LAGRANGIAN NAVIER-STOKES DIFFUSIONS ON MANIFOLDS: VARIATIONAL PRINCIPLE AND STABILITY

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Abstract.

We prove a variational principle for stochastic flows on manifolds. It extends V. Arnold's description of Lagrangian Euler flows, which are geodesics for the L^2 metric on the manifold, to the stochastic case. Here we obtain stochastic Lagrangian flows with mean velocity (drift) satisfying the Navier-Stokes equations.

We study the stability properties of such trajectories as well as the evolution in time of the rotation between the underlying particles. The case where the underlying manifold is the two-dimensional torus is described in detail.

Contents

1. Introduction

The Lagrangian approach to hydrodynamics in the non-viscous incompressible case consists in studying the configuration of the underlying particles, namely the solutions of equations

$$\frac{d}{dt}g(t)(x) = u(t, g(t)(x)),\ g(0) = x$$

where the velocity field u satisfies Euler equations

$$\frac{\partial}{\partial t}u + \nabla_u u = -\nabla p,\ div\ u = 0$$

and p is the pressure. The integral flows g are usually called Lagrangian flows.

V. I. Arnold ([3]) discovered that these flows can be characterized as geodesics on an (infinite-dimensional) group of diffeomorphisms. They are, in particular, critical paths of the action functional

$$S(g) = \frac{1}{2} \int_0^T \int_M ||\frac{d}{dt} g(t)(x)||^2 d\mu(x) dt$$

where μ is the volume element associated with the metric.

This point of view allows in particular to derive various properties of the geodesics (the Lagrangian flows) such as stability, through the study of the geometry of the group ([9], [4]).

When the fluid is viscous, namely for the Navier-Stokes equation, one can describe the Lagrangian trajectories as realizations of a stochastic process and interpret the associated drift, solving Navier-Stokes, as an expectation over this process. This probabilistic approach, which we follow here, is inspired by [11], [12]. It is intrinsically probabilistic in the sense that there is no random perturbation of the Navier-Stokes equations: in our model the velocity is, as it should be, deterministic; only the position is described by stochastic flows. Similar stochastic models are used for example in [6]. In this framework the trajectories remain, in an appropriate sense, geodesics as they are almost sure solutions of a variational principle. This was shown in [5] for the two-dimensional torus. We call these processes *stochastic Lagrangian flows.*

More recently an analogous stochastic least action principle was derived in [7]. The main differences are that there the author considers backward rather than forward semimartingales and also that the variations are assumed to be of bounded-variation type, which is not the case of those we use.

The purpose of this paper is twofold. On one hand we extend the variational principle for the Lagrangian Navier-Stokes diffusions, derived in [5] for the two-dimensional torus, to compact manifolds. Moreover we study the stability properties of these diffusion processes, more precisely the evolution in time of their distance. The behaviour of the stochastic Lagrangian flows concerning their (L^2) distance depends on the intensity of the noise as well as on the metric of the underlying manifold. The example of the torus is studied in detail, and in this case we observe that, at least for short times, the flows spread out more than the deterministic classical Navier-Stokes Lagrangian paths. This type of phenomenon was illustrated by some simulations in [2]. Finally we also describe the evolution in time of the rotation between stochastic Lagrangian particles.

The general outline of this paper is as follows. In Theorem 3.2 of Section 3 we prove the variational principle on a general compact oriented manifold without boundary. This principle gives rise to the Lagrangian stochastic flows to be analysed afterwards. The following three sections are devoted to the derivation of formulae for the distance of two flows. In Section 4 the case of the torus with the Euclidean distance is considered. Proposition 4.2 gives the Itô formula for the L^2 distance between two flows. Proposition 4.4 yields a lower bound for the equation of the distance. Finally Theorem 4.5 proves chaotic behaviour of trajectories, more precisely exponential growth of the L^2 distance, under the condition that the L^∞ one has a sufficiently small upper bound. This upper bound is needed due to the presence of cutlocus in the torus. To overcome the calculation with the cutlocus the torus is endowed in section 5 with the extrinsic distance. Example 5.1 shows that that without the uniform bound on the L^∞ distance it may happen that the L^2 distance between two flows is very small, however its drift is negative, meaning the conclusion of Theorem 4.5 is not valid here. In Proposition 6.1 of Section 6 the Itô differential of the L^2 distance between two flows in a general Riemannian manifold is computed. From this formula it can be deduced that negative curvature together with a uniform bound on the distance implies exponential growth of L^2-distance. In the last section we study the stochastic process that describes the evolution in time of the rotation between particles and how this rotation depends on the diffusion coefficients. Lemma 7.1 yields the Itô covariant differential of the rotation vector between the particles. It is proven in Proposition 7.2 that if a certain series involving the coefficients diverges then the rotation becomes faster and faster when the distance between the particles converges to zero.

2. General setting

Let $(M, \mathbf{g})$ be a compact oriented Riemannian manifold without boundary.

Recall that the Itô differential of an M-valued semimartingale Y is defined by

$$dY_t = P\left(Y\right)_t d\left(\int_0^\cdot P\left(Y\right)_s^{-1} \circ dY_s\right)_t \tag{2.1}$$

where

$$P\left(Y\right)_t : T_{Y_0}M \to T_{Y_t}M \tag{2.2}$$

is the parallel transport along $t \mapsto Y_t$. Alternatively, in local coordinates,

$$dY_t = \left(dY_t^i + \frac{1}{2}\Gamma^i_{jk}(Y_t)dY_t^j \otimes dY_t^k\right)\partial_i \tag{2.3}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection.

If the semimartingale Y_t has an absolutely continuous drift, we denote it by $DY_t\,dt$: for every 1-form $\alpha \in \Gamma(T^*M)$, the finite variation part of

$$\int_0^\cdot \langle \alpha(Y_t),\ dY_t\rangle \tag{2.4}$$

is

$$\int_0^\cdot \langle \alpha(Y_t),\ DY_t\,dt\rangle \tag{2.5}$$

Let G^s, $s \geq 0$ be the infinite dimensional group of homeomorphisms on M which belong to H^s, the Sobolev space of order s. For $s > \frac{m}{2} + 1$, $m = \mathrm{dim}M$, G^s is a C^∞ Hilbert manifold. The volume preserving homeomorphism subgroup will be denoted by G_V^s:

$$G_V^s = \{g \in G^s,\ :\ g_*\mu = \mu\},$$

with μ the volume element associated to the Riemannian metric. We denote by $\mathscr{G}^s$ (resp. $\mathscr{G}_V^s$) the Lie algebra of G^s (resp. G_V^s). See [9] for example.

On M we consider an incompressible Brownian flow $g_u(t) \in G_V^0$ with covariance $a \in \Gamma(TM \odot TM)$ and time dependent drift $u(t,\cdot) \in \Gamma(TM)$. We assume that for all $x \in M$, $a(x,x) = 2\nu\mathbf{g}^{-1}(x)$ for some $\nu > 0$. This means that

$$dg_u(t)(x) \otimes dg_u(t)(y) = a\left(g_u(t)(x), g_u(t)(y)\right)\,dt, \tag{2.6}$$

$$dg_u(t)(x) \otimes dg_u(t)(x) = 2\nu\mathbf{g}^{-1}\left(g_u(t)(x)\right)\,dt, \tag{2.7}$$

the drift of $g_u(t)(x)$ is absolutely continuous and satisfies $Dg_u(t)(x) = u(t, g_u(t)(x))$. The generator of this process is

$$L_u = \nu\Delta + \partial_u$$

where Δ is the Laplace-Beltrami operator on M. The parameter ν will be called the speed of the Brownian flow.

Such incompressible flows are known to exist on compact symmetric spaces and on compact Lie groups.

If the time is indexed by $[0,T]$ for some $T>0$, we define the action functional by

$$S(g_u) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(\int_M \|Dg_u(t)(x)\|^2 \, d\mu(x)\right) dt\right].$$

From now on, for simplicity, we shall simply write dx for integration on the manifold.

3. The variational principle

Define

$$\mathscr{H} = \left\{v \in C^1([0,T],\ \mathscr{G}_V^\infty),\ v(0,\cdot)=0,\ v(T,\cdot)=0\right\} \tag{3.1}$$

Given $v \in \mathscr{H}$, consider the following ordinary differential equation

$$\begin{aligned} \frac{de_t(v)}{dt} &= \dot{v}(t, e_t(v)) \\ e_0(v) &= e \end{aligned} \tag{3.2}$$

where e is the identity of G_V^∞. Since v is divergence free, $e.(v)$ is a G_V^∞-valued deterministic path.

We denote by $\mathscr{P}$ the set of continuous G_V^0-valued semimartingales $g(t)$ such that $g(0)=e$. Then for all $v \in \mathscr{H}$, we have $e_t(v)\circ g_u(t) \in \mathscr{P}$.

Definition 3.1. Let J be a functional defined on $\mathscr{P}$ and taking values in $\mathbb{R}$. We define its left and right derivatives in the direction of $h(\cdot) = e.(v)$, $v \in \mathscr{H}$ at a process $g \in \mathscr{P}$ respectively, by

$$\begin{aligned} (D_L)_h J[g] &= \frac{d}{d\varepsilon} J[e.(\varepsilon v)\circ g(\cdot)]|_{\varepsilon=0}, \\ (D_R)_h J[g] &= \frac{d}{d\varepsilon} J[g(\cdot)\circ e.(\varepsilon v)]|_{\varepsilon=0}. \end{aligned} \tag{3.3}$$

A process $g \in \mathscr{P}$ wil be called a critical point of the functional J if

$$(D_L)_h J[g] = (D_R)_h J[g] = 0,\ \forall h = e(v),\ v \in \mathscr{H}. \tag{3.4}$$

Theorem 3.2. *Let $(t,x) \mapsto u(t,x)$ be a smooth time-dependent divergence-free vector field on M, defined on $[0,T]\times M$. Let $g_u(t)$ a stochastic Brownian flow with speed $\nu>0$ and drift u. The stochastic process $g_u(t)$ is a critical point of the energy functional S if and only if the vector field $u(t)$ verifies the Navier-Stokes equation*

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \Box u - \nabla p \tag{3.5}$$

where $\Box = dd^* + d^*d$ is the damped Laplacian.

