SHARP LARGE DEVIATIONS FOR HYPERBOLIC FLOWS

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Abstract. For hyperbolic flows φ_t we examine the Gibbs measure of points w for which

$$\int_0^T G(\varphi_t w)dt - aT \in (-e^{-\epsilon n}, e^{-\epsilon n})$$

as $n \to \infty$ and $T \geq n$, provided $\epsilon > 0$ is sufficiently small. This is similar to local central limit theorems. The fact that the interval $(-e^{-\epsilon n}, e^{-\epsilon n})$ is exponentially shrinking as $n \to \infty$ leads to several difficulties. Under some geometric assumptions we establish a sharp large deviation result with leading term $C(a)\epsilon_n e^{\gamma(a)T}$ and rate function $\gamma(a) \leq 0$. The proof is based on the spectral estimates for the iterations of the Ruelle operators with two complex parameters and on a new Tauberian theorem for sequence of functions $g_n(t)$ having an asymptotic as $n \to \infty$ and $t \geq n$.

1. Introduction

Let $\varphi_t : M \longrightarrow M$ be a C^2 weak mixing Axiom A flow on a compact Riemannian manifold M, and let Λ be a basic set for φ_t. The restriction of the flow on Λ is a hyperbolic flow [11]. For any $x \in M$ let $W^s_\epsilon(x), W^u_\epsilon(x)$ be the local stable and unstable manifolds through x, respectively (see [2], [7], [11]). It follows from the hyperbolicity of Λ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\varphi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x, y] \in \Lambda$ (cf. [7]). That is, there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\varphi_t([x, y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x, y) = t$, defines the so called *temporal distance function*. Here and throughout the whole paper we denote by $d(\cdot, \cdot)$ the *distance* on M determined by the Riemannian metric.

Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a fixed *(pseudo) Markov family* of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$ (see Section 2). Set $R = \cup_{i=1}^k R_i$, $U = \cup_{i=1}^k U_i$. Consider the *Poincaré map* $\mathcal{P} : R \longrightarrow R$, defined by $\mathcal{P}(x) = \varphi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\varphi_{\tau(x)}(x) \in R$ (*first return time* function). The *shift map* $\sigma : R \longrightarrow U$ is given by $\sigma = \pi_U \circ \mathcal{P}$, where $\pi_U : R \longrightarrow U$ is the *projection* along stable leaves.

Define a $(k \times k)$ matrix $A = \{A(i, j)\}_{i,j=1}^k$ by

$$A(i,j) = \begin{cases} 1 \ \text{ if } \mathcal{P}(\text{Int } R_i) \cap \text{Int } R_j \neq \emptyset, \\ 0 \ \text{ otherwise.} \end{cases}$$

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Following [2], it is possible to construct a Markov family $\mathcal{R}$ so that A is irreducible and aperiodic.

Consider the suspension space

$$R^\tau = \{(x,t) \in R \times \mathbb{R} : \ 0 \leq t \leq \tau(x)\}/\sim,$$

where by $\sim$ we identify the points $(x, \tau(x))$ and $(\mathcal{P}x, 0)$. The suspension flow on R^τ is defined by $\varphi_t^\tau(x,s) = (x, s+t)$ taking into account the identification $\sim$. For a Hölder continuous function f on R, the *topological pressure* $\mathrm{Pr}_{\mathcal{P}}(f)$ with respect to $\mathcal{P}$ is defined by

$$\mathrm{Pr}_{\mathcal{P}}(f) = \sup_{m \in \mathcal{M}_{\mathcal{P}}} \big\{ h(\mathcal{P}, m) + \int f dm \big\},$$

where $\mathcal{M}_{\mathcal{P}}$ denotes the space of all $\mathcal{P}$-invariant Borel probability measures and $h(\mathcal{P}, m)$ is the entropy of $\mathcal{P}$ with respect to m. We say that u and v are *cohomologous* and we denote this by $u \sim v$ if there exists a continuous function w such that $u = v + w \circ \mathcal{P} - w$. The flow φ_t on Λ is naturally related to the suspension flow φ_t^τ on R^τ. There exists a natural semi-conjugacy projection $\pi(x,t) : R^\tau \longrightarrow \Lambda$ which is one-to-one on a residual set (see [2]) such that $\pi(x,t) \circ \varphi_s^\tau = \varphi_s^\tau \circ \pi(x,t)$. For $z \in R$ set

$$\tau^n(z) := \tau(z) + \tau(\mathcal{P}(z)) + ... + \tau(\mathcal{P}^{n-1}(z)).$$

Notice that since $\tau(x)$ is constant along stable leaves for $x = \pi_U(z)$ we have

$$\tau^n(z) = \tau^n(x) = \tau(x) + \tau(\sigma(x)) + ... + \tau(\sigma^n(x)).$$

Denote by $\widehat{U}$ (or $\widehat{R}$) the set of those $x \in U$ (resp. $x \in R$) such that $\mathcal{P}^m(x)$ does not belong to the bounder of any rectangle R_i for all $m \in \mathbb{Z}$. In a similar way define $\widehat{R}^\tau$. It is well-known (see [1]) that $\widehat{U}$ (resp. $\widehat{R}$) is a residual subset of U (resp. R) and has full measure with respect to any Gibbs measure on U (resp. R). Clearly in general τ is not continuous on U, however τ is *essentially Hölder* on U, i.e. there exist constants $L > 0$ and $\alpha > 0$ such that $|\tau(x) - \tau(y)| \leq L\,(d(x,y))^\alpha$ whenever $x, y \in U_i$ and $\sigma(x), \sigma(y) \in U_j$ for some i, j. The same applies to $\sigma : U \longrightarrow U$ and to $\mathcal{P} : R \longrightarrow R$. Throughout we will mainly work with the restrictions of τ and σ to $\widehat{U}$ and also with the restrictions of τ and $\mathcal{P}$ to $\widehat{R}$.

