# RESONANCES AND SPECTRAL SHIFT FUNCTION FOR THE SEMI-CLASSICAL DIRAC OPERATOR

ABDALLAH KHOCHMAN

ABSTRACT. We consider the self-adjoint operator $H = H_0 + V$, where $H_0$ is the free semi-classical Dirac operator on $\mathbb{R}^3$. We suppose that the smooth matrix-valued potential $V = O(\langle x\rangle^{-\delta})$, $\delta > 0$, has an analytic continuation in a complex sector outside a compact. We define the resonances as the eigenvalues of the non-selfadjoint operator obtained from the Dirac operator $H$ by complex distortions of $\mathbb{R}^3$. We establish an upper bound $O(h^{-3})$ for the number of resonances in any compact domain. For $\delta > 3$, a representation of the derivative of the spectral shift function $\xi(\lambda, h)$ related to the semi-classical resonances of $H$ and a local trace formula are obtained. In particular, if $V$ is an electro-magnetic potential, we deduce a Weyl-type asymptotics of the spectral shift function. As a by-product, we obtain an upper bound $O(h^{-2})$ for the number of resonances close to non-critical energy levels in domains of width $h$ and a Breit-Wigner approximation formula for the derivative of the spectral shift function.

*Keywords:* Semi-classical Dirac operator - Resonances - Trace formula - Spectral shift function - Weyl-type asymptotics - Breit-Wigner approximation.
*Mathematics classification:* 35B34 - 35P05 - 34L40 - 81Q20 - 81Q10.

## 1. INTRODUCTION

The resonance theory for the Schrödinger equation has been developed following several approaches. Among them we can mention the analytic dilation (see [1]) or the analytic distortion (see [22]) and, in the semi-classical regime, that related to the work of Helffer-Sjöstrand [21]. In [19] Helffer-Martinez showed that the different definitions give the same resonances when one can simultaneously apply them to an operator. For the three dimensional Dirac operator, Seba [42] defined the resonances as complex eigenvalues of the operator obtained by a complex dilation. Applying the approach of Helffer-Sjöstrand [21], Parisse [30] has studied the Dirac resonances in the semi-classical regime, with some scaling functions. The last two works deal with analytic perturbations near the real axis.

The concept of the spectral shift function has been introduced by Lifshits [26] in connection with problems in quantum statistics and solid physics. Thereafter, a mathematical theory of the spectral shift function has been constructed by Krein [25]. Moreover, in [3] Birman-Krein found a connection between scattering theory and the theory of the spectral shift function. A detailed presentation of the theory of the spectral shift function can be found in [45]. For a survey concerning the spectral shift function (SSF) for Schrödinger and Dirac operators or the asymptotic expansion of this function, we refer to Robert [36] and to the references given there.

A representation of the derivative of the scattering phase in terms of resonances has been established for Schrödinger operators. Such representations have been successively obtained by Melrose [27] for obstacle problems in the high energy case, by Petkov-Zworski [32], [33] for "black box" scattering with compact perturbations in the classical and the semi-classical cases and by Bruneau-Petkov [8] for long-range perturbations in the semi-classical "black box" framework. The results in [8] have been generalized by Dimassi-Petkov [13] for non-semi-bounded Schrödinger type operators. As a by-product, they prove a Weyl type asymptotics for the scattering phase. Moreover, Weyl asymptotics can also be obtained by representation of the derivative of the spectral shift function involving the trace of the cut-off resolvent (see Robert [37], Bruneau-Petkov [7] and Nakamura [29]).

Concerning the Breit-Wigner approximation for the derivative of the spectral shift function in the Schrödinger case, similar results have been obtained in a particular semi-classical set-up by C.Gérard-Martinez-Robert [16] for short range potentials on $\mathbb{R}^n$ and by Petkov-Zworski [33] for a general compactly supported perturbation (see also [7], [6]).

For Dirac operators, Bruneau-Robert [10] established an asymptotic expansion of the scattering phase $s(\lambda)$ and its derivatives in the high energy regime and in the semi-classical regime for $\lambda$ in a non-trapping energy interval. For an interval $I \subset ]-mc^2, mc^2[$ with non critical extremities, Helffer and Robert in [20] gave an asymptotics of the number of the eigenvalues in $I$ for scalar potentials. Nevertheless, we are neither aware of works dealing with the link between the derivative of the SSF and resonances for the semi-classical Dirac operators (in the spirit of Petkov-Zworski [33] and Bruneau-Petkov [8]), nor of papers giving the Weyl asymptotics of the spectral shift function for Dirac operators in any interval $I$.

The purpose of this work is to extend the definition of resonance for analytic perturbations outside a compact set. We define the resonances for the semi-classical Dirac operator as the discrete eigenvalues of the non-selfadjoint operator obtained from the Dirac operator $H$ by a general class of complex distortions of $\mathbb{R}^3$. We prove that the resonances are independent of the distortion (see Section 4). We establish an upper bound for the number of resonances in a compact domain $\Omega$ (see Section 5). The second goal of this work is to obtain a meromorphic continuation of the derivative of the spectral shift function $\xi(\lambda, h)$ related to the resonances for the semi-classical Dirac operator (see Section 6). The latter is closely related to trace formulae (see [8], [39], [40], [32], [33], [41]) and to resonance expansions (see [43], [11]). Thereafter, in the case where the potential is an electro-magnetic potential, we deduce a Weyl-type asymptotics of the spectral shift function (see Section 7). As a by-product, we obtain an upper bound $O(h^{-2})$ for the number of resonances close to non-critical energy levels in domains of width $h$ (see Subsection 8.1), as well as a Breit-Wigner approximation for the derivative of the SSF (see Subsection 8.2).

## 2. STATEMENT OF THE RESULTS

We consider the selfadjoint Dirac operator

$$H_0 = -ich \sum_{j=1}^{3} \alpha_j \frac{\partial}{\partial x_j} + \beta mc^2, \tag{1}$$

with domain $D(H_0) = H^1(\mathbb{R}^3) \otimes \mathbb{C}^4 \subset \mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$, where $h \searrow 0$ is the semi-classical parameter, $m > 0$ is the mass of the Dirac particle and $c$ is the speed of light. The quantities $\alpha_1, \ \alpha_2, \ \alpha_3$ and $\beta$ are $4 \times 4$ Dirac matrices satisfying the anti-commutation relations

$$\begin{cases} \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} I_4, & \text{for} \quad i, j = 1, 2, 3, \\ \alpha_i\beta + \beta\alpha_i = 0, & \text{for} \quad i = 1, 2, 3, \end{cases} \tag{2}$$

and $\beta^2 = I_4$. Here $I_n$ is the $n \times n$ identity matrix. For example, we choose the standard (or Dirac-Pauli) representation of these matrices

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where $(\sigma_j)_{1\leq j\leq 3}$ are the $2\times 2$ Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**Remark 1.** *Most calculations with Dirac matrices can be done without referring to a particular representation (see Appendix 1.A* [44, Chap. 1]*).*

Let $H_1 = H := H_0 + V$, where $V$ is the multiplication operator by a $4\times 4$-matrix potential $V$. We suppose that $V \in C^\infty(\mathbb{R}^3)$ and satisfies the following assumption

$(\mathbf{A_V})$ : *$V$ is Hermitian on $\mathbb{R}^3$ and has an analytic extension in the sector*

(3) $$C_{\epsilon,0} := \{z \in \mathbb{C}^3, |\mathrm{Im}(z)| \leq \epsilon\,|\mathrm{Re}(z)|, \quad |\mathrm{Re}(z)| > R_0\}, \quad for\ 0 < \epsilon < 1.$$

*Moreover, for $x \in C_{\epsilon,0}$ it satisfies*

(4) $$\|V(x)\| = O(\langle x\rangle^{-\delta}), \quad \delta > 0, \quad \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}\ .$$

The free Dirac operator $H_0$ has essential spectrum $\sigma_{ess}(H_0) = ]-\infty, -mc^2] \cup [mc^2, +\infty[$ and its spectrum is purely absolutely continuous. Under the assumption $(\mathbf{A_V})$ the operator $H_1$ is selfadjoint. Using Weyl's theorem, we have $\sigma_{ess}(H_1) = \sigma_{ess}(H_0)$.

For $\theta \in D_\epsilon := \{\theta \in \mathbb{C}, |\theta| \leq r_\epsilon := \frac{\epsilon}{\sqrt{1+\epsilon^2}}\}$, we denote

$$H_{1,\theta} = H_\theta := U_\theta H_0 U_\theta^{-1} + U_\theta V U_\theta^{-1} = H_{0,\theta} + U_\theta V U_\theta^{-1},$$

where $U_\theta$ is the one-parameter family of distortions defined below (see Section 3).

For $\theta_0$ fixed in $D_\epsilon^+ := D_\epsilon \cap \{\theta \in \mathbb{C},\ \mathrm{Im}(\theta) \geq 0\}$, we define

$$\Gamma_{\theta_0} := \{\pm c\sqrt{\frac{\lambda}{(1+\theta_0)^2} + m^2c^2} \in \mathbb{C}, \quad \lambda \in [0, +\infty[\},$$

and

$$S_{\theta_0} := \{z \in \bigcup_{\theta \in D_\epsilon^+} \Gamma_\theta; \quad \arg(1+\theta) < \arg(1+\theta_0), \quad \frac{1}{|1+\theta|} < \frac{1}{|1+\theta_0|}\ \}.$$

The square root $\sqrt{z}$ is defined such that for $z \in \mathbb{C}\backslash]-\infty, 0]$, $\mathrm{Re}(\sqrt{z}) > 0$.

For $\theta \in D_\epsilon^+$, $\arg(1+\theta_0) \leq \arg(1+\theta)$, $\frac{1}{|1+\theta_0|} \leq \frac{1}{|1+\theta|}$, we prove that the spectrum of $H_\theta$ is discrete in $S_\theta$ and independent of $\theta$ in $S_{\theta_0}$. This justifies the following definition.

**Definition 1.** *The resonances of $H$ in $S_{\theta_0} \cup \mathbb{R}$ are the eigenvalues of $H_{\theta_0}$. The multiplicity of a resonance $z_0$ is defined by*

$$\mathrm{mult}(z_0) := \mathrm{rank}\ \frac{1}{2i\pi}\int_{\Gamma_0} (z - H_{\theta_0})^{-1} dz,$$

*where $\Gamma_0$ is a small positively oriented circle centered at $z_0$. We will denote* $\mathrm{Res}(H)$ *the set of resonances.*

The most important advantage of this definition is that the resonances can be calculated by solving a non-selfadjoint eigenvalue problem.

**Remark 2.** *The resonances of $H$ in $\{z \in \mathbb{C};\ \mathrm{Re}(z) \in ]-mc^2, mc^2[\}$ are the real eigenvalues of $H$.*

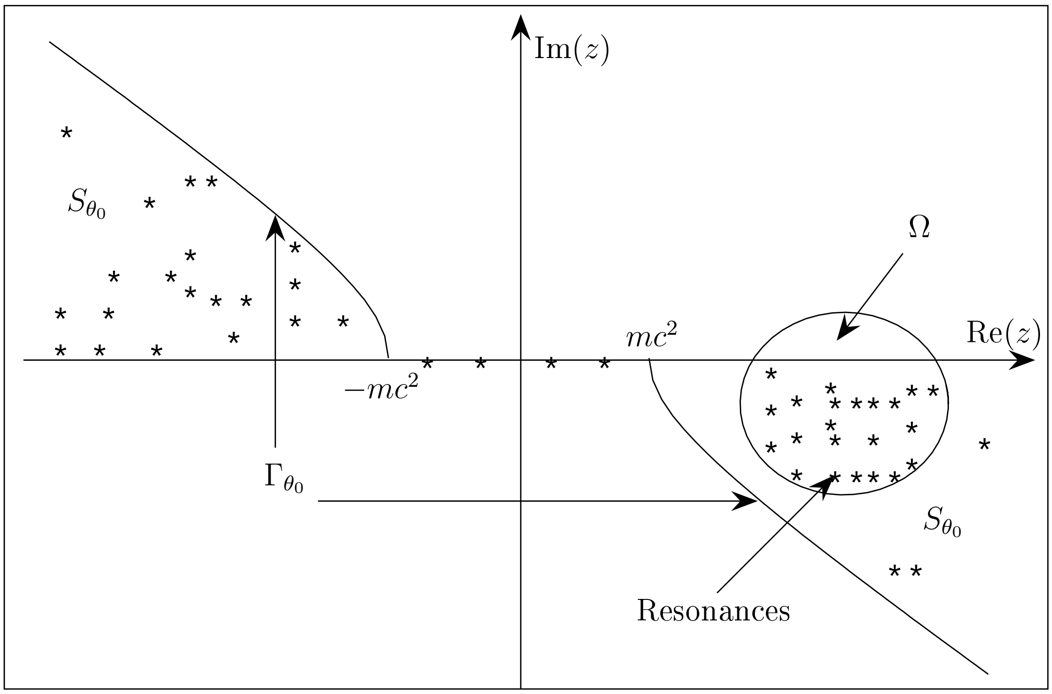


Fig.1. The set $S_{\theta_0}$

Now, we would like to find symmetry properties so that we can limit our study of the resonances to a domain $\Omega$ which satisfies $(\mathbf{A}^+_{\mathbf{\Omega}})$, with

$(\mathbf{A}^{\pm}_{\mathbf{\Omega}})$: *$\Omega$ is an open simply connected and relatively compact subset of $\{z \in \mathbb{C}; \pm\mathrm{Re}(z) > mc^2\}$ such that $\Omega \cap \{\pm\mathrm{Im}(z) > 0\} \neq \emptyset$ and there exists $\theta_0 \in D^+_\epsilon$ such that $\overline{\Omega} \cap \Gamma_{\theta_0} = \emptyset$.*

**Proposition 1.** *Let $H^-$ be the selfadjoint Hamiltonian*

$$H^- = H_0 - U_c\overline{V}(x)U_c^{-1},$$

*where $U_c = i\beta\alpha_2$ is a $4\times 4$ unitary matrix and $\overline{V}$ is the conjugate of $V$. Then the following assertions are equivalent:*

*(i) The complex value $z$ is a resonance of $H$.*
*(ii) The symmetric of the conjugate $-\bar{z}$ is a complex eigenvalue of $U_{\bar\theta}H^-U_{\bar\theta}^{-1}$.*

*Moreover, the multiplicity of $z$ is equal to the multiplicity of $-\bar{z}$ considered as an eigenvalue of $U_{\bar\theta}H^-U_{\bar\theta}^{-1}$.*

**Proposition 2.** *Let $\overline{H}$ be the selfadjoint Hamiltonian*

$$\overline{H} = -ic\sum_{j=1}^{3} \alpha'_j \partial_{x_j} + \beta' mc^2 + \overline{V},$$

*where $\alpha'_1 = -\alpha_1,\ \alpha'_2 = \alpha_2,\ \alpha'_3 = -\alpha_3,\ \beta' = \beta$ are matrices which satisfy the anti-commutation relations (2) and $\overline{V}$ is the conjugate of $V$. Then the following assertions are equivalent:*

*(i) The complex value $z$ is a resonance of $H$.*
*(ii) The conjugate $\bar{z}$ is a complex eigenvalue of $U_{\bar\theta}\overline{H}U_{\bar\theta}^{-1}$.*

*Moreover, the multiplicity of $z$ is equal to the multiplicity of $\bar{z}$ considered as an eigenvalue of $U_{\bar\theta}\overline{H}U_{\bar\theta}^{-1}$.*

Using the same type of approach as in [40], we construct an operator $\widehat{H}_{j,\theta}$ for $j = 0, 1$, so that

$$\widehat{H}_{j,\theta} - H_{j,\theta} = K_j = O(1), \quad \text{has finite rank } O(h^{-3}),$$

and $\|(\widehat{H}_{j,\theta} - z)^{-1}\| = O(1)$, uniformly for $z \in \overline{\Omega}$ (see Subsection 5.1).

Using this construction we establish an upper bound of the number of resonances:

**Theorem 1. (Upper bound)** *Assume that $V$ satisfies the assumption $(\mathbf{A_V})$ with $\delta > 0$. Let $\Omega$ be a complex domain satisfying the assumption $(\mathbf{A}^{\pm}_{\mathbf{\Omega}})$, then*

$$\#\mathrm{Res}(H) \cap \Omega \le C(\Omega) h^{-3}.$$

For a pair of self-adjoint operators $(H_0, H_0 + V)$ where $V$ satisfies the assumption $(\mathbf{A_V})$ with $\delta > 3$, (see [10], [37], [36]), the spectral shift function $\xi(\lambda, h)$ is a distribution in $\mathcal{D}'(\mathbb{R})$ such that its derivative satisfies:

$$\langle \xi'(\lambda, h), f(\lambda)\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} = \mathrm{tr}(f(H_1) - f(H_0)), \quad f(\lambda) \in C_0^\infty(\mathbb{R}). \tag{5}$$

By the Birman-Krein theory, the SSF is in $L^1_{loc}(\mathbb{R})$ and coincides with the scattering phase: for almost every $\lambda$ in the absolutely continuous spectrum of $H_0$ we have det $S(\lambda) = e^{-2i\pi\xi(\lambda,h)}$ where $S(\lambda)$ is the scattering matrix for the operator pairs $(H, H_0)$ (see [3] or [45, Chapter 8]). On the other hand, in the standard definition, the scattering phase is equal to $\arg\det S(\lambda)$.

The SSF for the operator pair $(H_0,\ H_0 + V)$ satisfies the following general gauge invariance (for the proof we refer to Section 6):

**Proposition 3. (Gauge invariance)** *Let $V$ be a potential and $\Phi_g$ be a scalar function such that $V$ and $\alpha \cdot \nabla\Phi_g$ satisfy $(\mathbf{A_V})$ with $\delta > 3$. Then, the SSF for the operator pair $(H_0,\ H_0 + V - \alpha\cdot\nabla\Phi_g)$ coincides with the SSF for the operator pair $(H_0,\ H_0 + V)$.*

Our principal result is a meromorphic continuation of the derivative of the spectral shift function $\xi(\lambda, h)$.

**Theorem 2. (Representation formula)** *Assume that $V$ satisfies the assumption $(\mathbf{A_V})$ with $\delta > 3$. Let $\Omega$ be a complex domain satisfying the assumption $(\mathbf{A}^{\pm}_{\mathbf{\Omega}})$ and $W \Subset \Omega$ be an open simply connected set which is symmetric with respect to the real axis. Assume that $I = W \cap \mathbb{R}$*

*is an interval. Then for all $\lambda \in I$ we have the representation:*

$$\xi'(\lambda, h) = \frac{1}{\pi}\mathrm{Im}\, r(\lambda, h) + \sum_{\substack{w\in \mathrm{Res}(H_1)\cap\Omega \\ \mathrm{Im}\, w\neq 0}} \frac{-\mathrm{Im}\, w}{\pi|\lambda - w|^2} + \sum_{w\in \mathrm{Res}(H_1)\cap I} \delta_w(\lambda), \tag{6}$$

*where $r(z, h) = g(z, h) - \overline{g(\bar{z}, h)}$, $g(z, h)$ is a holomorphic function in $\Omega$ which satisfies the following estimate:*

$$|g(z, h)| \leq C(W)h^{-3}, \quad z \in W, \tag{7}$$

*with $C(W) > 0$ independent of $h \in ]0,\ h_0]$. Here $\delta_w(\cdot)$ is the Dirac mass at $w \in \mathbb{R}$.*

**Remark 3.** *This theorem can be extended to the operator pairs $(H_0 + V_1,\ H_0 + V_2)$, where $V_1,\ V_2$ are two $4\times 4$ Hermitian potential matrices satisfying $(\mathbf{A_V})$ with $\delta > 0$ and $V = V_2 - V_1$ satisfies the assumption $(\mathbf{A_V})$ with $\delta > 3$ (see Theorem 5).*

As a corollary of the last theorem, we have a Sjöstrand type local trace formula (see Theorem 6).