The damped Laplacian, associated to the damped connection ∇^c, is also known as the Laplace-de Rham operator. We recall that when computed on forms and, in particular, on vector fields, it differs from the usual Levi-Civita Laplacian by a Ricci curvature term (this is the content of the Weitzenböck formula). Therefore, on flat manifolds such as the torus, as the curvature vanishes, the two Laplacians coincide and reduce to the usual one.

For the construction of weak solutions of Navier-Stokes equations on Riemannian manifolds we refer to [10].

Proof. of Theorem 3.2. Since the functional S is right invariant, it is enough to consider the left derivative. So we need to compute

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} S(e.(\varepsilon v)(g_u)). \tag{3.6}$$

We let

$$f(\varepsilon) = S(e.(\varepsilon v)(g_u)). \tag{3.7}$$

Then

$$f(\varepsilon) = \frac{1}{2}\int_M \left(\mathbb{E}\left[\int_0^T \left(\|De_t(\varepsilon v)(g_u)(t)(x)\|^2 \right) dt \right] \right) dx \tag{3.8}$$

which yields

$$f'(0) = \int_M \left(\mathbb{E}\left[\int_0^T \left(\langle \nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)\left(g_u(t)(x)\right), u(t, g_u(t)(x))\rangle \right) dt \right] \right) dx. \tag{3.9}$$

We need to compute

$$\nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)\left(g_u(t)(x)\right). \tag{3.10}$$

We have

$$\begin{aligned} \nabla_t \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v) &= \nabla_\varepsilon|_{\varepsilon=0} \frac{de_t(\varepsilon v)}{dt} \\ &= \nabla_\varepsilon|_{\varepsilon=0} \varepsilon \dot{v}(t, e_t(\varepsilon v)) \\ &= \dot{v}(t, e). \end{aligned}$$

Together with $v(0, \cdot) = 0$, this implies

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)(x) = v(t, x). \tag{3.11}$$

Consequently

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)\left(g_u(t)(x)\right) = v\left(t, g_u(t)(x)\right). \tag{3.12}$$

By Itô equation,

$$\begin{aligned} & de_t(\varepsilon v)(g_u(t)(x)) \\ &= \langle de_t(\varepsilon v)(\cdot),\ dg_u(t)(x)\rangle + \frac{1}{2}\nabla de_t(\varepsilon v)(g_u(t)(x))\left(dg_u(t)(x) \otimes dg_u(t)(x)\right) \\ &= \langle de_t(\varepsilon v)(\cdot),\ dg_u(t)(x)\rangle + \nu \Delta e_t(\varepsilon v)(g_u(t)(x))\, dt. \end{aligned} \tag{3.13}$$

Here $\Delta e_t(\varepsilon v)(\cdot)$ denotes the tension field of the map $e_t(\varepsilon v) : M \to M$. This yields

$$\begin{aligned} De_t(\varepsilon v)(g_u(t)(x)) &= \langle de_t(\varepsilon v)(\cdot),\ Dg_u(t)(x)\rangle + \nu \Delta e_t(\varepsilon v)(g_u(t)(x)) \\ &\quad + \varepsilon \dot{v}(t, e_t(\varepsilon v)(g_u(t)(x))) \\ &= \langle de_t(\varepsilon v)(\cdot),\ u(t, g_u(t)(x))\rangle + \nu \Delta e_t(\varepsilon v)(g_u(t)(x)) \\ &\quad + \varepsilon \dot{v}(t, e_t(\varepsilon v)(g_u(t)(x))). \end{aligned} \tag{3.14}$$

Differentiating with respect to ε at $\varepsilon = 0$, we get

$$
\begin{aligned}
&\nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)(g_u(t)(x)) \\
&= \langle \nabla_\varepsilon|_{\varepsilon=0} de_t(\varepsilon v)(\cdot),\ u(t, g_u(t)(x))\rangle + \nu \nabla_\varepsilon|_{\varepsilon=0} \Delta e_t(\varepsilon v)(g_u(t)(x)) \\
&\quad + \frac{\partial v}{\partial t}(t, g_u(t)(x)) \\
&= \left\langle \nabla . \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)(\cdot),\ u(t, g_u(t)(x)) \right\rangle + \nu \Box \frac{d}{d\varepsilon}|_{\varepsilon=0} e_t(\varepsilon v)(g_u(t)(x)) \\
&\quad + \frac{\partial v}{\partial t}(t, g_u(t)(x)) \\
&= \langle \nabla . v(t,\cdot),\ u(t, g_u(t)(x))\rangle + \nu \Box v(t,\cdot)(g_u(t)(x)) + \frac{\partial v}{\partial t}(t, g_u(t)(x)) \\
&= \nabla_{u(t, g_u(t)(x))} v(t,\cdot) + \nu \Box v(t,\cdot)(g_u(t)(x)) + \frac{\partial v}{\partial t}(t, g_u(t)(x)).
\end{aligned} \tag{3.15}
$$

We used the commutation formula $\nabla_\varepsilon|_{\varepsilon=0}\Delta = \Box \frac{d}{d\varepsilon}$. Alternatively,

$$
\Box v = \Delta^h v + \mathrm{Ric}^\sharp(v). \tag{3.16}
$$

For a TM-valued semimartingale J_t which projects onto the M-valued semimartingale Y_t, we denote by $\mathscr{D} J_t$ the Itô covariant derivative:

$$
\mathscr{D} J_t = P(Y)_t d\left(P(Y)_t^{-1} J_t\right). \tag{3.17}
$$

Then Itô equation yields

$$
\mathscr{D} u(t, g_u(t)(x)) \simeq \frac{\partial u}{\partial t}(t, g_u(t)(x))\, dt + \nabla_{dg_u(t)(x)} u + \nu \Delta^h u(t, g_u(t)(x))\, dt \tag{3.18}
$$

and

$$
\mathscr{D} v(t, g_u(t)(x)) \simeq \frac{\partial v}{\partial t}(t, g_u(t)(x))\, dt + \nabla_{dg_u(t)(x)} v + \nu \Delta^h v(t, g_u(t)(x))\, dt. \tag{3.19}
$$

where the notation $\simeq$ means "equal up to a martingale"; and

$$
\begin{aligned}
&\int_0^\cdot P(g_u(\cdot))_t^{-1} \mathscr{D} u(t, g_u(t)(x)) \\
&- \int_0^\cdot P(g_u(\cdot))_t^{-1} \left(\frac{\partial u}{\partial t}(t, g_u(t)(x))\, dt + \nabla_{dg_u(t)(x)} u + \nu \Delta^h u(t, g_u(t)(x))\, dt\right)
\end{aligned}
$$

is a local martingale.

On the other hand, writing $u_t = u(t, g_u(t)(x))$ and $v_t = v(t, g_u(t)(x))$ we have

$$
\langle u_T, v_T\rangle = \int_0^T \langle \mathscr{D} u_t,\ v_t\rangle + \int_0^T \langle u_t,\ \mathscr{D} v_t\rangle + \int_0^T \langle \mathscr{D} u_t,\ \mathscr{D} v_t\rangle. \tag{3.20}
$$

Let us denote by Dv_t the drift of v_t with respect to the damped connection ∇^c on TM, whose geodesics are the Jacobi fields. It is known that,

$$
\left(Du_t - \nu\, \mathrm{Ric}^\sharp(u_t)\right) dt \quad \text{is the drift of} \quad \mathscr{D} u_t \tag{3.21}
$$

and

$$
\left(Dv_t - \nu\, \mathrm{Ric}^\sharp(v_t)\right) dt \quad \text{is the drift of} \quad \mathscr{D} v_t. \tag{3.22}
$$

As can be seen from (3.15), (3.19) and (3.22), the drift Dv_t commutes with the derivative with respect to a parameter, so it satisfies

$$
Dv_t = \nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)(g_u(t)(x)). \tag{3.23}
$$

Taking the expectation in (3.20) and using (3.23), (3.21) and (3.22), we get by removing the martingale parts

$$\begin{aligned}\mathbb{E}\left[\langle u_T, v_T\rangle\right] &= \mathbb{E}\left[\int_0^T \langle \frac{\partial u}{\partial t}(t, g_u(t)(x)) + \nabla_{u_t} u + \nu \Delta^h u(t, g_u(t)(x)),\ v_t\rangle\, dt\right]\\ &+ \mathbb{E}\left[\int_0^T \langle u_t,\ \nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)(g_u(t)(x)) - \nu\, \mathrm{Ric}^\sharp(v_t)\rangle\, dt\right]\\ &+ \mathbb{E}\left[2\nu \int_0^T \mathrm{tr}\,\langle \nabla . u,\ \nabla . v\rangle\,(t, g_u(t)(x))\, dt\right].\end{aligned} \tag{3.24}$$

Then using the facts that $v_T = 0$, together with

$$\langle u_t, \mathrm{Ric}^\sharp(v_t)\rangle = \langle \mathrm{Ric}^\sharp(u_t), v_t\rangle \tag{3.25}$$

and (3.16), we get

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \langle u_t,\ \nabla_\varepsilon|_{\varepsilon=0} De_t(\varepsilon v)(g_u(t)(x))\rangle\, dt\right]\\ &= -\mathbb{E}\left[\int_0^T \langle \frac{\partial u}{\partial t}(t, g_u(t)(x)) + \nabla_{u_t} u + \nu \Box u(t, g_u(t)(x)),\ v_t\rangle\, dt\right]\\ &\quad - \mathbb{E}\left[2\nu \int_0^T \mathrm{tr}\,\langle \nabla . u_t,\ \nabla . v_t\rangle\,(t, g_u(t)(x))\, dt\right].\end{aligned} \tag{3.26}$$

