Mixed Spectral Elements for the Helmholtz Equation

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Introduction

- First use of these elements for the transient wave equation

  $\Rightarrow$ G. COHEN, S. FAUQUEUX, *Mixed finite elements with mass-lumping for the transient wave equation*, J. Comp. Acous. **8** (1), pp. 171-188, 2000.

- Advantages of these elements for the Helmholtz equation

  $\Rightarrow$ Low storage of the matrix coming from the discretization

  $\Rightarrow$ Gain of time for the product matrix vector

- Contents of this presentation

  $\Rightarrow$ Short presentation of these elements

  $\Rightarrow$ Comparison with "classical elements" for direct and iterative solvers

Model Problem

$$-\omega^2 \rho\, u(\mathbf{x}) - \nabla \cdot (\mu \nabla u(\mathbf{x})) = f(\mathbf{x}) \quad \text{in } \Omega \tag{1}$$

We transform 1 to the following first-order system:

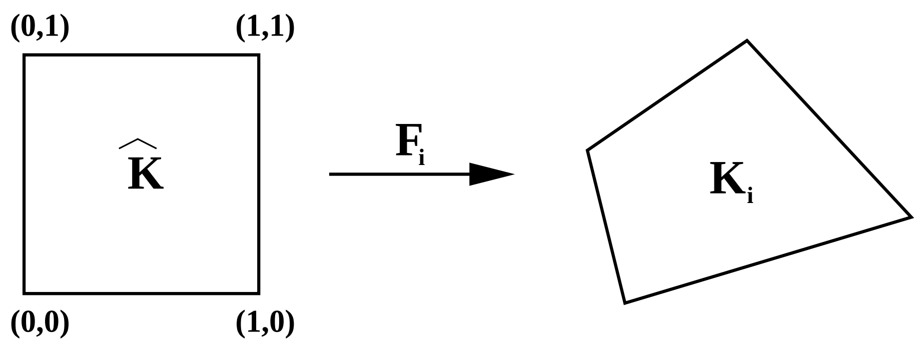
$$\begin{cases} -\omega^2 \rho(\vec{x})\, u(\vec{x}) &= div(-\mathrm{i}\,\omega \vec{v}(\vec{x})) + f(\vec{x}) \\ -\mathrm{i}\,\omega \vec{v}(\vec{x}) &= \mu(\vec{x})\, \nabla u(\vec{x}) \end{cases} \tag{2}$$

To this system we add the first-order absorbing boundary condition:

$$\frac{\partial u}{\partial n} = \mathrm{i} \sqrt{\frac{\rho}{\mu}}\, \omega\, u \quad \text{on } \partial\Omega \tag{3}$$

Approximation

Quadrilateral mesh : $\mathcal{M}_h = \bigcup_{j=1}^{N_e} K_j$, $\widehat{K} = [0,1]^2$, K_j : a quadrilateral j



The transformation $\vec{F}_j$

DF_i is the Jacobian matrix of $\vec{F}_i$, $J_i = \det DF_i$.

Q_r is the space of polynomials in $\vec{\hat{x}} \in \hat{K}$ of order less or equal to r in each variable.

Approximate Variational Formulation

$$U_h^r = \left\{ \varphi \in H^1(\Omega) \text{ so that } \varphi_{|K_i} \circ F_i \in Q_r(\hat{K}) \right\} \quad (4)$$

$$V_h^r = \left\{ \vec{\psi} \in \left[L^2(\Omega) \right]^2 \text{ so that } |J_i|\, DF_i^{-1} \vec{\psi}|_{K_i} \circ F_i \in [Q_r(\hat{K})]^2 \right\} \quad (5)$$

$$\begin{aligned} -\omega^2 \int_\Omega \rho\, u_h\, \varphi_h\, d\vec{x} - i\,\omega \int_{\partial\Omega} \sqrt{\frac{\rho}{\mu}}\, u_h\, \varphi_h\, d\sigma &= -\int_\Omega (-i\,\omega\, \vec{v}_h) \cdot \nabla \varphi_h\, d\vec{x} + \int_\Omega f\, \varphi_h\, d\vec{x} \\ \int_\Omega \frac{1}{\mu} (-i\,\omega\, \vec{v}_h)\, \vec{\psi}_h\, d\vec{x} &= \int_\Omega \nabla u_h\, \vec{\psi}_h\, d\vec{x} \end{aligned} \quad (6)$$

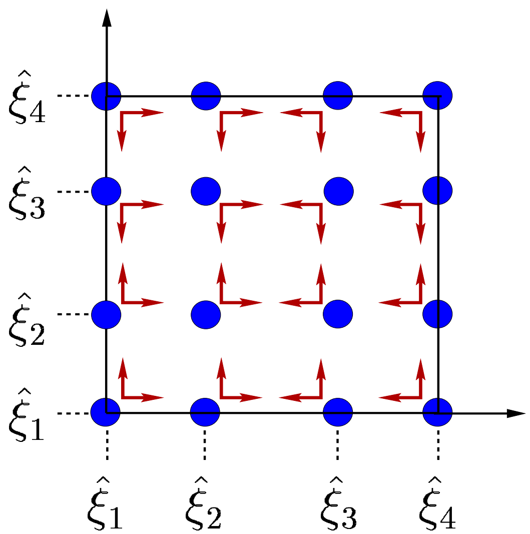
$$
\begin{cases}
-\omega^2 D_h \vec{U} - i\,\omega\,\widetilde{D}_h \vec{\widetilde{U}} = -R_h \vec{V} + \vec{F}_h \\
B_h \vec{V} = R_h^* \vec{U}
\end{cases}
\tag{7}
$$

