RESONANCES AND SPECTRAL SHIFT FUNCTION FOR A MAGNETIC SCHRÖDINGER OPERATOR

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Abstract. We consider the 3D Schrödinger operator H_0 with constant magnetic field and subject to an electric potential v_0 depending only on the variable along the magnetic field x_3. The operator H_0 has infinitely many eigenvalues of infinite multiplicity embedded in its continuous spectrum. We perturb H_0 by smooth scalar potentials $V = O(\langle (x_1, x_2) \rangle^{-\delta_\perp} \langle x_3 \rangle^{-\delta_\parallel})$, $\delta_\perp > 2$, $\delta_\parallel > 1$. We assume also that V and v_0 have an analytic continuation, in the magnetic field direction, in a complex sector outside a compact set. We define the resonances of $H = H_0 + V$ as the eigenvalues of the non-selfadjoint operator obtained from H by analytic distortions of $\mathbb{R}_{x_3}$. We study their distribution near any fixed real eigenvalue of H_0, $2bq + \lambda$ for $q \in \mathbb{N}$. In a ring centered at $2bq + \lambda$ with radiuses $(r, 2r)$, we establish an upper bound, as r tends to 0, of the number of resonances. This upper bound depends on the decay of V at infinity only in the directions (x_1, x_2). Finally, we deduce a representation of the derivative of the spectral shift function (SSF) for the operator pair (H_0, H) in terms of resonances. This representation justifies the Breit-Wigner approximation and implies a local trace formula.

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1. Introduction

The resonance theory for non-relativistic particles satisfying the Schrödinger equation has been developed following several approaches. Among them we can mention the analytic dilation (see Aguilar-Combes [1]) or the analytic distortion (see Hunziker [10]) and meromorphic continuation of the resolvent or of the scattering matrix (see Lax-Philips [14] and Vainberg [20]). For Schrödinger operators with constant magnetic field, the resonances can be defined by analytic dilation (only) with respect to the variable along the magnetic field (see Avron-Herbst-Simon [3], Wang [21], Astaburuaga-Briet-Bruneau-Fernández-Raikov [2]) and by meromorphic continuation of the resolvent (see J.F.Bony-Bruneau-Raikov [4]).

The link between the resonances and the spectral shift function (SSF) by the so-called Breit-Wigner approximation has been developed in different situations. Such a representation of the derivative of the spectral shift function related to the resonances, implies trace formulas. In the semi-classical regime we can mention Sjöstrand [18], [19], Petkov-Zworski [15], J.F.Bony-Sjöstrand [5], Bruneau-Petkov [6] and Dimassi-Zerzeri [8] for the Schrödinger operator and [12] for the Dirac operator. In [4], J.-F.Bony, Bruneau and Raikov obtain a Breit-Wigner approximation of the spectral shift function near a Landau level for the 3D Schrödinger operator with constant magnetic field. For the last operator, under more general assumptions, Fernández-Raikov [9] studied the singularities of the spectral shift function at a

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Landau level. These singularities has been analysed at eigenvalues of infinite multiplicity by Astaburuaga, Briet, Bruneau, Fernández and Raikov [2] for a magnetic Schrödinger operator having electric potential depending (only) on the variable along the magnetic field.

In this paper we consider the magnetic Schrödinger operator H_0 with an electromagnetic field introduced in [2]. We suppose that the magnetic field is constant and that the electric potential v_0 depends only on the variable x_3 and is analytic outside a compact set. This operator is remarkable because of the generic presence of infinitely many eigenvalues of infinite multiplicity, embedded in the continuous spectrum of H_0. We perturb the operator H_0 by a smooth scalar potential V analytic outside a compact set with respect to the variable x_3.

The purpose of this work is to define the resonances of the electromagnetic Schrödinger operator $H = H_0 + V$ for analytic perturbation outside a compact set in the third direction x_3. We define the resonances for H as the discrete eigenvalues of the non-selfadjoint operator H_θ obtained from the magnetic Schrödinger operator by a general class of complex distortions of $\mathbb{R}_{x_3}$. In Section 3, we prove that the discrete eigenvalues of H_θ are the zeros of a regularized determinant $\det_2(\cdot)$ which is independent of the distortion. This justifies the definition of the resonances. We calculate the essential spectrum of the distorted operator to determine the sector where we can define the resonances. In Section 4, we establish an upper bound for the number of resonances of H in a domain of size $r \to 0$ near an embedded eigenvalue of H_0. The second goal of this work is to obtain a Breit-Wigner approximation for the derivative of the spectral shift function $\xi(\lambda)$ related to the resonances of the operator H, as well as a local trace formula (see Section 5).

2. Assumptions and results

In this section, we summarize some spectral properties of the 3D Schrödinger operator H_0 with constant magnetic field $\mathbf{B} = (0, 0, b)$, $b > 0$ and subject to a non-constant electric field $E = -(0, 0, v_0'(x_3))$ depending only on the variable x_3 (see [2]). We also state the main results. Let

$$H_0 = H_{0,\perp} \otimes I_\parallel + I_\perp \otimes H_{0,\parallel}, \tag{2.1}$$

where $I_\parallel$ and $I_\perp$ are the identity operators in $L^2(\mathbb{R}_{x_3})$ and $L^2(\mathbb{R}^2_{x_1,x_2})$ respectively,

$$H_{0,\perp} := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \quad (x_1, x_2) \in \mathbb{R}^2, \tag{2.2}$$

is the Landau Hamiltonian shifted by the constant b, self-adjoint in $L^2(\mathbb{R}^2)$, and

$$H_{0,\parallel} := -\frac{d^2}{dx_3^2} + v_0, \quad x_3 \in \mathbb{R}. \tag{2.3}$$

The operator v_0 is the multiplication operator by an one dimensional scalar potential $v_0(x_3)$. We suppose that $v_0 \in L^\infty(\mathbb{R})$ and satisfies

$$|v_0| = O(\langle x_3 \rangle^{-\delta_0}), \tag{2.4}$$

with $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ and $\delta_0 > 1$. Then using Weyl theorem, we have

$$\sigma_{ess}(H_{0,\parallel}) = \sigma_{ess}(-\frac{d^2}{dx_3^2}) = [0, +\infty[. \tag{2.5}$$

It is well known that the spectrum of the operator $H_{0,\perp}$ consists of the Landau levels $2bq$, $q \in \mathbb{N} := \{0, 1, 2 \dots\}$, and the multiplicity of each eigenvalue $2qb$ is infinite (see [3]). Consequently, the eigenvalues of H_0 have the form $2bq + \lambda$ where $q \in \mathbb{N}$ and λ is an eigenvalue of the one dimensional Schrödinger operator $H_{0,\|} = -\frac{d^2}{dx_3^2} + v_0(x_3)$. For simplicity, throughout the article we suppose also that

$$\inf \sigma(H_{0,\|}) > -2b. \tag{2.6}$$

Note that, (2.6) holds true if $v_0 > -2b$. The eigenvalues of H_0, $2bq + \lambda$, $q \in \mathbb{N}^*$, are embedded in its continuous spectrum $[0, +\infty[= \cup_{q=0}^{\infty}[2bq, \infty[$ and are of infinite multiplicity.

Now, we introduce the perturbed operator $H = H_0 + V$ where V is the multiplication operator by the potential $V(x)$. Assume that $V \in L^\infty(\mathbb{R}^3)$ and satisfies

$$|V(x)| = O(\langle X_\perp \rangle^{-\delta_\perp} \langle x_3 \rangle^{-\delta_\|}), \quad X_\perp = (x_1, x_2), \tag{2.7}$$

with $\delta_\perp > 2$ and $\delta_\| > 1$. We suppose also that V and v_0 have holomorphic extensions in the magnetic field direction x_3 in the sector

$$C_{\epsilon,0} := \{z \in \mathbb{C}; |\mathrm{Im}\ (z)| \le \epsilon |\mathrm{Re}\ (z)|, \quad |\mathrm{Re}\ (z)| \ge R_0 > 0\}, \quad \text{for } 0 < \epsilon < 1, \tag{2.8}$$

and satisfy respectively (2.7) and (2.4) for $x_3 \in C_{\epsilon,0}$.

For $\theta \in D_\epsilon \cap \mathbb{R}$ with $D_\epsilon := \{\theta \in \mathbb{C};\ |\theta| \le r_\epsilon := \frac{\epsilon}{\sqrt{1+\epsilon^2}}\}$, we denote

$$H_\theta \ := \ (I_\perp \otimes U_\theta) H (I_\perp \otimes U_\theta^{-1}) = H_{0,\theta} + V_\theta,$$

where

$$H_{0,\theta} := (I_\perp \otimes U_\theta) H_0 (I_\perp \otimes U_\theta^{-1}) = H_{0,\perp} \otimes I_\| + I_\perp \otimes H_{0,\|}(\theta) \tag{2.9}$$

and $H_{0,\|}(\theta) = U_\theta H_{0,\|} U_\theta^{-1}$ (see (3.2) for the definition of U_θ). We will prouve in the next section that the operator H_θ has an analytic extension for $\theta \in D_\epsilon$.