Integrating with respect to x yields

(3.27)

$$\begin{aligned}&f'(0)\\ &= -\mathbb{E}\left[\int_0^T \left(\int_M \left\langle \left(\left(\frac{\partial}{\partial t} + \nabla_u + \nu\Box\right) u\right)(t, g_u(t)(x)),\ v(t, g_u(t)(x))\right\rangle dx\right) dt\right]\\ &\quad - \mathbb{E}\left[2\nu \int_0^T \left(\int_M \mathrm{tr}\,\langle \nabla . u,\ \nabla . v\rangle\,(t, g_u(t)(x))\, dx\right) dt\right].\end{aligned}$$

Now we use the fact that $g_u(t)(\cdot)$ is volume preserving:

$$\begin{aligned}&f'(0)\\ &= -\mathbb{E}\left[\int_0^T \left(\int_M \left\langle \left(\left(\frac{\partial}{\partial t} + \nabla_u + \nu\Box\right) u\right)(t, x),\ v(t, x)\right\rangle dx\right) dt\right]\\ &\quad - \mathbb{E}\left[2\nu \int_0^T \left(\int_M \mathrm{tr}\,\langle \nabla . u,\ \nabla . v\rangle\,(t, x)\, dx\right) dt\right].\end{aligned} \tag{3.28}$$

Since M is compact and orientable, an integration by parts gives

$$\int_M \mathrm{tr}\,\langle \nabla . u,\ \nabla . v\rangle\,(t, x)\, dx = -\int_M \langle \Box u, v\rangle\,(t, x)\, dx. \tag{3.29}$$

Replacing in (3.28) we obtain

$$f'(0) = -\mathbb{E}\left[\int_0^T \left(\int_M \left\langle \left(\left(\frac{\partial}{\partial t} + \nabla_u - \nu\Box\right) u\right)(t, x),\ v(t, x)\right\rangle dx\right) dt\right]. \tag{3.30}$$

The process $g_u(t)$ is a critical point of the energy functional S if and only if $f'(0) = 0$, which by equation (3.30) is equivalent to

$$\left(\frac{\partial}{\partial t} + \nabla_u - \nu\square\right) u = -\nabla p \tag{3.31}$$

for some function p on $[0,T] \times M$. This achieves the proof. □

4. Stability: the two-dimensional torus endowed with the Euclidean distance

We study the evolution in time of the L^2 distance between two Lagrangian flows in the two dimensional torus. Notice that, in order to interpret the diffusion processes as a solution of the variational principle described in section 2, there is no canonical choice for the Brownian motion, as far as it corresponds to the same generator. We make here a particular choice.

On the two-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ we consider the following vector fields

$$A_k(\theta) = (k_2, -k_1)\cos k.\theta, \quad B_k(\theta) = (k_2, -k_1)\sin k.\theta$$

and the Brownian motion

$$dW(t) = \sum_{k\in\mathbb{Z}} \lambda_k \sqrt{\nu}(A_k dx_k + B_k dy_k) \tag{4.1}$$

where x_k, y_k are independent copies of real Brownian motions. We assume that $\sum_k |k|^2\lambda_k^2 < \infty$, a necessary and sufficient condition for the Brownian flow to be defined in $L^2(\mathbb{T})$. Furthermore we consider $\lambda_k = \lambda(|k|)$ to be nonzero for a equal number of k_1 and k_2 components. In this case the generator of the process is equal to

$$L_u = C\nu\Delta + \frac{\partial}{\partial t} + \partial_u$$

with $2C = \sum_k \lambda_k^2$ (c.f.[5] Theorem 2.2). We shall assume C to be equal to one. Let us take two Lagrangian stochastic trajectories starting from different diffeomorphisms ϕ and ψ and write

$$dg_t = (odW(t)) + u(t, g_t)dt, \qquad d\tilde{g}_t = (odW(t)) + u(t, \tilde{g}_t)dt \tag{4.2}$$

with

$$g_0 = \phi, \qquad \tilde{g}_0 = \psi, \qquad \phi \neq \psi$$

We consider the L^2 distance of the particles defined by

$$\rho^2(\phi,\psi) = \int_{\mathbb{T}} |\phi(\theta) - \psi(\theta)|^2 \, d\theta.$$

where $d\theta$ stands for the normalized Lebesgue measure on the torus.

Denoting $\rho_t = \rho(g_t, \tilde{g}_t)$ and $\tau(g,\tilde{g}) = \inf\{t > 0 : \rho_t = 0\}$, we have the following result:

Lemma 4.1. *The stopping time $\tau(g,\tilde{g})$ is infinite.*

Proof. By uniqueness of the solution of the sde for $\tilde{g}_t$, for all $t > 0$ we can write $\tilde{g}_t(\theta) = g_t((\phi^{-1} \circ \psi)(\theta))$. Since g_t, φ and ψ are diffeomorphisms, if $\varphi(\theta) \neq \psi(\theta)$ then $g_t(\theta) \neq g_t((\phi^{-1} \circ \psi)(\theta))$.

As $\phi \neq \psi$, the set $\{\theta \in \mathbb{T},\ \tilde{g}_t(\theta) \neq g_t(\theta)\}$ has positive measure and this implies that $\rho_t > 0$, which in turn implies that $\tau(g, \tilde{g})$ is infinite. □

Denote by $L_t(\theta)$ the local time of the process $|g_t(\theta) - \tilde{g}_t(\theta)|$ when $(g_t(\theta), \tilde{g}_t(\theta))$ reaches the cutlocus of $\mathbb{T}$. By Itô calculus we have

$$\begin{aligned} d\rho_t =& \frac{1}{\rho_t} \sum_k \lambda_k \sqrt{\nu} \, \langle g_t - \tilde{g}_t, (A_k(g_t) - A_k(\tilde{g}_t)) \, dx_k(t) + (B_k(g_t) - B_k(\tilde{g}_t)) \, dy_k(t) \rangle_{\mathbb{T}} \\ &+ \frac{1}{\rho_t} \langle g_t - \tilde{g}_t, u(t, g_t) - u(t, \tilde{g}_t) \rangle_{\mathbb{T}} \, dt - \frac{1}{\rho_t} \int_{\mathbb{T}} |g_t - \tilde{g}_t|(\theta) dL_t(\theta) \\ &+ \frac{1}{2\rho_t} \sum_k \lambda_k^2 \nu \left(\|A_k(g_t) - A_k(\tilde{g}_t)\|_{\mathbb{T}}^2 + \|B_k(g_t) - B_k(\tilde{g}_t)\|_{\mathbb{T}}^2 \right) dt \\ &- \frac{1}{2\rho_t^3} \sum_k \lambda_k^2 \nu \left(\langle g_t - \tilde{g}_t, A_k(g_t) - A_k(\tilde{g}_t) \rangle_{\mathbb{T}}^2 + \langle g_t - \tilde{g}_t, B_k(g_t) - B_k(\tilde{g}_t) \rangle_{\mathbb{T}}^2 \right) dt \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{T}}$ and $\| \cdot \|_{\mathbb{T}}$ denote, resp., the L^2 inner product and norm. We shall use the following notation,

$$\delta u(t) = \frac{1}{\rho_t} \left(u(t, g_t) - u(t, \tilde{g}_t) \right). \tag{4.3}$$

We have

$$A_k(g_t) - A_k(\tilde{g}_t) = -2 \sin \frac{k \cdot (g_t + \tilde{g}_t)}{2} \sin \left(\frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) k^{\perp}, \tag{4.4}$$

$$B_k(g_t) - B_k(\tilde{g}_t) = 2 \cos \frac{k \cdot (g_t + \tilde{g}_t)}{2} \sin \left(\frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) k^{\perp}, \tag{4.5}$$

where we have noted $k^{\perp} = (k_2, -k_1)$. Then, for $k \neq 0$ we define

$$n_k = \frac{k}{|k|}, \qquad \text{and} \qquad n_g(t) = \frac{1}{\rho_t} (g_t - \tilde{g}_t). \tag{4.6}$$

This yields

$$A_k(g_t) - A_k(\tilde{g}_t) = -2|k|^2 \rho_t \sin \frac{k \cdot (g_t + \tilde{g}_t)}{2} \frac{\sin}{|k| \rho_t} \left(\frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) n_{k^{\perp}}, \tag{4.7}$$

$$B_k(g_t) - B_k(\tilde{g}_t) = 2|k|^2 \rho_t (\cos \frac{k \cdot (g_t + \tilde{g}_t)}{2} \frac{\sin}{|k| \rho_t} \left(\frac{k \cdot (g_t - \tilde{g}_t)}{2} \right) n_{k^{\perp}}. \tag{4.8}$$

With these notations we get

$$
\begin{aligned}
&d\rho_t\\
&= \rho_t\sqrt{\nu}\sum_k \lambda_k|k|^2\int_{\mathbb{T}} 2\left(n_{k^\perp}\cdot n_g(t,\theta)\right)\frac{\sin}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\\
&\times\left(-\sin\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}dx_k(t)+\cos\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}dy_k(t)\right)\,d\theta\\
&+\rho_t\left\langle n_g(t),\delta u(t)\right\rangle_{\mathbb{T}}\,dt-\rho_t\int_{\mathbb{T}}|n_g(t,\theta)|\frac{1}{\rho_t}dL_t(\theta)\\
&+2\nu\rho_t\sum_k\lambda_k^2|k|^4\left\|\frac{\sin}{|k|\rho_t}\left(\frac{k\cdot(g_t-\tilde g_t)}{2}\right)\right\|_{\mathbb{T}}^2\,dt\\
&-2\nu\rho_t\sum_k\lambda_k^2|k|^4\\
&\times\left(\int_{\mathbb{T}}\left(n_{k^\perp}\cdot n_g(t,\theta)\right)\sin\left(\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}\right)\frac{\sin}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\,d\theta\right)^2\,dt\\
&-2\nu\rho_t\sum_k\lambda_k^2|k|^4\\
&\times\left(\int_{\mathbb{T}}\left(n_{k^\perp}\cdot n_g(t,\theta)\right)\cos\left(\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}\right)\frac{\sin}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\,d\theta\right)^2\,dt.
\end{aligned}
$$

