Image decomposition
Application to SAR images

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Abstract. We construct an algorithm to split an image into a sum $u + v$ of a bounded variation component and a component containing the textures and the noise. This decomposition is inspired from a recent work of Y. Meyer. We find this decomposition by minimizing a convex functional which depends on the two variables u and v, alternatively in each variable. Each minimization is based on a projection algorithm to minimize the total variation. We carry out the mathematical study of our method. We present some numerical results. In particular, we show how the u component can be used in nontextured SAR image restoration.

Key-words: Total variation minimization, BV, texture, classification, restoration, SAR images, speckle.

1 Introduction

1.1 Preliminaries

Image restoration is one of the major goals of image processing. A classical approach consists in considering that an image f can be decomposed into two components $u + v$. The first component u is well-structured, and has a simple geometric description: it models the homogeneous objects which are present in the image. The second component v contains both textures and noise. An ideal model would split an image into three components $u + v + w$, where v should contain the textures of the original image, and w the noise.

In Section 1, we begin by recalling some models proposed in the literature. Then our model is introduced in Section 2. We give a powerful algorithm to compute the image decomposition we want to get. We carry out the mathematical

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study of our model in Section 3. We then show some experimental results. In Section 4, we give an application to SAR images, the u component being a way to carry out efficient restoration.

1.2 Related works

Rudin-Osher-Fatemi's model: Images are often assumed to be in BV, the space of functions with bounded variation (even if it is known that such an assumption is too restrictive [1]). We recall here the definition of BV:

Definition 1. *$BV(\Omega)$ is the subspace of functions $u \in L^1(\Omega)$ such that the following quantity is finite:*

$$J(u) = \sup \left\{ \int_\Omega u(x) \operatorname{div}(\xi(x)) dx / \xi \in C_c^1(\Omega; \mathbb{R}^2), \|\xi\|_{L^\infty(\Omega)} \leq 1 \right\} \tag{1}$$

where $C_c^1(\Omega;\mathbb{R}^2)$ is the space of functions in $C^1(\Omega;\mathbb{R}^2)$ with compact support in Ω. $BV(\Omega)$ endowed with the norm $\|u\|_{BV} = \|u\|_{L^1} + J(u)$ is a Banach space.

If $u \in BV(\Omega)$, the distributional derivative Du is a bounded Radon measure and (1) corresponds to the total variation $|Du|(\Omega)$.

In [2], the authors decompose an image f into a component u belonging to $BV(\Omega)$ and a component v in $L^2(\Omega)$. In this model v is supposed to be the noise. In such an approach, they minimize (see [2]):

$$\inf_{(u,v)\in BV(\Omega)\times L^2(\Omega)/f=u+v} \left(J(u) + \frac{1}{2\lambda} \|v\|^2_{L^2(\Omega)} \right) \tag{2}$$

In practice, they try to compute a numerical solution of the Euler-Lagrange equation associated to (2). The mathematical study of (2) has been done in [4].

Meyer's model: In [3], Y. Meyer points out some limitations of the model developed in [2]. He proposes a variant which he believes is more adapted:

$$\inf_{(u,v)\in BV(\mathbb{R}^2)\times G(\mathbb{R}^2)/f=u+v} \left(J(u) + \lambda \|v\|_G \right) \tag{3}$$

The Banach space $G(\mathbb{R}^2)$ contains signals with large oscillations, and thus in particular textures and noise. We give here the definition of $G(\mathbb{R}^2)$.

Definition 2. *$G(\mathbb{R}^2)$ is the Banach space composed of the distributions f which can be written*

$$f = \partial_x g_1 + \partial_y g_2 = \operatorname{div}(g) \tag{4}$$

with g_1 and g_2 in $L^\infty(\mathbb{R}^2)$. On G, the following norm is defined:

$$\|v\|_G = \inf \left\{ \|g\|_{L^\infty(\mathbb{R}^2)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^2} |g(x)| \; / v = \operatorname{div}(g), \; g = (g_1, g_2), \right.$$
$$\left. g_1 \in L^\infty(\mathbb{R}^2), g_2 \in L^\infty(\mathbb{R}^2), |g(x)| = \sqrt{|g_1|^2 + |g_2|^2}(x) \right\} \tag{5}$$

In the space G, very oscillating functions have a small norm (see [3]) (and large oscillations are linked with textures and noises).

Vese-Osher's model: L. Vese and S. Osher have first proposed an approach for the resolution of Meyer's program. They have studied the problem (see [5]):

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)} \left(\int |Du| + \lambda \|f-u-v\|_2^2 + \mu \|v\|_{G(\Omega)} \right) \tag{6}$$

where Ω is a bounded open set. To compute their solution, they replace the term $\|v\|_{G(\Omega)}$ by $\|\sqrt{g_1^2+g_2^2}\|_p$ (where $v = \operatorname{div}(g_1, g_2)$). It approximates (6) when p goes to $+\infty$. For numerical reasons, the authors use the value $p = 1$ and they claim they did not see any visual difference when they used larger values for p. Then they formally derive the Euler-Lagrange equations. They report good numerical results.

These two authors, together with A. Solé, have proposed another approach to this problem in [6], where they propose a more direct algorithm in the case $\lambda = +\infty$ and $p = 2$.

2 Our approach

In this section we introduce our model.It is inspired from the formulation of [5]. We first present it in the continuous setting. Then we propose a discretization, and provide a mathematical study and an algorithm for the discretized model.

