SUMMARY OF LECTURE #2

WHEN f IS A SOLUTION OF THE BGK EQUATION

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon}\left(M[\rho, u, T] - f \right)$$

<u>THEN</u> ρ, u, T SATISFY THE CONSERVATION EQUATIONS

$$\begin{cases} \partial_t \rho + \nabla \cdot \rho u = 0 \\ \partial_t \rho u + \nabla \cdot \left(\rho u \otimes u + \mathbb{P} \right) = 0 \\ \partial_t E + \nabla \cdot \left(E u + \mathbb{P} u + q \right) = 0 \end{cases}$$

<u>NON CLOSED SYSTEM</u>

$$\mathbb{P} = \int (v-u) \otimes (v-u) f \, dv \qquad q = \int \frac{1}{2} \|v-u\|^2 (v-u) f \, dv$$

ARE NOT FUNCTIONS OF ρ, u, T

IF f IS CLOSE TO A MAXWELLIAN:

$$f = M[\rho, u, T]$$

THEN $\begin{cases} \mathbb{P} = p \, \mathrm{Id} \\ q = 0 \end{cases}$ WHERE $p = \rho R T$ (PRESSURE)

AND THE CONSERVATION LAWS ARE:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t \rho u + \nabla \cdot (\rho u \otimes u + p \, \mathrm{Id}) = 0 \\ \partial_t E + \nabla \cdot (E u + p u) = 0 \end{cases}$$

EULER SYSTEM OF GAS DYNAMICS
(CLOSED SYSTEM)

3. FROM KINETIC TO MACROSCOPIC (FLUID) MODELS

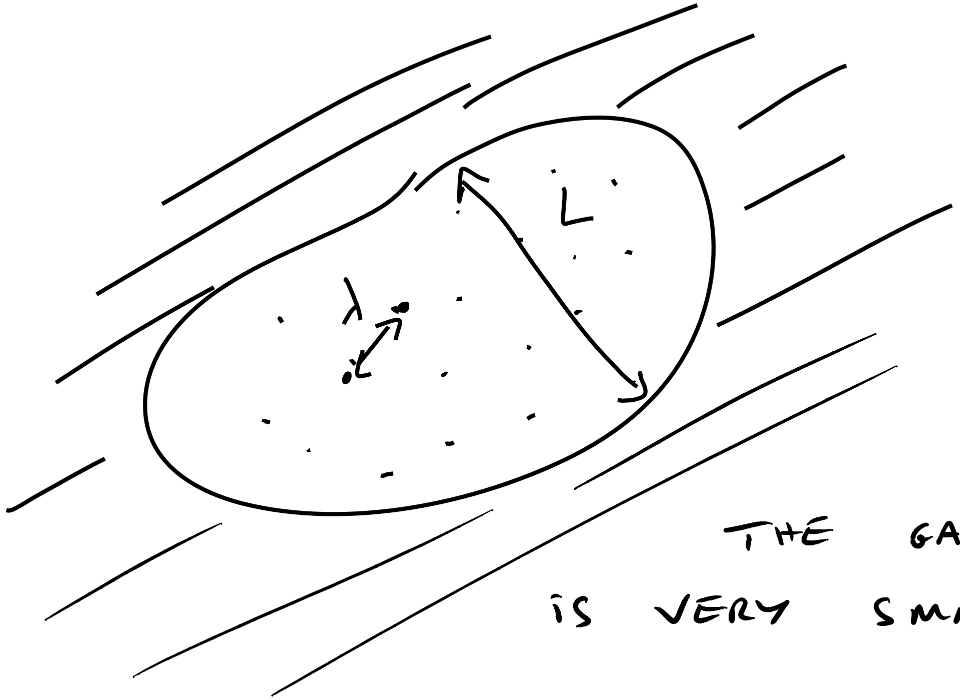
3.1 DIMENSIONAL ANALYSIS

WE EXPECT THE EVOLUTION OF THE GAS TO BE ACCURATELY DESCRIBED BY THE EULER EQUATIONS WHEN THE GAS IS DENSE

↳ RELATIVE NOTION !

DENSE GAS: RELATIVE NOTION CONNECTING TWO DIFFERENT QUANTITIES

- THE AVERAGED DISTANCE BETWEEN PARTICLES (MEAN FREE PATH)
- SOME REFERENCE MACROSCOPIC LENGTH



L = CHARACTERISTIC MACROSCOPIC LENGTH (E.G. : DIAMETER)

λ = MEAN DISTANCE BETWEEN TWO PARTICLES

THE GAS IS DENSE WHEN λ IS VERY SMALL AS COMPARED TO L ($\lambda \ll L$)

WE DEFINE THE RATIO :

$$\boxed{K_N = \frac{\lambda}{L}}$$

(DENOTED BY ε)

KNUDSEN NUMBER

(NON DIMENSIONAL NUMBER)

- $\varepsilon \ll 1$: THE GAS IS DENSE
- $\varepsilon \gg 1$: ——— IS DILUTED

HOW TO MAKE THIS NUMBER APPEAR IN THE KINETIC EQUATION ?