$\vec{U}$ and $\vec{F}_h$ are the vectors of the components of u and f respectively on the basis of U_h^r

$\vec{\widetilde{U}}$ the restriction of $\vec{U}$ to the boundary of Ω

$\vec{V}$ the vector of the components of $-i\,\omega\vec{v}$ on the basis of V_h^r

Degrees of Freedom



Degrees of freedom for u (circles) and $\vec{v}$ (arrows)

$\xi_k, k = 1..(r+1)$ are the Gauss-Lobatto quadrature points , r the order of approximation

$\Rightarrow$ Scalar Lagrange basis functions $\varphi_{\ell,m} \circ F_i = \hat{\varphi}_{\ell,m}$

$$\hat{\varphi}_{\ell,m}(\hat{x},\hat{y}) = \prod_{i=1\, i\neq\ell}^{r+1} \frac{\hat{x}-\xi_\ell}{\xi_i-\xi_\ell} \prod_{j=1\, j\neq m}^{r+1} \frac{\hat{y}-\xi_m}{\xi_j-\xi_m} \tag{8}$$

Properties of mass matrices

$$\left\{\begin{array}{llll} -\omega^2 D_h \vec{U} & - \mathrm{i}\,\omega\, \widetilde{D}_h \vec{\widetilde{U}} & = & -R_h \vec{V} + \vec{F}_h \\ B_h \vec{V} & & = & R_h^* \vec{U} \end{array}\right. \tag{9}$$

- Use of Gauss-Lobatto quadrature formulas to compute all the integrals

$\Rightarrow$ D_h and $\widetilde{D}_h$ are diagonal, B_h block-diagonal (2x2 in dimension 2)

Properties of Stiffness Matrices

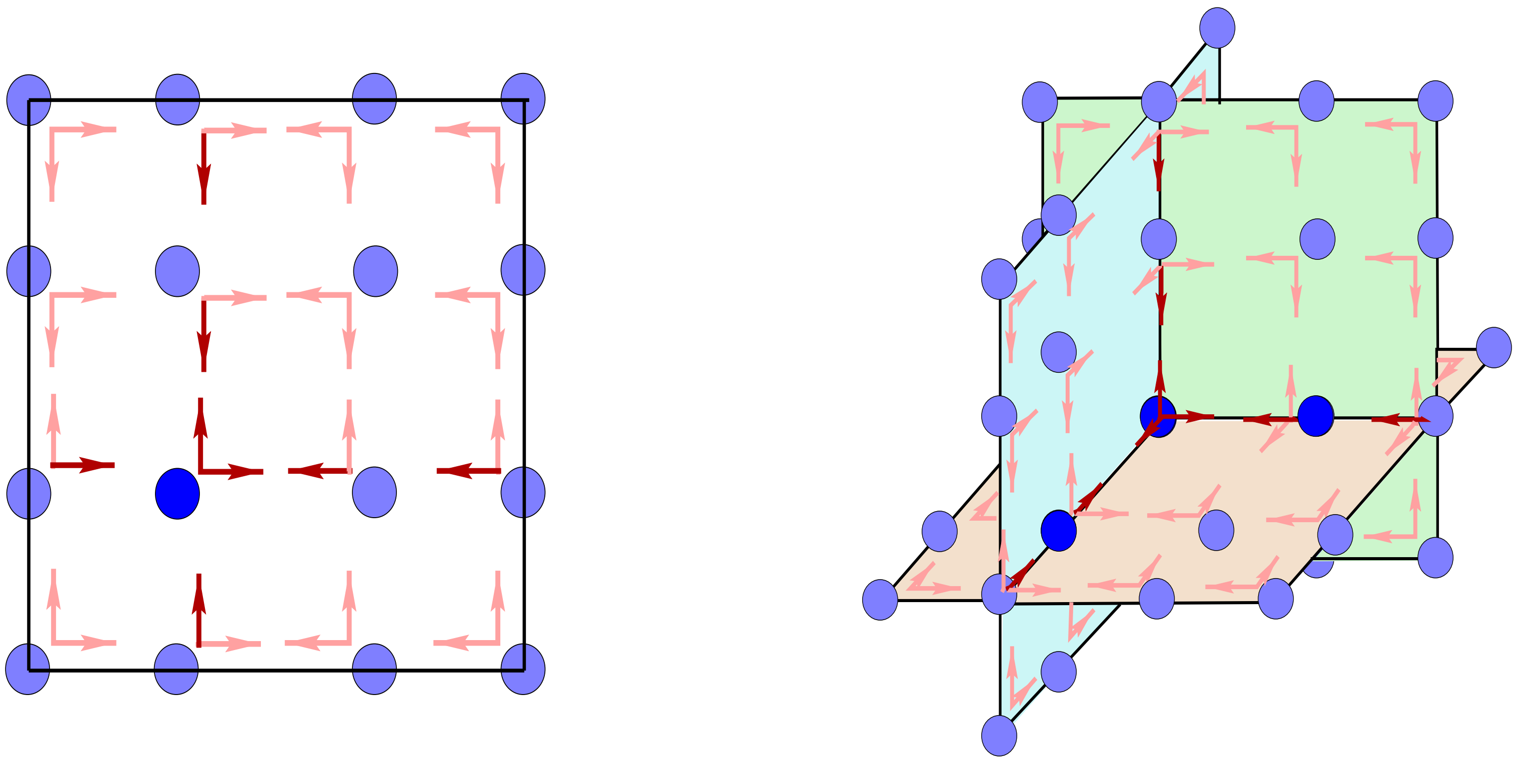
$$\int_{K_i} \psi \cdot \nabla\varphi \, dx = \int_{\hat{K}} J_i \frac{1}{J_i} DF_i \hat{\psi} \quad DF_i^{*-1} \nabla\hat{\varphi} \, dx = \int_{\hat{K}} \hat{\psi} \, \nabla\hat{\varphi} \, dx \tag{10}$$

