High Order Edge Finite Element Method for Vlasov-Maxwell Equation on Hexahedral Meshes

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22 June 2009

Bibliography and motivation

- G. Cohen, M. Duruflé
  Non Spurious Spectral-Like Element Methods for Maxwell's Equations
- C.K. Birdsall, A.B. Langdon
  Plasma Physics via Computer Simulation
- G. B. Jacobs, J. S. Hesthaven
  High-order nodal discontinuous Galerkin particle-in-cell method on unstructured grids
- Use of finite difference code : QuickSilver developed in Sandia Labs

Vlasov-Maxwell equations

Maxwell system

$$\begin{aligned} \varepsilon \frac{\partial E}{\partial t}(x,t) - \text{curl}\, H(x,t) &= -J(x,t) \\ \mu \frac{\partial H}{\partial t}(x,t) + \text{curl}\, E(x,t) &= 0 \end{aligned}$$

Relativistic motion of particles

$$\begin{aligned} \frac{\mathrm{d}x_k}{\mathrm{d}t}(t) &= v_k(t) \\ \frac{\mathrm{d}p_k}{\mathrm{d}t}(t) &= \frac{q}{m}(E(x_k(t),t) + \mu v_k(t) \times H(x_k(t),t)) \end{aligned}$$

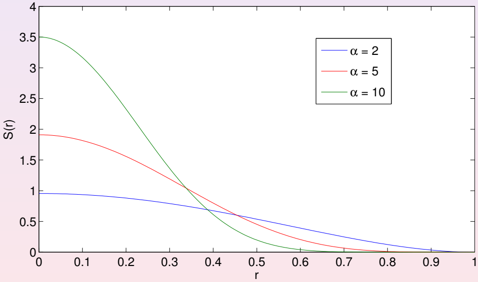
Relation between current J and particles

$$J(x,t) = \sum_k \omega_k\, q_k\, v_k(t)\, S(x - x_k(t))$$

Distribution function for particles

Radial distribution function with influence radius R

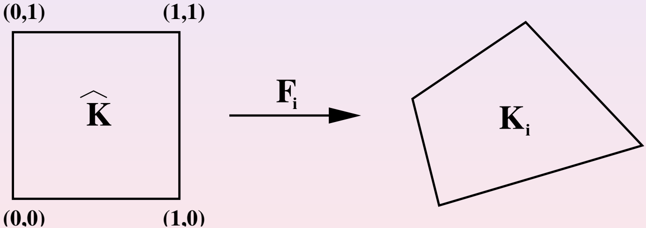
$$S(x - x_k) = \hat{S}(|x - x_k|) = \beta(1 - (\frac{r}{R})^2)^\alpha$$



Finite element variational formulation

Mesh including hexahedra/quadrilaterals

$$\Omega = \bigcup_e K_e$$



Finite element variational formulation

Edge finite elements :

$$V_E = \{u \in H(\text{curl}, \Omega) \text{ so that } DF_e^* \, u \in Q_r^d\}$$

$$V_H = \{u \in L^2(\Omega) \text{ so that } u \in Q_r^d\}$$

$$\frac{\partial}{\partial t} \int_\Omega \varepsilon \, E \cdot \varphi \; - \; \int_\Omega H \cdot \nabla \times \varphi + \gamma \sum_e \int_{\Sigma_e} [E \cdot n][\varphi \cdot n] \quad = \quad - \int_\Omega J \cdot \varphi$$

$$\frac{\partial}{\partial t} \int_\Omega \mu \, H \cdot \psi \; + \; \int_\Omega \nabla \times E \cdot \psi + \delta \sum_e \int_{\Sigma_e} [H \times n][\varphi \times n] \quad = \quad 0$$

Finite element variational formulation

$$\begin{aligned} B_h \frac{\mathrm{d}E}{\mathrm{d}t} - R_h H + S_h^1 E &= C_h V \\ D_h \frac{\mathrm{d}H}{\mathrm{d}t} + R_h^* E + S_h^2 H &= 0 \end{aligned}$$

- Mass lumping : $\Rightarrow B_h, D_h, S_h^1, S_h^2$ block-diagonal
- Tensorial basis functions : efficient matrix-vector product with R_h
- Coupling term C_h strategic to have an efficient method

Coupling strategy with particles

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}(t) = v_k(t)$$

$$\frac{\mathrm{d}p_k}{\mathrm{d}t}(t) = \frac{q_k}{m_k}(E_k + \mu(x_k(t))\, v_k(t) \times H_k)$$