For θ_0 fixed in $D_\epsilon^+ := D_\epsilon \cap \{\theta \in \mathbb{C};\ \mathrm{Im}\ (\theta) \ge 0\}$, $q \in \mathbb{N}$ and $r \in \mathbb{R}$, we define

$$\Gamma_{r,\theta_0} := 2br + (1 + \theta_0)^{-2}[0, +\infty[\tag{2.10}$$

and

$$S_{q,\theta_0} := \bigcup_{q<r<q+1} \Gamma_{r,\theta_0}. \tag{2.11}$$

The spectrum of H_{0,θ_0} is purely essential and we have

$$\begin{aligned} \sigma(H_{0,\theta_0}) = \sigma_{ess}(H_{0,\theta_0}) \ &= \ \bigcup_{q\in\mathbb{N}} \left(2bq + \sigma(H_{0,\|}(\theta_0))\right) \\ &= \ \bigcup_{q\in\mathbb{N}} \left(\Gamma_{q,\theta_0} \cup \left(2bq + \sigma_{disc}(H_{0,\|}(\theta_0))\right)\right), \end{aligned} \tag{2.12}$$

where $\sigma_{disc}(H_{0,\|}(\theta_0)) = \sigma_{disc}(H_{0,\|}) \cup \{z_1, z_2, \dots\}$, $\sigma_{disc}(H_{0,\|})$ denotes the discrete spectrum of $H_{0,\|}$ and $z_1, z_2, \dots$ are the complex eigenvalues of $H_{0,\|}(\theta_0)$. In the following, we assume that $\sigma_{disc}(H_{0,\|}) = \{\lambda\}$. Note that λ is necessarily simple.

The essential spectrum of H_θ coincides with that of $H_{0,\theta}$. We prove also that the discrete spectrum of H_θ in $S_\theta = \bigcup_{q\in\mathbb{N}} S_{q,\theta}$ is independent of θ in D_ϵ^+ (i.e. for two values θ_1, θ_2, the discrete spectrum of H_{θ_1} and H_{θ_2} coincide on $S_{\theta_1} \cap S_{\theta_2}$), (see Proposition 3.2). This justifies the following definition.

Definition 2.1. *The resonances of H in S_{θ_0} are the discrete eigenvalues of H_{θ_0}. The multiplicity of a resonance z_0 is defined by*

$$\text{mult}(z_0) := \text{rank}\frac{1}{2i\pi}\int_{\Gamma_0}(z - H_{\theta_0})^{-1}dz, \tag{2.13}$$

where Γ_0 is a small positively oriented circle centered at z_0. We will denote $\text{Res}(H)$ *the set of resonances.*

Remark 2.1. *The resonances of H in $\{z \in \mathbb{C};\ \ \text{Re}(z) < 0\}$ are the real discrete eigenvalues of H.*

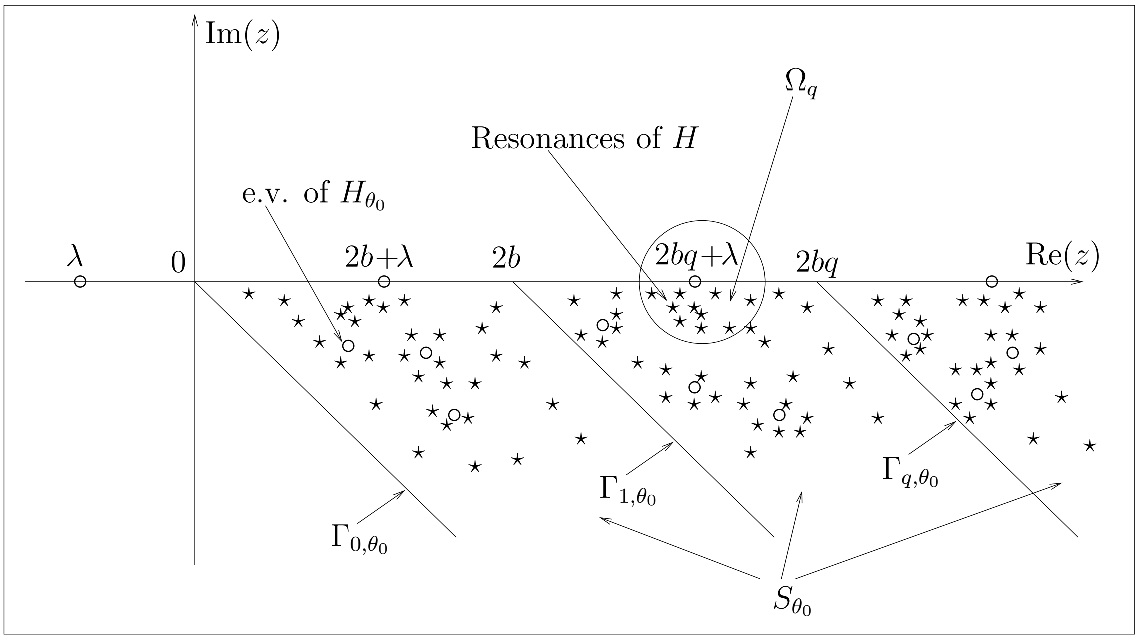

Fig.1. The set S_{θ_0}

Now, we state an upper bound as $r \to 0$ on the number of resonances of H in a ring in Ω_q with radiuses $(r, 2r)$ and centered at $2bq + \lambda$, $q \in \mathbb{N}^*$ fixed.

Theorem 2.1. [Upper bound] *Suppose that V and v_0 satisfy the above hypotheses. Then there exist $r_0 > 0$ and $\nu > 0$, such that, for any $0 < r < r_0$,*

$$\#\{z \in \text{Res}(H) \cap \Omega_q; r < |z - 2bq - \lambda| < 2r\} = O(n_+(r, \nu p_q W p_q)|\ln r|), \tag{2.14}$$

where $W = \sup_{x_3 \in C_{\epsilon,0}} |\langle x_3\rangle^{\delta_\parallel} V|$, p_q is the orthogonal projection onto $\mathcal{H}_q := ker(H_{0,\perp} - 2bq)$ and $n_+(r, p_q W p_q)$ is the counting function of the eigenvalues larger than r of the Toeplitz operator $p_q W p_q$. In particular, under our assumption we have always $n_+(r, p_q W p_q) = O(r^{-2/\delta_\perp})$.

The counting function $n_+(r, p_q W p_q) := \text{rank}\ \mathbf{1}_{(r,+\infty)}(p_q W p_q)$ satisfies asymptotic relations depending on the decay of W at infinity. The following three lemmas give an upper bound of $n_+(r, p_q W p_q)$ in the case power-like decay, exponential decay, or compact support of W, respectively. For more precise results concerning the asymptotic properties, we refer to the cited theorem.

Lemma 2.1. (Theorem 2.6 of [16]) *Let the function* $U \in L^\infty(\mathbb{R}^2)$ *satisfy the estimate*

$$U(X_\perp) \leq C\langle X_\perp\rangle^{-\alpha}, \qquad X_\perp \in \mathbb{R}^2,$$

for some $\alpha > 0$*. Then for each* $q \in \mathbb{N}$*, we have*

$$n_+(r, p_q U p_q) = O(r^{-2/\alpha}).$$

Lemma 2.2. (Theorem 2.1 of [17]) *Let* $U \in L^\infty(\mathbb{R}^2)$*. Assume that*

$$\limsup_{|X_\perp|\to\infty} \frac{\ln U(X_\perp)}{|X_\perp|^{2\beta}} < 0, \qquad X_\perp \in \mathbb{R}^2,$$

for some $\beta > 0$ *(with the convention* $\ln(u) = -\infty$ *if* $u \leq 0$*). Then for each* $q \in \mathbb{N}$*, we have*

$$n_+(r, p_q U p_q) = O(\varphi_\beta(r))$$

where, for $0 < r < e^{-1}$*,*

$$\varphi_\beta(r) := \begin{cases} |\ln r|^{\frac{1}{\beta}} & \text{if} \quad 0 < \beta < 1, \\ |\ln r| & \text{if} \quad \beta = 1, \\ (\ln|\ln r|)^{-1}|\ln r| & \text{if} \quad \beta > 1. \end{cases} \tag{2.15}$$

Lemma 2.3. (Theorem 2.4 of [17]) *Let* $U \in L^\infty(\mathbb{R}^2)$*. Assume that the support of* U *is compact. Then for each* $q \in \mathbb{N}$*, we have*

$$n_+(r, p_q U p_q) = O(\varphi_\infty(r)),$$

where, for $0 < r < e^{-1}$*,*

$$\varphi_\infty(r) := (\ln|\ln r|)^{-1}|\ln r|.$$

Now, we study the spectral shift function (SSF) for the pair (H, H_0). The SSF $\xi(\lambda)$ for a pair of self-adjoint operators $(H,\, H_0)$ is a distribution in $\mathcal{D}'(\mathbb{R})$ whose derivative is

$$\xi' : f \in C_0^\infty(\mathbb{R}) \longmapsto -\mathrm{tr}\,(f(H) - f(H_0))\,. \tag{2.16}$$

In our case, $|V|^{\frac{1}{2}}(H_0 + i)^{-1}$ is in the Hilbert-Schmidt class, (2.16) is well defined and the SSF $\xi(\lambda)$ is a function in $L^1_{loc}(\mathbb{R})$.