2.1 Presentation

We propose to solve the problem:

$$\inf_{(u,v)\in BV(\Omega)\times G_\mu(\Omega)} \left(J(u) + \frac{1}{2\lambda} \|f-u-v\|_{L^2(\Omega)}^2 \right) \tag{7}$$

where

$$G_\mu(\Omega) = \{v \in G(\Omega) / \|v\|_G \leq \mu\} \tag{8}$$

We recall that $\|v\|_G$ is defined by (5). The parameter μ plays the same role as the one in problem (6). The larger μ is, the more v contains information, and therefore the more u is averaged. The smaller λ is, the smaller the L^2 norm of the residual $f-u-v$ is. We will render more precisely the link of our model with Meyer's one later. Let us introduce the following functional defined on $BV(\Omega) \times G(\Omega)$:

$$F(u,v) = \begin{cases} J(u) + \frac{1}{2\lambda} \|f-u-v\|_{L^2(\Omega)}^2 & \text{if } v \in G_\mu(\Omega) \\ +\infty & \text{if } v \in G(\Omega) \backslash G_\mu(\Omega) \end{cases} \tag{9}$$

$F(u,v)$ is finite if and only if (u,v) belongs to $BV(\Omega) \times G_\mu(\Omega)$. Problem (7) can thus be written:

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)} F(u,v) \tag{10}$$

2.2 Discretization

We study (10) in the discrete case. We take here the same notations as in [7]. The image is a two dimensional array of size $N \times N$. We denote by X the Euclidean space $\mathbb{R}^{N \times N}$, and $Y = X \times X$. The space X will be endowed with the scalar product $(u,v)_X = \sum_{1 \leq i,j \leq N} u_{i,j} v_{i,j}$ and the norm $\|u\|_X = \sqrt{(u,u)_X}$. In Y, we use the Euclidean scalar product $(p,q)_Y = \sum_{1 \leq i,j \leq N} p^1_{i,j} q^1_{i,j} + p^2_{i,j} q^2_{i,j}$ with $p = (p^1, p^2)$ and $q = (q^1, q^2)$ in Y. To define a discrete total variation, we introduce a discrete version of the gradient operator. If $u \in X$, the gradient ∇u is a vector in Y given by: $(\nabla u)_{i,j} = ((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j})$. with

$$(\nabla u)^1_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N \end{cases} \text{ and } (\nabla u)^2_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < N \\ 0 & \text{if } j = N \end{cases}$$

The discrete total variation of u is then defined by:

$$J(u) = \sum_{1 \leq i,j \leq N} |(\nabla u)_{i,j}| \tag{11}$$

We also introduce a discrete version of the divergence operator. We define it by analogy with the continuous setting by div $= -\nabla^*$ where ∇^* is the adjoint of ∇: that is, for every $p \in Y$ and $u \in X$, $(-\text{div}\, p, u)_X = (p, \nabla u)_Y$. It is easy to check that:

$$(\text{div}\,(p))_{i,j} = \begin{cases} p^1_{i,j} - p^1_{i-1,j} & \text{if } 1 < i < N \\ p^1_{i,j} & \text{if i=1} \\ -p^1_{i-1,j} & \text{if i=N} \end{cases} + \begin{cases} p^2_{i,j} - p^2_{i,j-1} & \text{if } 1 < j < N \\ p^2_{i,j} & \text{if j=1} \\ -p^2_{i,j-1} & \text{if j=N} \end{cases} \tag{12}$$

We are now in position to introduce the discrete version of the space G.

Definition 3.

$$G^d = \{v \in X \ / \ \exists g \in Y \ \textit{such that } v = \text{div}\,(g)\} \tag{13}$$

and if $v \in G^d$*:*

$$\|v\|_{G^d} = \inf \{\|g\|_\infty \ / \ v = \text{div}\,(g), \quad g = (g^1, g^2) \in Y, |g_{i,j}| = \sqrt{(g^1_{i,j})^2 + (g^2_{i,j})^2}\} \tag{14}$$

where $\|g\|_\infty = \max_{i,j} |g_{i,j}|$.

Moreover, we will denote:

$$G^d_\mu = \left\{v \in G^d \ / \ \|v\|_{G^d} \leq \mu\right\} \tag{15}$$

We notice that

$$J(u) = \sup_{v \in G^d_1} (v,u)_X \tag{16}$$

and

$$\|v\|_{G^d} = \sup_{u \in X, J(u) \leq 1} (u,v)_X \tag{17}$$

Proposition 1. *The space G^d identifies with the following subspace:*

$$X_0 = \{v \in X \;/\; \sum_{i,j} v_{i,j} = 0\} \tag{18}$$

Proof: Choose $v \in G^d$. There exists $g \in Y$ such that: $v = \mathrm{div}\,(g)$. But $\sum_{i,j}(\mathrm{div}\, g)_{i,j} = (-\nabla^* g, 1)_Y = (g, \nabla 1)_X = 0$ i.e. $v \in X_0$. Hence $G^d \subset X_0$.