Mean values of E and H

$$E_k = \int_\Omega E(x)\, S_k(x)$$

$$H_k = \int_\Omega H(x)\, S_k(x)$$

Linear system :

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V$$

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t} = C_h^* E + bv(V, H)$$

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Presence of C_h^* $\Rightarrow$ conservation of a discrete energy :

$$\frac{1}{2} B_h\, E \cdot E + \frac{1}{2} D_h H \cdot H + \sum_k \omega_k\, m_k\, c_0^2\, (\gamma_k - 1) = \text{Constant}$$

Coupling strategy with particles by using interpolation

Instead of a direct integration, we use interpolation first :

$$J(x) = \sum J_i \, \psi_i(x)$$

where J_i the value of J on interpolation point ζ_i :

$$J_i = \sum_k \omega_k \, q_k \, v_k \, S(\zeta_i - x_k)$$

Mean values of E and H with the interpolate of S_k instead of S_k

$$E_k = \int_\Omega E(x) \, \Pi S_k(x)$$

$$H_k = \int_\Omega H(x) \, \Pi S_k(x)$$

Linear system :

$$\frac{\mathrm{d}X}{\mathrm{d}t} = V$$

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- Presence of C_h^* $\Rightarrow$ conservation of a discrete energy
- Less interpolation points needed (than quadrature points), more flexible

Time scheme

$$
\begin{aligned}
B_h \, \frac{E^{n+1} - E^n}{\Delta t} &= R_h \, H^{n+1/2} + C_h \, V^{n+1/2} \\
D_h \, \frac{H^{n+3/2} - H^{n+1/2}}{\Delta t} &= -R_h^* \, E^{n+1} \\
\frac{X^{n+1} - X^n}{\Delta t} &= V^{n+1/2} \\
\frac{\tilde{P}^{n+3/2} - \tilde{P}^{n+1/2}}{\Delta t} &= -C_h^* \, E^{n+1} + bv\left(\frac{V^{n+3/2} + V^{n+1/2}}{2}, \bar{H}^{n+1}\right)
\end{aligned}
$$

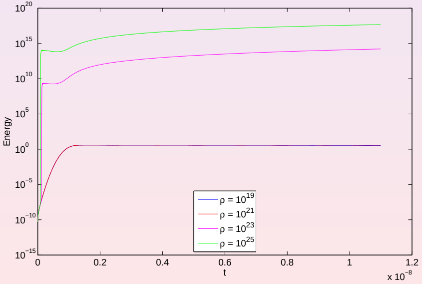
Time scheme

$$\text{CFL}_{\text{Maxwell}} = \max|\lambda({D_h}^{-1/2}\, R_h^* {B_h}^{1/2})|$$

$\Rightarrow$ Maxwell CFL is the most restrictive except for high density plasmas

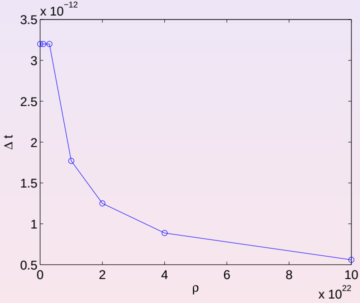
Low-storage Runge-Kutta about four times more expensive and $\frac{\text{CFL}_{RK}}{\text{CFL}_{LF}} = 1.68$

Evolution of energy for a highly dense plasma



Time scheme

Evolution of CFL according to plasma density



Computation of C_h

matrix vector-product $C_h\, V$:

- Computation of

$$S_{k,m} = \hat{S}(|\zeta_m - x_k|)$$

for each interpolation point ζ_m so that

$$|\zeta_m - x_k| \leq R$$

- Computation of current J on interpolation points ζ_m :

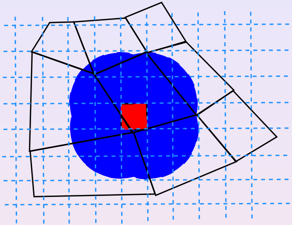
$$J_m = \sum_k q_k\, \omega_k\, v_k\, S_{k,m}$$