And, finally, we obtain the following formula for the L^2 distance ρ_t between two Lagrangian flows g_t and $\tilde g_t$:

Proposition 4.2. *The Itô equation for the distance ρ_t between the diffeomorphisms g_t and $\tilde g_t$ satisfies the equation*

$$
d\rho_t=\rho_t\left(\sigma_t dz_t+b_t\,dt+\left\langle n_g(t),\delta u(t)\right\rangle_{\mathbb{T}}\,dt-da_t\right)\tag{4.9}
$$

where z_t is a real valued Brownian motion, $\sigma_t>0$ is given by

(4.10)

$$
\begin{aligned}
\sigma_t^2 =&4\nu\sum_k\lambda_k^2|k|^4\\
&\times\left(\int_{\mathbb{T}}\left(n_{k^\perp}\cdot n_g(t,\theta)\right)\sin\left(\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}\right)\frac{\sin}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\,d\theta\right)^2\\
&+4\nu\sum_k\lambda_k^2|k|^4\\
&\times\left(\int_{\mathbb{T}}\left(n_{k^\perp}\cdot n_g(t,\theta)\right)\cos\left(\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}\right)\frac{\sin}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\,d\theta\right)^2,
\end{aligned}
$$

the process b_t satisfies

$$
b_t+\frac{1}{2}\sigma_t^2 =2\nu\rho_t\sum_k\lambda_k^2|k|^4\left\|\frac{\sin}{|k|\rho_t}\left(\frac{k\cdot(g_t-\tilde g_t)}{2}\right)\right\|_{\mathbb{T}}^2\,dt\tag{4.11}
$$

and a_t is defined by

$$
a_0=0,\qquad da_t=\int_{\mathbb{T}}|n_g(t,\theta)|\frac{1}{\rho_t}dL_t(\theta).\tag{4.12}
$$

From the sde satisfied by the distance ρ_t and Girsanov's Theorem, we deduce that for all $0 < t_0 < t$,

(4.13)
$$\rho_t = \rho_{t_0} \exp\left(\int_{t_0}^t \sigma_s\, dz_s + \int_{t_0}^t \left(b_s - \frac{1}{2}\sigma_s^2 + \langle n_g(s), \delta u(s)\rangle_{\mathbb{T}}\right) ds - (a_t - a_{t_0})\right).$$

We introduce the notation

$$\delta_k = \delta_k(t,\theta) = \frac{\rho_t(n_g \cdot n_k)}{|g_t(\theta) - \tilde{g}_t\theta)|}. \tag{4.14}$$

and notice that

$$\delta_k^2 + \delta_{k^\perp}^2 = 1.$$

We obtain the following estimates,

Lemma 4.3. *We have*

$$\sigma_t^2 \leq 4\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} \delta_{k^\perp}^2 \frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta) - \tilde{g}_t(\theta))}{2}\right) d\theta \tag{4.15}$$

and

$$b_t \geq 2\nu \sum_k \lambda_k^2 |k|^4 \int_{\mathbb{T}} (n_g \cdot n_k)^2\, d\theta \int_{\mathbb{T}} \frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta) - \tilde{g}_t(\theta))}{2}\right) d\theta, \tag{4.16}$$

In particular $b_t \geq 0$.

Let $R > 0$. Assuming that $\lambda_k = 0$ for all k such that $|k| > R$ then on the set

$$\left\{\omega \mid \forall\theta \in \mathbb{T},\ |g_t(\theta) - \tilde{g}_t(\theta)| \leq \frac{\pi}{R}\right\}$$

we have

$$b_t - \frac{1}{2}\sigma_t^2 \geq 0.$$

Proof. Using Cauchy Schwartz inequality,

$$\begin{aligned}
\sigma_t^2 &\le 4\sum_k \lambda_k^2|k|^4\nu\int_{\mathbb{T}} |(n_g\cdot n_{k^\perp})|\sin^2\left(\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}\right)\frac{|\sin|}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\\
&\times\int_{\mathbb{T}}|(n_g\cdot n_{k^\perp})|)\frac{|\sin|}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\\
&+4\sum_k \lambda_k^2|k|^4\nu\int_{\mathbb{T}} |(n_g\cdot n_{k^\perp})|\cos^2\left(\frac{k\cdot(g_t(\theta)+\tilde g_t(\theta))}{2}\right)\frac{|\sin|}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\\
&\times\int_{\mathbb{T}}|(n_g\cdot n_{k^\perp})|\frac{|\sin|}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\\
&=4\nu\sum_k\lambda_k^2|k|^4\left(\int_{\mathbb{T}}|(n_g\cdot n_{k^\perp})|\frac{|\sin|}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\right)^2\\
&=4\nu\sum_k\lambda_k^2|k|^4\left(\int_{\mathbb{T}}\frac{\delta_{k^\perp}|g_t(\theta)-\tilde g_t\theta)|}{\rho_t}\frac{|\sin|}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\right)^2\\
&\le 4\nu\sum_k\lambda_k^2|k|^4\int_{\mathbb{T}}\frac{|g_t(\theta)-\tilde g_t(\theta)|^2}{\rho_t^2}\,d\theta\int_{\mathbb{T}}\delta_{k^\perp}^2\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\\
&=4\nu\sum_k\lambda_k^2|k|^4\int_{\mathbb{T}}\delta_{k^\perp}^2\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta.
\end{aligned}$$

On the other hand,

$$b_t+\frac12\sigma_t^2=2\nu\sum_k\lambda_k^2|k|^4\int_{\mathbb{T}}\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta,$$

Hence, using the bound

$$\begin{aligned}
\sigma_t^2&\le 4\nu\sum_k\lambda_k^2|k|^4\left(\int_{\mathbb{T}}|(n_g\cdot n_{k^\perp})|\frac{|\sin|}{|k|\rho_t}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta\right)^2\\
&\le 4\nu\sum_k\lambda_k^2|k|^4\int_{\mathbb{T}}(n_g\cdot n_{k^\perp})^2\,d\theta\int_{\mathbb{T}}\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta
\end{aligned}$$

we deduce that

$$b_t\ge 2\nu\sum_k\lambda_k^2|k|^4\int_{\mathbb{T}}(n_g\cdot n_k)^2\,d\theta\int_{\mathbb{T}}\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)d\theta$$

where we have used the identity

$$\int_{\mathbb{T}}(n_g\cdot n_k)^2\,d\theta+\int_{\mathbb{T}}(n_g\cdot n_{k^\perp})^2\,d\theta=1.$$

Since λ_k depends only on $|k|$, we have $\lambda_k=\lambda_{k^\perp}$ for all k. Then combining the terms corresponding to k and $k^\perp$ we obtain

$$\begin{aligned}
b_t+\frac12\sigma_t^2&=\nu\sum_k\lambda_k^2|k|^4\\
&\times\int_{\mathbb{T}}\left(\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)+\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k^\perp\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\right)d\theta,
\end{aligned}$$

From this equality, using the bound for σ_t^2 as well as the identity $\delta_k^2+\delta_{k^\perp}^2=1$, we derive

$$b_t-\frac{1}{2}\sigma_t^2\geq \nu\sum_k \lambda_k^2|k|^4$$
$$\times\int_{\mathbb{T}}(\delta_k^2-\delta_{k^\perp}^2)\left(\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)-\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k^\perp\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\right)d\theta$$
$$=\nu\sum_k\lambda_k^2|k|^4$$
$$\times\int_{\mathbb{T}}(\delta_k^2-\delta_{k^\perp}^2)\left(\frac{\sin^2}{|k|^2\rho_t^2}\left(\delta_k\frac{|k||g_t-\tilde g_t|(\theta)}{2}\right)-\frac{\sin^2}{|k|^2\rho_t^2}\left(\delta_{k^\perp}\frac{|k||g_t-\tilde g_t|(\theta)}{2}\right)\right)d\theta$$

Assuming that $\lambda_k=0$ whenever $|k|>R$, on the set

$$\left\{\omega|\forall\theta\in\mathbb{T},\ |g_t(\theta)-\tilde g_t(\theta)|\leq\frac{\pi}{R}\right\}$$

the functions inside the integral in the expression are nonegative. As a result,

$$b_t-\frac{1}{2}\sigma_t^2\geq 0.$$

□

Let us write

$$\ell(x)=\frac{\sin x}{x}\ \text{ for }\ x\neq 0,\quad \ell(0)=1. \tag{4.17}$$

From Lemma 4.3 we easily obtain the following result,

Proposition 4.4. *Let $R\geq 1$. On the set*

$$\left\{\omega|\forall\theta\in\mathbb{T},\ |g_t(\theta)-\tilde g_t(\theta)|\leq\frac{\pi\sqrt{2}}{R}\right\}$$

we have,

$$d\rho_t\geq\rho_t\left(\sigma_t dz_t-\|\delta u(t)\|_{\mathbb{T}}\,dt-\int_{\mathbb{T}}|n_g(t,\theta)|\frac{1}{\rho_t}dL_t(\theta)+c_R\,dt\right) \tag{4.18}$$

where

$$c_R=\frac{\nu}{8}\ell^2\left(\frac{\pi}{\sqrt{2}}\right)\sum_{|k|\leq R}\lambda_k^2|k|^4.$$

Moreover assuming that $\lambda_k=0$ whenever $|k|>R$, on the set

$$\left\{\omega|\forall\theta\in\mathbb{T},\ |g_t(\theta)-\tilde g_t(\theta)|\leq\frac{\pi}{2R}\right\}$$

we have,

$$d\rho_t\geq\rho_t\left(\sigma_t dz_t+\frac{1}{2}\sigma_t^2\,dt-\|\delta u(t)\|_{\mathbb{T}}\,dt+c_R'\,dt\right) \tag{4.19}$$

where

$$c_R'=\frac{1}{8}\nu\inf_{|v|=1}\sum_{|k|\leq R}\lambda_k^2|k|^4\left((n_k\cdot v)^2-(n_{k^\perp}\cdot v)^2\right)^2.$$