Conversely, let $v \in X_0$. Since the kernel of ∇ is the constant images, i.e. the vectors $x \in X$ such that $x_{i,j} = x_{i',j'}$ for all i, j, i', j', it is clear that a discrete Poincaré inequality holds: $\|x - \frac{1}{N^2}\sum_{i,j} x_{i,j}\|_X \le c\|\nabla x\|_Y$. Hence one shows easily that the problem $\min_{x\in X} A(x)$, with $A(x) = \|\nabla x\|_Y^2 + 2(x,v)_X$, has a solution. This solution satisfies $A'(x) = 0$, that is, $-2\mathrm{div}\,(\nabla x) + 2v = 0$. Hence $v = \mathrm{div}\,(\nabla x) \in G^d$, and we conclude that $X_0 \subset G^d$.

■

The discretized functional associated to (9), defined on $X \times X$, is given by:

$$F(u,v) = \begin{cases} J(u) + \frac{1}{2\lambda}\|f - u - v\|_X^2 & \text{if } v \in G_\mu^d \\ +\infty & \text{if } v \in X \backslash G_\mu^d \end{cases} \tag{19}$$

The problem we want to solve is:

$$\inf_{(u,v)\in X\times X} F(u,v) \tag{20}$$

2.3 Total variation minimization as a projection

Introduction: We recall that the Legendre-Fenchel transform of J is:

$$J^*(v) = \sup_u \left((u,v)_X - J(u)\right) \tag{21}$$

Since here J defined by (1) is homogeneous of degree one (i.e. $J(\lambda u) = \lambda J(u)\ \forall u$ and $\lambda > 0$), it is then standard (see [8]) that J^* is the indicator function of some closed convex set, which turns out to be the set G_1^d defined by (15):

$$J^*(v) = \chi_{G_1^d}(v) = \begin{cases} 0 & \text{if } v \in G_1^d \\ +\infty & \text{otherwise} \end{cases} \tag{22}$$

This can be checked out easily (see [7] for details). In [7], A. Chambolle proposes a nonlinear projection algorithm to minimize the total variation. The problem is:

$$\inf_{u\in X}\left(J(u) + \frac{1}{2\lambda}\|f-u\|_X^2\right) \tag{23}$$

The following result is shown:

Proposition 2. *The solution of (23) is given by:*

$$u = f - P_{G_\lambda^d}(f) \tag{24}$$

where P is the orthogonal projector on G_λ^d (defined by (15)).

Algorithm: [7] gives an algorithm to compute $P_{G_\lambda^d}(f)$. It indeed amounts to finding:

$$\min \left\{ \|\lambda \text{div}\,(p) - f\|_X^2 \ / \ p \in Y \ , \ |p_{i,j}| \leq 1 \ \forall i,j = 1,\dots,N \right\} \tag{25}$$

This problem can be solved by a fixed point method:

$$p^0 = 0 \tag{26}$$

and

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau(\nabla(\text{div}\,(p^n) - f/\lambda))_{i,j}}{1 + \tau|(\nabla(\text{div}\,(p^n) - f/\lambda))_{i,j}|} \tag{27}$$

In [7] is given a sufficient condition ensuring the convergence of the algorithm:

Theorem 1 (Thm 1 [7]). *Assume that the parameter τ in (27) verifies $\tau \leq 1/8$. Then $\lambda \text{div}\,(p^n)$ converges to $P_{G_\lambda^d}(f)$ as $n \to +\infty$.*

2.4 Application to problem (20)

Since J^* is the indicator function of G_1^d (see (16,22)), we can rewrite (19) as

$$F(u,v) = \frac{1}{2\lambda}\|f - u - v\|_X^2 + J(u) + J^*\left(\frac{v}{\mu}\right) \tag{28}$$

With this formulation, we see the symmetric roles played by u and v. And the problem we want to solve is:

$$\inf_{(u,v)\in X\times X} F(u,v) \tag{29}$$

To solve (29), we consider the two following problems:

- v being fixed, we search for u as a solution of:

$$\inf_{u\in X}\left(J(u) + \frac{1}{2\lambda}\|f - u - v\|_X^2\right) \tag{30}$$

- u being fixed, we search for v as a solution of:

$$\inf_{v\in G_\mu^d} \|f - u - v\|_X^2 \tag{31}$$

From Proposition 2, we know that the solution of (30) is given by: $\hat{u} = f - v - P_{G_\lambda^d}(f - v)$. And the solution of (31) is simply given by: $\hat{v} = P_{G_\mu^d}(f - u)$.

2.5 Algorithm

1. Initialization:

$$u_0 = v_0 = 0 \tag{32}$$

2. Iterations:

$$v_{n+1} = P_{G_\mu^d}(f - u_n) \tag{33}$$

$$u_{n+1} = f - v_{n+1} - P_{G_\lambda^d}(f - v_{n+1}) \tag{34}$$

3. Stopping test: we stop if

$$\max(|u_{n+1} - u_n|, |v_{n+1} - v_n|) \leq \epsilon \tag{35}$$

3 Mathematical results

In this section we carry out the mathematical study of the algorithm (32)–(35). We first show its convergence when λ is fixed. We then state more precisely the link of the limit of our model (when λ goes to 0) with Meyer's one.

