Solution of time-harmonic Galbrun's equations in the context of helioseismology.

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Model problem

Time-harmonic Galbrun's equations (pulsation ω)

$$\left\{ \begin{array}{l} -\rho_0 \, (-i\omega + \sigma + M \cdot \nabla)^2 \, u - \nabla \left(\rho_0 \, c_0^2 \, \text{div} \, u \right) \\ \qquad + (\text{div} \, u) \, \nabla p_0 - (\nabla u)^T \nabla p_0 = f, \quad \text{in } \Omega \end{array} \right.$$

u: Lagrangian fluid displacement (unknown)

ρ_0, p_0, c_0: background density, pressure and sound speed

M, σ: flow velocity and damping.

$$(\nabla u)^T = \begin{pmatrix} \partial_x u_x & \partial_x u_y \\ \partial_y u_x & \partial_y u_y \end{pmatrix}$$

Model problem

Time-harmonic Galbrun's equations (pulsation ω)

$$\left\{ \begin{array}{l} -\rho_0 \, (-i\omega + \sigma + M \cdot \nabla)^2 \, u - \nabla \left(\rho_0 \, c_0^2 \, \text{div} \, u \right) \\ \qquad + (\text{div} \, u) \, \nabla p_0 - (\nabla u)^T \nabla p_0 = f, \quad \text{in } \Omega \end{array} \right.$$

u: Lagrangian fluid displacement (unknown)

ρ_0, p_0, c_0: background density, pressure and sound speed

M, σ: flow velocity and damping.

Assumptions:

- Coefficients $\rho_0, p_0, c_0, \sigma, M$ smooth functions of x (at least continuous).
- The flow satisfies the condition $\text{div}(\rho_0 M) = 0$

Discontinuous Galerkin Method

Linear equation:

$$A(x)\, u(x) \; - \mathrm{Div}\left[C(x)\, \nabla u(x) + B(x)\, u(x)\right] + E(x)\, \nabla u(x) = f(x)$$

where $A(x), C(x), B(x), E(x)$ are tensors.

Discontinuous Galerkin Method

Discontinuous Galerkin formulation

$$\sum_{K \text{ element}} \left(\int_K A\, u \cdot \varphi + (C\, \nabla u + B\, u) \cdot \nabla \varphi + E\, \nabla u \cdot \varphi \, dx \right)$$

$$+ \sum_{F \text{ face}} \left(\int_F \{C\, \nabla u\, \nu\}[\varphi] + [u]\{C\, \nabla \varphi\, \nu\} + \{B\, u\, \nu\}[\varphi] + [u]\{E^*\, \varphi\, \nu\} \right.$$

$$\left. + \frac{1}{2}[P\, u]\,[\varphi]\, dx \right) + \int_\Gamma N\, u \cdot \varphi\, dx \; = \sum_{K \text{ element}} \int_K f \cdot \varphi\, dx$$

where

$$\{u\} = \frac{u^+ + u^-}{2}, \quad [u] = (u^+ - u^-)$$

ν : outward normale

P, N : penalty matrix and boundary condition matrix

Symmetric Interior Penalty Galerkin

Discontinuous Galerkin applied directly $\Rightarrow$ **SIPG**

$$\left\{ \begin{array}{l} -\rho_0 \, (-i\omega + \sigma + M \cdot \nabla)^2 \, u - \nabla \left(\rho_0 \, c_0^2 \, \text{div} \, u \right) \\ \qquad +(\text{div} \, u) \, \nabla p_0 - (\nabla u)^T \nabla p_0 = f, \quad \text{in } \Omega \end{array} \right.$$

Penalty matrix:

$$P \; = \; \alpha \frac{r(r+1)}{h^2} \nu \, \nu^T$$

r: order of approximation
h: length of the edge
In practice, we took $\alpha = 10$ and observed numerically a positive stiffness matrix for a null flow.

Local Discontinuous Galerkin

Equivalent first-order formulation:

$$\left\{ \begin{array}{l} \rho_0 \left(-i\omega + \sigma + M \cdot \nabla\right) u - \rho_0 v = 0 \\ \rho_0 \left(-i\omega + \sigma + M \cdot \nabla\right) v - \nabla(\rho_0 \, c_0^2 \, p) + (\mathrm{div} u) \, \nabla p_0 - (\nabla u)^T \nabla p_0 = f \\ p - \mathrm{div} \, u = 0 \end{array} \right.$$

- Upwind fluxes difficult to implement
- Not adapted for explicit time-stepping

Local Discontinuous Galerkin

Better equivalent first-order formulation ⇒ **LDG**

$$\left\{\begin{array}{l} \rho_0\,(-i\omega + \sigma + M\cdot\nabla)\,u - \nabla p - \rho_0\,q = 0 \\ \rho_0\,(-i\omega + \sigma + M\cdot\nabla)\,q - (\nabla\sigma)\,p - (\nabla M)^T\nabla p - \dfrac{M\cdot\nabla\rho_0}{\rho_0}\nabla p \\ \qquad +(\mathrm{div}\,u)\nabla p_0 - (\nabla u)^T\nabla p_0 = f \\ \rho_0\,(-i\omega + \sigma + M\cdot\nabla)\,p - \rho_0^2\,c_0^2\,\mathrm{div}\,u = 0 \end{array}\right.$$

- Well adapted for explicit time-stepping
- Form close to an hyperbolic system

Upwind fluxes

Considered equation

$$A(x)\, u(x) \; - \mathrm{Div}\,[B(x)\, u(x)] + E(x)\, \nabla u(x) = f(x)$$

Matrix $D(x)$ defined as:

$$D(x) = -B(x)\nu + E(x)\nu$$

$A(x)$ decomposed as

$$A(x) = -i\omega M(x) + A_0(x)$$

Λ, V eigenvalues and eigenvectors of $M(x)^{-1} D(x)$

Upwind fluxes

Λ, V eigenvalues and eigenvectors of $M^{-1}D$

$$M^{-1}D = V \Lambda V^{-1}$$

Absolute value of D defined as:

$$|D| = M\, V |\Lambda| V^{-1}$$

Upwind fluxes $\Rightarrow$ penalty matrix P:

$$P = |D|$$

Eigenvalues Λ real if the system is hyperbolic

Upwind fluxes for Galbrun

Uniform flow:

$$M = \rho_0 I, \quad D = \begin{pmatrix} \rho_0\alpha & 0 & 0 & 0 & -\nu_x \\ 0 & \rho_0\alpha & 0 & 0 & -\nu_y \\ 0 & 0 & \rho_0\alpha & 0 & 0 \\ 0 & 0 & 0 & \rho_0\alpha & 0 \\ -(\rho_0 c_0)^2\nu_x & -(\rho_0 c_0)^2\nu_y & 0 & 0 & \rho_0\alpha \end{pmatrix}$$

where

$$\alpha = M \cdot \nu$$

Upwind fluxes for Galbrun

Uniform flow :

$$\Lambda = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha + c_0 \\ \alpha - c_0 \end{pmatrix}, \quad |D| = \rho_0 \begin{pmatrix} |\alpha|\mathbb{I} + (s - |\alpha|)\,\nu\,\nu^T & 0 & -\dfrac{d}{\rho_0\, c_0}\nu \\ 0 & |\alpha|\mathbb{I} & 0 \\ -\rho_0\, c_0\, d\nu^T & 0 & s \end{pmatrix}$$

where

$$s = \frac{1}{2}\left(|\alpha + c_0| + |\alpha - c_0|\right), \quad d = \frac{1}{2}\left(|\alpha + c_0| - |\alpha - c_0|\right)$$