Consider the space $C^\alpha(\widehat{R}^\tau)$ of all α-Hölder functions on $\widehat{R}^\tau$ with norm $\|w\|_\alpha = |w|_\alpha + \|w\|_\infty$. We should stress that throughout the whole paper the Hölder norms $|w|_\alpha$ for functions on $\widehat{U}$, $\widehat{R}$ or $\widehat{R}^\tau$ are always determined with respect to distance $d(\cdot,\cdot)$ on M determined by the Riemannian metric.

For $F \in C^\alpha(\widehat{R}^\tau)$, define the function $f : R \longrightarrow \mathbb{R}$ by

$$f(z) = \int_0^{\tau(z)} F(z,t)\; dt, \quad z \in R.$$

Here and in the following we use the notation $F(z,t) = F(\sigma_t^\tau(z,0))$. Given a function $G \in C(\widehat{R}^\tau)$, we define

$$G^T(w) = \int_0^T G(\varphi_t^\tau(w))\; dt, \quad w \in R^\tau.$$

Throughout the paper we assume that $F, G \in C^\alpha(\widehat{R}^\tau)$. Let $a = \int_{R^\tau} G dm_{F+tG}$, where m_{F+tG} is the equilibrium state of $F + tG$ for some $t \in \mathbb{R}$. More precisely, for a function A on $\widehat{R}^\tau$ the equilibrium state m_A of A is a σ_t^τ-invariant probability measure on R^τ such that

$$\Pr_{\varphi_t^\tau}(A) = \sup_{m \in \mathcal{M}_\tau} \{h(\varphi_1^\tau, m) + \int A(w)\, dm(w)\},$$

$h(\varphi_1^\tau, m)$ being the *entropy* of φ_1^τ with respect to m and $\mathcal{M}_\tau$ the space of all φ_t^τ invariant Borel probability mesures on R^τ. The supremum above is given by a measure m_A called equilibrium state of A. Moreover, if G is not cohomologous to a constant, we have

$$0 < \sigma_m^2(G) = \lim_{T \to \infty} \frac{1}{T} \left(\int_0^T (G \circ \sigma_t^\tau) dt - T \int G dm \right)^2 < \infty$$

and $\frac{d^2 \Pr(F+tG)}{dt^2} = \sigma^2_{m_{F+tG}}(G)$, m_H being the *equilibrium state* of H.

Introduce the rate function

$$\gamma(a) := \inf_{t \in \mathbb{R}} \{\Pr(F + tG) - \Pr(F) - ta\} = \Pr(F + \xi(a)G) - \Pr(F) - \xi(a)a,$$

where $\xi(a)$ is the unique real number such that

$$\left.\frac{d\Pr(F + tG)}{dt}\right|_{t=\xi(a)} = \int G\, dm_{F+\xi(a)G} = a$$

and $\Pr(A) = \Pr_{\sigma_t^\tau}(A)$ is the *pressure* of A with respect to the flow σ_t^τ on R^τ.

For simplicity of the notations we will write $\Pr(A)$ instead of $\Pr_{\varphi_t^\tau}(A)$. Let

$$\beta(t) = \Pr(F + tG) - \Pr(F).$$

Clearly,

$$\gamma'(a) = \beta'(\xi(a))\xi'(a) - \xi'(a)a - \xi(a) = -\xi(a),$$

and

$$\xi'(a) = \frac{1}{\beta''(\xi(a))} = \frac{1}{\sigma^2_{m_{F+\xi(a)G}}(G)}.$$

Consequently, $\gamma'(a) = 0$ if and only if $\xi(a) = 0$ which is equivalent to $\int G\, dm_F = a$. Thus $\gamma(a)$ is a *non-positive* concave function and $\gamma(a) = 0$ if only if $\int G\, dm_F = a$.

In this paper we continue the analysis of sharp large deviations in [12], [14]. Our purpose is to improve the results in [14] and to study for a fixed $q \geq 0$ the asymptotic of

$$m_F \Big\{ w \in R^\tau : \forall T \geq n - q \text{ we have } \int_0^T G(\varphi_t^\tau(w)) dt - aT \in \left(-e^{-\epsilon n}, e^{-\epsilon n}\right) \Big\}$$

as $n \to \infty$. Here $0 < \epsilon \leq \mu_0/8$ is a small constant, where $\mu_0 > 0$ defined in Section 4 is related to the meromorphic continuation of the function $Z(s, \omega, a)$ across the line $\operatorname{Re} s = \gamma(a)$.

Recall the subset $\widehat{U}$ of $\widehat{R}$ defined earlier. We will regard $\widehat{U}$ as a subset of $\widehat{R}^\tau$ using the identification $x \longleftrightarrow (x, 0)$ for $x \in R$. Now we introduce two definitions of independence.

Definition 1.1. Two functions $f_1, f_2 \in C^\alpha(\widehat{U})$ are called σ-independent if whenever there are constants $t_1, t_2 \in \mathbb{R}$ such that $t_1 f_1 + t_2 f_2$ is cohomologous to a function in $C(\widehat{U} : 2\pi\mathbb{Z})$, we have $t_1 = t_2 = 0$.

For a function $G \in C^\alpha(\widehat{R}^\tau)$ consider the skew product flow S_t^G on $\mathbb{S}^1 \times R^\tau$ defined by

$$S_t^G(e^{2\pi \mathbf{i}\alpha}, y) = \Big(e^{2\pi \mathbf{i}(\alpha + G^t(y))}, \varphi_t^\tau(y)\Big).$$

Definition 1.2 ([10])**.** Let $G \in C^\alpha(\widehat{R}^\tau)$. Then G and φ_t^τ are *flow independent* if the following condition is satisfied. If $t_0, t_1 \in \mathbb{R}$ are constants such that the skew product flow S_t^H with $H = t_0 + t_1 G$ is not topologically transitive, then $t_0 = t_1 = 0$.