- Integration against basis functions

$$(C_h\, V)_i = \sum_m \int_\Omega \psi_m\, J_m \cdot \varphi_i$$

Complexity of C_h

Use of a regular grid to localize both interpolation points and particles



- $\Rightarrow$ no need to localize particles in the elements (need of the inverse of F_i)
- Easy to detect when the particle is completely outside the mesh (useful to eliminate particles)
- Cost of C_h in $N_{part} k + 2r^4 N_{elt}$ in 3-D (r : order of approximation)

Charge conservation

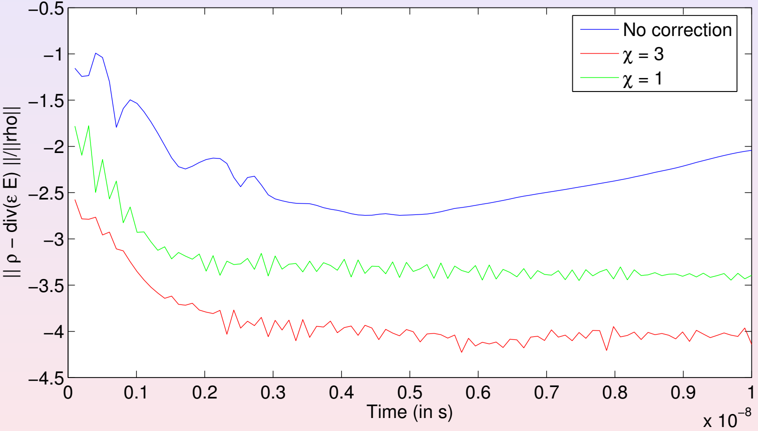
- Boris correction $\Rightarrow$ resolution of a Poisson equation

$$\Delta \phi = \rho - \mathrm{div}(\varepsilon E)$$

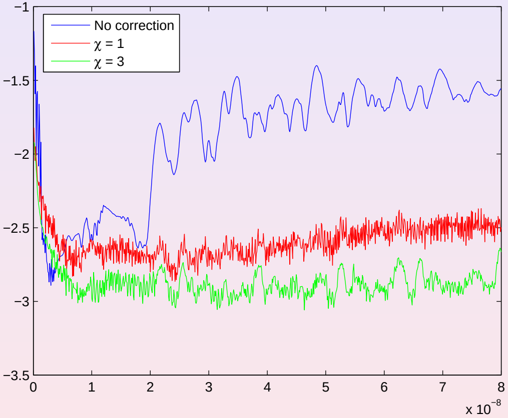
- Hyperbolic correction

$$\begin{aligned}
\varepsilon \frac{\partial E}{\partial t} - \nabla \times H + \nabla \phi &= -J \\
\mu \frac{\partial H}{\partial t} + \nabla \times E &= 0 \\
\frac{1}{\chi^2 \, c_0^2} \frac{\partial \phi}{\partial t} &= \rho - \mathrm{div}(\varepsilon E)
\end{aligned}$$

Charge conservation



Charge conservation



Destruction/Creation of particles

- For beams, particles created with a negative shift compared to the emitting surface, in order to have null intersection
- When $|E \cdot n| > E_{\text{breakdown}}$, creation of np particules with

$$np \, q_0 \, \omega_k = \int_\Gamma E \cdot n$$

Particles created with a small random heigh and a small random initial velocity
$\Rightarrow$ Boris Correction in this case

- Destruction of particles when the influence area does not intersect with the mesh

$\Rightarrow$ No correction for the destruction

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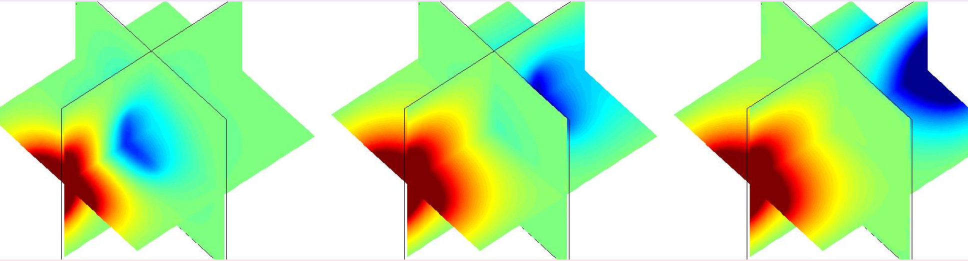
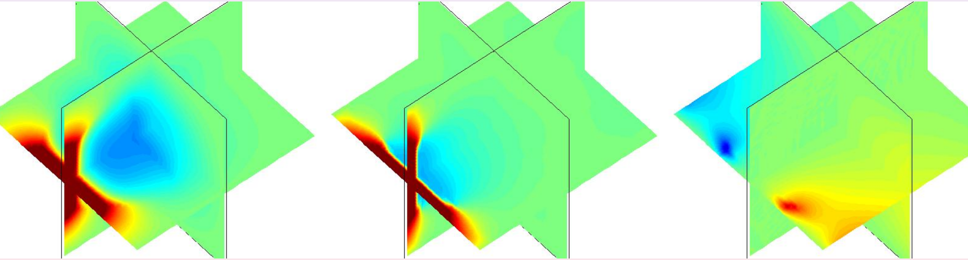
Electron beam