- Stiffness matrices independent of the element K_i

  $\Rightarrow$ No storage needed for these matrices

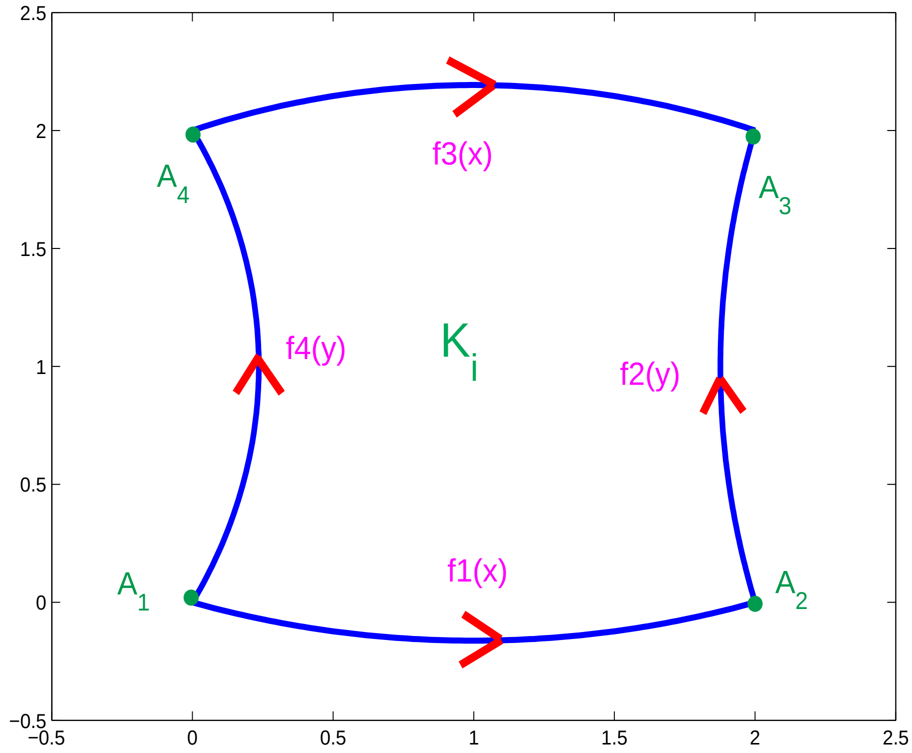
- Elementary stiffness matrices are sparse :

  $\Rightarrow$ Gain of time expected



Many interactions in elementary stiffness matrices are null, particularly in 3D .

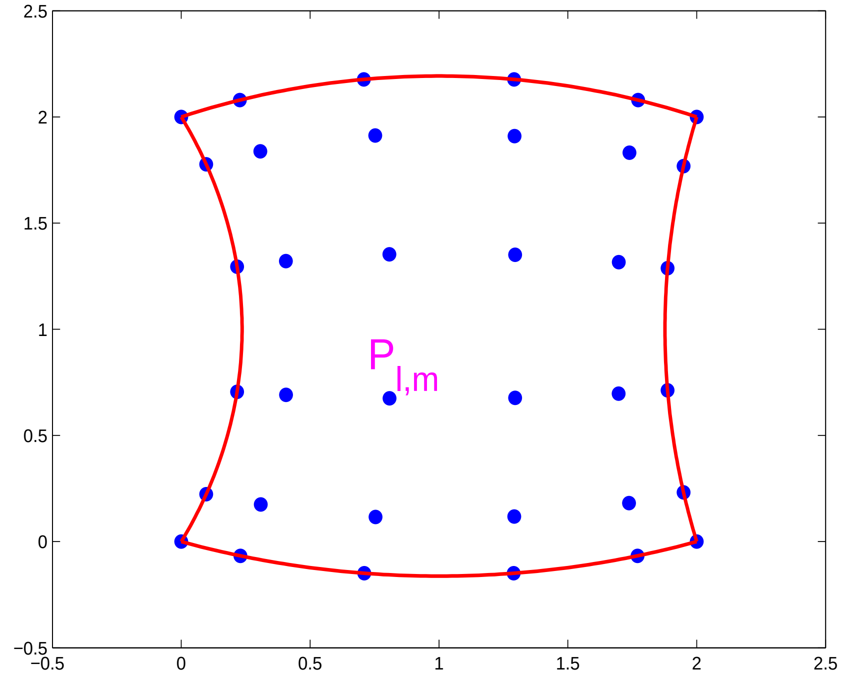
Curved Elements



Transformation of Gordon-Hall from $\hat{K} = [0\,1]^2$ to K_i

$$\widetilde{F}_i(\hat{x},\hat{y}) = \hat{y}\, f_3(\hat{x}) + (1-\hat{y})\, f_1(\hat{x}) + \hat{x}(\, f_2(\hat{y}) - \hat{y}A_3 - (1-\hat{y})A_2) + (1-\hat{x})(\, f_4(\hat{y}) - \hat{y}A_4 - (1-\hat{y})A_1) \quad (11)$$