Now we discuss a Weyl type asymptotics of the spectral shift function $\xi(\lambda, h)$ in the case where $V$ is an electro-magnetic potential

$$V(x) = e(-\alpha\cdot A + v)(x) = -\sum_{j=1}^{3}\alpha_j \cdot eA_j(x) + ev(x), \tag{8}$$

satisfying the assumption $(\mathbf{A_V})$ with $\delta > 3$. Here $e < 0$ is the charge of the Dirac particle. We assume that, the electric potential $v(x) = \begin{pmatrix} v_+(x)I_2 & 0 \\ 0 & v_-(x)I_2 \end{pmatrix}$ where $v_+,\ v_-$ are $C^\infty$ scalar functions satisfying

$$|e(v_+ - v_-)(x)| < 2mc^2, \tag{9}$$

and $A = (A_1,\ A_2,\ A_3)$ is a magnetic vector potential where $A_1,\ A_2,\ A_3$ are $C^\infty$ scalar functions.

For any $(x, \xi) \in \mathbb{R}^6$, the semi-classical symbols of $H_\nu$, $\nu = 0, 1$ are the matrices

$$\mathcal{D}_\nu(x, \xi) = \alpha\cdot(c\xi - \nu eA(x)) + \beta mc^2 + \nu ev(x), \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \tag{10}$$

which are Hermitian and each have two eigenvalues

$$H_\nu^\pm(x, \xi) = \pm\left(|c\xi - \nu eA(x)|^2 + \left(mc^2 + \nu\frac{e}{2}(v_+ - v_-)\right)^2\right)^{\frac{1}{2}} + \nu\frac{e}{2}(v_+ + v_-), \tag{11}$$

of multiplicity two. The function $H_1^+(x, \xi)$ is the Hamiltonian for a relativistic classical particle and $H_1^-(x, \xi)$ can be considered as the Hamiltonian for the corresponding anti-particle (see [14], [44], [47]). Moreover, from (9) the two Hamiltonians $H_1^\pm(x, \xi)$ are smooth functions.

For $\nu = 0, 1$, the matrix

$$\Pi_\nu^\pm(x, \xi) = \frac{1}{2}\left(1 + \frac{\mathcal{D}_\nu(x, \xi) - \nu ev(x) + \nu\frac{e}{2}(v_+ - v_-)\beta}{H_\nu^\pm(x, \xi) - \nu\frac{e}{2}(v_+ + v_-)}\right), \tag{12}$$

is the orthogonal projection onto the eigenspace $\mathcal{E}_\nu^\pm(x, \xi)$ of $\mathcal{D}_\nu(x, \xi)$ corresponding to the eigenvalue $H_\nu^\pm(x, \xi)$.

**Definition 2.** *A real $\lambda$ is a noncritical energy level for $H_1$ if for all $(x,\xi) \in \mathbb{R}^6$, with $H_1^{\pm}(x,\xi)=\lambda$, we have $\nabla_{x,\xi} H_1^{\pm}(x,\xi) \neq 0$.*

**Theorem 3. (Weyl formula)** *Assume that the potential $V$ is an electro-magnetic potential given by* (8) *and satisfying the assumption* $(\mathbf{A_V})$ *with $\delta > 3$. For all noncritical energy levels $\lambda$, $\lambda_1$ for $H_1$ such that $\pm mc^2 \notin ]\lambda_1, \lambda[$ and $h \in ]0, h_0[$, we have the asymptotic expansion*

$$\xi(\lambda,h) - \xi(\lambda_1,h) = w(\lambda,\lambda_1)h^{-3} + O(h^{-2}). \tag{13}$$

*Here the $O(h^{-2})$ is uniform for $\lambda$ (resp.$\lambda_1$) in a small interval $I_2$ (resp. $I_1$). The first term $w(\lambda,\lambda_1) \in C^\infty(I_2 \times I_1)$ is given by*

$$w(\lambda,\lambda_1) = w(\lambda) - w(\lambda_1),$$

*with:*

$$w(\lambda) = \frac{1}{3\pi^2}\int_{\mathbb{R}^3} W_+(\lambda,v_+,v_-) - W_+(\lambda,0,0) - W_-(\lambda,v_+,v_-) + W_-(\lambda,0,0)\ dx \tag{14}$$

*where* $W_\pm(\lambda,a,b) = \left(\left(\lambda - \frac{e(a+b)}{2}\right)_\pm^2 - \left(mc^2 + \frac{e(a-b)}{2}\right)^2\right)_+^{\frac{3}{2}}.$

Theorem 3 can be extended to the operator pairs $(H_0 + V_1,\ H_0 + V_2)$ where $V_1,\ V_2$ satisfy $(\mathbf{A_V})$ with $\delta > 0$ (or $\|V_j(x)\| \longrightarrow 0$) and $V_2 - V_1$ satisfies $(\mathbf{A_V})$ with $\delta > 3$ (see Remark 10). Furthermore, using Proposition 3, magnetic potentials $A^1$ and $A^2$ which are gauge equivalent (i.e. $A^1 - A^2 = \nabla\Phi_g$ with $\Phi_g$ as in Proposition 3) generate the same SSF.

**Remark 4.** *The two formulae* (13) *and* (14) *give in particular a Weyl type asymptotics of the counting function of the number of eigenvalues of $H_1$ between two values in the interval $]-mc^2, mc^2[$ . In the case of a scalar potential $v$ $(v_+ = v_-)$, this result was proved by Helffer-Robert* [20] *without the analyticity assumption at infinity.*

To prove Theorem 3 we construct, in Appendix A (see Theorem 9), a parametrix at small times of the propagator of the Dirac equation in an external electro-magnetic field (see also Yajima [47] for scalar electric potential cases).

As a direct consequence of the last theorem we deduce an upper bound $O(h^{-2})$ for the number of resonances close to non-critical energy levels in domains of width $h$ (see Proposition 12) and a Breit-Wigner approximation for the derivative of the spectral shift function $\xi(\lambda,h)$ (see Theorem 8).

## 3. Distortion for the free Dirac operator

In this section, we start with the definition of the deformation for the free Dirac operator by analytic distortion (in the spirit of Hunziker [22]) and we calculate the essential spectrum for the distorded free Dirac operator. Here, $h$ does not play any role, and can be taken equal to 1. Let us now introduce the one-parameter family of unitary distortions

$$U_\theta f(x) = J_{\phi_\theta(x)}^{\frac{1}{2}} f(\phi_\theta(x)),\quad \theta \in \mathbb{R},\quad f \in (S(\mathbb{R}^3))^4,$$

where $\phi_\theta(x) = x + \theta g(x)$ and $g : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ is a smooth function. Let $J_{\phi_\theta(x)} = \det(I + \theta\nabla g(x))$ be the Jacobian of $\phi_\theta(x)$.

We suppose that $g$ satisfies the assumption

$$(\mathbf{A_g})\begin{cases}\text{(i) } \sup_{x\in\mathbb{R}^3}\|\nabla g(x)\| = M^{-1} < +\infty.\\[2ex] \text{(ii) } g(x) = 0, \ \textit{in the compact set } B(0,R_0), \ (\textit{see } (3)).\\[2ex] \text{(iii) } g(x) = x, \ \ \textit{outside a compact set } K(\supset B(0,R_0)).\end{cases}$$

**Lemma 1.** *For $\theta \in ]-M, M[$, $U_\theta$ can be extended as an unitary operator on $\mathcal{H}$.*

**Proof.** Since $|\theta| < M$, we have $\|\theta\nabla g(x)\| < 1$, and

$$(\nabla\phi_\theta(x))^{-1} = (I + \theta\nabla g(x))^{-1} = \sum_{n=0}^{\infty}(-1)^n(\theta)^n(\nabla g(x))^n.$$

The function $\phi_\theta(x)$ is injective and $\phi_\theta(\mathbb{R}^3) = \mathbb{R}^3$, consequently $\phi_\theta(x)$ is a diffeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$. The inverse of $U_\theta$ is given by

$$U_\theta^{-1}u = J_{\phi_\theta(x)}^{\frac{-1}{2}}u(\phi_\theta^{-1}(x)) : \ (L^2(\mathbb{R}^3))^4 \longmapsto (L^2(\mathbb{R}^3))^4.$$

The lemma follows from the relations

$$U_\theta U_\theta^{-1} = U_\theta^{-1}U_\theta = I_{(L^2(\mathbb{R}^3))^4} \text{ and } \|U_\theta f\|_{\mathcal{H}} = \|U_\theta^{-1}f\|_{\mathcal{H}} = \|f\|, \ \ \forall f \in (L^2(\mathbb{R}^3))^4.$$

$\square$

**Definition 3.** *We denote by $\mathcal{B}$, the space of functions $f = (f_i)_{1\le i\le 4}$ such that $f_i(x)$ has an analytic continuation in $C_{\epsilon,0}$ and $\lim_{\substack{|z|\to\infty\\ z\in C_{\epsilon,0}}} |z|^k f_i(z) = 0$, for all $k \in \mathbb{N}$ and $\epsilon \in ]0,1[$ (see (3)).*

**Lemma 2.** *The subspace $\mathcal{B}$ is dense in $\mathcal{H}$.*

**Proof.** The subspace $\mathcal{B}$ contains vectors of Hermite functions and the linear combinations of Hermite functions are dense in $L^2(\mathbb{R}^3)$. $\square$

**Proposition 4.** *Let be $D_{\epsilon,M} = D_\epsilon \cap \{\theta \in \mathbb{C}; |\theta| < M\}$. We have the two assertions:*

*(i) For all $f \in \mathcal{B}$, $\theta \in D_{\epsilon,M} \longmapsto U_\theta f$ is analytic.*
*(ii) For all $\theta \in D_{\epsilon,M}, U_\theta\mathcal{B}$ is dense in $\mathcal{H}$.*

**Proof.** In order to prove (i), we show that $\theta \longmapsto \langle U_\theta f, g\rangle$ is analytic for all $g \in \mathcal{H}$. Let $R \gg 1$ be such that $K \subset B(0,R) = \{x, |x| < R\}$.

- In $\{x, |x| < R\}$: since $f \in \mathcal{B}$, then $\theta \longmapsto \int_{|x|<R} J_{\phi_\theta}^{\frac{1}{2}} f(\phi_\theta(x))\overline{g}(x)dx$, is clearly analytic for all $g \in \mathcal{H}$.
- In $\{x, |x| > R\}$: we have $g(x) = x$, consequently $\phi_\theta(x) = (1+\theta)x$. We remark that

$$|\mathrm{Im}(\phi_\theta(x))| = |\mathrm{Im}(\theta x)| \le |\mathrm{Im}(\theta)||x| = \frac{|\mathrm{Im}(\theta)||\mathrm{Re}(\phi_\theta(x))|}{|1+\mathrm{Re}(\theta)|}.$$

If $|\theta| \le r_\epsilon = \frac{\epsilon}{\sqrt{1+\epsilon^2}}$, $0 < \epsilon < 1$, then

$$|\mathrm{Im}(\phi_\theta(x))| \le \epsilon|\mathrm{Re}(\phi_\theta(x))|.$$

According to the definition of $\mathcal{B}$ we have

$$|f(\phi_\theta(x))| \le \frac{C_k}{|\phi_\theta(x)|^k} \le \frac{C_{k,\epsilon}}{|x|^k}, \ \ \forall|x| \ge R, \ \ \forall\,\theta \in D_{\epsilon,M}, \ k \in \mathbb{N}$$

then, $\theta \longmapsto \int_{|x|\geq R} J_{\phi_\theta}^{\frac{1}{2}} f(\phi_\theta)\overline{g(x)}dx$ is analytic.

(ii) Let $h(x) \in (C_0^\infty(\mathbb{R}^3))^4$. We denote

$$h_k(x) = (\frac{k}{\pi})^{\frac{3}{2}} \int e^{-k(x-y-\theta g(y))^2} h(y) J_{\phi_\theta(y)} dy,$$

which is clearly in $\mathcal{B}$.
Using $(\frac{k}{\pi})^{\frac{3}{2}} \int e^{-k(x-y-\theta g(y))^2} J_{\phi_\theta(y)} dy = (\frac{k}{\pi})^{\frac{3}{2}} \int e^{-k\, z^2} dz = 1$, we get

$$h(x) - h_k(\phi_\theta(x)) = (\frac{k}{\pi})^{\frac{3}{2}} \int e^{-k(\phi_\theta(x)-\phi_\theta(y))^2} (h(x)-h(y)) J_{\phi_\theta(y)} dy.$$

The last term tends to 0 when $k \to +\infty$. Consequently, we have

$$h_k \circ \phi_\theta(x) \overset{k\to+\infty}{\longrightarrow} h(x), \quad \text{in } \mathcal{H}.$$

□

**Remark 5.** *One can always choose $g$ satisfying the assumption $(\mathbf{A_g})$ with $M > r_\epsilon = \frac{\epsilon}{\sqrt{1+\epsilon^2}}$. In that case, we have $D_{\epsilon,M} = D_\epsilon$.*

**Lemma 3.** *For $\theta \in D_\epsilon$, we have*

$$H_{0,\theta} := U_\theta H_0 U_\theta^{-1} = \frac{1}{1+\theta}(-ic\sum_{j=1}^{3} \alpha_j \frac{\partial}{\partial x_j}) + \beta mc^2 + Q_\theta(x, \partial_{x_j}),$$

*where $Q_\theta(x,\partial_{x_j}) = \sum_{|\alpha|\leq 1} a_\alpha(x,\theta)\partial_{x_j}^\alpha$ is such that:*

*(i) $\theta \longmapsto a_\alpha(x,\theta)$ is an analytic function bounded by $O(\theta)$.*

*(ii) $x \longmapsto a_\alpha(x,\theta) \in (C_0^\infty(\mathbb{R}^3))^4$.*

*In particular $\theta \longmapsto H_{0,\theta}$ is an analytic family of type A with domain $D(H_0)$ (see Kato* [24]*, for the definition of an analytic family of type A).*

**Proof.** We denote $\partial_j = \frac{\partial}{\partial x_j}$ and we calculate the term $U_\theta \partial_j U_\theta^{-1}$.

$$\begin{aligned} U_\theta \partial_j U_\theta^{-1} f(x) &= U_\theta \partial_j \left( J_{\phi_\theta}^{\frac{-1}{2}} f(\phi_\theta^{-1}(x)) \right) \\ &= U_\theta \left( \partial_j \left( J_{\phi_\theta}^{\frac{-1}{2}} \right) .f\left(\phi_\theta^{-1}(x)\right) + J_{\phi_\theta}^{\frac{-1}{2}} \left(\partial_j f(\phi_\theta^{-1}(x))\right) \right) \\ &= -\frac{1}{2} J_{\phi_\theta}^{-1} \partial_j J_{\phi_\theta(x)} f(x) + \sum_{k=1}^{3} \partial_k f(x) \left(\partial_j \phi_{\theta,k}^{-1}\right)(\phi_\theta(x)), \end{aligned}$$

with $\phi_\theta^{-1}(x) = \left(\ \phi_{\theta,1}^{-1}(x),\ \phi_{\theta,2}^{-1}(x),\ \phi_{\theta,3}^{-1}(x)\right)$.
We remark that $\phi_\theta^{-1}(x) = \frac{x}{1+\theta}$ outside the compact set $K$, then

$$\sum_{k=1}^{3} \partial_k f(x) \left(\partial_j \phi_{\theta,k}^{-1}\right)(\phi_\theta(x)) = \frac{1}{1+\theta} \partial_j f(x), \quad \text{outside } K.$$

Let $\chi_K \in C_0^\infty(\mathbb{R}^3)$, $0 \leq \chi_K \leq 1$, be equal to 1 on $K$ and 0 outside a compact set which contains $K$. We have

$$U_\theta \partial_j U_\theta^{-1} f(x) = -\frac{1}{2} J_{\phi_\theta}^{-1} \partial_j J_{\phi_\theta(x)} f(x) + \frac{1}{1+\theta} \partial_j f(x)(1-\chi_K) + \sum_{k=1}^{3} \partial_k f(x) \left(\partial_j \phi_{\theta,k}^{-1}\right)(\phi_\theta(x))\chi_K.$$

Since $\partial_j J_{\phi_\theta(x)}$ has compact support,

$$U_\theta \partial_j U_\theta^{-1} = \frac{1}{1+\theta}\partial_j + q_\theta(x, \partial_{x_j}) \tag{15}$$

with $q_\theta(x, \partial_{x_j})$ satisfying same hypothesis as $Q_\theta(x, \partial_{x_j})$.
Now we just have to multiply (15) by $-ic\alpha_j$, take the sum on all values of $j$ and add $\beta mc^2$ to both hands. The estimate $a_\alpha(x, \theta) = O(\theta)$ is clear using that $Q_0(x, \partial_{x_j}) = 0$ and the analyticity of $\theta \longmapsto a_\alpha(\cdot, \theta)$. □

**Lemma 4.** *Let* $P_\theta = \frac{1}{1+\theta}(-ic\sum_j \alpha_j \frac{\partial}{\partial x_j}) + \beta mc^2$. *Then*

$$\sigma(P_\theta) = \sigma_{ess}(P_\theta) = \Gamma_\theta = \{z \in \mathbb{C}; z = \pm c(\frac{\lambda}{(1+\theta)^2} + m^2c^2)^{\frac{1}{2}}, \lambda \in [0, +\infty[\}.$$

**Proof.** Let $\mathcal{F}$ be the Fourier transform and

$$\begin{aligned} K(\theta) = \mathcal{F}\, P_\theta\, \mathcal{F}^{-1} &= \frac{c}{1+\theta}\sum_j \alpha_j \xi_j + mc^2\beta \\ &= \begin{pmatrix} mc^2 I_2 & (\frac{c}{1+\theta})(\sigma_1\xi_1 + \sigma_2\xi_2 + \sigma_3\xi_3) \\ (\frac{c}{1+\theta})(\sigma_1\xi_1 + \sigma_2\xi_2 + \sigma_3\xi_3) & -mc^2 I_2 \end{pmatrix} \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $\alpha_j\xi_j$ is the multiplication operator by the $4\times 4$ matrix $\alpha_j\xi_j$. Here $\sigma_1, \sigma_2, \sigma_3$ are the $2\times 2$ Pauli matrices. The spectrum of $P_\theta$ coincides with the spectrum of the multiplication operator $K(\theta)$. We easily prove that

$$\sigma(K(\theta)) = \sigma_{ess}(K(\theta)) = \Gamma_\theta = \{z \in \mathbb{C}; z = \pm c(\frac{\lambda}{(1+\theta)^2} + m^2c^2)^{\frac{1}{2}}, \lambda \in [0, +\infty[\},$$

and we deduce the lemma. □

The principal branch of the square root function is holomorphic on the set $\mathbb{C}\backslash ]-\infty, 0]$. Let $S_{D_\epsilon} = \{z = \frac{\lambda}{(1+\theta)^2} + m^2c^2,\ \theta \in D_\epsilon,\ \lambda \in [0, +\infty[\}$. Since,

$$S_{D_\epsilon} \subset ]0, +\infty[e^{]-\frac{\pi}{2}, \frac{\pi}{2}[},$$

the square root $z \longmapsto z^{\frac{1}{2}}$ is holomorphic on $S_{D_\epsilon}$.

**Lemma 5.** *For* $H_{0,\theta}$, $P_\theta$, *defined as above, we have* $\sigma_{ess}(H_{0,\theta}) = \sigma_{ess}(P_\theta)$.