We will see further that the resonances of H in S_{θ_0} are the zeros of the holomorphic extension of

$$z \in \{z \in \mathbb{C},\ \mathrm{Im}\ z > 0\} \longmapsto D(z) = \det{}_2((H - z)(H_0 - z)^{-1})$$

into S_{θ_0} (see (3.1) for the definition of $\det_2$). Thus in order to obtain a link between the SSF and the resonances, it will be convenient to introduce the regularized spectral shift function

$$\xi_2(\nu) = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \arg\det{}_2\left((H - \nu - i\varepsilon)(H_0 - \nu - i\varepsilon)^{-1}\right), \tag{2.17}$$

whose derivative is the following distribution (see [4])

$$\xi_2' : f \in C_0^\infty(\mathbb{R}) \longmapsto -\mathrm{tr}\left(f(H) - f(H_0) - \frac{d}{d\varepsilon} f(H_0 + \varepsilon V)|_{\varepsilon=0}\right). \tag{2.18}$$

We will deduce the properties of the SSF from those of the regularized SSF using the relation

$$\xi' = \xi_2' + \frac{1}{\pi}\mathrm{Im\ tr}\left(V_\theta(H_{0,\theta} - z)^{-2}\right). \tag{2.19}$$

We represent now, the derivative of the spectral shift function near $2bq + \lambda$ as a sum of a harmonic measure related to the resonances and the imaginary part of a holomorphic function.

Let $\widetilde{\Omega} \subset\subset \Omega$ be open relatively compact subsets of $\mathbb{C} \setminus \{0\}$. We assume that these sets are independent of r and that $\widetilde{\Omega}$ is simply connected. Also assume that the intersections between $\widetilde{\Omega}$ and $\mathbb{R}$ is a non-empty interval I.

Theorem 2.2. [Breit-Wigner approximation] *We suppose V and v_0 satisfy the above hypothesis. For $\widetilde{\Omega} \subset\subset \Omega$ and I as above, there exists a function g holomorphic in Ω, such that for $\mu \in 2bq + \lambda + rI$, we have*

$$\xi'(\mu) = \frac{1}{\pi r}\mathrm{Im}\, g'(\frac{\mu - 2bq - \lambda}{r}, r) - \sum_{\substack{w \in \mathrm{Res}(H) \cap 2bq+\lambda+r\Omega \\ \mathrm{Im}\, w \neq 0}} \frac{-\mathrm{Im} w}{\pi |\mu - w|^2} - \sum_{w \in \mathrm{Res}(H) \cap 2bq+\lambda+rI} \delta(\mu - w)$$

where $g(z, r)$ satisfies the estimate

$$g(z, r) = O\left(n_+(r, \nu p_q W p_q)|\ln r| + \widetilde{n}_1(r/\nu) + \widetilde{n}_2(r/\nu)\right) = O(|\ln r| r^{-\frac{2}{\delta_\perp}}), \quad \nu > 0, \tag{2.20}$$

uniformly with respect to $0 < r < r_0$ and $z \in \widetilde{\Omega}$, with $\widetilde{n}_p$, $p = 1, 2$, defined by

$$\widetilde{n}_p(r) := \left\| \frac{p_q W p_q}{r} \mathbf{1}_{[0,r]}(p_q W p_q) \right\|_p^p, \quad r > 0. \tag{2.21}$$

Here, $\|\cdot\|_p$ stands for the trace-class norms ($p = 1$) and Hilbert-Schmidt norms ($p = 2$).

Using [4, Corollary1], for W defined above satisfying the assumption of Lemma 2.1 with $\alpha \geq 2$, we have

$$\widetilde{n}_p(r) = O(r^{-\frac{2}{\alpha}}), \; p = 1, 2. \tag{2.22}$$

Finally, if the assumption of Lemma 2.2 or 2.3 hold for $W = U$, we have

$$\widetilde{n}_p(r) = o(\varphi_\beta(r)) \quad r \searrow 0, \tag{2.23}$$

the function $\varphi_\beta(r)$ being defined in Lemma 2.2 or 2.3.

As in [15], [6] or [4] and repeating the arguments used in the proof of [4, Corollary 3], we deduce from Theorem 2.1-2.2 the following theorem

Theorem 2.3. [Trace formula] *Let $\widetilde{\Omega} \subset\subset \Omega$ be as in Theorem 2.2. Suppose that f is holomorphic on a neighborhood of Ω and that $\phi \in C_0^\infty(\Omega \cap \mathbb{R})$ satisfies $\phi = 1$ near $\widetilde{\Omega} \cap \mathbb{R}$. Then, under the assumptions of Theorem 2.2, we have the following trace formula*

$$\mathrm{tr}\left((\phi f)(\frac{H - 2bq - \lambda}{r}) - (\phi f)(\frac{H_0 - 2bq - \lambda}{r}) \right) = \sum_{w \in \mathrm{Res}(H) \cap 2bq+\lambda+r\widetilde{\Omega}} f(\frac{w - 2bq - \lambda}{r}) + E_{f,\phi}(r)$$

with

$$|E_{f,\phi}(r)| \leq M_\phi \sup\{|f(z)| : z \in \Omega \setminus \widetilde{\Omega}, \mathrm{Im} z \leq 0\} \times N_q(r),$$

where $N_q(r) = n_+(r, \nu p_q W p_q)|\ln r| + \widetilde{n}_1(r/\nu) + \widetilde{n}_2(r/\nu) = O(|\ln r| r^{-\frac{2}{\delta_\perp}})$, and M_ϕ depends only on ϕ.

3. Definition of resonances via distortion analyticity

In this section, we start with the definition of the deformation for the electromagnetic Schrödinger operator by analytic distortion on $\mathbb{R}_{x_3}$. We calculate the essential spectrum of the distorted Schrödinger operator H_θ. We prove that the discrete eigenvalues of H_θ are independent of the distortion, this justifies the definition of a resonance as a discrete eigenvalue of the distorted operator H_θ. We will also prove that the resonances of H repeated with their multiplicity coincide with the zeros of a regularized determinant $\det_2(I + A)$ defined for a Hilbert-Schmidt operator A by

$$\det_2(I + A) := \det((I + A)e^{-A}), \tag{3.1}$$

(see Krein [13]). Let us now introduce the one-parameter family of unitary distortions in the magnetic field direction x_3:

$$U_\theta f(x) = J^{\frac{1}{2}}_{\phi_\theta(x)} f(\phi_\theta(x)), \quad \theta \in \mathbb{R}, \quad f \in S(\mathbb{R}), \tag{3.2}$$

where $\phi_\theta(x) = x + \theta g(x)$, $g : \mathbb{R} \longmapsto \mathbb{R}$ is a smooth function and $J_{\phi_\theta(x)} = \det(I + \theta g'(x))$ is the Jacobian of $\phi_\theta(x)$. We suppose that g satisfies the assumption

$$(\mathbf{A_g}) \begin{cases} \text{(i)} \;\; \sup_{x\in\mathbb{R}} |g'(x)| < 1, \\ \text{(ii)} \;\; g(x) = 0, \;\; \text{in the compact set } [-R_0, R_0], \;\; \text{(see (2.8))}, \\ \text{(iii)} \;\; g(x) = x, \;\; \text{outside a compact set } K(\supset [-R_0, R_0]). \end{cases}$$

We recall that

$$H_\theta := (I_\perp \otimes U_\theta) H (I_\perp \otimes U_\theta^{-1}) = H_{0,\theta} + V_\theta.$$

From (2.9) and using Kato's theorem [11, Theorem 4.5.35] we have the following (see also Hunziker [10] for Schrödinger operator and [12, Section 3] for the Dirac operator).