Following the result in [10], if G and φ_t^τ are flow independent, then the flow φ_t^τ is topologically weak-mixing and the function G is not cohomologous to a constant function. On the other hand, if G and φ_t^τ are flow independent, then $g(x) = \int_0^{\tau(x)} G(x,t)dt$, $x \in U$ and $\tau(x)$ are σ-independent.

Introduce the set

$$\Gamma_G = \Big\{ \int G dm_{F+tG} : \; t \in \mathbb{R}\Big\}.$$

Our first result is the following

Theorem 1.3. *Assume that the Standing Assumptions stated in Section* 2 *below are satisfied. Let* $G : \Lambda \longrightarrow (0, \infty)$ *be a Hölder function function for which there exists a Markov family* $\mathcal{R} = \{R_i\}_{i=1}^k$ *for the flow* φ_t *on* Λ *such that* G *is constant on the stable leaves of all "rectangular boxes"*

$$B_i = \{\varphi_t(x) : x \in R_i, 0 \le t \le \tau(x)\},$$

$i = 1, \dots, k$. *Assume in addition that* G *and* φ_t^τ *are flow independent. Let* $q \ge 0$ *be fixed, let* $0 < \mu_0 < 1$ *be the constant in Proposition* 3 *and let* $\epsilon_n = e^{-\epsilon n}$, $0 < \epsilon \le \mu_0/8$. *Then for any compact set* $J \Subset \Gamma_G$ *and* $0 < \eta \ll 1$ *there exists* $n_0(\eta) \in \mathbb{N}$ *such that for* $a \in J$, $n \ge n_0(\eta) + q$ *and* $T \ge n_0(\eta) - q$ *we have*

$$\frac{\sqrt{2}\epsilon_n C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a)T}(1-\eta) \le m_F\Big\{w \in R^\tau : \int_0^T G(\varphi_t^\tau(w))\, dt - aT \in \Big(-e^{-\epsilon n}, e^{-\epsilon n}\Big)\Big\}$$

$$\le \frac{\sqrt{2}\epsilon_n C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a)T}(1+\eta), \tag{1.1}$$

where $C(a) > 0$ *is a constant defined in Section* 6.

The above theorem says that for $T \ge n - q$ and $n \to \infty$ we have

$$m_F\Big\{w \in R^\tau : \int_0^T G(\varphi_t^\tau(w))\, dt - aT \in \Big(-e^{-\epsilon n}, e^{-\epsilon n}\Big)\Big\}$$

$$\sim \frac{\sqrt{2}\epsilon_n C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a)T}. \tag{1.2}$$

Notice that the proof of Theorem 1.3 works if we assume that G is not cohomologous to a constant and $\tau(x)$ and $g(x) = \int_0^{\tau(x)} G(x,t)dt$ are $\sigma-$ independent. As it was mentioned above, these properties are satisfied if G and φ_t^τ are flow independent.

Theorem 1.3 is an improvement of a result in [14], where the asymptotic of

$$\mu_f\Big\{x \in U : \int_0^{\tau^n(x)} G(\varphi_t^\tau(x,0))dt - a\tau^n(x) \in \Big(-e^{-\epsilon n}, e^{-\epsilon n}\Big)\Big\},\; n \to \infty$$

has been investigated, μ_f being the equilibrium state of $f(x)$. When we replace $\tau^n(x)$ by T, we have to study two limits: $n \to \infty$ and $T \to \infty$, and the condition $T \geq n - q$ is natural. It is easer to study the case when n is fixed and we take $T \geq T(\eta, n)$ to arrange (1.1). However assuming $n \geq n_0(\eta)$, for fixed $0 < \eta \ll 1$ we could have $\lim_{n\to\infty} T(\eta, n) = \infty$ and we cannot arrange an uniformity with respect to n. Thus, our result is much more precise and to prove it we follow a strategy based on Tauberian theorems with two parameters n, T examining the asymptotic of a sequence of functions $g_n(T)$. We discuss briefly this approach below.

Our second result concerns the function

$$\zeta(T; a) = m_F\Big\{w \in R^\tau : \int_0^T G(\varphi_t^\tau(w))\, dt - aT \in \Big(-e^{-\epsilon T}, -e^{-\epsilon T}\Big)\Big\}.$$

Applying Theorem 1, we prove the following

Theorem 1.4. *Under the assumptions of Theorem* 1.3*, for* $a \in J$ *and any* $0 < \eta \ll 1$ *there exists* $n_0(\eta) \in \mathbb{N}$ *such that for* $T \geq n_0(\eta) + 1$ *we have*

$$\frac{\sqrt{2}e^{-\epsilon}e^{-\epsilon T}C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a)T}(1-\eta) \leq \zeta(T; a) \leq \frac{\sqrt{2}e^{\epsilon}e^{-\epsilon T}C(a)}{\sqrt{\pi T \beta''(\xi(a))}} e^{\gamma(a)T}(1+\eta). \tag{1.3}$$

It is possible to obtain a slightly better result assuming one can generalise Theorem 1.3 for sequences $n_k \to \infty$ instead of a sequence of integers $n \to \infty$, however we are not going to discuss such generalisations.

