Optimal High-Order Elements in H(curl) and H(div) for Pyramids, Prisms and Hexahedra

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Bibliography and motivation

- Nedelec's first family not optimal on non-affine hexahedra and prisms
  R.S. Falk, P. Gatto and P. Monk
  Hexahedral $H(div)$ and $H(curl)$ finite elements

- Difficult case of finite elements on pyramids
  N. Nigam, J. Phillips
  Higher-order finite elements on pyramids
  J.-L. Coulomb, F.-X. Zgainski and Y. Maréchal
  A pyramidal element to link hexahedral, prismatic and tetrahedral edge finite elements

- Is it possible to construct finite elements providing an optimal H(div) / H(curl) estimate in $O(h^r)$?

Polynomial spaces

$$\mathbb{Q}_{m,n,p}(x,y,z) \;=\; Span\left\{x^i y^j z^k,\; 0 \le i \le m,\; 0 \le j \le n,\; 0 \le k \le p\right\}$$

$$\mathbb{P}_r(x,y,z) \;=\; Span\left\{x^i y^j z^k,\; i,j,k \ge 0,\; i+j+k \le r\right\}$$

$$\mathbb{B}_r \;=\; \mathbb{P}_r(x,y,z) \;\oplus\; \sum_{k=0}^{r-1} \mathbb{P}_k(x,y) \left(\frac{xy}{1-z}\right)^{r-k}$$

$$\widetilde{\mathbb{P}}_r(x,y,z) \;=\; Span\left\{x^i y^j z^k,\; i,j,k \ge 0,\; i+j+k = r\right\}$$

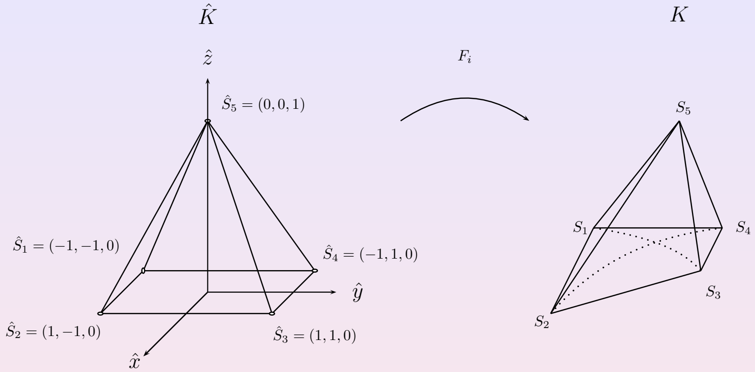
$$\mathbb{D}_r(x,y,z) \;=\; \mathbb{P}^3_{r-1} \oplus \begin{bmatrix} x \\ y \\ z \end{bmatrix} \widetilde{\mathbb{P}}_{r-1}(x,y,z)$$

$$\mathcal{S}_r(x,y,z) \;=\; \left\{u \in \widetilde{\mathbb{P}}_r^{\,3} \text{ so that } u_1\,x + u_2\,y + u_3\,z = 0\right\}$$

$$\mathcal{R}_r(x,y,z) \;=\; \mathbb{P}^3_{r-1} \oplus \mathcal{S}_r$$

$\mathcal{R}_r(x,y,z), \mathbb{D}_r(x,y,z)$: Nedelec's first family on tetrahedra

Condition of optimality for edge elements



Expression of F for the pyramid :

$$F(\hat{x}, \hat{y}, \hat{z}) \;=\; A + B\hat{x} + C\hat{y} + D\hat{z} + \frac{\hat{x}\,\hat{y}}{4(1-\hat{z})}\,(S_1 + S_3 - S_2 - S_4)$$

F affine if the basis of the pyramid is a parallelogramm.

Condition of optimality for edge elements

Finite element space :

$$V_h = \{u \in H(\mathrm{curl}, \Omega) \text{ so that } u|_K \in P_r^F(K)\}$$

Condition of optimality for edge elements

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$$V_h = \{u \in H(\mathrm{curl}, \Omega) \text{ so that } u|_K \in P_r^F(K)\}$$

Use of Piola transform to write the space in the reference element $\hat{K}$:

$$P_r^F(K) = \{u \text{ such that } DF^* \, u \circ F \in \hat{P}_r(\hat{K})\}$$

Condition of optimality for edge elements

Finite element space :

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Use of Piola transform to write the space in the reference element $\hat{K}$:

$$P_r^F(K) = \{u \text{ such that } DF^* \, u \circ F \in \hat{P}_r(\hat{K})\}$$

Condition of optimality for a given choice of $\hat{P}_r(\hat{K})$:

$$\forall K, \quad \mathcal{R}_r(x, y, z) \subset P_r^F(K)$$

This condition is sufficient to obtain optimal estimates in $O(h^r)$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Hexahedra

$$\mathbb{Q}_{r-1,r+1,r+1} \times \mathbb{Q}_{r+1,r-1,r+1} \times \mathbb{Q}_{r+1,r+1,r-1}$$