Proof. If $|g_t(\theta)-\tilde g_t(\theta)|\le \frac{\pi\sqrt2}{R}$ then for all k such that $|k|\le R$,

$$\ell^2\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\ge \ell^2\left(\frac{\pi}{\sqrt2}\right)$$

and this implies

$$\frac{\sin^2}{|k|^2\rho_t^2}\left(\frac{k\cdot(g_t(\theta)-\tilde g_t(\theta))}{2}\right)\ge \frac14\ell^2\left(\frac{\pi}{\sqrt2}\right)(n_k\cdot n_g)^2.$$

Therefore using (4.16) we get

$$\begin{aligned}
b_t &\ge \frac12\ell^2\left(\frac{\pi}{\sqrt2}\right)\nu\sum_{|k|\le R}\lambda_k^2|k|^4\left(\int_{\mathbb T}(n_g\cdot n_k)^2\,d\theta\right)^2\\
&\ge \frac14\ell^2\left(\frac{\pi}{\sqrt2}\right)\nu\sum_{|k|\le R}\lambda_k^2|k|^4\left(\left(\int_{\mathbb T}(n_g\cdot n_k)^2\,d\theta\right)^2+\left(\int_{\mathbb T}(n_g\cdot n_{k^\perp})^2\,d\theta\right)^2\right)\\
&\ge \frac18\ell^2\left(\frac{\pi}{\sqrt2}\right)\nu\sum_{|k|\le R}\lambda_k^2|k|^4
\end{aligned}$$

(again we have used equality $\int_{\mathbb T}(n_g\cdot n_k)^2\,d\theta+\int_{\mathbb T}(n_g\cdot n_{k^\perp})^2\,d\theta=1$). This establishes (4.18).

If $|g_t(\theta)-\tilde g_t(\theta)|\le \frac{\pi}{2R}$, from the proof of Lemma 4.3 we deduce the following inequalities,

$$\begin{aligned}
b_t-\frac12\sigma_t^2 &\ge \nu\sum_{|k|\le R}\lambda_k^2|k|^4\\
&\times\int_{\mathbb T}(\delta_k^2-\delta_{k^\perp}^2)\left(\frac{\sin^2}{|k|^2\rho_t^2}\left(\delta_k\frac{|k||g_t-\tilde g_t|(\theta)}{2}\right)-\frac{\sin^2}{|k|^2\rho_t^2}\left(\delta_{k^\perp}\frac{|k||g_t-\tilde g_t|(\theta)}{2}\right)\right)d\theta\\
&\ge \nu\sum_{|k|\le R}\lambda_k^2|k|^4\int_{\mathbb T}(\delta_k^2-\delta_{k^\perp}^2)^2\frac{|g_t-\tilde g_t|^2(\theta)}{8\rho_t^2}\,d\theta\\
&\ge \int_{\mathbb T}\frac{|g_t-\tilde g_t|^2(\theta)}{\rho_t^2}c_R'\,d\theta=c_R'.
\end{aligned}$$

This establishes (4.19). □

We can now describe how the distance of two stochastic Lagrangian flows evolve along the time. They will get exponentially apart, thus exibiting a kind of chaotic behavior, at least during some time interval. This is the content of next theorem. When one compares the separation of the stochastic trajectories with the one of the deterministic integral flows for the Navier-Stokes equations, the spread out of the stochastic trajectories is larger, at least for small times. We refer to [2] for examples and simulations.

Theorem 4.5. *Let $t>0$, $R\ge1$ and*

$$\Omega_t=\left\{\omega\in\Omega,\ \forall s\le t,\ \forall\theta\in\mathbb T,\ |(g_s(\theta)(\omega)-\tilde g_s(\theta)(\omega))|\le\frac{\pi}{2R}\right\}.$$

If we assume the initial conditions for the L^2 distance and the L^2 norm of the initial velocity to be related as $c=\rho_0-2\|u_0\|_{\mathbb T}>0$, and suppose that $\int_{\mathbb T}u=0$, then on *the set Ω_t we have,*

$$\forall s \leq t, \quad \rho_s \geq e^{\int_0^t \sigma_s\, dz_s + c'_R t} \left(\rho_0 - 2\|u_0\|_{\mathbb{T}} \int_0^t e^{-\int_0^s \sigma_r\, dz_r - (c'_R + \frac{\nu}{2})s}\, ds \right) \tag{4.20}$$

as long as the right hand side stays positive.

On the other hand if we assume that there exist constants $c_1, c_2 > 0$ such that for all $\theta \in \mathbb{T}$ and $s \in [0,t]$,

$$|\nabla u(t,\theta)| \leq c_1 e^{-c_2 t}, \tag{4.21}$$

then on Ω_t we have the more precise lower bound, holding $\forall s \leq t$,

$$\rho_s \geq \rho_0 \exp\left(\int_0^t \sigma_s\, dz_s + c'_R t - \frac{c_1}{c_2}\left(1 - e^{-c_2 t}\right) \right). \tag{4.22}$$

Proof. Assume that $\rho_0 - 2\|u_0\|_{\mathbb{T}} > 0$. From inequality (4.19) we deduce that on the set Ω_t, for $s \leq t$,

$$d\rho_s \geq \rho_s \left(\sigma_s dz_s + (c'_R + \frac{1}{2}\sigma_s^2)\, ds \right) - 2\|u(s,\cdot)\|_{\mathbb{T}}\, ds. \tag{4.23}$$

Using the fact that $u(t,.)$ satisfies Navier-Stokes equation together with Poincaré inequality,

$$\begin{aligned} \frac{d}{ds}||u(s,.)||_{\mathbb{T}}^2 &= -2\nu||\nabla u(s,.)||_{\mathbb{T}}^2 \\ &\leq -\nu||u(s,.)||_{\mathbb{T}}^2. \end{aligned}$$

Therefore we have

$$||u(s,.)||_{\mathbb{T}} \leq e^{-\frac{\nu}{2}s}||u_0||_{\mathbb{T}}.$$

We obtain

$$d\rho_s \geq \rho_s \left(\sigma_s dz_s + (c'_R + \frac{1}{2}\sigma_s^2)\, ds \right) - 2e^{-\frac{\nu}{2}s}||u_0||_{\mathbb{T}}\, ds. \tag{4.24}$$

Comparison theorems for solutions of sde's yield (4.20).

Now assume (4.21). To prove (4.22) we start with (4.19), and remark that $\|\delta u(t)\|_{\mathbb{T}} \leq \sup_{\theta \in \mathbb{T}} |\nabla u(t,\theta)|$. Then from the bound on $\nabla u(t,\theta)$ we derive

$$d\rho_t \geq \rho_t \left(\sigma_t dz_t + \frac{1}{2}\sigma_t^2\, dt - c_1 e^{-c_2 t}\, dt + c'_R\, dt \right).$$

Integrating the right hand side between t_0 and t gives the result. □

Remark 4.6. The bound (4.21) is satisfied for instance for solutions $u(t,\cdot)$ of the form $e^{-\nu|k|^2 t}A_k$.

Remark 4.7. Assumption (4.21) implies that the velocity decays to zero at exponential rate. On the contrary the stochastic Lagrangian flows, describing the position of the fluid, get apart exponentially, at least for short times.

Remark 4.8. Also notice that, by the expression of the constant c'_R, the stochastic Lagrangian trajectories for a fluid with a given viscosity constant tend to get apart faster when the higher Fourier modes (and therefore the smaller lenght scales) are randomly excited.

5. Stability: the two-dimensional torus endowed with the extrinsic distance

It seems difficult to deal with the local time term in Proposition 4.2. To circumvent this problem we propose here to endow the torus $\mathbb{T}$ with a distance $\rho_{\mathbb{T}}$ which is equivalent to the one of section 5, but such that $\rho_{\mathbb{T}}^2$ is smooth on $\mathbb{T}\times\mathbb{T}$. The assumptions of Theorem 4.5 are not fulfilled and the behaviour of the distance of two diffeomorphisms can be completely different even if their distance is small. The uniform control of the distance in Theorem 4.5 looks therefore as a necessary condition for obtaining an exponential growth of the distance.

The map

$$\begin{aligned}\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z} &\to [0,2]\\ (\theta_1,\theta_2) &\mapsto 2\left|\sin\left(\frac{\theta_2-\theta_1}{2}\right)\right|\end{aligned}$$

defines a distance on the circle $\mathbb{R}/2\pi\mathbb{Z}$: it is the extrinsic distance on the circle embedded in the plane. We can define the corresponding product distance on the torus $\mathbb{T}$ as

$$\rho_{\mathbb{T}}((\theta_1,\theta_2),(\theta_1',\theta_2'))=2\left(\sin^2\left(\frac{\theta_1'-\theta_1}{2}\right)+\sin^2\left(\frac{\theta_2'-\theta_2}{2}\right)\right)^{1/2}.$$

Notice that

$$\rho_{\mathbb{T}}^2((\theta_1,\theta_2),(\theta_1',\theta_2'))=2\left(2-\cos(\theta_1'-\theta_1)-\cos(\theta_2'-\theta_2)\right).$$

The distance $\rho_{\mathbb{T}}^2$ is smooth on $\mathbb{T}\times\mathbb{T}$.

