A NUMERICAL STUDY OF VARIABLE DEPTH KDV EQUATIONS AND GENERALIZATIONS OF CAMASSA-HOLM-LIKE EQUATIONS

MARC DURUFLÉ AND SAMER ISRAWI

Abstract. In this paper we study numerically the KdV-top equation and compare it with the Boussinesq equations over uneven bottom. We use here a finite-difference scheme that conserves a discrete energy for the fully discrete scheme. We also compare this approach with the discontinuous Galerkin method. For the equations obtained in the case of stronger nonlinearities and related to the Camassa-Holm equation, we find several finite difference schemes that conserve a discrete energy for the fully discrete scheme. Because of its accuracy for the conservation of energy, our numerical scheme is also of interest even in the simple case of flat bottoms. We compare this approach with the discontinuous Galerkin method.

Keywords : Kdv equation, Camassa-Holm equation, Boussinesq system, Topography effect, Finite Difference, Local Discontinuous Galerkin

1. Introduction

1.1. **General setting.** This article is devoted to the numerical comparison of different asymptotic models for the water waves problem for uneven bottoms. These equations describe the motion of the free surface and the evolution of the velocity field of a layer of fluid under the following assumptions: the fluid is ideal, incompressible, irrotationnal, and under the influence of gravity. The solutions of these equations are very difficult to describe, because of their complexity. We thus look for approximate models and hence for approximate solutions. The main asymptotical models used in coastal oceanography, including shallow-water equations, Boussinesq systems, Green-Naghdi equations (GN) have been rigorously justified in [1]. Some of these models capture the existence of solitary water waves and the associated phenomenon of soliton manifestation [21]. The Korteweg-de Vries (KdV) equation originally derived over flat bottoms [24] is an approximation of the Boussinesq equations, and this relation has been rigorously justified in [8, 29, 4, 18]. When the bottom is uneven, various generalizations of the KdV equation with non constant coefficients (called KdV-top) have been proposed [23, 30, 10, 26, 13, 34, 27, 12, 22, 28], and rigorously justified in [19]. One of the aims of this article is to study numerically these KdV-top equations, and to compare them with the Boussinesq equations over uneven bottom. The KdV equation on flat bottom can be numerically solved by using finite difference schemes [31, 35], or discontinuous Galerkin schemes [32]. It is treated with finite differences in [6] by using a Crank-Nicolson relaxation method in time introduced by Besse-Bruneau in [2] and justified by Besse in [3]. Our finite-difference scheme is inspired from these earlier works. We propose a modification so that the numerical scheme conserves a discrete energy for the fully discrete

scheme (in space and time). We also compare this approach with the discontinuous Galerkin method of [32].

The generalization of the KdV-top equation to more nonlinear regimes (related to Camassa-Holm [5] and Degasperis-Procesi [9] equations) contains higher order nonlinear dispersive/nonlocal balances not present in the KdV and BBM equations. In 2008 Constantin and Lannes [7], rigorously justified these generalizations of the KdV equation in the case of flat bottoms . They proved that these equations can be used to furnish approximations to the governing equations for water waves. These Camassa-Holm (CH) equations on flat bottom can be numerically studied by using finite difference schemes [15, 16, 7, 17], or discontinous Galerkin schemes [33]. In 2009 S. Israwi [19], investigated the case of variable bottoms in the same scaling as in [7]. He derived a new variable coefficients class of equations which takes into account topographic effects and generalizes the CH-like equations of Constantin-Lannes [7]. In the present article, we find many finite difference schemes for these new models, so that the numerical scheme conserves a discrete energy for the fully discrete scheme (in space and time). Therefore, we compare numerically these models with the Green-Naghdi equations for a flat bottom. We also compare this approach with the discontinuous Galerkin method [33].

1.2. **Presentation of two-ways models : Boussinesq and Green-Naghdi equations.** Parameterizing the free surface by $z = \varepsilon\zeta(t,x)$ (with $x \in \mathbb{R}$) and the bottom by $z = -1 + \beta b^{(\alpha)}(x)$ (with $b^{(\alpha)}(x) = b(\alpha x)$), one can use the incompressibility and irrotationality conditions to write the classical adimensionalized water waves in terms of a velocity potential φ associated with the flow, and where $\varphi(t,.)$ is defined on $\Omega_t = \{(x,z), -1 + \beta b^{(\alpha)}(x) < z < \varepsilon\zeta(t,x)\}$ (i.e. the velocity field is given by $v = \nabla_{x,z}\varphi$):

$$
(1) \quad
\begin{cases}
\mu\partial_x^2\varphi + \partial_z^2\varphi = 0, & \text{at} \quad -1 + \beta b^{(\alpha)} < z < \varepsilon\zeta, \\
\partial_z\varphi - \mu\beta\alpha\partial_x b^{(\alpha)}\partial_x\varphi = 0 & \text{at} \quad z = -1 + \beta b^{(\alpha)}, \\
\partial_t\zeta - \dfrac{1}{\mu}(-\mu\varepsilon\partial_x\zeta\partial_x\varphi + \partial_z\varphi) = 0, & \text{at} \quad z = \varepsilon\zeta, \\
\partial_t\varphi + \dfrac{1}{2}(\varepsilon(\partial_x\varphi)^2 + \dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2) + \zeta = 0 & \text{at} \quad z = \varepsilon\zeta.
\end{cases}
$$

The dimensionless parameters are defined as :

$$
\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2}, \quad \beta = \frac{b_0}{h_0};
$$

where a is a typical amplitude of the waves; λ is the wavelength, b_0 is the order of amplitude of the variations of the bottom topography; λ/α is the wavelength of the bottom variations; h_0 is the reference depth. We also recall that $b^{(\alpha)}(x) = b(\alpha x)$.

The parameter ε is often called nonlinearity parameter; while μ is the shallowness parameter. Asymptotic models from (1) are derived by making assumptions on the size of ε, β, α, and μ. In the shallow-water scaling ($\mu \ll 1$), one can derive (ε, β and α do not need to be small) the so-called Green-Naghdi equations (see [11, 25] for a derivation and [1, 20] for a rigorous justification). For one-dimensional surfaces and over uneven bottoms these equations couple the free surface elevation ζ to the vertically averaged horizontal component of the velocity,

$$
(2) \quad u(t,x) = \frac{1}{1 + \varepsilon\zeta - \beta b^{(\alpha)}} \int_{-1+\beta b^{(\alpha)}}^{\varepsilon\zeta} \partial_x\varphi(t,x,z)dz
$$

and can be written as:

$$
(3) \quad
\begin{cases}
\partial_t \zeta + \partial_x(hu) = 0, \\[2mm]
(1 + \dfrac{\mu}{h}\mathcal{T}[h, \beta b^{(\alpha)}])\partial_t u + \partial_x \zeta + \varepsilon u \partial_x u \\[2mm]
\qquad + \mu\varepsilon\{ -\dfrac{1}{3h}\partial_x(h^3(u\partial_x^2 u) - (\partial_x u)^2) + \Im[h, \beta b^{(\alpha)}]u \} = 0
\end{cases}
$$

where $h = 1 + \varepsilon\zeta - \beta b^{(\alpha)}$ and

$$
\mathcal{T}[h, \beta b^{(\alpha)}]W = -\frac{1}{3}\partial_x(h^3 \partial_x W) + \frac{\beta}{2}\partial_x(h^2 \partial_x b^{(\alpha)})W + \beta^2 h(\partial_x b^{(\alpha)})^2 W,
$$

while the purely topographical term $\Im[h, \beta b^{(\alpha)}]u$ is defined as:

$$
\begin{aligned}
\Im[h, \beta b^{(\alpha)}]u &= \frac{\beta}{2h}[\partial_x(h^2(u\partial_x)^2 b^{(\alpha)}) - h^2((u\partial_x^2 u) - (\partial_x u)^2)\partial_x b^{(\alpha)}] \\
&\quad + \beta^2((u\partial_x)^2 b^{(\alpha)})\partial_x b^{(\alpha)}.
\end{aligned}
$$

We remark that the Green-Naghdi equations can then be simplified over uneven bottoms into

$$
(4) \quad
\begin{cases}
\zeta_t + [hu]_x = 0 \\[2mm]
u_t + \zeta_x + \varepsilon u u_x = \dfrac{\mu}{3h}[h^3(u_{xt} + \varepsilon u u_{xx} - \varepsilon u_x^2)]_x,
\end{cases}
$$

where $O(\mu^2)$ terms have been discarded, and provided that the parameters satisfy $\theta = (\alpha, \beta, \varepsilon, \mu) \in \wp$, where the set $\wp$ is defined as

$$
\begin{aligned}
(5) \quad \wp &= \{(\alpha, \beta, \varepsilon, \mu) \text{ such that } \varepsilon = O(\sqrt{\mu}),\ \beta\alpha = O(\mu),\ \beta\alpha = O(\varepsilon), \\
&\qquad \beta\alpha^{3/2} = O(\mu^2),\ \beta\alpha\varepsilon = O(\mu^2)\}.
\end{aligned}
$$

In order to obtain the KdV equation (called KdV-top) originally derived in [23, 30, 10], stronger assumptions on ε, β, α and μ must be made namely that the paramters belong to the subset $\wp' \subset \wp$ defined as:

$$
(6) \quad \wp' = \{(\alpha, \beta, \varepsilon, \mu) \text{ such that } \varepsilon = O(\mu),\quad \alpha\beta = O(\varepsilon),\quad \alpha^{3/2}\beta = O(\varepsilon^2)\}.
$$

Neglecting the $O(\mu^2)$ terms, one obtains from (4) the following Boussinesq system:

$$
(7) \quad
\begin{cases}
\zeta_t + [hu]_x = 0 \\[2mm]
u_t + \zeta_x + \varepsilon u u_x = \dfrac{\mu}{3}(c^4 u_{xt})_x,
\end{cases}
$$

where $c = \sqrt{1 - \beta b^{(\alpha)}}$.

2. Numerical scheme for the KdV-top equations

In this section, attention is given to the regime of slow variations of the bottom topography under the long-wave scaling $\varepsilon = O(\mu)$. We investigate several situations satisfying the condition (6) on the parameters ε, β, α and μ.