Upwind fluxes for Galbrun

Non-uniform flow :

$$M = \rho_0 I, \quad D = \begin{pmatrix} \rho_0 \alpha & 0 & 0 & 0 & -\nu_x \\ 0 & \rho_0 \alpha & 0 & 0 & -\nu_y \\ 0 & \rho_0 \gamma & \rho_0 \alpha & 0 & \beta_x \\ -\rho_0 \gamma & 0 & 0 & \rho_0 \alpha & \beta_y \\ -(\rho_0 c_0)^2 \nu_x & -(\rho_0 c_0)^2 \nu_y & 0 & 0 & \rho_0 \alpha \end{pmatrix}$$

where

$$\alpha = M \cdot \nu, \quad \rho_0 \gamma = \nabla p_0 \times \nu$$

$$(\beta_x, \beta_y) = (-\nabla M)^T \, \nu - \frac{M \cdot \nabla \rho_0}{\rho_0} \nu$$

Issue : D is not diagonalizable if $\gamma, \beta_x, \beta_y \neq 0$

Upwind fluxes for Galbrun

Non-uniform flow :

$$M = \rho_0 I, \quad D = \begin{pmatrix} \rho_0\alpha & 0 & 0 & 0 & -\nu_x \\ 0 & \rho_0\alpha & 0 & 0 & -\nu_y \\ 0 & \rho_0\gamma & \rho_0\alpha_1 & 0 & \beta_x \\ -\rho_0\gamma & 0 & 0 & \rho_0\alpha_1 & \beta_y \\ -(\rho_0 c_0)^2\nu_x & -(\rho_0 c_0)^2\nu_y & 0 & 0 & \rho_0\alpha \end{pmatrix}$$

D diagonalizable if $\alpha_1 \notin \{\alpha, \alpha + c_0, \alpha - c_0\}$

Penalty matrix:

$$P = \lim_{\alpha_1 \to \alpha} |D|_{\alpha_1}$$

H^1 formulation

Formulation proposed by Bonnet Ben Dhia et al (in 2-D):

$$\left\{ \begin{array}{l} \rho_0 \left(-i\omega + \sigma + M\cdot\nabla\right)^2 u - \nabla\left(\rho_0 c_0^2 \,\mathrm{div}\, u\right) + \mathrm{curl}\left(\rho_0 c_0^2 \left(\mathrm{curl}(u) - \psi\right)\right) \\ \qquad +(\mathrm{div}\, u)\nabla p_0 - (\nabla u)^T \nabla p_0 = f \\ \rho_0 \left(-i\omega + \sigma + M\cdot\nabla\right)^2 \psi + 2\rho_0 \left(-i\omega + \sigma + M\cdot\nabla\right) \mathcal{B}(u) + \rho_0\, \mathcal{C}(u) \\ \qquad = -\mathrm{curl}(f) + \dfrac{1}{\rho_0 c_0^2} f \wedge \nabla p_0 \end{array} \right.$$

$$\mathcal{B}(u) = \sum_{j=1}^{2} \nabla M_j \wedge \frac{\partial u}{\partial x_j}$$

$$\mathcal{C}(u) = \sum_{j,k=1}^{2} \left(\frac{\partial M_k}{\partial x_j} \nabla M_j \wedge \frac{\partial u}{\partial x_k} - M_j \nabla \frac{\partial M_k}{\partial x_j} \wedge \frac{\partial u}{\partial x_k} \right) + \frac{1}{\rho_0} \sum_{j=1}^{2} \left(\frac{1}{\rho_0 c_0^2} \frac{\partial p_0}{\partial x_j} \nabla p_0 - \nabla\left(\frac{\partial p_0}{\partial x_j}\right)\right) \wedge \nabla u_j$$

H^1 formulation

Formulation proposed by Bonnet Ben Dhia et al (in 2-D):

$$\left\{ \begin{array}{l} \rho_0 \, (-i\omega + \sigma + M \cdot \nabla)^2 \, u - \nabla \left(\rho_0 c_0^2 \, \text{div} \, u \right) + \text{curl} \left(\rho_0 c_0^2 \, (\text{curl}(u) - \psi) \right) \\ \qquad + (\text{div} \, u) \nabla p_0 - (\nabla u)^T \nabla p_0 = f \\ \rho_0 \, (-i\omega + \sigma + M \cdot \nabla)^2 \, \psi + 2\rho_0 \, (-i\omega + \sigma + M \cdot \nabla) \, \mathcal{B}(u) + \rho_0 \, \mathcal{C}(u) \\ \qquad = -\text{curl}(f) + \dfrac{1}{\rho_0 c_0^2} f \wedge \nabla p_0 \end{array} \right.$$

- Continuous finite elements for $u, \psi \Rightarrow \mathbf{H^1}$
- Discontinuous finite elements for $u, \psi \Rightarrow \mathbf{H^1}(\mathbf{DG})$

Results for an uniform flow

Uniform coefficients:

$$M = (m_x, 0),\ \rho_0 = 2.5,\ c_0 = 0.8,\ p_0 = 1,\ \omega = 0.78 \times 2\pi,\ \sigma = 0.1$$

Computational domain:

$$\Omega = [-4, 4]^2$$

Gaussian source:

$$f = \beta_0\, exp(-\alpha_0(x^2 + y^2))e_x$$

Periodic boundary conditions

h: mesh size, r: degree of polynomial space

Results for an uniform flow

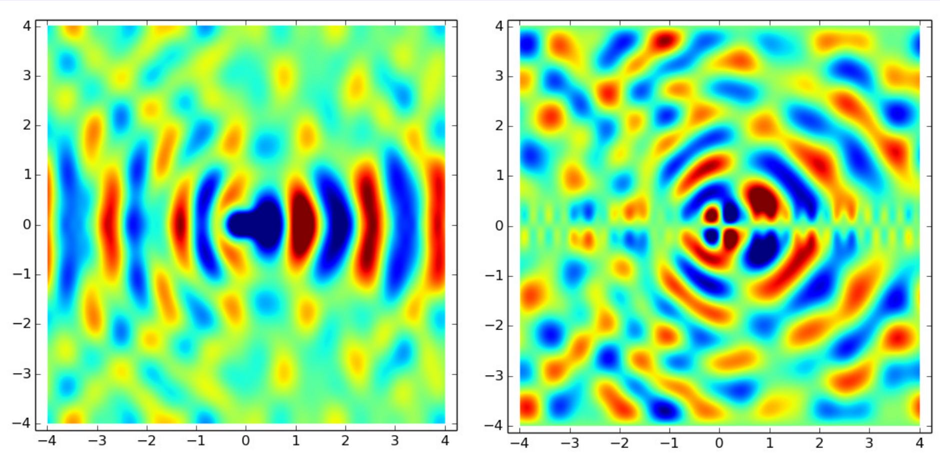


Figure: Real part of u_x (left) and u_y (right) for an uniform flow $m_x = 0.25$

Results for an uniform flow

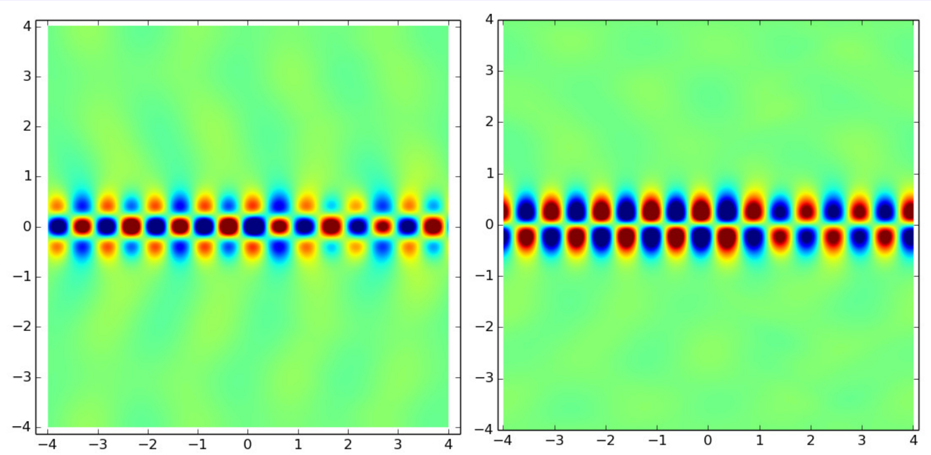


Figure: Real part of u_x (left) and u_y (right) for an uniform flow $m_x = 0.75$

Results for an uniform flow

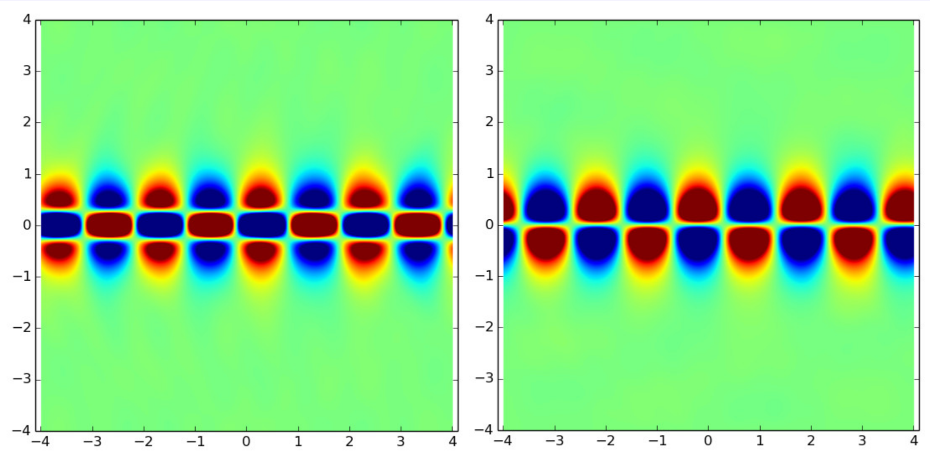


Figure: Real part of u_x (left) and u_y (right) for an uniform flow $m_x = 1.5$

Convergence for an uniform flow

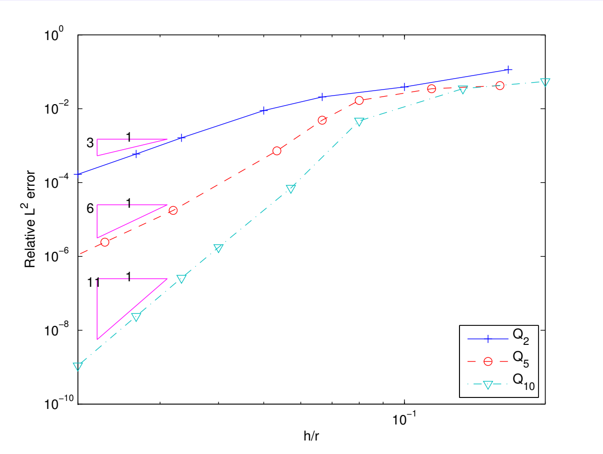


Figure: Relative L^2 error vs h/r for LDG quadrilateral elements and an uniform flow ($m_x = 0.25$).

Convergence for an uniform flow

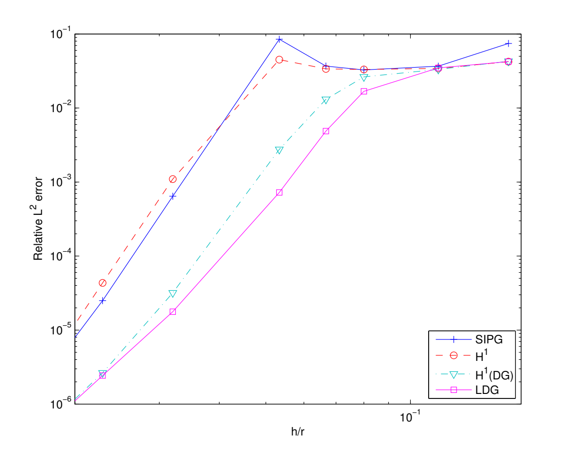


Figure: Comparison of the different formulations for an uniform flow ($r = 5, m_x = 0.25$)

Convergence for an uniform flow

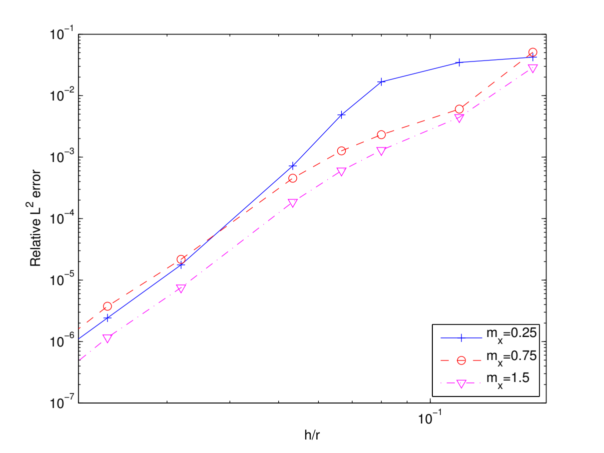


Figure: Convergence observed for any value m_x.

Numerical results for a non-uniform flow

Physical coefficients are chosen periodic :

$$\rho_0 = 1.5 + 0.2 \cos\left(\frac{\pi x}{4}\right) \sin\left(\frac{\pi y}{2}\right)$$

$$p_0 = 1.44 \rho_0 + 0.08 \rho_0^2$$

$$c_0^2 = 1.44 + 0.16 \rho_0$$

$$\omega = 0.78 \times 2\pi, \quad \sigma = 0.1$$

The flow M satisfies $\text{div}(\rho_0 M) = 0$:

$$m_x = \text{coeff} \left(\frac{0.3 + 0.1 \cos\left(\frac{\pi y}{4}\right)}{\rho_0} \right)$$

$$m_y = \text{coeff} \left(\frac{0.2 + 0.08 \sin\left(\frac{\pi x}{4}\right)}{\rho_0} \right)$$

Numerical results for a non-uniform flow

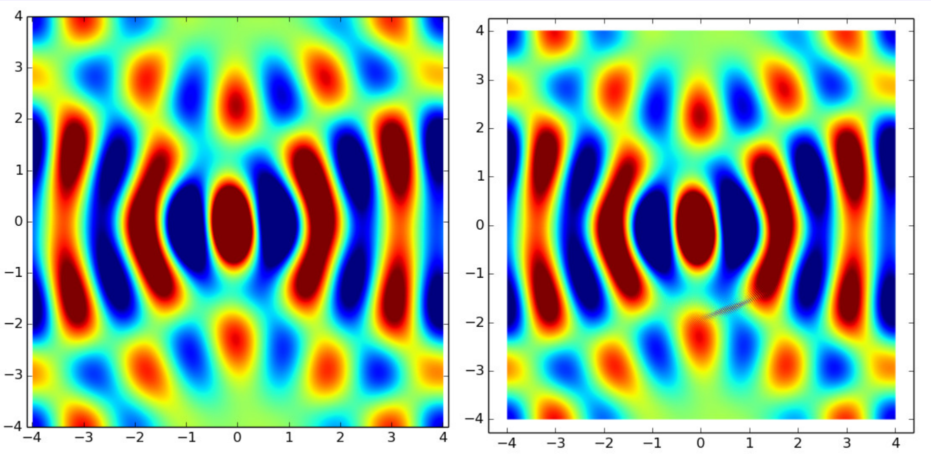


Figure: Numerical solution obtained with H^1 formulation, with $N = 61$ points (left) and $N = 81$ points (right) and $r = 10, \mathrm{coeff} = 0.1$.