Proposition 3.1. *We suppose that the potential V satisfies all the assumptions of Section 2. Then we have*

(i) $\theta \in D_\epsilon \longmapsto H_\theta = H_{0,\theta} + V_\theta$ *is an analytic family of type A.*
(ii) $\sigma_{ess}(H_\theta) = \sigma_{ess}(H_{0,\theta}) = 2b\mathbb{N} + \sigma(H_{0,\|}(\theta))$.

Lemma 3.1. *The essential spectrum of $H_{0,\|}(\theta)$ is*

$$\sigma_{ess}(H_{0,\|}(\theta)) = \Big\{ \frac{\mu}{(1+\theta)^2} \in \mathbb{C}; \quad \mu \in [0, +\infty[\Big\}. \tag{3.3}$$

The rest of the spectrum is

$$\sigma_{disc}(H_{0,\|}) \cup \{z_1, z_2, \dots\},$$

where $\sigma_{disc}(H_{0,\|})$ denotes the discrete spectrum of $H_{0,\|}$ and $z_1, z_2, \dots$ are the complex eigenvalues of $H_{0,\|}(\theta)$.

Remark 3.1. *The part $\sigma_{disc}(H_{0,\|}) \cup \{z_1, z_2, \dots\}$ of the spectrum of $H_{0,\|}(\theta)$ correspond to eigenvalues with infinite multiplicity of $I_\perp \otimes H_{0,\|}(\theta)$.*

In the following we fix $q \in \mathbb{N}$ and a compact set Ω_q centered at $2bq + \lambda$ such that

$$\Omega_q \cap \sigma_{ess}(H_\theta) = \{2bq + \lambda\}. \tag{3.4}$$

Repeating arguments in the proof of [4, Proposition 1], [4, Lemma 1] and using the resolvent equation

$$(H_0 - z)^{-1} = (H_0 - v_0 - z)^{-1} \left(I - v_0 (H_0 - z)^{-1} \right),$$

we obtain the following lemma.

Lemma 3.2. *The operators* $V(H_0 - z)^{-1}$ *and* $\partial_z(V(H_0 - z)^{-1})$ *are holomorphic on* $\{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$ *with values in the Hilbert-Schmidt class* S_2 *and in the trace class* S_1 *respectively.*

Lemma 3.3. *The operators* $V_\theta(H_{0,\theta} - z)^{-1}$ *and* $\partial_z(V_\theta(H_{0,\theta} - z)^{-1})$ *are holomorphic for* $z \in \Omega_q \backslash \{2bq + \lambda\}$ *with values in the Hilbert-Schmidt class* S_2 *and in the trace class* S_1 *respectively.*

Proof. According to (i) of Proposition 3.1 and to the definition of Ω_q, the function $z \longmapsto V_\theta(H_{0,\theta} - z)^{-1}$ is analytic for $z \in \Omega_q \backslash \{2bq + \lambda\}$. Moreover, from the resolvent equation, for $z \in \Omega_q \backslash \{2bq + \lambda\}$, we have

$$V_\theta(H_{0,\theta} - z)^{-1} = V_\theta(H_0 - i)^{-1} \left(1 + (H_0 - H_{0,\theta} + z - i)(H_{0,\theta} - z)^{-1}\right). \tag{3.5}$$

If we denote by p_q the orthogonal projection onto $\mathcal{H}_q := \ker(H_{0,\perp} - 2bq)$ we have

$$(H_{0,\theta} - z)^{-1} = \sum_{q\in\mathbb{N}} p_q \otimes (H_{0,\parallel}(\theta) + 2bq - z)^{-1}, \quad z \in \Omega_q \backslash \{2bq + \lambda\}. \tag{3.6}$$

Since

$$H_0 - H_{0,\theta} = I_\perp \otimes (H_{0,\parallel} - H_{0,\parallel}(\theta))$$

the operator $(H_0 - H_{0,\theta})(H_{0,\theta} - z)^{-1}$ is bounded. Consequently, according to Lemma 3.2 and equation (3.5), we obtain the lemma. □

Let us now introduce the function

$$z \in \Omega_q \setminus \{2bq + \lambda\} \longrightarrow d_\theta(z) = \det{}_2(I + T_{V,\theta}(z)), \tag{3.7}$$

with $T_{V,\theta}(z) = V_\theta(H_{0,\theta} - z)^{-1}$. The determinant $d_\theta(z)$ is well defined according to Lemma 3.3

Proposition 3.2. *Let* V *and* v_0 *as in Section 2. The resonances of* H *in* Ω_q *are the zeros of the regularized determinant* $d_\theta(z) = \det_2(I + T_{V,\theta}(z))$ *in* $\Omega_q \backslash \{2bq + \lambda\}$*, and are independent on* $\theta \in D_\epsilon$ *such that* $\Omega_q \cap \sigma_{ess}(H_\theta) = \{2bq + \lambda\}$*.*

If z_0 *is a resonance, there exists a holomorphic function* $f(z)$*, for* z *close to* z_0*, such that* $f(z_0) \neq 0$ *and*

$$\det{}_2(I + T_{V,\theta}(z)) = (z - z_0)^{l(z_0)} f(z),$$

with $0 < l(z_0) = \mathrm{mult}(z_0)$ *where* $\mathrm{mult}(z_0)$ *is the multiplicity of the resonance defined by (2.13).*

Proof. Since the operator $H_{0,\theta}$ has no spectrum in $\Omega_q \backslash \{2bq + \lambda\}$, we have

$$H_\theta - z = \left(I + V_\theta(H_{0,\theta} - z)^{-1}\right)(H_{0,\theta} - z). \tag{3.8}$$

Then, if $z \in \Omega_q \backslash \{2bq + \lambda\}$ is a resonance of H which is by definition a discrete eigenvalue of H_θ, the determinant $d_\theta(z) = \det_2(I + V_\theta(H_{0,\theta} - z)^{-1}) = \det_2(I + T_{V,\theta}(z))$ vanishes.

Let us recall that, if A is a bounded operator and B is a trace class operator on some separable Hilbert space, we have $\det(I + AB) = \det(I + BA)$. Moreover, for A bounded and B Hilbert-Schmidt, we have

$$\det{}_2(I + AB) = \det{}_2(I + BA). \tag{3.9}$$

Then, the function $d_\theta(z) = \det_2(I + T_{V,\theta}(z))$ coincide with $\det_2(I + T_{V,0}(z)) = \det_2(I + V(H_0 - z)^{-1})$ for $\theta \in \mathbb{R}$, $\mathrm{Im}\ z > 0$ and by uniqueness of the extension, it is independent on $\theta \in D_\epsilon$. Since the resonances of H are the zero of $d_\theta(z)$ the resonances are independents on $\theta \in D_\epsilon$.

In a neighborhood of a zero z_0 of $d_\theta(z)$ with multiplicity $l(z_0)$, we write $d_\theta(z) = (z - z_0)^{l(z_0)} G(z)$, where $G(z)$ is a holomorphic function in a neighborhood of z_0 with $G(z_0) \neq 0$. Then, by the definition of $l_0(z)$,

$$l_0(z) = \frac{1}{2i\pi} \int_\Gamma \partial_z \ln \det{}_2 \left(1 + T_{V,\theta}(z)\right) dz,$$

where Γ is a small positively oriented circle centered at z_0. Further, we have

$$\partial_z \ln \det \left(1 + T(z)\right) = \operatorname{tr} \left((1 + T(z))^{-1} \partial_z T(z)\right), \quad z \in \Omega_q$$

for any operator-valued holomorphic function $T(z)$ in the trace class S_1. Therefore,

$$\partial_z \ln \det{}_2 \left(1 + T_{V,\theta}(z)\right) = \operatorname{tr} \left((1 + T_{V,\theta}(z))^{-1} \partial_z T_{V,\theta}(z)\right) - \operatorname{tr} \left(\partial_z T_{V,\theta}(z)\right).$$

According to Lemma 3.3, $\partial_z T_{V,\theta}(z)$ is holomorphic in the trace class, then its integral on Γ vanishes and (3.8) yields

$$\begin{aligned} l_0(z) &= \frac{1}{2i\pi} \int_\Gamma \operatorname{tr} \left((H_\theta - z)^{-1} V_\theta (H_{0,\theta} - z)^{-1}\right) dz \\ &= -\frac{1}{2i\pi} \int_\Gamma \operatorname{tr} \left((H_\theta - z)^{-1} - (H_{0,\theta} - z)^{-1}\right) dz \\ &= \operatorname{rank} \frac{1}{2i\pi} \int_\Gamma \left((z - H_\theta)^{-1} - (z - H_{0,\theta})^{-1}\right) dz \\ &= \operatorname{rank} \frac{1}{2i\pi} \int_\Gamma (z - H_\theta)^{-1} dz. \end{aligned}$$

In the two latter equalities, we have used that the trace of the projector coincide with its rank and the integral of $(H_{0,\theta} - z)^{-1}$ on Γ vanishes since it is holomorphic in Ω_q. □

