a solution of problem (3.4).

■

Remark: In particular, we have shown that:

$$\inf_{\phi\in Lip(\Omega)} \left(\int_{\phi=0} ds + \int_\Omega BH(\phi) \right) = \inf_{\psi\in BV(\Omega),\ 0\leq\psi\leq 1} \left(\int_\Omega |D\psi| + \int_\Omega B\psi \right) \tag{3.26}$$

We have also proved the following result:

Lemma 3.4. *Let ϕ a solution of problem (3.2), and let $\psi = H(\phi)$. Then: $\int_{\phi=0} ds = \int_\Omega |D\psi|$.*

We have shown an equivalence between problems (3.2), (3.3) and (3.4), equivalence that we sum up with the following theorem:

Theorem 3.2.

1. *If ϕ is a solution of problem (3.2), then $E = \{\phi > 0\}$ is a solution of problem (3.3), and $\psi = H(\phi)$ is a solution of problem (3.4).*

2. *If E is a solution of problem (3.3), then $\psi = \mathbf{1}_E$ is a solution of problem (3.4), and ϕ the signed distance function to ∂E is a solution of problem (3.2).*

3. *If ψ is a solution of problem (3.4), and if we can write $\psi = \mathbf{1}_E$, then E is a solution of problem (3.3), and ϕ the signed distance function to ∂E is a solution of problem (3.2).*

Proof: This a consequence of Propositions 3.2, 3.3 and 3.4, and of Theorem 3.1.

■

In order to have a complete equivalence between problems (3.2), (3.3) and (3.4), we would need that any solution of (3.4) is a characteristic function, i.e. that it can be written $\psi = \mathbf{1}_E$. Unfortunately, this result does not hold in general.

Indeed, it is immediate to check that if B has zero mean, then any constant function (provided this constant lies between 0 and 1) is a solution of problem (3.4).

In fact, even when $\int_\Omega B \neq 0$, problem (3.4) can have solutions which are not characteristic functions, as shown by the following example.

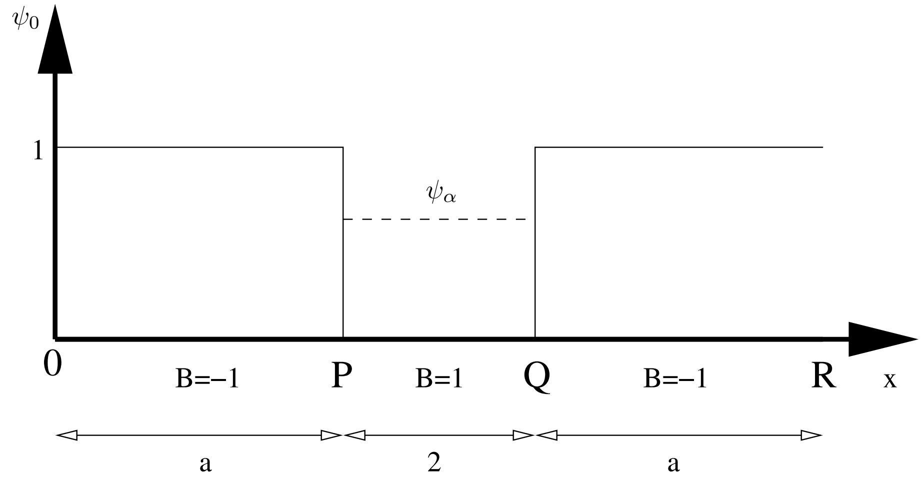

Figure 1: Example of solutions of problem (3.4) which are not characteristic functions

Example: We give here an example in dimension 1, of an open set Ω and of a data term function B such that problem (3.4) has solutions which are not characteristic functions. We recall that the functional we want to minimize in problem (3.4) is the following:

$$F(\psi) = \int_\Omega B\psi + \int_\Omega |D\psi| \tag{3.27}$$

We consider the case depicted on Figure 1. We choose $B = -1$ on the segments $[O, P]$ and $[Q, R]$ which have both the same length $a > 1$. And we choose $B = 1$ on the segment $[P, Q]$ which has a length equal to 2. Thanks to Proposition 3.2, we know that the infimum in (3.4) is a minimum, and that it is reached by at least one characteristic function. If ψ is a characteristic function, then: $F(\psi) = nb + \int_\Omega B\psi$, where nb stands for the number of jumps of ψ on Ω.