**Proof.** We want to use Kato's theorem [24, Th.4.5.35]. For $\lambda \gg 1$, $\lambda \in \mathbb{R}$ and $Q_\theta$ defined in Lemma 3, we have

$$(H_{0,\theta} - i\lambda) = (1 + Q_\theta(P_\theta - i\lambda)^{-1})(P_\theta - i\lambda).$$

Since $(P_\theta - i\lambda)^{-1} \in \mathcal{L}(\mathcal{H}, (H^1)^4)$ and $Q_\theta(P_\theta - i\lambda)^{-1} = O(\frac{\theta}{\lambda})$, we obtain that $i\lambda \in \rho(H_{0,\theta}) = \mathbb{C} \setminus \sigma(H_{0,\theta})$. To apply Kato's Theorem, it is enough to show that

$$(H_{0,\theta} - i\lambda)^{-1} - (P_\theta - i\lambda)^{-1} \text{ is compact.} \tag{16}$$

Using the resolvent equation, we have

$$(H_{0,\theta} - i\lambda)^{-1} - (P_\theta - i\lambda)^{-1} = (H_{0,\theta} - i\lambda)^{-1} Q_\theta (P_\theta - i\lambda)^{-1}.$$

with $Q_\theta(x, \partial_{x_j}) = \sum_{|\alpha|\leq 1} a_\alpha(x,\theta)\partial_{x_j}^\alpha$ compactly supported. Since the operaror $(H_{0,\theta} - i\lambda)^{-1} Q_\theta$ is bounded and $\mathbf{1}_{\mathrm{supp}(Q_\theta)}(P_\theta - i\lambda)^{-1}$ is compact, assertion (16) holds. $\square$

## 4. Definition of resonances

In this section we distort the perturbed Dirac operator $H = H_0 + V$, where the potential $V$ satisfies the assumption $(\mathbf{A_V})$ and we define the resonances for the semi-classical Dirac operator.
The distorted Dirac operator is denoted by

$$H_\theta = U_\theta H_0 U_\theta^{-1} + U_\theta V U_\theta^{-1} = H_{0,\theta} + V(\phi_\theta(x)).$$

**Proposition 5.** *We suppose that the potential $V$ satisfies the assumption $(\mathbf{A_V})$, then*
- *(i) $\theta \in D_\epsilon \longmapsto H_\theta = H_{0,\theta} + V(\phi_\theta(x))$ is an analytic family of type A.*
- *(ii) $\sigma_{ess}(H_\theta) = \Gamma_\theta$.*

**Proof.** The assertion (i) is clear since $H_{0,\theta}$ is an analytic family of type A and $V$ satisfies the assumption $(\mathbf{A_V})$.
Now, we prove (ii) as in the proof of Lemma 5. For $\lambda \gg 1$, $i\lambda \in \rho(H_\theta)$ and

$$(H_\theta - i\lambda)^{-1} - (H_{0,\theta} - i\lambda)^{-1} = (H_\theta - i\lambda)^{-1} V(\phi_\theta(x))(H_{0,\theta} - i\lambda)^{-1}. \tag{17}$$

Since the operator $V(\phi_\theta(x))(H_{0,\theta} - i\lambda)^{-1}$ is compact (see $(\mathbf{A_V})$) and the resolvent $(H_\theta - i\lambda)^{-1}$ is bounded, the difference $(H_\theta - i\lambda)^{-1} - (H_{0,\theta} - i\lambda)^{-1}$ is compact. According to Kato's theorem [24, Theorem.4.5.35] and to lemmas 4, 5, we obtain (ii). $\square$

We denote

$$\Sigma = \{z \in \mathbb{C};\ \mathrm{Im}(z) \geq 0,\ \mathrm{Re}(z) > -mc^2\} \cup \{z \in \mathbb{C};\ \mathrm{Im}\,(z) \leq 0,\ \mathrm{Re}(z) < mc^2\} \setminus \sigma(H).$$

**Theorem 4.** *With the notations used above, taking $\theta_0 \in D_\epsilon^+ = D_\epsilon \cap \{\mathrm{Im}(\theta) \geq 0\}$, we have:*
- *(i) For all $f, g \in \mathcal{B}$, the function: $z \in \Sigma \longmapsto M_{f,g}(z) = \langle (z - H)^{-1} f, g\rangle$ has a meromorphic extension on $S_{\theta_0}$.*
- *(ii) The poles of $M_{f,g}(z)$ are the eigenvalues of $H_{\theta_0}$.*
- *(iii) These poles are independent of the family $U_{\theta_0}$.*
- *(iv) $\sigma_d(H_{\theta_0}) \cap \Sigma = \emptyset$, where $\sigma_d(H_{\theta_0})$ is the discrete spectrum of the operator $H_{\theta_0}$.*

**Proof.** (i) Since $U_\theta$ is unitary for $\theta \in \mathbb{R}$,

$$M_{f,g}(z) = \langle (z - H)^{-1} f, g\rangle = \langle (z - H_\theta)^{-1} U_\theta f, U_\theta g\rangle.$$

We denote

$$M_{f,g,\theta}(z) = \langle (z - H_\theta)^{-1} U_\theta f, U_{\bar\theta} g\rangle, \quad \text{for } \theta \in D_\epsilon. \tag{18}$$

According to (i) of Proposition 5 and to the definition of $U_\theta$, the functions $\theta \mapsto (z - H_\theta)^{-1}$, $\theta \mapsto U_\theta f$ and $\theta \mapsto \langle \psi, U_{\bar\theta} g\rangle$ are analytic on $D_\epsilon$ for all $\psi \in \mathcal{H}$ and any $z \in \Sigma$.

Thus, for $z \in \Sigma$, the function $\theta \mapsto M_{f,g,\theta}(z)$ is analytic on $D_\epsilon$. Since $M_{f,g,\theta}(z)$ is independent of $\theta$ on the real axis and by uniqueness of the extension, it is independent of $\theta$.

Now, we fix $\theta_0 \in D_\epsilon^+$. Since $S_{\theta_0} \cap \sigma_{ess}(H_{\theta_0}) = \emptyset$, the function $z \in \Sigma \longmapsto M_{f,g,\theta_0}(z)$ has a meromorphic extension in $S_{\theta_0}$.

(ii) First, let $z \in S_{\theta_0}$ be a pole of $M_{f,g}(z)$ which is equal to $M_{f,g,\theta_0}(z)$ for $\theta_0 \in D_\epsilon^+$. Then $z \in \sigma_d(H_{\theta_0}) \cap S_{\theta_0}$ (see proof of (i)).

Now, let $\omega \in \sigma_d(H_{\theta_0}) \cap S_{\theta_0}$. There exists $u \in \mathcal{H}$ such that $\|u\| = 1$ and $H_{\theta_0} u = \omega u$. Let $\gamma$ be a small disk centered at $\omega$ such that $\mathring{\gamma} \cap \sigma(H_{\theta_0}) = \{\omega\}$ and $\Gamma$ be the positively oriented boundary of $\gamma$.
Let us introduce the projector

$$\Pi = \frac{1}{2i\pi} \int_\Gamma (z - H_{\theta_0})^{-1} dz; \quad \Pi u = u.$$

Since $\overline{U_{\theta_0}\mathcal{B}} = \mathcal{H} = \overline{U_{\bar{\theta}_0}\ \mathcal{B}}$ (see Proposition 4), there exist $f_n,\ g_n \in \mathcal{B}$ such that

$$|u - U_{\theta_0} f_n| \leq \frac{1}{n} \text{ and } |u - U_{\bar{\theta}_0} g_n| \leq \frac{1}{n}, \quad n \in \mathbb{N}.$$

Therefore, as $n$ goes to infinity, we have

$$\begin{aligned} \frac{1}{2i\pi} \int_\Gamma \langle (z - H_{\theta_0})^{-1} U_{\theta_0} f_n, U_{\bar{\theta}_0} g_n \rangle dz &= \frac{1}{2i\pi} \int_\Gamma \langle (z - H_{\theta_0})^{-1} u, u \rangle dz + o(1) \\ &= \langle \Pi u, u \rangle + o(1) \\ &= \|u\|^2 + o(1) \\ &= 1 + o(1), \end{aligned}$$

and then,

$$\frac{1}{2i\pi} \int_\Gamma \langle (z - H)^{-1} f_n, g_n \rangle dz = 1 + o(1).$$

So that $M_{f_n, g_n}(z)$ admits $\omega$ as a pole in $\gamma$.

The assertion (iii) follows from (ii) because $M_{f,g}(z)$ is independent of $U_\theta$.

(iv) If there exists $z \in \sigma_d(H_{\theta_0}) \cap \Sigma$, then $z$ is a pole of $\langle (z - H)^{-1} f, g \rangle$, for $f, g \in \mathcal{B}$, but $\langle (z - H)^{-1} f, g \rangle$ is analytic on this domain. We conclude that such $z$ does not exist. □

**Remark 6.** (i) *It follows from (ii) of Theorem 4 that for all $\theta \in D_\epsilon^+$, the discrete spectrum $\sigma_d(H)$ is a subset of $\sigma_d(H_\theta)$.*

(ii) *The previous theorem justifies the definition of the resonances (Definition 1) and using Lemma 4, $H_0$ has no resonances.*

**Remark 7.** *If $\theta \in D_\epsilon$, then its conjugate $\bar{\theta} \in D_\epsilon$. Repeating the arguments of the proof of Theorem 4, we have*

(i) *The function $\theta \longmapsto M_{f,g,\bar{\theta}}(z)$ has a analytic extension for $\theta \in D_\epsilon$.*

(ii) *The function $z \in \bar{\Sigma} \longmapsto M_{f,g,\bar{\theta}}(z)$ has a meromorphic extension on $S_{\bar{\theta}_0}$,*

*where*

$$\bar{\Sigma} = \{z \in \mathbb{C}; \mathrm{Im}(z) \geq 0,\ \mathrm{Re}(z) < mc^2\} \cup \{z \in \mathbb{C}; \mathrm{Im}(z) \leq 0,\ \mathrm{Re}(z) > -mc^2\} \setminus \sigma(H),$$

*and $S_{\bar{\theta}_0}$ is the symmetric of $S_{\theta_0}$ with respect to the real axis*

$$S_{\bar{\theta}_0} = \{z \in \bigcup_{\theta \in D_\epsilon^+} \Gamma_{\bar{\theta}};\ \ arg(1+\theta) < arg(1+\theta_0),\ \frac{1}{|1+\theta|} < \frac{1}{|1+\theta_0|}\ \}.$$

*Consequently, we obtain (see Theorem 4):*

1) *The poles of $M_{f,g}(z)$ in $S_{\bar{\theta}_0}$ are the eigenvalues of $H_{\bar{\theta}_0}$.*
2) *These poles are independent of the family $U_{\bar{\theta}_0}$.*

3) $\sigma_d(H_{\bar\theta_0}) \cap \bar\Sigma = \emptyset$.
*The assertions* 3) *and* (ii) *prove that the operator* $H_{\bar\theta}$ *has purely discrete spectrum in* $S_{\bar\theta_0}$.

**Proof of Proposition 1.** We consider the anti-linear application on $\mathcal{H}$,

$$C : \psi \longmapsto U_c\overline{\psi} = i\beta\alpha_2\overline{\psi}.$$

Then, we have

$$\begin{aligned} C\ H_\theta\ C^{-1} &= -H_{0,\bar\theta} + CV \circ \phi_\theta(x)C^{-1} \\ &= -\left(H_{0,\bar\theta} - U_c\overline{V \circ \phi_\theta(x)}U_c^{-1}\right). \end{aligned}$$

Using that $V$ is analytic, we get $\overline{V \circ \phi_\theta(x)} = \overline{V} \circ \phi_{\bar\theta}(x)$. Then,

$$\begin{aligned} C\ H_\theta\ C^{-1} &= -\left(H_{0,\bar\theta} - U_c\overline{V} \circ \phi_{\bar\theta}(x)U_c^{-1}\right) \\ &= -\left(H_{0,\bar\theta} - U_cU_{\bar\theta}\overline{V}(x)U_{\bar\theta}^{-1}U_c^{-1}\right). \end{aligned}$$

We recall that $U_\theta f(x) = J_{\phi_\theta(x)}^{\frac12} f(\phi_\theta(x))$. Since $U_cU_\theta = U_\theta U_c$, we obtain

$$\begin{aligned} C\ H_\theta\ C^{-1} &= -\left(H_{0,\bar\theta} - U_{\bar\theta}U_c\overline{V}(x)U_c^{-1}U_{\bar\theta}^{-1}\right) \\ &= -U_{\bar\theta}\left(H_0 - U_c\overline{V}(x)U_c^{-1}\right)U_{\bar\theta}^{-1} = -H_{\bar\theta}^{-}. \end{aligned}$$

Consequently, $C\ (H_\theta - z)\ C^{-1} = -(H_{\bar\theta}^{-} + \bar z)$, and the property follows. □

**Proof of Proposition 2.**
By definition of $H_\theta$, we have

$$\overline{H_\theta - z} = U_{\bar\theta}\left(\overline{H}_0 + \overline{V} - \bar z\right)U_{\bar\theta}^{-1}. \tag{19}$$

Using that $\bar\alpha_1 = \alpha_1,\ \bar\alpha_2 = -\alpha_2,\ \bar\alpha_3 = \alpha_3,\ \bar\beta = \beta$, we find

$$\overline{H}_0 = ic\sum_{j=1}^{3}\overline{\alpha}_j\frac{\partial}{\partial x_j} + \beta mc^2 = -ic\sum_{j=1}^{3}\alpha'_j\frac{\partial}{\partial x_j} + \beta' mc^2$$

and

$$\overline{H}_0 + \overline{V} = \overline{H}.$$

Using the last relation and equation (19), we obtain Proposition 2. □

Finally, the study of resonances in a domain of the complex plane $\mathbb{C}$ is reduced to the study of resonances in $\Omega \cap \{z \in \mathbb{C},\ \mathrm{Im}(z) < 0\}$, with $\Omega$ satisfying assumption $(\mathbf{A}^+_{\mathbf{\Omega}})$ (see Fig.1).

## 5. Upper bound for the number of resonances

In this section, we establish an upper bound on the number of resonances in a compact domain $\Omega$. To this purpose we construct an operator $\widehat{H}_\theta : D(H_0) \to \mathcal{H}$ with some properties (see Proposition 7). According to Section 4, it is sufficient to treat the case where $\Omega$ satisfies assumption $(\mathbf{A}^+_{\mathbf{\Omega}})$.

We shall use the theory of $h$-pseudo-differential operators (see [12], [35]). Let $m$ be an order function on $\mathbb{R}^{2n}$ (i.e. there are $C_0, N_0 > 0$, such that $m(x) \leq C_0\langle x - y\rangle^{N_0} m(y)$). The space

$\mathcal{S}^p(m)$ is the set of $a(x,\xi;h) \in C^\infty(\mathbb{R}^{2n})\otimes\mathbb{C}^4$ such that for every $\alpha \in \mathbb{N}^{2n}$, there exists $C_\alpha > 0$, such that

$$\|\nabla^\alpha_{x,\xi} a(x,\xi;h)\| \leq C_\alpha m(x,\xi) h^{-p}.$$

For a symbol $a(x,\xi;h)$, we define the Weyl quantization, $a^w(x, h\nabla_x; h) := Op^\omega_h(a)$ by

$$Op^\omega_h(a)u(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ih^{-1}(x-y)\cdot\xi} a(\frac{x+y}{2},\xi;h)u(y)dyd\xi,$$

where $u(x)$ is in the Schwartz space.

5.1. **Construction of $\widehat{H}_\theta$.** We follow the approach of Sjöstrand [40]. Let $\Omega$ be a complex domain satisfying the assumption $(\mathbf{A}^+_{\mathbf{\Omega}})$ and $\psi \in C^\infty_0(\mathbb{R}^3)$ be such that $\psi(x) \geq 0$, $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$. We recall the notations of Section 3: $\phi_\theta(x) = x + \theta g(x)$ with $g(x) = 0$ in the compact set $B(0, R_0) \subset K$ and $g(x) = x$ outside $K \subset B(0, \alpha_0)$ where $\alpha_0 > 0$ is sufficiently large.

Using Lemma 3, the semi-classical principal symbol of $H_\theta$ is given by:

$$h_\theta(x,\xi) = \alpha\cdot\zeta_\theta(x,\xi) + mc^2\beta + V(\phi_\theta(x)),$$

with

$$\zeta_\theta(x,\xi) = (\zeta_{\theta,1}(x,\xi),\ \zeta_{\theta,2}(x,\xi),\ \zeta_{\theta,3}(x,\xi)) \quad\text{and}\quad \zeta_{\theta,j}(x,\xi) = c\sum_{k=1}^{3}\xi_k\left(\partial_j\phi^{-1}_{\theta,k}\right)(\phi_\theta(x)).$$

For all $(x,\xi)$, the matrix $M = \alpha\cdot\zeta_\theta(x,\xi) + mc^2\beta$, has two eigenvalues

$$\lambda^\pm_\theta = \pm\sqrt{\zeta_\theta(x,\xi)^2 + m^2c^4}.$$

Consequently, there exists an invertible matrix $U$ such that

$$U^{-1}MU = d_\theta := \begin{pmatrix} \lambda^+_\theta I_2 & 0 \\ 0 & \lambda^-_\theta I_2 \end{pmatrix},$$

where

$$U = \begin{pmatrix} I_2 & \frac{-1}{\lambda^+_\theta + mc^2}\sigma\cdot\zeta_\theta(x,\xi) \\ \frac{1}{\lambda^+_\theta + mc^2}\sigma\cdot\zeta_\theta(x,\xi) & I_2 \end{pmatrix},$$

with $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ and $(\sigma_j)_{1\leq j\leq 3}$ the $2\times 2$ Pauli matrices.
One can easily prove that the norms of $U$, $U^{-1}$ and their derivatives are bounded in the following way:

$$\begin{aligned} \|\partial^\alpha_x\partial^\beta_\xi U(x,\xi)\| &< C\langle\xi\rangle^{-\beta}, \\ \|\partial^\alpha_x\partial^\beta_\xi U^{-1}(x,\xi)\| &< C\langle\xi\rangle^{-\beta}, \quad \forall\alpha,\ \beta\in\mathbb{N}. \end{aligned} \tag{20}$$

Applying $U^{-1}$ on the left and $U$ on the right of $h_\theta(x,\xi)$, we obtain

$$U^{-1}h_\theta(x,\xi)U = d_\theta + \widetilde{V}_\theta(x,\xi),$$

where $\widetilde{V}_\theta(x,\xi) = U^{-1}V(\phi_\theta(x))U$. Since the matrix $V(\phi_\theta(x))$, $U$, $U^{-1}$, and their derivatives are uniformly bounded, $\widetilde{V}_\theta(x,\xi)$ and its derivatives are uniformly bounded.

In order to construct $\widehat{H}_\theta$, we introduce an intermediate function $f(x,\xi)$:

We denote $|\Omega|$ the diameter of $\Omega$. Let us choose $\beta_0 > 0$ and $C_0 > 0$, sufficiently large such that

$$\forall\,\xi \in \mathbb{R}^3;\ \sup_{x\in\mathbb{R}^3} \|\widetilde{V}_\theta(x,\xi)\| + |\Omega| \le \frac{1}{2}\,|\lambda_\theta^\pm - iC_0\psi(\frac{\xi}{\beta_0})| = \frac{1}{2}\sqrt{\left(\mathrm{Re}(\lambda_\theta^\pm)\right)^2 + \left(\mathrm{Im}(\lambda_\theta^\pm) - C_0\psi(\frac{\xi}{\beta_0})\right)^2}. \tag{21}$$

We prove (21) considering the two cases:

- For $|\xi| > \beta_0$, with $\beta_0 > 0$, sufficiently large we have

$$\sup_{x\in\mathbb{R}^3} \|\widetilde{V}_\theta(x,\xi)\| + |\Omega| \le \frac{1}{2}\,|\mathrm{Re}(\lambda_\theta^\pm)|.$$

- For $|\xi| \le \beta_0$, since $\lambda_\theta^\pm$ is bounded, we choose $C_0 > 0$, sufficiently large such that

$$\sup_{x\in\mathbb{R}^3} \|\widetilde{V}_\theta(x,\xi)\| + |\Omega| \le \frac{1}{2}\,|\mathrm{Im}(\lambda_\theta^\pm) - C_0|.$$

For $|x| > \alpha_0 > 0$, sufficiently large we have $\zeta_{\theta,j}(x,\xi) = \frac{c\xi_j}{1+\theta}$ and $\lambda_\theta^\pm = \pm c\sqrt{\frac{\xi^2}{(1+\theta)^2} + m^2c^2}$. Since the domain $\Omega$ satisfies the assumption $(\mathbf{A}_{\mathbf{\Omega}}^+)$, we have

$$\min\{\mathrm{dist}(\overline{\Omega}, \lambda_\theta^+),\ \mathrm{dist}(\overline{\Omega}, \lambda_\theta^-)\} \neq 0,$$

hence we can choose $\alpha_0 > 0$ sufficiently large such that

$$\forall |x| > \alpha_0, \quad \|\widetilde{V}_\theta(x,\xi)\| \le \frac{1}{2}\,\mathrm{dist}(\overline{\Omega}, \lambda_\theta^\pm) := \frac{1}{2}\min\{\mathrm{dist}(\overline{\Omega}, \lambda_\theta^+),\ \mathrm{dist}(\overline{\Omega}, \lambda_\theta^-)\}. \tag{22}$$

Now, we define $f(x,\xi)$ in the following way:

$$f(x,\xi) = C_0\psi(\frac{x}{\alpha_0})\psi(\frac{\xi}{\beta_0}). \tag{23}$$

**Lemma 6.** *The matrix $h_\theta(x,\xi) - if(x,\xi) - z$ is invertible for all $z \in \Omega$ and satisfies*

$$\|\partial_x^\alpha \partial_\xi^\beta \left(h_\theta(x,\xi) - if(x,\xi) - z\right)^{-1}\| < C\langle\xi\rangle^{-1-\beta}, \quad \forall \alpha,\, \beta \in \mathbb{N}. \tag{24}$$

**Proof.** Applying $U^{-1}$ on the left and $U$ on the right of $h_\theta(x,\xi) - if(x,\xi) - z$, we obtain

$$U^{-1}\left(h_\theta(x,\xi) - if(x,\xi) - z\right)U = d_\theta - if(x,\xi) - z + \widetilde{V}_\theta(x,\xi).$$

1) Let us prove that the symbol $\sigma := d_\theta - if(x,\xi) - z + \widetilde{V}_\theta(x,\xi)$ is invertible.