Numerical results for a non-uniform flow

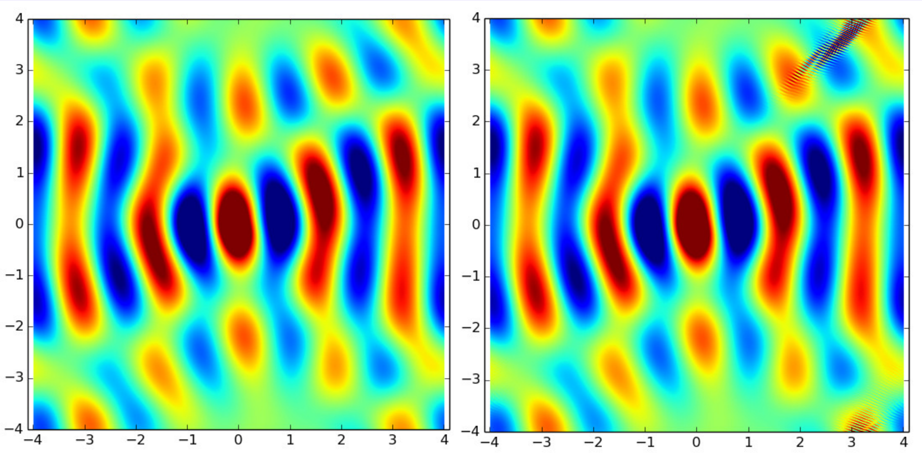


Figure: Numerical solution obtained with H^1 formulation with $N = 41$ points (left) and $N = 61$ points (right) and $r = 10, \text{coeff} = 0.2$

Numerical results for a non-uniform flow

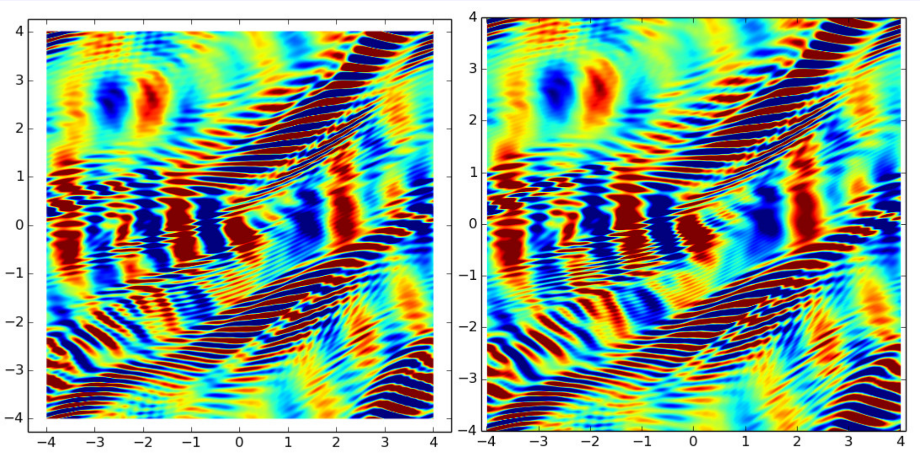


Figure: Numerical solution obtained with H^1 formulation with $N = 41$ points (left) and $N = 61$ points (right) and $r = 10$, coeff $= 1.5$

Convergence for a non-uniform flow

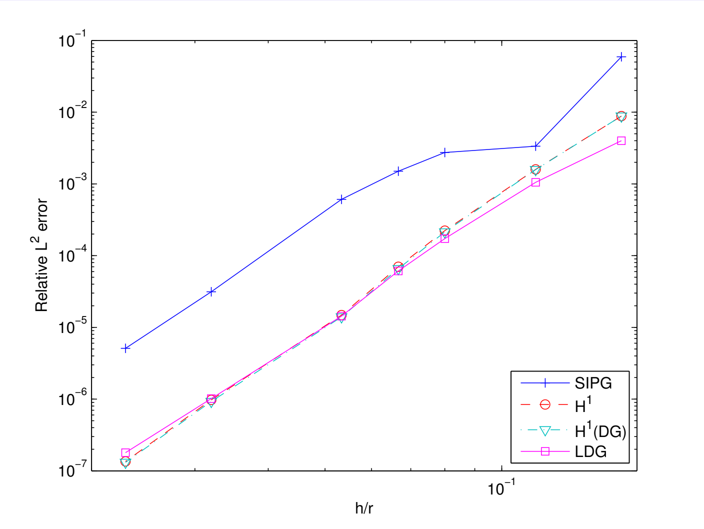


Figure: Convergence for non-uniform flow coeff $= 0.1, r = 5, ||M||_\infty \approx 0.033$.

Convergence for a non-uniform flow

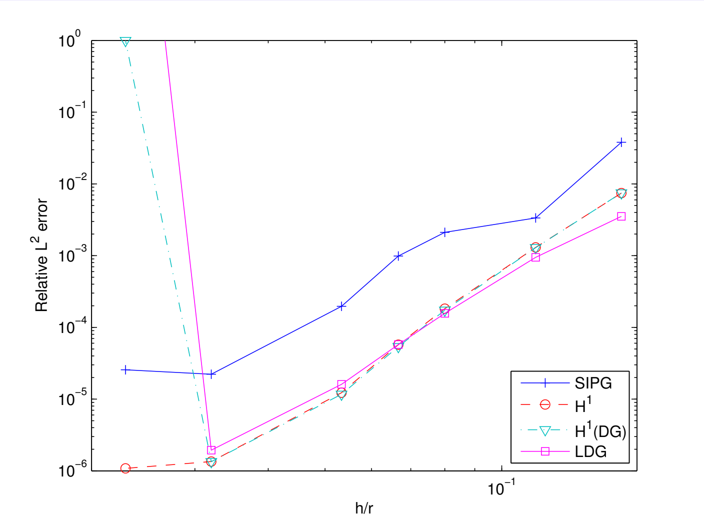


Figure: Convergence for non-uniform flow coeff $= 0.2, r = 5, ||M||_\infty \approx 0.067$.

Convergence for a non-uniform flow

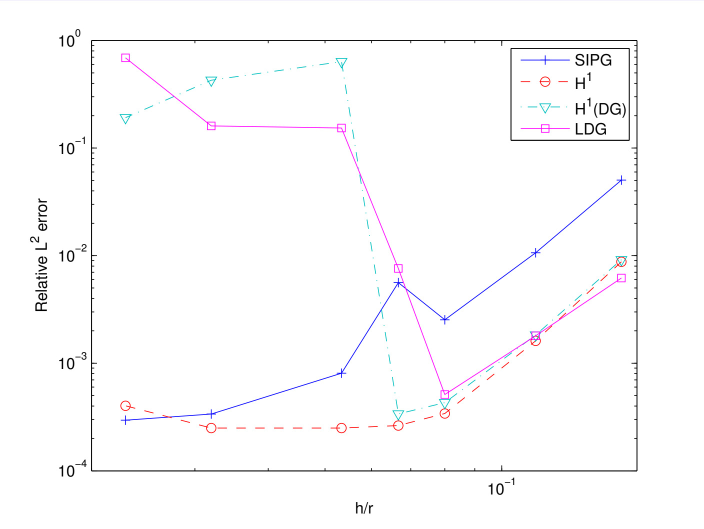


Figure: Convergence for non-uniform flow coeff $= 0.5$, $r = 5$, $||M||_\infty \approx 0.15$.