4. Upper bound for the number of resonances near $2bq + \lambda$

In this section, we establish an upper bound on the number of resonances in a ring of Ω_q centered at $2bq + \lambda$ (see (3.4)). For $z \in \Omega_q$, we write $z = 2bq + \lambda + \eta$ where η is a complex number in a domain centered at 0 and $0 < r < |\eta|$. Let $W = \sup_{x_3 \in C_{\epsilon,0}} |\langle x_3 \rangle^{\delta_\parallel} V|$. There exists a bounded function $M(x)$ such that

$$V(x) = W(X_\perp) \langle x_3 \rangle^{-2\delta_3} M(x), \quad \text{for} \quad \delta_3 = \delta_\parallel / 2.$$

According to the previous section, the resonances in Ω_q can be identified with the points $z \in \Omega_q$ where the determinant $d_\theta(z) = \det_2(I + T_{V,\theta}(z))$ vanishes. Using (3.9), we have

$$d_\theta(z) = \det{}_2(I + \mathcal{T}_{V,\theta}(z))$$

with

$$\mathcal{T}_{V,\theta}(z) = W^{\frac{1}{2}} M_\theta \langle x_3 \rangle_\theta^{-\delta_3} (H_{0,\theta} - z)^{-1} W^{\frac{1}{2}} \langle x_3 \rangle_\theta^{-\delta_3} \tag{4.1}$$

where $M_\theta = M(X_\perp, \phi_\theta(x_3))$ and $\langle x_3 \rangle_\theta := U_\theta \langle x_3 \rangle U_\theta^{-1} = (1 + (\phi_\theta(x_3))^2)^{\frac{1}{2}}$.

Using spectral theorem, for Im $z > 0$, we can write

$$(H_{0,\theta} - z)^{-1} = \sum_{j \in \mathbb{N}} p_j \otimes (H_{0,\parallel}(\theta) - z + 2bj)^{-1}. \tag{4.2}$$

In order to study the resonances near $2bq+\lambda$, we split $\mathcal{T}_{V,\theta}(z)$ into two parts:

$$\mathcal{T}_{V,\theta}(z) = \mathcal{T}^-_{J,\theta} + \mathcal{T}^+_{J,\theta},$$

where $\mathcal{T}^-_{J,\theta}(z) = \sum_{j\le J} \mathcal{T}_{j,\theta}$ and $\mathcal{T}^+_{J,\theta} = \sum_{j>J} \mathcal{T}_{j,\theta}$ for $J>q$ sufficiently large such that $\|\mathcal{T}^+_{J,\theta}\| < \frac{1}{8}$ and $\|\mathcal{T}^+_{J,0}\| < \frac{1}{8}$ (for that we use the h-pseudo-differential calculus and the spectral theorem). Here,

$$\mathcal{T}_{j,\theta} = M_\theta B_j \otimes \langle x_3\rangle_\theta^{-\delta_3} R_{j,\theta} \langle x_3\rangle_\theta^{-\delta_3},$$

with $B_j = W^{\frac{1}{2}} p_j W^{\frac{1}{2}}$ and $R_{j,\theta} = (H_{0,\|}(\theta) - z + 2bj)^{-1}$. The operator $\langle x_3\rangle^{-\delta_3} R_{j,0} \langle x_3\rangle^{-\delta_3}$ is of class trace (see [2]).

Further, let us decompose the self-adjoint operator B_j into a trace-class operator whose norm is bounded by $\varepsilon/2$ for some $\varepsilon>0$ and an operator of finite-rank independent on r, namely

$$B_j = B_j \mathbf{1}_{[0,\varepsilon/2]}(B_j) + B_j \mathbf{1}_{]\varepsilon/2,+\infty[}(B_j). \tag{4.3}$$

Then, for $j\neq q$, we have

$$\begin{aligned} \mathcal{T}_{j,\theta} &= M_\theta B_j \mathbf{1}_{[0,\varepsilon/2]}(B_j) \otimes \langle x_3\rangle_\theta^{-\delta_3} R_{j,\theta} \langle x_3\rangle_\theta^{-\delta_3} + M_\theta B_j \mathbf{1}_{]\varepsilon/2,+\infty[}(B_j) \otimes \langle x_3\rangle_\theta^{-\delta_3} R_{j,\theta} \langle x_3\rangle_\theta^{-\delta_3} \\ &= \mathcal{T}^<_{j,\theta} + \mathcal{T}^>_{j,\theta}. \end{aligned}$$

Let us now analyze the term $\mathcal{T}_{q,\theta}$. Denote by $p_\|(\theta)$ the spectral projection onto $\mathrm{Ker}(H_{0,\|}(\theta) - \lambda)$. We have $p_\|(\theta)\cdot = \langle \cdot, \psi_{\bar\theta}\rangle \psi_\theta$ with $\psi_\theta = U_\theta^{-1}\psi$ and ψ is an eigenfunction satisfying

$$H_{0,\|}\psi = \lambda\psi, \quad \|\psi\|_{L^2(\mathbb{R})} = 1, \quad \psi = \bar\psi \text{ on } \mathbb{R}.$$

Then we have, for $\eta = z - 2bq - \lambda$

$$\begin{aligned} \mathcal{T}_{q,\theta} &= M_\theta B_q \otimes \langle x_3\rangle_\theta^{-\delta_3} R_{q,\theta} p_\|(\theta) \langle x_3\rangle_\theta^{-\delta_3} + M_\theta B_q \otimes \langle x_3\rangle_\theta^{-\delta_3} R_{q,\theta}(I - p_\|(\theta)) \langle x_3\rangle_\theta^{-\delta_3} \\ &= -\frac{1}{\eta}\tau_q + \tilde{\mathcal{T}}_{q,\theta}, \end{aligned}$$

with $\tau_q = M_\theta B_q \otimes \langle x_3\rangle_\theta^{-\delta_3} p_\|(\theta) \langle x_3\rangle_\theta^{-\delta_3}$. We also have

$$\begin{aligned} \tilde{\mathcal{T}}_{q,\theta} &= M_\theta B_q \mathbf{1}_{[0,\varepsilon/2]}(B_q) \otimes \langle x_3\rangle_\theta^{-\delta_3} R_{q,\theta}(I - p_\|(\theta)) \langle x_3\rangle_\theta^{-\delta_3} \\ &\quad + M_\theta B_q \mathbf{1}_{]\varepsilon/2,+\infty[}(B_q) \otimes \langle x_3\rangle_\theta^{-\delta_3} R_{q,\theta}(I - p_\|(\theta)) \langle x_3\rangle_\theta^{-\delta_3} \\ &= \tilde{\mathcal{T}}^<_{q,\theta} + \tilde{\mathcal{T}}^>_{q,\theta}. \end{aligned}$$

We denote by $A^>(z) = \tilde{\mathcal{T}}^>_{q,\theta} + \sum_{j\neq q,\ j\le J} \mathcal{T}^>_{j,\theta}$, and $A^<(z) = \tilde{\mathcal{T}}^<_{q,\theta} + \sum_{j\neq q,\ j\le J} \mathcal{T}^<_{j,\theta} + \mathcal{T}^+_{J,\theta}$. Then, we have

$$\mathcal{T}_{V,\theta}(z) = -\frac{1}{\eta}\tau_q + A^>(z) + A^<(z). \tag{4.4}$$

We decompose the operator τ_q into a trace-class operator whose norm is bounded by $r\nu^{-1}$ for $\nu>0$ and an operator of finite-rank:

$$\begin{aligned} \tau_q &= M_\theta B_q \mathbf{1}_{[0,r\nu^{-1}]}(B_q) \otimes \langle x_3\rangle_\theta^{-\delta_3} p_\|(\theta) \langle x_3\rangle_\theta^{-\delta_3} + M_\theta B_q \mathbf{1}_{]r\nu^{-1},+\infty[}(B_q) \otimes \langle x_3\rangle_\theta^{-\delta_3} p_\|(\theta) \langle x_3\rangle_\theta^{-\delta_3} \\ &= \tau_{q,1} + \tau_{q,2}. \end{aligned}$$

If we take ε sufficiently small and $\nu>0$ sufficiently large, we have the two following lemmas