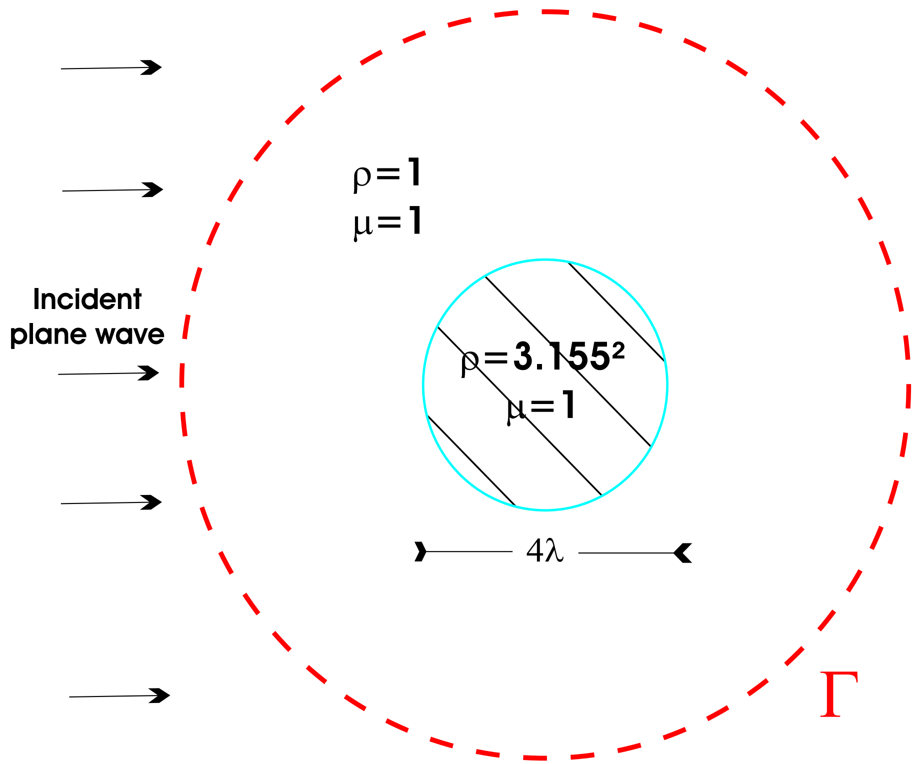
Curved Elements



- $P_{l,m}$ Projection of Gauss-Lobatto points from $\hat{K}$ to K_i by the Gordon-Hall transformation $\widetilde{F}_i$

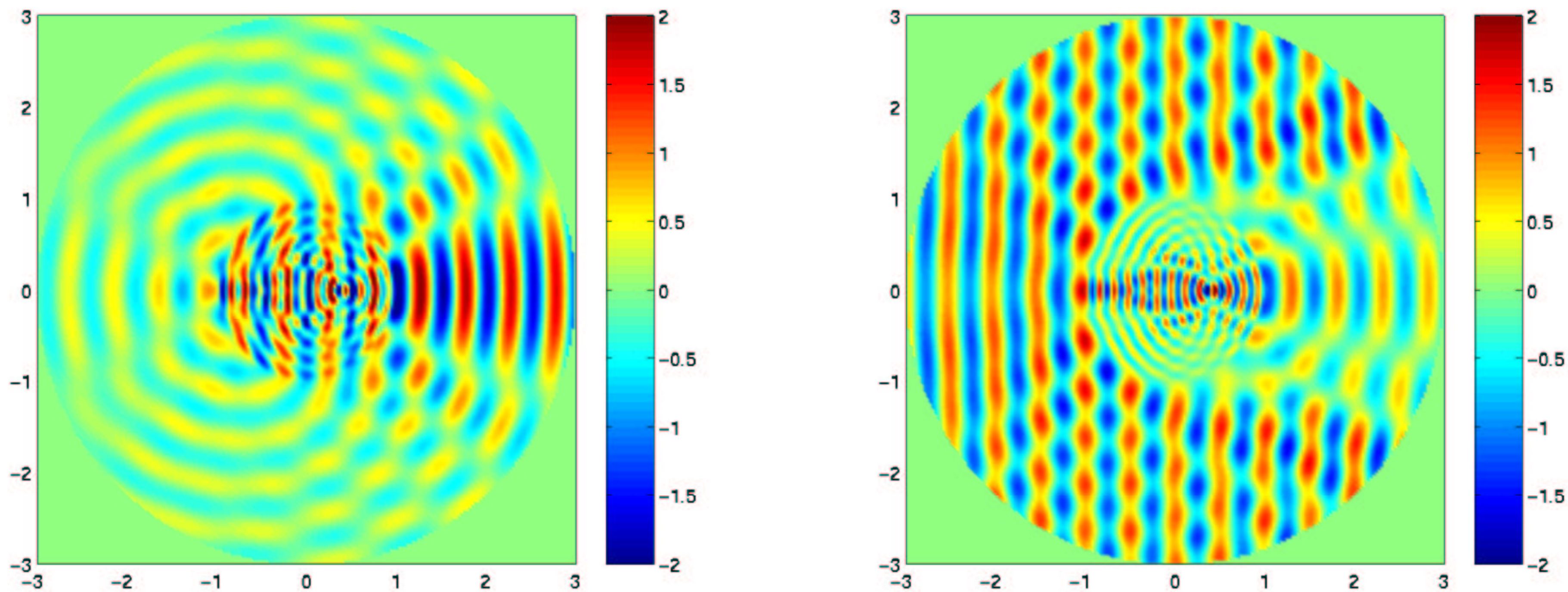
- F_i is a Lagrangian interpolation $$F_i(\hat{x},\hat{y}) = \sum_{l,m=1}^{r+1} \hat{\varphi}_{\ell,m}(\hat{x},\hat{y}) \quad P_{l,m}$$