2.1. **The continuous case.**

2.1.1. *The KDV-top (or original) model.* The model studied in this section is the following (ζ is the elevation)

(8)
$$\zeta_t + \Gamma_1\zeta + \frac{3}{2c}\varepsilon\zeta\zeta_x + \frac{1}{6}\mu c^5\zeta_{xxx} = 0,$$

and

$$\Gamma_1\zeta = \frac{1}{2}(c\zeta_x + \partial_x(c\zeta)),$$

where $c = \sqrt{1 - \beta b^{(\alpha)}}$. We assume that (6) is satisfied without any further assumptions; to this regime corresponds the so-called KDV-top (or original) model (8). It is related to the Boussinesq equations in the meaning of consistency (see below) and it was originally derived in [23, 30, 10]. We list here some of the properties of this model. The proof of all the results below can be found in [19]. Let us first define two different kinds of consistency, namely, L^∞ and H^s consistency.

Definition 1. *Let $\wp_0 \subset \wp$ be a family of parameters (with $\wp$ as in (5)). A family $(\zeta^\theta, u^\theta)_{\theta\in\wp_0}$ is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the GN equations (4), if for all $\theta \in \wp_0$ (and denoting $h^\theta = 1 + \varepsilon\zeta^\theta - \beta b^{(\alpha)}$),*

$$\begin{cases} \zeta_t^\theta + [h^\theta u^\theta]_x = \mu^2 r_1^\theta \\ u_t^\theta + \zeta_x^\theta + \varepsilon u^\theta u_x^\theta = \dfrac{\mu}{3h^\theta}[(h^\theta)^3(u_{xt}^\theta + \varepsilon u^\theta u_{xx}^\theta - \varepsilon(u_x^\theta)^2)]_x + \mu^2 r_2^\theta \end{cases}$$

with $(r_1^\theta, r_2^\theta)_{\theta\in\wp_0}$ bounded in $L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R})$.

When the residual is bounded in H^s and not in L^∞, we speak of H^s-consistency. When $s > 1/2$, this H^s-consistency is obviously stronger then the L^∞-consistency.

Definition 2. *Let $\wp_0 \subset \wp$ be a family of parameters (with $\wp$ as in (5)). A family $(\zeta^\theta, u^\theta)_{\theta\in\wp_0}$ is H^s-consistent of order $s \geq 0$ and on $[0, \frac{T}{\varepsilon}]$ with the GN equations (4), if for all $\theta \in \wp_0$, (and denoting $h^\theta = 1 + \varepsilon\zeta^\theta - \beta b^{(\alpha)}$),*

$$\begin{cases} \zeta_t^\theta + [h^\theta u]_x = \mu^2 r_1^\theta \\ u_t^\theta + \zeta_x^\theta + \varepsilon u^\theta u_x^\theta = \dfrac{\mu}{3}\dfrac{1}{h^\theta}[(h^\theta)^3(u_{xt}^\theta + \varepsilon u^\theta u_{xx}^\theta - \varepsilon(u_x^\theta)^2)]_x + \mu^2 r_2^\theta \end{cases}$$

with $(r_1^\theta, r_2^\theta)_{\theta\in\wp_0}$ bounded in $L^\infty([0, \frac{T}{\varepsilon}], H^s(\mathbb{R})^2)$.

Remark 1. *The definitions can be adapted to define L^∞ and H^s consistency with the Boussinesq equations (7) rather then the GN equations (4).*

For the KdV-top model (8), H^s-consistency cannot be established, but L^∞-consistency holds as shown below:

Theorem 1. *Let $s > \frac{3}{2}$, $b \in H^\infty(\mathbb{R})$ and $\zeta_0 \in H^{s+1}(\mathbb{R})$. For all $\theta \in \wp'$*

$$\wp' = \{(\alpha, \beta, \varepsilon, \mu) \;\; such \; that \;\; \varepsilon = O(\mu), \; \alpha\beta = O(\varepsilon), \; \alpha^2\beta = O(\varepsilon^2)\},$$

we obtain the following properties :
- *there exists $T > 0$ and a unique family of solutions $(\zeta^\theta)_{\theta\in\wp'}$ to (8) bounded in $C([0, \frac{T}{\varepsilon}]; H^{s+1}(\mathbb{R}))$ with initial condition ζ_0 ;*
- *the familily $(\zeta^\theta, u^\theta)_{\theta\in\wp'}$ with (omitting the index θ)*

(9)
$$u := \frac{1}{c}\left(\zeta - \frac{1}{2}\int_{-\infty}^x \frac{c_x}{c}\zeta\, ds - \frac{\varepsilon}{4c^2}\zeta^2 + \mu\frac{1}{6}c^4\zeta_{xx}\right)$$

is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the equations (7).

Remark 2. *The term $\int_{-\infty}^{x} \frac{c_x}{c}\zeta\, ds$ does not necessarily decay at infinity, and this the reason why H^s-consistency does not hold in general. The problem of the con-vergence of the solution of (8) to the solution of (7) remains open; numerical sim-ulations are performed in §2.2 to being some insight on this matter.*

2.1.2. *The gentle model.* In a first stage, we restrict here our attention to parame-ters ε, β, α and μ such that

$$(10) \qquad\qquad \varepsilon = \mu, \quad \beta = O(\varepsilon), \quad \alpha = O(\varepsilon).$$

These conditions are stronger than $(\alpha, \beta, \varepsilon, \mu) \in \wp'$; we remark in particular that under the condition (10), the model (8) can be written after neglecting the $O(\mu^2)$ terms as :

$$(11) \qquad\qquad \zeta_t + c\zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x + \frac{1}{6}\varepsilon\zeta_{xxx} + \frac{1}{2}c_x\zeta = 0,$$

we keep here the term $\frac{1}{2}c_x\zeta$ which is of order $O(\mu^2)$, to obtain a conservative scheme, and in that case, we are able to deduce an energy preserved by this model. This model (11) will be called gentle model since it is only able to handle gentle variations of bottom topography.

Proposition 1. *Let b and ζ_0 be given by the above theorem and ζ solve (11). Then, for all $t \in [0, \frac{T}{\varepsilon}]$,*

$$\int_{\mathbb{R}} |\zeta(t)|^2\, dx = \int_{\mathbb{R}} |\zeta_0|^2\, dx.$$

Remark 3. *With the choice of parameters (10), the model (11), is H^s-consistent on $[0, \frac{T}{\varepsilon}]$ with the equations (7), and a full justification (convergence) can given for this model (see [19]).*

2.1.3. *The strong model.* We consider here stronger variations of the topography, i.e. :

$$(12) \qquad\qquad \mu = \varepsilon, \quad \beta = O(1), \quad \alpha = O(\varepsilon^{4/3}).$$

In order to obtain model with better properties, we add terms of order $O(\mu^2)$, so that we get equation (13) :

$$(13) \qquad \zeta_t + \Gamma_1\zeta + \varepsilon\frac{3}{2}\left(\frac{1}{c}\right)^{2/3}\zeta\left(\left(\frac{1}{c}\right)^{1/3}\zeta\right)_x + \frac{\mu}{6}\Gamma_3\zeta = 0,$$

where, the skew-symmetric operators Γ_1 and Γ_3 are defined as

$$\Gamma_1\zeta = \frac{1}{2}(c\zeta_x + \partial_x(c\zeta)),$$

and

$$\Gamma_3\zeta = c^5\zeta_{xxx} + \frac{3}{2}(c^5)_x\zeta_{xx} + \frac{3}{4}(c^5)_{xx}\zeta_x + \frac{1}{8}(c^5)_{xxx}\zeta.$$

It is remarked that (13) differ from (8) only up to terms of order $O(\mu^2)$ under the condition (12), indeed

$$\frac{3}{2c}\zeta\zeta_x = \frac{3}{2}\left(\frac{1}{c}\right)^{2/3}\zeta\left(\left(\frac{1}{c}\right)^{1/3}\zeta\right)_x + O(\mu^2),$$

$$\frac{\mu}{6}c^5\zeta_{xxx} = \frac{\mu}{6}\Gamma_3\zeta + O(\mu^2).$$

The interest of this formulation of the nonlinear and of the dispersive term is that it allows for exact conservation of the energy.

Proposition 2. *Let b and ζ_0 be given by the above theorem and ζ solve (13). Then, for all $t \in [0, \frac{T}{\varepsilon}]$,*

$$\int_{\mathbb{R}} |\zeta(t)|^2 \, dx = \int_{\mathbb{R}} |\zeta_0|^2 \, dx.$$

2.2. **The numerical case.** For any function f, let us denote by $f^n(x)$ the approximation of $f(t, x)$ with $t = n\Delta t$, $f^{n+1/2}(x)$ the one of $f(t, x)$ with $t = (n + 1/2)\Delta t$ and by $f_i(t)$ the approximation of $f(t, x)$ with $x = i\Delta x$, $f_{i+1/2}(t)$ the one of $f(t, x)$ with $x = (i + 1/2)\Delta x$.

2.2.1. L^2 *conservative finite-difference schemes.* We derive in the Lemma below a spatial discretization for the following nonlinear terms

$$u_t + u^p u_x \quad \text{and} \quad u_t + 2f[u]u_x + f[u]_x u;$$

where, $p \in \mathbb{N}^*$, $f[u] = u_{xx}$ and $f[u]_x = u_{xxx}$ so that the finite difference schemes conserve the discrete L^2 norm.

Lemma 1. *The following schemes for $u_t + u^p u_x$ and $u_t + 2f[u]u_x + f[u]_x u$:*

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{p+2}\left(D_1\left(\frac{u^{n+1} + u^n}{2}\right)_i (u^{n+1/2})_i^p + D_1\left((u^{n+1/2})^p \frac{u^{n+1} + u^n}{2}\right)_i\right),$$

and

$$\frac{u^{n+1} - u^n}{\Delta t} + D_1\left(\frac{u^{n+1} + u^n}{2}\right)_i f[u^{n+1/2}]]_i + D_1\left(f[u^{n+1/2}]\frac{u^{n+1} + u^n}{2}\right)_i,$$

respectively are conservatives, that is to say we have the equality :

$$\sum_i (u_i^n)^2 = \sum_i (u_i^0)^2,$$

where, the matrix D_1 is the classical centered discretizations of the derivative ∂_x.