Convergence for a non-uniform flow

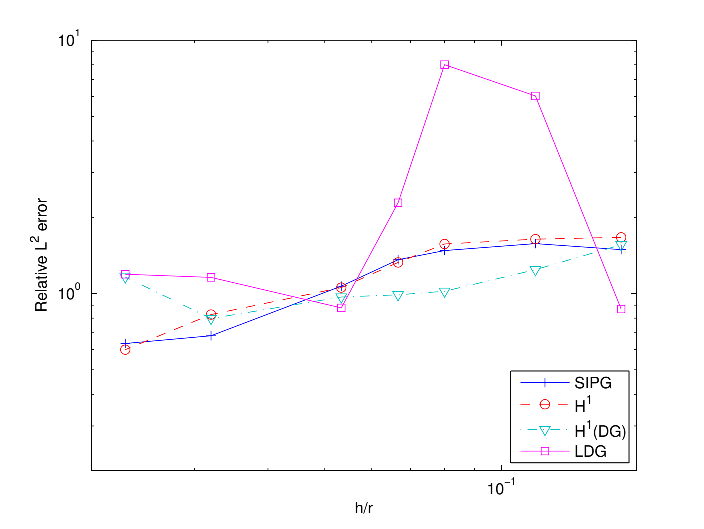


Figure: Convergence for non-uniform flow coeff $= 1.5$, $r = 5$, $||M||_\infty \approx 0.5$.

Convergence for a non-uniform flow

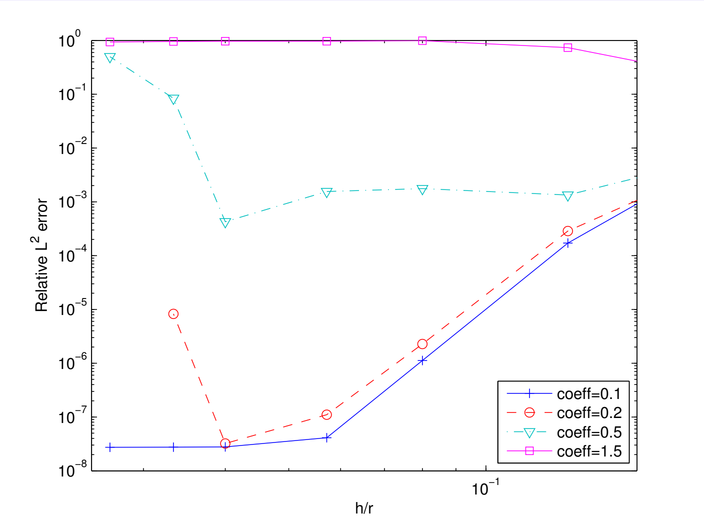


Figure: Consistency error for LDG formulation ($r = 10$), upwind fluxes.

Convergence for a non-uniform flow

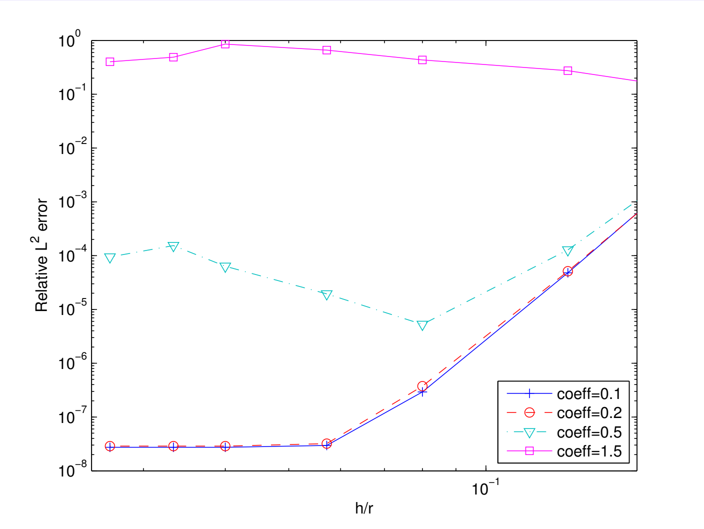


Figure: Consistency error for H^1 formulation ($r = 10$).

Linearized Euler Equations

$$\begin{cases} (-i\omega + \sigma + M \cdot \nabla) p + \text{div}(c_0^2\, u) + \gamma (\text{div} M)\, p - \dfrac{(\gamma - 1)}{\rho_0} u \cdot \nabla p_0 = 0 \\ (-i\omega + \sigma + M \cdot \nabla) \rho + \rho\, \text{div}\, M + \text{div}\, u = 0 \\ (-i\omega + \sigma + M \cdot \nabla) u + (\text{div} M) u + \nabla p + \nabla M (u + \rho\, M) = \dfrac{g}{\rho_0} \end{cases}$$

ρ, p, u perburbations $(\rho', \rho_0 u', p')$

γ defined by $c_0^2 = \dfrac{\gamma p_0}{\rho_0}$

Equivalence with Galbrun's equations for uniform flow if:

$$f \;=\; (-i\omega + \sigma + M \cdot \nabla) g$$

Numerical results for LEE

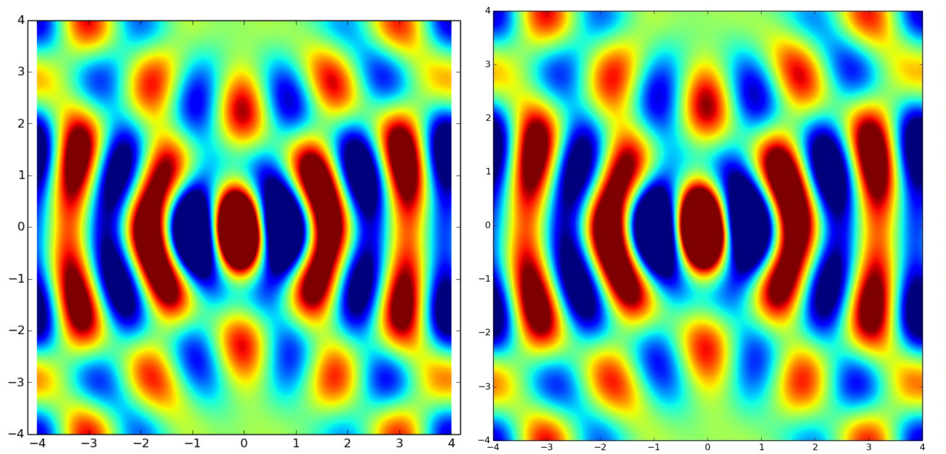


Figure: Solution obtained with Galbrun's equation (left) and LEE (right) (coeff $= 0.1$, real part of u_x).