Prisms

$$(\mathcal{R}_r(\hat{x},\hat{y}) \otimes \mathbb{P}_{r+1}(\hat{z})) \; \times \; (\mathbb{P}_{r+1}(\hat{x},\hat{y}) \otimes \mathbb{P}_{r-1}(\hat{z}))$$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Pyramids

$$\mathbb{B}^3_{r-1} \oplus \left\{ \frac{\hat{x}^p \hat{y}^p}{(1-\hat{z})^{p+2}} \begin{bmatrix} \hat{y}(1-\hat{z}) \\ \hat{x}(1-\hat{z}) \\ \hat{x}\hat{y} \end{bmatrix}, \quad 0 \le p \le r-1 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^m \hat{y}^{n+2}}{(1-\hat{z})^{m+2}} \begin{bmatrix} (1-\hat{z}) \\ 0 \\ \hat{x} \end{bmatrix}, \frac{\hat{x}^{n+2} \hat{y}^m}{(1-\hat{z})^{m+2}} \begin{bmatrix} 0 \\ (1-\hat{z}) \\ \hat{y} \end{bmatrix}, \quad 0 \le m \le n \le r-2 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^p \hat{y}^q}{(1-\hat{z})^{p+q+1-r}} \begin{bmatrix} (1-\hat{z}) \\ 0 \\ \hat{x} \end{bmatrix}, \frac{\hat{x}^q \hat{y}^p}{(1-\hat{z})^{p+q+1-r}} \begin{bmatrix} 0 \\ (1-\hat{z}) \\ \hat{y} \end{bmatrix}, \quad \begin{matrix} 0 \le p \le r-1 \\ 0 \le q \le r+1 \end{matrix} \right\}$$

Optimal Finite Element spaces

When expressed on the cube $[-1, 1]^3$, it is more friendly :

Pyramids (expressed on the cube)

$$\left(\mathbb{B}_{r-1} \circ T(\tilde{x}, \tilde{y}, \tilde{z})\right)^3 \oplus \left\{ \tilde{x}^p \tilde{y}^p (1-\tilde{z})^p \begin{bmatrix} \tilde{y} \\ \tilde{x} \\ \tilde{x}\tilde{y} \end{bmatrix}, \quad 0 \leq p \leq r-1 \right\}$$

$$\oplus \left\{ \tilde{x}^m \tilde{y}^{n+2} (1-\tilde{z})^{n+1} \begin{bmatrix} 1 \\ 0 \\ \tilde{x} \end{bmatrix}, \tilde{x}^{n+2} \tilde{y}^m (1-\tilde{z})^{n+1} \begin{bmatrix} 0 \\ 1 \\ \tilde{y} \end{bmatrix}, \quad 0 \leq m \leq n \leq r-2 \right\}$$

$$\oplus \left\{ \tilde{x}^p \tilde{y}^q (1-\tilde{z})^r \begin{bmatrix} 1 \\ 0 \\ \tilde{x} \end{bmatrix}, \tilde{x}^q \tilde{y}^p (1-\tilde{z})^r \begin{bmatrix} 0 \\ 1 \\ \tilde{y} \end{bmatrix}, \quad \begin{matrix} 0 \leq p \leq r-1 \\ 0 \leq q \leq r+1 \end{matrix} \right\}$$

Nedelec's first family

Nedelec's first family $\hat{P}_r^1(\hat{K})$:

Hexahedra

$$\mathbb{Q}_{r-1,r,r} \times \mathbb{Q}_{r,r-1,r} \times \mathbb{Q}_{r,r,r-1}$$

Prisms

$$(\mathcal{R}_r(\hat{x}, \hat{y}) \otimes \mathbb{P}_r(\hat{z})) \ \times \ (\mathbb{P}_r(\hat{x}, \hat{y}) \otimes \mathbb{P}_{r-1}(\hat{z}))$$

Pyramids

Same expression with $0 \leq p \leq r-1, \ 0 \leq q \leq r$
This finite element space is new.

H(curl) conformity

Tangential restrictions on triangular faces :

$$\mathcal{R}_r(x, y)$$

Tangential restrictions on quadrilateral faces :

$$\mathbb{Q}_{r-1,r+1}(x, y) \times \mathbb{Q}_{r+1,r-1}(x, y)$$

Condition of optimality for facet elements

Finite element space :

$$V_h = \{u \in H(\mathrm{div}, \Omega) \text{ so that } u|_K \in P_r^F(K)\}$$

Use of Piola transform to write the space in the reference element $\hat{K}$:

$$P_r^F(K) = \{u \text{ such that } |DF| DF^{-1}\, u \circ F \in \hat{P}_r(\hat{K})\}$$

Condition of optimality for a given choice of $\hat{P}_r(\hat{K})$:

$$\forall K, \quad \mathbb{D}_r(x, y, z) \subset P_r^F(K)$$

This condition is sufficient to obtain optimal estimates in $O(h^r)$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Hexahedra

$$\mathbb{Q}_{r+2,r,r}(\hat{x},\hat{y},\hat{z}) \times \mathbb{Q}_{r,r+2,r}(\hat{x},\hat{y},\hat{z}) \times \mathbb{Q}_{r,r,r+2}(\hat{x},\hat{y},\hat{z})$$