BGK EQUATION : $\partial_t f + v \cdot \nabla_x f = \frac{1}{\tau}\left(M[\rho, u, T] - f \right)$

WHERE $\tau = \dfrac{1}{c\rho}$, c IS A CONSTANT

DIMENSIONAL ANALYSIS :

1. FIND CHARACTERISTIC VALUES OF THE VARIABLES (t, x, v, ρ, u, T, f)
2. DEFINE RESCALED VARIABLES (SO AS TO HAVE VARIABLES OF ORDER 1)
3. REWRITE THE BGK EQUATION WITH THESE NEW VARIABLES
4. LOOK AT THE NON DIMENSIONAL NUMBERS IN THE EQUATION

1. TYPICAL VALUES OF THE VARIABLES

$\bar{\rho} = \rho_0$ (INITIAL DENSITY)

$\bar{T} = T_0$ (—— TEMPERATURE)

$\bar{x} = L$ (MACROSCOPIC DISTANCE)

$\bar{v} = \sqrt{RT_0}$ (THERMAL VELOCITY AT INITIAL TIME
= AVERAGED RELATIVE VELOCITY OF PARTICLES IN EQUILIBRIUM STATE)

$\bar{t} = \dfrac{\bar{x}}{\bar{v}} = \dfrac{L}{\sqrt{RT_0}}$ (TIME FOR PARTICLES WITH VELOCITY $\bar{v}$ TO CROSS THE DOMAIN)

$\bar{u} = \bar{v} = \sqrt{RT_0}$

$\bar{\tau} = \dfrac{1}{c\bar{\rho}} = \dfrac{1}{c\rho_0}$

$\bar{f} = M[\rho_0, \bar{v}, T_0](v = \bar{v}) = \dfrac{m\rho_0}{(2\pi R T_0)^{3/2}}$

ⓒ NON DIMENSIONAL VARIABLES

$$x' = \frac{x}{\bar{x}} \qquad t' = \frac{t}{\bar{t}} \qquad v' = \frac{v}{\bar{v}} \qquad \rho' = \frac{\rho}{\bar{\rho}} \qquad u' = \frac{u}{\bar{u}}$$

$$T' = \frac{T}{\bar{T}} \qquad \tau' = \frac{\tau}{\bar{\tau}} \qquad f' = \frac{f}{\bar{f}}$$

<u>EXERCISE:</u> PROVE THAT

(a)
$$\int \begin{pmatrix} 1 \\ v' \\ \frac{1}{2}\|v'\|^2 \end{pmatrix} f' \, dv' = \begin{pmatrix} \rho' \\ \rho' u' \\ E' \end{pmatrix}$$

WHERE $E' = \frac{1}{2}\rho' \|u'\|^2 + \frac{3}{2}\rho' R T'$

(b)
$$M[\rho, u, T](v) = \bar{f} \underbrace{\frac{\rho'}{(2\pi T')^{3/2}} \exp\left(-\frac{\|v' - u'\|^2}{2T'}\right)}_{\text{DENOTED BY } M'[\rho', u', T']}$$

③ KINETIC EQUATIONS IN NON-DIMENSIONAL VARIABLES

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\tau}\left(M[\rho,u,T] - f\right)$$

- $\displaystyle \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t'} \underbrace{\frac{\partial t'}{\partial t}}_{=\frac{1}{\bar t}} = \frac{1}{\bar t}\frac{\partial f}{\partial t'} = \frac{\bar f}{\bar t}\frac{\partial f'}{\partial t'}$

- $\displaystyle v \cdot \nabla_x f = \bar v\, v' \cdot \nabla_{x'} (\bar f f') \frac{1}{\bar x} = \frac{\bar v \bar f}{\bar x} v' \cdot \nabla_{x'} f'$

- $\displaystyle \frac{1}{\tau}\left(M[\rho,u,T] - f\right) = \frac{1}{\bar\tau \tau'}\left(\bar f M'[\rho',u',T'](v') - \bar f f'\right)$

THIS GIVES:

$$\cancel{\frac{\bar f}{\bar t}} \partial_{t'} f' + \frac{\bar v \cancel{\bar f}}{\bar x} v' \cdot \nabla_{x'} f' = \frac{\cancel{\bar f}}{\bar\tau}\frac{1}{\tau'}\left(M'[\rho',u',T'](v') - f'\right)$$

$$\Rightarrow \quad \partial_{t'} f' + \underbrace{\frac{\bar t \bar v}{\bar x}}_{=1} v' \cdot \nabla_{x'} f' = \underbrace{\frac{\bar t}{\bar\tau}}_{=\frac{1}{\varepsilon}} \frac{1}{\tau'}\left(M[\rho',u',T'](v') - f'\right)$$