Now let ϕ and ψ be two diffeomorphisms on the torus $\mathbb{T}$: we define $\rho(\phi,\psi)$ via the formula

$$\begin{aligned}\rho^2(\phi,\psi)&=\int_{\mathbb{T}}\rho_{\mathbb{T}}^2(\phi(\theta),\psi(\theta))\,d\theta\\ &=2\int_{\mathbb{T}}\left(2-\cos(\phi^1(\theta)-\psi^1(\theta))-\cos(\phi^2(\theta)-\psi^2(\theta))\right)\,d\theta\\ &=4\int_{\mathbb{T}}\left(\sin^2\left(\frac{\phi^1(\theta)-\psi^1(\theta)}{2}\right)+\sin^2\left(\frac{\phi^2(\theta)-\psi^2(\theta)}{2}\right)\right)\,d\theta\end{aligned}$$

Denote

$$\rho_t=\rho(g_t,\tilde g_t).$$

Because of the smoothness of $\rho_{\mathbb{T}}^2$, the formula for ρ_t does not involve a local time. More precisely, writing

$$\delta g=g_t(\theta)-\tilde g_t(\theta),$$

$$\delta\cos k\cdot g=\cos k\cdot g_t(\theta)-\cos k\cdot\tilde g_t(\theta),$$

$$\delta\sin k\cdot g=\sin k\cdot g_t(\theta)-\sin k\cdot\tilde g_t(\theta),$$

$$\sin\delta g=(\sin(\delta g_t)_1(\theta),\ \sin(\delta g_t)_2(\theta)),$$

$$\delta u=(u(t,g_t)-u(t,\tilde g_t))$$

we get from Itô calculus,

$$d\rho_t = \rho_t \sum_k \lambda_k \left\langle \frac{\sin \delta g}{\rho_t}, (k_2, -k_1) \left(\frac{\delta \cos k\cdot g}{\rho_t} dx_k + \frac{\delta \sin k \cdot g}{\rho_t} dy_k \right) \right\rangle_{\mathbb{T}}$$

$$+ \rho_t \left\langle \frac{\sin \delta g}{\rho_t}, \ \frac{\delta u}{\rho_t} \right\rangle_{\mathbb{T}} dt$$

$$+ \frac{\rho_t}{2} \left(\sum_k \lambda_k^2 \int_{\mathbb{T}} \left(k_2^2 \cos \delta g_1 + k_1^2 \cos \delta g_2\right) \frac{(\delta \cos k\cdot g)^2 + (\delta \sin k \cdot g)^2}{\rho_t^2} d\theta \right) dt$$

$$- \frac{\rho_t}{2} \sum_k \lambda_k^2 \left(\int_{\mathbb{T}} \left(k_2 \frac{\sin \delta g_1}{\rho_t} - k_1 \frac{\sin \delta g_2}{\rho_t} \right) \frac{\delta \cos k \cdot g}{\rho_t} d\theta \right)^2 dt$$

$$- \frac{\rho_t}{2} \sum_k \lambda_k^2 \left(\int_{\mathbb{T}} \left(k_2 \frac{\sin \delta g_1}{\rho_t} - k_1 \frac{\sin \delta g_2}{\rho_t} \right) \frac{\delta \sin k \cdot g}{\rho_t} d\theta \right)^2 dt.$$

Clearly the Itô differential of the distance ρ_t has the form

$$d\rho_t = \rho_t \left(\sigma_t \, dz_t + b_t \, dt \right)$$

where σ_t and b_t are bounded processes and z_t is a real- valued Brownian motion. However the drift may be negative even if ρ_t is small, as the following example shows.

Example 5.1. Let $\alpha > 0$ be small and $\varepsilon > 0$ satisfying $\varepsilon << \alpha$. Take $\phi = \mathrm{id}$ and assume that there exist two subsets E_1 and E_2 of $\mathbb{T}$ such that $E_1 \subset E_2$, E_1 has measure α, E_2 has measure $\alpha+\varepsilon$, $\psi(\theta) = \theta$ for all $\theta \in \mathbb{T}\backslash E_2$ and $\psi(\theta) = (\theta_1 + \pi, \theta_2)$ for all $\theta \in E_1$. Since ε can be as small as we want, we have

$$\rho_0^2 \simeq 4\alpha, \quad (\sin \delta g)_0 \simeq 0, \quad (\delta g_0)_2 \simeq 0, \quad (\delta \sin k \cdot g)_0 \simeq 0,$$

$$\text{on } \mathbb{T}\backslash E_2, \quad (\delta \cos k \cdot g)_0 = 0,$$

$$\text{on } E_1, \quad (\delta \cos k \cdot g)_0 = -2 \quad \text{if } k_1 \text{is odd}, \quad (\delta \cos k \cdot g)_0 = 0 \quad \text{if } k_1 \text{ is even},$$

Therefore at time $t = 0$ we have ,

$$d\rho_t \simeq -\frac{\rho_t}{2} \left(\sum_{k_1 \text{ odd}} \lambda_k^2 k_2^2 \right) dt.$$

In order to construct a diffeomorphism like ψ, one can cut an annulus E_1 of width $\dfrac{\alpha}{2\pi}$ in $\mathbb{T}$ and rotate it by π. This yields a one to one map on $\mathbb{T}$. Then smoothen it around the boundary of the annulus to get ψ. For the set E_2 one can take an annulus of width $\dfrac{\alpha + \varepsilon}{2\pi}$ containing E_1.

6. Stability: a formula for the distance of two particles on a general Riemannian manifold

Let $B_t = (B_t^\ell)_{\ell \geq 0}$ be a family of independent real Brownian motions, $\sigma = (\sigma_\ell)_{\ell \geq 0}$, with, for all $\ell \geq 0$, σ_ℓ a divergence free vector field on M. We furthermore assume that

$$\sigma(x)\sigma^*(y) = a(x,y). \tag{6.1}$$

In particular

$$\sigma(x)\sigma^*(x) = 2\nu \mathbf{g}^{-1}(x). \tag{6.2}$$

We let $\varphi, \psi \in G_V^0$. In this section we assume that

$$dg_t(x) = \sigma(g_t(x))\, dB_t + u(t, g_t(x))\, dt, \quad g_0 = \varphi \tag{6.3}$$

and

$$d\tilde{g}_t(x) = \sigma(\tilde{g}_t(x))\, dB_t + u(t, \tilde{g}_t(x))\, dt, \quad \tilde{g}_0 = \psi \tag{6.4}$$

For simplicity we write $x_t = g_t(x)$, $y_t = \tilde{g}_t(x)$ and

$$\rho_t(x) = \rho_M(x_t, y_t)$$

For $x, y \in M$ such that y does not belong to the cutlocus of x, we consider $a \mapsto \gamma_a(x, y)$, the minimal geodesic in time 1 from x to y ($\gamma_0(x, y) = x$, $\gamma_1(x, y) = y$)). For $a \in [0, 1]$ let $J_a = T\gamma_a$ be the tangent map to γ_a. In other words, for $v \in T_x M$ and $w \in T_y M$, $J_a(v, w)$ is the value at time a of the Jacobi field along $\gamma_\cdot$ which takes the values v at time 0 and w at time 1.

We first consider the case where y_t does not belong to the cutlocus of x_t. We note $T_a = T_a(t) = \dot{\gamma}_a(x_t, y_t)$ and $\gamma_a(t) = \gamma_a(x_t, y_t)$.

Letting $P(\gamma_a)_t$ be the parallel transport along $\gamma_a(t)$, the following formula for the Itô covariant differential holds,

$$\begin{aligned} \mathscr{D}\dot{\gamma}_a(t) &:= P(\gamma_a)_t d\left(P(\gamma_a)_t^{-1}\dot{\gamma}_a(t)\right) \\ &= \nabla_{d\gamma_a(t)}\dot{\gamma}_a + \frac{1}{2}\nabla_{d\gamma_a(t)} \cdot \nabla_{d\gamma_a(t)}\dot{\gamma}_a(t). \end{aligned}$$

On the other hand the Itô differential $d\gamma_a(t)$ satisfies

$$d\gamma_a(t) = J_a(dx_t, dy_t) + \frac{1}{2}\left(\nabla_{(dx_t, dy_t)} J_a\right)(dx_t, dy_t).$$

So we get

$$\mathscr{D}\dot{\gamma}_a(t) = \nabla_{J_a(dx_t, dy_t)}\dot{\gamma}_a + \nabla_{\frac{1}{2}\left(\nabla_{(dx_t, dy_t)} J_a\right)(dx_t, dy_t)}\dot{\gamma}_a + \frac{1}{2}\nabla_{d\gamma_a(t)} \cdot \nabla_{d\gamma_a(t)}\dot{\gamma}_a(t). \tag{6.5}$$

Let $e(t) \in T_{x_t} M$ be the unit vector satisfying $T_0(t) = \rho_t(x) e(t)$. For $\ell \geq 0$ we let $a \mapsto J_a^\ell(t, x)$ be the Jacobi field such that $J_0^\ell(t, x) = \sigma_\ell(g_t(x))$, $J_1^\ell(t) = \sigma_\ell(\tilde{g}_t(x))$. Moreover we assume that $\nabla_{J_0^\ell(t,x)} J_0^\ell(t, x) = 0$ and $\nabla_{J_1^\ell(t,x)} J_1^\ell(t, x) = 0$.