Prisms

$$\mathbb{D}_{r+1}(\hat{x},\hat{y}) \otimes \mathbb{P}_r(\hat{z}) \times \mathbb{W}_{r-1,r+2}(\hat{x},\hat{y},\hat{z})$$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Pyramids

$$\mathbb{B}_{r-1}(\hat{x},\hat{y},\hat{z})^3$$

$$\oplus \begin{bmatrix} \dfrac{\hat{x}^{n+1}\,\hat{y}^m}{(1-\hat{z})^{m+1}} \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \dfrac{\hat{x}^m\,\hat{y}^{n+1}}{(1-\hat{z})^{m+1}} \\ 0 \end{bmatrix} \quad 0 \le m \le n \le r-1$$

$$\oplus \begin{bmatrix} \dfrac{\hat{x}^{m+1}\,\hat{y}^{n+1}}{(1-\hat{z})^{m+2}} \\ 0 \\ -\dfrac{\hat{x}^m\hat{y}^{n+1}}{(1-\hat{z})^{m+1}} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \dfrac{\hat{x}^{n+1}\,\hat{y}^{m+1}}{(1-\hat{z})^{m+2}} \\ -\dfrac{\hat{x}^{n+1}\hat{y}^m}{(1-\hat{z})^{m+1}} \end{bmatrix} \quad 0 \le m \le n \le r-1$$

$$\oplus \frac{\hat{x}^i\,\hat{y}^j}{(1-\hat{z})^{i+j+1-r}} \begin{bmatrix} \hat{x} \\ \hat{y} \\ -(1-\hat{z}) \end{bmatrix}, \quad 0 \le i,j \le r$$

Optimal Finite Element spaces

When expressed on the cube $[-1,1]^3$, it is more friendly :

Pyramids (expressed on the cube)

$$\left(\mathbb{B}_{r-1} \circ T(\tilde{x}, \tilde{y}, \tilde{z})\right)^3$$

$$\oplus \begin{bmatrix} \tilde{x}^{k+1}\tilde{y}^{m}(1-\tilde{z})^k \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \tilde{x}^{m}\tilde{y}^{k+1}(1-\tilde{z})^k \\ 0 \end{bmatrix}, \quad 0 \leq m \leq k \leq r-1$$

$$\oplus \begin{bmatrix} \tilde{x}^{m+1}\tilde{y}^{n+1}(1-\tilde{z})^n \\ 0 \\ -\tilde{x}^{m}\tilde{y}^{n+1}(1-\tilde{z})^n \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \tilde{x}^{n+1}\tilde{y}^{m+1}(1-\tilde{z})^n \\ -\tilde{x}^{n+1}\tilde{y}^{m}(1-\tilde{z})^n \end{bmatrix}, \quad 0 \leq m \leq n \leq r-1$$

$$\oplus \begin{bmatrix} \tilde{x}^{i+1}\tilde{y}^{j}(1-\tilde{z})^r \\ \tilde{x}^{i}\tilde{y}^{j+1}(1-\tilde{z})^r \\ -\tilde{x}^{i}\tilde{y}^{j}(1-\tilde{z})^r \end{bmatrix}, \quad 0 \leq i, j \leq r$$

H(div) conformity

Normal components on triangular faces :

$$\mathbb{P}_{r-1}(x, y)$$

Normal components on quadrilateral faces :

$$\mathbb{Q}_{r,r}(x, y)$$

Super-optimality for facet elements

Necessary condition to obtain optimal estimates in $O(h^r)$:

$$\forall K, \quad \mathbb{P}^3_{r-1}(x, y, z) \subset P^F_r(K)$$

$$\forall K, \quad \mathbb{P}_{r-1}(x, y, z) \subset \text{div} P^F_r(K)$$

Super-optimal spaces are the minimal spaces $\hat{P}_r$ satisfying these conditions.

Because of the condition with the divergence, non-unicity of these spaces.

Super-optimal spaces for H(div)

Minimal spaces satisfying the super optimality condition :

Hexahedra

$$\begin{bmatrix} \hat{x}^i \hat{y}^j \hat{z}^k \\ 0 \\ 0 \end{bmatrix}, \; i \leq r+1; \; j,k \leq r; j+k \neq 2r, \quad \text{and symmetric counterparts}$$

$$\begin{bmatrix} \hat{x}^{r+2} \hat{y}^j \hat{z}^k \\ 0 \\ 0 \end{bmatrix}, \; 0 \leq j,k \leq r, \quad \text{and symmetric counterparts}$$

$$\begin{bmatrix} \hat{x}^{r+1} \hat{y}^r \hat{z}^r \\ \hat{x}^r \hat{y}^{r+1} \hat{z}^r \\ \hat{x}^r \hat{y}^r \hat{z}^{r+1} \end{bmatrix}$$

$$\text{Dimension} = \dim \hat{P}_r^{opt} \; - \; (3r+5)$$

Super-optimal spaces for H(div)

Minimal spaces satisfying the super optimality condition :

Prisms

$$\hat{P}_r^{opt} \setminus \left[\left(\widetilde{P}_r(\hat{x}, \hat{y}) \hat{z}^r \right)^2 \times \{0\} \right] + \widetilde{P}_{r-1}(\hat{x}, \hat{y}) \hat{z}^r \begin{bmatrix} \hat{x} \\ \hat{y} \\ 0 \end{bmatrix}$$

$$\text{Dimension} = \dim \hat{P}_r^{opt} - (r+2)$$

Pyramids

$\hat{P}_r^{opt}$ is minimal

Basis functions for super-optimal hexahedra

$P_i^{\alpha,\beta}$ being Jacobi polynomials :