We consider some particular characteristic functions:

$$\psi_0 = \begin{cases} 1 \text{ on } \{B = -1\} \\ 0 \text{ on } \{B = 1\} \end{cases} \tag{3.28}$$

as well as $\psi_1 = 1$ on Ω, and $\psi_2 = 0$ on Ω. We have $F(\psi_2) = 0$, and $F(\psi_0) = 2-2a = F(\psi_1)$. As we have chosen $a > 1$, we also have $F(\psi_0) < F(\psi_2)$.

If ψ is a characteristic function, then $\int_\Omega B\psi \geq -2a$. Thus $F(\psi) \geq nb - 2a$. But $F(\psi_0) = 2 - 2a$; therefore if ψ is a characteristic function and is a solution of (3.4), then we necessarily have: $nb \leq 2$. It is then easy to check that in fact ψ_0 and ψ_1 reach the minimum of (3.4), and that the value of this minimum is equal to $2-2a$.

Now, we consider the familly of functions defined for $0 \leq \alpha \leq 1$:

$$\psi_\alpha = \begin{cases} 1 \text{ on } \{B = -1\} \\ \alpha \text{ on } \{B = 1\} \end{cases} \tag{3.29}$$

We have $F(\psi_\alpha) = 2(1-\alpha) - 2a + 2\alpha = 2 - 2a = F(\psi_0)$. Thus ψ_α is a solution of problem (3.4) $\forall \alpha \in [0, 1]$. And if $0 < \alpha < 1$, then ψ_α is not a characteristic function.

■

4. The approximated problems

We get interested here in the numerical model (1.6), as well as its link with the theoretical model of Section 3. For the study of the numerical model, we need to state results about the existence of solutions, and we also need to get information about the regularity of these solutions. Indeed, in the numerical method, we assume the existence of continuous solutions. To do this, we are first going to regularize the problem. We will then pass to the limit ($\alpha \to 0$) to come back to our initial problem, the main argument being a perturbation theorem by R. Temam [23, 12].

When α is fixed, we will show the existence of a continuous solution of problem (1.6) (this is the main result of this section).

4.1 Smooth approximations of H and δ

We will use the following smooth approximations of the Heaviside function H and the Dirac distribution δ . We assume that $0 \leq \alpha \leq 1$.

$$\delta_\alpha(s) = \begin{cases} \frac{1}{2\alpha}\left(1+\cos\frac{\pi s}{\alpha}\right) & \text{if } |s| \leq \tilde{s} \\ \alpha & \text{if } |s| \geq \tilde{s} \end{cases} \tag{4.1}$$

$$H_\alpha(s) = \begin{cases} \frac{1}{2}\left(1+\frac{s}{\alpha}+\frac{1}{\pi}\sin\frac{\pi s}{\alpha}\right) & \text{if } |s| \leq \tilde{s} \\ \frac{1}{2}\left(1+\frac{\tilde{s}}{\alpha}+\frac{1}{\pi}\sin\frac{\pi \tilde{s}}{\alpha}\right)+\alpha(s-\tilde{s}) & \text{if } s > \tilde{s} \\ \frac{1}{2}\left(1+\frac{-\tilde{s}}{\alpha}+\frac{1}{\pi}\sin\frac{-\pi \tilde{s}}{\alpha}\right)+\alpha(s+\tilde{s}) & \text{if } s < -\tilde{s} \end{cases} \tag{4.2}$$

where:

$$\tilde{s} = \frac{2\alpha}{\pi}\arccos(\alpha) \tag{4.3}$$

With this definition of $\tilde{s}$, we have (when $\alpha \to 0$): $0 \leq \tilde{s} \leq \alpha$ and $\tilde{s} \sim \alpha$. Moreover, $\delta_\alpha(\tilde{s}) = \delta_\alpha(-\tilde{s}) = \alpha$, $\lim_{\alpha\to 0} H_\alpha(\tilde{s}) = 1$, and $\lim_{\alpha\to 0} H_\alpha(-\tilde{s}) = 0$.

As Ω is bounded, we have when $\alpha \to 0$: $\delta_\alpha \to \delta$ and $H_\alpha \to H$ (in the distributional sense).