With these notations, equation (6.5) can be written as

$$\begin{aligned} \mathscr{D}T_a &= \nabla_{J_a(dx_t, dy_t)} T_a + \frac{1}{2}\sum_{\ell \geq 0} \nabla_{\nabla_{J_a^\ell} J_a^\ell} T_a\, dt + \frac{1}{2}\sum_{\ell \geq 0} \nabla_{J_a^\ell} \cdot \nabla_{J_a^\ell} T_a\, dt \\ &= \dot{J}_a(dx_t, dy_t) + \frac{1}{2}\sum_{\ell \geq 0} \nabla_{J_a^\ell} \nabla_{J_a^\ell} T_a\, dt. \end{aligned}$$

We have

$$\begin{aligned}
d\rho_t(x) &= d\left(\left(\int_0^1 \langle T_a(t), T_a(t)\rangle \, da\right)^{1/2}\right) \\
&= \frac{1}{2\rho_t(x)}\left(2\int_0^1 \langle \mathscr{D}T_a(t), T_a(t)\rangle \, da + \int_0^1 \langle \mathscr{D}T_a(t), \mathscr{D}T_a(t)\rangle \, da\right) \\
&- \frac{1}{8\rho_t(x)^3} d\left(\|T_0\|^2\right)\cdot d\left(\|T_0\|^2\right) \\
&= \sum_{\ell\geq 0}\left\langle \dot J_0^\ell(t,x), e_t(x)\right\rangle dB_t^\ell + \left\langle \dot J_0(u(t,g_t(x)), u(t,\tilde g_t(x))), e_t(x)\right\rangle \\
&+ \frac{1}{2\rho_t(x)}\left(\int_0^1 \sum_{\ell\geq 0}\left\langle \nabla_{J_a^\ell}\nabla_{J_a^\ell}T_a, T_a\right\rangle da\, dt + \sum_{\ell\geq 0}\int_0^1 \|\dot J_a^\ell\|^2 \, da\right) \\
&- \frac{1}{2\rho_t(x)}\sum_{\ell\geq 0}\langle \dot J_0^\ell(t,x), e_t(x)\rangle^2.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^1 \left\langle \nabla_{J_a^\ell}\nabla_{J_a^\ell}T_a, T_a\right\rangle da &= \int_0^1 \left\langle \nabla_{J_a^\ell}\nabla_{T_a}J_a^\ell, T_a\right\rangle da \\
&= \int_0^1 \left\langle \nabla_{T_a}\nabla_{J_a^\ell}J_a^\ell, T_a\right\rangle da - \int_0^1 \left\langle R(T_a, J_a^\ell)J_a^\ell, T_a\right\rangle da \\
&= \int_0^1 T_a\left\langle \nabla_{J_a^\ell}J_a^\ell, T_a\right\rangle da - \int_0^1 \left\langle R(T_a, J_a^\ell)J_a^\ell, T_a\right\rangle da \\
&= \left[\left\langle \nabla_{J_a^\ell}J_a^\ell, T_a\right\rangle\right]_0^1 - \int_0^1 \left\langle R(T_a, J_a^\ell)J_a^\ell, T_a\right\rangle da \\
&= -\int_0^1 \left\langle R(T_a, J_a^\ell)J_a^\ell, T_a\right\rangle da
\end{aligned}$$

where we used the fact that $\nabla_{J_a^\ell}J_a^\ell = 0$ for $a = 0, 1$. Hence,

$$\begin{aligned}
d\rho_t(x) &= \sum_{\ell\geq 0}\left\langle \dot J_0^\ell(t,x), e_t(x)\right\rangle dB_t^\ell \\
&+ \left\langle \dot J_0(u(t,g_t(x)), u(t,\tilde g_t(x))), e_t(x)\right\rangle \\
&+ \frac{1}{2\rho_t(x)}\left(\int_0^1 \sum_{\ell\geq 0}\left(\|\dot J_a^{\ell,N}\|^2 - \left\langle R(T_a(t,x), J_a^{\ell,N}(t,x))J_a^{\ell,N}(t,x), T_a(t,x)\right\rangle\right) da\right) dt
\end{aligned}$$

with $J_a^{\ell,N}(t,x)$ the part of $J_a^\ell(t,x)$ normal to T_a.

Removing the assumption that y_t does not belong to the cutlocus of x_t, it is well known (see [8] for a similar argument) that the formula becomes

$$\begin{aligned} d\rho_t(x) &= \sum_{\ell\ge 0}\left\langle \dot J_0^\ell(t,x), e_t(x)\right\rangle\, dB_t^\ell \\ &+ \left\langle \dot J_0(u(t,g_t(x)), u(t,\tilde g_t(x))), e_t(x)\right\rangle - dL_t(x) \\ &+ \frac{1}{2\rho_t(x)}\left(\int_0^1 \sum_{\ell\ge 0}\left(\|\dot J_a^{\ell,N}\|^2 - \left\langle R(T_a(t,x), J_a^{\ell,N}(t,x))J_a^{\ell,N}(t,x), T_a(t,x)\right\rangle\right)\, da\right)\, dt \end{aligned}$$

where $-L_t(x)$ is the local time of $\rho_t(x)$ when $(g_t(x), \tilde g_t(x))$ visits the cutlocus. Then writing

$$\rho_t = \rho(g_t, \tilde g_t) = \left(\int_M \rho_t^2(x)\, dx\right)^{1/2},$$

we obtain

$$\begin{aligned} d\rho_t &= \frac{1}{\rho_t}\sum_{\ell\ge 0}\left(\int_M \rho_t(x)\left\langle \dot J_0^\ell(t,x), e_t(x)\right\rangle\, dx\right)\, dB_t^\ell \\ &+ \frac{1}{\rho_t}\int_M \rho_t(x)\left\langle \dot J_0(u(g_t(x)), u(\tilde g_t(x))), e_t(x)\right\rangle\, dx\, dt - \frac{1}{\rho_t}\int_M \rho_t(x) L_t(x)\, dx \\ &+ \frac{1}{2\rho_t}\left(\int_M \sum_{\ell\ge 0}\left(\int_0^1\left(\|\dot J_a^{\ell,N}\|^2 - \left\langle R(T_a(t,x), J_a^{\ell,N}(t,x))J_a^{\ell,N}(t,x), T_a(t,x)\right\rangle\right)\, da\right)\, dx\right)\, dt \\ &+ \frac{1}{2\rho_t}\int_M \sum_{\ell\ge 0}\left\langle \dot J_0^\ell(t,x), e_t(x)\right\rangle^2\, dx\, dt \\ &- \frac{1}{2\rho_t^3}\sum_{\ell\ge 0}\left(\int_M \rho_t(x)\left\langle \dot J_0^\ell(t,x), e_t(x)\right\rangle\, dx\right)^2\, dt. \end{aligned}$$

For a vector $w \in T_{g_t(x)}M$, we denote w^T the part of w which is tangential to $T_0(t,x)$. Writing

$$\cos\left(\dot J_0^{\ell,T}(t,\cdot), T_0(t,\cdot)\right) = \frac{\int_M \left\langle \dot J_0^{\ell,T}(t,x), T_0(t,x)\right\rangle\, dx}{\rho_t\left(\int_M \left\|\dot J_0^{\ell,T}(t,x)\right\|^2\, dx\right)^{1/2}}$$

(observe that $\rho_t^2 = \displaystyle\int_M \|T_0(t,x)\|^2\, dx$), we have therefore proved the following result:

Proposition 6.1. *The Itô differential of the distance ρ_t between g_t and $\tilde{g}_t$ is given by*

$$\begin{aligned} d\rho_t = &\frac{1}{\rho_t}\sum_{\ell\geq 0}\left(\int_M \rho_t(x)\left(P_{\tilde{g}_t(x),g_t(x)}(\sigma_\ell^T(\tilde{g}_t(x))) - \sigma_\ell^T(g_t(x)\right)\,dx\right)dB_t^\ell \\ &+\frac{1}{\rho_t}\int_M \rho_t(x)\left(P_{\tilde{g}_t(x),g_t(x)}(u^T(\tilde{g}_t(x)))) - u^T(g_t(x))\right)\,dx\,dt - \frac{1}{\rho_t}\int_M \rho_t(x)dL_t(x)\,dx \\ &+\frac{1}{2\rho_t}\left(\int_M\sum_{\ell\geq 0}\left(\int_0^1\left(\|\dot{J}_a^{\ell,N}\|^2 - \left\langle R(T_a(t,x),J_a^{\ell,N}(t,x))J_a^{\ell,N}(t,x),T_a(t,x)\right\rangle\right)da\right)dx\right)dt \\ &+\frac{1}{2\rho_t}\sum_{\ell\geq 0}\left(1-\cos^2\left(\dot{J}_0^{\ell,T}(t,\cdot),T_0(t,\cdot)\right)\right)\int_M\left\|\dot{J}_0^{\ell,T}(t,x)\right\|^2\,dx\,dt. \end{aligned}$$

In the case of manifolds with negative curvature we may observe a similar phenomena to the one of the torus with the Euclidean distance treated in Section 4: as long as the L^∞ norm stays sufficiently small to avoid the cut-locus of the manifold, the L^2 mean distance between the stochastic particles tends to increase exponentially fast.

Let us mention that in [1] another approach, using coupling methods, was developed for the study of the distance between stochastic Lagrangian flows.

7. The rotation process

In the following we would like to study the rotation of two particles $g_t(x)$ and $\tilde{g}_t(x)$ when their distance is small. Recall that we have denoted $x_t = g_t(x)$, $y_t = \tilde{g}_t(x)$. We shall keep here the definitions and notations of last section.

We always assume that the distance from x_t to y_t is small: we are interested in the behaviour of $e(t)$ as $\rho_t(x)$ converges to zero. We let

$$d_m x(t)^N = \sigma(x_t)dB_t - \langle\sigma(x_t)dB_t, e(t)\rangle e(t) \tag{7.1}$$

and

$$d_m y(t)^N = \sigma(y_t)dB_t - \langle\sigma(y_t)dB_t, P_{x_t,y_t}e(t)\rangle P_{x_t,y_t}e(t) \tag{7.2}$$

where $P_{x_t,\gamma_a(t)}$ denotes the parallel transport along γ_a.