- For $|x| \le \alpha_0$,

$$\sigma = (d_\theta - iC_0\psi(\frac{\xi}{\beta_0}))\left(I_4 + \left(d_\theta - iC_0\psi(\frac{\xi}{\beta_0})\right)^{-1}\left(\widetilde{V}_\theta(x,\xi) - z\right)\right).$$

  According to (21), we have

$$\|\left(d_\theta - iC_0\psi(\frac{\xi}{\beta_0})\right)^{-1}\left(\widetilde{V}_\theta(x,\xi) - z\right)\| < \frac{1}{2},$$

  thus $\sigma$ is invertible and satisfies

$$\|\sigma^{-1}\| < 2\left\|\left(d_\theta - iC_0\psi(\frac{\xi}{\beta_0})\right)^{-1}\right\| < C\langle\xi\rangle^{-1}. \tag{25}$$

- For $|x| > \alpha_0$, we have $\lambda_\theta^\pm = \pm c\sqrt{\frac{\xi^2}{(1+\theta)^2} + m^2c^2}$. Since $f(x,\xi) \geq 0$, we have

$$|\lambda_\theta^+ - (z + if(x,\xi))| > \mathrm{dist}(\overline{\Omega}, \lambda_\theta^\pm) > C\langle\xi\rangle > 0,$$

and

$$|\mathrm{Re}\left(\lambda_\theta^- - (z + if(x,\xi))\right)| = |\mathrm{Re}\left(\lambda_\theta^- - z\right)| > \mathrm{dist}(\overline{\Omega}, \lambda_\theta^\pm) > C\langle\xi\rangle > 0.$$

Since

$$\sigma = (d_\theta - if(x,\xi) - z)\left(I_4 + (d_\theta - if(x,\xi) - z)^{-1}\widetilde{V}_\theta(x,\xi)\right),$$

and

$$\|(d_\theta - if(x,\xi) - z)^{-1}\widetilde{V}_\theta(x,\xi)\| < \frac{1}{2}, \quad \text{(see (22))},$$

the matrix $\sigma$ is invertible and

$$\|\sigma^{-1}\| < 2\|(d_\theta - if(x,\xi) - z)^{-1}\| < C\langle\xi\rangle^{-1}. \tag{26}$$

2) According to 1), the matrix $U^{-1}(h_\theta(x,\xi) - if(x,\xi) - z)U$ is invertible. From (25), (26) and (20), we deduce that the matrix $h_\theta(x,\xi) - if(x,\xi) - z$ is invertible and

$$\begin{aligned}
\|(h_\theta(x,\xi) - if(x,\xi) - z)^{-1}\| &= \|U^{-1}\left(d_\theta - if(x,\xi)I_4 - z + \widetilde{V}_\theta(x,\xi)\right)^{-1}U\| \\
&\leq \|U\|\|U^{-1}\|\|\left(d_\theta - if(x,\xi)I_4 - z + \widetilde{V}_\theta(x,\xi)\right)^{-1}\| \\
&< C\langle\xi\rangle^{-1}. \qquad (27)
\end{aligned}$$

This gives (24) for $\alpha = \beta = 0$. Using (20) and (27) we obtain (24) for $(\alpha,\beta) \in \mathbb{N}^2$ by induction. $\square$

We denote $\widetilde{H}_\theta = H_\theta + \widetilde{T}$, with $\widetilde{T} = Op_h^\omega(-if(x,\xi))$, where $f(x,\xi)$ is defined in (23). It is clear that the semi-classical principal symbol of $(\widetilde{H}_\theta - z)$ is

$$\sigma_{\widetilde{H}_\theta} := h_\theta(x,\xi) - if(x,\xi) - z.$$

**Proposition 6.** *If $h > 0$ is small enough, the operator $(z - \widetilde{H}_\theta)$ is invertible for every $z \in \Omega$ and, for every $N \in \mathbb{N}$ its inverse satisfies:*

$$(z - \widetilde{H}_\theta)^{-1} = O_N(1) : D(H^N) \longmapsto D(H^{N+1}),$$

*uniformly for $z \in \Omega$. Here $D(H^N)$ denotes the domain of $H^N$ with the convention $D(H^0) = \mathcal{H}$.*

**Proof.** Let us prove that the operator $(z - \widetilde{H}_\theta)$ is a Fredholm operator of index 0. We have

$$\begin{aligned}
(z - \widetilde{H}_\theta)(z - H_{0,\theta})^{-1} &= (z - H_{0,\theta} + H_{0,\theta} - \widetilde{H}_\theta)(z - H_{0,\theta})^{-1} \\
&= I - (\widetilde{T} + V(\phi_\theta(x)))(z - H_{0,\theta})^{-1}.
\end{aligned}$$

Since the right-hand side is a perturbation of the identity by a compact operator and

$$(z - H_{0,\theta})^{-1} : (L^2(\mathbb{R}^3))^4 \mapsto D(H) \text{ is invertible},$$

the operator $(z - \widetilde{H}_\theta)$ is Fredholm of index 0. Consequently, it is enough to show that

$$\|u\|^2_{D(H^{N+1})} \leq C\|(z - \widetilde{H}_\theta)u\|^2_{D(H^N)}, \quad \text{for } u \in D(H^{N+1}). \tag{28}$$

According to Lemma 6, the symbol $q_0 = \sigma^{-1}_{\widetilde{H}_\theta}$ is well defined and satisfies

$$\|\partial_x^\alpha \partial_\xi^\beta q_0\| < C\langle\xi\rangle^{-1-\beta}.$$

Moreover, having

$$\|\partial_x^\alpha \partial_\xi^\beta (\sigma_{\widetilde{H}_\theta})\| < C\langle\xi\rangle^{+1-\beta},$$

the composition theorem of $h$-pseudo-differential operators implies

$$Op_h^\omega(q_0)Op_h^\omega(\sigma_{\widetilde{H}_\theta}) = Op_h^\omega(r)$$

where $(r-1)$ is in the space of symbols $\mathcal{S}^0(h)$. In particular the operator

$$Op_h^\omega(r):\ D(H^{N+1}) \longmapsto D(H^{N+1}), \quad \forall N \in \mathbb{N},$$

is invertible for $h$ small enough, then (28) follows. Therefore the operator $(\widetilde{H}_\theta - z)$ is also invertible and we have

$$(z - \widetilde{H}_\theta)^{-1} = O_N(1) : D(H^N) \longmapsto D(H^{N+1}).$$

□

**Proposition 7.** *There exists $\widehat{H}_\theta : D(H) \longmapsto \mathcal{H}$, with the following properties.*
*The difference $K := \widehat{H}_\theta - H_\theta$ is of finite rank $O(h^{-3})$, has compact support in the sense that $K = \chi_1 K \chi_1$ for some $\chi_1 \in C_0^\infty(\mathbb{R}^3)$ and*

$$K = O(1) : D(H^N) \longmapsto D(H^M)\ \forall N, M \in \mathbb{N}.$$

*Moreover, for every $N \in \mathbb{N}$, we have*

$$(\widehat{H}_\theta - z)^{-1} = O(1) : D(H^N) \longmapsto D(H^{N+1}),$$

*uniformly for $z \in \overline{\Omega}$.*

**Proof.** (We again use all the previous notations) We define

$$\widehat{H}_\theta := H_\theta + \chi_1 T \chi_1 = \widetilde{H}_\theta + \chi_1 T \chi_1 - \widetilde{T},$$

with $\chi_1(x) = \psi(\frac{x}{2\alpha_0})$ and

$$T := \chi(-h^2\Delta + x^2)\widetilde{T} = \chi(-h^2\Delta + x^2)Op_h^\omega(-if(x,\xi))$$

where $\chi \in C_0^\infty(\mathbb{R})$ is such that:

$$\chi(\xi^2 + x^2) = 1 \text{ on the support of } f(x,\xi) \text{ (see (23))}.$$

By the functional calculus (see [12]), we can prove that

$$\widetilde{H}_\theta - \widehat{H}_\theta = \widetilde{T} - \chi_1 T \chi_1 = O(h^\infty) : D(H^N) \longmapsto D(H^M),\ \forall\ M,\ N \in \mathbb{N}. \tag{29}$$

The last lemma, formula (29) and

$$(\widehat{H}_\theta - z)^{-1} = (\widetilde{H}_\theta - z)^{-1}\left(I + (\widehat{H}_\theta - \widetilde{H}_\theta)(\widetilde{H}_\theta - z)^{-1}\right)^{-1}$$

yield for all N $\in \mathbb{N}$

$$(\widehat{H}_\theta - z)^{-1} = O(1) : D(H^N) \longmapsto D(H^{N+1}).$$

According to the facts that $\chi(-h^2\Delta + x^2)$ is of finite rank $O(h^{-3})$, that the Weyl quantization $Op_h^\omega(-if(x,\xi))$ is bounded, and to the definition of $\chi_1$, the operator

$$K := \widehat{H}_\theta - H_\theta = \chi_1\left(\chi(-h^2\Delta + x^2)Op_h^\omega(-if(x,\xi))\right)\chi_1$$

is of finite rank $O(h^{-3})$ and compactly supported. $\square$

5.2. **Upper bound for the number of resonances.** In this section we establish the upper bound on the number of resonances given in Theorem 1.

**Lemma 7.** *Let $\rho > 0$, $\Omega$ be an open complex relatively compact subset of $\mathbb{C}$ and $H_\theta$ be defined as above. There exists $g$ satifying ($\mathbf{A_g}$) such that for $h$ small enough and $z \in \overline{\Omega} \cap \{\operatorname{Im} z \geq \rho > 0\}$, we have $(z - H_\theta)^{-1} = O(1)$.*

**Proof.** We again use the notations of Section 3: $\phi_\theta(x) = x + \theta g(x)$ with $g(x) = 0$ in the compact set $B(0, R_0)$, and the notations of Subsection 5.1 concerning $h_\theta(x,\xi)$, $U$, $U^{-1}$, $d_\theta$ and $\widetilde{V}_\theta(x,\xi)$ which satisfy
$$U^{-1} h_\theta(x,\xi) U = d_\theta + \widetilde{V}_\theta(x,\xi).$$
The matrix $h_\theta(x,\xi)$ is the semi-classical principal symbol of $H_\theta$.

According to Section 4, the resonances are independent of the family $U_\theta$. Then we can assume that $g(x) = 0$ in the ball $B(0, R_g) \supset B(0, R_0)$, with $R_g > 0$, sufficiently large such that
$$\forall x \in \mathbb{R}^3, \ \ |x| > R_g > 0, \ \ \|\widetilde{V}_\theta(x,\xi)\| \leq \frac{\rho}{2}.$$
Repeating arguments of Subsection 5.1, we can prove that $(d_\theta + \widetilde{V}_\theta(x,\xi) - z)$ is invertible, thus $(h_\theta(x,\xi) - z)$ is invertible and
$$\|\partial_x^\alpha \partial_\xi^\beta \left(h_\theta(x,\xi) - z\right)^{-1}\| < C\langle\xi\rangle^{-1-\beta}.$$
Since we have:
$$\|\partial_x^\alpha \partial_\xi^\beta \left(h_\theta(x,\xi) - z\right)\| < C\langle\xi\rangle^{+1-\beta},$$
the composition theorem of $h$-pseudo-differential operators implies
$$Op_h^\omega((h_\theta(x,\xi) - z)^{-1}) Op_h^\omega(h_\theta(x,\xi) - z) = 1 + O(h),$$
where $O(h)$ corresponds to the norm in $\mathcal{L}(L^2)$. $\square$

**Proof of Theorem 1.**
Let $\widehat{K}(z) = K(z - \widehat{H}_\theta)^{-1}$ with $\widehat{H}_\theta$, $K$ defined in Proposition 7. We remark that
$$(I + \widehat{K}(z))(z - \widehat{H}_\theta) = (z - \widehat{H}_\theta) + K = z - H_\theta.$$
Thus, the resonances $z \in \operatorname{Res}(H) \cap \Omega$ repeated with their multiplicities coincide with the zeros of the function
$$D(z) = \det(I + \widehat{K}(z)).$$
Indeed, in a neighborhood of a zero $z_0$ of $D(z)$ with multiplicity $l(z_0)$, we write $D(z) = (z - z_0)^{l(z_0)} G_0(z)$, where $G_0(z)$ is a holomorphic function in a neighborhood of $z_0$ with $G_0(z_0) \neq 0$. As [40, Equation (4.31)] we have
$$-\operatorname{tr}\left((H_\theta - z)^{-1} K (\widehat{H}_\theta - z)^{-1}\right) \ = \ \partial_z \log\, \det\left(1 + \widehat{K}(z)\right). \tag{30}$$
On the other hand, by the definition of $l(z_0)$,
$$l(z_0) = \frac{1}{2i\pi} \int_\Gamma \partial_z \log\, \det\left(1 + \widehat{K}(z)\right) dz,$$

where $\Gamma$ is a small positively oriented circle centered at $z_0$. From (30), we obtain

$$\begin{aligned} l(z_0) &= \frac{-1}{2i\pi}\int_\Gamma \operatorname{tr}\Big((H_\theta - z)^{-1}K(\widehat{H}_\theta - z)^{-1}\Big)\,dz \\ &= \frac{-1}{2i\pi}\int_\Gamma \operatorname{tr}\Big((H_\theta - z)^{-1} - (\widehat{H}_\theta - z)^{-1}\Big)\,dz \\ &= \operatorname{rank} \frac{1}{2i\pi}\int_\Gamma (z - H_\theta)^{-1}dz, \end{aligned}$$

In the latter equality, we have used that the trace of the projector coincides with its rank. Since $K$ is bounded and is of finite rank $O(h^{-3})$,

$$|D(z)| \le e^{\|\widehat{K}(z)\|_{tr}} \le e^{C_0 h^{-3}}, \text{ for all } z \in \overline{\Omega}.$$

Using Lemma 7, we get $(z - H_\theta)^{-1} = O(1)$ for $\mathrm{Im} z \ge \rho > 0$ and $z \in \overline{\Omega}$. Since

$$(I + \widehat{K}(z))^{-1} = (z - \widehat{H}_\theta)(z - H_\theta)^{-1}, \tag{31}$$

then

$$\|(I + \widehat{K}(z))^{-1}\| \le C_1, \quad \mathrm{Im} z \ge \rho > 0.$$

Writing the operator $(I + \widehat{K}(z))^{-1}$ in the form

$$(I + \widehat{K}(z))^{-1} = I - \widehat{K}(z)(I + \widehat{K}(z))^{-1},$$

we obtain

$$|\det\Big((I + \widehat{K}(z))^{-1}\Big)| \le e^{C_2 h^{-3}}, \ \ \mathrm{Im}\, z \ge \rho,$$

which implies

$$|D(z)| \ge Ce^{-C_3 h^{-3}}, z \in \overline{\Omega} \cap \{\mathrm{Im}\, z \ge \rho\}.$$

Now, applying Jensen's inequality in a slightly larger domain, we obtain Theorem 1. □

## 6. Representation of the derivative of the spectral shift function

In this section we prove our principal result given in Theorem 2 and a generalization (see Theorem 5). Moreover, we give a Sjöstrand type local trace formula.

The spectral shift function $\xi(\lambda, h)$ $(\in \mathcal{D}'(\mathbb{R}))$ associated to $H_0, H_1$ is defined (see [10], [37], [45]) by

$$\langle \xi'(\lambda, h), f(\lambda)\rangle = \operatorname{tr}(f(H_1) - f(H_0)), \quad f \in C_0^\infty(\mathbb{R}).$$

**Proof of Proposition 3.** The Dirac operator $H_0 + V$ is unitarily equivalent to the operator

$$H_0 + V - \alpha\cdot\nabla\Phi_g = e^{\frac{i}{hc}\Phi_g}(H_0 + V)e^{\frac{-i}{hc}\Phi_g}.$$

Then, for $f(\lambda) \in C_0^\infty(\mathbb{R})$, the SSF $\xi(\lambda, h)$ for the operator pair $(H_0,\ H_0 + V - \alpha\cdot\nabla\Phi_g)$ satisfies

$$\begin{aligned} \langle \xi'(\lambda, h), f(\lambda)\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} &= \operatorname{tr}(f(H_0 + V - \alpha\cdot\nabla\Phi_g) - f(H_0)) \\ &= \operatorname{tr}(f(e^{\frac{i}{hc}\Phi_g}(H_0 + V)e^{\frac{-i}{hc}\Phi_g}) - f(H_0)) \\ &= \operatorname{tr}(e^{\frac{i}{hc}\Phi_g}f(H_0 + V)e^{\frac{-i}{hc}\Phi_g} - f(H_0)). \end{aligned} \tag{32}$$

Let us now calculate $\mathrm{tr}\left(\chi_R\left[e^{\frac{i}{hc}\Phi_g}f(H_0+V)e^{\frac{-i}{hc}\Phi_g}-f(H_0)\right]\right)$ where $\chi_R(x)=\chi(\frac{x}{R})$, $\chi\in C_0^\infty(\mathbb{R}^3)$, $\chi(x)=1$ if $|x|\leq 1$ and $\chi(x)=0$ if $|x|\geq 2$. Using that $\chi_R f(H_0+V)$ and $\chi_R f(H_0)$ are trace class operators and the cyclicity of the trace, we get:

$$\begin{aligned}\mathrm{tr}\left(\chi_R\left[e^{\frac{i}{hc}\Phi_g}f(H_0+V)e^{\frac{-i}{hc}\Phi_g}-f(H_0)\right]\right) &= \mathrm{tr}\left(e^{\frac{i}{hc}\Phi_g}\chi_R f(H_0+V)e^{\frac{-i}{hc}\Phi}\right)-\mathrm{tr}\left(\chi_R f(H_0)\right)\\ &= \mathrm{tr}\left(\chi_R f(H_0+V)\right)-\mathrm{tr}\left(\chi_R f(H_0)\right)\\ &= \mathrm{tr}\left(\chi_R\left[f(H_0+V)-f(H_0)\right]\right).\end{aligned} \tag{33}$$

Using Theorem 6.3 of [17], we can take the limit $R\to\infty$ in (33). From (32), we obtain

$$\langle\xi'(\lambda,h),f(\lambda)\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}=\mathrm{tr}\left(f(H_0+V)-f(H_0)\right),$$

and the proposition follows. □

In the following, we will use the notations:

$$H_1=H,\quad K_1:=K=\widehat{H}_{1,\theta}-H_{1,\theta}:=\widehat{H}_\theta-H_\theta \text{ and } [a_.]_0^1=a_1-a_0.$$

For an integer $m>3$, we define the functions:

$$\sigma_\pm(z)=(z^2+1)^m\mathrm{tr}\Big[(H_.-i)^{-m}(H_.+i)^{-m}(z-H_.)^{-1}\Big]_0^1,\quad \pm\mathrm{Im}\,z>0. \tag{34}$$

The $\sigma_\pm$ satisfy the relation

$$\sigma_-(z)=\overline{\sigma_+(\bar z)},\quad \mathrm{Im}\,(z)<0. \tag{35}$$

**Proposition 8.** *For a potential $V$ satisfying the assumption $(\mathbf{A_V})$ with $\delta>3$, the function $\theta\longmapsto\Big[(H_{.,\theta}-i)^{-m}(H_{.,\theta}+i)^{-m}(z-H_{.,\theta})^{-1}\Big]_0^1$ is holomorphic from $D_\epsilon^+$ to the space of trace class operators. Moreover, for any $\theta\in D_\epsilon^+$, we have*

$$\sigma_\pm(z)=(z^2+1)^m\mathrm{tr}\Big[(H_{.,\theta}-i)^{-m}(H_{.,\theta}+i)^{-m}(z-H_{.,\theta})^{-1}\Big]_0^1,\quad \pm Im\,z>0. \tag{36}$$

**Proof.** For $\theta\in\mathbb{R}$, the operator

$$(H_.-i)^{-m}(H_.+i)^{-m}(z-H_.)^{-1},$$

is unitarily equivalent to the operator

$$(H_{.,\theta}-i)^{-m}(H_{.,\theta}+i)^{-m}(z-H_{.,\theta})^{-1}.$$

Using the cyclicity of the trace, we deduce

$$\sigma_\pm(z)=(z^2+1)^m\mathrm{tr}\Big[(H_{.,\theta}-i)^{-m}(H_{.,\theta}+i)^{-m}(z-H_{.,\theta})^{-1}\Big]_0^1,\quad \pm\mathrm{Im}\,z>0,\ \theta\in\mathbb{R}. \tag{37}$$

According to the proof of Theorem 4, the resolvent $(z-H_{.,\theta})^{-1}$ is analytic for $\theta\in D_\epsilon^+$ and $z\in\Omega\cap\{\mathrm{Im}z>0\}$. Then, the function $\theta\longmapsto(H_{.,\theta}-i)^{-m}(H_{.,\theta}+i)^{-m}(z-H_{.,\theta})^{-1}$ is also analytic on $D_\epsilon^+$.