Lemma 4.1. *For all α, $0 < \alpha \leq 1$, H_α is a C^1-nondecreasing diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Moreover, $\psi_\alpha = H_\alpha(\phi)$ is Lipschitz if and only if the function ϕ is Lipschitz.*

Proof: We have $H_\alpha^{'} = \delta_\alpha$, thus: $\alpha \leq H_\alpha^{'} \leq \frac{1}{\alpha}$. And $\left(H_\alpha^{-1}(y)\right)^{'} = \frac{1}{H_\alpha^{'}(H_\alpha(y))} = \frac{1}{\delta_\alpha(H_\alpha(y))}$. Therefore: $\alpha \leq \left(H_\alpha^{-1}\right)^{'} \leq \frac{1}{\alpha}$.

■

Remark: We recall that the problem we want to solve numerically in the two phases case is:

$$\inf_{\phi\in Lip(\Omega)} \left(\int_\Omega \delta_\alpha(\phi)|\nabla\phi| + \int_\Omega BH_\alpha(\phi)\right) \tag{4.4}$$

We can show that (see [6]):

$$\lim_{\alpha\to 0^+}\left\{\int_\Omega \delta_\alpha(\phi)|\nabla\phi|\right\} = \int_{\phi=0} d\,s \tag{4.5}$$

We easily deduce that (3.2) is the limit problem of (4.4) when $\alpha \to 0^+$.

There remains to check if problem (4.4) admits a solution for $\alpha > 0$. There exists a very simple link between problem (4.4) and problem (3.4) which we recall here:

$$\inf_{\psi\in BV(\Omega),\ 0\leq\psi\leq 1} \left(\int_\Omega |D\psi| + \int_\Omega B\psi\right) \tag{4.6}$$

Indeed, if we set: $\psi_\alpha = H_\alpha(\phi)$, than $D\psi_\alpha = \delta_\alpha(\phi)\nabla\phi$, and for all $\alpha > 0$:

$$\int_\Omega BH_\alpha(\phi) + \int_\Omega \delta_\alpha(\phi)|\nabla\phi| = \int_\Omega B\psi_\alpha + \int_\Omega |\nabla\psi_\alpha| \tag{4.7}$$

4.2 Locally continuous solutions

As we do not know how to show directly that (4.4) has solutions (when α is fixed), since it is difficult to get some properties for minimizing sequences, we consider an approximated problem, as suggested in [12, 23]:

$$\inf_{\phi\in W^{1,2}(\Omega)} F^\epsilon_{\alpha,\beta}(\phi) \tag{4.8}$$

where:

$$F^\epsilon_{\alpha,\beta}(\phi) = \left(\int_\Omega g_\beta(\delta_\alpha(\phi)|\nabla\phi|) + \int_\Omega BH_\alpha(\phi) + \int_\Omega h_\beta(H_\alpha(\phi))\right) + \frac{\epsilon}{2}\int_\Omega (\delta_\alpha(\phi)|\nabla\phi|)^2 \tag{4.9}$$

where the function $g_\beta : \mathbb{R}^2 \to \mathbb{R}$ is defined by:

$$g_\beta(\xi) = \sqrt{|\xi|^2 + \beta^2} \tag{4.10}$$

and the function $h_\beta : \mathbb{R} \to \mathbb{R}$ by:

$$h_\beta(x) = \begin{cases} \frac{x^4}{\beta^5} & \text{if } x \leq 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ \frac{(x-1)^4}{\beta^5} & \text{if } x \geq 1 \end{cases} \tag{4.11}$$

Here $W^{1,p}$ denotes the Sobolev space: $\{\phi \in L^p(\Omega);\ \nabla\phi \in L^p(\Omega);\ 1 \leq p \leq \infty\}$ [1]. The function h_β is convex and C^3 on $\mathbb{R}$. When we compute the Euler-Lagrange equation associated to functional (4.8), the term coming from $\int_\Omega g_\beta(\delta_\alpha(\phi)|\nabla\phi|)$ corresponds to the one used in the numerical scheme (we indeed introduce a parameter $\beta > 0$ to avoid to divide by 0) (see [22]). The function h_β will replace the condition $0 \leq \psi \leq 1$ in the approximation of problem (4.6) (or (3.4)).