Beam with a small current
Beam with current $J = 1$, and velocity $1e8$
Field E_x for $t = 4e-9$, $t = 8e-9$ and $t = 12e-9$

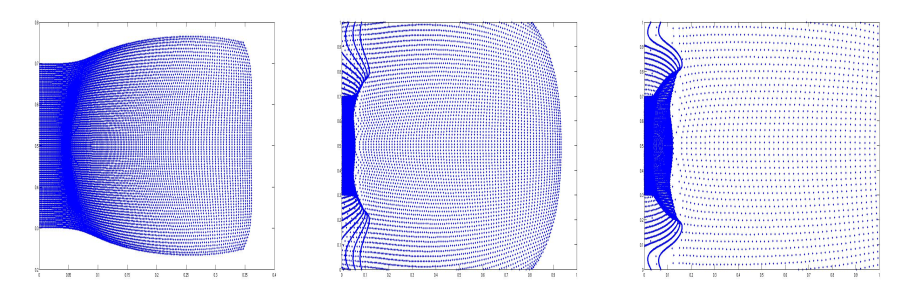


Electron beam

Beam with a high current
Beam with current $J = 3e3$, and velocity $1e8$
Field E_x for $t = 4e-9$, $t = 8e-9$ and field E_y for $t = 8e-9$

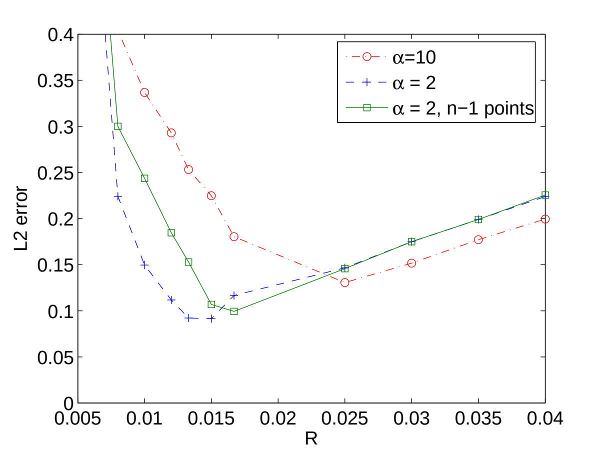


Motion of particles in a 2-D experiment

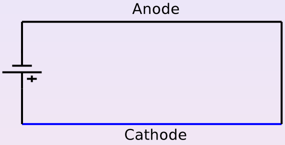


Electron beam

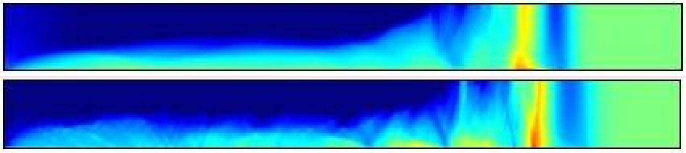
L2 Error versus influence radius R



Magnetic Insulated Transmission Line

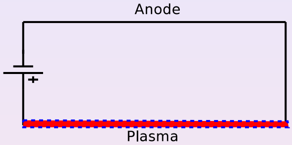


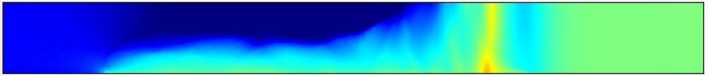
On top, finite difference solution; on bottom, finite element solution



