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On a Coupling of Solutions to the Interface Stochastic Differential Equation on a Star Graph

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Abstract Inspired by Tsirelson's proof of the non-Brownian character of Walsh Brownian motion filtration on three or more rays, we prove some results on a particular coupling of solutions to the interface stochastic differential equation on a star graph, recently introduced in Hajri and Raimond (Stoch Process Appl 126:33–65, 2016). This coupling consists of two solutions which are independent given the driving Brownian motion. As a consequence, we deduce that if the star graph contains three or more rays, the argument of the solution at a fixed time is independent of the driving Brownian motion.

Keywords Tsirelson theorem · Non-Brownian filtration · Walsh Brownian motion · Stochastic equations on graphs

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1 Introduction and Main Results

A filtration $(\mathcal{F}_t)_t$ has the Brownian representation property (BRP) if there exists a Brownian motion B such that every $(\mathcal{F}_t)_t$-martingale is a stochastic integral of B. In 1979 Yor posed the reverse problem, i.e., whether a filtration having the BRP is

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necessarily Brownian [13]. At the end of his paper [12], Walsh suggested the study of a Markov process with state space

$$G = \bigcup_{j=1}^{N} E_j;\ \ E_j = \{re^{i\theta_j} : r \geq 0\}$$

where θ_j are given angles. This process, called since then Walsh Brownian motion (WBM), behaves like a standard Brownian motion on each ray; and at 0 it makes excursions with probability p_j on $E_j \setminus \{0\}$. Later on, a detailed study of WBM was given in [1]. In particular, it was shown that WBM is a strong Markov process with Feller semigroup and that the natural filtration $(\mathcal{F}_t^Z)_t$ of a WBM Z has the BRP with respect to the Brownian motion B given by the martingale part of $|Z|$, the geodesic distance between Z and 0.

After nearly two decades, a negative answer to Yor's question was finally given by Tsirelson [11]. The result proved by Tsirelson is the following

Theorem 1.1 *If $(\mathcal{G}_t)_t$ is a Brownian filtration, i.e., a filtration generated by a finite or infinite family of independent standard Brownian motions, there does not exist any $(\mathcal{G}_t)_t$-WBM ($(\mathcal{G}_t)_t$-Markov process with semigroup P, the Feller semigroup of WBM) on a star graph with three or more rays.*

To prove Theorem 1.1, Tsirelson performs a beautiful reasoning by contradiction. Suppose there exists a Brownian motion B such $Z = F(B)$ is a WBM with $N \geq 3$ rays. Let $Z^r = F(B^r)$ where $B^r = rB + \sqrt{1-r^2}B'$ with B' an independent copy of B. Then, it is shown that $\mathbb{E}[d(Z_t^r, Z_t)]$ converges to 0. However, Tsirelson is able to prove that $\mathbb{E}[d(Z_t^r, Z_t)] > c > 0$ with c not depending on r.

In the present paper we are interested in a simple stochastic differential equation (SDE) on G whose solutions are WBMs. This SDE is the interface SDE introduced in [6] and driven by an N-dimensional Brownian motion $W = (W^1, \ldots, W^N)$. While moving inside E_i, a solution to this equation follows W^i so that the origin can be seen as an interface at the intersection of the half lines. For $N = 2$, the interface SDE is identified with

$$\mathrm{d}X_t = 1_{\{X_t > 0\}}\mathrm{d}W_t^1 + 1_{\{X_t \leq 0\}}\mathrm{d}W_t^2 \tag{1}$$

Equation (1) has a unique strong solution [7,10]. Not knowing Theorem 1.1, one could have the intuition that similarly to $N = 2$, solutions are also strong ones for $N \geq 3$. The theorem implies this cannot be the case.

The main result proved in [6] was the existence of a stochastic flow of mappings, unique in law and a Wiener stochastic flow [8] which solve the interface SDE. The problem of finding the flows of kernels which "interpolate" between these two particular flows was left open in [6]. The answer to this question needs a complete understanding of weak solutions of this equation.

The purpose of the present paper is to establish new results on weak solutions of the interface SDE in the case $N \geq 3$. These results are very different from the case $N = 2$. Our proofs are largely inspired by Tsirelson proof of Theorem 1.1.

1.1 Notations

This paragraph contains the main notations and definitions which will be used throughout the paper.

Let (G, d) be a metric star graph with a finite set of rays $(E_i)_{1\leq i\leq N}$ and origin denoted by 0. This means that (G, d) is a metric space, $E_i \cap E_j = \{0\}$ for all $i \neq j$ and for each i, there is an isometry $e_i : [0, \infty[\to E_i$. We assume d is the geodesic distance on G in the sense that $d(x, y) = d(x, 0) + d(0, y)$ if x and y do not belong to the same E_i.

For any subset A of G, we will use the notation A^* for $A\backslash\{0\}$. Also, we define the function $\varepsilon : G^* \to \{1, \ldots, N\}$ by $\varepsilon(x) = i$ if $x \in E_i^*$.

Let $C_b^2(G^*)$ denote the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in [1, N]$, $f \circ e_i$ is C^2 on $]0, \infty[$ with bounded first and second derivatives both with finite limits at $0+$. For $x = e_i(r) \in G^*$, set $f'(x) = (f \circ e_i)'(r)$ and $f''(x) = (f \circ e_i)''(r)$.