It is still unclear to show directly that (4.8) admits a solution on $W^{1,2}(\Omega)$, since $F^\epsilon_{\alpha,\beta}(\phi)$ is only coercive on $W^{1,1}(\Omega)$. Let us set

$$\psi_\alpha = H_\alpha(\phi) \tag{4.12}$$

This implies $\nabla\psi_\alpha = \delta_\alpha(\phi)\nabla\phi$ (since $\phi \in W^{1,2}(\Omega)$). This leads us to consider the problem:

$$\inf_{\psi \in W^{1,2}(\Omega)} G^\epsilon_\beta(\psi) \tag{4.13}$$

where:

$$G^\epsilon_\beta(\psi) = \left(\int_\Omega g_\beta(|\nabla\psi|) + \int_\Omega B\psi + \int_\Omega h_\beta(\psi)\right) + \frac{\epsilon}{2}\int_\Omega |\nabla\psi|^2 \tag{4.14}$$

In fact, we will show later (Proposition 4.2) that problems (4.8) and (4.13) are equivalent.

In all the sequel, parameter β will depend on parameter α by the relation:

$$\beta(\alpha) = \left(\frac{4\alpha^3}{\|B\|_{L^\infty(\Omega)}}\right)^{\frac{1}{5}} \tag{4.15}$$

(we will see the justification of this relation in the proof of Proposition 4.1).

We have introduced problems (4.13) and (4.8) in order to get existence and regularity results about their solutions. We then want to let $\epsilon \to 0$ to get information about the solution of problems:

$$\inf_{\phi \in W^{1,1}(\Omega) \bigcap L^2(\Omega)} F^0_{\alpha,\beta}(\phi) \tag{4.16}$$

and:

$$\inf_{\phi \in W^{1,1}(\Omega) \bigcap L^2(\Omega)} G^0_\beta(\psi) \tag{4.17}$$

Remark: In problems (4.13) and (4.17), there is no constraint $0 \leq \psi \leq 1$. This constraint comes from the function h_β (see 4.11).

4.3 A perturbation result

We are going to use a perturbation result due to R. Temam [23, 12] (theorem 1.1 page 125 of [23], [12] page 140). This theorem is itself based on results by Ladyzenskaya and Ural'Ceva [15, 16].

The result by Temam has already been used successfully in image processing in [7] in a slightly modified form. This is this last result that we are going to use: we consider a functional of the form:

$$\inf_{\psi \in W^{1,2}(\Omega)} \left\{\int_\Omega g(|\nabla\psi|) + \int_\Omega p(x)\psi + \int_\Omega h(\psi) + \frac{\epsilon}{2}\int_\Omega |\nabla\psi|^2\right) \tag{4.18}$$

We assume that the following hypotheses hold:

(H1) The function $\xi \to g(\xi)$ is convex and C^3 from $\mathbb{R}^2$ into $\mathbb{R}$.

(H2) The function $x \to g(q(x))$ is measurable on Ω for all q in $L^1(\Omega)^2$.

There exist some constants $\mu_i \geq 0$, $i = 0, \dots, 8$, such that for all $\xi \in \mathbb{R}^2$:

(H3)

$$g(\xi) \geq \mu_0 |\xi| - \mu_1 \ , \ \mu_0 > 0$$

(H4)

$$\frac{\partial g}{\partial \xi_i}(\xi) \leq \mu_2 \ , \ i = 1, 2$$

(H5)

$$\sum_{i=1}^{2} \frac{\partial g}{\partial \xi_i}(\xi)\xi_i \geq \mu_3 \sqrt{1 + |\xi|^2} - \mu_4 \ , \ \mu_3 > 0$$

(H6)

$$\frac{\mu_6 |\eta'|^2}{\sqrt{1 + |\xi|^2}} \leq \sum_{i,j} \frac{\partial^2 g}{\partial \xi_i \partial \xi_j}(\xi)\eta_i \eta_j \leq \frac{\mu_7 |\eta'|^2}{\sqrt{1 + |\xi|^2}} \ \forall \eta \in \mathbb{R}^2, \ \mu_6, \mu_7 > 0$$

where $|\eta'|^2 = |\eta|^2 - \frac{(\eta.\xi)^2}{1+|\xi|^2}$.

(H7)

$$\|p\|_{W^{1,\infty}} \leq \mu_8$$

(H8)

$$\sum_{i=1}^{2} \frac{\partial g}{\partial \xi_i}(\xi)\xi_i \geq 0 \ \forall \xi \in \mathbb{R}^2$$

(H9) The function $t \to h(t)$ is convex and $h'(0) = 0$.