The results of the type discussed above are known as local central limit theorems (LCLT) (see [4] for recent results and references). In particular, (LCLT) in a very general setting are studied in [4] and asymptotics of the form

$$m_F\Big\{w \in R^\tau : \int_0^T G(\varphi_t^\tau(w))dt - T\int G dm_F - c\sqrt{T} \in I\Big\} \sim \frac{\mathbf{g}(c)}{\sqrt{T}}\mathrm{Leb}(I),\ T \to \infty$$

are proved when I is a bounded interval in $\mathbb{R}$, $\mathbf{g}(c)$ is a Gaussian density and $\mathrm{Leb}(I)$ is the Lebesgue measure of I. The case considered in the present paper, where we deal with exponentially shrinking intervals $I_n \subset \mathbb{R}$ and we want to have an asymptotic as $n \to \infty$ and $T \to \infty$, is more difficult. Large deviations for Anosov flows have been examined by Waddington [20], where for the measure

$$m_F\Big\{w \in R^\tau : \int_0^T G(\sigma_t^\tau(w))\, dt - aT \in [c, d]\Big\}$$

it was obtained an asymptotic similar to (1.2) with leading term having the form

$$\int_c^d e^{-\xi(a)t}dt \frac{C(a)}{\sqrt{2\pi T\beta''(\xi(a))}} e^{\gamma(a)T}.$$

In [20] there are several points presented without proofs. In the exposition in [20] the case when G is constant along stable leaves is treated, while in the general case no argument is provided. This gap is essential since for large deviations, applying a reduction based on Proposition 2.2 (see Section 2), new terms appear in the analysis of the Laplace transform when we work with the iterations of Ruelle operators. A second gap is related to Proposition 6(ii) in [20] which is also presented without proof. This Proposition concerns a Tauberian theorem for a nonnegative function which is not monotonic. In general, without a slowly decreasing condition (see

Section 10 in [9]), or without some condition on the growth of the derivative, it is not known if the result is true.

In the analysis of the large deviations in the case when a fixed interval $[c, d]$ is replaced by an exponentially shrinking interval $(-e^{-\epsilon n}, e^{-\epsilon n})$, several additional difficulties appear. In two previous papers [12], [14] some partial cases have been treated, but in these papers we have considered only limits $n \to \infty$ (see also [16], where the case of an interval $(-n^{-\kappa}, n^{-\kappa})$ with suitable $\kappa > 1$ has been studied). In this paper we deal with two limits $n \to \infty$, $T \geq n - q$. Our approach is based on spectral estimates for the iterations of a Ruelle operator

$$\mathcal{L}_{s,\omega,a} = \mathcal{L}_{f - s\tau + (\xi(a) + \mathbf{i}\omega)(g - a\tau)},\ s \in \mathbb{C},\ \omega \in \mathbb{R}$$

with *two complex parameters* s and $\mathbf{i}\omega$ which may have modulus going to ∞. We exploit the estimates obtained in [13], [14] (see Theorem 2.1) in order to obtain an analytic continuation of the Laplace transform $F_n(s)$ of the function $g_n(T)$ defined below for $|\operatorname{Im} s| \geq M$ and $|\omega| \geq \epsilon_0$. We establish the existence of an analytic continuation of $F_n(s)$ across the line $\operatorname{Re} s = \gamma(a)$ for $\gamma(a) - \mu_0 \leq \operatorname{Re} s$, $|\operatorname{Im} s| \geq M \gg 1$. This continuation and the corresponding estimates play a crucial role in the new type Tauberian theorems concerning double limits $n \to \infty$, $T \to \infty$. These Tauberian theorems are of independent interest.

For convenience of the reader we explain briefly the idea of the proof of Theorem 1.3. Let $\chi(t) \in C_0^\infty(\mathbb{R} : \mathbb{R}^+)$ be a nonnegative cut-off function and let

$$G^T(w) = \int_0^T G(\varphi_t(w))dt,\ w \in R^\tau.$$

Set $\chi_n(t) = \chi\Big(\frac{t}{\epsilon_n}\Big)$.We study the function

$$\begin{aligned} g_n(T) &:= \epsilon_n e^{-\gamma(a)T + T} \int_{R^\tau} \chi\Big(\frac{G^T - aT}{\epsilon_n}\Big)(w) dm_F(w) \\ &= \frac{\epsilon_n^2}{2\pi} e^{-\gamma(a)T + T} \int_{R^\tau} \int_{\mathbb{R}} e^{\mathbf{i}\omega(G^T - aT)(w)} \hat{\chi}(\epsilon_n \omega) d\omega dm_F(w), \end{aligned}$$

where $\hat{\chi}_n(\omega) = \epsilon_n \hat{\chi}(\epsilon_n \omega)$ is the Fourier transform of $\chi_n(t)$. We extend this function as 0 for $T < 0$ and examine the Laplace transform

$$F_n(s) = \frac{\epsilon_n^2}{2\pi} \int_{\mathbb{R}} \Big[\int_0^\infty e^{-sT - \gamma(a)T + T} \Big(\int_{R^\tau} e^{\mathbf{i}\omega(G^T - aT)(w)} dm_F(w)\Big) dT\Big] \hat{\chi}(\epsilon_n \omega) d\omega.$$

Our purpose is to prove that for fixed $q \geq 0$, as $n \to \infty$, $T \geq n - q$, we have

$$g_n(T) \sim \frac{\sqrt{2} C(a) \epsilon_n^2}{\sqrt{\pi \beta''(\xi(a)) T}} e^T \tag{1.4}$$

which yields the asymptotic (1.2). The factor ϵ_n in $g_n(T)$ is involved to have an independent of n bound for the derivative $g_n'(T)$ (see Proposition 5.6 and Lemma 5.7 in Section 5). Let

$$Z(s, \omega, a) = \int_0^\infty e^{-(s + \gamma(a) - 1)T} \Big(\int_{R^\tau} e^{\mathbf{i}\omega(G^T - aT)w} dm_F(w)\Big) dT.$$