Let $p_1, \ldots, p_N \in (0, 1)$ such that $\sum_{i=1}^N p_i = 1$ and define

$$\mathcal{D} = \left\{ f \in C_b^2(G^*) : \sum_{i=1}^{N} p_i (f \circ e_i)'(0+) = 0 \right\}.$$

For $f \in C_b^2(G^*)$, we will take the convention $f'(0) = \sum_{i=1}^N p_i (f \circ e_i)'(0+)$ and $f''(0) = \sum_{i=1}^N p_i (f \circ e_i)''(0+)$ so that $\mathcal{D}$ can be written as $\mathcal{D} = \left\{ f \in C_b^2(G^*) : f'(0) = 0 \right\}$. We are now in position to recall the following

Definition 1.2 *A solution of the interface SDE (I) on G with initial condition $X_0 = x$ is a pair of processes (X, W) defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$ such that*

(i) *$W = (W^1, \ldots, W^N)$ is a standard $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^N$;*
(ii) *X is an $(\mathcal{F}_t)$-adapted continuous process on G;*
(iii) *For all $f \in \mathcal{D}$,*

$$f(X_t) = f(x) + \sum_{i=1}^{N} \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} \mathrm{d}W_s^i + \frac{1}{2} \int_0^t f''(X_s) \mathrm{d}s \tag{2}$$

To emphasize on the filtration $(\mathcal{F}_t)_t$, we will sometimes say (X, W) is an $(\mathcal{F}_t)_t$-solution. It has been proved in [6] (Theorem 2.3) that for all $x \in G$, (I) admits a solution (X, W) with $X_0 = x$, the law of (X, W) is unique, and X is an $(\mathcal{F}_t)$- WBM on G. We will denote by Q_x the law of a solution (X, W) with $X_0 = x$.

Tsirelson theorem 1.1 combined with Theorem 2.3 in [6] shows that X is $\sigma(W)$-measurable if and only if $N \leq 2$.

Let us give an intuitive description of solutions to the previous equation. Given a WBM X started from x, we will denote from now on by B^X the martingale part of $|X| - |x|$. Freidlin–Sheu formula [4] says that for all $f \in C_b^2(G^*)$

$$f(X_t) = f(x) + \int_0^t f'(X_s)\mathrm{d}B_s^X + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}s + \sum_{i=1}^{N} p_i (f \circ e_i)'(0+) L_t(|X)| \tag{3}$$

with $L_t(|X|)$ denoting the local time of $|X|$. Comparing (2) with (3), one gets

$$B_t^X = \sum_{i=1}^{N} \int_0^t 1_{\{X_s \in E_i\}} \mathrm{d}W_s^i \tag{4}$$

Thus, while it moves inside E_i, X follows the Brownian motion W^i which shows that (2) extends (1) in a natural way.

Let us now introduce the following

Definition 1.3 *We say that* (X, Y, W) *is a coupling of solutions to* (I) *if* (X, W) *and* (Y, W) *satisfy Definition 1.2 on the same filtered probability space* $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$.

A trivial coupling of solutions to (I) is given by (X, X, W) where (X, W) solves (I). This is also the law unique coupling of solutions to (I) if $N \le 2$ as $\sigma(X) \subset \sigma(W)$ in this case. Let us now introduce another interesting coupling.

Definition 1.4 *A coupling* (X, Y, W) *of solutions to* (I) *is called the Wiener coupling if* X *and* Y *are independent given* W.

The existence of the Wiener coupling is easy to check. For this note there exists a law unique triplet (X, Y, W) such that (X, W) and (Y, W) are distributed, respectively, as Q_x and Q_y, and moreover, X and Y are independent given W. It remains to check that W is a standard $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^N$ where $\mathcal{F}_t = \sigma(X_u, Y_u, W_u, u \le t)$. This holds from the conditional independence between X and Y given W and the fact that W is a Brownian motion with respect to the natural filtrations of (X, W) and (Y, W). The reason for choosing the name Wiener for this coupling will be justified in Sect. 3 in connection with stochastic flows.

1.2 Main Results

Given a WBM X on G, we define the process $\overline{X}$ by

$$\overline{X}_t = 1_{\{X_t \neq 0\}} \sum_{i=1}^{N} 1_{\{\varepsilon(X_t)=i\}} \times e_i\left(\frac{e_i^{-1}(X_t)}{Np_i}\right)$$

Note that $\overline{X} = X$ if $p_i = \frac{1}{N}$ for all $1 \le i \le N$. Following the terminology used in [2], the process $\overline{X}$ is a spidermartingale ("martingale-araignée"). In fact, for all $1 \le i \le N$, define

$$\overline{X}_t^i = |\overline{X}_t| \;\text{ if }\; \overline{X}_t \in E_i \text{ and } \overline{X}_t^i = 0 \;\text{ if not} \tag{5}$$

Note that $\overline{X}_t^i = f^i(X_t)$, where $f^i(x) = \frac{|x|}{Np_i} 1_{\{x \in E_i\}}$. Applying Freidlin–Sheu formula (3) for X and the function f^i shows that

$$\overline{X}_t^i = \frac{1}{Np_i} \int_0^t 1_{\{X_s \in E_i\}} \mathrm{d}B_s^X + \frac{1}{N} L_t(|X|) \tag{6}$$

In particular, $\overline{X}_t^i - \overline{X}_t^j$ is a martingale for all $i, j \in [1, N]$. Proposition 5 in [2] shows that $\overline{X}$ is a spidermartingale.

Our main result in this paper is the following.

Theorem 1.5 *Assume $N \geq 3$. Let (X, Y, W) be the Wiener coupling of solutions to (I) with $X_0 = Y_0 = 0$. Then*

(i) $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$ *is a martingale. In particular,*

$$\mathbb{E}[d(\overline{X}_t, \overline{Y}_t)] = 2\frac{N-2}{N}\sqrt{\frac{2t}{\pi}}.$$

(ii) *Call g_t^X and g_t^Y the last zeroes before t of X and Y, then for all $t > 0$, $\mathbb{P}(g_t^X = g_t^Y) = 0$ and $\mathbb{P}(X_t = Y_t) = 0$.*

(iii) *$\varepsilon(X_t)$ and $\varepsilon(Y_t)$ are independent for all $t > 0$.*

Another important fact about (X, Y, W) proved in [6], also true for $N = 2$, says that (X, Y, W) is a strong Markov process associated with a Feller semigroup. This result is sketched in Sect. 3.

The claim (ii) says that common zeros of X and Y are rare. It has been proved in [6] that couplings (X, Y) to (I) have the same law before $T = \inf\{t \geq 0 : X_t = Y_t\}$ and that $T < \infty$ with probability one. The strong Markov property shows then that the set of common zeros of X and Y is infinite.