Theorem 4.1. *[Temam [12, 23]]. Ω is assumed to be C^2. Then problem (4.18) has a smooth solution ψ_ϵ bounded independently of ϵ in $L^\infty(\Omega) \bigcap W^{1,1}(\Omega)$. This solution is unique up to an additive constant. Moreover, for any relatively compact open set V in Ω, there exists a constant $K(V, \Omega)$ independent of ϵ such that:*

$$\|\psi_\epsilon\|_{W^{1,\infty}(V)} \leq K(V, \Omega) \tag{4.19}$$

$$\|\psi_\epsilon\|_{H^2(V)} \leq K(V, \Omega) \tag{4.20}$$

In particular, we can apply this theorem to problem (4.13).
We even have a maximum principle result for problem (4.13):

Proposition 4.1. *The solution $\psi_{\beta,\epsilon}$ of problem (4.13) given by Theorem 4.1 is such that:*

$$-A(\beta) \leq \psi_{\beta,\epsilon} \leq 1 + A(\beta) \tag{4.21}$$

where

$$A(\beta) = \left(\frac{\beta^5 \|B\|_{L^\infty(\Omega)}}{4} \right)^{\frac{1}{3}} \tag{4.22}$$

and

$$\|\psi_{\beta,\epsilon}\|_{W^{1,1}(\Omega)} \leq |\Omega| \left(1 + A(\beta) + \beta + (1 + A(\beta))\|B\|_{L^\infty(\Omega)}\right) \tag{4.23}$$

In particular, if $\beta \leq 1$ (which is equivalent to $\alpha \leq \frac{\|B\|^{1/3}_{L^\infty(\Omega)}}{4^{5/3}}$, see (4.15)):

$$-A(1) \leq \psi_{\beta,\epsilon} \leq 1 + A(1) \tag{4.24}$$

and

$$\|\psi_{\beta,\epsilon}\|_{W^{1,1}(\Omega)} \leq |\Omega| \left(A(1) + 2 + (1 + A(1))\|B\|_{L^\infty(\Omega)}\right) \tag{4.25}$$

This result will help us to see what happens when $\alpha \to 0$.

Proof of Proposition 4.1: We consider for ϵ and $\lambda > 0$, the problem:

$$\inf_{\phi \in W^{1,2}(\Omega) \bigcap L^2(\Omega)} G^\epsilon_\beta(\psi) + \lambda \int_\Omega |\psi|^2 \tag{4.26}$$

where G^ϵ_β is given by (4.14). The existence and the uniqueness of a solution $\psi^\lambda_{\epsilon,\beta}$ are standard. We then split the proof into three parts.

Step 1: We are going to show that

$$\|\psi^\lambda_{\epsilon,\beta}\|_{L^\infty(\Omega)} \leq M_0 \tag{4.27}$$

(we will set the constant M_0 in the sequel). To do this, we are first going to show that:

$$\psi^\lambda_{\epsilon,\beta}(x) \leq M_1 \text{ a.e. } x \in \Omega \tag{4.28}$$

where M_1 is a constant to be also precised later. For all $v \in W^{1,2}(\Omega) \bigcap L^2(\Omega)$, $\psi^\lambda_{\epsilon,\beta}$ verifies:

$$\int_\Omega \left[\epsilon(\nabla\psi^\lambda_{\epsilon,\beta}, \nabla v) + 2\lambda\psi^\lambda_{\epsilon,\beta} v + \sum_{i=1}^2 \frac{\partial g_\beta}{\partial \xi_i}(\nabla\psi^\lambda_{\epsilon,\beta}) \frac{\partial v}{\partial x_i} + (h'_\beta(\psi^\lambda_{\epsilon,\beta}) + B)v \right] dx = 0 \tag{4.29}$$