For fixed a the function $Z(s, \omega, a)$ depends on two complex parameters $s \in \mathbb{C}$ and $\mathbf{i}\omega$. Moreover, in $Z(s, \omega, a)$ we have no integration with respect to ω. First we show that this function is analytic for $\operatorname{Re} s > \gamma(a)$. Second we prove that in a small neighbourhood of $(\gamma(a), 0) \in \mathbb{C}^2$ this function has a pole $s(\omega, a)$ with residue $C(\omega, a) > 0$. To establish an analytic continuation across the line $\operatorname{Re} s = \gamma(a)$ for $|s - \gamma(a)| \geq \epsilon_0 > 0$, first we reduce the integration on R, and then by using the hypothesis that G is constant along stable leaves, we reduce once more the integration on U and write (see Section 3)

$$Z(s, \omega, a) = \int_U B_2(s, \omega, a, u) \sum_{m=0}^{\infty} \Big(\mathcal{L}^m_{f-s\tau+\mathbf{i}\omega(g-a\tau)} B_1(s, \omega, a, .)\Big)(u) h(u) d\nu(u),$$

$\mathcal{L}_{f-s\tau+\mathbf{i}\omega(g-a\tau)}$ being the Ruelle operator related to $f - s\tau + \mathbf{i}\omega(g - a\tau)$, where $f(x), g(x)$, $x \in U$, are determined by F, G, respectevely, and the measure $\nu(u)$ on U is determined by $f(u)$. We exploit the estimates for the iterations $\mathcal{L}^m_{f-s\tau+\mathbf{i}\omega(g-a\tau)}$ obtained in [14] (see Theorem 2.1 in Section 2) to obtain a meromorphic continuation of $Z(s, \omega, a)$ across $\operatorname{Re} s = \gamma(a)$.

The integration with respect to ω is not involved in the definition of $Z(s, \omega, a)$. Taking the inyegration in a small interval $[-\epsilon_0, \epsilon_0]$, writing $\hat{\chi}(\epsilon_n\omega) = \hat{\chi}(0) + \hat{\chi}'(0)\epsilon_n\omega + \mathcal{O}(\epsilon_n^2\omega^2)$, and repeating the calculus in [8], [20], we get

$$\epsilon_n^2 \int_{-\epsilon_0}^{\epsilon_0} \frac{\hat{\chi}(\epsilon_n\omega)}{s - 1 - s(\omega, a)} d\omega = \frac{C(a)\hat{\chi}(0)\epsilon_n^2}{\sqrt{2\beta''(\xi(a))(s-1)}} + \text{smoother terms}.$$

The leading term becomes $\frac{A_n}{\sqrt{s-1}}$ with $A_n = \mathcal{O}(\epsilon_n^2) = \mathcal{O}(e^{-2\epsilon n})$ and this difficulty corresponds to the type of Tauberain results proved in Section 5, where the remainder of the asymptotic has order $o(\epsilon_n^2)$.

The integral with respect to ω over $\mathbb{R}\backslash[-\epsilon_0, \epsilon_0]$ yields analytic functions, however we need precise estimates on theirs growth as $|\operatorname{Im} s| \to \infty$ independent on n in order to apply Proposition 5.6. For this purpose, applying the results of Section 4, we are going to estimate the integral

$$\epsilon_n^2 \int_{|\omega| \geq M \gg 1} (1 + |\operatorname{Im} s|^\nu + |\omega|^\nu) |\hat{\chi}(\epsilon_n\omega)| d\omega$$

with $0 < \nu < 1$, uniformly with respect to n. Here the presence of the factor ϵ_n^2 is crucial and our choice of ϵ_n in the definition of $g_n(T)$ is once more very convenient. To check the hypothesis of Proposition 5.6, we use in an essential way the analytic continuation of $Z(s, \omega, a)$ and its corresponding estimates.

The plan of the paper is as follows. In Section 2 we introduce some definitions and our Standing assumptions. A Sinai's lemma for the suspended flow is stated; it is proved in the Appendix. In Section 3 a representation of the Laplace transform $Z(s, \omega, a)$ of the function $g_n(T)$ is obtained. Section 4 is devoted to the analysis of the meromorphic continuation of $Z(s, \omega, a)$ based, as mentioned above, on the results in [14]. In Section 5 two Tauberain theorems are proved for a sequence of nonnegative functions $g_n(T)$. The novelty here is that these functions have singularities $\frac{A_n}{\sqrt{s-1}}$ with $0 < e^{-\mu n} \leq A_n \leq C_1$ and $0 < \mu \leq \mu_0/4$ and we have a double limit $n \to \infty$, $t \to \infty$. Theorems 1.3 and 1.4 are proved in Section 6.

By using Proposition 2.2, we can study also the general case when the function G is not constant on stable leaves. A part of Proposition 4.2 in Section 4 concerning the analytic continuation for $|\operatorname{Im} s| \geq M$ can be established. However, some new difficulties appear in the description of the singularities of $Z(s, \omega, a)$ given by Proposition 4.1 (ii). How to deal with this is an interesting open problem.