Proof. Taking in the above schemes the inner product with $\frac{u_i^{n+1} + u_i^n}{2}$, using the fact that for all $v, w \in \mathbb{R}^m$

$$(D_1 v, w) = -(v, D_1 w),$$

where $m = \dim(D_1)$, one easily obtains :

$$\sum_i (u_i^{n+1})^2 = \sum_i (u_i^n)^2.$$

$\square$

2.2.2. *The numerical scheme of the gentle model.* We choose here a spatial discretization for the gentle model (11) so that the discrete L^2-norm is preserved by the fully discrete scheme. Lemma 1 shows how to discretize the nonlinear term $\frac{3}{2}\varepsilon\zeta\zeta_x$ in a conservative way, and the third order term $\frac{\mu}{6}\zeta_{xxx}$ does not raise any difficulties. For the variable coefficients linear terms $\Gamma_1\zeta = \frac{1}{2}(c\,\zeta_x + \partial_x(c\,\zeta))$, the situation is more delicate, we propose a special conservative discretization that

allows a discrete version of Proposition 1, which gives the final discretization of (11):

$$
\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \left(D_1^v \frac{\zeta^{n+1} + \zeta^n}{2}\right)_i + \varepsilon\left[\frac{1}{2}\left(\zeta_i^{n+\frac{1}{2}} + \frac{\zeta_{i+1}^{n+\frac{1}{2}} + \zeta_{i-1}^{n+\frac{1}{2}}}{2}\right)\left(D_1\frac{\zeta^{n+1} + \zeta^n}{2}\right)_i\right.
$$

(14)

$$
\left. + \frac{1}{2}\frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n + \zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{4}\left(D_1\zeta^{n+\frac{1}{2}}\right)_i + \frac{1}{6}\left(D_3\frac{\zeta^{n+1} + \zeta^n}{2}\right)_i\right] = 0,
$$

where the matrices D_1 and D_3 are the classical centered discretizations of the derivatives ∂_x and ∂_x^3, while the skew-symmetric matrix D_1^v is as follows:

$$
\left(D_1^v \zeta^n\right)_i = \frac{c_{i+1/2}\,\zeta_{i+1}^n - c_{i-1/2}\,\zeta_{i-1}^n}{2\Delta x},
$$

(the index v stands for "variable coefficients", if $c = 1$ one has $D_1^v = D_1$.) Throughout this section, we will denote by $(\zeta^n)_{n\in\mathbb{N}}$ the unique sequence which solves (14) for all $n \in \mathbb{N}$. We obtain the conservation of a discrete energy (whose continuous version is stated in Propostion 1).

Theorem 2. *The L^2-norm of ζ^n is conserved, that is,*

$$
\forall n \in \mathbb{N}, \quad \sum_i (\zeta_i^n)^2 = \sum_i (\zeta_i^0)^2.
$$

Therefore, the finite-difference scheme (14) is stable.

Proof. Taking in (14) the inner product with $\frac{\zeta_i^{n+1} + \zeta_i^n}{2}$, using the fact that D_1, D_3 and D_1^v are skew-symmetric matrices, we obtain

$$
\sum_i \left(\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t}\frac{\zeta_i^{n+1} + \zeta_i^n}{2}\right) + \varepsilon\sum_i\left(\left[\frac{1}{2}\left(\zeta_i^{n+\frac{1}{2}} + \frac{\zeta_{i+1}^{n+\frac{1}{2}} + \zeta_{i-1}^{n+\frac{1}{2}}}{2}\right)\left(D_1\frac{\zeta^{n+1} + \zeta^n}{2}\right)_i\right.\right.
$$

$$
\left.\left. + \frac{1}{2}\frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n + \zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{4}\left(D_1\zeta^{n+\frac{1}{2}}\right)_i\right]\frac{\zeta_i^{n+1} + \zeta_i^n}{2}\right) = S_1(\zeta) + \varepsilon S_2(\zeta) = 0.
$$

We show first that $S_2 = 0$. In order to do this, we remark that

$$
S_2(\zeta) = \frac{1}{2}\sum_i \zeta_i^{n+\frac{1}{2}}\left(\frac{\zeta_{i+1}^{n+1} - \zeta_{i-1}^{n+1} + \zeta_{i+1}^n - \zeta_{i-1}^n}{4\Delta x}\right)\frac{\zeta_i^{n+1} + \zeta_i^n}{2}
$$

$$
+ \frac{1}{2}\sum_i \frac{\zeta_{i+1}^{n+\frac{1}{2}} + \zeta_{i-1}^{n+\frac{1}{2}}}{2}\left(\frac{\zeta_{i+1}^{n+1} - \zeta_{i-1}^{n+1} + \zeta_{i+1}^n - \zeta_{i-1}^n}{4\Delta x}\right)\frac{\zeta_i^{n+1} + \zeta_i^n}{2}
$$

$$
+ \frac{1}{2}\sum_i \frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n + \zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{4}\left(D_1\zeta^{n+\frac{1}{2}}\right)_i\frac{\zeta_i^{n+1} + \zeta_i^n}{2}
$$

$$
= S_{21} + S_{22} + S_{23};
$$

with a change of subscripts in S_{21}, we get

$$
\begin{aligned}
S_{21} &= \frac{1}{8\Delta x} \sum_i \zeta_{i-1}^{n+\frac{1}{2}} \zeta_i^{n+1} \frac{\zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{2} - \zeta_{i+1}^{n+\frac{1}{2}} \zeta_i^{n+1} \frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n}{2} \\[2mm]
&\quad + \zeta_{i-1}^{n+\frac{1}{2}} \zeta_i^n \frac{\zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{2} - \zeta_{i+1}^{n+\frac{1}{2}} \zeta_i^n \frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n}{2} \\[2mm]
&= \frac{1}{16\Delta x} \sum_i \frac{\zeta_i^{n+1} + \zeta_i^n}{2} \left[2\zeta_{i-1}^{n+\frac{1}{2}} \left(\zeta_{i-1}^{n+1} + \zeta_{i-1}^n \right) - 2\zeta_{i+1}^{n+\frac{1}{2}} \left(\zeta_{i+1}^{n+1} + \zeta_{i+1}^n \right) \right],
\end{aligned}
$$

and we remind that

$$
\begin{aligned}
S_{22} &= \frac{1}{16\Delta x} \sum_i \left[\zeta_{i-1}^{n+\frac{1}{2}} \left(\zeta_{i+1}^{n+1} + \zeta_{i+1}^n - \zeta_{i-1}^{n+1} - \zeta_{i-1}^n \right) + \zeta_{i+1}^{n+\frac{1}{2}} \left(\zeta_{i+1}^{n+1} + \zeta_{i+1}^n - \zeta_{i-1}^{n+1} \right. \right. \\[2mm]
&\quad \left. \left. - \zeta_{i-1}^n \right) \right] \frac{\zeta_i^{n+1} + \zeta_i^n}{2}.
\end{aligned}
$$

Summing S_{21} and S_{22}, we get

$$
\begin{aligned}
S_{21} + S_{22} &= \frac{1}{16\Delta x} \sum_i \left[(\zeta_{i-1}^{n+\frac{1}{2}} - \zeta_{i+1}^{n+\frac{1}{2}})(\zeta_{i+1}^{n+1} + \zeta_{i+1}^n + \zeta_{i-1}^{n+1} + \zeta_{i-1}^n) \right] \frac{\zeta_i^{n+1} + \zeta_i^n}{2} \\[2mm]
&= -S_{23}.
\end{aligned}
$$

Finally, we deduce that $S_2(\zeta) = 0$. It follows now that $S_1(\zeta) = 0$, which implies easily the result.

$\square$

2.2.3. *The numerical scheme of the strong model.* We recall that

$$
\Gamma_1 \zeta = \frac{1}{2}(c\,\zeta_x + \partial_x(c\,\zeta)),
$$

and

$$
\Gamma_3 \zeta = c^5\,\zeta_{xxx} + \frac{3}{2}(c^5)_x\,\zeta_{xx} + \frac{3}{4}(c^5)_{xx}\,\zeta_x + \frac{1}{8}(c^5)_{xxx}\,\zeta.
$$

These two operators are discretized by matrices D_1^v and D_3^v :

$$
\left(D_1^v \zeta^n \right)_i = \frac{c_{i+1/2}\,\zeta_{i+1}^n - c_{i-1/2}\,\zeta_{i-1}^n}{2\Delta x}.
$$

$$
\left(D_3^v \zeta^n \right)_i = \frac{c_{i+1}^5\,\zeta_{i+2}^n - 2\,c_{i+1/2}^5\,\zeta_{i+1}^n + 2\,c_{i-1/2}^5\,\zeta_{i-1}^n - c_{i-1}^5\,\zeta_{i-2}^n}{2\Delta x^3}.
$$

These two matrices are skew-symmetric and do not modify the stability of the scheme. We choose for the strong model (13) a fully discrete scheme, similar to the

previous section :

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \Big(D_1^v \frac{\zeta^{n+1} + \zeta^n}{2}\Big)_i + \varepsilon\Big(\frac{1}{c}\Big)_i^{1/3}\Big[\frac{1}{2}\Big(\Big(\frac{1}{c}\Big)_i^{1/3}\zeta_i^{n+\frac{1}{2}}$$

$$+ \frac{\Big(\frac{1}{c}\Big)_{i+1}^{1/3}\zeta_{i+1}^{n+\frac{1}{2}} + \Big(\frac{1}{c}\Big)_{i-1}^{1/3}\zeta_{i-1}^{n+\frac{1}{2}}}{2}\Big)\Big(D_1\Big(\frac{1}{c}\Big)^{1/3}\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i$$

(15)

$$+ \frac{1}{2}\frac{\Big(\frac{1}{c}\Big)_{i+1}^{1/3}\zeta_{i+1}^{n+1} + \Big(\frac{1}{c}\Big)_{i+1}^{1/3}\zeta_{i+1}^n + \Big(\frac{1}{c}\Big)_{i-1}^{1/3}\zeta_{i-1}^{n+1} + \Big(\frac{1}{c}\Big)_{i-1}^{1/3}\zeta_{i-1}^n}{4} \times$$

$$\Big(D_1\Big(\frac{1}{c}\Big)^{1/3}\zeta^{n+\frac{1}{2}}\Big)_i\Big] + \frac{\mu}{6}\Big(D_3^v\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i = 0.$$