The case $N = 2$. Point (i) in Theorem 1.5 is also true for $N = 2$ since $X = Y$ in this case [6]. This can also be deduced from the proofs below. In fact, Proposition 2.1 claims that Λ_t defined as the local time of the semimartingale $d(\overline{X}_t, \overline{Y}_t)$ is zero for all $N \geq 2$. By the usual Tanaka formula (see also Proposition 2.6),

$$d(\overline{X}_t, \overline{Y}_t) = M_t + \frac{1}{2}\Lambda_t$$

where M is a martingale. Taking the expectation shows that $\overline{X} = \overline{Y}$ and so $X = Y$. The same reasoning applies to any coupling (X, Y) and in particular pathwise uniqueness holds for (2) in the case $N = 2$. Since weak uniqueness is also satisfied, this yields the strong solvability of (2) when $N = 2$ (or equivalently (1)).

Theorem 1.5 yields the following important

Corollary 1.6 *Assume $N \geq 3$. Let (X, W) be a solution of (I) with $X_0 = 0$. Then for each $t > 0$, $\varepsilon(X_t)$ is independent of W.*

This corollary seems to us quite remarkable. In fact, admitting Tsirelson theorem 1.1 and using (4), it can be deduced that $\varepsilon(X_t)$ is not $\sigma(W)$-measurable (actually neither $\varepsilon(X_t)$ nor $|X_t|$ are $\sigma(W)$-measurable). However, Corollary (1.6) gives a much stronger result than this non-measurability. Comparing this with the case $N = 2$, in which $\epsilon(X_t)$ is $\sigma(W)$-measurable, shows that stochastic differential equations on star graphs with $N \geq 3$ rays involve interesting "phase transitions."

Corollary 1.6 is easy to deduce from Theorem 1.5. For this, define $C_t = \mathbb{P}(\varepsilon(X_t) = i|W)$. Since X and Y are independent given W and (X, W), (Y, W) have the same law,

$$\mathbb{P}\left(\varepsilon(X_t) = i\right)^2 = \mathbb{P}(\varepsilon(X_t) = i, \varepsilon(Y_t) = i) = \mathbb{E}[C_t^2]$$

Thus, $\mathbb{E}[C_t] = \mathbb{E}[C_t^2]^{\frac{1}{2}}$ and so there exists a constant c_t such that $C_t = c_t$ a.s. Taking the expectation shows that $c_t = p_i$.

Let us now explain our arguments to prove Theorem 1.5.

In Sect. 2.1, we prove that for any coupling (X, Y, W) of solutions to (I) such that $X_0 = Y_0 = 0$, we have $L_t(D) = 0$ where $D_t = d(\overline{X}_t, \overline{Y}_t)$.

Next, inspired by Tsirelson arguments [11], we consider a perturbation $W^r = rW + \sqrt{1-r^2}\hat{W}$, $r < 1$ of W, $\hat{W}$ is an independent copy of W, and X^r, Y^r such that

- (X^r, W^r) and (Y^r, W) are solutions to (I).
- X^r and Y^r are independent given $(W, \hat{W})$.

The coupling (X^r, Y^r) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |X^r|, |Y^r|\rangle_t \leq r < 1 \tag{7}$$

A crucial result proved in [11] (see also [2,3]) which will be used below says that, since (7) holds, $L_t(|X^r|)$ and $L_t(|Y^r|)$ have rare common points of increase (see (ii) in Proposition 2.4 for more precision). The process (X^r, Y^r, W) is shown to converge in law as $r \to 1$ to the Wiener coupling (X, Y, W) described above. The passage to the limit $r \to 1$ allows to deduce the properties mentioned in Theorem 1.5.

Section 3 is a complement based on stochastic flows to the previous results. We consider the Wiener stochastic flow of kernels K constructed in [6] which is a strong solution to the flows of kernels version of (2) driven by a real white noise $(W_{s,t})_{s\leq t}$. The Wiener coupling (X, Y, W) is shown to be the strong Markov process associated with a Feller semigroup Q obtained from K.

2 Proofs

2.1 The Local Time of the Distance

The subject of this paragraph is to prove the following result which, in the case $N = 2$, is proved in [6].

Proposition 2.1 *Assume $N \geq 2$. Let (X, Y, W) be a coupling of two solutions to (I) with $X_0 = Y_0 = 0$ and let $D_t = d(\overline{X}_t, \overline{Y}_t)$. Then D is a semimartingale with $L_t(D) = 0$.*

Proof The fact that D is a semimartingale is shown in [11] (see [2] and Proposition 2.6 below for more details). We follow the proof of Proposition 4.5 in [6] and first prove that a.s.

$$\int_{]0,+\infty]} L_t^a(D)\frac{\mathrm{d}a}{a} < \infty \tag{8}$$

where $L_t^a(D)$ is the local time of D at level a and time t. Recall that by the occupation formula

$$\int_{]0,+\infty]} L_t^a(D)\frac{\mathrm{d}a}{a} = \int_0^t 1_{\{D_s>0\}}\frac{\mathrm{d}\langle D\rangle_s}{D_s}$$

By (6),

$$|\overline{X}_t| = \sum_{i=1}^N \overline{X}_t^i = M_t^1 + L_t(|X|)$$
$$|\overline{Y}_t| = \sum_{i=1}^N \overline{Y}_t^i = M_t^2 + L_t(|Y|)$$

with

$$M_t^1 = \sum_{i=1}^N \frac{1}{Np_i}\int_0^t 1_{\{X_s\in E_i\}}\mathrm{d}B_s^X,\ \ M_t^2 = \sum_{i=1}^N \frac{1}{Np_i}\int_0^t 1_{\{Y_s\in E_i\}}\mathrm{d}B_s^Y$$

In particular,

$$\langle M^1\rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2}\int_0^t 1_{\{X_s\in E_i\}}\mathrm{d}s,\ \ \langle M^2\rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2}\int_0^t 1_{\{Y_s\in E_i\}}\mathrm{d}s$$

and

$$\langle M^1, M^2\rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2}\int_0^t 1_{\{X_s\in E_i,\ Y_s\in E_i\}}\mathrm{d}s.$$