2. Definitions and Standing Assumptions

2.1. **Standing assumptions.** Throughout we use the notation and assumptions from the beginning of Section 1. In particular, $\varphi_t : M \longrightarrow M$ is a C^2 weak mixing Axiom A flow and Λ is a basic set for φ_t. As in [13] and [14], we will work under the following rather general non-integrability condition about the flow on Λ:

(LNIC): *There exist $z_0 \in \Lambda$, $\epsilon_0 > 0$ and $\theta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, any $\hat{z} \in \Lambda \cap W^u_\epsilon(z_0)$ and any tangent vector $\eta \in E^u(\hat{z})$ to Λ at $\hat{z}$ with $\|\eta\| = 1$ there exist $\tilde{z} \in \Lambda \cap W^u_\epsilon(\hat{z})$, $\tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z})$ with $\tilde{y}_1 \neq \tilde{y}_2$, $\delta = \delta(\tilde{z}, \tilde{y}_1, \tilde{y}_2) > 0$ and $\epsilon' = \epsilon'(\tilde{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon]$ such that*

$$|\Delta(\exp^u_z(v), \pi_{\tilde{y}_1}(z)) - \Delta(\exp^u_z(v), \pi_{\tilde{y}_2}(z))| \geq \delta \, \|v\|$$

for all $z \in W^u_{\epsilon'}(\tilde{z}) \cap \Lambda$ and $v \in E^u(z; \epsilon')$ with $\exp^u_z(v) \in \Lambda$ and $\langle \frac{v}{\|v\|}, \eta_z \rangle \geq \theta_0$, where η_z is the parallel translate of η along the geodesic in $W^u_{\epsilon_0}(z_0)$ from $\hat{z}$ to z.

Next, given $x \in \Lambda$, $T > 0$ and $\delta \in (0, \epsilon]$ set

$$B^u_T(x, \delta) = \{y \in W^u_\epsilon(x) : d(\varphi_t(x), \varphi_t(y)) \leq \delta \ , \ 0 \leq t \leq T\}.$$

We will say that φ_t has a *regular distortion along unstable manifolds* over the basic set Λ if there exists a constant $\epsilon_0 > 0$ with the following properties:

(a) For any $0 < \delta \leq \epsilon \leq \epsilon_0$ there exists a constant $R = R(\delta, \epsilon) > 0$ such that

$$\mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leq R \, \mathrm{diam}(\Lambda \cap B^u_T(z, \delta))$$

for any $z \in \Lambda$ and any $T > 0$.

(b) For any $\epsilon \in (0, \epsilon_0]$ and any $\rho \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\mathrm{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho \, \mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon))$.

Standing Assumptions:

(A) φ_t has Lipschitz local holonomy maps over Λ,

(B) the local non-integrability condition (LNIC) holds for φ_t on Λ,

(C) φ_t has a regular distortion along unstable manifolds over the basic set Λ.

In this paper we will work under the above Standing Assumptions[1].

A large class of examples satisfying the conditions (A) – (C) is provided by imposing the following *pinching condition*:

(P): *There exist constants $C > 0$ and $\beta \geq \alpha > 0$ such that for every $x \in M$ we have*

$$\frac{1}{C} \, e^{\alpha_x \, t} \, \|u\| \leq \|d\varphi_t(x) \cdot u\| \leq C \, e^{\beta_x \, t} \, \|u\| \quad , \quad u \in E^u(x) \ , t > 0$$

for some constants $\alpha_x, \beta_x > 0$ with $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in M$.

[1] These assumptions are needed to ensure the validity of certain strong spectral estimates for Ruelle transfer operators (see [14]). However recent developments in [19] suggest that similar estimates should be established in much higher generality. So, we expect that the methods of the present paper would apply without change under much more general assumptions.

It is well-known that **(P)** holds for geodesic flows on manifolds of strictly negative sectional curvature satisfying the so called $\frac{1}{4}$-pinching condition. **(P)** always holds when $\dim(M) = 3$.

Simplifying Assumptions: φ_t is a C^2 contact Anosov flow satisfying the condition **(P)**.

It follows from the results in [18], that the pinching condition **(P)** implies that φ_t has Lipschitz local holonomy maps and regular distortion along unstable manifolds and moreover:

THE SIMPLIFYING ASSUMPTIONS IMPLY THE STANDING ASSUMPTIONS.

Throughout this paper we work under the Standing Assumptions.

2.2. **Some definitions and Ruelle transfer operators.** As in Section 1, let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for φ_t over Λ consisting of rectangles $R_i = [U_i, S_i]$, where U_i (resp. S_i) are (admissible) subsets of $W^u_\epsilon(z_i) \cap \Lambda$ (resp. $W^s_\epsilon(z_i) \cap \Lambda$) for some $\epsilon > 0$ and $z_i \in \Lambda$. We will use the set-up and some arguments from [17] and [13]. As in these papers, fix a *(pseudo) Markov family* $\mathcal{R} = \{R_i\}_{i=1}^k$ of *pseudo-rectangles*

$$R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}.$$

Set

$$R = \cup_{i=1}^k R_i \quad , \quad U = \cup_{i=1}^k U_i.$$

Consider the *Poincaré map* $\mathcal{P} : R \longrightarrow R$, defined by $\mathcal{P}(x) = \varphi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\varphi_{\tau(x)}(x) \in R$ (*first return time* function). The *shift map* $\sigma : U \longrightarrow U$ is given by $\sigma = \pi_U \circ \mathcal{P}$, where $\pi_U : R \longrightarrow U$ is the *projection* along stable leaves.

Recall the subsets $\widehat{U}$ of U and $\widehat{R}$ of R introduced in Sect. 1. Throughout $\alpha > 0$ will be **fixed constant** such that $\tau \in C^\alpha(\widehat{U})$. We assume in Theorem 1 that $F, G \in C^\alpha(\widehat{R}^\tau)$. If $\tau \in C^{\tilde{\alpha}}(\widehat{U})$ for some $0 < \tilde{\alpha} < 1$, we may take $\alpha = \min\{\alpha, \tilde{\alpha}\}$, to arrange that F, G, τ are in the Hölder spaces with the same α. Fir simplicity of the notations we assume in the following that this is arranged. Since the local stable (and unstable) holonomy maps are uniformly Hölder ([5], [6]), we may assume α is chosen so that

$$d([x, y], [x', y]) \leq C(d(x, x'))^\alpha, \ x, x', y \in R_i \ , \ i = 1, \ldots, k. \tag{2.1}$$