Theorem 3. *The discrete scheme (15) of the model (13) conserves an energy i.e*

$$\forall n \in \mathbb{N}, \quad \sum_i (\zeta_i^n)^2 = \sum_i (\zeta_i^0)^2.$$

Proof. Taking in (15) the inner product with $\frac{\zeta_i^{n+1}+\zeta_i^n}{2}$, using the fact that D_1, D_3^v and D_1^v are skew-symmetric matrix, we obtain

$$\sum_i \Big(\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t}\frac{\zeta_i^{n+1} + \zeta_i^n}{2}\Big) + \varepsilon\sum_i\Big(\Big[\frac{1}{2}\Big(v_i^{n+\frac{1}{2}} + \frac{v_{i+1}^{n+\frac{1}{2}} + v_{i-1}^{n+\frac{1}{2}}}{2}\Big)\Big(D_1\frac{v^{n+1}+v^n}{2}\Big)_i$$

$$+ \frac{1}{2}\frac{v_{i+1}^{n+1} + v_{i+1}^n + v_{i-1}^{n+1} + v_{i-1}^n}{4}\Big(D_1 v^{n+\frac{1}{2}}\Big)_i\Big]\frac{v_i^{n+1} + v_i^n}{2}\Big) = S_1(\zeta) + \varepsilon S_2(v) = 0,$$

where

$$v = \Big(\frac{1}{c}\Big)^{1/3}\zeta.$$

$\square$

2.2.4. *Numerical validation.* Since there does not exist explicit solutions of (11) or (13), we have chosen a high-order Local Discontinuous Galerkin (LDG) in space and high-order Gauss-Runge-Kutta scheme in time (see [14]), in order to obtain very accurate of implementation results, that can be used as reference solutions to validate the finite difference method. The main advantage of finite difference schemes are their simplicity and quickness. It also gives very good conservation of energy. Details about the LDG method can be found in [32] (for a flat bottom), the extension to variable bottom does not raise important difficulties. In the case of flat bottoms, analytical solutions are well-known, and consist of solitary-waves. Let us consider the following initial condition parameterized by c_1

$$\zeta_0(x) = 2c_1 \operatorname{sech}^2\Big(\sqrt{\frac{3c_1\varepsilon}{2\mu}}\, x\Big),$$

Therefore, the analytical solution for a flat bottom of (8) is equal to

$$\zeta(x,t) = \zeta_0(x - c't),$$

with a real velocity c'

$$c' = 1 + \varepsilon c_1.$$

We can wonder what is the influence of the bottom for this solitary-wave. To this aim, we consider a sinusoidal bottom

$$b(x) \,=\, \sin(2\pi\alpha x).$$

In the figure 1, we have displayed the solution for this initial condition ($c_1 \,=\, 0.5$) for a sinusoidal bottom and with different models.
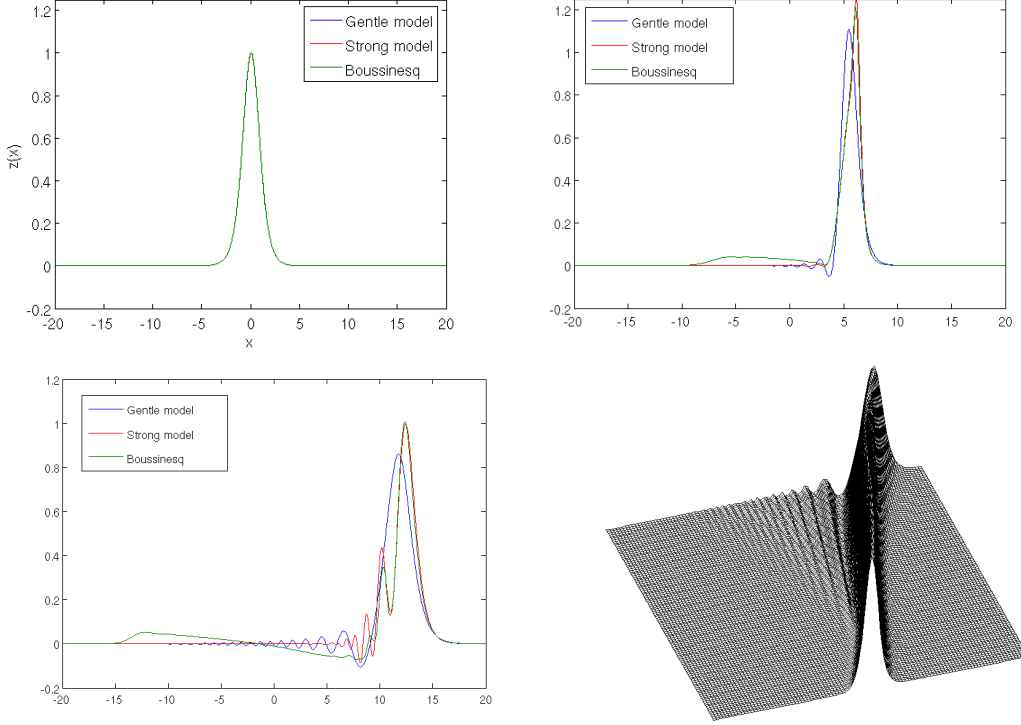

FIGURE 1. Solution of KdV equation for a sinusoidal bottom, with gentle, strong model and Boussinesq model, for $t = 0$, $t = 6.67$, $t = 13.33$. The 3-D graph represents the solution for the gentle model (11) for any time t in the case of a sinusoidal bottom. ($c_1 = 0.5$ $\beta = 0.5$, $\varepsilon = \mu = 0.1$ $\alpha = 0.05$)

In the tables 1, 2, 3, the L^2 error has been computed for the LDG method and the finite-difference method for a flat bottom and a sinusoidal bottom for the two models (11), (13) presented. In these tables, the time step Δt has been chosen small enough so that there is no error due to time-discretization, the error comes only because of space discretization.

In figure 2, we displayed the variation of the L^2-norm for the different proposed schemes. In this figure, we observe that the discrete energy of (14) and (15) is conserved (the magnitude of the variations is 10^{-15} due to machine precision), whereas the energy of LDG scheme is decreasing (in the figure, we see that the variation of energy is increasing, so that the total energy is strictly decreasing).

As a final test, we propose to check numerically the accuracy of the approximation provided by the KdV-top models (11), and the strong model (13) (we recall

	LDG, order 1			LDG, order 3			LDG, order 7		Finite Difference	
N	Error	Order		Error	Order		Error	Order	Error	Order
160	1.083e-1	-		1.889e-3	-		2.168e-4	-	2.129e-1	-
320	2.859e-2	1.92		1.489e-4	3.67		8.367e-7	8.02	6.195e-2	1.78
640	5.049e-3	2.50		6.231e-6	4.58		1.882e-9	8.80	1.634e-2	1.92
1280	8.414e-4	2.59		4.469e-7	3.80		1.328e-11	7.15	4.392e-3	1.90
2560	1.545e-4	2.46		2.606e-8	4.10		7.580e-14	7.45	1.100e-3	1.997

TABLE 1. L^2 errors for the solitary-wave and flat bottom between the numerical solution and the analytical one for $t = 13.333$. N denotes here the number of degrees of freedom. ($c_1 = 0.5 \quad \varepsilon = \mu = 0.1$)

	LDG, order 1			LDG, order 3			LDG, order 7		Finite Difference	
N	Error	Order		Error	Order		Error	Order	Error	Order
160	1.350e-1	-		1.770e-2	-		9.192e-3	-	2.395e-1	-
320	4.469e-2	1.59		2.124e-3	3.06		3.247e-5	8.15	6.804e-2	1.82
640	1.119e-2	2.00		4.347e-5	5.61		1.785e-7	7.51	1.743e-2	1.97
1280	1.950e-3	2.52		2.460e-6	4.14		7.071e-10	7.98	4.367e-3	1.997
2560	2.908e-4	2.75		1.600e-7	3.95		2.926e-12	7.92	1.092e-3	2.00

TABLE 2. L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 13.33$ and for the gentle model (11). N denotes here the number of degrees of freedom. ($c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05$)

	LDG, order 1			LDG, order 3			LDG, order 7		Finite Difference	
N	Error	Order		Error	Order		Error	Order	Error	Order
160	2.419e-1	-		8.4212e-2	-		2.263e-2	-	4.263e-1	-
320	1.312e-1	0.88		8.751e-3	3.27		3.780e-4	5.90	2.202e-1	0.95
640	4.514e-2	1.54		2.718e-4	5.01		1.819e-6	7.70	7.546e-2	1.55
1280	8.523e-3	2.41		8.208e-6	5.05		4.460e-9	8.67	2.294e-2	1.72
2560	1.218e-3	2.81		5.246e-7	3.97		4.131e-11	6.75	8.924e-3	1.36

TABLE 3. L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 13.33$ and for the strong model (13). N denotes here the number of degrees of freedom. ($c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05$)

here that the strong model (13) is not fully justified mathematically, and that we are only able to get L^∞ consistency) in comparison to the Boussinesq equations (7). The initial condition for ζ is the solitary wave (the same as previously, but centered at $x = -10$), and the expression of initial condition for u in the Boussinesq equations is given by (9). The computational domain is $[-100, 100]$ and we have computed the relative error for $T = 50$. For a flat bottom (see Fig. 3), we see that the solutions of the KdV and Boussinesq equations differ from $O(\varepsilon^2)$. For an uneven bottom (see Fig. 4) the relative error seems to be in $O(\varepsilon^2)$ when $\alpha = \beta = O(\varepsilon)$, whereas it seems to be in $O(\varepsilon)$ when $\beta = O(1)$. We can see that the strong model
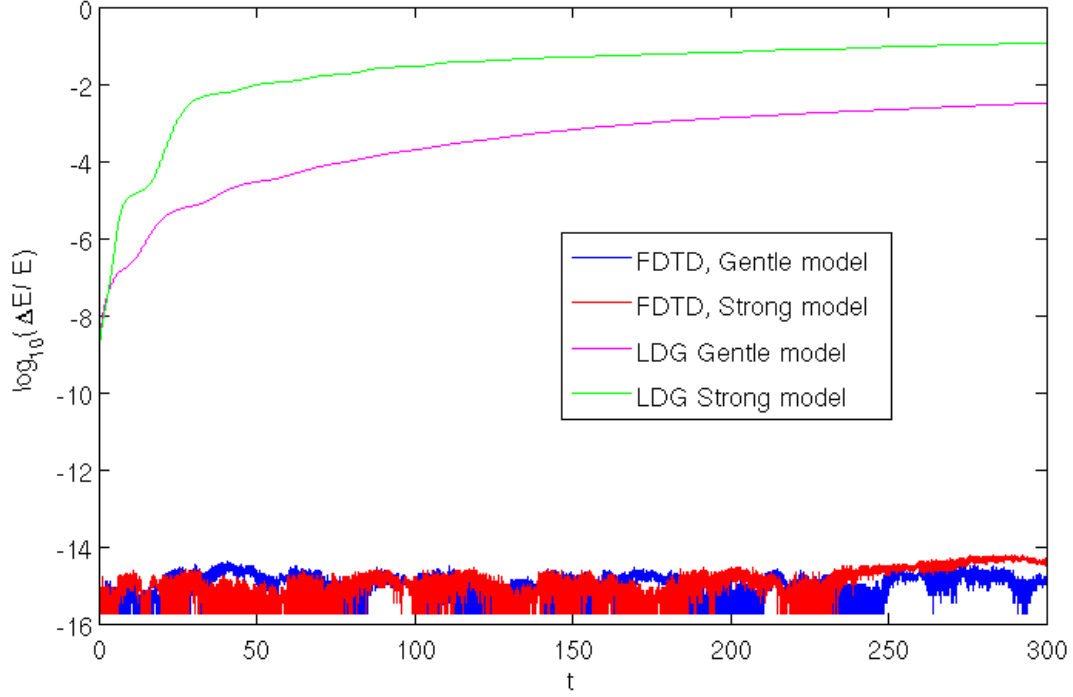
FIGURE 2. Logarithm of variation of energy versus time for gentle
scheme (14) and strong scheme (15) for solitary-wave and sinu-
soidal bottom ($c_1 = 0.5 \quad \beta = 0.5, \quad \varepsilon = \mu = 0.1 \quad \alpha = 0.05$).
Finite difference and third order LDG with 640 degrees of freedom.