Now, we treat the difference

$$\begin{aligned}\Big[(z-H_{.,\theta})^{-1}(H_{.,\theta}-i)^{-m}(H_{.,\theta}+i)^{-m}\Big]_0^1 &= A_1B_1C_1-A_0B_0C_0\\ &= A_1B_1(C_1-C_0)+A_1(B_1-B_0)C_0\\ &+ (A_1-A_0)B_0C_0.\end{aligned} \tag{38}$$

Clearly, the terms $A_\cdot := (z - H_{\cdot,\theta})^{-1}$ for Im $z > 0$, $B_\cdot := (H_{\cdot,\theta} - i)^{-m}$ and $C_\cdot := (H_{\cdot,\theta} + i)^{-m}$ are bounded.

For any integer $m > 3$, the term

$$B_1(C_1 - C_0) = \Big(B_1(C_1 - C_0)\langle x\rangle^{\delta}\langle h\nabla_x\rangle^{m}\Big)\Big(\langle h\nabla_x\rangle^{-m}\langle x\rangle^{-\delta}\Big), \tag{39}$$

is analytic for $\theta \in D_\epsilon^+$ with values in the space of trace class operators. This can be proved using functional calculus in the framework of $h$-pseudo-differential operators (see [12]): The first factor $B_1(C_1 - C_0)\langle x\rangle^{\delta}\langle h\nabla_x\rangle^{m}$ is analytic for $\theta \in D_\epsilon^+$, the second factor $\big(\langle h\nabla_x\rangle^{-m}\langle x\rangle^{-\delta}\big)$ is in the space of trace class operators and its trace norm is bounded by $O(h^{-3})$. Then, the left-hand side of equation (39) is in the space of trace class operators and its trace norms is bounded by $O(h^{-3})$. The same argument can be used for the terms $A_1(B_1 - B_0)$ and $(A_1 - A_0)B_0$, then their trace norm are bounded by $O(h^{-3})$.

Since the function $\operatorname{tr}\Big[(H_{\cdot,\theta} - i)^{-m}(H_{\cdot,\theta} + i)^{-m}(z - H_{\cdot,\theta})^{-1}\Big]_0^1$ is analytic with respect to $\theta \in D_\epsilon^+$ and independent of $\theta$ on the real axis, formula (36) follows. □

Repeating the construction of $\widehat{H}_{1,\theta}$, we can construct an operator $\widehat{H}_{0,\theta} : D(H_0) \to \mathcal{H}$ with the properties of $\widehat{H}_{0,\theta}$ such that the difference $K_0 := \widehat{H}_{0,\theta} - H_{0,\theta}$ satisfies the properties of $K_1$ (see Proposition 7).

**Proposition 9.** *There exists a function $a_+(z,h)$ holomorphic in $\Omega$, such that for all $z \in \Omega \cap \{\mathrm{Im}(z) > 0\}$, we have:*

$$\sigma_+(z) = \operatorname{tr}\Big[(H_{\cdot,\theta} - z)^{-1}K_\cdot(\widehat{H}_{\cdot,\theta} - z)^{-1}\Big]_0^1 + a_+(z,h), \tag{40}$$

$$|a_+(z,h)| \le C(\Omega)h^{-3}, \quad z \in \Omega,$$

*with $C(\Omega)$ a constant independent of $h$.*

**Proof.** For $z \in \Omega \cap \{\mathrm{Im} z > 0\}$, we have

$$(H_{\cdot,\theta} - z)^{-1} = (\widehat{H}_{\cdot,\theta} - z)^{-1} + (H_{\cdot,\theta} - z)^{-1}K_\cdot(\widehat{H}_{\cdot,\theta} - z)^{-1}. \tag{41}$$

From the equations (41) and (36), we deduce:

$$\sigma_+(z) = ((z-i)(z+i))^m \operatorname{tr}\Big[\Big((\widehat{H}_{\cdot,\theta} - z)^{-1}(H_{\cdot,\theta} - i)^{-m}(H_{\cdot,\theta} + i)^{-m}\Big)\Big]_0^1$$

$$+ ((z-i)(z+i))^m \operatorname{tr}\Big[\Big((H_{\cdot,\theta} - z)^{-1}K_\cdot(\widehat{H}_{\cdot,\theta} - z)^{-1}(H_{\cdot,\theta} - i)^{-m}(H_{\cdot,\theta} + i)^{-m}\Big)\Big]_0^1$$

$$= A(z) + B(z).$$

Starting with the resolvent equation, we obtain:

$$\begin{aligned}
((z-i)(z+i))^m(H_{\cdot,\theta} - i)^{-m}(H_{\cdot,\theta} + i)^{-m}(H_{\cdot,\theta} - z)^{-1}&\\
= \quad & (H_{\cdot,\theta} - z)^{-1} - \sum_{k=1}^{m}(z+i)^{k-1}(H_{\cdot,\theta} + i)^{-k}\\
- \quad & (z+i)\sum_{k=1}^{m}(z-i)^{k-1}(H_{\cdot,\theta} + i)^{-m}(H_{\cdot,\theta} - i)^{-k}.
\end{aligned}$$

Using the last equation, the cyclicity of the trace and Proposition 7 we obtain

$$B(z) = \operatorname{tr}\Big[K.(\widehat{H}_{\cdot,\theta} - z)^{-1}\Big((H_{\cdot,\theta} - z)^{-1} - \sum_{k=1}^{m}(z+i)^{k-1}(H_{\cdot,\theta}+i)^{-k}$$

$$-(z+i)\sum_{k=1}^{m}(z-i)^{k-1}(H_{\cdot,\theta}+i)^{-m}(H_{\cdot,\theta}-i)^{-k}\Big)\Big]_0^1$$

$$= \operatorname{tr}\Big[(H_{\cdot,\theta} - z)^{-1}K.(\widehat{H}_{\cdot,\theta} - z)^{-1}\Big)\Big]_0^1 + b(z).$$

Since the operator $(\widehat{H}_{\cdot,\theta} - z)^{-1}$ is bounded and holomorphic in $\Omega$ by construction, $b(z)$ is holomorphic and bounded by $O(h^{-3})$.

It remains to show that

$$\begin{aligned} A(z) &= ((z-i)(z+i))^m \operatorname{tr}\Big[(\widehat{H}_{\cdot,\theta} - z)^{-1}(H_{\cdot,\theta}-i)^{-m}(H_{\cdot,\theta}+i)^{-m}\Big]_0^1 \\ &= ((z-i)(z+i))^m \operatorname{tr}(\widehat{A}_1B_1C_1 - \widehat{A}_0B_0C_0), \end{aligned}$$

is holomorphic and bounded by $O(h^{-3})$.

We recall that the terms $\widehat{A}. := (\widehat{H}_{\cdot,\theta} - z)^{-1}$ for $z \in \Omega$, $B. := (H_{\cdot,\theta} - i)^{-m}$ and $C. := (H_{\cdot,\theta} + i)^{-m}$ are bounded. Using the assumption $(\mathbf{A_V})$ with $\delta > 3$, we treat the difference $(\widehat{A}_1B_1C_1 - \widehat{A}_0B_0C_0)$ as (38). The only difference is for the term $(\widehat{A}_1 - \widehat{A}_0)B_0$. We write

$$\begin{aligned} (\widehat{A}_1 - \widehat{A}_0)B_0 &= (\widehat{H}_{1,\theta} - z)^{-1}(\widehat{H}_{0,\theta} - \widehat{H}_{1,\theta})(\widehat{H}_{0,\theta} - z)^{-1}(H_{0,\theta} - i)^{-m}, \\ \text{with} \quad \widehat{H}_{0,\theta} - \widehat{H}_{1,\theta} &= H_{0,\theta} - H_{1,\theta} + K_0 - K_1. \end{aligned}$$

Then, modulo a trace class operator uniformly bounded, with trace norm bounded by $O(h^{-3})$, we have

$$\begin{aligned} (\widehat{A}_1 - \widehat{A}_0)B_0 &= (\widehat{H}_{1,\theta} - z)^{-1} \circ \big((H_{0,\theta} - H_{1,\theta})(H_{1,\theta} - i)^{-m}\big) \\ &\quad \circ \Big((H_{1,\theta} - i)^m(\widehat{H}_{0,\theta} - z)^{-1}(H_{0,\theta} - i)^{-m}\Big). \end{aligned}$$

The second factor $(H_{0,\theta} - H_{1,\theta})(H_{1,\theta} - i)^{-m}$ is trace class and its trace is $O(h^{-3})$, the first and the third factors are bounded. Then, the term $(\widehat{A}_1 - \widehat{A}_0)B_0$ is analytic for $z \in \Omega$ with values in the space of trace class operators and its trace is bounded by $O(h^{-3})$ and so is the difference $(\widehat{A}_1B_1C_1 - \widehat{A}_0B_0C_0)$. $\square$

**Lemma 8.** *For $f \in C_0^\infty(\mathbb{R})$, we have*

$$\langle \xi', f\rangle = \lim_{\varepsilon \to 0} \frac{i}{2\pi} \int f(\lambda)[\sigma_+(\lambda + i\varepsilon) - \sigma_-(\lambda - i\varepsilon)]d\lambda. \tag{42}$$

*This limit is taken in the sense of distributions.*

**Proof.** We follow the proof of [13, Lemma 1]. Let $f \in C_0^\infty(\mathbb{R})$, $\tilde{f}(z) \in C_0^\infty(\mathbb{R}^2)$ be an almost analytic extension of $f$ and

$$g(x) = f(x)(x^2+1)^m.$$

Then

$$g(H.) = -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z^2+1)^m(z - H.)^{-1}L(dz),$$

where $L(dz)$ is the Lebesgue measure on $\mathbb{C}$. Clearly

$$\begin{aligned} f(H_\cdot) &= (H_\cdot - i)^{-m}(H_\cdot + i)^{-m} g(H_\cdot) \\ &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z^2+1)^m (H_\cdot - i)^{-m}(H_\cdot + i)^{-m}(z - H_\cdot)^{-1} L(dz) \end{aligned}$$

which implies:

$$\begin{aligned} \operatorname{tr}\left(f(H_1) - f(H_0)\right) &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z^2+1)^m \\ &\times \ \operatorname{tr}\Big[(H_\cdot - i)^{-m}(H_\cdot + i)^{-m}(z - H_\cdot)^{-1}\Big]_0^1 L(dz). \end{aligned} \tag{43}$$

We have $\sigma_\pm(z) = O(h^{-3}|\operatorname{Im} z|^{-2})$ and the derivative $\bar{\partial}_z \tilde{f} = O(|\operatorname{Im} z|^N)$ for all $N \in \mathbb{N}$ ($f \in C_0^\infty(\mathbb{R})$), so we write the right-hand side of (43) as

$$\langle \xi', f \rangle = \operatorname{tr}\left(f(H_1) - f(H_0)\right)$$

$$= -\frac{1}{\pi}\lim_{\varepsilon \to 0}\left(\int_{\operatorname{Im} z > 0} \bar{\partial}_z \tilde{f}(z)\sigma_+(z + i\varepsilon)L(dz) + \int_{\operatorname{Im} z < 0} \bar{\partial}_z \tilde{f}(z)\sigma_-(z - i\varepsilon)L(dz)\right).$$

According to Proposition 9, the functions $\sigma_+(z + i\varepsilon)$ and $\sigma_-(z - i\varepsilon)$ are holomorphic in $\{z \in \Omega;\ \operatorname{Im} z > 0\}$ and $\{z \in \Omega;\ \operatorname{Im} z < 0\}$ respectively. Applying the Green formula, we obtain the lemma. □

Before the proof of Theorem 2, let us give the following proposition:

**Proposition 10.** (*see* [39], [40]) *Let $F(z,h)$ be a holomorphic function in an open simply connected domain $\Omega$ containing a number $N(h)$ of zeros. We suppose that,*

$$F(z,h) = O(1)e^{O(1)N(h)}, \quad z \in \Omega,$$

*and for all $\rho > 0$ small enough, there exists $C > 0$ such that for all $z \in \Omega_\rho := \Omega \cap \{\operatorname{Im} z > \rho\}$ we have*

$$|F(z,h)| \geq e^{-CN(h)}.$$

*Then for each open simply connected subset $\tilde{\Omega} \Subset \Omega$ there exists $g(.,h)$ holomorphic in $\tilde{\Omega}$ such that*

$$F(z,h) = \prod_{j=1}^{N(h)} (z - z_j)e^{g(z,h)}, \quad \partial_z g(z,h) = O(N(h)), \quad z \in \tilde{\Omega}.$$

**Proof of Theorem 2.** We follow the argument of Sjöstrand ([40]). Let

$$\widehat{K}_\cdot(z) = K_\cdot(z - \widehat{H}_{\cdot,\theta})^{-1}.$$

From formula (30) and Proposition 9, we have, modulo a holomorphic function that is $O(h^{-3})$ in $\Omega$,

$$\sigma_+(z) = -\Big[\partial_z \log\det\Big(1 + \widehat{K}_\cdot(z)\Big)\Big]_0^1, \quad \text{for all } z \in \Omega \cap \{\operatorname{Im}(z) > 0\}.$$

From Subsection 5.2 the resonances are the zeros of the function

$$D(z,h) = \det\Big(I + \widehat{K}_1(z)\Big) = O(1)e^{ch^{-3}}.$$

Since the function $\det(1 + \widehat{K}_0(z))$ has no zeros in $\Omega$ (see (31) and Remark 6) , the term $\partial_z \log\det\Big(1 + \widehat{K}_0(z)\Big)$ is analytic and using Proposition 10, it is bounded by $O(h^{-3})$.

We recall that $\mathrm{Res}(H)$ is the set of resonances of $H$ and let

$$D(z,h) = G(z,h) \prod_{w \in \mathrm{Res}(H)\cap\Omega} (z-w),$$

where, $G(z,h)$ and its inverse are holomorphic functions in $\Omega$. Obviously,

$$\partial_z \log D(z,h) = \partial_z \log G(z,h) + \sum_{w\in \mathrm{Res}(H)\cap\Omega} \frac{1}{z-w}. \tag{44}$$

Using Proposition 10, we have

$$|\partial_z \log G(z,h)| \le C(\tilde{\Omega})h^{-3}, \quad z \in \tilde{\Omega},$$

where $\tilde{\Omega} \subset\subset \Omega$ is an open simply connected set and $C(\tilde{\Omega})$ is independent of $h$.

Now, we treat the non-holomorphic term in $(\sigma_+(\lambda+i\varepsilon) - \sigma_-(\lambda - i\varepsilon))$ when $\varepsilon \to 0$, which is

$$\sum_{w\in \mathrm{Res}(H)\cap\Omega} \left( \frac{1}{\lambda + i\varepsilon - w} - \frac{1}{\lambda - i\varepsilon - \overline{w}} \right), \ \text{for} \lambda \in I.$$

If $\mathrm{Im}(w) \neq 0$, we have

$$\frac{-1}{2i\pi} \lim_{\varepsilon\to 0} \left( \frac{1}{\lambda + i\varepsilon - w} - \frac{1}{\lambda - i\varepsilon - \overline{w}} \right) = \frac{-\mathrm{Im}(w)}{\pi|\lambda - w|^2},$$

while for $w \in \mathbb{R}$ we get

$$\frac{-1}{2i\pi} \lim_{\varepsilon\to 0} \left( \frac{1}{\lambda + i\varepsilon - w} - \frac{1}{\lambda - i\varepsilon - w} \right) = \delta(\lambda - w) = \delta_w(\lambda).$$

The second limit is taken in the sense of distributions.
Lemma 8 and Proposition 9 show that the function $r(z,h) = g(z,h) - \overline{g}(\bar{z},h)$, with $g(z,h) = a_+(z,h) + \partial_z \log G(z,h) + \partial_z \log \det \left(1 + \widehat{K}_0(z)\right)$ a holomorphic function in $\Omega$ and satisfying the following estimate:

$$|g(z,h)| \le C(\Omega)h^{-3}, \ \ z \in W, \tag{45}$$

with $C(\Omega) > 0$ independent of $h$. $\square$

Theorem 2 can be extended to a more general situation:

**Theorem 5.** *Assume that $H_1 = H_0 + V_1$, $H_2 = H_0 + V_2$. The potentials $V_1, V_2$ (resp. $V = V_1 - V_2$) satisfy the assumption $(\mathbf{A_V})$ with $\delta > 0$ (resp. $\delta > 3$). Let $\Omega$ be a complex domain satisfying the assumption $(\mathbf{A}^{\pm}_{\mathbf{\Omega}})$, $W \Subset \Omega$ be an open simply connected and relatively compact set which is symmetric with respect to $\mathbb{R}$. Assume that $I = W \cap \mathbb{R}$ is an interval. Then for all $\lambda \in I$ we have a representation of the derivative of the spectral shift function associated to the operator pairs $(H_2,\ H_1)$ of the form:*

$$\xi'(\lambda,h) = \frac{1}{\pi}\mathrm{Im}\, r(\lambda,h) + \Big[ \sum_{\substack{w\in \mathrm{Res}(H.)\cap\Omega \\ \mathrm{Im}\, w \neq 0}} \frac{-\mathrm{Im}\, w}{\pi|\lambda - w|^2} + \sum_{w \in \mathrm{Res}(H.)\cap I} \delta_w(\lambda) \Big]_2^1, \tag{46}$$

*where* $r(z,h) = g(z,h) - \bar{g}(\bar{z},h)$*,* $g(z,h)$ *is a holomorphic function in* $\Omega$ *which satisfies the following estimate:*

$$|g(z,h)| \leq C(W)h^{-3}, \quad z \in W, \tag{47}$$

*with* $C(W) > 0$ *independent of* $h$*. Here* $\delta_w(\cdot)$ *is the Dirac mass at* $w \in \mathbb{R}$*.*

**Proof.** We denote $H_{2,\theta} = U_\theta H_2 U_\theta^{-1}$ ($U_\theta$ defined in Section 3). As in Subsection 5.1, one constructs $\widehat{H}_{2,\theta} : D(H) \longmapsto \mathcal{H}$ with the following properties:
$K_2 := \widehat{H}_{2,\theta} - H_{2,\theta}$ is of finite rank $O(h^{-3})$, has compact support in the sense that $K_2 = \chi_2 K_2 \chi_2$ if $\chi_2 \in C_0^\infty$ is equal to 1 on $B(0,R)$ for some sufficiently large $R$, and

$$(\widehat{H}_{2,\theta} - z)^{-1} = O(1) : \mathcal{H} \longmapsto D(H), \text{ uniformly for } z \in \overline{\Omega}.$$

We repeat the proof of Theorem 2 replacing $K_0$ by $K_2$ and $\widehat{K}_0(z)$ by $\widehat{K}_2(z) = K_2(z - \widehat{H}_{2,\theta})^{-1}$. Consequently $\partial_z \log\det\Big(1 + \widehat{K}_0(z)\Big)$ is replaced by $\partial_z \log\det\Big(1 + \widehat{K}_2(z)\Big)$ which is a non-holomorphic function. We treat this term as the term $\partial_z \log\det\Big(1 + \widehat{K}_1(z)\Big)$ in the proof of Theorem 2. □

**Remark 8.** *Equation* (6) *shows that the spectral shift function* $\xi(\lambda,h)$ *satisfies*

$$\begin{aligned} \xi(\lambda,h) - \xi(\lambda_0,h) \quad &= \sum_{\substack{w \in \mathrm{Res}(H_1)\cap\Omega \\ \mathrm{Im}\, w \neq 0}} \frac{1}{\pi}\int_{\lambda_0}^{\lambda} \frac{-\mathrm{Im}\, w}{|\mu - w|^2} d\mu + \frac{1}{\pi}\int_{\lambda_0}^{\lambda} \mathrm{Im}\, r(\mu,h) d\mu \\ &+ \quad \#\{\mu \in [\lambda_0,\lambda];\ \mu \in \sigma_d(H_1)\}. \end{aligned} \tag{48}$$

*In particular, for* $\lambda \in I \backslash \sigma_d(H_1)$ *the distribution* $\xi(\lambda,h)$ *is continuous, and the function*

$$\eta(\lambda,h) - \eta(\lambda_0,h) = \xi(\lambda,h) - \xi(\lambda_0,h) - \#\{\mu \in [\lambda_0,\lambda];\ \mu \in \sigma_d(H_1)\}$$

*is real analytic in* $I$*.* □

Repeating the argument used in the proof of [8, Theorem 4], the following theorem is a direct consequence of Theorem 2.