Functions associated with a face

$$\begin{bmatrix} (1-x)P_j^{0,0}(2y-1)P_k^{0,0}(2z-1) \\ 0 \\ 0 \end{bmatrix}, \; j,k \leq r; \; j+k \neq 2r$$

Interior separated functions

$$\begin{bmatrix} x(1-x)P_i^{1,1}(2x-1)P_j^{0,0}(2y-1)P_k^{0,0}(2z-1) \\ 0 \\ 0 \end{bmatrix}$$

$$i,j,k \leq r; \; i = r \text{ or } j \neq r \text{ or } k \neq r$$

Basis functions for super-optimal hexahedra

$P_i^{\alpha,\beta}$ being Jacobi polynomials :

Functions associated with a face

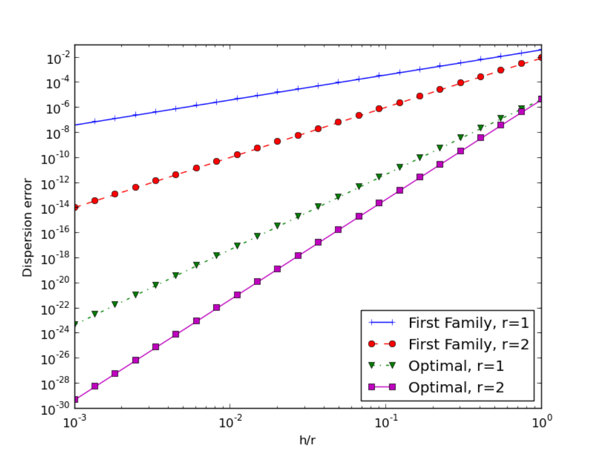
$$\begin{bmatrix} (1-x) P_j^{0,0}(2y-1) P_k^{0,0}(2z-1) \\ 0 \\ 0 \end{bmatrix}, \; j,k \leq r; \; j+k \neq 2r$$

Interior linked function

$$\begin{bmatrix} x(1-x) P_{r-1}^{1,1}(2x-1) P_r^{0,0}(2y-1) P_r^{0,0}(2z-1) \\ y(1-y) P_{r-1}^{1,1}(2y-1) P_r^{0,0}(2x-1) P_r^{0,0}(2z-1) \\ z(1-z) P_{r-1}^{1,1}(2z-1) P_r^{0,0}(2x-1) P_r^{0,0}(2y-1) \end{bmatrix}$$

Dispersion error for wave equation

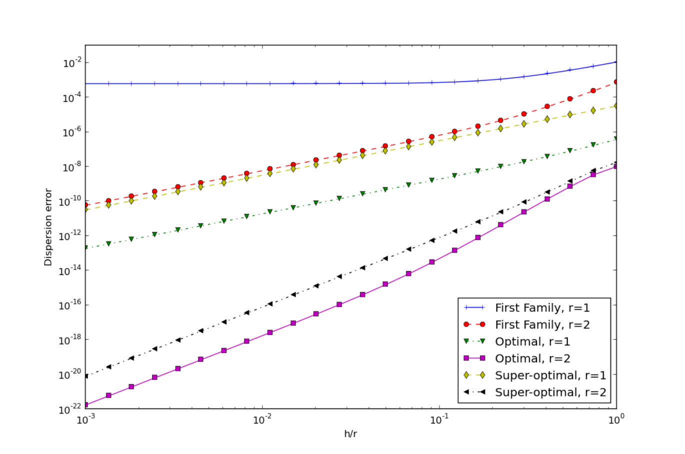
For regular hexahedra :



Dispersion error equal to $O(h^{2r})$ for first family, to $O(h^{2r+4})$ for optimal and super-optimal elements

Dispersion error for wave equation

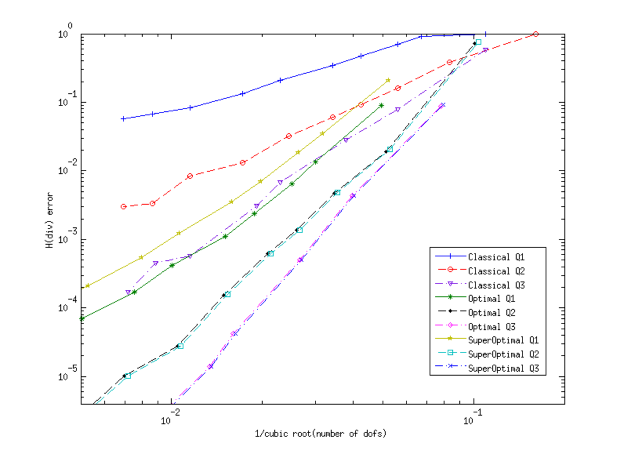
For non-affine hexahedra :



Dispersion error equal to $O(h^{2r-2})$ for first family, to $O(h^{2r})$ for optimal and super-optimal elements

H(div) error for wave equation

For non-affine hexahedra :