Lemma 4.1. *Let $r_0 > 0$. For $z = 2bq + \lambda + \eta \in \Omega_q$ and $0 < r < \operatorname{Im} \eta < r_0$, we have*

$$\mathcal{T}_{V,\theta}(z) = \mathcal{R}_\theta(z) + \mathcal{E}_\theta(z), \tag{4.5}$$

where the operator $\mathcal{R}_\theta(z) \in S_1$ is the holomorphic operator defined by

$$\mathcal{R}_\theta(z) = -\frac{1}{\eta}\tau_{q,2} + A^{>}(z). \tag{4.6}$$

The operator $\mathcal{E}_\theta(z) \in S_2$ is the holomorphic operator defined by

$$\mathcal{E}_\theta(z) = -\frac{1}{\eta}\tau_{q,1} + A^{<}(z).$$

Moreover, $\mathcal{E}_\theta(z)$ satisfies the following estimate

$$\|\mathcal{E}_\theta(z)\| < \frac{3}{4}. \tag{4.7}$$

Using the limiting absorption principle for $\langle x_3\rangle^{-\delta_3} R_{j,0} \langle x_3\rangle^{-\delta_3}$, $j < q$, the decomposition (4.4) is also available for $\theta = 0$, and we have

Lemma 4.2. *For $z = 2bq + \lambda + \eta \in \Omega_q$ and $0 < r < \operatorname{Im} \eta < r_0$, we have*

$$\mathcal{T}_{V,0}(z) = \mathcal{R}_0(z) + \mathcal{E}_0(z), \tag{4.8}$$

with $\mathcal{R}_0(z) = \mathcal{R}_\alpha(z)\Big|_{\alpha=0}$ and $\mathcal{E}_0(z) = \mathcal{E}_\alpha(z)\Big|_{\alpha=0}$. Moreover, $\mathcal{E}_0(z)$ satisfies the following estimate

$$\|\mathcal{E}_0(z)\| < \frac{3}{4}. \tag{4.9}$$

Proposition 4.1. *Let V and v_0 as in Section 2. For $0 < r < |\eta| < r_0$ with r_0 sufficiently small, $z = 2bq + \lambda + \eta \in \Omega_q$ is a resonance of H if and only if z is a zero of*

$$\mathcal{D}_\theta(z, r) = \det\left(I + \mathcal{R}_\theta(z)(I + \mathcal{E}_\theta(z))^{-1}\right), \tag{4.10}$$

where $\mathcal{R}_\theta(z)$ is a class trace operator. Moreover, for $\operatorname{Im} z > 0$, the determinant $\mathcal{D}_\theta(z, s)$ coincides with

$$\mathcal{D}_0(z, s) = \det\left(I + \mathcal{R}_0(z)(I + \mathcal{E}_0(z))^{-1}\right).$$

Proof. By Proposition 3.2, for $r < |\eta| < r_0$, z is a resonance of H if and only if z is a zero of $d_\theta(z) = \det_2(I + \mathcal{R}_\theta(z) + \mathcal{E}_\theta(z))$. We can write

$$d_\theta(z) = \det(I + \mathcal{R}_\theta(z)(I + \mathcal{E}_\theta(z))^{-1}) \det((I + \mathcal{E}_\theta(z))e^{-\mathcal{T}_{V,\theta}(z)}).$$

According to (4.7), we have $\det((I + \mathcal{E}_\theta(z))e^{-\mathcal{T}_{V,\theta}(z)}) \neq 0$, and then the zeros of $d_\theta(z)$ are the zeros of $\mathcal{D}_\theta(z, r)$ with the same multiplicity.

Using the theory of h-pseudo-differential operators (see [7]), the resolvent $(H_{0,\|}(\alpha) - z + 2bj)^{-1}$ is uniformly bounded for $\alpha \in D_\epsilon^+$, $j \leq J$ and $\operatorname{Im} z > 0$ sufficiently large. Then for z fixed with $\operatorname{Im} z \gg 1$, $\theta \to \mathcal{D}_\theta(z, r)$ is a holomorphic function on D_ϵ^+ (since the construction of $\mathcal{E}_\theta(z)$ is not uniform with respect to θ, this property is not clear for $\operatorname{Im} z > 0$ near the real axis). Using that for $\theta \in \mathbb{R}$

$$\mathcal{D}_\theta(z, r) = \det\left(I + U_\theta \mathcal{R}_0(z)(I + \mathcal{E}_0(z))^{-1} U_\theta^{-1}\right) = \det\left(I + \mathcal{R}_0(z)(I + \mathcal{E}_0(z))^{-1}\right),$$

the function $\theta \longmapsto \mathcal{D}_\theta(z, r)$ is constant on the real axis. Thus, by uniqueness of the extension on θ, the determinant $\mathcal{D}_\theta(z, r)$ coincides with $\mathcal{D}_0(z, r)$ for $\operatorname{Im} z \gg 1$ and $\theta \in D_\epsilon^+$. Moreover, since for θ fixed in D_ϵ^+, $z \longmapsto \mathcal{D}_\theta(z,r)$ and $z \longmapsto \mathcal{D}_0(z,r)$ are well defined and holomorphic for Im $z>0$ (see Lemmas 4.1, 4.2), $\mathcal{D}_\theta(z,r)$ coincides with $\mathcal{D}_0(z,r)$ for Im $z>0$. □

Since there exists an operator $C: \ L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ such that $B_q = C^*C$ and $CC^* = p_q W(X_\perp) p_q$ (see [9] and [4]), then for any $r>0$ we have

$$n_+(r, B_q) = n_+(r, p_q W p_q), \tag{4.11}$$

where, for a compact self-adjoint operator A and $r>0$, we set $n_+(r,A) = \operatorname{rank} \mathbf{1}_{(r,+\infty)}(A)$.

Lemma 4.3. *For $z = 2bq + \lambda + \eta \in \Omega_q$ and $0 < r < |\eta| < r_0$, there exists $\nu > 0$ such that*

$$\mathcal{D}_\theta(z,r) = O(1) \exp\left(O(n_+(r, \nu p_q W p_q) + 1)|\ln r|\right). \tag{4.12}$$

Proof. Since $z \to A^>(z)$ is holomorphic near $z = 2bq+\lambda$ or $\eta = 0$ with values in S_1, for r_0 sufficiently small, there exist a finite-rank operator $A_0^>$ independent of z and $\tilde{A}^>(z)$ holomorphic in S_1 near $z = 2bq + \lambda$ with $\|\tilde{A}^>(z)\|_{\mathrm{tr}} \le \frac{1}{8}$, $|\eta| \le r_0$ such that

$$A^>(z) = A_0^> + \tilde{A}^>(z). \tag{4.13}$$

Since we have $\|\tilde{A}^>(z)\|_{\mathrm{tr}} \le \frac{1}{8}$, for $0 < r < |\eta| < r_0$,

$$\det\Big(I + \tilde{A}^>(z)(I+\mathcal{E}_\theta)^{-1}\Big) \ne 0.$$

It follows that for $0 < r < |\eta| < r_0$, the zeros of $\mathcal{D}_\theta(z,r)$ are the zeros of

$$D_\theta(z,r) = \det\left(I + K_\theta(z,r)\right), \tag{4.14}$$

with

$$K_\theta(z,r) = \big(-\frac{1}{\eta}\tau_{q,2} + A_0^>\big)\Big(I + \mathcal{E}_\theta + \tilde{A}^>(z)\Big)^{-1}. \tag{4.15}$$

We recall that $\tau_{q,2} = M_\theta B_q \mathbf{1}_{]r\nu^{-1},+\infty[}(B_q) \otimes \langle x_3\rangle_\theta^{-\delta_3} p_\parallel(\theta) \langle x_3\rangle_\theta^{-\delta_3}$. Since the rank of the projector $p_\parallel(\theta)$ is equal to 1, the rank of the operator $K_\theta(z,r)$ is bounded by $O(n_+(r\nu^{-1}, B_q)+1) = O(n_+(r, \nu p_q W p_q) + 1)$ (see (4.11)) and its norm is bounded by $O(|\eta|^{-1}) = O(r^{-1})$ (see also Proposition 4.1).

By the properties of $K_\theta(z,r)$ for $0 < r < |\eta| = |z - 2bq - \lambda| < r_0$, we have

$$D_\theta(z,r) = \prod_{j=1}^{O(n_+(r,\nu p_q W p_q)+1)} (1 + \lambda_j(z,r)) = O(1)\exp\left(O(n_+(r,\nu p_q W p_q)+1)|\ln r|\right), \tag{4.16}$$

uniformly with respect to (z,r), where $\lambda_j(z,r)$ are the eigenvalues of $K_\theta(z,r)$ which satisfy $\lambda_j(z,r) = O(|r|^{-1})$. Since

$$\mathcal{D}_\theta(z,r) = D_\theta(z,r)\det\Big(I + \tilde{A}^>(z)(I+\mathcal{E}_\theta)^{-1}\Big), \tag{4.17}$$

and the norm of $\det\Big(I + \tilde{A}^>(z)(I+\mathcal{E}_\theta)^{-1}\Big)$ is uniformly bounded, the lemma follows. □

Lemma 4.4. *For $z = 2bq + \lambda + \eta \in \Omega_q$, and $0 < r < \mathrm{Im}\ \eta < r_0$, there exists $\nu > 0$ such that*

$$|\mathcal{D}_0(z,r)| \ge C\exp\left(-C(n_+(r,\nu p_q W p_q)+1)|\ln r|\right), \tag{4.18}$$

uniformly with respect to (z,r).