Numerical results for LEE

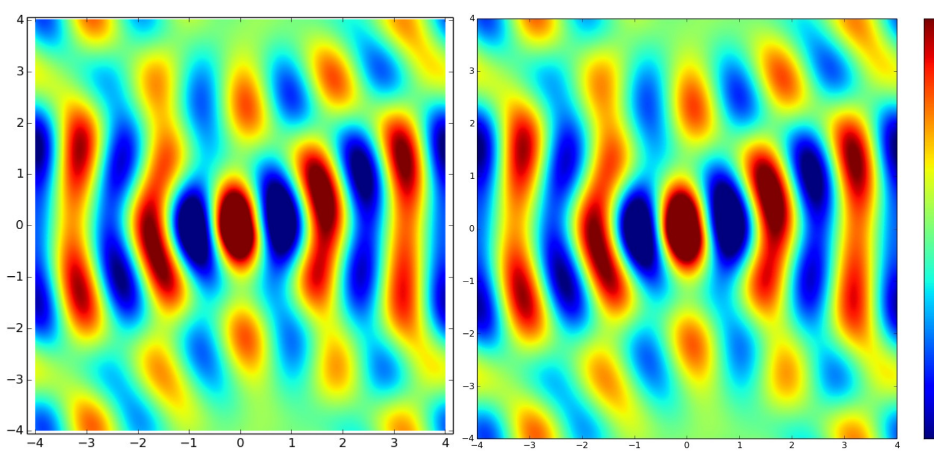


Figure: Solution obtained with Galbrun's equation (left) and LEE (right) (coeff $= 0.2$, real part of u_x).

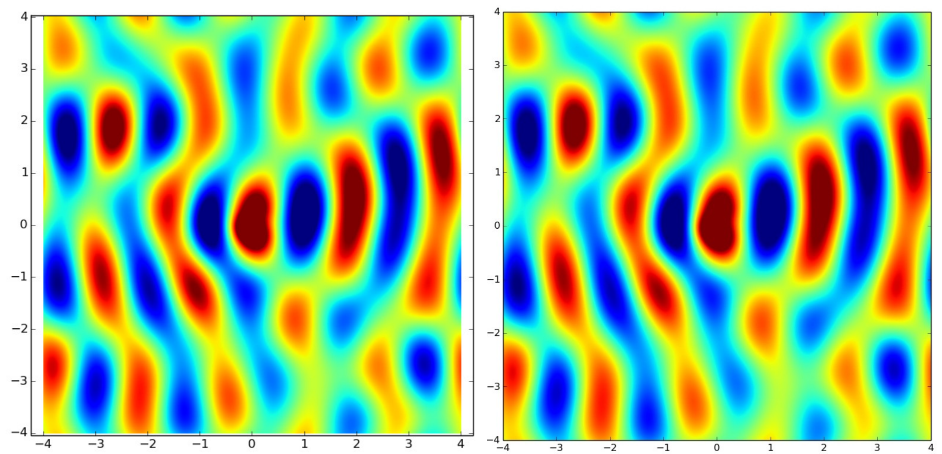


Figure: Solution obtained with Galbrun's equation (left) and LEE (right) (coeff $= 0.5$, real part of u_x).

Numerical results for LEE

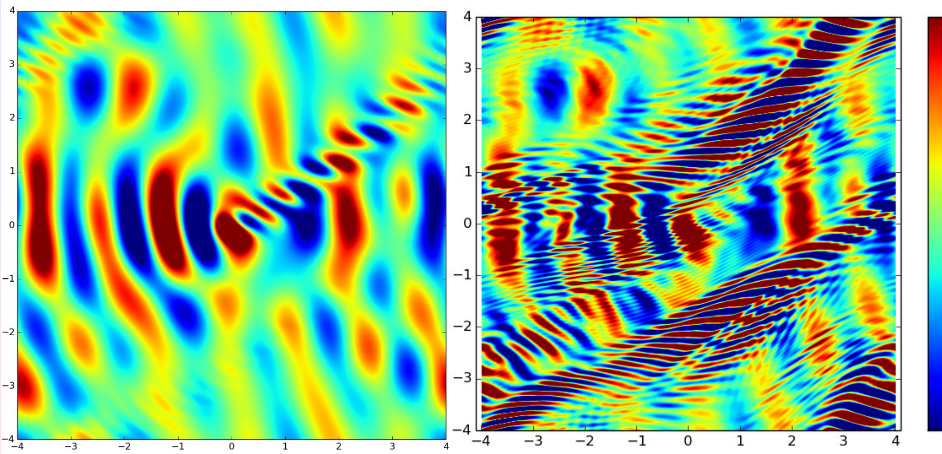


Figure: Comparison for coeff=1.5.

Convergence for LEE

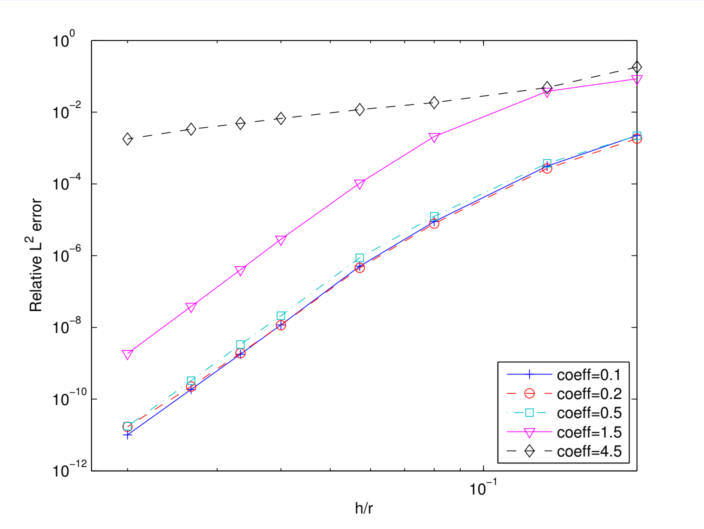


Figure: Convergence for non-uniform flow and LEE ($r = 10$).

Stabilization of Galbrun's equations

Convective stabilization :

$$
\left\{
\begin{array}{l}
\rho_0 \left(-i\omega + \sigma + M \cdot \nabla\right) u - \nabla p - \rho_0 \, q = 0 \\[2ex]
\rho_0 \left(-i\omega + \sigma\right) q - (\nabla \sigma)\, p - (\nabla M)^T \nabla p - \dfrac{M \cdot \nabla \rho_0}{\rho_0} \nabla p \\[1ex]
\qquad + (\operatorname{div} u) \nabla p_0 - (\nabla u)^T \nabla p_0 = f \\[2ex]
\rho_0 \left(-i\omega + \sigma + M \cdot \nabla\right) p - \rho_0^2 \, c_0^2 \operatorname{div} u = 0
\end{array}
\right.
$$

Stabilization of Galbrun's equations

Non-uniform stabilization:

$$\left\{ \begin{aligned} & \rho_0 \left(-i\omega + \sigma + M \cdot \nabla\right) u - \nabla p - \rho_0 \, q = 0 \\ & \rho_0 \left(-i\omega + \sigma + M \cdot \nabla\right) q = f \\ & \rho_0 \left(-i\omega + \sigma + M \cdot \nabla\right) p - \rho_0^2 \, c_0^2 \, \mathrm{div} \, u = 0 \end{aligned} \right.$$