H(div) convergence versus the cubic root of number of degrees of freedom

Hierarchical basis functions for the pyramid

Parameters β_i are associated with triangular faces
Parameters λ_i are associated with vertices of the pyramid

$$\left\{ \begin{array}{l} \beta_1 = \dfrac{1 - \hat{x} - \hat{z}}{2} \\ \beta_2 = \dfrac{1 - \hat{y} - \hat{z}}{2} \\ \beta_3 = \dfrac{1 + \hat{x} - \hat{z}}{2} \\ \beta_4 = \dfrac{1 + \hat{y} - \hat{z}}{2} \end{array} \right. \qquad \left\{ \begin{array}{l} \lambda_1 = \dfrac{\beta_1 \, \beta_2}{1 - \hat{z}} \\ \lambda_2 = \dfrac{\beta_2 \, \beta_3}{1 - \hat{z}} \\ \lambda_3 = \dfrac{\beta_3 \, \beta_4}{1 - \hat{z}} \\ \lambda_4 = \dfrac{\beta_4 \, \beta_1}{1 - \hat{z}} \\ \lambda_5 = \hat{z} \end{array} \right.$$

Hierarchical basis functions for the pyramid

γ_i are parametrizations of vertical edges
γ_i are parametrizations of horizontal edges

$$\left\{\begin{aligned} \gamma_1 &= \frac{2\hat{z} + \hat{x} + \hat{y}}{2} \\ \gamma_2 &= \frac{2\hat{z} - \hat{x} + \hat{y}}{2} \\ \gamma_3 &= \frac{2\hat{z} - \hat{x} - \hat{y}}{2} \\ \gamma_4 &= \frac{2\hat{z} + \hat{x} - \hat{y}}{2} \end{aligned}\right. \qquad \left\{\begin{aligned} \delta_1 &= \delta_3 = \hat{x} \\ \delta_2 &= \delta_4 = \hat{y} \end{aligned}\right.$$

Hierarchical basis functions for the pyramid

Use of Jacobi polynomials $P_i^{\alpha,\beta}(x)$ orthogonal with respect to weight $(1-x)^\alpha(1+x)^\beta$

For two horizontal edges :

$$(\lambda_1 \nabla(\lambda_2+\lambda_3) - \lambda_2 \nabla(\lambda_1+\lambda_4))\, P_i^{0,0}(\delta_1),\ 0 \leq i \leq r-1$$

$$(\lambda_1 \nabla(\lambda_3+\lambda_4) - \lambda_4 \nabla(\lambda_1+\lambda_2))\, P_i^{0,0}(\delta_2),\ 0 \leq i \leq r-1$$

For a vertical edge :

$$(\lambda_1 \nabla \lambda_5 - \lambda_5 \nabla \lambda_1)\, P_i^{0,0}(\gamma_1), \quad 0 \leq i \leq r-1$$

Hierarchical basis functions for the pyramid

For the base:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4))\ \beta_4\, P_i^{0,0}(\frac{\beta_3 - \beta_1}{1 - \hat{z}}) P_j^{1,1}(\frac{\beta_4 - \beta_2}{1 - \hat{z}})(1 - \hat{z})^{\max(i,j)-1}$$

$$(\lambda_1 \nabla(\lambda_3 + \lambda_4) - \lambda_4 \nabla(\lambda_2 + \lambda_1))\ \beta_3\, P_j^{1,1}(\frac{\beta_3 - \beta_1}{1 - \hat{z}}) P_i^{0,0}(\frac{\beta_4 - \beta_2}{1 - \hat{z}})(1 - \hat{z})^{\max(i,j)-1}$$

$$0 \leq i, j \leq r - 1$$

For a triangular face:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4))\ \lambda_5\, P_i^{0,0}(\delta_1) P_j^{0,0}(\gamma_1)$$

$$(\lambda_1 \nabla \lambda_5 - \lambda_5 \nabla \lambda_1)\ \beta_3\, P_i^{0,0}(\delta_1) P_j^{0,0}(\gamma_1)$$

$$0 \leq i + j \leq r - 2$$

Hierarchical basis functions for the pyramid

For interior functions:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4))\ \beta_4\, \lambda_5\, P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$(\lambda_1 \nabla(\lambda_3 + \lambda_4) - \lambda_4 \nabla(\lambda_2 + \lambda_1))\ \beta_3\, \lambda_5\, P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$(\lambda_1 \nabla \lambda_5 - \lambda_5 \nabla \lambda_1)\ \beta_3\, \beta_4\, P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$0 \le i, j \le r - 2, \quad 0 \le k \le r - 2 - max(i, j)$$

$$P_{ijk}(\hat{x}, \hat{y}, \hat{z}) = P_i^{0,0}\left(\frac{\beta_3 - \beta_1}{1 - \hat{z}}\right) P_j^{0,0}\left(\frac{\beta_4 - \beta_2}{1 - \hat{z}}\right) P_k^{2\max(i,j)+2,0}(2\hat{z} - 1)(1 - \hat{z})^{\max(i,j)-1}$$

Comparison with other pyramidal edge elements

- $\hat{P}_r^1$ is the same space as proposed by Coulomb et al, Graglia and Gheorma, Gradinaru and Hiptmair, Doucet et al, Nigam and Phillips for $r = 1$.
- First space proposed by Nigam and Phillips contains more degrees of freedom than $\hat{P}_r^1$ while providing the same order of convergence
- Second space proposed by Nigam and Phillips contains $r(r-1)$ less degrees of freedom but is not consistent for non-affine pyramids
- Basis functions of Coulomb et al, Graglia and Gheorma for $r = 2$ induce spurious modes and are providing only first-order convergence even for affine pyramids