3.1 Existence and uniqueness of a solution for (20)

Lemma 1. *There exists a unique couple $(\hat{u}, \hat{v}) \in X \times G_\mu^d$ minimizing F on $X \times X$.*

Proof: We split the proof into two steps.
Step 1: Existence

1. We first remark that the set $X \times G_\mu^d$ is convex, and then that F is convex on $X \times G_\mu^d$. We thus deduce that F is convex on $X \times X$.
2. It is immediate to see that F is continuous on $X \times G_\mu^d$. We then deduce that F is lower semi-continuous on $X \times X$.
3. Let $(u, v) \in X \times G_\mu^d$. We have $\|v\|_{G^d} \leq \mu$. Moreover, since X is of finite dimension, there exists $g \in X$ such that $v = \text{div}\,(g)$ and $\|g\|_{L^\infty} = \|v\|_{G^d} \leq \mu$. We deduce from (12) that (N^2 is the size of the image):

$$\|v\|_X = \leq 4\mu N^2 \tag{36}$$

   We recall that $X \times X$ is endowed with the Euclidean norm.

$$\|(u, v)\|_{X \times X} = \sqrt{\|u\|_X^2 + \|v\|_X^2} \tag{37}$$

   Thus, if $\|(u, v)\|_{X \times X} \to +\infty$, then we get from (36) that $\|u\|_X \to +\infty$. We therefore deduce, since f is fixed, and since (36) holds, that $\|f - u - v\|_X^2 \to +\infty$. And since $F(u, v) \geq \frac{1}{2\lambda}\|f - u - v\|_2^2$, we get $F(u, v) \to +\infty$. Hence we deduce that F is coercive on $X \times G_\mu^d$. We therefore conclude that F is coercive on $X \times X$.

We deduce the existence of a minimizer $(\hat{u}, \hat{v})$.

Step 2: Uniqueness
To get the uniqueness, we first remark that F is strictly convex on $X \times G_\mu^d$, as the sum of a convex function and of a strictly convex function, except in the direction $(u, -u)$. Hence it suffices to check that if $(\hat{u}, \hat{v})$ is a minimizer of F then for $t \neq 0$, $(\hat{u} + t\hat{u}, \hat{v} - t\hat{u})$ is not a minimizer of F. The result is obvious if $\hat{v} - t\hat{u} \in X \backslash G_\mu^d$. Let us show that if $\hat{v} - t\hat{u} \in G_\mu^d$ then the result is still true. Indeed, if $\hat{v} - t\hat{u} \in G_\mu^d$, we have:

$$F(\hat{u} + t\hat{u}, \hat{v} - t\hat{u}) = F(\hat{u}, \hat{v}) + (|1 + t| - 1)J(\hat{u}) \tag{38}$$

By contradiction, let us assume that there exists $\hat{t} \neq \{-2, 0\}$ such that $\hat{v} - \hat{t}\hat{u} \in G_\mu^d$ and

$$F(\hat{u} + \hat{t}\hat{u}, \hat{v} - \hat{t}\hat{u}) \leq F(\hat{u}, \hat{v}) \tag{39}$$

As $(\hat{u}, \hat{v})$ minimizes F, (39) is an equality. From (38), we deduce that $(|1 + \hat{t}| - 1)J(\hat{u}) = 0$. And as $\hat{t} \neq \{-2, 0\}$, we get that $J(\hat{u}) = 0$. There exists therefore $\gamma \in \mathbb{R}$ such that for all (i, j), $\hat{u}_{i,j} = \gamma$.

1. If $\gamma = 0$, then $\hat{u} = 0$. Thus $(\hat{u} + \hat{t}\hat{u}, \hat{v} - \hat{t}\hat{u}) = (\hat{u}, \hat{v})$.
2. If $\gamma \neq 0$, then $\hat{v} - \hat{t}\hat{u}$ cannot belong to G_μ^d since its mean is not 0 (see Proposition 1). This contradicts our assumption.

There remains to check what happens in the case when $\hat{t} = -2$. In this case, we have: $F(-\hat{u}, \hat{v} + 2\hat{u}) \leq F(\hat{u}, \hat{v})$, i.e. $(-\hat{u}, \hat{v} + 2\hat{u})$ is also a minimizer of F. As we assume $\hat{v} + 2\hat{u} \in G_\mu^d$, and as F convex (and as G_μ^d convex), we get:

$$F(0, \hat{u} + \hat{v}) \leq \frac{1}{2}F(\hat{u}, \hat{v}) + \frac{1}{2}F(-\hat{u}, \hat{v} + 2\hat{u}) \tag{40}$$

And we deduce that $(0, \hat{u} + \hat{v})$ is also a minimizer of F. But $F(0, \hat{u} + \hat{v}) = F(\hat{u}, \hat{v})$, i.e. $\frac{1}{2\lambda}\|f - \hat{u} - \hat{v}\|_X^2 = J(\hat{u}) + \frac{1}{2\lambda}\|f - \hat{u} - \hat{v}\|_X^2$. We thus get that $J(\hat{u}) = 0$, and we conclude as before. Hence there exists a unique couple $(\hat{u}, \hat{v}) \in X \times G_\mu^d$ minimizing F on $X \times X$.

■

3.2 Convergence of the algorithm

We show here that our algorithm gives asymptotically the solution of the discrete problem associated to (29).