Proposition 7 [2] tells us that

$$D_t - \int_0^t 1_{\{\varepsilon(X_s)\neq\varepsilon(Y_s)\}}\left(\mathrm{d}M_s^1 + \mathrm{d}M_s^2\right)$$
$$- \int_0^t 1_{\{\varepsilon(X_s)=\varepsilon(Y_s)\}}\mathrm{sgn}\left(M_s^1 - M_s^2\right)\left(\mathrm{d}M_s^1 - \mathrm{d}M_s^2\right)$$

is a continuous increasing process. Consequently,

$$
\begin{aligned}
\mathrm{d}\langle D\rangle_s &= \sum_{i=1}^{N} \frac{1}{(Np_i)^2} 1_{\{\varepsilon(X_s)\neq\varepsilon(Y_s)\}}(1_{\{X_s\in E_i\}} \\
&\quad + 1_{\{Y_s\in E_i\}})\mathrm{d}s \le C 1_{\{\varepsilon(X_s)\neq\varepsilon(Y_s)\}}\mathrm{d}s
\end{aligned}
$$

where C is a positive constant. Note there exists $C' > 0$ such that $D_s \ge C'(|X_s|+|Y_s|)$ for all s such that $\varepsilon(X_s) \neq \varepsilon(Y_s)$. Thus, to get (8), it is sufficient to prove

$$
\int_0^t 1_{\{X_s\neq 0, Y_s\neq 0\}} 1_{\{\epsilon(X_s)\neq\epsilon(Y_s)\}} \frac{\mathrm{d}s}{|X|_s + |Y_s|} < \infty
$$

Let us prove for instance that

$$
(1) = \int_0^t \frac{1}{|X_s| + |Y_s|} 1_{\{X_s\in E_1^*, Y_s\notin E_1\}}\mathrm{d}s < \infty
$$

Define $f(z) = |z|$ if $z \in E_1$ and $f(z) = -|z|$ if not and set $x_t = f(X_t)$, $y_t = f(Y_t)$. Clearly

$$
\frac{1}{|X_s| + |Y_s|} 1_{\{X_s\in E_1^*, Y_s\notin E_1\}} = \frac{1}{2}\frac{|\mathrm{sgn}(x_s) - \mathrm{sgn}(y_s)|}{|x_s - y_s|} 1_{\{y_s<0<x_s\}}.
$$

As in [6], let $(f_n)_n \subset C^1(\mathbb{R})$ such that $f_n \to$ sgn pointwise and $(f_n)_n$ is uniformly bounded in total variation. Defining $z_s^u = (1-u)x_s + uy_s$, we have by Fatou's Lemma

$$
\begin{aligned}
(1) &\le \liminf_n \int_0^t 1_{\{y_s<0<x_s\}} \frac{|f_n(x_s) - f_n(y_s)|}{|x_s - y_s|}\frac{\mathrm{d}s}{2} \\
&\le \liminf_n \int_0^t 1_{\{y_s<0<x_s\}} \int_0^1 \left|f_n'(z_s^u)\right|\mathrm{d}u\frac{\mathrm{d}s}{2}
\end{aligned}
$$

Writing Freidlin–Sheu formula for the function f applied to X and Y shows that on $\{y_s < 0 < x_s\}$,

$$
\frac{\mathrm{d}}{\mathrm{d}s}\langle z^u\rangle_s = u^2 + (1-u)^2 \ge \frac{1}{2}
$$

Thus,

$$
\begin{aligned}
(1) &\le \liminf_n \int_0^1 \int_0^t 1_{\{y_s<0<x_s\}} \left|f_n'(z_s^u)\right|\mathrm{d}\langle z^u\rangle_s \mathrm{d}u \\
&\le \liminf_n \int_0^1 \int_{\mathbb{R}} \left|f_n'(a)\right| L_t^a(z^u)\mathrm{d}a\mathrm{d}u
\end{aligned}
$$

So a sufficient condition for (1) to be finite is

$$\sup_{a\in\mathbb{R},u\in[0,1]} \mathbb{E}\big[L_t^a(z^u)\big] < \infty$$

By Tanaka's formula

$$\begin{aligned}\mathbb{E}\big[L_t^a(z^u)\big] &= \mathbb{E}\big[\big|z_t^u - a\big|\big] - \mathbb{E}\big[\big|z_0^u - a\big|\big] - \mathbb{E}\left[\int_0^t \operatorname{sgn}(z_s^u - a)\mathrm{d}z_s^u\right]\\ &\le \mathbb{E}[\big|z_t^u - z_0^u\big|] - \mathbb{E}\left[\int_0^t \operatorname{sgn}(z_s^u - a)\mathrm{d}z_s^u\right]\end{aligned}$$

Since x and y are two skew Brownian motions, it is easily seen that $\sup_{u\in[0,1]} \mathbb{E}[\big|z_t^u - z_0^u\big|] < \infty$. The same argument shows that

$$\mathbb{E}\left[\int_0^t \operatorname{sgn}(z_s^u - a)\mathrm{d}z_s^u\right]$$

is uniformly bounded with respect to (u, a) and consequently (1) is finite. Finally, $\int_{]0,+\infty]} L_t^a(D)\frac{\mathrm{d}a}{a}$ is finite a.s. Since $\lim_{a\downarrow 0} L^a(D) = L^0(D)$, we deduce $L_t^0(D) = 0$. $\square$

2.2 Proof of Theorem 1.5

This section gives the proof of our main result. First, we define the perturbation of the Wiener coupling as described in the introduction and then perform a passage to the limit.

Lemma 2.2 *For all $r \in [0, 1]$, there exists a law unique process $(X, W, \hat{W})$ such that, denoting $\mathcal{F}_t = \sigma(X_u, W_u, \hat{W}_u, u \le t)$,*

- *W and $\hat{W}$ are two independent $(\mathcal{F}_t)_t$-Brownian motions in $\mathbb{R}^N$.*
- *(X, W^r) is an $(\mathcal{F}_t)_t$-solution to (I) with $X_0 = 0$ and where $W^r = rW + \sqrt{1-r^2}\hat{W}$.*