From Itô formula we have

$$\mathscr{D}T_0 = \rho_t(x)\mathscr{D}e(t) + d\rho_t(x)e(t) + d\rho_t(x)\mathscr{D}e(t) \tag{7.3}$$

and this yields

$$\begin{aligned}\mathscr{D}e(t) &= \frac{1}{\rho_t(x)}\mathscr{D}T_0 - \frac{1}{\rho_t(x)}d\rho_t(x)e(t) - \frac{1}{2}\frac{1}{\rho_t(x)}d\rho_t(x)\mathscr{D}e(t)\\
&= \frac{1}{\rho_t(x)}\dot J_0(d_m x(t)^N, d_m y(t)^N)\\
&+ \frac{1}{\rho_t(x)}\dot J_0(u(t,x_t),u(t,y_t))\,dt + \frac{1}{2\rho_t(x)}\sum_{\ell\ge 0}\nabla_{J_0^\ell}\nabla_{J_0^\ell}T_0\,dt\\
&- \frac{1}{\rho_t(x)}\left\langle P_{y_t,x_t}(u(t,y_t)) - u(t,x_t), e(t)\right\rangle e(t)\\
&- \frac{1}{2\rho_t(x)^2}\left(\int_0^1\sum_{\ell\ge 0}\left(\|\nabla_{T_a}J_a^\ell\|^2 - R(T_a,J_a^\ell)J_a^\ell,T_a\right)\,da\right)e(t)\\
&- \frac{1}{2}\frac{1}{\rho_t(x)}d\rho_t(x)\mathscr{D}e(t)\\
&= \frac{1}{\rho_t(x)}\dot J_0(d_m x(t)^N, d_m y(t)^N) + \frac{1}{\rho_t(x)}\dot J_0(u^N(t,x_t),u^N(t,y_t))\\
&+ \frac{1}{2\rho_t(x)}\sum_{\ell\ge 0}\nabla_{J_0^\ell}\nabla_{J_0^\ell}T_0\,dt\\
&- \frac{1}{2\rho_t(x)^2}\left(\int_0^1\sum_{\ell\ge 0}\left(\|\nabla_{T_a}J_a^\ell\|^2 - R(T_a,J_a^\ell)J_a^\ell,T_a\right)\,da\right)e(t)\end{aligned}$$

where we used the fact that $d\rho_t(x)\mathscr{D}e(t) = 0$, and where u^N denotes the part of u which is normal to the geodesic γ_a. As before,

$$\nabla_{J_0^\ell}\nabla_{J_0^\ell}T_0 = \nabla_{T_0}\nabla_{J_0^\ell}J_0^\ell - R(T_0,J_0^\ell)J_0^\ell.$$

Finally we obtain the following

Lemma 7.1.

$$\begin{aligned}\mathscr{D}e(t) &= \frac{1}{\rho_t(x)}\dot J_0(d_m x(t)^N, d_m y(t)^N) + \frac{1}{\rho_t(x)}\dot J_0(u^N(t,x_t),u^N(t,y_t))\\
&+ \frac{1}{2\rho_t(x)}\sum_{\ell\ge 0}\nabla_{T_0}\nabla_{J^\ell}J^\ell - R(T_0,J_0^\ell)J_0^\ell\,dt\\
&- \frac{1}{2\rho_t(x)^2}\left(\int_0^1\sum_{\ell\ge 0}\left(\|\nabla_{T_a}J_a^\ell\|^2 - R(T_a,J_a^\ell)J_a^\ell,T_a\right)\,da\right)e(t).\end{aligned}$$

From now on we assume that $M = \mathbb{T}$ the two dimensional torus.

In this situation the curvature tensor vanishes and the following formulas hold:

$$J_a(v,w) = v + a(w-v), \qquad \dot J_a(v,w) = w - v.$$

We immediately get

$$\begin{aligned}de(t) = \mathscr{D}e(t) =&\frac{1}{\rho_t(x)}\left(d_m y(t)^N - d_m x(t)^N\right) + \frac{1}{\rho_t(x)}\left((u^N(t,y_t) - u^N(t,x_t)\right)\,dt\\
&- \frac{1}{2\rho_t(x)^2}\sum_{\ell\ge 0}\|\sigma_\ell(y_t) - \sigma_\ell(x_t)\|^2\,dt\,e(t)\end{aligned}$$

where we used the fact that $\nabla_{T_0}\nabla_{J^\ell}J^\ell = 0$, as a consequence of $\nabla_{J_0^\ell}J_0^\ell = 0$, $\nabla_{J_1^\ell}J_1^\ell = 0$, and $R \equiv 0$.

Let us specialize again to the case where the vector fields are given by

$$A_k(\theta) = (k_2, -k_1)\cos k.\theta, \quad B_k(\theta) = (k_2, -k_1)\sin k.\theta$$

and the Brownian motion is of the form

$$dW(t) = \sum_{k\in\mathbb{Z}} \lambda_k \sqrt{\nu}(A_k dx_k + B_k dy_k) \tag{7.4}$$

where x_k, y_k are independent copies of real Brownian motions. As in section 5 we assume that $\sum_k |k|^2\lambda_k^2 < \infty$ and we consider $\lambda_k = \lambda(|k|)$ to be nonzero for a equal number of k_1 and k_2 components. Again we write

$$dg_t = (odW(t)) + u(t, g_t)dt, \qquad d\tilde{g}_t = (odW(t)) + u(t, \tilde{g}_t)dt \tag{7.5}$$

with

$$g_0 = \phi, \qquad \tilde{g}_0 = \psi, \qquad \phi \neq \psi.$$

Changing the notation to $g_t = g_t(\theta) = x_t$, $\tilde{g}_t = \tilde{g}_t(\theta) = y_t$, we get

$$\begin{aligned}
de(t) &= \frac{1}{\rho_t(\theta)} \sum_{|k|\neq 0} \lambda_k \sqrt{\nu}\,(\cos k\cdot\tilde{g}_t - \cos k\cdot g_t)\, k^{\perp,N} dx_k \\
&+ \frac{1}{\rho_t(\theta)} \sum_{|k|\neq 0} \lambda_k \sqrt{\nu}\,(\sin k\cdot\tilde{g}_t - \sin k\cdot g_t)\, k^{\perp,N} dy_k \\
&+ \frac{1}{\rho_t(\theta)} \big((u^N(t,\tilde{g}_t) - u^N(t,g_t)\big)\, dt \\
&- \frac{1}{2\rho_t^2(\theta)} \sum_{|k|\neq 0} \lambda_k^2 \nu |k^{\perp,N}|^2 \Big((\cos k\cdot\tilde{g}_t - \cos k\cdot g_t)^2 + (\sin k\cdot\tilde{g}_t - \sin k\cdot g_t)^2\Big)\, e(t)\, dt \\
&= \frac{1}{\rho_t(\theta)} \sum_{|k|\neq 0} \lambda_k \sqrt{\nu} k^{\perp,N} \left(2\sin\frac{k\cdot(\tilde{g}_t - g_t)}{2}\right) dz_k \\
&+ \frac{1}{\rho_t(\theta)} \big((u^N(t,\tilde{g}_t) - u^N(t,g_t)\big)\, dt \\
&- \frac{2}{\rho_t^2(\theta)} \sum_{|k|\neq 0} \lambda_k^2 \nu |k^{\perp,N}|^2 \sin^2(\frac{k\cdot(\tilde{g}_t - g_t)}{2}) e(t)\, dt
\end{aligned}$$

where z_k is the Brownian motion defined by

$$dz_k = -\sin\frac{k\cdot(\tilde{g}_t + g_t)}{2}\, dx_k + \cos\frac{k\cdot(\tilde{g}_t + g_t)}{2}\, dy_k.$$

Denoting $|k^{\perp,N}|^2 = |k|^2(n_k\cdot e(t))^2$, we obtain

$$\begin{aligned} de(t) &= \frac{1}{\rho_t(\theta)}\sum_{|k|\neq 0}|k|\lambda_k\sqrt{\nu}(n_k\cdot e(t))e'(t)\left(2\sin\frac{k\cdot(\tilde g_t-g_t)}{2}\right)dz_k \\ &+\frac{1}{\rho_t(\theta)}\left((u^N(t,\tilde g_t)-u^N(t,g_t)\right)\,dt \\ &-\frac{2}{\rho_t^2(\theta)}\sum_{|k|\neq 0}|k|^2\lambda_k^2\nu(n_k\cdot e(t))^2\sin^2\frac{k\cdot(\tilde g_t-g_t)}{2}e(t)\,dt \end{aligned} \tag{7.6}$$

where $e'(t)$ is a unit vector in $\mathbb{T}$ orthonormal to $e(t)$. Now for every $K>0$, if $\rho_t(\theta)\le\dfrac{\pi}{2K}$, then for all k such that $|k|\le K$,

$$\frac{\sin^2\frac{k\cdot(\tilde g_t-g_t)}{2}}{|k|^2\rho_t^2(\theta)(n_k\cdot e(t))^2}\ge\frac{1}{\pi^2}.$$

Since $|k|=|k^\perp|$ and $(n_k\cdot e(t))^2+(n_{k^\perp}\cdot e(t))^2=1$, we get

$$\frac{2}{\rho_t^2(\theta)}\sum_{|k|\neq 0}|k|^2\lambda_k^2\nu(n_k\cdot e(t))^2\sin^2\frac{k\cdot(\tilde g_t-g_t)}{2}\ge\frac{\nu}{2\pi^2}\sum_{0<|k|<K}\lambda_k^2|k|^4. \tag{7.7}$$

Observe that the term in the left is the second part of the drift in equation (7.6) as well as the derivative of the quadratic variation of $e(t)$. This yields the following result,

Proposition 7.2. *Identifying $T\mathbb{T}$ with $\mathbb{C}$, we have $e(t)=e^{iX_t}$ where X_t is a real-valued semimartingale with quadratic variation satisfying*

$$d[X,X]_t=\frac{4}{\rho_t^2(\theta)}\sum_{|k|\neq 0}|k|^2\lambda_k^2\nu(n_k\cdot e(t))^2\sin^2\frac{k\cdot(\tilde g_t-g_t)}{2}\,dt \tag{7.8}$$

and drift given by

$$\int_0^t\frac{1}{\rho_s(\theta)}\left\langle u(s,\tilde g_s)-u(s,g_s),ie(s)\right\rangle\,ds. \tag{7.9}$$

For all $K>0$, on the set $\left\{\rho_t(\theta)\le\dfrac{\pi}{2K}\right\}$, we have

$$d[X,X]_t\ge\frac{\nu}{\pi^2}\sum_{0<|k|<K}\lambda_k^2|k|^4. \tag{7.10}$$

If $\displaystyle\sum_{|k|\neq 0}\lambda_k^2|k|^4=+\infty$, then as $\tilde g_t(\theta)$ gets closer and closer to $g_t(\theta)$, the rotation $e(t)$ becomes more and more irregular in the sense that the derivative of the quadratic variation of X_t tends to infinity.

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