We then choose $v = (\psi^\lambda_{\epsilon,\beta} - M_1)_+$ (where the function $(x)_+$ is equal to x if $x \geq 0$, and 0 otherwise). We then get:

$$\int_{\Omega \bigcap \{\psi^\lambda_{\epsilon,\beta} > M_1\}} \underbrace{[\epsilon|\nabla\psi^\lambda_{\epsilon,\beta}|^2 + 2\lambda\psi^\lambda_{\epsilon,\beta}(\psi^\lambda_{\epsilon,\beta} - M_1)]}_{\geq 0} dx$$

$$= \int_{\Omega \bigcap \{\psi^\lambda_{\epsilon,\beta} > M_1\}} [\underbrace{- \sum_{i=1}^2 \frac{\partial g_\beta}{\partial \xi_i}(\nabla\psi^\lambda_{\epsilon,\beta}) \frac{\partial \psi^\lambda_{\epsilon,\beta}}{\partial x_i}}_{\leq 0 \text{ thanks to } (\mathrm{H}\ \mathbf{8})} - (h'_\beta(\psi^\lambda_{\epsilon,\beta}) + B)(\psi^\lambda_{\epsilon,\beta} - M_1)]\, dx$$

Hence:

$$\left(-h'_\beta(M_1) + \|B\|_{L^\infty(\Omega)}\right)(\psi^\lambda_{\epsilon,\beta} - M_1)_+ \geq 0 \tag{4.30}$$

If we impose

$$h'_\beta(M_1) > \|B\|_{L^\infty(\Omega)} \tag{4.31}$$

we then have $(\psi^\lambda_{\epsilon,\beta} - M_1)_+ = 0$, which proves (4.28). We just need to set M_1 so that (4.31) be fulfilled. But:

$$h'_\beta(x) = \begin{cases} \frac{4x^3}{\beta^5} & \text{if } x \leq 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ \frac{4(x-1)^3}{\beta^5} & \text{if } x \geq 1 \end{cases} \tag{4.32}$$

We deduce that (4.31) holds if and only if ($M_1 \geq 1$):

$$\frac{4(M_1 - 1)^3}{\beta^5} > \|B\|_{L^\infty(\Omega)} \tag{4.33}$$

i.e. if and only if:

$$M_1 > 1 + \left(\frac{\beta^5\|B\|_{L^\infty(\Omega)}}{4}\right)^{\frac{1}{3}} \tag{4.34}$$

In the same way as above, we also get:

$$\psi^\lambda_{\epsilon,\beta}(x) \geq M_2 \text{ a.e. } x \in \Omega \tag{4.35}$$

if

$$M_2 < -\left(\frac{\beta^5\|B\|_{L^\infty(\Omega)}}{4}\right)^{\frac{1}{3}} \tag{4.36}$$

By setting $M_0 = \max(M_1, -M_2) = M_1$, we deduce (4.27).

Step 2: We want to bound: $\|\nabla\psi^{\lambda}_{\epsilon,\beta}\|_{L^1(\Omega)}$ We have:

$$G^{\epsilon}_{\beta}(\psi^{\lambda}_{\epsilon,\beta}) + \lambda \int_{\Omega} |\psi^{\lambda}_{\epsilon,\beta}|^2 \leq G^{\epsilon}_{\beta}(0) = \beta|\Omega| \tag{4.37}$$

where $|\Omega| = \int_{\Omega} dx$. But (thanks to (4.27))

$$G^{\epsilon}_{\beta}(\psi^{\lambda}_{\epsilon,\beta}) + \lambda \int_{\Omega} |\psi^{\lambda}_{\epsilon,\beta}|^2 \geq \|\nabla\psi^{\lambda}_{\epsilon,\beta}\|_{L^1(\Omega)} - \|B\|_{L^{\infty}(\Omega)}|\Omega|M_0 \tag{4.38}$$

Hence

$$\|\nabla\psi^{\lambda}_{\epsilon,\beta}\|_{L^1(\Omega)} \leq |\Omega| \left(\beta + M_0\|B\|_{L^{\infty}(\Omega)}\right) \tag{4.39}$$

We show in the same way as above that:

$$\|\nabla\psi^{\lambda}_{\epsilon,\beta}\|_{L^2(\Omega)} \leq \sqrt{\frac{2M_0\|B\|_{L^{\infty}(\Omega)}|\Omega|}{\epsilon}} \tag{4.40}$$

On the other hand, thanks to (4.27), we get:

$$\|\psi^{\lambda}_{\epsilon,\beta}\|_{W^{1,2}(\Omega)} \leq \frac{c}{\sqrt{\epsilon}} \tag{4.41}$$

where c is a constant which does not depend from λ.