Dispersion analysis

Maxwell's equations

$$-\omega^2 E + \mathrm{curl}(\mathrm{curl} E) = f$$

$$-\omega^2 \int_\Omega E \cdot \varphi_i + \mathrm{curl} E \cdot \mathrm{curl} \varphi_i \, dx = \int_\Omega f \cdot \varphi_i \, dx$$

$$-\omega^2 M_h E + K_h E = F_h$$

Dispersion analysis

Maxwell's equations

$$-\omega^2 E + \mathrm{curl}(\mathrm{curl} E) = f$$

$$-\omega^2 \int_\Omega E \cdot \varphi_i + \mathrm{curl} E \cdot \mathrm{curl} \varphi_i \, dx = \int_\Omega f \cdot \varphi_i \, dx$$

$$-\omega^2 M_h E + K_h E = F_h$$

Research of eigenvalues (ω, E) with quasi-periodic conditions

$$E(\vec{x} + \vec{h}) = exp^{i\vec{k}\cdot\vec{h}} E(\vec{x})$$

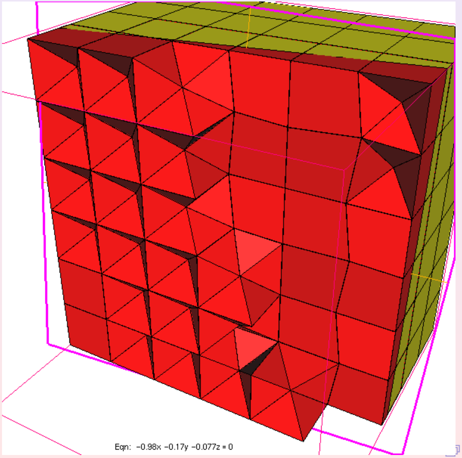
Dispersion error = $\dfrac{\omega - ||\vec{k}||}{\omega}$

Dispersion analysis

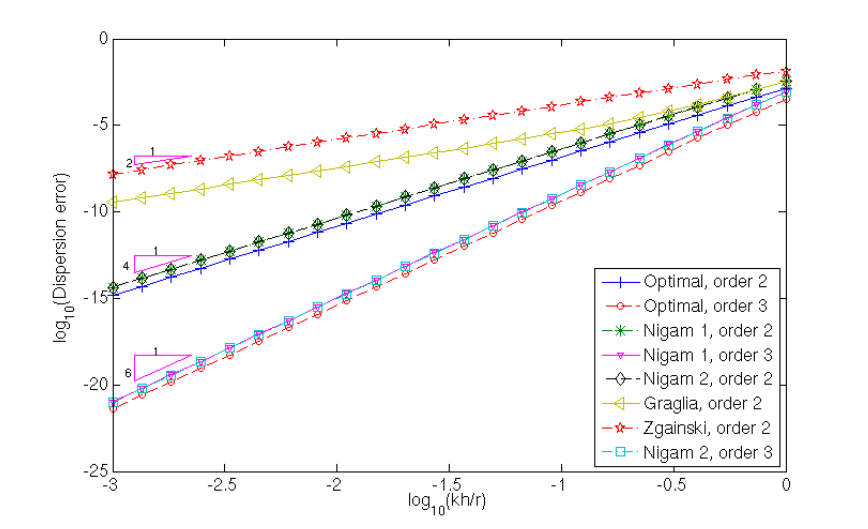
Research of eigenvalues (ω, E) with quasi-periodic conditions

$$E(\vec{x} + \vec{h}) \;=\; exp^{i\vec{k}\cdot\vec{h}} E(\vec{x})$$

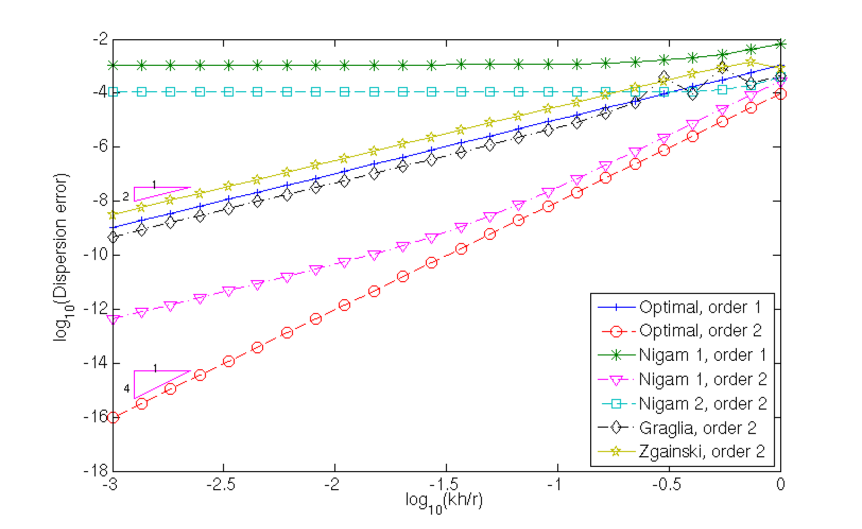
Dispersion analysis on purely pyramidal mesh