Stabilization of Galbrun's equations

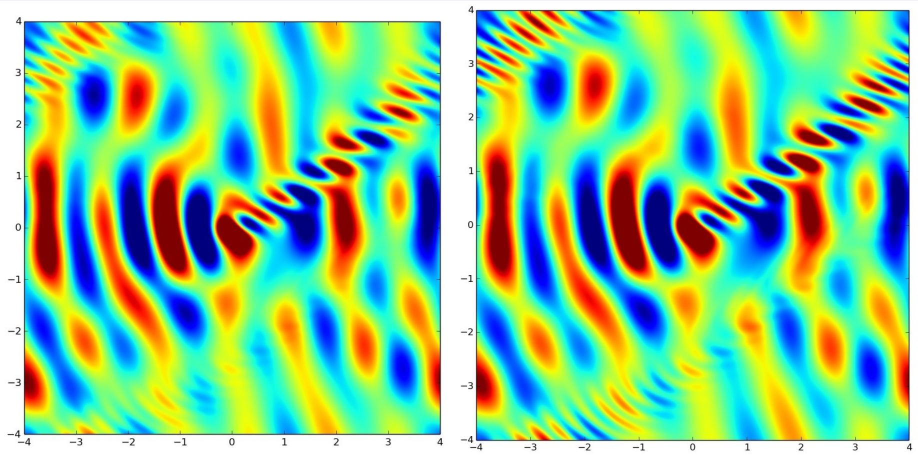


Figure: Real part of u_x for a non-uniform flow (coeff=1.5) and stabilized Galbrun's equations (on left, convective stabilization, on right non-uniform stabilization)

Simplified Galbrun's equations

Equivalent to Galbrun's equations when $M = 0$

$$\left\{ \begin{array}{l} \rho_0(-i\omega + \sigma + M \cdot \nabla)\, p + \rho_0^2\, c_0^2 \,\mathrm{div}\, u = 0 \\ \rho_0(-i\omega + \sigma + M \cdot \nabla)\, u + \nabla p \\ \qquad + \dfrac{1}{-i\omega + \sigma} \Big((\mathrm{div}\, u) \nabla p_0 - (\nabla u)^T \nabla p_0 \Big) = g \end{array} \right.$$

Simplified Galbrun's equations

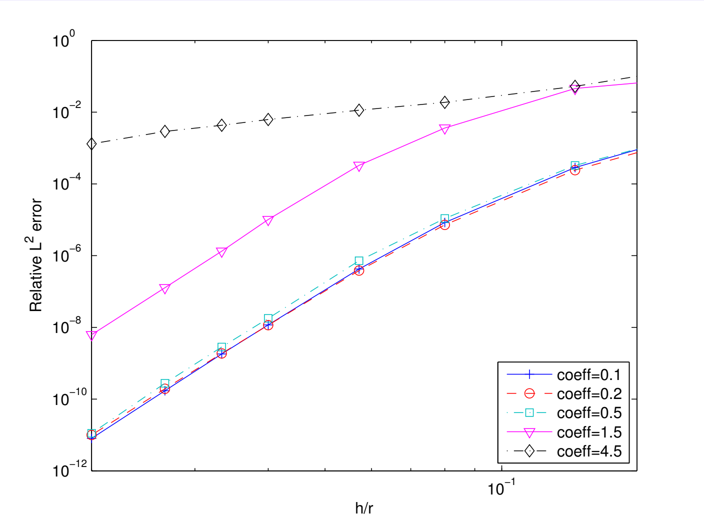


Figure: Convergence for non-uniform flow and simplified Galbrun ($r = 10$)

Computation of Green's function

$$g = \delta\, e_y,\; M = (0, 0.3)$$

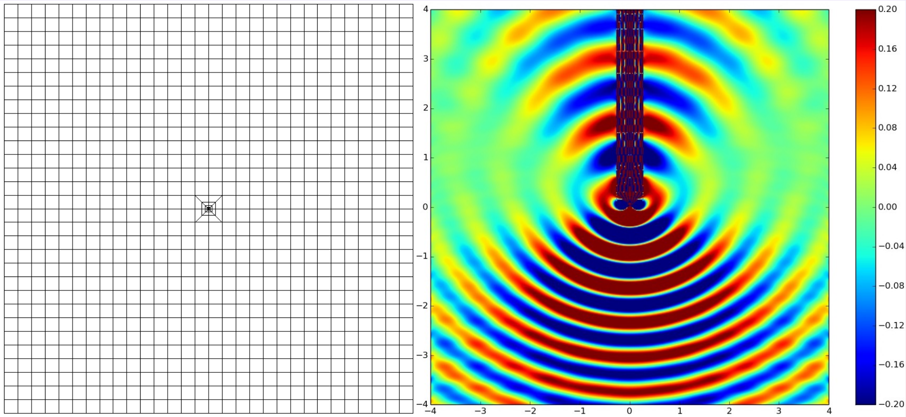


Figure: Imaginary part of u_y and associated mesh.

Computation of Green's function

$$g = \delta\, e_y, M = (0, 0.3)$$

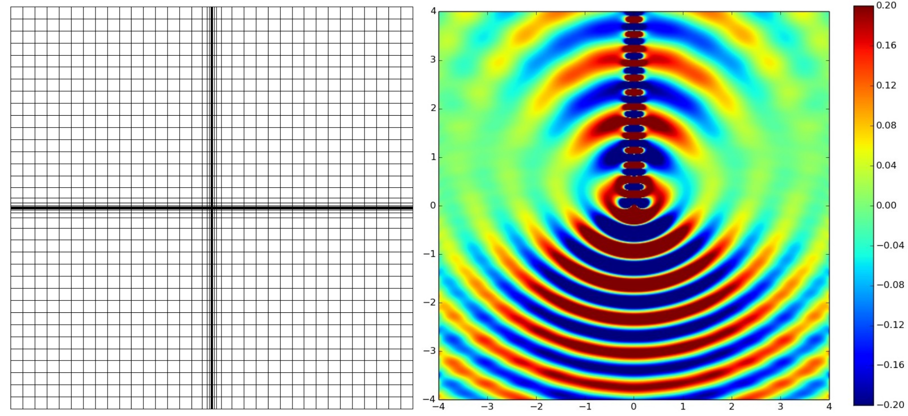


Figure: Imaginary part of u_y and associated mesh.

Computation of Green's function

Non-uniform flow and mesh only refined at the center

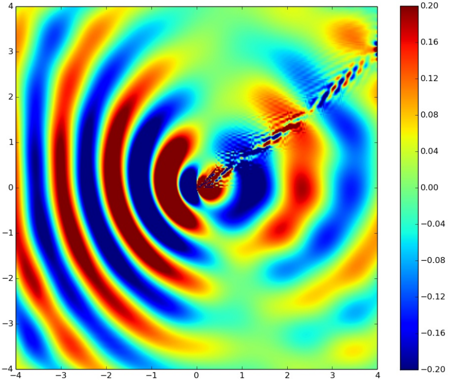
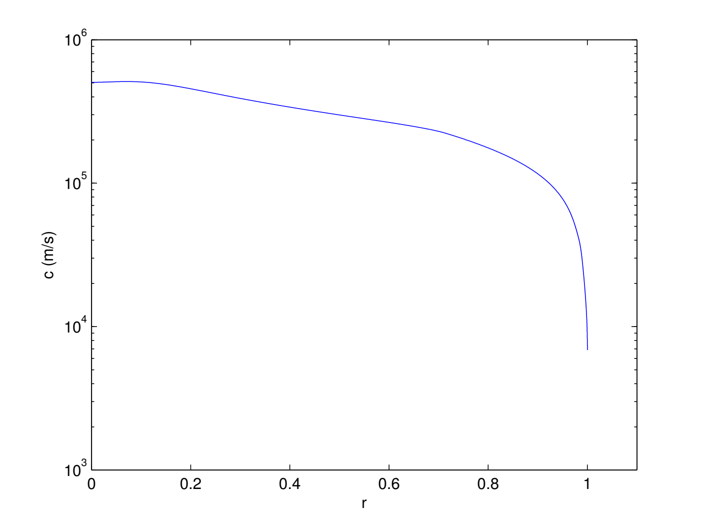