Step 3: By letting $\lambda \to 0$, we get the existence of $\psi_{\epsilon,\beta} \in W^{1,2}(\Omega)\bigcap L^{\infty}(\Omega)$, and $\psi_{\epsilon,\beta}$ vérifie (4.27). Moreover, we show in the same way as in the preceeding step that $\nabla\psi_{\epsilon,\beta}$ still verifies (4.39). We then have (thanks to (4.27)):

$$\|\psi^{\lambda}_{\epsilon,\beta}\|_{W^{1,1}(\Omega)} \leq M_0|\Omega| + |\Omega| \left(\beta + M_0\|B\|_{L^{\infty}(\Omega)}\right) \tag{4.42}$$

where we recall that $M_0 \geq 1 + \left(\frac{\beta^5\|B\|_{L^{\infty}(\Omega)}}{4}\right)^{\frac{1}{3}}$.

■

Corollary 4.1. *We assume that parameter β is given by (4.15). We then have:*

$$-\alpha \leq \psi_{\beta,\epsilon} \leq 1 + \alpha \tag{4.43}$$

Proof: This is an immediate consequence of Proposition 4.1.

■

Thanks to Theorem 4.1, we can let $\epsilon \to 0$ in problem (4.13). We thus get the following result (see [12, 23]):

Corollary 4.2. *We assume that $B \in W^{1,\infty}(\Omega)$, and Ω is C^2. Problem (4.17) admits a solution ψ_β in $L^{\infty}(\Omega)\bigcap W^{1,1}(\Omega)$, and ψ_β is unique up to an additive constant. Moreover, for any relatively compact open set V in Ω, there exists a constant $K(V,\Omega)$ such that:*

$$\|\psi_\beta\|_{W^{1,\infty}(V)} \leq K(V,\Omega) \tag{4.44}$$

$$\|\psi_\beta\|_{H^2(V)} \leq K(V,\Omega) \tag{4.45}$$

Remark: From (4.45), we deduce that ψ_β belongs to $Lip(V)$ for any relatively compact open set V in Ω (thanks to Sobolev injections [1, 9]).

4.4 "Equivalence" between problems (4.8) and (4.13)

We can now show that when α is fixed, problems (4.16) and (4.17) have the same value.

Proposition 4.2. *We have:*

$$\inf_{\phi} F^0_{\alpha,\beta(\alpha)}(\phi) = \inf_{\psi} G^0_{\beta}(\psi) \tag{4.46}$$

Proof: Let $\phi \in W^{1,1}(\Omega) \bigcap L^2(\Omega)$. We set $\psi_\alpha = H_\alpha(\phi)$. We have:

$$F^0_{\alpha,\beta}(\phi) = G^0_\beta(\psi_\alpha) \geq \inf_\psi G^0_\beta(\psi) \tag{4.47}$$

And with Theorem 4.1, we know that problem (4.13) admits a solution ψ_β. Therefore:

$$F^0_{\alpha,\beta}(\phi) \geq \inf_\phi F^0_{\alpha,\beta}(\phi) \geq \underbrace{\inf_\psi G^0_\beta(\psi)}_{=G^0_\beta(\psi_\beta)} \tag{4.48}$$

Thanks to Lemma 4.1, we can define $\phi_{\alpha,\beta}$ such that $H_\alpha(\phi_{\alpha,\beta}) = \psi_\beta$. We then get:

$$\underbrace{F^0_{\alpha,\beta}(\phi_{\alpha,\beta})}_{=G^0_\beta(\psi_\beta)} \geq \inf_\phi F^0_{\alpha,\beta}(\phi) \geq \underbrace{\inf_\psi G^0_\beta(\psi)}_{=G^0_\beta(\psi_\beta)} \tag{4.49}$$

from which we deduce (4.46).

■

4.5 Back to ϕ

We work here with $\alpha > 0$ fixed. We have got (4.6) from (4.4) with $\psi = H_\alpha(\phi)$. To come back to ϕ, we just need to do the reverse operation.

Corollary 4.3. *Problem (4.16) has a solution $\phi_\alpha \in L^\infty(\Omega) \bigcap W^{1,1}(\Omega)$. Moreover, for any relatively compact open set V in Ω, we have: $\phi_\alpha \in Lip(V)$.*

Proof: We consider ψ_β a solution of (4.13) given by Corollary 4.2. We set $\phi_\alpha = H_\alpha^{-1}(\psi_\beta)$ (which is possible thanks to Lemma 4.1). And we conclude thanks to Proposition 4.2.