Proposition 3. *The sequence $F(u_n, v_n)$ built in Section 2.5 converges to the minimum of F on $X \times X$.*

Proof: We first remark that, as we solve successive minimization problems, we have:

$$F(u_n, v_n) \geq F(u_n, v_{n+1}) \geq F(u_{n+1}, v_{n+1}) \tag{41}$$

In particular, the sequence $F(u_n, v_n)$ is nonincreasing. As it is bounded from below by 0, it thus converges in $\mathbb{R}$. We denote by m its limit. We want to show that

$$m = \inf_{(u,v) \in X \times X} F(u, v) \tag{42}$$

Without any restriction, we can assume that, $\forall n$, $(u_n, v_n) \in X \times G_\mu^d$. As F is coercive and as the sequence $F(u_n, v_n)$ converges, we deduce that the sequence (u_n, v_n) is bounded in $X \times G_\mu^d$. We can thus extract a subsequence (u_{n_k}, v_{n_k}) which converges to $(\hat{u}, \hat{v})$ as $n_k \to +\infty$, with $(\hat{u}, \hat{v}) \in X \times G_\mu^d$. Moreover, we have, for all $n_k \in \mathbb{N}$ and all v in X:

$$F(u_{n_k}, v_{n_k+1}) \leq F(u_{n_k}, v) \tag{43}$$

and for all $n_k \in \mathbb{N}$ and all u in X:

$$F(u_{n_k}, v_{n_k}) \leq F(u, v_{n_k}) \tag{44}$$

Let us denote by $\bar{v}$ a cluster point of (v_{n_k+1}). Considering (41), we get (since F is continuous on $X \times G_\mu^d$):

$$m = F(\hat{u}, \hat{v}) = F(\hat{u}, \bar{v}) \tag{45}$$

By passing to the limit in (33), we get: $\bar{v} = P_{G_\mu^d}(f - \hat{u})$. But from (45), we know that: $\|f - \hat{u} - \hat{v}\| = \|f - \hat{u} - \bar{v}\|$. By uniqueness of the projection, we conclude that $\bar{v} = \hat{v}$. Hence $v_{n_k+1} \to \hat{v}$. By passing to the limit in (43) (F is continuous on $X \times G_\mu^d$), we therefore have for all v:

$$F(\hat{u}, \hat{v}) \leq F(\hat{u}, v) \tag{46}$$

And by passing to the limit in (44), for all u:

$$F(\hat{u}, \hat{v}) \leq F(u, \hat{v}) \tag{47}$$

(46) and (47) can respectively be rewritten:

$$F(\hat{u}, \hat{v}) = \inf_{v \in X} F(\hat{u}, v) \tag{48}$$

$$F(\hat{u}, \hat{v}) = \inf_{u \in X} F(u, \hat{v}) \tag{49}$$

But, from the definition of $F(u, v)$ (see (28)), (49) is equivalent to (see [8]):

$$0 \in -f + \hat{u} + \hat{v} + \lambda \partial J(\hat{u}) \tag{50}$$

and (48) to:

$$0 \in -f + \hat{u} + \hat{v} + \lambda \partial J^* \left(\frac{\hat{v}}{\mu} \right) \tag{51}$$

The subdifferential ∂F of F at $(\hat{u}, \hat{v})$ is given by:

$$\partial F(\hat{u}, \hat{v}) = \frac{1}{\lambda} \begin{pmatrix} -f + \hat{u} + \hat{v} + \lambda \partial J(\hat{u}) \\ -f + \hat{u} + \hat{v} + \lambda \partial J^* \left(\frac{\hat{v}}{\mu} \right) \end{pmatrix} \tag{52}$$

And thus, according to (50) and (51), we have:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \partial F(\hat{u}, \hat{v}) \tag{53}$$

which is equivalent to: $F(\hat{u}, \hat{v}) = \inf_{(u,v) \in X^2} F(u, v) = m$. Hence the whole sequence $F(u_n, v_n)$ converges towards m, the unique minimum of F on $X \times G_\mu^d$. We deduce that the sequence (u_n, v_n) converges to $(\hat{u}, \hat{v})$, the minimizer of F, when n tends to $+\infty$.

■

3.3 Link with Meyer's model

We examine here the link between the discrete model (29) and Meyer's problem. We first recall the discrete version of Meyer's problem:

$$\inf_{(u,v)\in X\times G^d/f=u+v} H_\alpha(u,v) \tag{54}$$

with

$$H_\alpha(u,v) = (J(u) + \alpha\|v\|_{G^d}) \tag{55}$$

The following result is straightforward:

Lemma 2. *There exists a solution $(\hat{u},\hat{v}) \in X \times G^d$ of problem (54).*

Remark: We do not know if a uniqueness result holds for problem (54).
We then recall problem (29):

$$\inf_{(u,v)\in X\times X} F_{\lambda,\mu}(u,v) \tag{56}$$

with

$$F_{\lambda,\mu}(u,v) = \frac{1}{2\lambda}\|f-u-v\|^2 + J(u) + J^*\left(\frac{v}{\mu}\right) \tag{57}$$

Let us consider the problem

$$\inf_{(u,v)\in X\times X/f=u+v} J(u) + J^*\left(\frac{v}{\mu}\right) \tag{58}$$

One easily shows the next result:

Lemma 3. *There exists a solution $(\bar{u},\bar{v}) \in X \times X$ of problem (58).*

Proposition 4. *Let us fix $\alpha > 0$ in problem (54). Let $(\hat{u},\hat{v})$ a solution of problem (54). We fix $\mu = \|\hat{v}\|_{G^d}$ in (58). Then:*

