SUMMARY :

KINETIC EQUATION : $\partial_t f + v \cdot \nabla_x f = Q(f)$

MACROSCOPIC QUANTITIES :

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} = \int m \begin{pmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{pmatrix} f \, dv$$ (MOMENTS OF f W.R.T v)

$$E = \frac{1}{2}\rho\|u\|^2 + \frac{3}{2}\rho R T \qquad \rho R T = p$$ (PRESSURE)

EQUILIBRIUM : MAXWELLIAN DISTRIBUTION

$$M[\rho, u, T](v) = \frac{\rho}{(2\pi R T)^{3/2}} \exp\left(-\frac{\|v-u\|^2}{2RT}\right)$$

ONLY SOLUTIONS OF $Q(f) = 0$

CONSERVATION PROP. OF $Q(f)$: $\int m \begin{pmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{pmatrix} Q(f) \, dv = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

MOMENTUM CONSERVATION: $\quad (K) \quad \partial_t f + v \cdot \nabla_x f = Q(f)$

$$\int mv\,(K)\,dv \longrightarrow \partial_t \rho u + \int mv\,(v \cdot \nabla_x f)\,dv = 0$$

i^{th} component: $\displaystyle\int m v_i \,(v \cdot \nabla_x f)\,dv$

$$= \int m v_i \sum_{j=1}^{3} v_j \,\partial_{x_j} f \; dv = \sum_{j=1}^{3} \partial_{x_j} \int m v_i v_j \, f \, dv$$

TENSOR PRODUCT: $\quad a \otimes b = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = (a_i b_j)$

a, b: VECTORS

DIVERGENCE OF A TENSOR: $\quad T = (T_{ij}) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

$$\nabla \cdot T = \left(\sum_{j=1}^{3} \partial_{x_j} T_{ij} \right) \quad \text{VECTOR}$$

$$= \nabla \cdot \left(\int m\, v \otimes v \, f \, dv \right) \quad \text{MOMENTUM FLUX.}$$

THEN THE MOMENTUM CONSERVATION LAW IS

$$\partial_t \rho u + \nabla \cdot \left(\int m v \otimes v \, f \, dv \right) = 0$$

EXERCISE P.29 : SOLUTION

$$\int m v \otimes v \, f \, dv = \int m \left[(v-u) + u \right] \otimes \left[(v-u) + u \right] f \, dv$$

$$= \int m (v-u) \otimes (v-u) f \, dv + \underbrace{\cancel{\left(\int m (v-u) f \, dv \right)}}_{=0} \otimes u + u \otimes \underbrace{\cancel{\left(\int m (v-u) f \, dv \right)}}_{=0}$$

$$+ (u \otimes u) \int m f \, dv$$

$$= \mathbb{P} + \rho \, u \otimes u$$

(BECAUSE : $\int m v f \, dv = \rho u$, $\int m f \, dv = \rho$)

THEN :

$$\boxed{\partial_t \rho + \nabla \cdot \left(\mathbb{P} + \rho u \otimes u \right) = 0}$$

ENERGY CONSERVATION:

$$\hookrightarrow \quad \partial_t E + \nabla_x \cdot \underbrace{\left(\int \frac{1}{2} m \|v\|^2 v f \, dv \right)}_{\text{ENERGY FLUX}} = 0$$

EXERCISE P. 31 : SOLUTION

① EASY

② $$\int \frac{1}{2} m \|v\|^2 v f \, dv = \int \frac{1}{2} m \|v\|^2 \left[(v-u) + u \right] f \, dv$$

$$= \underbrace{\int \frac{1}{2} m \|v\|^2 (v-u) f \, dv}_{\parallel} + u \underbrace{\int \frac{1}{2} m \|v\|^2 f \, dv}_{=E}$$

$$\int \frac{1}{2} m \|(v-u) + u\|^2 (v-u) f \, dv$$

$$= \underbrace{\int \frac{1}{2} m \|v-u\|^2 (v-u) f \, dv}_{\substack{\parallel \\ q}} + \frac{1}{2} \|u\|^2 \overbrace{\cancel{\int m (v-u) f \, dv}}^{=0} + \underbrace{\int m \big(u \cdot (v-u)\big) (v-u) f \, dv}_{= \mathbb{P} u}$$