The hyperbolicity of the flow implies the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma > 1$ such that

$$c_0 \gamma^m \ (d(x, y))^{1/\alpha} \leq d(\mathcal{P}^m(x)), \mathcal{P}^m(y)) \leq \frac{\gamma_1^m}{c_0} (d(x, y))^\alpha \tag{2.2}$$

for all $x, y \in R$ such that $\mathcal{P}^j(x), \mathcal{P}^j(y)$ belong to the same R_{i_j} for all $j = 0, 1, \ldots, m$. Moreover, we choose the constants so that if $y \in W^s(x)$ for some $x, y \in R_i$, then

$$d(\mathcal{P}^n(x), \mathcal{P}^n(y)) \leq \frac{c_0}{\gamma^n} \quad , \quad n \geq 0. \tag{2.3}$$

Fix a constant $\alpha' \in (0, \alpha]$ so that

$$\rho = \frac{\gamma_1^{\alpha'}}{\gamma} < 1.$$

Let $\|h\|_0$ denote the *standard* sup *norm* of h on U. For $|b| \geq 1$, and $\beta > 0$, as in [3], define the norm

$$\|h\|_{\beta,b} = \|h\|_\infty + \frac{|h|_\beta}{|b|}$$

on the space $C^\beta(\widehat{U})$ of β-Hölder functions on $\widehat{U}$.

As in [13] and [14], in this paper we will frequently use *Ruelle transfer operators* of the form

$$L_{f-s\tau+zg}v(y) = \sum_{\sigma x = y} e^{f(x)-s\tau(x)+zg(x)}v(x),\ s, z \in \mathbb{C},\ y \in U,$$

depending on two complex parameters s and z. The following theorem was proved in [14].

Theorem 2.1. *Let $\varphi_t : M \longrightarrow M$ satisfy the Standing Assumptions over the basic set Λ, and let $\alpha > \beta > 0$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for φ_t over Λ as above. Then for any real-valued functions $f, g \in C^\alpha(\widehat{U})$ and any constants $\nu > 0$ and $B > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \nu) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$ then*

$$\|L^m_{f-(a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g}h\|_{\beta,b} \leq C\, e^{m\mathrm{Pr}_\sigma(f)}\, \rho^m\, |b|^\nu\, \|h\|_{\beta,b} \tag{2.4}$$

for all $h \in C^\beta(U)$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B\,|b|$.

2.3. **Sinai's lemma for suspension flows.** Given functions G and $\tilde{G}$ on $\widehat{R}^\tau$, define

$$g(x) = \int_0^{\tau(x)} G(x,t)\,dt,\ \tilde{g}(x) = \int_0^{\tau(x)} \tilde{G}(x,t)\,dt$$

for all $x \in \widehat{R}$. Here $G(x,t) = G(\pi(x,t)) = G(\sigma_t^\tau(x,0))$. It is easy to see that if $G \in C^\alpha(\widehat{R}^\tau)$, then $g \in C^\alpha(\widehat{R})$.

Consider the function (defined as in the proof of Proposition 1.2 in [PP])

$$p(x) = \sum_{n=0}^\infty \left[g(\mathcal{P}^n(x)) - g(\mathcal{P}^n(\pi_U x)))\right], \quad x \in \widehat{R}. \tag{2.5}$$

Since $x, \tilde{x} = \pi_U(x)$ belong to the same stable leaf in $\widehat{R}$, (2.3) implies

$$d(\mathcal{P}^n(x), \mathcal{P}^n(\pi_U(x))) \leq \frac{c_0}{\gamma^n}$$

for all $n \geq 0$. Thus, the series in (2.5) is convergent. Now define

$$\begin{aligned}\tilde{G}(x,t) &= G(\pi_U(x), t)\\ &+ \sum_{n=0}^\infty \Big(G(\mathcal{P}^{n+1}(\pi_U(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\\ &\quad - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\Big)\end{aligned}$$

for $x \in \widehat{R}$ and $0 \leq t < \tau(x)$. Notice that since $\tau(x)$ is constant on stable leaves, we have

$$\tau(\mathcal{P}^{n+1}(x)) = \tau(\mathcal{P}^n(\pi_U(\mathcal{P}(x))).$$

The following is the analogue of the well-known Sinai's lemma (see e.g. Proposition 1.2 in [11]) for suspension flows.

Proposition 2.2. (a) *The function* p *defined above belongs to* $C^\beta(\widehat{R})$ *for some* $\beta > 0$ *and*

$$g(x) = \tilde{g}(x) + p(x) - p(\mathcal{P}(x)) \tag{2.6}$$

for all $x \in \widehat{R}$. *Moreover* $\tilde{G}$ *is constant on stable leaves of* $\widehat{R}^\tau$ *and is* β*-Hölder, where*

$$\beta = \alpha^2\alpha'/2 > 0. \tag{2.7}$$

(b) *The function*

$$P(x,t) = \sum_{n=0}^{\infty} [G(\mathcal{P}^n(x), t\,\tau(\mathcal{P}^n(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(x)), t\,\tau(\mathcal{P}^n(x))/\tau(x))] \tag{2.8}$$

$(x \in \widehat{R},\ 0 \le t < \tau(x))$ *is also* β*-Hölder on* $\widehat{R}^\tau$,

$$p(x) = \int_0^{\tau(x)} P(x,t)\,dt, \quad x \in \widehat{R}, \tag{2.9}$$

and

$$G(x,t) = \tilde{G}(x,t) + P(x,t) - P(\mathcal{P}(x), t\,\tau(\mathcal{P}(x))/\tau(x)) \tag{2.10}$$

for all $x \in \widehat{R}$ *and* $0 \le t < \tau(x)$.