- *$(\hat{u},\hat{v})$ is also a solution of problem (58).*
- *Conversely, any solution $(\bar{u},\bar{v})$ of (58) (with $\mu = \|\hat{v}\|_{G^d}$) is a solution of (54).*

Proof: We split the proof into two steps.
Step 1:
We first want to show that $(\hat{u},\hat{v})$ is a solution of (58) (with $\mu = \|\hat{v}\|_{G^d}$). As $(\hat{u},\hat{v})$ is a solution of (54) (the existence of $(\hat{u},\hat{v})$ is given by Lemma 2) and as $\|\hat{v}\|_{G^d} = \mu$, then $\hat{u}$ is solution of

$$\inf_{u\in X/u=f-v,\|v\|_{G^d}=\mu} J(u) + \alpha\mu \tag{59}$$

i.e. $\hat{u}$ is solution of

$$\inf_{u\in X/u=f-v,\|v\|_{G^d}=\mu} J(u) \tag{60}$$

Since the set $\{u \in X/u = f - v, \|v\|_{G^d} = \mu\}$ is contained in $\{u \in X/u = f - v, \|v\|_{G^d} \leq \mu\}$, we have:

$$\inf_{u \in X/u=f-v, \|v\|_{G^d}=\mu} J(u) \geq \inf_{u \in X/u=f-v, \|v\|_{G^d}\leq\mu} J(u) \tag{61}$$

By contradiction, let us assume that

$$\inf_{u \in X/u=f-v, \|v\|_{G^d}=\mu} J(u) > \inf_{u \in X/u=f-v, \|v\|_{G^d}\leq\mu} J(u) \tag{62}$$

Thus, there exists $v' \in X$ such that $\|v'\|_{G^d} < \mu$ and

$$J(f - v') < \inf_{u \in X/u=f-v, \|v\|_{G^d}=\mu} J(u) \tag{63}$$

Denoting by $u' = f - v'$, we have: $J(u') + \alpha\|v'\|_{G^d} < J(u') + \alpha\mu$. But since $(\hat{u}, \hat{v})$ is a solution of (54):

$$J(\hat{u}) + \alpha\|\hat{v}\|_{G^d} \leq J(u') + \alpha\|v'\|_{G^d} < J(u') + \alpha\mu \tag{64}$$

Hence (we recall that $\|\hat{v}\|_{G^d} = \mu$), we get from (64) that $J(\hat{u}) < J(u')$. This contradicts (63). We conclude that (62) cannot hold. Hence:

$$\inf_{u \in X/u=f-v, \|v\|_{G^d}=\mu} J(u) = \inf_{u \in X/u=f-v, \|v\|_{G^d}\leq\mu} J(u) \tag{65}$$

From (60), we see that $\hat{u}$ is solution of $\inf_{u \in X/u=f-v, \|v\|_{G^d}\leq\mu} J(u)$, i.e. $\hat{u}$ is solution of

$$\inf_{u \in X/u=f-v} J(u) + J^*\left(\frac{v}{\mu}\right) \tag{66}$$

Hence $(\hat{u}, \hat{v})$ is also a solution of (58).
Step 2:
Let us now consider $(\bar{u}, \bar{v})$ a solution of (58) (the existence of $(\bar{u}, \bar{v})$ is given by Lemma 3). We can repeat the computations we made in Step 1. We get that $\bar{u}$ is a solution of:

$$\inf_{u \in X/u=f-v, \|v\|_{G^d}=\mu} J(u) + \alpha\mu \tag{67}$$

We therefore have: $J(\bar{u}) + \alpha\mu = J(\hat{u}) + \alpha\|\hat{v}\|_{G^d}$. But as $(\bar{u}, \bar{v})$ is a solution of (58), we have $\|\bar{v}\|_{G^d} \leq \mu$. Hence $J(\bar{u}) + \alpha\|\bar{v}\|_{G^d} \leq J(\hat{u}) + \alpha\|\hat{v}\|_{G^d}$. And since $(\hat{u}, \hat{v})$ is a solution of (54), we get that:

$$J(\bar{u}) + \alpha\|\bar{v}\|_{G^d} = J(\hat{u}) + \alpha\|\hat{v}\|_{G^d} \tag{68}$$

We thus conclude that $(\bar{u}, \bar{v})$ is a solution of (54).

■

In particular, we have thus shown that, when μ is correctly tuned, a solution of the limit problem (58) is in fact a solution of Meyer's problem (54).

3.4 Role of λ

We show here that problem (58) is obtained by passing to the limit λ goes to 0^+ in (56).

Proposition 5. *Let us fix $\alpha > 0$ in (54). Let us assume that problem (54) has a unique solution $(\hat{u}, \hat{v})$. Set $\mu = \|\hat{v}\|_{G^d}$ in (56) and (58). Let us denote (u_λ, v_λ) the solution of problem (56). Then (u_λ, v_λ) converges to $(u_0, v_0) \in X \times X$ as λ goes to 0. Moreover, $(u_0, v_0) = (\hat{u}, \hat{v})$ is the solution of problem (58).*

Remark: In the case when the solution of problem (54) is not unique, the result of Proposition 5 does not hold. We can just show that any cluster point of $(u_{\lambda_n}, v_{\lambda_n})$ is a solution of problem (58) and thus of (54)

Proof of Proposition 5: The existence of $(\hat{u}, \hat{v})$ is given by Lemma 3. The existence and uniqueness of (u_λ, v_λ) is given by Lemma 1.