PROPERTY: a, b VECTORS THEN

$$(a \cdot b)\, b = (b \otimes b)\, a$$

THEN

$$\int \frac{1}{2} m \|v\|^2 f \, dv = \mathbb{P} u + E u + q$$

FINALLY: CONSERVATION LAWS :

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t \rho u + \nabla \cdot (\rho u \otimes u + \mathbb{P}) = 0 \\ \partial_t E + \nabla \cdot (E u + \mathbb{P} u + q) = 0 \end{cases}$$

NON-CLOSED SYSTEM

UNKNOWNS : $\underset{1}{\rho}, \underset{3}{u}, \underset{1}{E}, \underset{6}{\mathbb{P}}, \underset{3}{q} \longrightarrow \boxed{14}$

EQUATIONS : $\longrightarrow \boxed{5}$

CONSERVATION LAWS FOR EQUILIBRIUM STATE :

ASSUMPTION : $f = M[\rho, u, T]$

- $$\mathbb{P} = \int m (v-u) \otimes (v-u) \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{\|v-u\|^2}{2RT}\right) dv$$

CHANGE OF VARIABLES : $v \mapsto v + u$

$$= \int m\, v \otimes v \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{\|v\|^2}{2RT}\right) dv$$

$$\mathbb{P}_{ij} = \int m\, v_i v_j \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{v_1^2 + v_2^2 + v_3^3}{2RT}\right) dv_1\, dv_2\, dv_3$$

$\rightarrow i \neq j$: ODD FUNCTION W.R.T v_i AND v_j $\longrightarrow \underline{0}$

$\rightarrow i = j$: $= \int m\, v_i^2 \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{\|v\|^2}{2RT}\right) dv$ INDEPENDENT OF i

$$= \frac{1}{3} \int m (v_1^2 + v_2^2 + v_3^2) \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{\|v\|^2}{2RT}\right) dv$$

(IMPAIRE)

$$= \frac{1}{3} \int m \|v\|^2 \underline{\qquad\qquad}$$

CHANGE OF VARIABLES $v \mapsto v - u$

$$= \frac{1}{3} \int m \, \|v - u\|^2 \, M[\rho, u, T](v) \, dv$$

$$= p \quad \text{(PRESSURE)} \quad (= \rho R T)$$

FINALLY:
$$\mathbb{P}_{ij} = \begin{cases} 0 & \text{IF } i \neq j \\ p & \text{IF } i = j \end{cases}$$

THEN:
$$\boxed{\begin{aligned} \mathbb{P} &= p\, I \longrightarrow \text{UNIT TENSOR} \\ &= p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}}$$

$$\bullet \quad q = \int \frac{1}{2} m \|v-u\|^2 (v-u)\, M[\rho, u, T](v)\, dv$$

$$v \mapsto v+u$$

$i^{\underline{th}}$ COMPONENT OF q :

$$q_i = \int \underbrace{\frac{1}{2} m \left(v_1^2+v_2^2+v_3^3\right) v_i \frac{\rho}{(2\pi RT)^{3/2}} \exp\left(-\frac{v_1^2+v_2^2+v_3^2}{2RT}\right)}_{\text{ODD FUNCTION W.R.T } v_i} dv$$

$$= 0$$

THEN : $\boxed{q = 0}$

<u>CONSEQUENTLY:</u> THE CONSERVATION LAWS ARE CLOSED

THE CONSERVATION LAWS FOR

$$f = M[\rho, u, T]$$

ARE :

$$\begin{cases} \partial_t \rho + \nabla \cdot \rho u = 0 \\ \partial_t \rho u + \nabla \cdot (\rho u \otimes u + p I) = 0 \\ \partial_t E + \nabla \cdot (E u + p u) = 0 \end{cases}$$

$$p = \rho R T \quad \text{AND} \quad E = \frac{1}{2} \rho \|u\|^2 + \frac{3}{2} \rho R T$$