FINALLY WE GET :

$$\partial_{t'} f' + v' \cdot \nabla_{x'} f' = \frac{1}{\varepsilon} \frac{1}{\tau'} \left(M'[\rho', u', T'](v') - f' \right)$$

WHERE $\varepsilon = \dfrac{\bar{\tau}}{\bar{t}}$

④ NON DIMENSIONAL NUMBER :

$$\varepsilon = \frac{\bar{\tau}}{\bar{t}} = \frac{\text{MICROSCOPIC TIME}}{\text{MACROSCOPIC TIME}}$$) RATIO BETWEEN TWO TIMES

$$= \frac{\bar{\tau}}{\bar{x}/\bar{v}} = \frac{\bar{\tau}\bar{v}}{\bar{x}} = \frac{\sqrt{RT_0}/c\rho_0}{L}$$

$$= \frac{\lambda}{L}$$ WHERE $\lambda = \dfrac{\sqrt{RT_0}}{c\rho_0}$ MICROSCOPIC LENGTH

$\simeq$ MEAN FREE PATH

$\Rightarrow$ ε = <u>KNUDSEN NUMBER</u>

BGK EQUATION IN NON DIMENSIONAL VARIABLES:

$$\boxed{\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \frac{1}{\tau}\left(M[\rho, u, T](v) - f\right)}$$

(WE DROP THE PRIMES ')

WE EXPECT THE GAS TO BE WELL DESCRIBED BY THE EULER EQUATIONS WHEN THE GAS IS DENSE, THAT IS WHEN $\varepsilon << 1$

↳ MATHEMATICAL PROBLEM:

WHAT DOES HAPPEND IN THE EQUATION IF $\varepsilon \longrightarrow 0$?

3.2 EULER LIMIT

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \times \frac{1}{\tau}\left(M[\rho, u, T] - f\right)$$

ASSUME $\varepsilon \ll 1$

$$f = M[\rho, u, T] - \varepsilon \times \underbrace{\left[\tau\left(\partial_t f + v \cdot \nabla_x f\right)\right]}_{\text{ASSUME THIS IS BOUNDED W.R.T } \varepsilon}$$

THIS MEANS THAT

$$\boxed{f = M[\rho, u, T] + O(\varepsilon)}$$

- WE INSERT THIS INTO THE CONSERVATION EQUATIONS :

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0 \\ \partial_t \rho u + \nabla\cdot(\rho u \otimes u + \mathbb{P}) = 0 \\ \partial_t E + \nabla\cdot(E u + \mathbb{P} u + q) = 0 \end{cases}$$

$$\mathbb{P} = \int (v-u)\otimes(v-u) f \, dv \qquad q = \int \tfrac{1}{2}\|v-u\|^2 (v-u) f \, dv$$

SINCE $f = M[\rho, u, T] + O(\varepsilon)$

WE GET :

- $\mathbb{P} = \underbrace{\int (v-u)\otimes(v-u) M[\rho,u,T] \, dv}_{\rho T \, Id} + O(\varepsilon)$

- $q = \underbrace{\int \tfrac{1}{2}\|v-u\|^2 (v-u) M[\rho,u,T] \, dv}_{0} + O(\varepsilon)$

THIS GIVES.

$$\begin{cases} \partial_t \rho + \nabla \cdot \rho u = 0 \\ \partial_t \rho u + \nabla \cdot (\rho u \otimes u + p \, Id) = O(\varepsilon) \\ \partial_t E + \nabla \cdot (E u + p u) = O(\varepsilon) \end{cases}$$

- IF WE NEGLECT $O(\varepsilon)$ TERMS, WE GET THE EULER EQUATIONS

- IN OTHER WORDS: WHEN $\varepsilon \longrightarrow 0$

$$f \xrightarrow[\varepsilon \to 0]{} M[\rho, u, T]$$

$$(\rho, u, T) \xrightarrow[\varepsilon \to 0]{} \text{SOLUTION OF THE EULER EQUATIONS}$$

3.7 COMPRESSIBLE NAVIER-STOKES EQUATIONS OF FLUID MECHANICS

$$f = M[\rho, u, T] + \underbrace{O(\varepsilon)}_{\text{SMALL WHEN } \varepsilon \ll 1}$$

↳ THIS GIVES $1^{\underline{ST}}$ ORDER APPROXIMATION W.R.T ε

↳ CAN WE FIND A $2^{\underline{ND}}$ ORDER APPROXIMATION?