**Theorem 6. (Local trace formula)** *Let* $\Omega$ *be an open, complex, simply connected and relatively compact set satisfying assumption* $(\mathbf{A}_{\mathbf{\Omega}}^{\pm})$ *such that* $I = \Omega \cap \mathbb{R}$ *is an interval.*
*We suppose that* $f$ *is a holomorphic function in* $\overline{\Omega}$ *and* $\psi \in C_0^\infty(\mathbb{R})$ *satisfies*

$$\psi(\lambda) = \begin{cases} 0, & d(I,\lambda) > 2\varepsilon, \\ 1, & d(I,\lambda) < \varepsilon, \end{cases}$$

*where* $\varepsilon > 0$ *and sufficiently small. Then*

$$tr[(\psi f)(H.)]_0^1 = \sum_{z \in \mathrm{Res}(H_1)\cap\Omega} f(z) + E_{\Omega,f,\psi}(h), \ \textit{with}$$

$$|E_{\Omega,f,\psi}(h)| \leq M(\psi,\Omega) \sup\{|f(z)|;\ 0 \leq d(\partial\Omega,z) \leq 2\varepsilon,\ \mathrm{Im}\,(z) \leq 0\} h^{-3}.$$

## 7. Weyl Asymptotics

In this section we obtain a Weyl-type asymptotics for the spectral shift function $\xi(\lambda,h)$ associated to the operators $H_0$ and $H_1 = H_0 + V$. Here we assume that $V$ is an electro-magnetic potential (8),

$$H_1 = -\sum_{j=1}^{3} \alpha_j(ich\partial_j + eA_j) + mc^2\beta + ev.$$

In the following, we fix $I_0 \subset \mathbb{R}\backslash\{\pm mc^2\}$ and choose $W_0$ an open simply connected, relatively compact subset of $\Omega$ satisfying assumption $(\mathbf{A}_{\mathbf{\Omega}}^{+})$ such that $I_0 = W_0 \cap \mathbb{R}$.

For the $h$-pseudo-differential and functional calculus for the Dirac operator, we refer to ([12],[10],[35],[20]). We recall that $H_\nu = Op_h^{\omega}(\mathcal{D}_\nu)$ and $\varphi(H_\nu)$ are $h$-pseudo-differential operators for a smooth function $\varphi$. The semi-classical symbol $\mathcal{D}_\nu$ is defined in (10).

Let us introduce the intervals $I_1,\ I_2 \subset I_0$ neighborhoods of $\lambda_1,\ \lambda$ respectively such that, each $\lambda \in I_1 \cup I_2$ is a noncritical energy level for $H$ (see Definition 2). Let $\varphi_j \in C_0^\infty(\mathbb{R},\mathbb{R}^+)$ be such that

$$\varphi_1 = 1 \text{ on } I_1, \quad \varphi_2 = 1 \text{ on } I_2 \text{ and } \quad \varphi_1+\varphi_2+\varphi_3 = 1 \text{ on } I_0. \tag{49}$$

Consider a function $\theta(t) \in C_0^\infty(]-\delta_1,\delta_1[)$, $\theta(0) = 1$, $\theta(-t) = \theta(t)$, so that the Fourier transform $\hat{\theta}$ of $\theta$ satisfies $\hat{\theta}(\lambda) \geq 0$ on $\mathbb{R}$, and assume that there exist $0 < \epsilon_0 < 1,\ \delta_0 > 0$, such that $\hat{\theta}(\lambda) \geq \delta_0 > 0$ for $|\lambda| \leq \epsilon_0$. Next, we introduce

$$(\mathcal{F}_h^{-1}\theta)(\lambda) = (2\pi h)^{-1}\int e^{it\lambda h^{-1}}\theta(t)dt = (2\pi h)^{-1}\hat{\theta}(-h^{-1}\lambda).$$

To prove Theorem 3, we need the proposition:

**Proposition 11.** *For the trace involving $H_\nu$, $\nu = 0,1$, we have for $\lambda \in I_j$,*

$$\operatorname{tr}\left(\Big[(\mathcal{F}_h^{-1}\theta)(\lambda - H_.)\varphi_j(H_.)\Big]_0^1\right) = w_j(\lambda)h^{-3} + O(h^{-2}), \quad j = 1,2, \tag{50}$$

*with $w_j(\lambda) \in C_0^\infty(I_j)$ and $O(h^{-2})$ uniform with respect to $\lambda \in I_j$.*

**Proof.** Proposition 11 is close to the calculation of the trace in [5, Section 4] and to the appendix of [8] for the Schrödinger operator. But, here we use a trick of Robert [10]. We fix $j = 2$ (it is similar for $j = 1$). The proof of (50) is obtained following these two steps:

- First, we recall that $\lambda \in I_2$ and $\operatorname{Supp}\theta(t) \subset [-\delta_1,\delta_1]$. Let us write

$$\begin{aligned}\mathcal{T} = \operatorname{tr}\Big[(\mathcal{F}_h^{-1}\theta)(\lambda - H_.)\varphi_2(H_.)\Big]_0^1 &= \operatorname{tr}\Big[\int \frac{\theta(t)}{2\pi h} e^{it(\lambda - H_.)h^{-1}}\varphi_2(H_.)dt\Big]_0^1 \\ &= \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\theta(t)\operatorname{tr}[e^{-itH_.h^{-1}}\varphi_2(H_.)]_0^1 dt\end{aligned}$$

In the order to calculate the trace

$$\operatorname{tr}(f(H_1) - f(H_0)), \quad \text{for all } f \in C_0^\infty(\mathbb{R}\backslash\{\pm mc^2\})$$

we use [10, Proposition 3.2]. If we note $W(h) = Q - \frac{1}{2}i[Q,\mathcal{A}(h)]$ with $Q = H_1^2 - H_0^2$, $\mathcal{A}(h) = \frac{1}{2}(x\cdot h\partial_x + h\partial_x \cdot x)$ and $[Q,\mathcal{A}(h)] = Q\mathcal{A}(h) - \mathcal{A}(h)Q$, we have

$$\operatorname{tr}(f(H_1) - f(H_0)) = \operatorname{tr}(W(h)(H_1^2 - m^2c^4)^{-1}f(H_1)). \tag{51}$$

Applying formula (51) for $f(\lambda) = e^{-it\lambda h^{-1}}\varphi_2(\lambda)$, we have

$$\mathcal{T} = \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\theta(t)\mathrm{tr}\left(W(h)(H_1^2 - m^2c^4)^{-1}e^{-itH_1h^{-1}}\varphi_2(H_1)\right)dt.$$

**Remark 9.** *Of course $(H_1^2-m^2c^4)^{-1}$ is not well defined, however for $f \in C_0^\infty(\mathbb{R}\backslash\{\pm mc^2\})$, we can define $(H_1^2-m^2c^4)^{-1}f(H_1)$ as the self-adjoint operator $\varphi(H_1)$ where $\varphi \in C_0^\infty(\mathbb{R})$ satisfies:*

$$\varphi(\lambda) = \begin{cases} (\lambda^2 - m^2c^4)^{-1}f(\lambda) & \text{for } \lambda \neq \pm mc^2, \\ 0 & \text{for } \lambda = \pm mc^2. \end{cases}$$

• Now, we treat $\mathcal{T}$ following the analysis of [5, Section 4.2]. By the $h$-pseudo-differential calculus, we obtain the existence of a $h$-pseudo-differential operator $S$ which is trace class with symbol

$$s(x, y, \xi, h) \in \mathcal{S}^0(\langle x\rangle^{-\delta}\langle\xi\rangle^{-N}), \quad \forall N \in \mathbb{N},\ \delta > 3, \tag{52}$$

having compact support in $\xi$ and in $(x - y)$ (i.e. $\mathrm{supp}_{(x-y)}(s) = \{x - y,\ \exists\ \xi\ ;\ (x, y, \xi, h) \in \mathrm{supp}(s)\}$ is compact) and support in $\{(x, \xi);\ |x| > R,\ (x, \xi) \in \mathcal{D}_1^{-1}(I_2)\}$, with $\mathcal{D}_1$ the semi-classical symbol of $H_1$, so that

$$\mathcal{T} = \frac{1}{2\pi h}\mathrm{tr}\left(\int e^{it\lambda h^{-1}}\theta(t)e^{-itH_1h^{-1}}Sdt\right) + O(h^\infty).$$

Using Theorem 9 in Appendix A and the hypothesis on $S$ by composition of Fourier integral operators, we obtain a Fourier integral operator $\widetilde{\mathcal{U}}_t = \widetilde{\mathcal{U}}_t^+ + \widetilde{\mathcal{U}}_t^-$, such that for $|t| \leq \delta_1$ and $\delta_1$ sufficiently small, we have

$$\|\widetilde{\mathcal{U}}_t - e^{-itH_1h^{-1}}S\|_{tr} = O(h^\infty), \tag{53}$$

where the kernel of the operator $\int e^{it\lambda h^{-1}}\theta(t)\widetilde{\mathcal{U}}_t dt$ is equal to $\widetilde{K}^+(x, y; h) + \widetilde{K}^-(x, y; h)$ with

$$\widetilde{K}^\pm(x, y; h) = \frac{1}{(2\pi h)^3}\iint e^{i(t\lambda+\Phi^\pm(t,x,\xi)-y\cdot\xi)h^{-1}}\theta(t)\widetilde{E}^\pm(t, x, y, \xi; h)dtd\xi.$$

The amplitudes $\widetilde{E}^\pm$, satisfy

$$\widetilde{E}^\pm(t, x, y, \xi; h) \in \mathcal{S}^0(\langle x\rangle^{-\delta}\langle\xi\rangle^{-N}), \quad \forall N \in \mathbb{N}$$

and are compactly supported in $\xi$ and in $(x - y)$.
Using the Taylor formula for the functions $\Phi^\pm(t, x, \xi)$ in a neighborhood of $t = 0$, we have:

$$\Phi^\pm(t, x, \xi) = x \cdot \xi - tH_1^\pm(x, \xi) + O(t^2).$$

We will deduce that $\mathcal{T} = \mathcal{T}^+ + \mathcal{T}^-$, with

$$\mathcal{T}^\pm = \frac{1}{(2\pi h)^4}\iiint e^{i(t\lambda+\Phi^\pm(t,x,\xi)-x\cdot\xi)h^{-1}}\theta(t)\widetilde{E}^\pm(t, x, x, \xi; h)dtdxd\xi + O(h^\infty).$$

Moreover, the symbol $\widetilde{E}^\pm(t, x, x, \xi; h)$ has support in $\{(x, \xi);\ |x| > R,\ |\xi| \leq C_1,\ (x, \xi) \in \mathcal{D}_1^{-1}(I_2)\}$, so that for all $\alpha$ and $|t| \leq \delta_1$, we have

$$|\partial^\alpha\widetilde{E}^\pm(t, x, x, \xi; h)| \leq C_\alpha\langle x\rangle^{-\delta}, \quad \delta > 3. \tag{54}$$

The last estimate enables us to calculate $\mathcal{T}$ by using an infinite partition of unity

$$\sum_{\alpha\in\mathbb{N}^3}\Psi(x - \alpha) = 1, \quad \forall x \in \mathbb{R}^3,$$

where $\Psi \in C_0^\infty(K), \Psi \geq 0$, $K$ being a neighborhood of the unit cube. Consequently, for every fixed $h \in ]0, h_0]$, we have

$$\begin{aligned}\mathcal{T}^\pm &= \frac{1}{(2\pi h)^4} \lim_{m\to\infty} \iiint e^{i(t\lambda+\Phi^\pm(t,x,\xi)-x\cdot\xi)h^{-1}} \theta(t) \\ &\times \sum_{|\alpha|\leq m} \Psi(x-\alpha) \widetilde{E}^\pm(t,x,x,\xi;h) dt dx d\xi + O(h^\infty) = \lim_{m\to\infty} I_m^\pm + O(h^\infty),\end{aligned}$$

and we reduce the problem to the analysis of the integrals $I_m^\pm$. Concerning the phase function, we observe that

$$t\lambda + \Phi^\pm(t,x,\xi) - x\cdot\xi = t(\lambda - H_1^\pm(x,\xi) + O(t)), \tag{55}$$

where $O(t)$ and its derivatives are uniformly bounded on the support of $\theta(t)\widetilde{E}^\pm(t,x,x,\xi;h)$ since the derivatives of $(\Phi^\pm(t,x,\xi) - x\cdot\xi)$ are bounded on this set.

Now we look for critical points of the phase function $(t\lambda + \Phi^\pm(t,x,\xi) - x\cdot\xi)$. Putting the derivative with respect to $t$ equal to 0, we see that $H_1^\pm(x,\xi) = \lambda + O(t)$. Since $\partial_{x,\xi} H_1^\pm(x,\xi) \neq 0$, when $H_1^\pm(x,\xi) = \lambda$, and putting the derivative of the phase function $t(\lambda - H_1^\pm(x,\xi) + O(t))$ with respect to $H_1^\pm(x,\xi)$ equal to 0, we have

$$t = O(t^2).$$

Then the phase is critical for $|t|$ small precisely when $t = 0, \ \ \lambda = H_1^\pm$. Near any such critical point we choose local coordinates $t$, $H_1^\pm(x,\xi)$, $w_1, \cdots, w_5$ and consider the Hessian of (55) with respect to $t$, $H_1^\pm(x,\xi)$ at the critical point:

$$\begin{pmatrix} \star & -1 \\ -1 & 0 \end{pmatrix}$$

This is a non-degenerate matrix of determinant $-1$ and of signature 0. By the stationary phase method we obtain

$$I_m^\pm = \frac{\psi^\pm(\lambda)}{(2\pi h)^3} \int_{\lambda = H_1^\pm} \sum_{|\alpha|\leq m} \Psi(x-\alpha) \widetilde{E}^\pm(0,x,\xi,\lambda;h) L_\lambda^\pm(dw) + O(h^2),$$

where $L_\lambda^\pm(dw)$ is the Liouville measure on $\lambda = H_1^\pm$ and the remainder $O(h^{-2})$ is uniform with respect to $\lambda \in I_2$ and $m \in \mathbb{N}$. Here $\psi^\pm(\lambda) \in C_0^\infty(I_2)$. Taking the limit $\lim_{m\to\infty} I_m^\pm$, we obtain an asymptotics of $\mathcal{T}$. $\square$

**Lemma 9.** *With the above definitions of* $\theta(t)$, $\xi(\lambda,h)$, $\varphi_j(\lambda)$, $I_j$, $\ j = 1,2$, *we have*

$$\int_{-\infty}^{\lambda} \mathcal{F}_h^{-1}\theta * \varphi_j \xi'(\mu,h) d\mu - \int_{-\infty}^{\lambda} \varphi_j(\mu)\xi'(\mu) d\mu = O(h^{-2}), \ \lambda \in I_j. \tag{56}$$

**Proof.** We deal only with the analysis of (56) for $j = 2$ since that for $j = 1$ is similar. According to Theorem 2, there exists a holomorphic function $r(z,h)$ in $\Omega$ such that for all $\lambda \in I_0 = W_0 \cap \mathbb{R}$, we have

$$\xi'(\lambda,h) = \frac{1}{\pi} \mathrm{Im}\, r(\lambda,h) + \sum_{\substack{w\in \mathrm{Res}(H_1)\cap\Omega \\ \mathrm{Im} w \neq 0}} \frac{-\mathrm{Im} w}{\pi|\lambda - w|^2} + \sum_{w\in\mathrm{Res}(H_1)\cap I_0} \delta_w(\lambda),$$

where $r(z,h)$ satisfies the following estimate:

$$|r(z,h)| \leq C(W)h^{-3}, \quad z \in W, \tag{57}$$

with $C(W) > 0$ independent of $h$. Let us denote

$$G_{\varphi_2}(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\lambda} \mathrm{Im}\ r(\mu,h)\varphi_2(\mu)d\mu,$$

$$M_{\varphi_2}(\lambda) = \sum_{\substack{w\in \mathrm{Res}(H_1)\cap\Omega \\ \mathrm{Im} w\neq 0}} \int_{-\infty}^{\lambda} \frac{-\mathrm{Im}\ w}{\pi|\lambda - w|^2}\varphi_2(\mu)d\mu + \sum_{w\in \mathrm{Res}(H_1)\cap ]c_0,\lambda]} \varphi_2(w). \tag{58}$$

Using the Cauchy inequality and (57), it follows easily that

$$G'_{\varphi_2}(\lambda) = O(h^{-3}) \text{ and } G''_{\varphi_2}(\lambda) = O(h^{-3}),$$

and we immediately obtain

$$\mathcal{F}_h^{-1}\theta * G'_{\varphi_2} - G'_{\varphi_2} = O(h^{-2}). \tag{59}$$

Now, we want to apply a Tauberian theorem (see [35, Theorem V-13 ]) for the increasing function $M_{\varphi_2}(\lambda)$. For this purpose, we need the estimates

$$M_{\varphi_2}(\lambda) = O(h^{-3}), \quad \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_{\varphi_2})(\lambda) = O(h^{-3}), \quad \forall\lambda \in \mathbb{R}, \tag{60}$$

and the equality $M_{\varphi_2}(\mu) = G_{\varphi_2}(\mu) = 0, \quad \mu \leq inf I_2$.
The first estimate in (60) follows easily from equation (58) with the upper bound of the number of the resonances in $\Omega$ (see Theorem 1), and the second follows from (50) and the equation

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_{\varphi_2})(\lambda) = \mathcal{F}_h^{-1}\theta * \varphi_2\xi'(\lambda) - \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * G_{\varphi_2})(\lambda).$$

Then, according to the Tauberian theorem we have

$$(\mathcal{F}_h^{-1}\theta * M_{\varphi_2})(\lambda) = M_{\varphi_2}(\lambda) + O(h^{-2}),$$

this enables us to obtain

$$\begin{aligned}
\int_{-\infty}^{\lambda} \varphi_2(\mu)\xi'(\mu)d\mu &= M_{\varphi_2}(\lambda) + \int_{-\infty}^{\lambda} G'_{\varphi_2}(\mu)d\mu \\
&= \int_{-\infty}^{\lambda} \frac{d}{d\mu}(\mathcal{F}_h^{-1}\theta * M_{\varphi_2} + \mathcal{F}_h^{-1}\theta * G_{\varphi_2})(\mu)d\mu + O(h^{-2}) \\
&= \int_{-\infty}^{\lambda} \mathcal{F}_h^{-1}\theta * \varphi_2\xi'(\mu,h)d\mu + O(h^{-2}).
\end{aligned}$$

□

**Proof of Theorem 3.** For $\lambda_1 \in I_1,\ \lambda \in I_2$, using the functions defined in (49), we have

$$\begin{aligned}\xi(\lambda,h)-\xi(\lambda_1,h) &= \int_{-\infty}^{\lambda}\varphi_1(\mu)\xi'(\mu,h)d\mu-\int_{-\infty}^{\lambda_1}\varphi_2(\mu)\xi'(\mu,h)d\mu\\ &- \int_{-\infty}^{\lambda_1}\varphi_1(\mu)\xi'(\mu,h)d\mu+\int_{-\infty}^{\lambda}\varphi_2(\mu)\xi'(\mu,h)d\mu\\ &+ \int_{\lambda_1}^{\lambda}\varphi_3(\mu)\xi'(\mu,h)d\mu. \end{aligned} \tag{61}$$

Since $\varphi_j = 0$ on $I_{3-j}$ for $j=1,2$, the first term (resp. the second term) is independent of $\lambda \in I_2$ (resp. $\lambda_1 \in I_1$) and is equal to $\operatorname{tr}[\varphi_1(H_.)]_0^1 = C(\varphi_1)h^{-3}+O(h^{-2})$ (resp. $\operatorname{tr}[\varphi_2(H_.)]_0^1 = C(\varphi_2)h^{-3}+O(h^{-2})$), where $C(\varphi_j)$ is a constant depending on $\varphi_j$ for $j=1,2$. Since $\varphi_3 = 0$ on $I_j$, $j=1,2$, the last term is independent of $\lambda\in I_2$, $\lambda_1 \in I_1$ and is equal to $C(\varphi_3)h^{-3}+O(h^{-2})$, where $C(\varphi_3)$ is a constant depending on $\varphi_3$. The proof of these results is based on the functional calculus in the framework of $h$-pseudo-differential operators.