Since (u_λ, v_λ) is the solution of problem (56), we have $v_\lambda \in G^d_\mu$, i.e. $\|v_\lambda\|_{G^d} \leq \mu$. As we saw in the proof of Lemma 1, this inequality implies:

$$\|v_\lambda\|_X \leq 4\mu N^2 \tag{69}$$

Since (u_λ, v_λ) is the solution of problem (56), we have:

$$F_{\lambda,\mu}(u_\lambda, v_\lambda) \leq F_{\lambda,\mu}(f, 0) \tag{70}$$

which means

$$F_{\lambda,\mu}(u_\lambda, v_\lambda) \leq J(f) \tag{71}$$

And the left hand-side of (71) is given by:

$$F_{\lambda,\mu}(u_\lambda, v_\lambda) = J(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|_X^2 + J^*\left(\frac{v_\lambda}{\mu}\right) = J(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|_X^2 \tag{72}$$

Hence $J(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|_X^2 \leq J(f)$, and

$$\|f - u_\lambda - v_\lambda\|^2 \leq 2\lambda J(f) \tag{73}$$

As $\|v_\lambda\|_X$ is bounded (from (69)), we conclude that if $\lambda \in [0; 1]$, u_λ is bounded by a constant $C > 0$ which does not depend on λ.

Consider a sequence (λ_n) which goes to 0^+ as $n \to +\infty$. Then, up to an extraction (since $(u_{\lambda_n}, v_{\lambda_n})$ is bounded in $X \times X$), there exists $(u_0, v_0) \in X \times X$ such that $(u_{\lambda_n}, v_{\lambda_n})$ converges to (u_0, v_0). By passing to the limit in (73), we get: $\|f - u_0 - v_0\|_X = 0$, i.e. $f = u_0 + v_0$.

To conclude the proof of the proposition, there remains to show that (u_0, v_0) is a solution of problem (58). We first notice that as $\lambda > 0$, and since $\|v_\lambda\|_{G^d} \leq \mu$, we get: $\|v_0\|_{G^d} \leq \mu$. Let $(u,v) \in X \times X$ such that $f = u + v$. We have:

$$\begin{aligned} & J(u) + J^*\left(\frac{v}{\mu}\right) + \frac{1}{2\lambda} \underbrace{\|f-u-v\|^2}_{=0} \\ \geq\ & J(u_{\lambda_n}) + J^*\left(\frac{v_{\lambda_n}}{\mu}\right) + \frac{1}{2\lambda_n}\|f - u_{\lambda_n} - v_{\lambda_n}\|^2 \\ \geq\ & \underbrace{J(u_{\lambda_n}) + J^*\left(\frac{v_{\lambda_n}}{\mu}\right)}_{\to J(u_0)+J^*\left(\frac{v_0}{\mu}\right)} \end{aligned}$$

Hence (u_0, v_0) is a solution of problem (58). And as we have assumed that problem (58) has a unique solution, we deduce that $(u_0, v_0) = (\hat{u}, \hat{v})$, i.e. (u_0, v_0) is the solution of problem (58).

■

4 SAR images restoration

4.1 Introduction

Synthetic Aperture Radar (SAR) images are strongly corrupted by a noise called speckle. A radar sends a coherent wave which is reflected on the ground, and then registered by the radar sensor [9]. When one cares with the reflection of a coherent wave on a coarse surface, then one can see that the observed image is degraded by a noise of large amplitude. This gives a speckled aspect to the image. That is why such a noise is called speckle.

Link with our approach: Contrary to the usual modeling in SAR, the noise in our model is considered to be additive: the image f is decomposed into a component u belonging to BV, and a component v in G. But it is to be noticed that our model is completely different from the classical additive models: in these, v is often considered to be a Gaussian white noise, and therefore has a constant variance all over the image. Here, v belongs to G, a space in which signals can have large oscillations but small norm. Moreover the variance of the oscillations of v may not be uniform on the whole image. Note that by considering u as the restored image (without speckle) we assume that there is no texture in the SAR image.

4.2 Results on synthetic images

Restoration: Figure 1 shows why for a SAR image the decomposition proposed by Meyer is very interesting. Indeed, one checks that the v component contains the speckle, and the u component can be regarded as a restoration of the original image (if it does not contain textures). It is difficult to make comparisons with other methods [10], since the main criterion remains the visual interpretation.

Nevertheless, the results we achieve appear promising in comparison with existing methods. And above all, our approach being a variational one, computation time are very short. With a processor of 800 MHz and 128 kByte of RAM, it takes less than one minute to deal with an image of size 256*256.