Test-problem studied



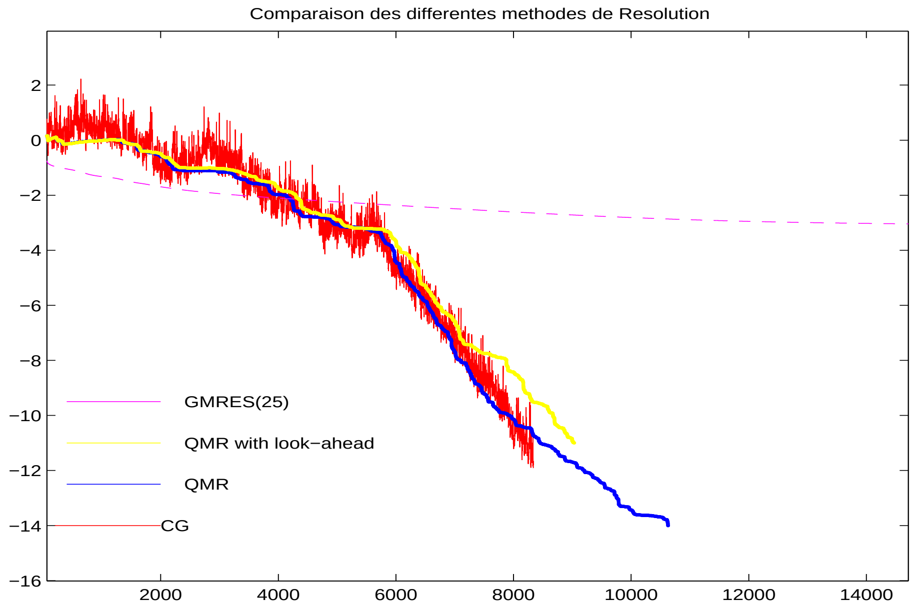
Scattering of an incident plane wave by a dielectric disk of diameter 4λ

Numerical Solution of the problem



The real part of the diffracted field on the left, and the total field on the right for the dielectric disk of diameter 4λ

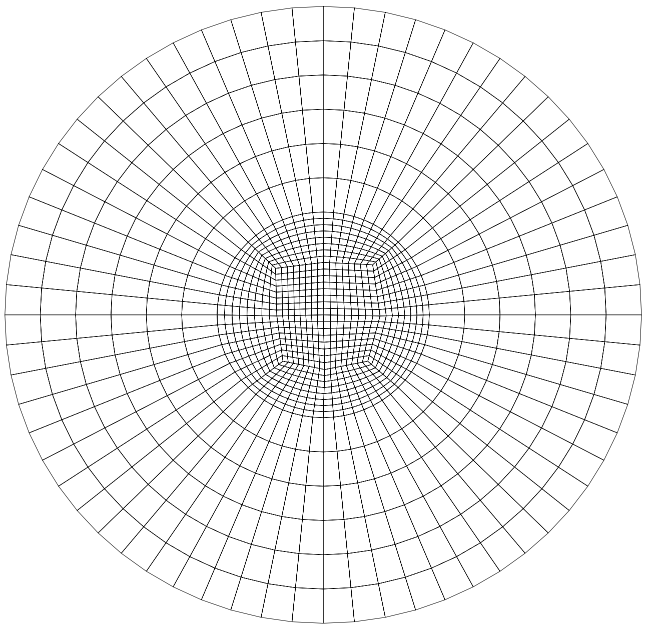
Comparison of different iterative solvers

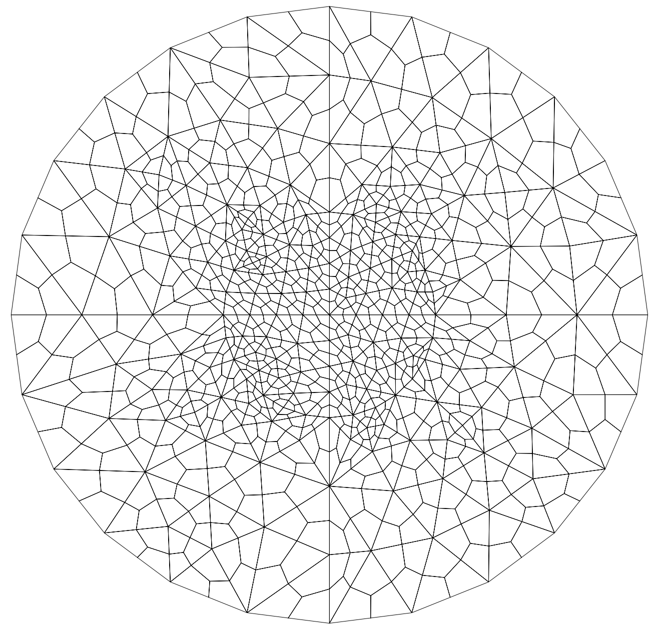


Evolution of the logarithm of the criterion versus the number of iterations for the different solvers

$\Rightarrow$ Conjugate Gradient is the most efficient, despite its fluctuating convergence.