■

Corollary 4.4. *β being given by (4.15), we have:*

$$-1 - \alpha \leq \phi_\alpha \leq 1 + \alpha \tag{4.50}$$

Proof: This an immediate consequence of Corollary 4.1 and of the definition of H_α (4.2).

■

4.6 Letting $\alpha \to 0$

Proposition 4.3. *There exists $\tilde{\psi} \in BV(\Omega)$ and a sequence β_n, with $\beta_n \to 0$ as $n \to +\infty$, such that $\psi_{\beta_n} \to \tilde{\psi}$ in BV weak * (where ψ_β is a solution of (4.13) with $\beta = \beta_n$ given by Corollary 4.2). Moreover, $\tilde{\psi}$ is a solution of problem (4.6).*

Proof: We just need to let $\beta(\alpha) \to 0$ (i.e. $\alpha \to 0$) in (4.17) (with $\epsilon = 0$) thanks to Proposition 4.1.

■

Remark: Letting $\beta(\alpha) \to 0$ does not preserve more regularity for ψ. In this case, we do not know how to come back to ϕ.

In practice, we set α to a fixed value α_0 in the numerical model. Indeed, if α is too small, the narrow band introduced by the support of δ_α is then itself too small to make any discretization of the equations. In fact, as α decreases to 0, we would also need to make the discretization step (of the evoluting equation) decreases (this last one is equal to the pixel size) (see [5]).

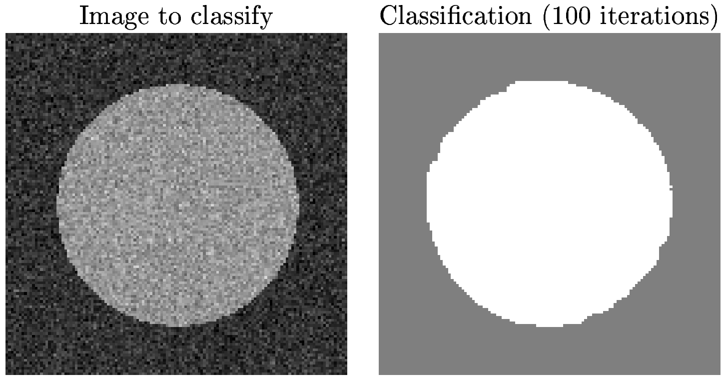

Figure 2: Classification of a synthetic image composed of two classes (algorithm of [22])

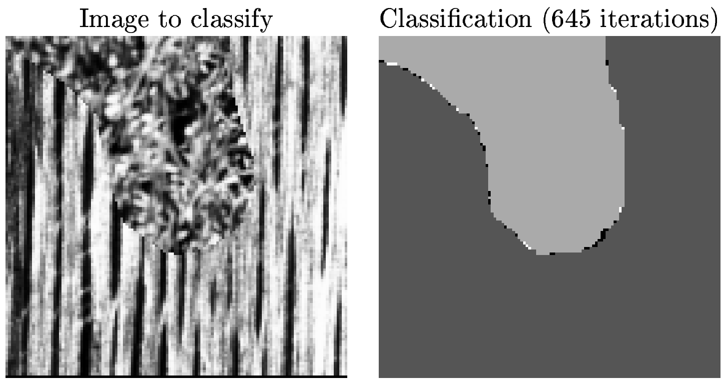

Figure 3: Classification of a synthetic image composed of two textures (algorithm of [8])

5. Experimental classification results

In this final section, we want to show some experimental classification result. We follow the numerical approach used in [8, 22, 21, 10, 24, 6]. To minimize (1.6), we consider the associated Euler-Lagrange equation (we assume that Neumann conditions are verified):

$$0 = \delta_\alpha(\phi)\left(\frac{\nabla\phi}{|\nabla\phi|} + B(x)\right) \tag{5.1}$$

We embed it in the following dynamical scheme:

$$\frac{\partial\phi}{\partial t} = -\delta_\alpha(\phi)\left(\text{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + B(x)\right) \tag{5.2}$$

We discretize this system with finite differences (see [22, 8] for more details).

We display here two examples. The first one (Figure 2) is obtained with the algorithm of [22]. This algorithm deals with the classification of non-textured images. The second one (Figure 3) is obtained with the algorithm of [8]. This algorithm deals with the classification of textured images.

In [22, 8], one can find examples of classification results in the case of $I > 2$ phases.

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