Using the equations (50), (56) and (61) we complete the proof of asymptotic expansion (13) by writing

$$\begin{aligned}(\mathcal{F}_h^{-1}\theta*(\varphi_j\xi'))(\lambda) &= \langle(\mathcal{F}_h^{-1}\theta)(\lambda-\cdot)\varphi_j(\cdot),\xi'\rangle\\ &= \operatorname{tr}\left(\left[(\mathcal{F}_h^{-1}\theta)(\lambda-H_.)\varphi_j(H_.)\right]_0^1\right)\\ &= w_j(\lambda)h^{-3}+O(h^{-2}),\ \ j=1,2.\end{aligned}\tag{62}$$

It remains to compute the Weyl term (14).

According to the definition of the spectral shift function $\xi(\lambda,h)$ in (5), we have:

$$\langle\xi'(\lambda,h),\varphi(\lambda)\rangle = \operatorname{tr}(\varphi(H_1)-\varphi(H_0)),\ \ \varphi(\lambda)\in C_0^\infty(\mathbb{R}). \tag{63}$$

We use weak asymptotics which is a direct consequence of the functional calculus in the framework of $h$-pseudo-differential operators, as established in [12], [35], [10]. We find

$$H_\nu = Op_h^\omega(\mathcal{D}_\nu),\quad \nu=0,1,\ (\mathcal{D}_\nu \text{ defined in (10)}),$$

and

$$\begin{aligned}\operatorname{tr}(\varphi(H_1)-\varphi(H_0)) &= h^{-3}\sum_{j\geq 0}\gamma_j(\varphi)h^j\\ &= h^{-3}\gamma_0(\varphi)+O(h^{-2}),\end{aligned}$$

with $\gamma_0(\varphi) = (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\operatorname{tr}(\varphi(\mathcal{D}_1(x,\xi))-\varphi(\mathcal{D}_0(x,\xi)))\,dxd\xi$.
($\operatorname{tr}(A)$ is the trace of the matrix $A$).

The matrix $\mathcal{D}_\nu(x,\xi)$ is Hermitian and has two eigenvalues $H_\nu^{\pm}(x,\xi)$ (see (11)), whence

$$\operatorname{tr}(\varphi(\mathcal{D}_1)-\varphi(\mathcal{D}_0)) = 2\left(\varphi(H_1^+)+\varphi(H_1^-)-\varphi(H_0^+)-\varphi(H_0^-)\right).$$

According to the asymptotic expansions (13) and (63) we obtain

$$w(\lambda,\lambda_1) = w(\lambda)-w(\lambda_1),$$

with

$$w(\lambda)=\frac{1}{4\pi^3}\int_{\mathbb{R}^3}\left(\int_{H_1^+(x,\xi)\le\lambda}d\xi-\int_{H_0^+(x,\xi)\le\lambda}d\xi-\int_{H_1^-(x,\xi)\ge\lambda}d\xi+\int_{H_0^-(x,\xi)\ge\lambda}d\xi\right)dx.$$

Putting $\zeta_\nu=c\xi-\nu eA(x)$ for $\nu=0,1$ and $\zeta=r\omega$ ($\omega\in S^2$), we get

$$\begin{aligned}\pm H_\nu^\pm\le\pm\lambda\quad&\Leftrightarrow\quad\left(\zeta_\nu^2+(mc^2+\nu\frac{e(v_+-v_-)}{2})^2\right)^{\frac12}\pm\nu\frac{e(v_++v_-)}{2}\le\pm\lambda\\&\Leftrightarrow\quad\left(\zeta_\nu^2+(mc^2+\nu\frac{e(v_+-v_-)}{2})^2\right)^{\frac12}\le\pm\left(\lambda-\nu\frac{e(v_++v_-)}{2}\right),\end{aligned}$$

thus

$$\begin{aligned}\int_{H_1^+(x,\xi)\le\lambda}d\xi-\int_{H_1^-(x,\xi)\ge\lambda}d\xi=&\frac{4\pi}{3}\left((\lambda-\frac{e(v_++v_-)}{2})_+^2-(mc^2+\frac{e(v_+-v_-)}{2})^2\right)_+^{\frac32}\\&-\frac{4\pi}{3}\left((\lambda-\frac{e(v_++v_-)}{2})_-^2-(mc^2+\frac{e(v_+-v_-)}{2})^2\right)_+^{\frac32},\end{aligned}$$

and

$$-\int_{H_0^+(x,\xi)\le\lambda}d\xi+\int_{H_0^-(x,\xi)\ge\lambda}d\xi=\mp\frac{4\pi}{3}\left(\lambda^2-(mc^2)^2\right)_+^{\frac32},\quad\text{for }\pm\lambda\ge0,$$

with $(x)_+=max(x,0)$ and $(x)_-=max(-x,0)$ for $x\in\mathbb{R}$. □

**Remark 10.** *Theorem 3 can be extended to the operator pairs* $(H_1=H_0+V_1,\ H_2=H_0+V_2)$, *where the potentials* $V_1,V_2$ *are electro-magnetic potentials*

$$V_\cdot(x)=e(-\alpha\cdot A^\cdot+v^\cdot)(x)=-\sum_{j=1}^{3}\alpha_j\cdot eA_j^\cdot(x)+e\begin{pmatrix}v_+^\cdot(x)I_2&0\\0&v_-^\cdot(x)I_2\end{pmatrix}$$

*satisfying assumption* $(\mathbf{A_V})$ *with* $\delta>0$ *(or* $\|V_j(x)\|\longrightarrow0$*) and the potential* $V=V_2-V_1$ *satisfies assumption* $(\mathbf{A_V})$ *with* $\delta>3$*:*
*For all* $\lambda,\ \lambda_1$ *noncritical energy levels for* $H_1,\ H_2$ *such that* $\pm mc^2\notin]\lambda_1,\lambda[$ *and* $h\in]0,h_0[$, *we have the asymptotic expansion*

$$\xi(\lambda,h)-\xi(\lambda_1,h)=w(\lambda,\lambda_1)h^{-3}+O(h^{-2}).\tag{64}$$

*Here the* $O(h^{-2})$ *is uniform for* $\lambda$ *(resp.*$\lambda_1$*) in a small interval* $I_2$ *(resp.* $I_1$*). The first term* $w(\lambda,\lambda_1)\ \in C^\infty(I_2\times I_1)$ *is given by*

$$w(\lambda,\lambda_1)=w(\lambda)-w(\lambda_1)$$

*with,*

$$w(\lambda)=\frac{1}{3\pi^2}\int_{\mathbb{R}^3}\left[W_+(\lambda,v_+^\cdot,v_-^\cdot)-W_+(\lambda,v_+^\cdot,v_-^\cdot)\right]_1^2dx\tag{65}$$

*where* $W_\pm(\lambda,a,b)=\left(\left(\lambda-\frac{e(a+b)}{2}\right)_\pm^2-\left(mc^2+\frac{e(a-b)}{2}\right)^2\right)_+^{\frac32}.$

*In this setting, we do not have a formula like (51). But it could be possible to use the approach due to Bruneau-Petkov in* [8]. *For this we need more information on the approximation of the propagator* $e^{-itH_jh^{-1}}$ *by the Fourier integral operator* $\mathcal{U}_t$.

## 8. Resonances in small domains

In this section, we assume that the Hamiltonian is $H = H_0 + V$, where $V$ is an electro-magnetic potential:

$$H = -\sum_{j=1}^{3} \alpha_j (ich\partial_j + eA_j) + mc^2\beta + ev.$$

8.1. **Upper bound for the number of resonances in domains of width $h$.** We adapt, for the Dirac operator, Theorem 1 of [9] which is based on a representation formula for the spectral shift function (see Theorem 2).

**Theorem 7.** *Suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for $H$. Then for $h \in ]0, h_0]$, the following assertions are equivalent:*

*(i) There exist positive constants $B$, $C$, $b$, $h_0$, such that for any $\lambda \in [E_0 - b, E_1 + b]$, $h \in ]0, h_0]$ and $h/B \leq \rho \leq B$, we have*

$$\#\{z \in \mathbb{C} : \ z \in \mathrm{Res}(H), |z - \lambda| \leq \rho\} \leq C\rho h^{-3}.$$

*(ii) There exist positive constants $B_1, C_1, \varepsilon_1, h_1$, such that for any $\lambda \in [E_0 - \varepsilon_1, E_1 + \varepsilon_1]$, $h \in ]0, h_1]$ and $h/B_1 \leq \rho \leq B_1$, we have*

$$|\xi(\lambda + \rho, h) - \xi(\lambda - \rho, h)| \leq C\rho h^{-3}.$$

As a consequence of Theorem 7, we have an upper bound $O(h^{-2})$ for the number of resonances for the semi-classical Dirac operator close to a non-critical energy level in a domain of width $h$:

**Proposition 12.** *Assume that $V$ is the electro-magnetic potential* (8) *satisfying the assumption* $(\mathbf{A_V})$ *with $\delta > 3$. We suppose also that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for $H$. There are positive constants $C$, $B$, $b$, $h_0$ such that for any $\lambda \in [E_0 - b, E_1 + b]$, $h \in ]0, h_0]$ and $h/B \leq \rho \leq B$, we have*

$$\#\{z \in \mathbb{C} : \ z \in \mathrm{Res}(H), |z - \lambda| \leq \rho\} \leq C\rho h^{-3}.$$

**Proof.** It follows from Theorem 7 and equation (13). □

8.2. **Breit-Wigner approximation.** In this part, we consider small domains of width $h$, and we prove a Breit-Wigner approximation for $\xi(\lambda, h)$ (see [32], [33], [16], [6], [8]). Let $\eta(\lambda, h)$ be the real analytic function defined by

$$\eta(\lambda, h) = \xi(\lambda, h) - \#\{\mu \in [E_0, \lambda] : \ \mu \in \sigma_d(H)\}.$$

Using Proposition 12 and the arguments used in [8, Section 6], we obtain a Breit-Wigner approximation for the derivative of the spectral shift function $\xi(\lambda, h)$.

**Theorem 8. (Breit-Wigner)** *Assume that $V$ is an electro-magnetic potential (*8*), for any $\lambda \in [E_0, E_1]$ a non-critical energy level for $H$, $0 < \rho < h/B$, $0 < B_1 < B$, and $h$ sufficiently small, we have*

$$\eta(\lambda + \rho, h) - \eta(\lambda - \rho, h) = \sum_{\substack{w \in \mathrm{Res}(H) \\ \mathrm{Im}\, w \neq 0, \ |w - \lambda| < h/B_1}} \omega_{\mathbb{C}_-}(w, [\lambda - \rho, \lambda + \rho]) + O(\rho) h^{-3},$$

*where $B > 0$ is a positive constant and $\omega_{\mathbb{C}_-}$ is the harmonic measure*

$$\omega_{\mathbb{C}_-}(w, E) = -\frac{1}{\pi} \int_E \frac{\mathrm{Im}(w)}{|t - w|^2} dt, \quad E \subset \mathbb{R} = \partial \mathbb{C}_-.$$

Using Theorem 7 and repeating with little modifications the arguments used in [7, Section 6], we obtain the following corollary which entails also a trace formula in small domains.

**Corollary 1.** *Under the assumptions of Theorem 8 and supposing that $[E_0, E_1]$ contains only non-critical energy levels for $H$, for each $E \in [E_0, E_1]$ there exist constants $C_2 > C_1 > 0$, $h_0 > 0$ so that for $|\lambda - E| \leq C_1 h,\ h \in ]0, h_0]$, we have*

$$\xi'(\lambda, h) = -\frac{1}{\pi} \sum_{\substack{w \in \mathrm{Res}(H) \\ |E - w| \leq C_2 h}} \frac{\mathrm{Im}(w)}{|\lambda - w|^2} + \sum_{\substack{w \in \sigma_d(H) \\ |E - w| \leq C_1 h}} \delta_w(\lambda) + O(h^{-3}). \tag{66}$$

*Here $\delta_w(\cdot)$ is the Dirac mass at $w \in \mathbb{R}$.*

## Appendix A. Construction of $\mathcal{U}_t$

In this appendix, we construct a parametrix at small time of the propagator for the Dirac equation in an external electro-magnetic field

$$ih\partial_t \psi = H_1 \psi,$$

with $H_1 = H_0 + V$. Here $H_0$ is the selfadjoint operator defined in (1) and $V$ is an electro-magnetic potential (8).

**Theorem 9. (Approximation of the propagator)** *There exist $\delta_1 > 0$ small enough and a Fourier integral operator $\mathcal{U}_t = \mathcal{U}_t^+ + \mathcal{U}_t^-$ with*

$$\mathcal{U}_t^{\pm} f(y) = \frac{1}{(2\pi h)^3} \iint e^{i(\Phi^{\pm}(t,x,\xi) - y \cdot \xi)h^{-1}} E^{\pm}(t, x, y, \xi; h) f(y) d\xi dy,$$

*defined for $|t| < \delta_1$ such that:*

- *The amplitudes $E^{\pm}(t, x, y, \xi; h) \in \mathcal{S}^0(1)$.*
- *$\|\mathcal{U}_t - e^{-itH_1 h^{-1}}\| = O(h^\infty)$, uniformly for $|t| < \delta_1$.*
- *The phase function $\Phi^{\pm}(t, x, \xi) - x \cdot \xi$ and its derivatives $\partial_t^\alpha \partial_x^\beta \partial_\xi^\gamma (\Phi^{\pm}(t, x, \xi) - x \cdot \xi)$ are uniformly bounded for $(t, x, \xi) \in [-\delta_1, \delta_1] \times \mathbb{R}^3 \times B(0, C_1)$, $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ and $C_1 > 0$ (see(71)).*

With a different approach, a similar result has been obtained by Yajima [47] for a scalar electric potential ($v_+ = v_-$).

**Proof.** We consider the equivalent problem for $\mathcal{U}_t$

$$\begin{cases} ih\partial_t \mathcal{U}_t - H_1 \mathcal{U}_t &= 0, \\ \mathcal{U}_0 &= I. \end{cases} \tag{67}$$

We solve this problem using the B.K.W. method. We assume that the kernel of the operator $\mathcal{U}_t$ is $K_t$, where

$$K_t(x, y; h) = \frac{1}{(2\pi h)^3} \int e^{i(\Phi(t,x,\xi) - y \cdot \xi)h^{-1}} E(t, x, y, \xi; h) d\xi,$$

with $E(t,x,y,\xi;h) = E_0(t,x,y,\xi) + hE_1(t,x,y,\xi) + \cdots$.

Thus, if we look for $E(t,x,y,\xi;h)$ having the asymptotic expansion above, it is enough to solve (in some fixed neighborhood of $t=0$) the sequence of equations

$$\left\{\begin{array}{rl} 0 & = \big(\partial_t\Phi(t,x,\xi) + c\alpha\cdot\nabla_x\Phi - e\alpha\cdot A + mc^2\beta + ev\big)\, E_0, \\ i(\partial_t + c\alpha\cdot\nabla_x)E_j & = \big(\partial_t\Phi(t,x,\xi) + c\alpha\cdot\nabla_x\Phi - e\alpha\cdot A + mc^2\beta + ev\big)\, E_{j+1}, \\ E_0(0,x,\xi) & = I_4, \\ E_j(0,x,\xi) & = 0, \quad \text{for } j\geq 1. \end{array}\right. \tag{68}$$

On the support of $E_0$, we deduce the eikonal equation

$$\left\{\begin{array}{c} \det\big(\partial_t\Phi(t,x,\xi) + c\alpha\cdot\nabla_x\Phi - e\alpha\cdot A + mc^2\beta + ev\big) = 0, \\ \Phi(0,x,\xi) = x\cdot\xi. \end{array}\right. \tag{69}$$

The system (69) is equivalent to

$$\left\{\begin{array}{c} \partial_t\Phi^\pm(t,x,\xi) + H_1^\pm(x,\nabla_x\Phi) = 0, \quad \text{(see (11))}, \\ \Phi^\pm(0,x,\xi) = x\cdot\xi. \end{array}\right. \tag{70}$$

The latter system can be solved using the Hamilton-Jacobi method (see [2]) and all derivatives

$$\partial_t^\alpha\partial_x^\beta\partial_\xi^\gamma\left(\Phi^\pm(t,x,\xi) - x\cdot\xi\right) \tag{71}$$

are uniformly bounded for $(t,x,\xi)\in[-\delta_1,\delta_1]\times\mathbb{R}^3\times B(0,C_1)$ and $(\alpha,\beta,\gamma)\neq(0,0,0)$. Using the Taylor formula in a neighborhood of $t=0$, the two solutions of (70) satisfy:

$$\Phi^\pm(t,x,\xi) = x\cdot\xi - tH_1^\pm(x,\xi) + O(t^2).$$

Then $\mathcal{U}_t = \mathcal{U}_t^+ + \mathcal{U}_t^-$, and the kernel of the operator $\mathcal{U}_t$ is $K_t = K_t^+ + K_t^-$, with

$$K_t^\pm(x,y;h) = \frac{1}{(2\pi h)^3}\int e^{i(\Phi^\pm(t,x,\xi) - y\cdot\xi)h^{-1}} E^\pm(t,x,y,\xi;h)\,d\xi.$$

We look for the amplitude $E^\pm(t,x,y,\xi;h)$ having an asymptotic expansion in powers of $h$:

$$E_0^\pm(t,x,y,\xi) + hE_1^\pm(t,x,y,\xi) + \cdots.$$

Consequently, the coefficients $E_j^\pm(t,x,y,\xi)$ are the solutions of the transport equations

$$\left\{\begin{array}{rl} 0 = & \big(\partial_t\Phi^\pm + c\alpha\cdot\nabla_x\Phi^\pm - e\alpha\cdot A + mc^2\beta + ev\big)\, E_0^\pm, \\ i(\partial_t + c\alpha\cdot\nabla_x)E_j^\pm = & \big(\partial_t\Phi^\pm + c\alpha\cdot\nabla_x\Phi^\pm - e\alpha\cdot A + mc^2\beta + ev\big)\, E_{j+1}^\pm, \\ E_j^+(0,x,\xi) + E_j^-(0,x,\xi) & = 0 \quad \text{for } j\geq 1, \\ E_0^\pm(0,x,\xi) & = \Pi_1^\pm(x,\xi), \end{array}\right. \tag{72}$$

with $\Pi_1^\pm(x,\xi)$ defined by (12).