CLOSED SYSTEM

↳ THIS THE EULER EQUATIONS OF GAS DYNAMICS

2.5 A SIMPLE COLLISION MODEL.

THE BGK MODEL

$$\boxed{Q(f) = \frac{1}{\tau}\left(M[\rho, u, T](v) - f \right)}$$ BGK COLLISION OPERATOR

WHERE ρ, u, T ARE THE MACROSCOPIC QUANTITIES OF f $\left(\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} = \int m \begin{pmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{pmatrix} f \, dv \right)$

-> MODELS THE EFFECT OF COLLISIONS BY A RELAXATION OF f TO THE CORRESPONDING LOCAL EQUILIBRIUM $M[\rho, u, T]$

-> RELAXATION TIME : τ

(GENERALLY $\tau = \frac{1}{c\rho}$, c IS A CONSTANT)

$\simeq$ MEAN TIME BETWEEN TWO COLLISIONS

→ RELAXATION PHENOMENON : DIFFERENTIAL EQUATION

$$\frac{d}{dt} y(t) = \frac{1}{\tau}\left(\bar{y} - y(t)\right)$$

$\bar{y}$ IS A CONSTANT

SOLUTION: $y(t) = e^{-t/\tau}\, y(0) + \left(1 - e^{-t/\tau}\right)\bar{y}$

$$y(t) \xrightarrow[t \to +\infty]{} \bar{y}$$

$y(t)$ RELAXES TO $\bar{y}$

$\rightarrow$ PROPERTIES OF $Q(f) = \frac{1}{\varepsilon}(M[\rho,u,T](v) - f)$

- CONSERVATION PROPERTIES: $\displaystyle\int m \begin{pmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{pmatrix} Q(f)\, dv = 0$

($\Leftrightarrow$ f AND $M[\rho,u,T]$ HAVE THE SAME FIRST MOMENTS)

- ENTROPY: $\displaystyle\int Q(f) \log f\, dv = \frac{1}{\varepsilon}\int (M-f) \log f\, dv$

$$= \frac{1}{\varepsilon}\underbrace{\int (M-f)(\log f - \log M)\, dv}_{\leq 0 \text{ SINCE } \log \nearrow} + \frac{1}{\varepsilon}\underbrace{\int (M-f) \log M\, dv}$$

$$\hookrightarrow \log M = \log\left(\frac{\rho}{(2\pi RT)^{3/2}}\right) - \frac{\|v-u\|^2}{2RT}$$

$$= \alpha + \beta \cdot v + \gamma \|v\|^2$$

THEN: $\displaystyle\frac{1}{\varepsilon}\int (M-f) \log M\, dv = 0$

THEREFORE: $\int Q(f) \log f \, dv \leq 0$

- EQUILIBRIUM STATES:

$$Q(f) = 0 \iff f = M[\rho, u, T] \quad \text{(OBVIOUS)}$$

MAXWELLIAN EQUILIBRIUM STATES

- TRENDS TO EQUILIBRIUM:

$$\partial_t f = Q(f) = \frac{1}{\tau}\left(M[\rho, u, T] - f \right)$$

$$\int m \begin{pmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{pmatrix} \longrightarrow \partial_t \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

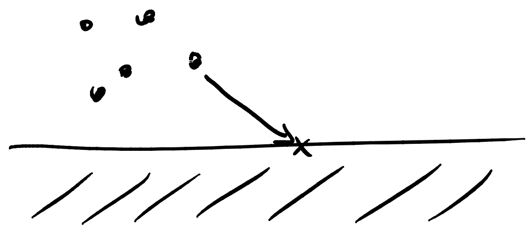
$\Rightarrow \rho, u, E$ (AND HENCE T) ARE CONSTANT

$$\Rightarrow \partial_t f = \frac{1}{\tau}\Big(\underbrace{M[\rho, u, T]}_{\bar{f}} - f \Big) \Rightarrow f \xrightarrow[t \to +\infty]{} M[\rho, u, T]$$

f RELAXES TO $M[\rho, u, T]$

2.6 COLLISIONS WITH BOUNDARIES

PROBLEM: SOLID WALL



PARTICLES HIT THE WALL

QUESTION: • HOW ARE THEY REFLECTED?