Proof The proof of this lemma is similar to that of Theorem 2.3 in [6]. For the existence part, take independent processes $X, V^1, \ldots, V^N, \ldots, V^{2N}$ where X is a WBM started from 0 and each V^i is a standard Brownian motion. Denote by $(\mathcal{G}_t)_t$ the natural filtration of $(X, V^1, \ldots, V^N, \ldots, V^{2N})$ and for $1 \le i \le N$, define

$$\mathrm{d}\Gamma_t^i = 1_{\{X_t\in E_i\}}\mathrm{d}B_t^X + 1_{\{X_t\notin E_i\}}\mathrm{d}V_t^i,\ \mathrm{d}W_t^i = r\mathrm{d}\Gamma_t^i + \sqrt{1-r^2}\mathrm{d}V_t^{i+N}$$

and

$$\mathrm{d}\hat{W}_t^i = \sqrt{1-r^2}\mathrm{d}\Gamma_t^i - r\mathrm{d}V_t^{i+N}.$$

Then $W = (W^1, \dots, W^N)$, $\hat{W} = (\hat{W}^1, \dots, \hat{W}^N)$ are two independent $(\mathcal{G}_t)_t$-Brownian motions in $\mathbb{R}^N$, $(X, \Gamma^1, \dots, \Gamma^N)$ is a $(\mathcal{G}_t)_t$-solution to (I) and since $\mathrm{d}\Gamma_t^i = r\mathrm{d}W_t^i + \sqrt{1-r^2}\mathrm{d}\hat{W}_t^i$, existence holds.

Now let $(X, W, \hat{W})$ and $(\mathcal{F}_t)_t$ be as in the lemma. Introduce a Brownian motion B independent of $(X, W, \hat{W})$ and define $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_u, u \leq t)$. Write $\hat{W} = (\hat{W}^1, \dots, \hat{W}^N)$ and for $1 \leq i \leq N$, define

$$\mathrm{d}\Gamma_t^i = r\mathrm{d}W_t^i + \sqrt{1-r^2}\mathrm{d}\hat{W}_t^i, \quad \mathrm{d}V_t^{i+N} = \sqrt{1-r^2}\mathrm{d}W_t^i - r\mathrm{d}\hat{W}_t^i$$

and

$$\mathrm{d}V_t^i = 1_{\{X_t \notin E_i\}}\mathrm{d}\Gamma_t^i + 1_{\{X_t \in E_i\}}\mathrm{d}B_t.$$

Note that $V^1, \dots, V^N, \dots, V^{2N}$ are independent $(\mathcal{G}_t)_t$-Brownian motions. Using $1_{\{X_t \in E_i\}}\mathrm{d}B_t^X = 1_{\{X_t \in E_i\}}\mathrm{d}\Gamma_t^i$, simple calculations show that $(V^1, \dots, V^N, \dots, V^{2N})$ is independent of B^X. Since X is a $(\mathcal{G}_t)_t$-WBM, Lemma 4.3 in [6] claims that $X, V^1, \dots, V^N, \dots, V^{2N}$ are independent. Finally, $(X^+, W^+, \hat{W}^+)$ constructed from $X, V^1, \dots, V^N, \dots, V^{2N}$ as in the existence part coincides with $(X, W, \hat{W})$. This finishes the proof. □

An immediate consequence of the previous lemma is the following

Lemma 2.3 *For all $r \in [0, 1]$, there exists a law unique process $(X, Y, W, \hat{W})$ such that, denoting $\mathcal{F}_t = \sigma(X_u, Y_u, W_u, \hat{W}_u, u \leq t)$,*

- *W and $\hat{W}$ are two independent $(\mathcal{F}_t)_t$-Brownian motions in $\mathbb{R}^N$.*
- *(X, W^r) and (Y, W) are two $(\mathcal{F}_t)_t$-solutions to (I) with $X_0 = Y_0 = 0$ and where $W^r = rW + \sqrt{1-r^2}\hat{W}$.*
- *X and Y are independent given $(W, \hat{W})$.*

The proof of this lemma is similar to the existence and law uniqueness of the Wiener coupling and is left as an exercise.

In the sequel, we will denote $(X, Y, W, \hat{W})$ by $(X^r, Y^r, W, \hat{W})$ and use the notation W^r to denote $rW + \sqrt{1-r^2}\hat{W}$.

Proposition 2.4 *The following assertions hold*

(i) $\mathrm{d}\langle B^{X^r}, B^{Y^r}\rangle_t = r1_{\{\varepsilon(X_t^r)=\varepsilon(Y_t^r)\}}\mathrm{d}t$.
(ii) $\int_0^t 1_{\{Y_s^r \neq 0\}}\mathrm{d}L_s(|X^r|) = L_t(|X^r|)$ *and* $\int_0^t 1_{\{X_s^r \neq 0\}}\mathrm{d}L_s(|Y^r|) = L_t(|Y^r|)$.

Proof Write $W^r = (W^{r,1}, \dots, W^{r,N})$. By the previous lemma

$$\mathrm{d}B_t^{X^r} = \sum_{i=1}^N 1_{\{X_t^r \in E_i\}}\mathrm{d}W_t^{r,i} \text{ and } \mathrm{d}B_t^{Y^r} = \sum_{i=1}^N 1_{\{Y_t^r \in E_i\}}\mathrm{d}W_t^i$$

which yields (i). (ii) is Lemma 4.12 in [11] (see also [2,3]). □

The next lemma establishes the convergence in law of (X^r, Y^r, W) to the Wiener coupling (X, Y, W).

Lemma 2.5 *As $r \to 1$, (X^r, Y^r, W) converges in law to (X, Y, W), the Wiener coupling of solutions to (I) with $X_0 = Y_0 = 0$.*

Proof Let $(r_n)_n$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty} r_n = 1$. For any $p \geq 1$, $(f_i, g_i, h_i)_{1\leq i\leq p}$ bounded, $(t_i)_{1\leq i\leq p}$

$$\mathbb{E}\left[\prod_{i=1}^{p} f_i(X_{t_i}^{r_n})g_i(Y_{t_i}^{r_n})h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^{p} \mathbb{E}[f_i(X_{t_i}^{r_n})|W, \hat{W}]\mathbb{E}[g_i(Y_{t_i}^{r_n})|W, \hat{W}]h_i(W_{t_i})\right]$$