Different kind of Meshes

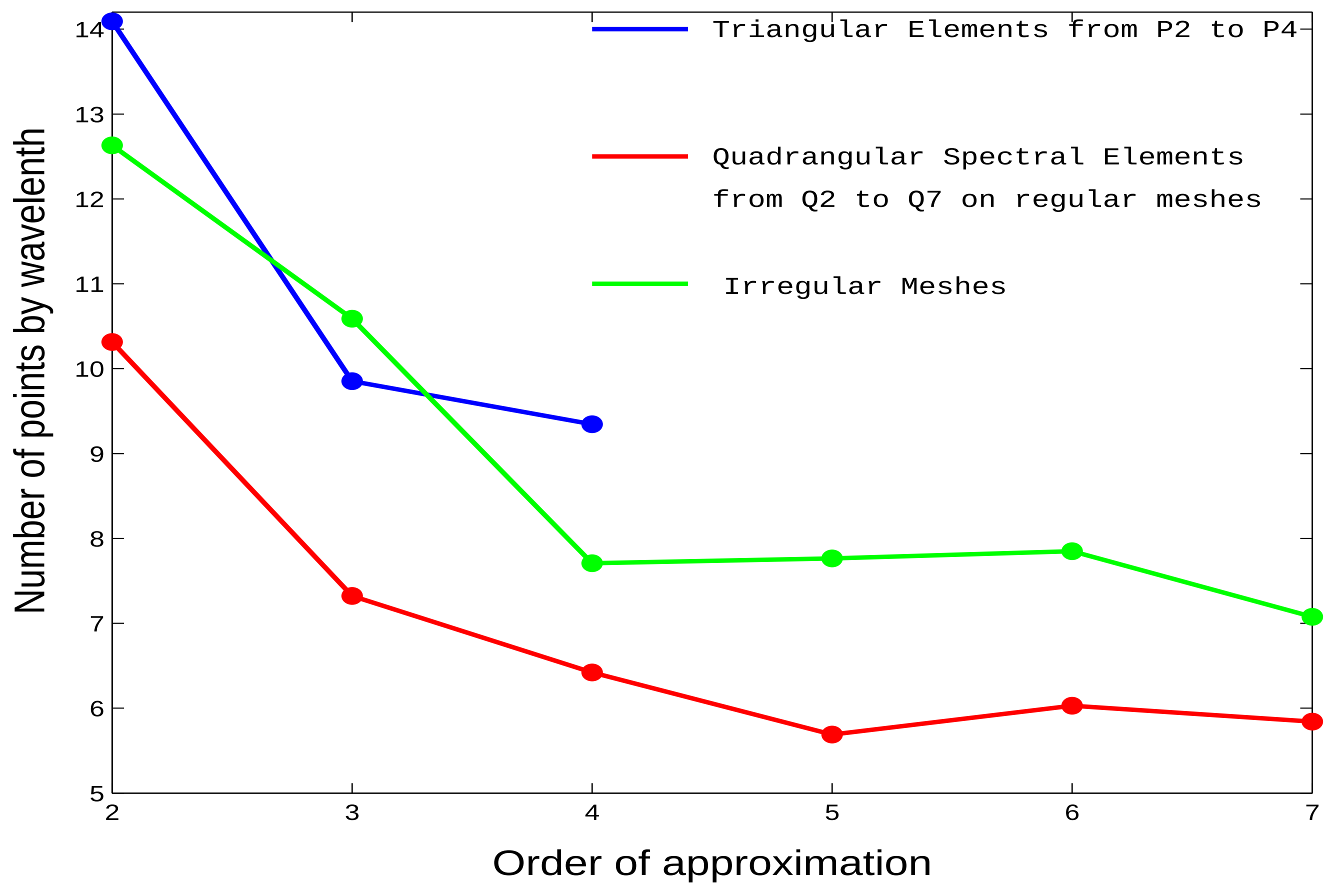




Left : a quasi-regular mesh of quadrilaterals

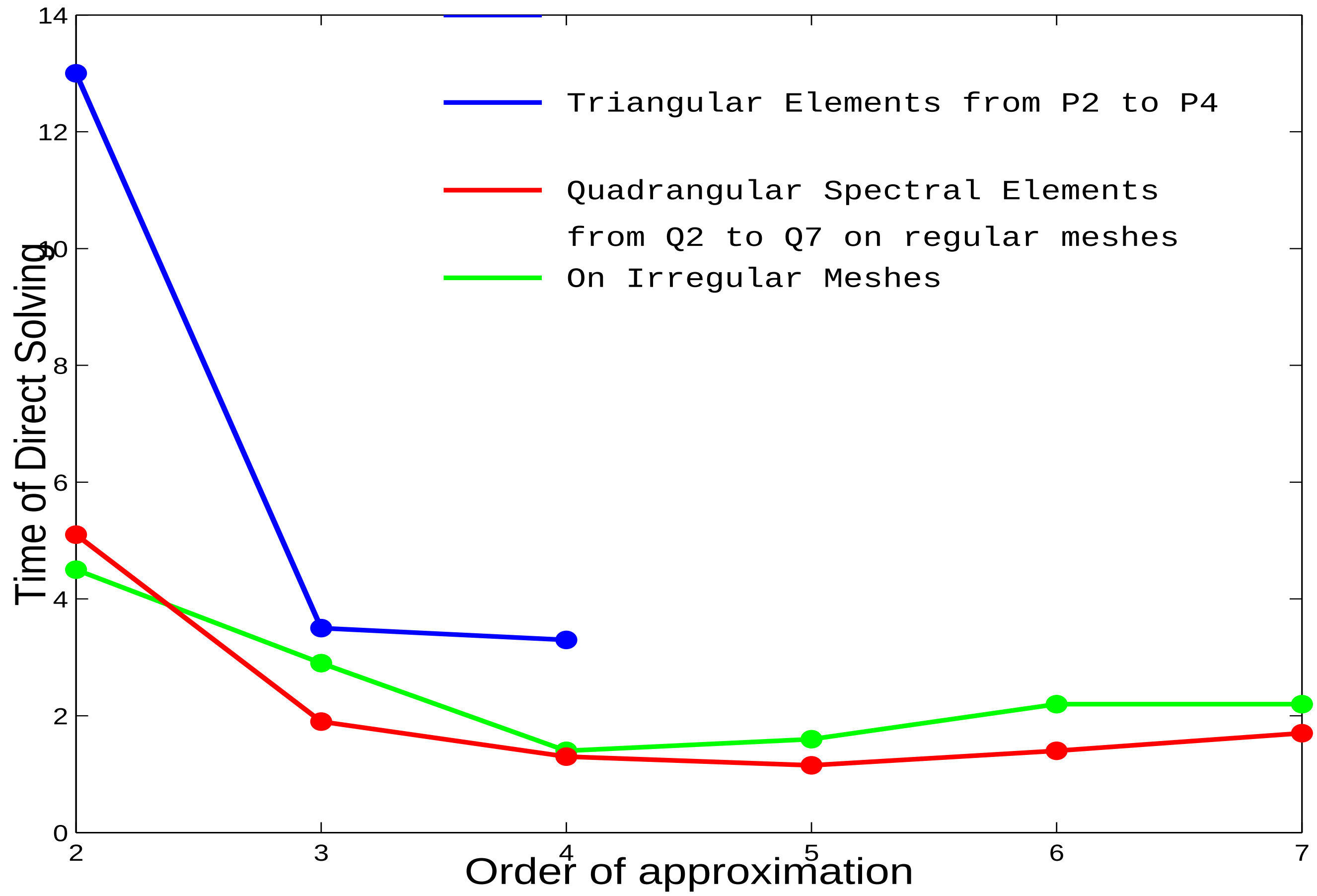
Right : a triangular mesh splitted in quadrilaterals