Magnetic Insulated Transmission Line

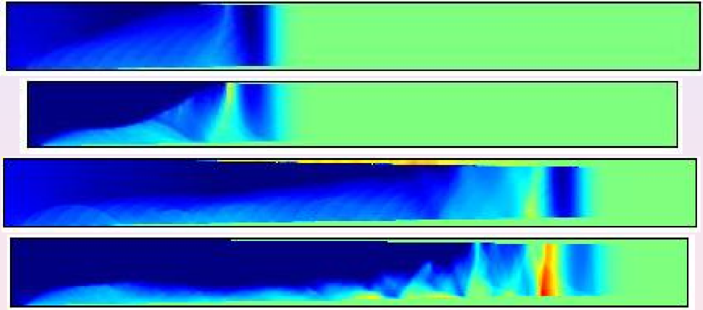
Alternative approach : replace perfect conductor boundary by a plasma





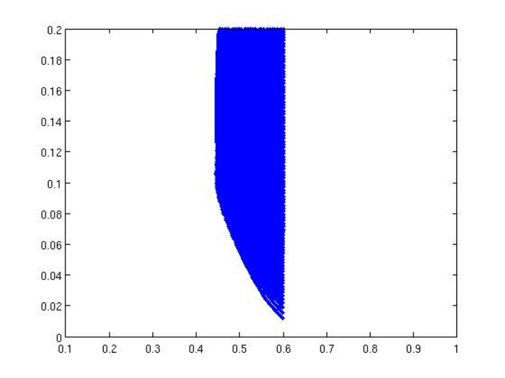
Magnetic Insulated Transmission Line

Particles trapped by staircase finite difference



Plasma Opening Switch

Cold plasma in the box $[0.4, 0.6] \times [0, 0.2]$, with density 10^{17}



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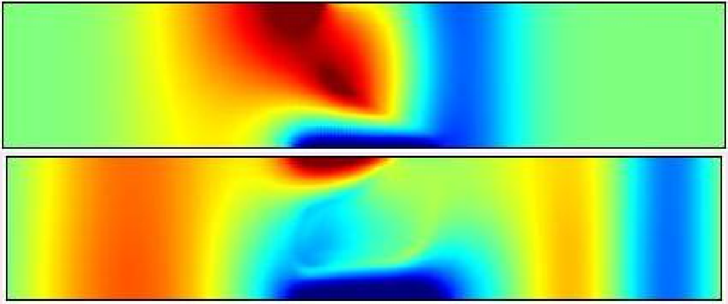
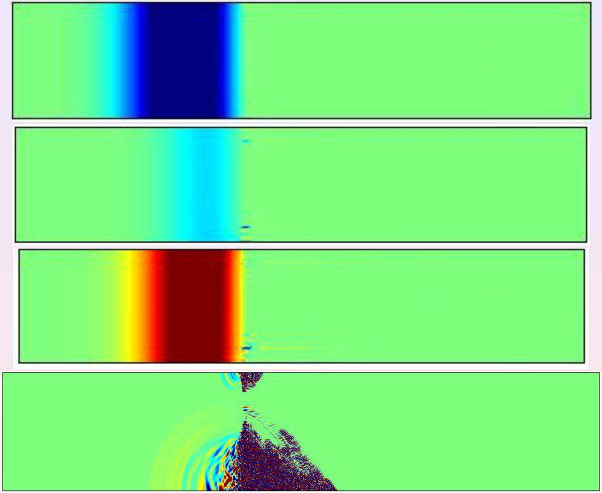


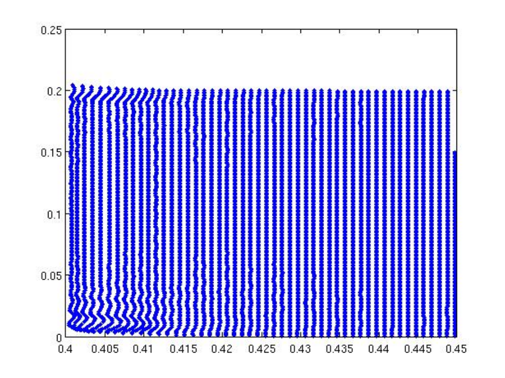
Plasma Opening Switch

Cold plasma in the box $[0.4, 0.6] \times [0, 0.2]$, density 10^{21}



Plasma Opening Switch

Cold plasma in the box $[0.4, 0.6] \times [0, 0.2]$, density 10^{21}



Summary

- Particle in cell method using efficient high order finite element for the solution of Maxwell equations
- Almost constant cost when order is increased, because of the use of hexahedral finite element
- No conservation charge technique needed if no particle is created inside the domain
- Energy conservation with proposed scheme, no grid heating