↳ YES (CHAPMANN-ENSKOG EXPANSION)

DECOMPOSE f AS : $f = M[\rho, u, T] + \varepsilon g$

(WHERE g IS ASSUMED TO BE AN $O(1)$)

WHAT WE WANT IS 2ND ORDER APPROXIMATIONS OF $\mathbb{P}$ AND q.

WITH $f = M[\rho, u, T] + \varepsilon g$ WE HAVE:

$$\mathbb{P} = \rho \, Id + \varepsilon \underbrace{\int (v-u) \otimes (v-u) \, g \, dv}_{\sigma_1 \text{ SHEAR STRESS TENSOR}}$$

$$q = 0 + \varepsilon \underbrace{\int \tfrac{1}{2} \|v-u\|^2 (v-u) \, g \, dv}_{q_1}$$

IDEA OF THE CHAPMAN-ENSKOG EXPANSION

↳ COMPUTE 1ST ORDER APPROXIMATIONS OF σ_1 AND q_1

$\Rightarrow$ GIVE 2ND ORDER APPROXIMATIONS OF $\mathbb{P}$ AND q.

INSERT $f = M(\rho, u, T) + \varepsilon g$ INSIDE THE BGK EQUATION

$$\left(\partial_t M + v \cdot \nabla_x M\right) + \varepsilon\left(\partial_t g + v \cdot \nabla_x g\right)$$

$$= \frac{1}{\cancel{\varepsilon}} \frac{1}{\tau}\left(-\cancel{\varepsilon} g\right) = -\frac{1}{\tau} g$$

$$\Rightarrow \quad g = -\tau\left(\partial_t M + v \cdot \nabla_x M\right.$$

$$\left. + \varepsilon \underbrace{\left(\partial_t g + v \cdot \nabla_x g\right)}_{\text{ASSUME THIS IS BOUNDED W.R.T } \varepsilon}\right)$$

$$\Rightarrow \quad \boxed{g = \underbrace{-\tau\left(\partial_t M + v \cdot \nabla_x M\right)}_{1^{\text{ST}} \text{ APPROXIMATION OF } g} + O(\varepsilon)}$$

THEREFORE :

$$\sigma_1 = \int (v-u)\otimes(v-u)\left[-\tau\left(\partial_t M + v\cdot\nabla_x M\right)\right]dv + O(\varepsilon)$$

$$q_1 = \int \frac{1}{2}\|v-u\|^2 (v-u)\left[-\tau\left(\partial_t M + v\cdot\nabla_x M\right)\right]dv + O(\varepsilon)$$

↳ 1^{st} ORDER APPROX OF σ_1 AND q_1

SINCE M DEPENDS ON ρ, u, T

THEN σ_1, q_1 DEPEND ONLY ON

$\rho, u, T, \partial_t\rho, \partial_t u, \partial_t T, \nabla\rho, \nabla u, \nabla T$

↳ WE GET THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

<u>NOTE</u>: WE GET

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0 \\ \partial_t \rho u + \nabla\cdot\Big(\rho u\otimes u + p\, Id + \varepsilon\sigma_1(\rho, u, T, \partial_t\rho, \partial_t u, \partial_t T, \nabla\rho, \nabla u, \nabla T)\Big) \\ \qquad = O(\varepsilon^2) \\ \partial_t E + \nabla\cdot\Big(Eu + p\,u \\ \qquad + \varepsilon\sigma_1(\text{---})\,u + \varepsilon q_1(\rho, \partial_t\rho, \nabla\rho, \dots)\Big) \\ \qquad = O(\varepsilon^2) \end{cases}$$

<u>QUESTION</u>: HOW TO REMOVE THESE TIME DERIVATIVES?

① $\partial_t \rho = -\nabla\cdot(\rho u)$ (EASY)

② $\partial_t u = \frac{1}{\rho}\underbrace{\partial_t \rho u}_{} - u\frac{\partial_t \rho}{\rho}$

$$= -\nabla\cdot(\rho u\otimes u + p\, Id) + O(\varepsilon)$$

$\Rightarrow$ THE TIME DERIVATIVES ARE ELIMINATED UP TO 2^{nd}-ORDER W.R.T ε.

EXERCISE: PROVE THAT

$$\sigma_1 = -\mu \left(\nabla u + (\nabla u)^T - \frac{2}{3} \nabla \cdot u \right) + O(\varepsilon)$$

$$q_1 = -\kappa \, \nabla T + O(\varepsilon)$$

WHERE $\mu = p\,\tau$ AND $\kappa = \frac{5}{2} p\,\tau$

VISCOSITY COEFF. HEAT CONDUCTIVITY.

N.B: NOT DIFFICULT

BUT VERY LONG AND TEDIOUS