gives almost the same solutions as the original model 8 . We have displayed the
solution obtained for $\beta = 0.5$, $\varepsilon = 0.018$ and $T = 50$ on the figure Fig. 5. We
can see that the solution of the Boussinesq equations is non-null for a large range
of x, on the interval $[-60, 40]$, whereas solutions of KdV models are non-null for a
smaller range $[-20, 40]$. And we clearly see that in this case, the strong and original
model give much better solution (relative L^2 error is respectively equal to 6.4 %
and 7.4%) than gentle model (relative L^2 error is equal to 62.2 %).
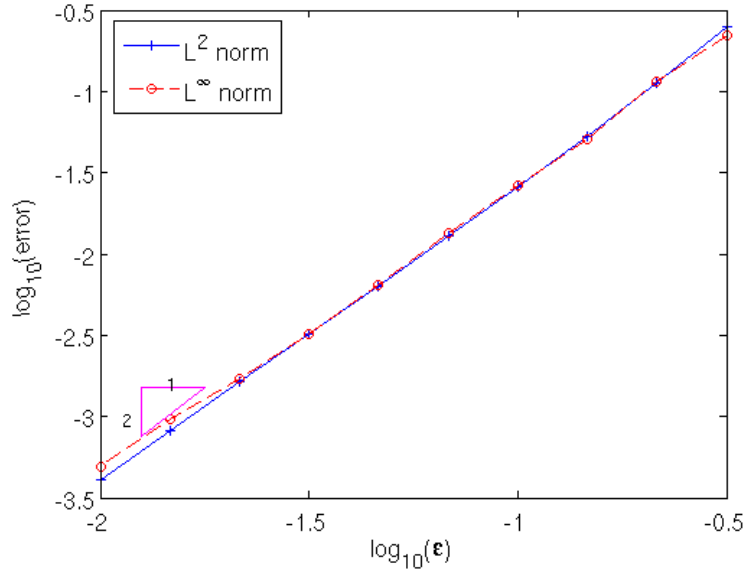
FIGURE 3. Relative error between solutions of Boussinesq equations and KdV for a flat bottom (Log-log scale)
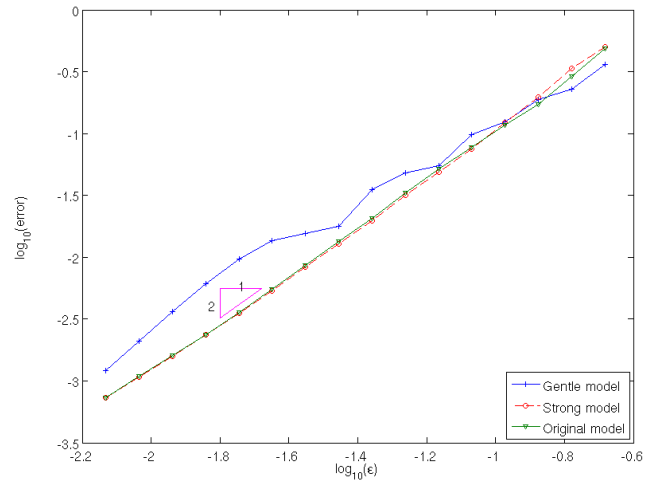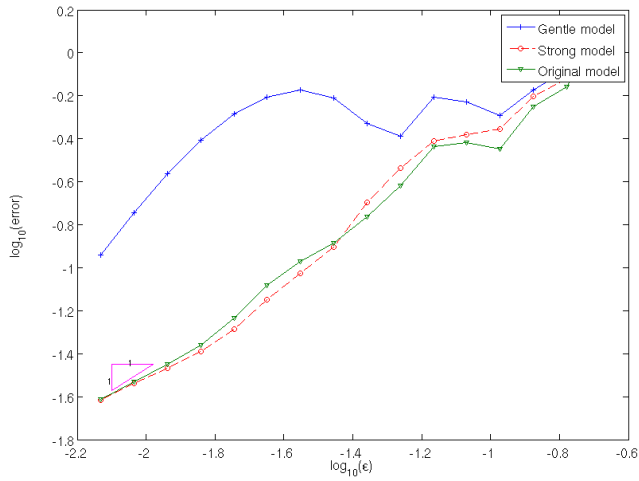

FIGURE 4. Relative error between solutions of Boussinesq equations and KdV for a sinusoidal bottom (Log-log scale). We have considered gentle model (11), strong model (13) and original model (8). On left $\alpha = 0.5\,\varepsilon$, $\beta = 0.5$, on right $\alpha = 0.5\varepsilon$, $\beta = \varepsilon$.
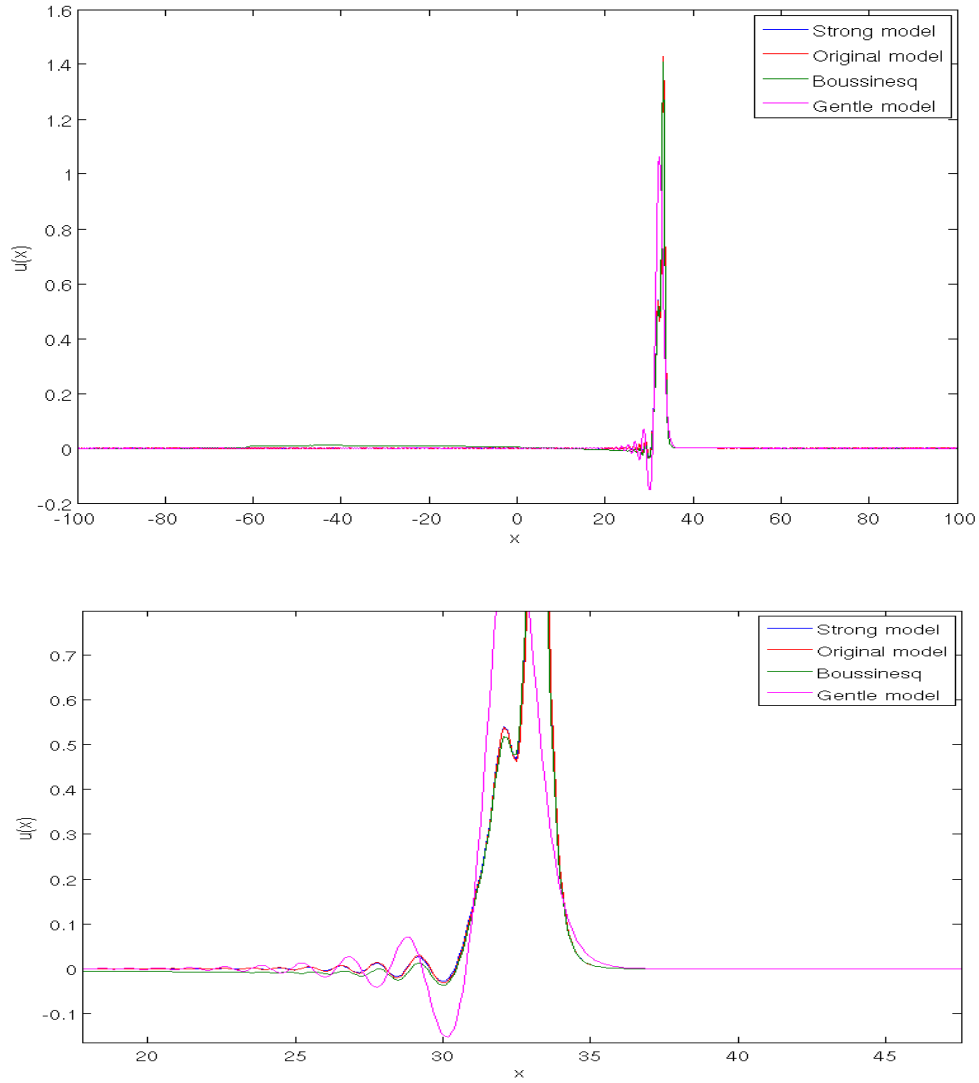
FIGURE 5. Solution obtained for a solitary-wave with a sinusoidal bottom for $\beta = 0.5$, $\varepsilon = 0.018$, $\alpha = 0.5\varepsilon$, and $T = 50$. On bottom, you can see the solution on a reduced interval so that you can observe the differences.

3. Numerical scheme for the Camassa-Holm-like equations

Now we consider the generalizations to more nonlinear regimes of the KdV-top equation derived in [7] for flat bottoms and [19] for variable bottoms.

3.1. **The continuous case.**

3.1.1. *The original model.* The family of equations on the surface elevation ζ (see [19]) :

$$\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2c}\varepsilon\zeta\zeta_x - \frac{3}{8c^3}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16c^5}\varepsilon^3\zeta^3\zeta_x$$

(16)
$$+\mu(\tilde{A}\zeta_{xxx} + B\zeta_{xxt}) = \varepsilon\mu\tilde{E}\zeta\zeta_{xxx} + \varepsilon\mu\Big(\partial_x(\frac{\tilde{F}}{2}\zeta)\zeta_{xx} + \zeta_x\partial_x^2(\frac{\tilde{F}}{2}\zeta)\Big),$$

where

$$\begin{aligned}
\tilde{A} &= Ac^5 - Bc^5 + Bc \\
\tilde{E} &= Ec^3 - \frac{3}{2}Bc^3 + \frac{3}{2c}B \\
\tilde{F} &= Fc^3 - \frac{9}{2}Bc^3 + \frac{9}{2c}B,
\end{aligned}$$

called here original model, can be used to construct an approximate solution consistent with the Green-Naghdi equations.