Dispersion on affine pyramids

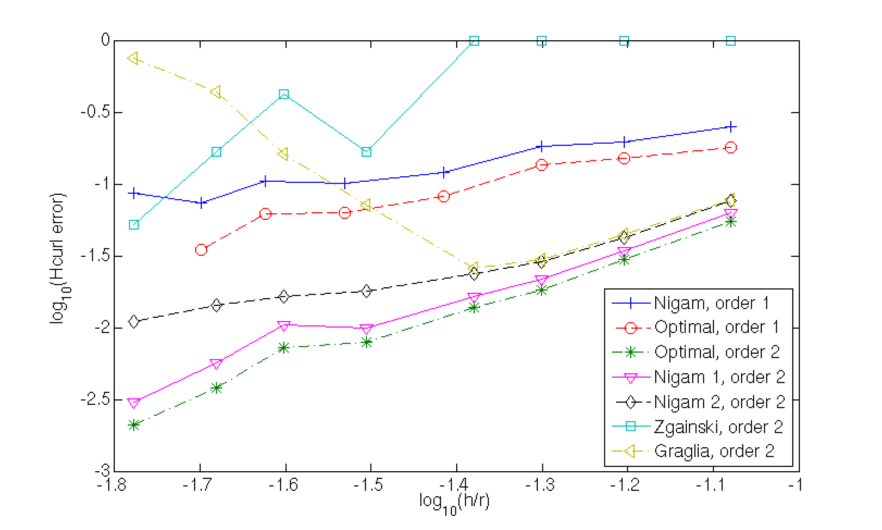


Dispersion on non-affine pyramids



Convergence for the cube

Gaussian source inside a cube and non-affine pyramids :

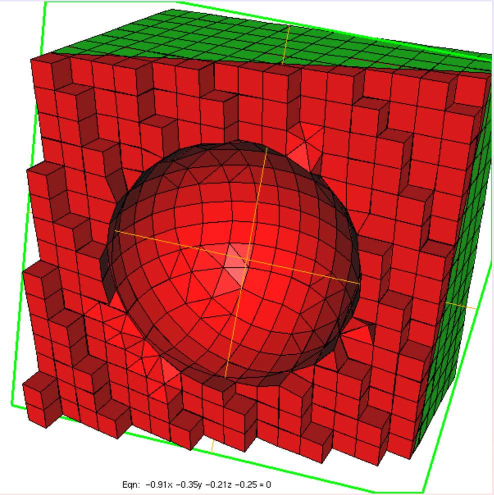


Convergence for the sphere

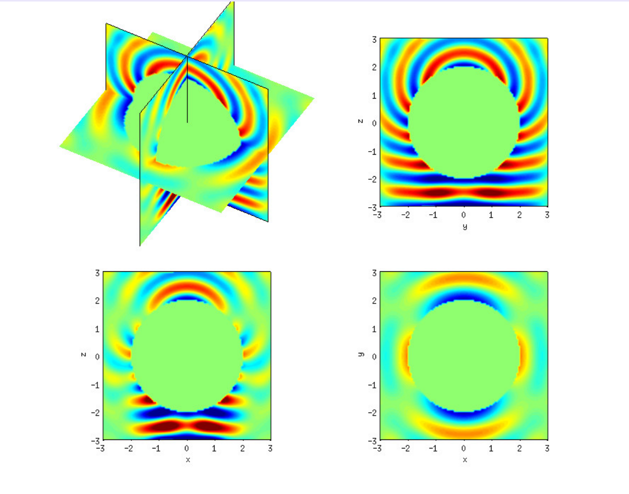
Scattering of a perfectly conducting object

$$\begin{cases} -\omega^2 \varepsilon\, E + \text{curl}(\dfrac{1}{\mu}\,\text{curl}\, E) \;=\; f \text{ in } \Omega \\ E \times n = -E^{\text{inc}} \times n \text{ on } \Gamma \\ \text{curl} E \times n \;=\; i\, k (n \times E) \times n \text{ on } \Sigma \end{cases}$$