Note that (Y^{r_n}, W) is independent of $\hat{W}$. This can be deduced from the uniqueness part in Lemma 2.2 by taking $r = 1$. Consequently, $\mathbb{E}[g_i(Y_{t_i}^{r_n})|W, \hat{W}] = \mathbb{E}[g_i(Y_{t_i}^{r_n})|W]$ a.s. Now $\sigma(W, \hat{W}) = \sigma(W^{r_n}, \overline{W}^{r_n})$ where $\overline{W}^{r_n}$ is the independent complement to W^{r_n} given by

$$\overline{W}^{r_n} = \sqrt{1 - r_n^2}W - r_n\hat{W}.$$

By the proof of Lemma 2.2, $(X^{r_n}, W^{r_n}, \overline{W}^{r_n})$ and $(Y^{r_n}, W, \hat{W})$ have the same law. Consequently, (X^{r_n}, W^{r_n}) is also independent of $\overline{W}^{r_n}$ and so

$$\mathbb{E}[f_i(X_{t_i}^{r_n})|W, \hat{W}] = \mathbb{E}[f_i(X_{t_i}^{r_n})|W^{r_n}, \overline{W}^{r_n}] = \mathbb{E}[f_i(X_{t_i}^{r_n})|W^{r_n}].$$

Slutsky lemma (see Theorem 1 in [2]) shows that for all $f : G \to \mathbb{R}$ measurable bounded and $t > 0$, as $n \to \infty$,

$$\mathbb{E}[f(X_t^{r_n})|W^{r_n}] \longrightarrow Q_t f(W)$$

in probability where $Q_t f(W) = \int f(y)Q_t(W, dy)$ and $Q_t(W, dy)$ is a regular conditional expectation of X_t given W. Finally,

$$\begin{aligned}\lim_n \mathbb{E}\left[\prod_{i=1}^{p} f_i(X_{t_i}^{r_n})g_i(Y_{t_i}^{r_n})h_i(W_{t_i})\right] &= \mathbb{E}\left[\prod_{i=1}^{p} Q_{t_i} f_i(W)Q_{t_i} g_i(W)h_i(W_{t_i})\right]\\ &= \mathbb{E}\left[\prod_{i=1}^{p} f_i(X_{t_i})g_i(Y_{t_i})h_i(W_{t_i})\right]\end{aligned}$$

and the lemma is proved. □

Let us now recall Proposition 7 in [2].

Proposition 2.6 *Let Z^1 and Z^2 be two WBMs with respect to the same filtration such that $Z_0^1 = Z_0^2 = 0$. Denote by Λ the local time of $D_t = d(\overline{Z_t^1}, \overline{Z_t^2})$. Then*

$$D_t = M_t + \frac{1}{2}\Lambda_t + (N-2)\left(\int_0^t 1_{\{\overline{Z_s^1}\neq 0\}}\mathrm{d}L_s^2 + \int_0^t 1_{\{\overline{Z_s^2}\neq 0\}}\mathrm{d}L_s^1\right)$$

with M a martingale, $M_0 = 0$ and L^1, L^2 are (see Proposition 5 in [2]) the bounded variation parts of $\overline{X}^i_t$ (defined by (5)) and $\overline{Y}^i_t$.

Note that $L^1_t = \frac{1}{N} L_t(|Z^1|)$ and $L^2_t = \frac{1}{N} L_t(|Z^2|)$ by (6).

Applying the previous proposition to $(Z^1, Z^2) = (X^r, Y^r)$ and using Proposition 2.4 (ii), we get

$$d(\overline{X^r_t}, \overline{Y^r_t}) = M^r_t + \frac{1}{2}\Lambda^r_t + \frac{(N-2)}{N}\left(L_t(|X^r|) + L_t(|Y^r|)\right)$$

with M^r a martingale and Λ^r the local time of $d(\overline{X^r_t}, \overline{Y^r_t})$. In particular,

$$\mathbb{E}[d(\overline{X^r_t}, \overline{Y^r_t})] \geq 2\frac{(N-2)}{N}\mathbb{E}[R_t] \tag{9}$$

with R a reflected Brownian motion started from 0.

Proposition 2.6 applied to the Wiener coupling $(Z^1, Z^2) = (X, Y)$ and the result of Sect. 2.1 show that

$$\mathrm{d}(\overline{X_t}, \overline{Y_t}) = M_t + \frac{(N-2)}{N}\left(\int_0^t 1_{\{X_s \neq 0\}} \mathrm{d}L_s(|Y|) + \int_0^t 1_{\{Y_s \neq 0\}} \mathrm{d}L_s(|X|)\right) \tag{10}$$

with M a martingale. By the Balayage formula (see [9] on page 111 or the proof of Proposition 8 in [2]) and the fact that $L_t(D) = 0$,

$$\mathrm{d}(\overline{X_t}, \overline{Y_t}) = \mathrm{Martingale} + \frac{N-2}{N}\left(1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}| + 1_{\{\overline{Y}_{g^1} \neq 0\}}|\overline{X_t}|\right) \tag{11}$$

where $g^1 := g^{\overline{X}}_t$ and $g^2 := g^{\overline{Y}}_t$. Admit for a moment that $\mathbb{E}[d(\overline{X^r_t}, \overline{Y^r_t})]$ converges to $\mathbb{E}[d(\overline{X_t}, \overline{Y_t})]$. It comes from (9), (11), (X, Y) has the same law as (Y, X) that

$$2\frac{N-2}{N}\mathbb{E}\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}|\right] \geq 2\frac{(N-2)}{N}\mathbb{E}[R_t]$$

Consequently,

$$\mathbb{E}\left[|\overline{Y_t}|\right] \geq \mathbb{E}\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}|\right] \geq \mathbb{E}[R_t]$$

Note that this consequence is true only if $N \geq 3$. But $\mathbb{E}\left[|\overline{Y_t}|\right] = \mathbb{E}[R_t]$ and so $\overline{X}_{g^2} \neq 0$ a.s. By symmetry $\overline{Y}_{g^2} \neq 0$. Returning back to (11), we deduce that $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$ is a martingale which proves Theorem 1.5 (i).

Note that $g^1 = g^X_t$, $g^2 = g^Y_t$ and for Z a WBM, the sets of zeros of Z and $\overline{Z}$ are equal. Consequently, $X_{g^Y_t} \neq 0$ and $Y_{g^X_t} \neq 0$ a.s. In particular, $g^X_t \neq g^Y_t$ a.s and since $\{X_t = Y_t\} \subset \{g^X_t = g^Y_t\}$ (as X, Y follow the same Brownian motion on the same ray), Theorem 1.5 (ii) is also proved.