Figure: Imaginary part of u_x for non-uniform flow

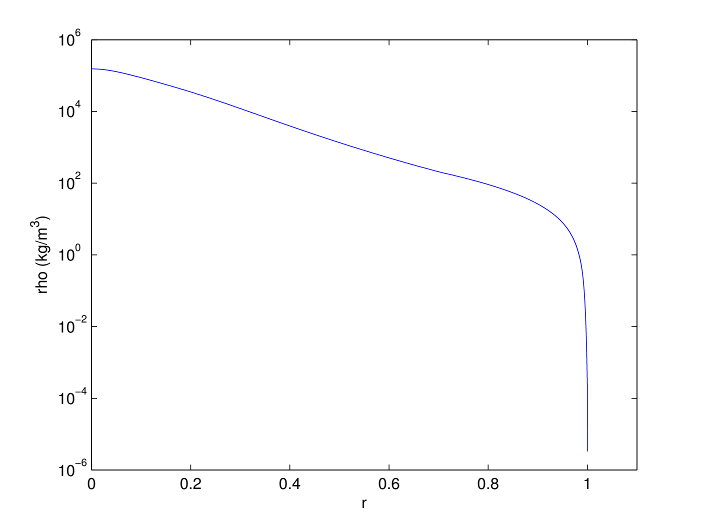
Application to the sun

Profile of the sound speed c_0 for the sun in log-scale:



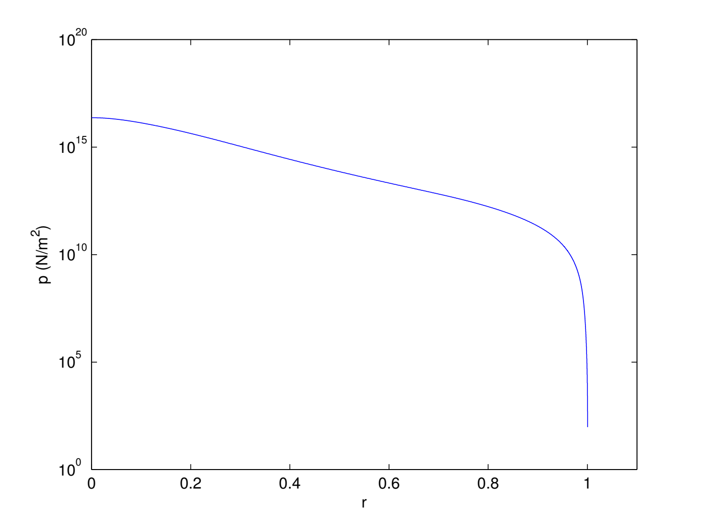
Application to the sun

Profile of the backgroound density ρ_0 for the sun in log-scale:



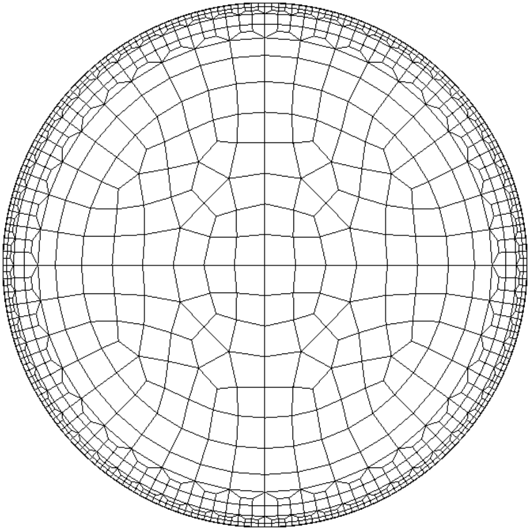
Application to the sun

Profile of the background pressure p_0 for the sun in log-scale:



Application to the sun

Example of mesh used for 2-D experiments



Numerical results for the sun

Source g gaussian centered around $(0.5, 0.5)$ and f given as:

$$f \;=\; (-i\omega + \sigma + M \cdot \nabla) g$$

Rotating flow:

$$M = \frac{\text{Coeff}}{R} c_0(r) \begin{bmatrix} -y \\ x \end{bmatrix}$$

Uniform damping:

$$\sigma = \frac{\omega}{100}$$

Frequency:

$$\text{freq} = 3mHz, \quad \omega = 2\pi \times \text{freq}$$

Numerical results for the sun

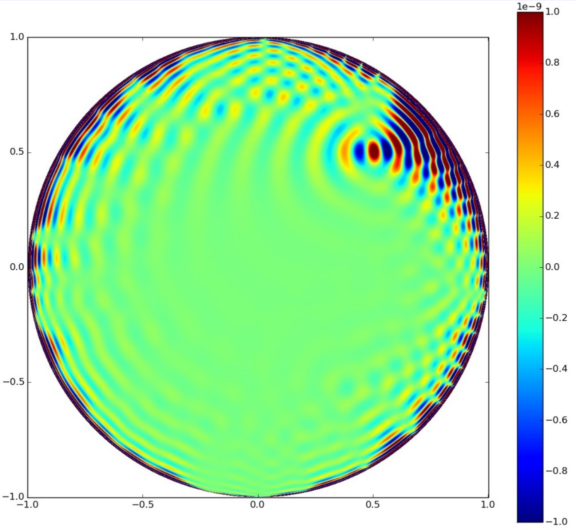


Figure: Real part of u_x for the sun (Coeff $= 0$, Galbrun).

Numerical results for the sun

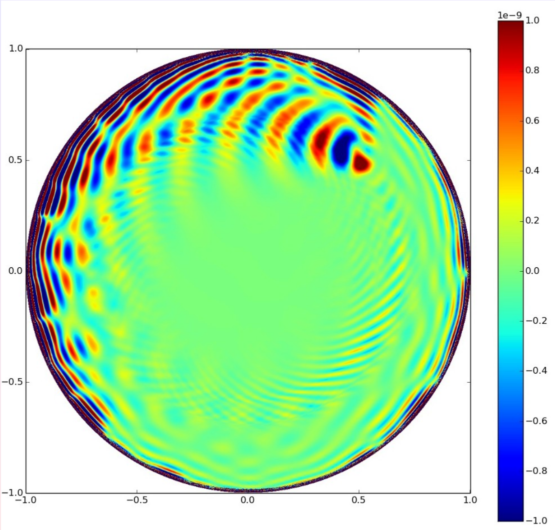


Figure: Real part of u_x for the sun (Coeff $= 1$, Galbrun).

Numerical results for the sun

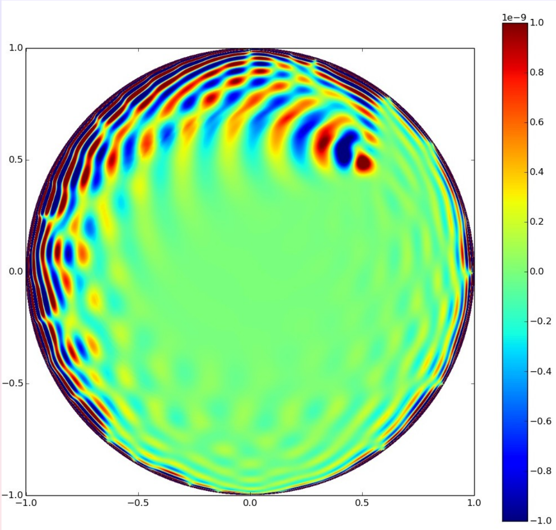


Figure: Real part of u_x for the sun (Coeff $= 1$, simplified Galbrun).