Theorem 4. *Let $s > \frac{3}{2}$, $b \in H^\infty(\mathbb{R})$ and $\zeta_0 \in H^{s+1}(\mathbb{R})$. Assume that*

$$A = q, \quad B = q - \frac{1}{6} \quad E = -\frac{3}{2}q - \frac{1}{6}, \quad F = -\frac{9}{2}q - \frac{5}{24}.$$

For all $\theta \in \wp$ such that

$$\begin{aligned}
\wp &= \{(\alpha, \beta, \varepsilon, \mu) \text{ such that } \varepsilon = O(\sqrt{\mu}), \ \beta\alpha = O(\mu), \ \beta\alpha = O(\varepsilon), \\
&\qquad \beta\alpha^2 = O(\mu^2), \ \beta\alpha\varepsilon = O(\mu^2)\}
\end{aligned}$$

we obtain

- *there exists $T > 0$ and a unique family of solutions $(\zeta^\theta)_{\theta \in \wp}$ to (16) bounded in $C([0, \frac{T}{\varepsilon}]; H^{s+1}(\mathbb{R}))$ with initial condition ζ_0 ;*
- *the familly $(\zeta^\theta, u^\theta)_{\theta \in \wp}$ with (omitting the index θ)*

(17)
$$u := \frac{1}{c}\Big(\zeta + \frac{c^2}{c^2 + \varepsilon\zeta}\Big(-\frac{1}{2}\int_{-\infty}^x \frac{c_x}{c}\zeta - \frac{\varepsilon}{4c^2}\zeta^2 - \frac{\varepsilon^2}{8c^4}\zeta^3 + \frac{3\varepsilon^3}{64c^6}\zeta^4$$

$$-\mu\frac{1}{6}c^3\zeta_{xt} + \varepsilon\mu c^2\Big[\frac{1}{6}\zeta\zeta_{xx} + \frac{1}{48}\zeta_x^2\Big]\Big)\Big)$$

is L^∞-consistent on $[0, \frac{T}{\varepsilon}]$ with the GN equations (4).

Remark 4. *If we take $q = \frac{1}{12}$, $b = 0$, i.e if we consider a flat bottom, then one can recover the equation (19) of [7]:*

$$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16}\varepsilon^3\zeta^3\zeta_x$$

(18)
$$+\frac{\mu}{12}(\zeta_{xxx} - \zeta_{xxt}) = -\frac{7}{24}\varepsilon\mu(\zeta\zeta_{xxx} + 2\zeta_x\zeta_{xx}).$$

The ratio $2 : 1$ between the coefficients of $\zeta_x\zeta_{xx}$ and $\zeta\zeta_{xxx}$ is crucial in our considerations.

3.1.2. *The gentle model.* Choosing $q = \frac{1}{12}$, $\alpha = \varepsilon$ and $\beta = \mu^{3/2}$ the equation (16) reads after neglecting the $O(\mu^2)$ terms:

$$\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2}\varepsilon\zeta\zeta_x - \frac{3}{8}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16}\varepsilon^3\zeta^3\zeta_x$$

$$\text{(19)} \qquad + \frac{\mu}{12}(\zeta_{xxx} - \zeta_{xxt}) = -\frac{7}{24}\varepsilon\mu(\zeta\zeta_{xxx} + 2\zeta_x\zeta_{xx}).$$

This model (19) is called gentle model since it is only able to handle gentle variations of bottom topography. It is more advantageous to use the equations (18) and (19), to study numerically the Camassa-Holm-like equations. In that case, we are able to deduce in the following Proposition (see [19], in the case of (19)) an energy preserved by these two models.

Proposition 3. *Let b and ζ_0 be given by the above theorem and ζ solves (18) or (19). Then, for all $t \in [0, \frac{T}{\varepsilon}]$,*

$$\int_{\mathbb{R}} |\zeta|^2 + \frac{\mu}{12}|\zeta_x|^2 \, dx = \int_{\mathbb{R}} |\zeta_0|^2 + \frac{\mu}{12}|\zeta_{0_x}|^2 \, dx.$$

3.1.3. *The strong model.* We consider here stronger variations of the parameters, i.e. :

$$\text{(20)} \qquad\qquad \varepsilon = \sqrt{\mu}, \quad \beta = O(\varepsilon), \quad \alpha = O(\mu).$$

In order to obtain a stable model, as in the KdV-scaling we add terms of order $O(\mu^2)$. Choosing $q = \frac{1}{12}$, so that we get equation (21) after neglecting the $O(\mu^2)$ terms of (16):

$$\zeta_t + c\zeta_x + \frac{1}{2}c_x\zeta + \frac{3}{2}\varepsilon\left(\frac{1}{c}\right)^{2/3}\zeta\left(\left(\frac{1}{c}\right)^{1/3}\zeta\right)_x$$

$$- \frac{3\varepsilon^2}{8}\left(\frac{1}{c^3}\right)^{1/4}\left(\left(\frac{1}{c^3}\right)^{1/4}\zeta\right)^2\left(\left(\frac{1}{c^3}\right)^{1/4}\zeta\right)_x + \frac{3}{16}\varepsilon^3\frac{1}{c}\left(\frac{1}{c}\zeta\right)^3\left(\frac{1}{c}\zeta\right)_x$$

$$+ \mu(a_{1/12})^{1/2}\left((a_{1/12})^{1/2}\zeta\right)_{xxx} - \mu(b_{1/12})^{1/2}\left((b_{1/12})^{1/2}\zeta\right)_{xxx}$$

$$\text{(21)} \qquad - \frac{\mu}{12}\zeta_{xxt} = -\frac{7}{24}\varepsilon\mu(\zeta\zeta_{xxx} + 2\zeta_x\zeta_{xx}).$$

where, $a_{1/12} = \frac{1}{6}c^5$ and $b_{1/12} = \frac{1}{12}c$. This model (21) is called strong model since it is able to handle strong variations of bottom topography.

Proposition 4. *Let b and ζ_0 be given by the above theorem and ζ solves (21). Then, for all $t \in [0, \frac{T}{\varepsilon}]$,*

$$\int_{\mathbb{R}} |\zeta|^2 + \frac{\mu}{12}|\zeta_x|^2 \, dx = \int_{\mathbb{R}} |\zeta_0|^2 + \frac{\mu}{12}|\zeta_{0_x}|^2 \, dx.$$

3.2. The numerical case.

3.2.1. *The numerical scheme of the model (18).* In this subsection, we propose a numerical scheme such that the discrete version of the scalar product

$$\left(\left(1 - \frac{\mu}{12}\partial_x^2\right)\zeta, \zeta\right),$$

is preserved as in Propostion 3. The numerical scheme used here is a simple finite difference scheme whose final discretized version reads

$$M \frac{\zeta^{n+1} - \zeta^n}{\Delta t} + D_1 \frac{\zeta^{n+1} + \zeta^n}{2} + \varepsilon D_{\frac{3}{2}uu_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2})$$

$$- \frac{3\varepsilon^2}{8} D_{u^2 u_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) + \frac{3\varepsilon^3}{16} D_{u^3 u_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2})$$

$$(22) \qquad + \frac{\mu}{12} D_3 \frac{\zeta^{n+1} + \zeta^n}{2} = -\frac{7}{24} \varepsilon \mu \, D_{2u_x u_{xx} + uu_{xxx}}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2})$$

where

$$M = (1 - \frac{\mu}{12} D_2),$$

and (see Lemma 1 in order to justify these choice of $D_{\frac{3}{2}uu_x}$, $D_{u^2 u_x}$ and $D_{u^3 u_x}$)

$$\left(D_{\frac{3}{2}uu_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) \right)_i = \frac{1}{2}\left(\zeta_i^{n+\frac{1}{2}} + \frac{\zeta_{i+1}^{n+\frac{1}{2}} + \zeta_{i-1}^{n+\frac{1}{2}}}{2} \right)\left(D_1 \frac{\zeta^{n+1} + \zeta^n}{2} \right)_i$$

$$+ \frac{1}{2} \frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n + \zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{4} \left(D_1 \zeta^{n+\frac{1}{2}} \right)_i,$$

$$\left(D_{u^2 u_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) \right)_i = \frac{1}{4}\left((\zeta_i^{n+\frac{1}{2}})^2 + \frac{(\zeta_{i+1}^{n+\frac{1}{2}})^2 + (\zeta_{i-1}^{n+\frac{1}{2}})^2}{2} \right)\left(D_1 \frac{\zeta^{n+1} + \zeta^n}{2} \right)_i$$

$$+ \frac{1}{2} \frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n + \zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{4} \frac{\zeta_{i+1}^{n+\frac{1}{2}} + \zeta_{i-1}^{n+\frac{1}{2}}}{2} \times$$

$$\left(D_1 \zeta^{n+\frac{1}{2}} \right)_i,$$

$$\left(D_{u^3 u_x}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) \right)_i = \frac{1}{5}\left((\zeta_i^{n+\frac{1}{2}})^3 + \frac{(\zeta_{i+1}^{n+\frac{1}{2}})^3 + (\zeta_{i-1}^{n+\frac{1}{2}})^3}{2} \right)\left(D_1 \frac{\zeta^{n+1} + \zeta^n}{2} \right)_i$$

$$+ \frac{3}{5} \frac{\zeta_{i+1}^{n+1} + \zeta_{i+1}^n + \zeta_{i-1}^{n+1} + \zeta_{i-1}^n}{4} \times$$

$$\frac{(\zeta_{i+1}^{n+\frac{1}{2}})^2 + \zeta_{i+1}^{n+\frac{1}{2}} \zeta_{i-1}^{n+\frac{1}{2}} + (\zeta_{i-1}^{n+\frac{1}{2}})^2}{3} \left(D_1 \zeta^{n+\frac{1}{2}} \right)_i,$$

for the term $2\zeta_x \zeta_{xx} + \zeta \zeta_{xxx}$ we propose the following special conservative discretizations

$$\left(D_{2u_x u_{xx} + uu_{xxx}}(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2}) \right)_i = \left(D_1 \frac{\zeta^{n+1} + \zeta^n}{2} \right)_i (D_2 \zeta^{n+\frac{1}{2}})_i$$

$$+ \left[D_1\left(D_2 \zeta^{n+\frac{1}{2}}\left(\frac{\zeta^{n+1} + \zeta^n}{2} \right) \right) \right]_i,$$

or one can use the Lemma 1 to get a simple conservative discretizations of this term.