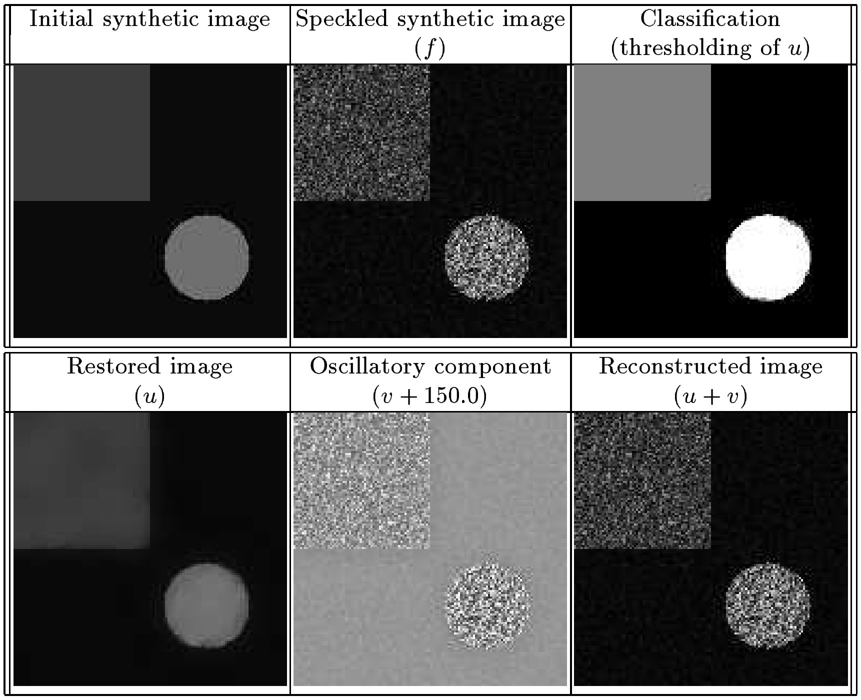

Fig. 1. Simple synthetic image ($\lambda = 0.01$ and $\mu = 80$)

4.3 Results on real images

We use SAR images of Bourges' area provided by the CNES. The reference image (also furnished by the CNES) has been obtained by amplitude summation. Image 2 shows the effect of parameter μ on the restoration process. The larger μ is, the more v contains information, and therefore the more u is averaged. According to the value of μ, we can thus get a more or less restored image, and also more or less of a smoother image.

5 Conclusion

In this article, we present a new algorithm to decompose a given image f into a component u belonging to BV and a component v containing the noise and

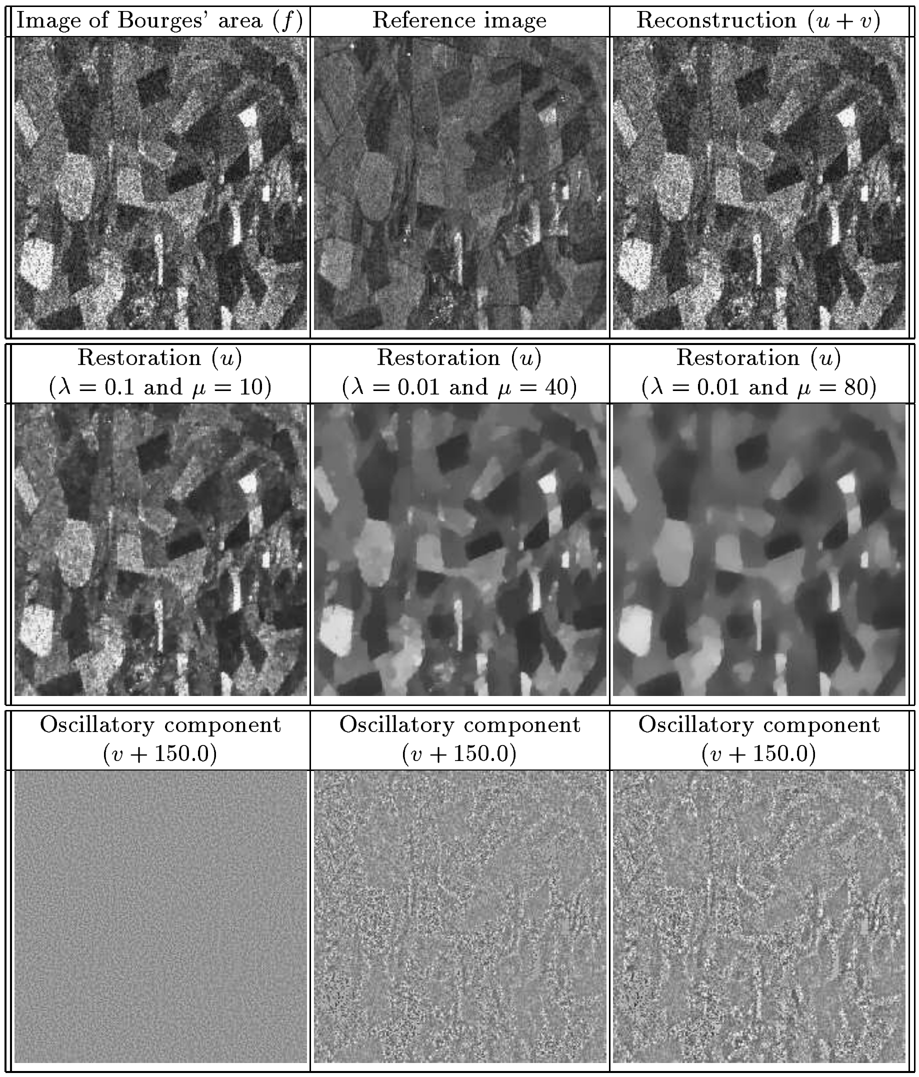

Fig. 2. Image of Bourges' area

the textures of the initial image. Our algorithm performs Meyer's program [3] when μ is suitably tuned. Moreover, we carry out the mathematical study of our model. We also show how the u component can be used for SAR image restoration. Further details about this work as well as comparisons with the standard BV filtering and with the Vese-Osher model [5] can be found in [11].

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