Remark 2.7 *Using the convergence of* $\mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})]$ *to* $\mathbb{E}[d(\overline{X_t}, \overline{Y_t})]$*, (9) and (10), we easily deduce that*

$$\int_0^t 1_{\{X_s \neq 0\}} \mathrm{d}L_s(|Y|) = L_t(|Y|); \int_0^t 1_{\{Y_s \neq 0\}} \mathrm{d}L_s(|X|) = L_t(|X|)$$

which is similar to Proposition 2.4 (ii).

Now it remains to prove the following

Lemma 2.8 *We have*

$$\lim_{r \to 1} \mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})] = \mathbb{E}[d(\overline{X_t}, \overline{Y_t})].$$

Proof From the convergence in law given in Lemma 2.5, it is easily seen that $(\overline{X^r}, \overline{Y^r})$ converges in law to $(\overline{X}, \overline{Y})$. This is because $\overline{Z}$ is a continuous function of Z. Let r_n be a sequence converging to 1. Skorokhod representation theorem says that it is possible to construct on some probability space $(\Omega', \mathcal{A}', \mathbb{P}')$, random variables $(X^n, Y^n)_{n \geq 1}$ and (X^∞, Y^∞) such that for each n, (X^n, Y^n) has the same law as $(\overline{X^{r_n}}, \overline{Y^{r_n}})$ and (X^∞, Y^∞) has the same law as $(\overline{X}, \overline{Y})$ and moreover (X^n, Y^n) converges a.s. to (X^∞, Y^∞). The lemma holds as soon as we prove

$$\lim_{n \to \infty} \mathbb{E}[d(X_t^n, Y_t^n)] = \mathbb{E}[d(X_t^\infty, Y_t^\infty)].$$

For each $\epsilon > 0$,

$$\begin{aligned} \mathbb{E}[d(X_t^n, X_t^\infty)] &\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty) 1_{\{d(X_t^n, X_t^\infty) > \epsilon\}}] \\ &\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty)^2]^{1/2} \mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \\ &\leq \epsilon + C \times \mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \end{aligned}$$

for some finite constant C. Thus, $\limsup_n \mathbb{E}[d(X_t^n, X_t^\infty)] = 0$ and similarly $\limsup_n \mathbb{E}[d(Y_t^n, Y_t^\infty)] = 0$. The lemma follows now using the triangle inequality. □

Let us now prove Theorem 1.5 (iii).

Denote by $\mathcal{G}$ the natural filtration of the Wiener coupling (X, Y). For a random time R, let us recall the following σ-fields (see [2] on page 286)

$$\begin{aligned} \mathcal{G}_R &= \sigma(U_R : U \text{ is a } \mathcal{G} - \text{ optional process}), \\ \mathcal{G}_{R+} &= \sigma(U_R : U \text{ is a } \mathcal{G} - \text{ progressive process}). \end{aligned}$$

In the sequel, we will always consider the completions of these sigma fields by null sets. Let $g^1 = g_t^X, g^2 = g_t^Y$. It is known (see for example Proposition 19 in [2]) that $\varepsilon(X_t)$ is independent of $\mathcal{G}_{g^1}$ and $\varepsilon(X_t)$ is $\mathcal{G}_{g^1+}$ measurable (the same holds for Y). The event $\{g^1 < g^2\} \in \mathcal{G}_{g^2}$ (see Proposition 13 in [2]) and on this event, $\varepsilon(X_t) = \limsup_{\epsilon \to 0+} \varepsilon(X_{(g^1+\epsilon) \wedge g^2})$. Since $(g^1 + \epsilon) \wedge g^2 \leq g^2$, by Proposition 13 in [2] again,

$\mathcal{G}_{(g^1+\epsilon)\wedge g^2} \subset \mathcal{G}_{g^2}$ and so $\limsup_{\epsilon\to 0+} \varepsilon(X_{(g^1+\epsilon)\wedge g^2})$ is $\mathcal{G}_{g^2}$-measurable. Take f an indicator function on a subset of $\{1, \dots, N\}$. By conditioning with respect to $\mathcal{G}_{g^2}$, we deduce

$$\mathbb{E}\left[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^1<g^2\}}\right] = \mathbb{E}[f(\varepsilon(Y_t)]\mathbb{E}\left[f(\varepsilon(X_t))1_{\{g^1<g^2\}}\right]$$

and

$$\mathbb{E}\left[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^2<g^1\}}\right] = \mathbb{E}\left[f(\varepsilon(X_t)\right]\mathbb{E}\left[f(\varepsilon(Y_t))1_{\{g^2<g^1\}}\right]$$

Summing and using $\mathbb{P}(g^1 = g^2) = 0$, we get

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t)]\big(\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1<g^2\}}] + \mathbb{E}[f(\varepsilon(Y_t))1_{\{g^2<g^1\}}]\big)$$

But $\{g^1 < g^2\} = \{g^2 < g^1\}^c$ a.s. Since $\mathcal{G}_{g^1}$ is complete, $\{g^1 < g^2\} \in \mathcal{G}_{g^1}$ which is independent of $\varepsilon(X_t)$ so that

$$\mathbb{E}\left[f(\varepsilon(X_t))1_{\{g^1<g^2\}}\right] = \frac{1}{2}\mathbb{E}[f(\varepsilon(X_t))].$$

Using the symmetry, we arrive at $\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t))]\mathbb{E}[f(\varepsilon(Y_t))]$.

3 Interpretation Using Stochastic Flows

This section gives an interpretation of the Wiener coupling using the Wiener stochastic flow of kernels solving the generalized interface equation considered in [6]. For basic definitions of stochastic flows of mappings, kernels, and real white noises, the reader is referred to [8].

For a family of doubly indexed random variables $Z = (Z_{s,t})_{s\le t}$, define $\mathcal{F}^Z_{s,t} = \sigma(Z_{u,v}, s \le u \le v \le t)$ for all $s \le t$. The extension to flows of kernels of the interface SDE is the following.