Proof. Repeating the argument (4.13) in the proof of Lemma 4.3 for $\theta = 0$ and using Lemma 4.2, there exist a finite-rank operator $K_0(z, r)$ satisfying

$$\text{rank } K_0(z,r) = O\left(n_+(r, \nu p_q W p_q) + 1\right), \qquad \|K_0(z,r)\| = O(r^{-1}),$$

uniformly with respect to $r < |\eta| < r_0$ and an operator $\varepsilon(z)$ such that

$$\mathcal{D}_0(z,r) = \det\left(I + K_0(z,r)\right) \det(I + \varepsilon(z))$$

with $\|\varepsilon(z)\|_{\text{tr}} \leq \frac{3}{4}$ (see (4.17)).

Let us now estimate $D_0(z,r)^{-1} = \det\left(I + K_0(z,r)\right)^{-1}$. For $\text{Im} z > r$, we have

$$D_0(z,r)^{-1} = \det\left((I + K_0)^{-1}\right) = \det\left(I - K_0(I + K_0)^{-1}\right). \tag{4.19}$$

By the construction of K_0, it satisfies

$$I + K_0 = \Big(I + \mathcal{T}_{V,0}(z)\Big)\Big(I + \widetilde{\mathcal{E}}_0\Big)^{-1},$$

with $\widetilde{\mathcal{E}}_0$ an operator bounded as $\|\widetilde{\mathcal{E}}_0\| < \frac{7}{8}$ and

$$\mathcal{T}_{V,0}(z) = \mathcal{T}_{V,\theta}(z)\Big|_{\theta=0} = W^{\frac{1}{2}} \langle x_3\rangle^{-\delta_3} M(H_0 - z)^{-1} \langle x_3\rangle^{-\delta_3} W^{\frac{1}{2}}.$$

Using the resolvent equation, the operator $I + \mathcal{T}_{V,0}(z)$ is invertible for Im $z > r > 0$, and

$$\Big(I + \mathcal{T}_{V,0}(z)\Big)^{-1} = I - W^{\frac{1}{2}} \langle x_3\rangle^{-\delta_3} M(H - z)^{-1} \langle x_3\rangle^{-\delta_3} W^{\frac{1}{2}}.$$

Then $I + K_0$ is invertible for Im $z > r$ and from the spectral theorem

$$\|(I + K_0)^{-1}\| = O(1 + \|W^{\frac{1}{2}} \langle x_3\rangle^{-\delta_3} M(H - z)^{-1} \langle x_3\rangle^{-\delta_3} W^{\frac{1}{2}}\|) = O(1 + \frac{1}{|\text{Im } z|}). \tag{4.20}$$

Since the operator K_0 is of finite-rank $O(n_+(r, \nu p_q W p_q) + 1)$ and using (4.19) and (4.20), we obtain the lemma. □

The following lemma contains a version of the well known Jensen inequality which is suitable for our purposes (see [4] for the proof).

Lemma 4.5. *Let Ω be a simply connected domain of $\mathbb{C}$ and let g be a holomorphic function in Ω with continuous extension to $\bar{\Omega}$. Assume there exists $z_0 \in \Omega$ such that $g(z_0) \neq 0$ and $g(z) \neq 0$ for $z \in \partial\Omega$. Let $z_1, z_2, \ldots, z_N \in \Omega$ be the zeros of g repeated according to their multiplicity. For any domain $\Omega' \subset\subset \Omega$, there exists $C > 0$ such that $N(\Omega', g)$, the number of zeros z_j of g contained in Ω', satisfies*

$$N(\Omega', g) \leq C \left(\int_{\partial\Omega} |\ln|g(z)||dz + |\ln|g(z_0)|| \right).$$

Now, applying this lemma to the function $g(z) = \mathcal{D}_\theta(z,r)$, we deduce from (4.12), (4.18) and Proposition 4.1 the upper bound on the number of resonances near $2bq + \lambda$ stated in Theorem 2.1.

5. Spectral shift function and resonances

In this section, we represent the derivative of the spectral shift function (SSF) near $2bq+\lambda$ for $q \in \mathbb{N}$ as a sum of a harmonic measure related to resonances, and the imaginary part of a holomorphic function. As in [15], [6], [8], and [4] such representation justifies the Breit-Wigner approximation and implies a trace formula. We deduce also an asymptotic expansion of the SSF near $2bq+\lambda$; in the case of $v_0 = 0$, this expansion is given in [4]. For a positive potentials V which decay slowly enough as $\|X_\perp\| \to \infty$, this expansion yields a remainder estimate for the corresponding asymptotic relations obtained in [9].

In order to obtain such a representation formula, the first step is the factorization of the generalized perturbation determinant. To this end, we need some complex-analysis results due to Sjöstrand, summarized in the following

Proposition 5.1. *(see [18], [19]) Let Ω be an open simply connected domain in $\mathbb{C}\setminus\{0\}$ such that $\Omega\cap\mathbb{R}$ is an interval. Let $z \longrightarrow F(z,h)$, $0<h<h_0$, be a family of holomorphic functions in Ω containing a number $N(h)$ of zeros. We suppose that,*

$$F(z,h) = O(1)e^{O(1)N(h)}, \quad z \in \Omega,$$

and for all $\rho>0$ small enough, there exists $C>0$ such that for all $z \in \Omega_\rho := \Omega\cap\{\operatorname{Im} z > \rho\}$ we have

$$|F(z,h)| \geq e^{-CN(h)}.$$

Then for each open simply connected subset $\tilde{\Omega} \Subset \Omega$ there exists $g(.,h)$ holomorphic in $\tilde{\Omega}$ such that

$$F(z,h) = \prod_{j=1}^{N(h)} (z-z_j)e^{g(z,h)}, \quad \partial_z g(z,h) = O(N(h)), \quad z \in \tilde{\Omega}.$$

Let $\widetilde{\Omega} \subset\subset \Omega$ be open relatively compact subset of $\mathbb{C}\setminus\{0\}$. We assume that these sets are independent of r and that $\widetilde{\Omega}$ is simply connected. Also assume that the intersections between $\widetilde{\Omega}$ and $\mathbb{R}$ is a non empty interval I. With these hypotheses we can obtain the following representation of the regularized spectral shift function near $2bq+\lambda$.

Theorem 5.1. [Representation formula] *Suppose that V and v_0 satisfy the hypotheses of Section 2. For $\widetilde{\Omega} \subset\subset \Omega$ and I as above, there exists a function g holomorphic in Ω, such that for $\mu \in 2bq+\lambda+rI$, we have*

$$\xi_2'(\mu) = \frac{1}{\pi r}\operatorname{Im} g'\Big(\frac{\mu-2bq-\lambda}{r}, r\Big) - \sum_{\substack{w\in \operatorname{Res}(H)\cap 2bq+\lambda+r\Omega \\ \operatorname{Im} w\neq 0}} \frac{-\operatorname{Im} w}{\pi|\mu-w|^2} - \sum_{w\in \operatorname{Res}(H)\cap 2bq+\lambda+rI} \delta(\mu-w)$$

$$-\frac{1}{\pi}\operatorname{Im}\,\operatorname{tr}\left(\partial_z \mathcal{T}_{V,\theta}(\mu)\right), \tag{5.1}$$

where $g(z,r)$ satisfies the estimate

$$g(z,r) = O\left(n_+(r,\nu p_q W p_q)|\ln r| + \widetilde{n}_1(r/\nu) + \widetilde{n}_2(r/\nu)\right) = O(|\ln r| r^{-\frac{2}{\delta_\perp}}) \tag{5.2}$$

uniformly with respect to $0<r<r_0$ and $z \in \widetilde{\Omega}$, with $\widetilde{n}_p$, $p=1,2$, defined by (2.21).

Proof. First, using the resolvent equation, we have

$$\det{}_2\left((H-z)(H_0-z)^{-1}\right) = \det{}_2\left(I + T_{V,0}(z)\right).$$

Using (3.9), the last determinant coincides with $d_\theta(z) = \det_2(I + \mathcal{T}_{V,\theta}(z))$ for $\theta \in \mathbb{R}$, where $\mathcal{T}_{V,\theta}(z)$ is defined in (4.1). According to previous section, $\mathcal{T}_{V,\theta}(z)$ is extended on $\theta \in D_\epsilon$ and

$$d_\theta(z) = D_\theta(z,r)\det\left((I + \tilde{A}^{>}(z) + \mathcal{E}_\theta(z))e^{-\mathcal{T}_{V,\theta}(z)}\right)$$

where $D_\theta(z,r)$ is defined by (4.14).