3.2.2. *The numerical scheme of the gentle model (19).* We choose here a fully discrete scheme for the model (19) , similar to the previous scheme but taking into account the topography effects: (we replace the matrix D_1 by D_1^v)

$$M\frac{\zeta^{n+1}-\zeta^n}{\Delta t}+D_1^v\frac{\zeta^{n+1}+\zeta^n}{2}+\varepsilon D_{\frac{3}{2}uu_x}(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})$$

$$-\frac{3\varepsilon^2}{8}D_{u^2u_x}(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})+\frac{3\varepsilon^3}{16}D_{u^3u_x}(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})$$

(23) $$+\frac{\mu}{12}D_3\frac{\zeta^{n+1}+\zeta^n}{2}=-\frac{7}{24}\varepsilon\mu\,D_{2u_xu_{xx}+uu_{xxx}},(\zeta^{n+1/2},\frac{\zeta^n+\zeta^{n+1}}{2})$$

The following theorem proves that the discrete equations of the model (18) and (19) are stable.

Theorem 5. *The inner product*

$$\Big(M\zeta^n,\zeta^n\Big),$$

where ζ^n solves (22) or (23), is conserved.

Remark 5. *We used the discrete stable scheme (22) for the equation (18) and (23) for the equation (19) , but one can similarly choose a stable discrete scheme using the spatial discretizations for the $\zeta\zeta_{xxx}+2\zeta_x\zeta_{xx}$ found in Lemma 1. In practice, we chose this solution, since the discrete schemes are simpler to implement in that case.*

Proof. We only prove the theorem for (23), which is the most difficult one because of the topography effects. Taking in (23) the inner product with $\frac{\zeta_i^{n+1}+\zeta_i^n}{2}$, using the fact that D_1, D_3 and D_1^v are skew-symmetric matrices, we obtain

$$\sum_i M\Big(\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}+\varepsilon\sum_i\Big(\Big[\frac{1}{2}\Big(\zeta_i^{n+\frac{1}{2}}+\frac{\zeta_{i+1}^{n+\frac{1}{2}}+\zeta_{i-1}^{n+\frac{1}{2}}}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i$$

$$+\frac{1}{2}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i\Big]\frac{\zeta_i^{n+1}+\zeta_i^n}{2}\Big)$$

$$-\frac{3\varepsilon^2}{8}\sum_i\Big(\frac{1}{4}\Big((\zeta_i^{n+\frac{1}{2}})^2+\frac{(\zeta_{i+1}^{n+\frac{1}{2}})^2+(\zeta_{i-1}^{n+\frac{1}{2}})^2}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i$$

$$+\frac{1}{2}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\frac{\zeta_{i+1}^{n+\frac{1}{2}}+\zeta_{i-1}^{n+\frac{1}{2}}}{2}\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}$$

$$+\frac{3\varepsilon^3}{16}\sum_i\Big(\frac{1}{5}\Big((\zeta_i^{n+\frac{1}{2}})^3+\frac{(\zeta_{i+1}^{n+\frac{1}{2}})^3+(\zeta_{i-1}^{n+\frac{1}{2}})^3}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i$$

$$+\frac{3}{5}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\frac{(\zeta_{i+1}^{n+\frac{1}{2}})^2+\zeta_{i+1}^{n+\frac{1}{2}}\zeta_{i-1}^{n+\frac{1}{2}}+(\zeta_{i-1}^{n+\frac{1}{2}})^2}{3}\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}$$

$$+\frac{7\varepsilon\mu}{24}\sum_i\Big(\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i(D_2\zeta^{n+\frac{1}{2}})_i+\Big[D_1\Big(D_2\zeta^{n+\frac{1}{2}}\Big(\frac{\zeta^{n+1}+\zeta^n}{2}\Big)\Big)\Big]_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}$$

$$=S_1(\zeta)+\varepsilon S_2(\zeta)-\frac{3\varepsilon^2}{8}S_3(\zeta)+\frac{3\varepsilon^3}{16}S_4(\zeta)+\frac{7\varepsilon\mu}{24}S_5(\zeta)=0.$$

Proceeding exactly as in the proof of Theorem 2, one can show that $S_2(\zeta) = S_3(\zeta) = S_4(\zeta) = 0$.

Using now the fact that for all $u, v \in \mathbb{R}^m$, one has

$$(D_1 v, u) = -(v, D_1 u),$$

to obtain $S_5(\zeta) = 0$, and since M is a symmetric matrix one gets easily the result.
$\square$

3.2.3. *The numerical scheme of the strong model (21).* Here again, we use numerical scheme for the equation (21) so that the discrete quantity $\left(M\zeta^n, \zeta^n \right)$ is preserved.

$$M \frac{\zeta^{n+1} - \zeta^n}{\Delta t} + D_1^v \frac{\zeta^{n+1} + \zeta^n}{2}$$

$$+ \varepsilon \left(\frac{1}{c} \right)^{1/3} D_{\frac{3}{2} u u_x} \left(\left(\frac{1}{c} \right)^{1/3} \zeta^{n+1/2}, \left(\frac{1}{c} \right)^{1/3} \frac{\zeta^n + \zeta^{n+1}}{2} \right)$$

$$- \frac{3\varepsilon^2}{8} \left(\frac{1}{c^3} \right)^{1/4} D_{u^2 u_x} \left(\left(\frac{1}{c^3} \right)^{1/4} \zeta^{n+1/2}, \left(\frac{1}{c^3} \right)^{1/4} \frac{\zeta^n + \zeta^{n+1}}{2} \right)$$

$$+ \frac{3\varepsilon^3}{16} \frac{1}{c} D_{u^3 u_x} \left(\frac{1}{c} \zeta^{n+1/2}, \frac{1}{c} \frac{\zeta^n + \zeta^{n+1}}{2} \right)$$

$$+ \mu \left(a_{1/12} \right)^{1/2} D_3 (a_{1/12})^{1/2} \frac{\zeta^{n+1} + \zeta^n}{2}$$

$$- \mu \left(b_{1/12} \right)^{1/2} D_3 (b_{1/12})^{1/2} \frac{\zeta^{n+1} + \zeta^n}{2}$$

$$(24) \qquad = -\frac{7}{24} \varepsilon \mu \, D_{2 u_x u_{xx} + u u_{xxx}} \left(\zeta^{n+1/2}, \frac{\zeta^n + \zeta^{n+1}}{2} \right)$$

where, $a_{1/12} = \frac{1}{6} c^5$ and $b_{1/12} = \frac{1}{12} c$.

Theorem 6. *The inner product* $\left(M\zeta^n, \zeta^n \right)$, *where* ζ^n *solves (24), is conserved.*

Proof. Taking in (24) the inner product with $\frac{\zeta_i^{n+1} + \zeta_i^n}{2}$, remarking that

$$\left(D_3 (a_{1/12})^{1/2} \frac{\zeta^{n+1} + \zeta^n}{2}, (a_{1/12})^{1/2} \frac{\zeta^{n+1} + \zeta^n}{2} \right) = 0,$$

and

$$\left(D_3 (b_{1/12})^{1/2} \frac{\zeta^{n+1} + \zeta^n}{2}, (b_{1/12})^{1/2} \frac{\zeta^{n+1} + \zeta^n}{2} \right) = 0.$$

Using the fact that D_1, D_3 and D_1^v are skew-symmetric matrix, we obtain

$$\sum_i M\Big(\frac{\zeta_i^{n+1}-\zeta_i^n}{\Delta t}\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2} + \varepsilon\sum_i \Big(\Big[\frac{1}{2}\Big(v_i^{n+\frac{1}{2}} + \frac{v_{i+1}^{n+\frac{1}{2}}+v_{i-1}^{n+\frac{1}{2}}}{2}\Big)\Big(D_1\frac{v^{n+1}+v^n}{2}\Big)_i$$

$$+\frac{1}{2}\frac{v_{i+1}^{n+1}+v_{i+1}^n+v_{i-1}^{n+1}+v_{i-1}^n}{4}\Big(D_1 v^{n+\frac{1}{2}}\Big)_i\Big]\frac{v_i^{n+1}+v_i^n}{2}\Big)$$

$$-\frac{3\varepsilon^2}{8}\sum_i \Big(\frac{1}{4}\Big((w_i^{n+\frac{1}{2}})^2 + \frac{(w_{i+1}^{n+\frac{1}{2}})^2+(w_{i-1}^{n+\frac{1}{2}})^2}{2}\Big)\Big(D_1\frac{w^{n+1}+w^n}{2}\Big)_i$$

$$+\frac{1}{2}\frac{w_{i+1}^{n+1}+w_{i+1}^n+w_{i-1}^{n+1}+w_{i-1}^n}{4}\frac{w_{i+1}^{n+\frac{1}{2}}+w_{i-1}^{n+\frac{1}{2}}}{2}\Big(D_1 w^{n+\frac{1}{2}}\Big)_i\Big)\frac{w_i^{n+1}+w_i^n}{2}$$

$$+\frac{3\varepsilon^3}{16}\sum_i \Big(\frac{1}{5}\Big((\zeta_i^{n+\frac{1}{2}})^3 + \frac{(\zeta_{i+1}^{n+\frac{1}{2}})^3+(\zeta_{i-1}^{n+\frac{1}{2}})^3}{2}\Big)\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i$$

$$+\frac{3}{5}\frac{\zeta_{i+1}^{n+1}+\zeta_{i+1}^n+\zeta_{i-1}^{n+1}+\zeta_{i-1}^n}{4}\frac{(\zeta_{i+1}^{n+\frac{1}{2}})^2+\zeta_{i+1}^{n+\frac{1}{2}}\zeta_{i-1}^{n+\frac{1}{2}}+(\zeta_{i-1}^{n+\frac{1}{2}})^2}{3}\Big(D_1\zeta^{n+\frac{1}{2}}\Big)_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}$$

$$+\frac{7\varepsilon\mu}{24}\sum_i \Big(\Big(D_1\frac{\zeta^{n+1}+\zeta^n}{2}\Big)_i(D_2\zeta^{n+\frac{1}{2}})_i + \Big[D_1\Big(D_2\zeta^{n+\frac{1}{2}}\Big(\frac{\zeta^{n+1}+\zeta^n}{2}\Big)\Big)\Big]_i\Big)\frac{\zeta_i^{n+1}+\zeta_i^n}{2}$$

$$= S_1(\zeta) + \varepsilon S_2(v) - \frac{3\varepsilon^2}{8}S_3(w) + \frac{3\varepsilon^3}{16}S_4(\zeta) + \frac{7\varepsilon\mu}{24}S_5(\zeta) = 0.$$

where,

$$v = \Big(\frac{1}{c}\Big)^{1/3}\zeta, \quad w = \Big(\frac{1}{c^2}\Big)^{1/4}\zeta.$$