Definition 3.1 *Let K be a stochastic flow of kernels on G and $\mathcal{W} = (\mathcal{W}^i, 1 \le i \le N)$ be a family of independent real white noises. We say that $(K, \mathcal{W})$ solves (I) if for all $s \le t$, $f \in \mathcal{D}$ and $x \in G$, a.s.*

$$K_{s,t}f(x) = f(x) + \sum_{i=1}^{N} \int_s^t K_{s,u}(1_{E_i}f')(x)d\mathcal{W}^i_{s,u} + \frac{1}{2}\int_s^t K_{s,u}f''(x)\mathrm{d}u.$$

We say K is a Wiener solution if for all $s \le t$, $\mathcal{F}^K_{s,t} \subset \mathcal{F}^{\mathcal{W}}_{s,t}$. When K is induced by a stochastic flow of mappings φ ($K = \delta_\varphi$), we say $(\varphi, \mathcal{W})$ is a solution of (I).

Note that when $K = \delta_\varphi$, the flow φ defines a system of solutions to the interface SDE (1.2) for all possible time and position initial conditions.

If $(K, \mathcal{W})$ solves (I), then $\mathcal{F}^{\mathcal{W}}_{s,t} \subset \mathcal{F}^K_{s,t}$ for all $s \le t$ [6]. Therefore, Wiener solutions are characterized by $\mathcal{F}^{\mathcal{W}}_{s,t} = \mathcal{F}^K_{s,t}$ for all $s \le t$.

It has been proved in [6] that there exists a law unique stochastic flow of mappings φ and a real white noise $\mathcal{W}$ such that $(\varphi, \mathcal{W})$ solves (I). Filtering this flow with respect to $\mathcal{W}$ gives rise to a Wiener stochastic flow of kernels $K_{s,t}(x) = \mathbb{E}[\delta_{\varphi_{s,t}(x)}|\mathcal{F}^{\mathcal{W}}_{s,t}]$ solution of (I) which is unique up to modification.

In the case $N = 2$ the Wiener flow and the flow of mappings coincide ($K = \delta_\varphi$) while $K \neq \delta_\varphi$ if $N \geq 3$ and other flows solving (I) may exist [6].

Let $(K, \mathcal{W})$ be the Wiener stochastic flow which solves (I). Then

$$Q_t(f \otimes g \otimes h)(x, y, w) = \mathbb{E}[K_{0,t} f(x) K_{0,t} g(y) h(w + \mathcal{W}_{0,t})]$$

defines a Feller semigroup on $G^2 \times \mathbb{R}^N$. Denote by (X, Y, W) the Markov process associated with $(Q_t)_t$ and started from $(x, y, 0)$.

Proposition 3.2 *(X, Y, W) is the Wiener coupling solution of (I) with $X_0 = x$ and $Y_0 = y$.*

Proof Note that

$$\widetilde{Q}_t(f \otimes h)(x, w) := Q_t(f \otimes I \otimes h)(x, w) = \mathbb{E}[f(\varphi_{0,t}(x)) h(w + \mathcal{W}_{0,t})]$$

In particular, (X, W) has the same law as $(\varphi_{0,t}(x), \mathcal{W}_{0,t})_{t \geq 0}$ and so it is a solution to (I). The same holds for (Y, W). Now it remains to prove that X and Y are independent given W. We will check that

$$\mathbb{E}\left[\prod_{i=1}^n f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^n \mathbb{E}[f_i(X_{t_i})|W] \mathbb{E}[g_i(Y_{t_i})|W] h_i(W_{t_i})\right]$$

for all measurable and bounded test functions $(f_i, g_i, h_i)_i$. Since K is a measurable function of $\mathcal{W}$, we may assume K (and so $\mathcal{W}$) is defined on the same space as X and Y and that $W_t = \mathcal{W}_{0,t}$. By an easy induction (see the proof of Proposition 4.1 in [5]),

$$\mathbb{E}\left[\prod_{i=1}^n f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^n K_{0,t_i} f_i(x) K_{0,t_i} g_i(y) h_i(W_{t_i})\right] \tag{12}$$

From (12), we also deduce $K_{0,t_i} f_i(x) = \mathbb{E}\left[f_i(X_{t_i})|\mathcal{F}^W_{0,t_i}\right]$ and $K_{0,t_i} g_i(y) = \mathbb{E}\left[g_i(Y_{t_i})|\mathcal{F}^W_{0,t_i}\right]$. This completes the proof. ☐

Let $(\mathcal{W}_{s,t})_{s \leq t}$ and $(\hat{\mathcal{W}}_{s,t})_{s \leq t}$ be two independent real white noises and set $\mathcal{W}^r_{s,t} = r\mathcal{W}_{s,t} + \sqrt{1 - r^2}\hat{\mathcal{W}}_{s,t}$. Denote by K and K^r the Wiener flows solutions of (I), respectively, driven by $\mathcal{W}$ and $\mathcal{W}^r$ and define

$$Q^r_t(f \otimes g \otimes h)(x, y, w) = \mathbb{E}\left[K^r_{0,t} f(x) K_{0,t} g(y) h(w + \mathcal{W}_{0,t})\right] \tag{13}$$

Then Q^r is a Feller semigroup. Following the proof of Proposition 3.2, one can prove that (X^r, Y^r, W) given in Lemma 2.5 is the Markov process associated with Q^r and starting from $(0, 0, 0)$. In particular, this is also a Feller process.

Final remarks and open problems

There are interesting open problems related to the interface SDE. Let us mention some of them.

- What is the conditional law of $|X_t|$ (and more generally of X_t) given W?
- What are the couplings which "interpolate" between the coalescing coupling and the Wiener one?
- What are the stochastic flows which "interpolate" between the coalescing flow and the Wiener one? (see [6] for more details).

Let us finish with the following remark regarding the first question. Let W be a standard Brownian motion and let $X^1, X^2, \ldots$ be WBMs started from 0 such that (X^i, W) is solution to (I) with $X_0^i = 0$ for all i and $X^1, X^2, \ldots$ are independent given W. Then by the law of the large numbers for all $f \in C_0(G)$, a.s $\mathbb{E}[f(X_t^1)|W] = \lim_n \frac{1}{n}\sum_{i=1}^n f(X_t^i)$ (see Section 2.6 in [8]).

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