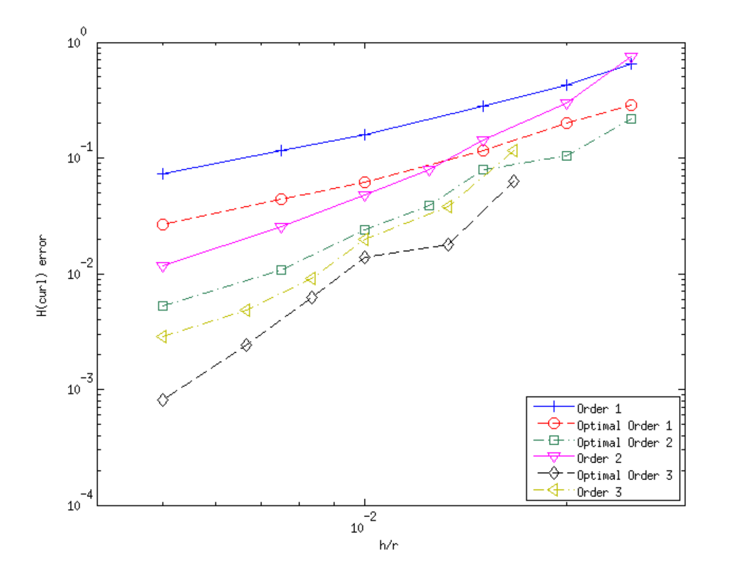
Convergence for the sphere



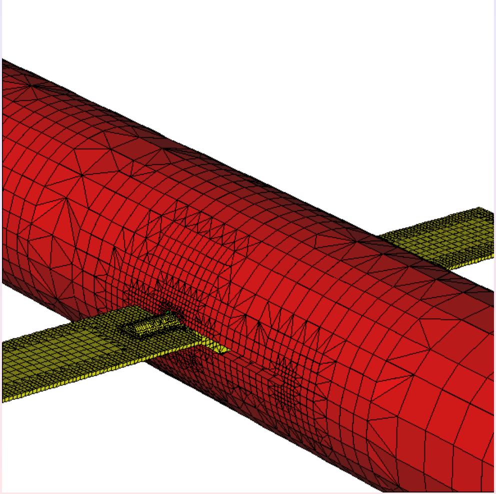
Convergence for the sphere



Convergence for the sphere

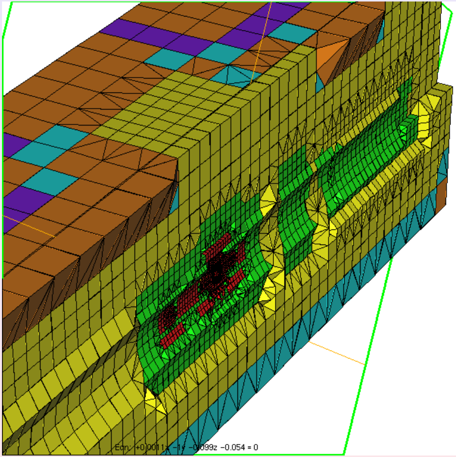


Scattering by a satellite

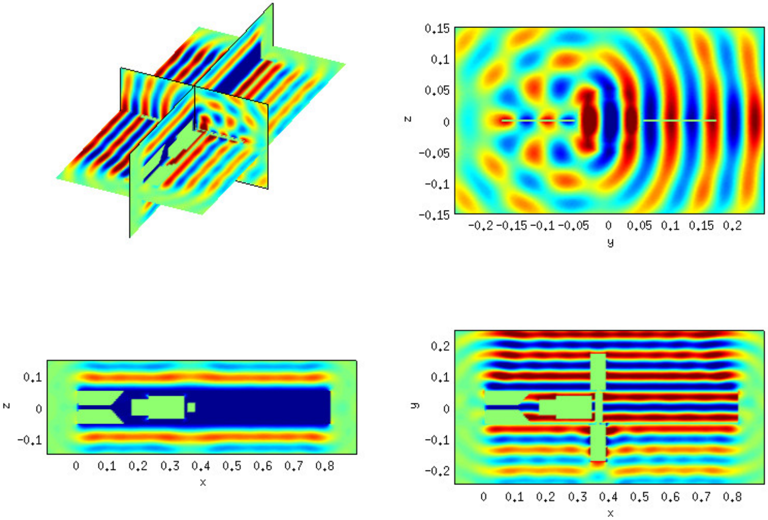


Scattering by a satellite

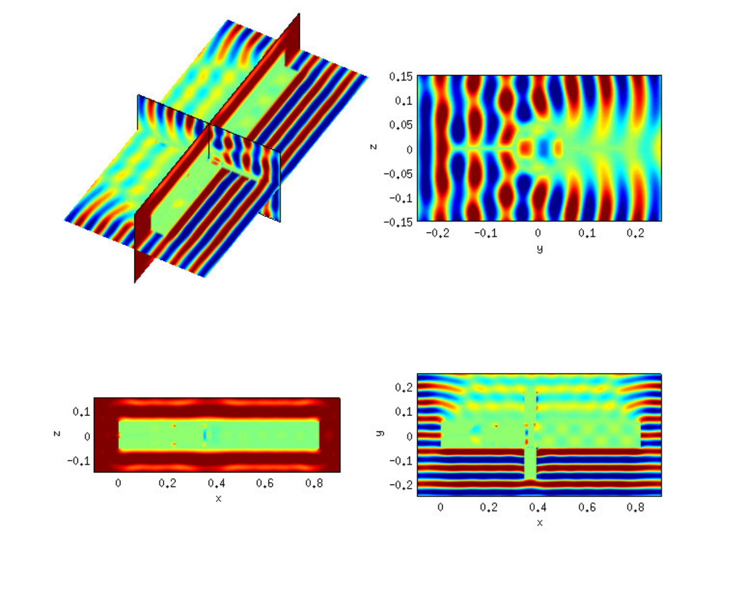
Each color is associated with an order of approximation



Scattering by a satellite



Scattering by a satellite



Scattering by a satellite

The mesh contains 35006 tetrahedra, 50390 hexahedra (40 659 affine hexahedra), 48865 pyramids (40 508 affine pyramids), 4582 wedges. We use $\hat{P}_r^1$ and there are 2 570 034 dofs.

END