• HOW CAN IT BE MODELLED AT THE KINETIC LEVEL (THAT IS WITH f)

↳ 3 DIFFERENT KINDS OF BOUNDARY COLLISIONS (B.C)

(a) SPECULAR REFLECTION

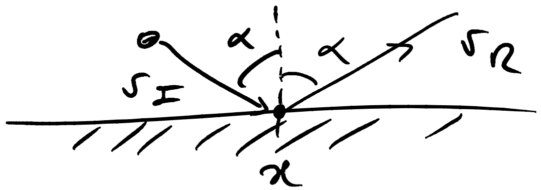
(b) DIFFUSE ——

(c) MAXWELL ——

(a) SPECULAR REFLECTION

PARTICLES ARE REFLECTED AS BILLARD BAWLS

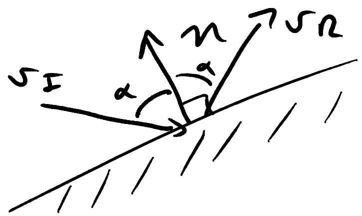
(DESCARTES LAW OF REFLECTION)



INCIDENCE ANGLE = REFLECTION ANGLE

AT THE KINETIC LEVEL :

$$f(t, x, v_R) = f(t, x, v_I)$$



n : UNIT VECTOR NORMAL TO THE WALL

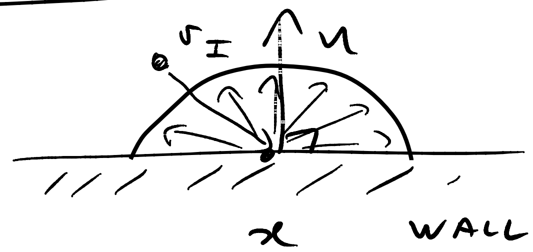
MATHEMATICALLY :

v_I IS SUCHTHAT

$$v_I \cdot n \leq 0$$

AND $\quad v_R = v_I - 2(v_I \cdot n)\, n$

(b) DIFFUSE REFLECTION



WALL TEMPERATURE T_w
VELOCITY u_w

PARTICLES ARE REFLECTED
WITH RANDOM VELOCITY UNIFORMLY DISTRIBUTED
(GAUSSIAN LAW) AROUND u_w :

IF $v \cdot n \geq 0$ (REFLECTED VELOCITY)

$$f(t, x, v) = M[\sigma, u_w, T_w](v)$$

DEFINITION OF σ: ALL THE INCIDENT PARTICLES ARE REFLECTED

$\Leftrightarrow$ NO NORMAL MASS FLUX ACROSS THE WALL

$$\Leftrightarrow \boxed{\int_v v \cdot n \, f(t, x, v) \, dv = 0}$$

EXERCISE: PROVE THAT THIS RELATION

IMLIES:

$$\sigma = - \frac{\displaystyle\int_{v \cdot n \le 0} v \cdot n \, f(t,x,v)\, dv \quad \text{(INCIDENT VELOCITIES)}}{\displaystyle\int_{v \cdot n \ge 0} v \cdot n \, M[1, u_w, T_w]\, dv \quad \text{(REFLECTED VELOCITIES)}}$$

(c) MAXWELL REFLECTION WITH PARTIAL ACCOMODATION

↳ CONVEX COMBINATION OF (a) AND (c)

FOR REFLECTED VELOCITIES : v . ST $v \cdot n \geq 0$

$$f(t,x,v) = \alpha f(t,x,v-2(v\cdot n)n) + (1-\alpha) M[\sigma, u_w, T_w](v)$$

WITH α S.T $0 \leq \alpha \leq 1$

$\alpha = 1$: SPECULAR REFLECTION

$\alpha = 0$: DIFFUSE —

<u>EXERCISE</u>: COMPUTE σ TO ENSURE THAT

THERE IS NO NORMAL MASS FLUX ACCROSS THE WALL