Comparison of numbers of degrees of freedom



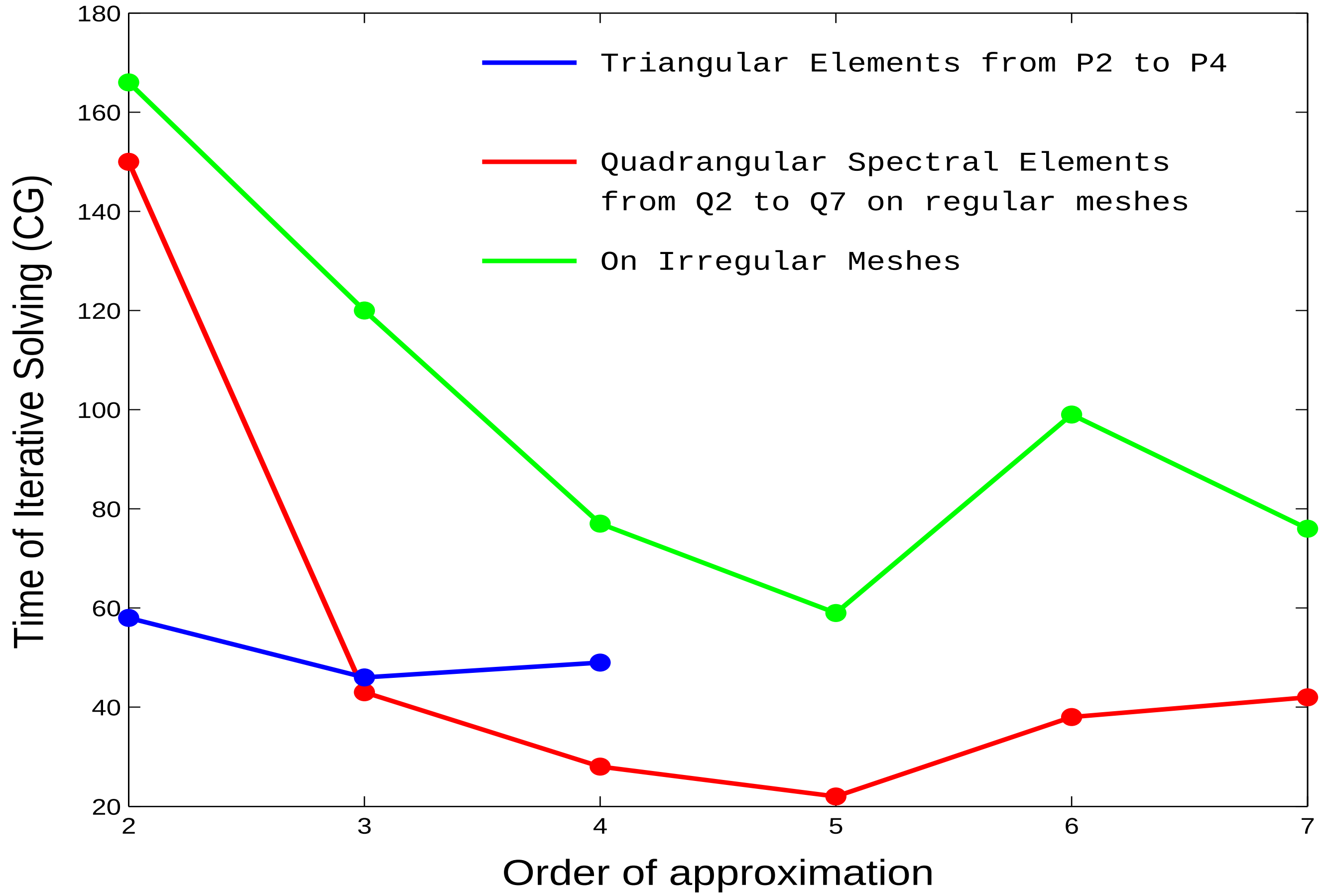
Comparison of numbers of points by wavelength between mixed spectral elements for two kinds of meshes and “classical” elements, for L2-error less or equal to 5%