By the properties of $\tilde{A}^{>}(z)$ (see (4.13)), for $\widetilde{K}(z) = \tilde{A}^{>}(z) + \mathcal{E}_\theta(z)$, the difference $\mathcal{T}_{V,\theta}(z) - \widetilde{K}(z) = -\frac{1}{\eta}\tau_{q,2} + A_0^{>}$ is a finite-rank operator. Using the fact that $\det_2(I+B) = \det(I+B)e^{-\mathrm{tr}B}$ for a trace-class operator B, we have

$$\det\left((I + \tilde{A}^{>}(z) + \mathcal{E}_\theta(z))e^{-\mathcal{T}_{V,\theta}(z)}\right) = \det{}_2(I + \widetilde{K}(z))e^{-\mathrm{tr}(\mathcal{T}_{V,\theta}(z) - \widetilde{K}(z))}, \tag{5.3}$$

where $\det_2(I + \widetilde{K}(z))$ is a non-vanishing holomorphic function. Since $\tilde{A}^{>}(z)$ is holomorphic in S_2 and

$$\|\frac{B_q}{r}\mathbf{1}_{[0,r]}(B_q)\|_2^2 = -\int_0^r \frac{u^2}{r^2}dn_+(u,B_q) = \widetilde{n}_2(r),$$

we have

$$\|\widetilde{K}(z)\|_2^2 = O(\widetilde{n}_2(r/\nu)),$$

which implies that $|\det_2(I + \widetilde{K}(z))| = O(\exp(\widetilde{n}_2(r/\nu)))$. Using moreover that $\|\widetilde{K}(z)\| < 1$, we have also $|\det(I + \widetilde{K}(z))|^{-1} = O(\exp(\widetilde{n}_+(r/\nu)))$. Then there exists $g_1(\cdot, r)$ holomorphic on Ω such that, $\frac{d}{dz}g_1(z,r) = O(\widetilde{n}_2(r/\nu))$, on $\widetilde{\Omega}$, and

$$\det{}_2(I + \widetilde{K}(z)) = e^{g_1(z,r)}.$$

We consider now the functions

$$F_\theta : \ z \in \Omega \longmapsto D_\theta(z,r).$$

The functions F_θ are holomorphic in Ω and $\widetilde{w} \in \Omega$ is a zero of F_θ if and only if $z = 2bd + \lambda + \widetilde{w}r$ is a resonance of H. Then applying Proposition 5.1 to $F = F_\theta$ with $h = r$, $N(r) = n_+(r, \nu p_q W p_q)|\ln r|$, we obtain existence of functions g_0 holomorphic in Ω such that for $z \in \Omega$, we have the following factorization:

$$F_\theta = \prod_{w \in \mathrm{Res}(H) \cap 2bq + \lambda + r\Omega} \left(\frac{zr + 2bq + \lambda - w}{r}\right) e^{g_0(z,r)}, \tag{5.4}$$

with

$$\frac{d}{dz}g_0(z,r) = O(n_+(r, \nu p_q W p_q)|\ln r|), \tag{5.5}$$

uniformly with respect to $z \in \widetilde{\Omega}$.

Then by definition of ξ_2 (see (2.17)), for $\mu \in 2bq+\lambda+r(\Omega\cap\mathbb{R})$ we obtain

$$\xi_2'(\mu) = \frac{1}{\pi r}\mathrm{Im}\partial_z(g_0+g_1)(\frac{\mu-2bq-\lambda}{r},r) - \sum_{\substack{w\in\mathrm{Res}(H)\cap 2bq+\lambda+r\Omega \\ \mathrm{Im}\, w\neq 0}} \frac{-\mathrm{Im}\ w}{\pi|\mu-w|^2} - \sum_{w\in\mathrm{Res}(H)\cap 2bq+\lambda+rI}\delta(\mu-w)$$
$$+\frac{1}{\pi r}\mathrm{Im\ tr}\left(\partial_z\widetilde{K}(\frac{\mu-2bq-\lambda}{r})\right) - \frac{1}{\pi}\mathrm{Im\ tr}\left(\partial_z\mathcal{T}_{V,\theta}(\mu)\right).$$

Then, we conclude the proof of Theorem 5.1 with $g=g_0+g_1+g_2$ taking

$$g_2(z,r) = \mathrm{tr}(\widetilde{K}(z)),$$

which satisfies $\frac{d}{dz}g_2(z,r) = O(\widetilde{n}_1(r/\nu))$. □

Lemma 5.1. *On* $\mathbb{R}\setminus(\{2b\mathbb{N}+\lambda\}\cup\{2b\mathbb{N}\})$*, for* $\theta\in D_\epsilon^+$*,* $\mathrm{Im}\ \theta>0$*, we have*

$$\xi' = \xi_2' + \frac{1}{\pi}\mathrm{Im\ tr}\left(\partial_z T_{V,\theta}(\cdot)\right), \tag{5.6}$$

where $T_{V,\theta}(z) = V_\theta(H_{0,\theta}-z)^{-1}$.

Proof. We follow the proof of [4, Lemma 8]. From (2.18), we have only to prove

$$\mathrm{tr}\left(\frac{d}{d\varepsilon}f(H_0+\varepsilon V)|_{\varepsilon=0}\right) = -\frac{1}{\pi}\int_{\mathbb{R}} f(\rho)\,\mathrm{Im\ tr}\left(\partial_z T_{V,\theta}(\rho)\right)d\rho,$$

for any $f\in C_0^\infty(\mathbb{R}\setminus(\{2b\mathbb{N}+\lambda\}\cup\{2b\mathbb{N}\}))$. As in [4, Lemma 8], we use the Helffer-Sjöstrand formula and we have

$$\frac{d}{d\varepsilon}f(H_0+\varepsilon V)|_{\varepsilon=0} = \frac{1}{\pi}\int_{\mathbb{C}}\overline{\partial}\widetilde{f}(z)(H_0-z)^{-1}V(H_0-z)^{-1}L(dz),$$

for $\widetilde{f}\in C_0^\infty(\mathbb{R}^2)$ an almost analytic extension of f, (i.e. $\widetilde{f}_{|\mathbb{R}}=f$ and $\overline{\partial}_\lambda\widetilde{f}(\lambda)=O(|\mathrm{Im}\lambda|^\infty)$) and $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}$.

Let us now define

$$\sigma_\pm(z) = \mathrm{tr}((H_0-z)^{-1}V(H_0-z)^{-1}),\quad \pm\mathrm{Im}\ (z)>0.$$

The functions $\sigma_\pm(z)$ satisfy the relation

$$\sigma_-(z) = \overline{\sigma_+(\overline{z})},\quad \mathrm{Im}\ (z)<0. \tag{5.7}$$

For $\theta\in\mathbb{R}$, the operator

$$(H_0-z)^{-1}V(H_0-z)^{-1},$$

is unitarily equivalent to the operator

$$(H_{0,\theta}-z)^{-1}V_\theta(H_{0,\theta}-z)^{-1}.$$

Using the cyclicity of the trace, we deduce

$$\sigma_\pm(z) = \mathrm{tr}(\partial_z T_{V,\theta}(z)),\quad \pm\mathrm{Im}\ (z)>0,\ \theta\in\mathbb{R}. \tag{5.8}$$

From Lemma 3.3, the function $\theta \longrightarrow \partial_z T_{V,\theta}(z)$ is holomorphic on D_ϵ^+ with value in the trace class for $\mathrm{Im}\ (z)>0$. Then, (5.8) is also available for $\theta\in D_\epsilon^+$ and taking $\mathrm{Im}\ \theta>0$, $z\longrightarrow\sigma_+(z)$ can be extended to $\mathbb{R}\setminus(\{2b\mathbb{N}+\lambda\}\cup\{2b\mathbb{N}\})$. According to (5.7), $\sigma_-(z)$ satisfies the same property of $\sigma_+(z)$.

Hence, $\frac{d}{d\varepsilon}f(H_0+\varepsilon V)|_{\varepsilon=0}$ is of trace class, and

$$\operatorname{tr}\left(\frac{d}{d\varepsilon}f(H_0+\varepsilon V)|_{\varepsilon=0}\right)=\frac{1}{\pi}\int_{\mathrm{Im}\ (z)>0}\overline{\partial}\widetilde{f}(z)\sigma_+(z)L(dz)+\frac{1}{\pi}\int_{\mathrm{Im}\ (z)<0}\overline{\partial}\widetilde{f}(z)\sigma_-(z)L(dz).$$

Then the Green formula yields the lemma. □

We will deduce Theorem 2.2 from Theorem 5.1 by using the previous lemma and the cyclicity of the trace.

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