**Resolution of (72).**
Let us denote by $L = \partial_t + c\alpha\cdot\nabla_x$, with $\alpha\cdot\nabla_x = \sum_{j=1}^3 \alpha_j\partial_{x_j}$. The matrix

$$\mathcal{M}^\pm = \partial_t\Phi^\pm + c\alpha\cdot\nabla_x\Phi^\pm - e\alpha\cdot A + mc^2\beta + ev,$$

is Hermitian and has two real eigenvalues which are linearly independent with multiplicity 2.

First, we multiply system (72) by the column-vector $N_1 = (1,0,0,0)^\dagger$, the superscript $\dagger$ indicates the complex conjugate of the transposed. We denote

$$E^{\pm}_{j,1} = E^{\pm}_j N_1 \text{ for } j = 1,2,\cdots, \quad E^{\pm}_{0,1}(0,x,\xi) = \Pi^{\pm}_1(x,\xi)N_1. \tag{73}$$

Since $\det(\mathcal{M}^{\pm}) = 0$, there exist $l^{\pm}_k$ and $r^{\pm}_k$, left and right eigenvectors of the matrix $\mathcal{M}^{\pm}$, corresponding to the eigenvalue zero, such that

$$\mathcal{M}^{\pm} r^{\pm}_k = 0, \quad l^{\pm}_k \mathcal{M}^{\pm} = 0, \quad l^{\pm}_k = (r^{\pm}_k)^\dagger, \quad k = 1,2, \tag{74}$$

(here $r^{\pm}_k$ is a column-vector and $l^{\pm}_k$ is a row-vector). We choose

$$r^+_1 = \begin{pmatrix} u^+ \\ 0 \\ v^+ \\ w^+_+ \end{pmatrix}, \quad r^+_2 = \begin{pmatrix} 0 \\ u^+ \\ w^+_- \\ -v^+ \end{pmatrix}, \quad r^-_1 = \begin{pmatrix} w^-_+ \\ v^- \\ 0 \\ u^- \end{pmatrix}, \quad r^-_2 = \begin{pmatrix} -v^- \\ w^-_- \\ u^- \\ 0 \end{pmatrix},$$

$$l^{\pm}_\nu r^{\pm}_k = (\mp 2p^{\pm}_5 u^{\pm})\delta_{\nu k}, \quad \nu, k = 1,2. \tag{75}$$

Here $u^{\pm}, v^{\pm}$ and $w^{\pm}_{\pm}$ are defined by

$$u^{\pm} = p^{\pm}_4 \mp p^{\pm}_5, \quad v^{\pm} = p^{\pm}_3, \quad w^{\pm}_+ = \pm p^{\pm}_1 + ip^{\pm}_2, \quad w^{\pm}_- = \pm p^{\pm}_1 - ip^{\pm}_2,$$

where $p^{\pm}_4 = mc^2 + \frac{e(v_+ - v_-)}{2}$, $\quad p^{\pm}_5 = \partial_t \Phi^{\pm} + \frac{e(v_+ + v_-)}{2}$, $\quad p^{\pm}_j = c\partial_{x_j}\Phi^{\pm} - eA_j$, for $j = 1,2,3$.

It is easy to see that the vector-valued functions $r^{\pm}_k(t,x,\xi)$ and $l^{\pm}_k(t,x,\xi)$ can be chosen smooth in $t$ and $x$ and nowhere vanishing. All the derivatives of $r^{\pm}_k, l^{\pm}_k$, $k = 1,2$, are uniformly bounded for $(t,x,\xi) \in [-\delta_1,\delta_1] \times \mathbb{R}^3 \times B(0,C_1)$. Then it follows from the first equation in (72) that

$$E^{\pm}_{0,1} = \sigma^{\pm}_{0,1}(t,x,\xi) r^{\pm}_1(t,x,\xi) + \sigma^{\pm}_{0,2}(t,x,\xi) r^{\pm}_2(t,x,\xi),$$

where $\sigma^{\pm}_{0,1}, \sigma^{\pm}_{0,2}$, are scalar-valued functions. If we multiply the second equation in (72) for $j = 0$ on the left by $l^{\pm}_k$ for $k = 1,2$, we deduce the following differential equations for $\sigma^{\pm}_{0,k}$:

$$\begin{cases} l^{\pm}_1 L(\sigma^{\pm}_{0,1} r^{\pm}_1) + l^{\pm}_1 L(\sigma^{\pm}_{0,2} r^{\pm}_2) &= 0, \\ l^{\pm}_2 L(\sigma^{\pm}_{0,1} r^{\pm}_1) + l^{\pm}_2 L(\sigma^{\pm}_{0,2} r^{\pm}_2) &= 0. \end{cases}$$

We conclude

$$\begin{cases} l^{\pm}_1 r^{\pm}_1 \partial_t(\sigma^{\pm}_{0,1}) + c\sum_{j=1}^3 l^{\pm}_1 \alpha_j r^{\pm}_1 \partial_{x_j}(\sigma^{\pm}_{0,1}) + c\sum_{j=1}^3 l^{\pm}_1 \alpha_j r^{\pm}_2 \partial_{x_j}(\sigma^{\pm}_{0,2}) \\ \qquad\qquad + l^{\pm}_1 L(r^{\pm}_1)\sigma^{\pm}_{0,1} + l^{\pm}_1 L(r^{\pm}_2)\sigma^{\pm}_{0,2} = 0, \\ l^{\pm}_2 r^{\pm}_2 \partial_t(\sigma^{\pm}_{0,2}) + c\sum_{j=1}^3 l^{\pm}_2 \alpha_j r^{\pm}_2 \partial_{x_j}(\sigma^{\pm}_{0,2}) + c\sum_{j=1}^3 l^{\pm}_2 \alpha_j r^{\pm}_1 \partial_{x_j}(\sigma^{\pm}_{0,1}) \\ \qquad\qquad + l^{\pm}_2 L(r^{\pm}_1)\sigma^{\pm}_{0,1} + l^{\pm}_2 L(r^{\pm}_2)\sigma^{\pm}_{0,2} = 0. \end{cases} \tag{76}$$

We now use Lemma 10 (see below) in system (76). Since $p^{\pm}_5 \neq 0$, $u^{\pm} = p^{\pm}_4 \mp p^{\pm}_5 \neq 0$ then, after multiplying (76) by $(\mp 2p^{\pm}_5 u^{\pm})^{-1}$, (76) can be written as

$$D^{\pm}\sigma^{\pm}_0 = M^{\pm}\sigma^{\pm}_0 := (\mp 2p^{\pm}_5 u^{\pm})^{-1} \begin{pmatrix} l^{\pm}_1 L(r^{\pm}_1) & l^{\pm}_1 L(r^{\pm}_2) \\ l^{\pm}_2 L(r^{\pm}_1) & l^{\pm}_2 L(r^{\pm}_2) \end{pmatrix} \begin{pmatrix} \sigma^{\pm}_{0,1} \\ \sigma^{\pm}_{0,2} \end{pmatrix}, \tag{77}$$

with $D^{\pm} = \partial_t + a^{\pm} \cdot \nabla_x = \partial_t + \sum_{j=1}^{3} a_j^{\pm}(t,x)\partial_{x_j}$, and

$$a^{\pm} = c(\mp 2p_5^{\pm} u^{\pm})^{-1}(l_1^{\pm}\alpha_1 r_1^{\pm},\ \ l_1^{\pm}\alpha_2 r_1^{\pm},\ \ l_1^{\pm}\alpha_3 r_1^{\pm}).$$

Thus the function $\sigma_{0,k}^{\pm}$ can be found provided its value is known for $t = 0$, and it is as smooth as $\sigma_{0,k}^{\pm}(0,x,\xi)$ (for more details, see a method for solving a similar equation in [38]). The equality

$$E_{0,1}^{\pm}(0,x,\xi) = \sigma_{0,1}^{\pm}(0,x,\xi)r_1^{\pm}(0,x,\xi) + \sigma_{0,2}^{\pm}(0,x,\xi)r_2^{\pm}(0,x,\xi) = \Pi_1^{\pm}(x,\xi)N_1,$$

gives the value of $\sigma_0^{\pm}$ at $t = 0$.

Since the derivatives of $\sigma_{0,k}^{\pm}$, $r_k^{\pm}$, for $k = 1,2$, are uniformly bounded, then all the derivatives $(\partial_t^{\alpha}\partial_x^{\beta}\partial_{\xi}^{\gamma}E_{0,1}^{\pm})$ are uniformly bounded for $(\alpha,\beta,\gamma) \in \mathbb{N}\times\mathbb{N}^3\times\mathbb{N}^3$.

It follows from the second equation in (72) for $j = 0$, that

$$iLE_{0,1}^{\pm} = \mathcal{M}^{\pm}E_{1,1}^{\pm},$$

i.e., $E_{1,1}^{\pm} = \sigma_{1,1}^{\pm}r_1^{\pm} + \sigma_{1,2}^{\pm}r_2^{\pm} + h_1^{\pm}$, where $\sigma_{1,k}^{\pm}$ is a scalar-valued function for $k = 1,2$, and $h_1^{\pm}$ is expressed in terms of $LE_{0,1}^{\pm}$. To find $\sigma_{1,k}^{\pm}$ it is sufficient to multiply the second equation in (72) for $j = 1$ on the left by $l_k^{\pm}$ for $k = 1,2$. Then

$$\begin{cases} l_1^{\pm}L(\sigma_{1,1}^{\pm}r_1^{\pm}) + l_1^{\pm}L(\sigma_{1,2}^{\pm}r_2^{\pm}) + l_1^{\pm}L(h_1^{\pm}) & = 0, \\[2ex] l_2^{\pm}L(\sigma_{1,1}^{\pm}r_1^{\pm}) + l_2^{\pm}L(\sigma_{1,2}^{\pm}r_2^{\pm}) + l_2^{\pm}L(h_1^{\pm}) & = 0. \end{cases}$$

From this equation, $\sigma_{1,k}^{\pm}$ can be found provided the function $\sigma_{1,k}(0,x)$ is known. By the same procedure, for all $j = 1,2,\cdots$, we obtain

$$\begin{cases} \sigma_{j,1}^{\pm}r_1^{\pm} + \sigma_{j,2}^{\pm}r_2^{\pm} + h_j^{\pm} & = E_{j,1}^{\pm}, \\ l_1^{\pm}L(\sigma_{j,1}^{\pm}r_1^{\pm}) + l_1^{\pm}L(\sigma_{j,2}^{\pm}r_2^{\pm}) + l_1^{\pm}L(h_j^{\pm}) & = 0, \\ l_2^{\pm}L(\sigma_{j,1}^{\pm}r_1^{\pm}) + l_2^{\pm}L(\sigma_{j,2}^{\pm}r_2^{\pm}) + l_2^{\pm}L(h_j^{\pm}) & = 0. \end{cases}$$

For $t = 0$, $\quad j = 1,2,\cdots$, we have

$$\sigma_{0,1}^{\pm}r_1^{\pm} + \sigma_{0,2}^{\pm}r_2^{\pm} = \Pi_1^{\pm}N_1, \quad \sigma_{j,1}^{+}r_1^{+} + \sigma_{j,1}^{-}r_1^{-} + \sigma_{j,2}^{+}r_2^{+} + \sigma_{j,2}^{-}r_2^{-} = -(h_j^{+} + h_j^{-}),$$

and the quantity $h_j^{\pm}$ is determined provided $E_{0,1}^{\pm}, E_{1,1}^{\pm}, \cdots, E_{j-1,1}^{\pm}$, are known. Solving the differential equation for $\sigma_j^{\pm} = \begin{pmatrix} \sigma_{j,1}^{\pm} \\ \sigma_{j,2}^{\pm} \end{pmatrix}$, we find these functions for all sufficiently small $t$.

Repeating this group of calculations, multiplying by $N_2 = (0,1,0,0)^{\dagger}$, $N_3 = (0,0,1,0)^{\dagger}$ and $N_4 = (0,0,0,1)^{\dagger}$ instead of $N_1$ in (73), we find $E_{j,2}^{\pm} = E_j^{\pm}N_2$, $E_{j,3}^{\pm} = E_j^{\pm}N_3$ and $E_{j,4}^{\pm} = E_j^{\pm}N_4$. Consequently, we have:

**Proposition 13.** *There exists a family of matrices*

$$E_j^{\pm} = (E_{j,1}^{\pm},\ E_{j,2}^{\pm},\ E_{j,3}^{\pm},\ E_{j,4}^{\pm}), \quad \textit{for } j \geq 0,$$

*solution of* (72). *Moreover, for all* $j \geq 0$, $E_j^{\pm} \in C^{\infty}$ *and all derivatives* $(\partial_t^{\alpha}\partial_x^{\beta}\partial_{\xi}^{\gamma}E_j^{\pm})$ *are uniformly bounded for all* $(t,x,\xi) \in [-\delta_1,\delta_1]\times\mathbb{R}^3\times B(0,C_1)$ *and* $(\alpha,\beta,\gamma) \in \mathbb{N}\times\mathbb{N}^3\times\mathbb{N}^3$.

Consequently, the Borel procedure provides a symbol $E^{\pm}(t,x,y,\xi;h) \in \mathcal{S}^0(1)$ with compact support in $\xi$ and $(x-y)$ with $E_0^{\pm}(t,x,y,\xi)+hE_1^{\pm}(t,x,y,\xi)+\cdots$ its asymptotic expansion.

**Desired estimate.**
Next, we remark that for all $N \in \mathbb{N}$:

$$\begin{aligned}(ih\partial_t - H_1)\left(e^{i(\Phi^{\pm}(t,x,\xi)-y\cdot\xi)h^{-1}}\sum_{j=0}^{N} h^j E_j^{\pm}\right) &= e^{i(\Phi^{\pm}(t,x,\xi)-y\cdot\xi)h^{-1}} \\ &\times \textstyle\sum_{j=0}^{N}(ihL(E_j^{\pm}) + \mathcal{M}^{\pm}E_j^{\pm})h^j \\ &= P_N(t,x,\xi;h)h^N, \end{aligned} \tag{78}$$

and all derivatives $D^{\alpha}_{x,\xi}P_N(t,x,\xi;h)$ are bounded as $h \to 0$ for all $\alpha$. Then for all $N \in \mathbb{N}$,

$$\left\{ \begin{aligned} ih\partial_t\mathcal{U}_t - H_1\mathcal{U}_t &= O(h^N), \\ \mathcal{U}_0 &= I + O(h^N), \end{aligned}\right. \tag{79}$$

thus

$$\left\{ \begin{aligned} \tfrac{d}{dt}(e^{+itH_1h^{-1}}\mathcal{U}_t) &= O(h^N), \\ \mathcal{U}_0 &= I + O(h^N), \end{aligned}\right. \tag{80}$$

where $O(h^N)$ is uniform in $t$ and corresponds to the norm in $\mathcal{L}(L^2)$. Then we get:

$$\|\mathcal{U}_t - e^{-itH_1h^{-1}}\| = O(h^{\infty}). \tag{81}$$

□

**Lemma 10.** *Under the notations used above, we have*

$$l_1^{\pm}\alpha_j r_1^{\pm} = l_2^{\pm}\alpha_j r_2^{\pm}, \qquad l_1^{\pm}\alpha_j r_2^{\pm} = l_2^{\pm}\alpha_j r_1^{\pm} = 0, \;\; j=1,2,3. \tag{82}$$

**Proof.** As Rubinow and Keller in [38] let us work in a general situation.
We consider the $n$ Hermitian matrices $M_\mu$ and $n$ real scalars $p_\mu$, $\mu = 1,\cdots,n$. Let $G$ be the Hermitian matrix defined by

$$G = \sum_{\mu=1}^{n} p_\mu M_\mu.$$

Let $\lambda$ be a multiple eigenvalue of $G$ and $B_1,\cdots,B_q$, a set of associated orthormal eigenvectors which are differentiable functions of $p_\mu$. Then

$$B_j^{\dagger}B_k = \delta_{jk}, \tag{83}$$

$$GB_k = \lambda B_k. \tag{84}$$

If $\lambda(p_1,\cdots,p_\mu)$ is differentiable, we differentiate (84) with respect to $p_\mu$ and obtain

$$M_\mu B_k + G\frac{\partial B_k}{\partial p_\mu} = \frac{\partial\lambda}{\partial p_\mu}B_k + \lambda\frac{\partial B_k}{\partial p_\mu}. \tag{85}$$

The multiplication of (85) on the left by $B_j^{\dagger}$, the use of (84), and the fact that $G$ is Hermitian yield

$$B_j^{\dagger}M_\mu B_k = \frac{\partial\lambda}{\partial p_\mu}\delta_{jk}. \tag{86}$$

In order to treat our case, we take

$$G = \mathcal{M}^{\pm} = \sum_{\mu=1}^{5} p_{\mu}^{\pm} M_{\mu},$$

where $M_j = \alpha_j$ for $j = 1, 2, 3$, $M_4 = \beta$ and $M_5 = I_4$ are Hermitian matrices ($\alpha_j, \beta$ are the Dirac matrices) and $p_{\mu}^{\pm}$ are five real scalars.
We also take $\lambda^{\pm} = p_5^{\pm} \pm \sqrt{(p_1^{\pm})^2 + (p_2^{\pm})^2 + (p_3^{\pm})^2 + (p_4^{\pm})^2}$ and $F^{\pm}$ the point with coordinates $p_{\mu}^{\pm}$: $p_j^{\pm} = c\partial_{x_j}\Phi^{\pm} - eA_j$ for $j = 1, 2, 3$, $p_4^{\pm} = mc^2 + \frac{e(v_+ - v_-)}{2}$, $p_5^{\pm} = \partial_t \Phi^{\pm} + \frac{e(v_+ + v_-)}{2}$.

When $\Phi^{\pm}$ satisfies (69) and (70), $r_1^{\pm}, r_2^{\pm}$ are two orthogonal eigenvectors of $\mathcal{M}^{\pm}$ corresponding to the eigenvalue $\lambda^{\pm} = \lambda^{\pm}(F^{\pm}) = 0$. Since $|e(v_+ - v_-)| < 2mc^2$ (see (9)), $\lambda^{\pm}$ is differentiable near the point $F^{\pm}$. Now, we apply (86) with $B_j^{\dagger} = l_j^{\pm}$ and $B_k = r_k^{\pm}$. After the normalization of $r_k^{\pm}, l_j^{\pm}$ we obtain

$$l_j^{\pm} M_{\mu} r_k^{\pm} = \frac{\partial \lambda^{\pm}\left(p_1^{\pm}, \cdots, p_5^{\pm}\right)}{\partial p_{\mu}^{\pm}}\Big|_{F^{\pm}} (\mp 2 p_5^{\pm} u^{\pm}) \delta_{jk},$$

and we have proved the lemma. □

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*E-mail address*: Abdallah.Khochman@math.u-bordeaux1.fr

Université Bordeaux I, Institut de Mathématiques, 351, cours de la Libération, 33405 Talence, France