Time for a direct solver



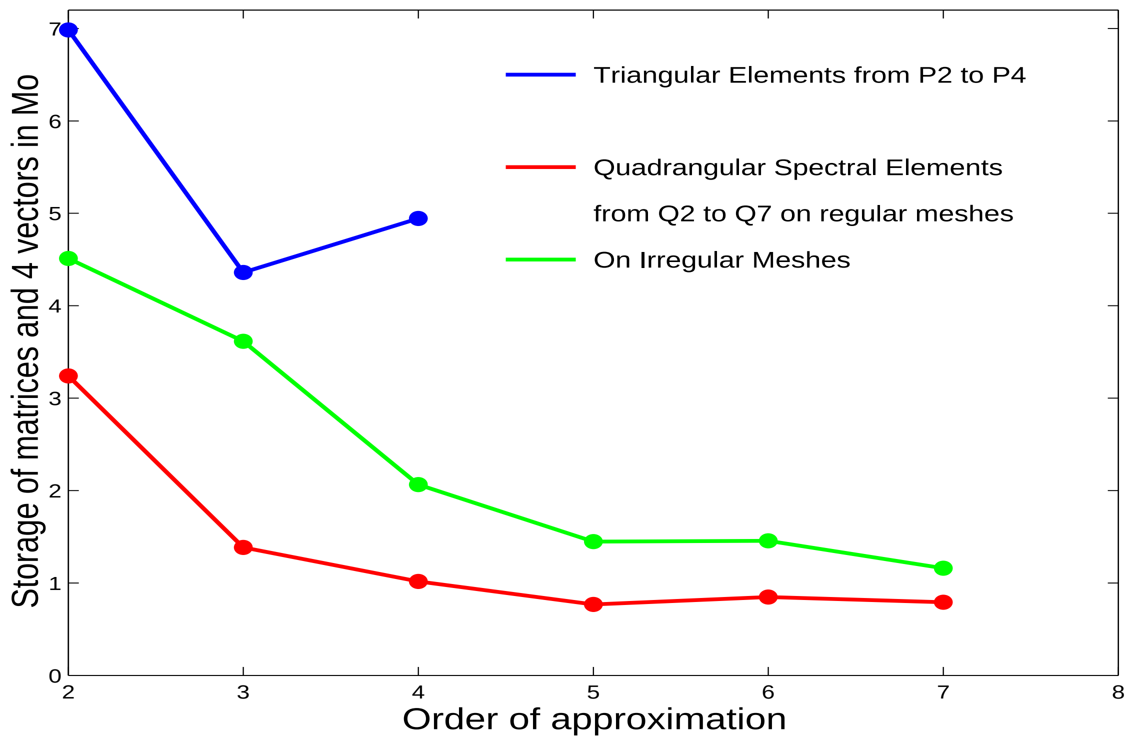
Comparison of time for a direct solver between mixed spectral elements for two kinds of meshes and “classical” elements, for an 5% L2-error

Time for an iterative solver - CG



Comparison of time for a conjugate gradient solver between mixed spectral elements in two kinds of meshes and “classical” elements, for a L2-error less or equal than 5 %

Gain of storage for mixed spectral elements



Comparison of the storage of the matrices and four vectors between mixed spectral elements for two kinds of meshes and “classical” elements, for a L2-error less or equal than 5 %

- Conjugate Gradient uses only four vectors to compute the solution

Concludings remarks on numerical results

- Non-regular meshes coming from splitting of triangular meshes give poor results
- Numbers of degrees of freedom decreases when order increases
- Q5 is an optimal order for this problem
- For an error less than 5 %, high order is more accurate

Conclusion

- Efficient method and low-storage accurate method
- More efficient in the 3D case
- A preconditioning method for an iterative solver is necessary
- Extensions to time-harmonic maxwell equations are studied