$\square$

3.2.4. *Numerical validation.* We consider the same initial condition as for KdV equation:

$$\zeta_0(x) = 2\,c_1\,\text{sech}^2\Big(\sqrt{\frac{3c_1\varepsilon}{2\mu}}\,x\Big).$$

We will produce the same experiment as for KdV equation with a sinusoidal bottom

$$b(x) = \sin(2\pi\alpha x).$$

In the figure 6, we have displayed the solution for this initial condition ($c_1 = 0.5$) for a sinusoidal bottom and with different models for $\varepsilon = \sqrt{\mu}$ and $T = 13.33$. We can see that strong and original models give very close solutions while the gentle model provides a different solution. For this problem, we have performed a study of the convergence in order to compare LDG method and the presented finite difference method. However, for LDG method, centered fluxes have been used, inducing a non-optimal convergence for odd orders. As in tables 4, 5, the convergence of LDG method seems to be in $O(h^{r+1})$ (r being the order of approximation) for even orders, while we observe a convergence of $O(h^{r-1})$ ($h = \Delta x$) for odd orders. For finite-difference code, we have used the following time step :

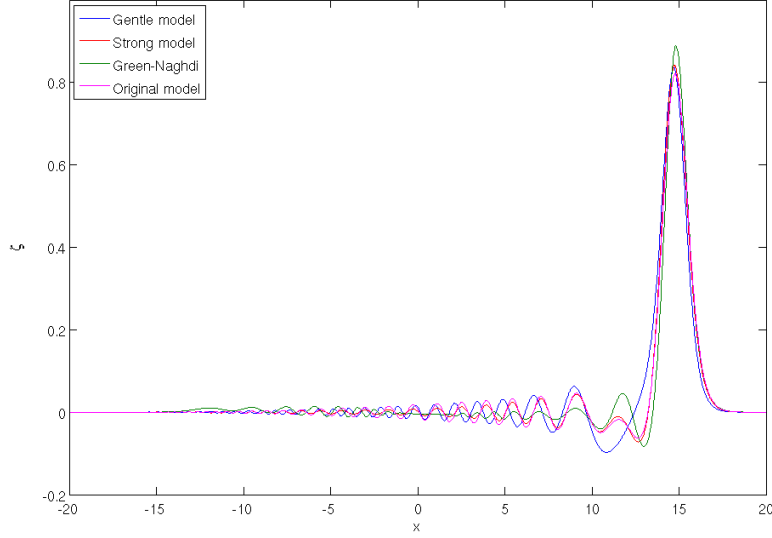$$\Delta t = 0.01\,\frac{320}{N}$$

where N denotes the number of points.

FIGURE 6. Solution of Camassa-Holm equation for a sinusoidal bottom, with gentle (19), strong (21) and original model (16) for $t = 13.33$. ($c_1 = 0.5$, $\mu = 0.1$, $\varepsilon = \sqrt{\mu}$, $\alpha = 0.5\,\mu$, $\beta = 0.5\varepsilon$)

LDG, order 1			LDG, order 3		LDG, order 4		Finite Difference	
N	Error	Order	Error	Order	Error	Order	Error	Order
320	1.05e-1	-	2.74e-2	-	2.32e-2	-	3.33e-1	-
640	5.79e-2	0.86	3.79e-3	2.85	7.97e-4	4.86	1.08e-1	1.62
1280	4.85e-2	0.26	2.55e-4	3.89	1.25e-5	5.99	2.95e-2	1.87
2560	4.54e-2	0.09	7.90e-5	1.69	2.04e-7	5.94	7.41e-3	1.99
5120	4.49e-2	0.02	2.28e-5	1.79	1.39e-8	3.88	1.85e-3	2.00

TABLE 4. L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 20$ and for the gentle model (19). N denotes here the number of degrees of freedom. ($c_1 = 0.5$, $\mu = 0.05$, $\varepsilon = \sqrt{\mu}$, $\alpha = 0.5\,\varepsilon$, $\beta = \varepsilon$)

In figure 7, we displayed the variation of the L^2-norm for the different proposed schemes. In this figure, we observe that the discrete energy of finite difference schemes is conserved, however the conservation is not as good as for KdV-top equation. In figure 8, it is difficult to observe a $O(\mu^2)$ error between the solutions of Green-Naghdi model and Camassa-Holm equation.

LDG, order 1			LDG, order 3		LDG, order 4		Finite Difference	
N	Error	Order	Error	Order	Error	Order	Error	Order
320	1.14e-1	-	1.58e-2	-	9.81e-3	-	3.51e-1	-
640	5.59e-2	1.03	3.12e-3	2.34	2.03e-3	2.27	1.12e-1	1.65
1280	4.78e-2	0.23	4.06e-4	2.94	6.40e-5	4.98	2.84e-2	1.98
2560	4.62e-2	0.05	8.23e-5	2.30	2.00e-6	5.00	7.58e-3	1.91
5120	4.59e-2	0.01	2.37e-5	1.80	5.11e-8	5.29	1.90e-3	2.00

TABLE 5. L^2 errors for the solitary-wave and sinusoidal bottom between the numerical solution and a reference solution for $t = 20$ and for the strong model (21). N denotes here the number of degrees of freedom. ($c_1 = 0.5$, $\mu = 0.05$, $\varepsilon = \sqrt{\mu}$, $\alpha = 0.5\,\varepsilon$, $\beta = \varepsilon$)
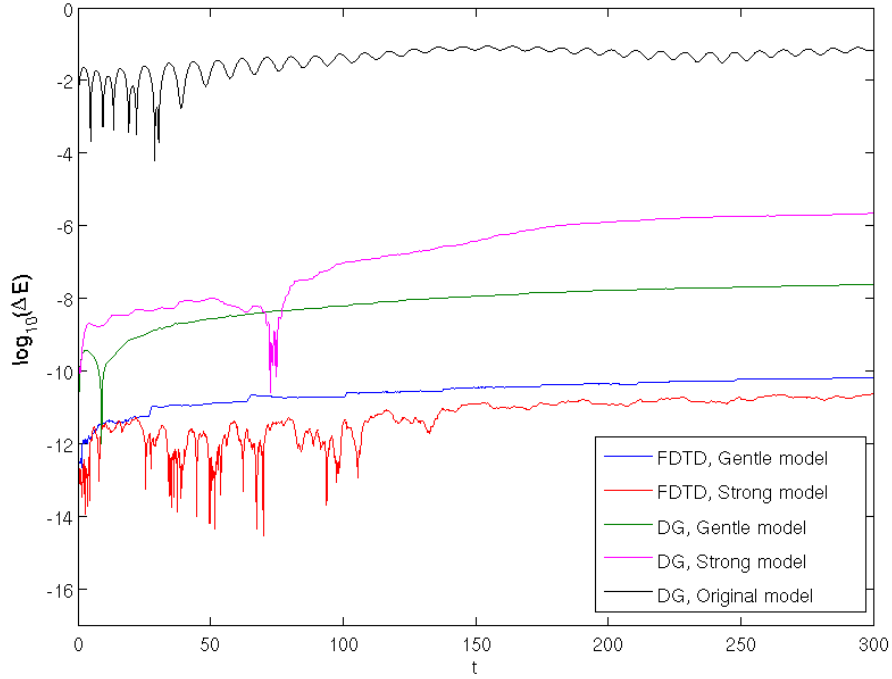

FIGURE 7. Logarithm of variation of energy versus time for gentle scheme, strong scheme and original scheme for solitary-wave and sinusoidal bottom ($c_1 = 0.5$, $\mu = 0.05$, $\varepsilon = \sqrt{\mu}$, $\alpha = 0.5\,\varepsilon$, $\beta = \varepsilon$) Finite difference and fourth order LDG with 5120 degrees of freedom.
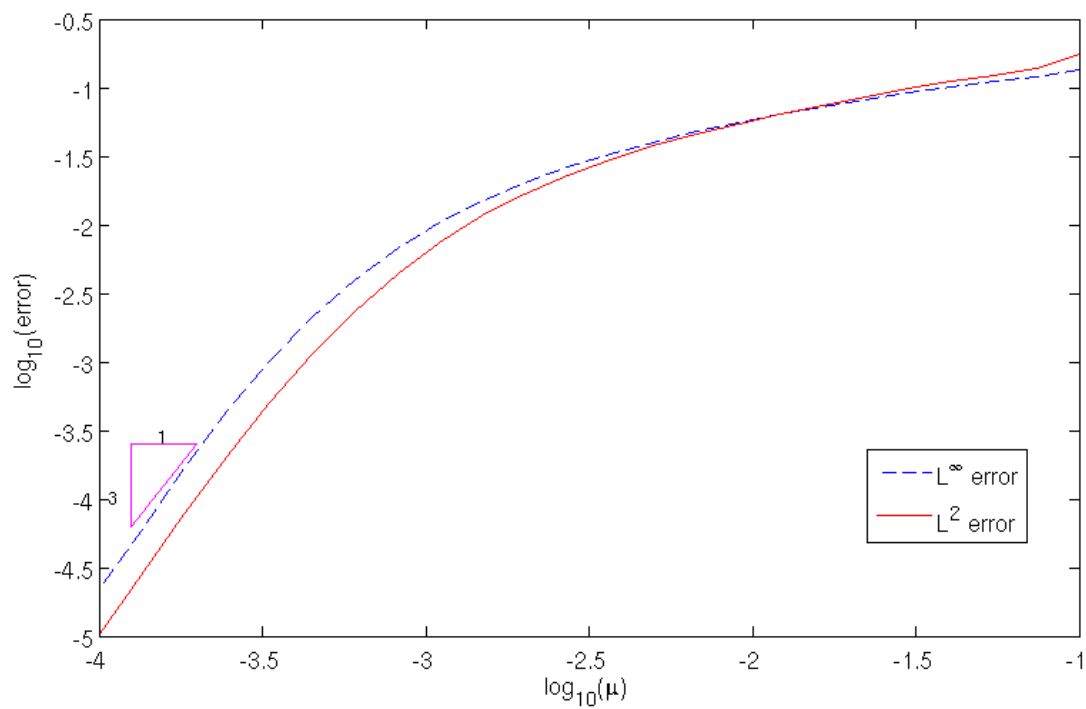
FIGURE 8. Logarithm of relative error between Green-Naghdi model and Camassa-Holm equation for a flat bottom and a gaussian initial condition ($\varepsilon = \sqrt{\mu}$)

Acknowledgments

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Université Bordeaux I; IMB, 351 Cours de la Libération, 33405 Talence Cedex, France
E-mail address: `Marc.Durufle@inria.fr, Samer.Israwi@math.u-bordeaux1.fr`