We prove Proposition 2.2 in the Appendix.

Remark 2.3. (a) It follows from the definition of g that it is α-Hölder with $|g|_\alpha \le C|G|_\alpha$. Then (2.5) and $d(\mathcal{P}^n(x), \mathcal{P}^n(\pi_U(x)) \le c_0/\gamma^n$, which follows from (2.3), imply

$$|p(x)| \le \sum_{n=0}^{\infty} C|G|_\alpha\, c_0/\gamma^n \le C\,|G|_\alpha$$

for all $x \in R$, so $|p|_\infty \le C\,|G|_\alpha$. Similarly, $|P|_\infty \le C\,|G|_\alpha$.

(b) Given $y \in R_i$ for some i, consider the function $w_y(x) = h([x,y])$ on R_i. Now (2.1) implies

$$|w_y(x) - w_y(x')| \le |h|_\alpha\,(d([x,y],[x',y]))^\alpha \le C|h|_\alpha\,(d(x,x'))^{\alpha^2}.$$

Thus, $w_y \in C_{\alpha^2}$ and $|w_y|_{\alpha^2} \le C\,|h|_\alpha \le C\,|g|_\alpha$. This can be written as

$$|p([\cdot,y])|_{\alpha^2} \le C\,|G|_\alpha,\ y \in R_i\ ,\ i = 1,\dots,k.$$

By (2.7) , $\beta < \alpha^2$, so the above is also true with α^2 replaced by β. With (a) this gives $\|h([\cdot,y])\|_{\alpha^2} \le C\,|G|_\alpha$ and so

$$\|p([\cdot,y])\|_\beta \le C\,|G|_\alpha,\ y \in R_i\ ,\ i = 1,\dots,k. \tag{2.11}$$

(c) (Following Ch. 1 in [2]; in particular sections 1B and 1C)

Let $\tilde{f} : R \longrightarrow \mathbb{R}$ depend only on $x \in U$, i.e. $\tilde{f}(x) = \tilde{f}(x')$ whenever $\pi_U(x) = \pi_U(x')$. Then we can regard $\tilde{f}$ as a function on U, $\tilde{f} \in C^\beta(U)$, and by the Ruelle-Perron-Frobenius Theorem there exist (unique) positive function $h_0 \in C^\beta(U)$ and a probability measure ν on U such that $(\mathcal{L}_{\tilde{f}})^*\nu = \nu$ and $\int_U h_0\,d\nu = 1$. Then

$$dm_{\tilde{f}}(x) = h_0(x)\,\nu(x) \tag{2.12}$$

is a σ-invariant probability measure on U, called the Gibbs measure determined by $\tilde{f}$. It gives rise to a $\mathcal{P}$-invariant probability measure m on R as follows. Given a continuous real valued function w on R, define

$$w^*(x) = \min\{w(y) : y \in W^s_R(x)\},\ x \in R.$$

Then $w^* \in C(R)$ and w^* is constant on stable leaves of R, so it can be considered as a function in $C(U)$. Define

$$\int_R w(z)\, d\mathsf{m}(z) = \int_U w^*(x)\, dm_{\tilde{f}(x)}.$$

As in section 1C in [2], one checks that this defines a $\mathcal{P}$-invariant probability measure m on R. Usually one denotes $\mathsf{m} = dm_{\tilde{f}}$ and calls this the *Gibbs measure* determined by $\tilde{f}$ on R. The above definition shows that if $w \in C(R)$ depends only on $x \in U$, i.e. $w(x) = w(x')$ whenever $\pi_U(x) = \pi_U(x')$, then we have

$$\int_R w(z)\, d\mathsf{m}(z) = \int_U w(x)\, dm_{\tilde{f}}(x).$$

(d) To estimate $\|e^{p([\cdot:y])}h_0\|_\beta$, first we have

$$|e^{p([\cdot,y])}h_0|_\infty \leq e^{C|G|_\alpha}|h_0|_\infty \leq Ce^{C|G|_\alpha}.$$

Using (a),

$$|e^{p([\cdot:y])}|_\beta \leq e^{C|G|_\alpha}\ |p([\cdot,y])|_\beta \leq C|G|_\alpha\ e^{C|G|_\alpha}.$$

This implies

$$|e^{p([\cdot,y])}h_0|_\beta \leq |e^{p([\cdot:y])}|_\infty\, |h_0|_\beta + |e^{p([\cdot:y])}|_\beta\, |h_0|_\infty \leq C,$$
$$e^{C|G|_\alpha} + C|G|_\alpha\ e^{C|G|_\alpha} \leq C|G|_\alpha\ e^{C|G|_\alpha}.$$

Combining the above estimates, yields

$$\|e^{p([\cdot,y])}h_0\|_\beta \leq C|G|_\alpha\ e^{C|G|_\alpha}$$

and this estimate is uniform in $y \in R$.

2.4. **Application of Proposition 2.2.** By Proposition 2.2 there exist functions $\tilde{F}(w,t)$ and $Y(w,t)$ such that

$$F(w,t) = \tilde{F}(w,t) + Y(w,t) - Y\Big(\mathcal{P}(w), \frac{t\tau(\mathcal{P}(w))}{\tau(w)}\Big),$$

where $\tilde{F}(w,t)$ is constant on stable leaves. Let

$$\tilde{f}(w) = \int_0^{\tau(w)} \tilde{F}(w,t)dt, \quad y(w) = \int_0^{\tau(w)} Y(w,t)dt.$$

By a change of variables $t\frac{\tau(\mathcal{P}(w))}{\tau(w)} = s$, one deduces

$$\int_0^{\tau(w)} Y\Big(\mathcal{P}(w), \frac{t\tau(\mathcal{P}(w))}{\tau(w)}\Big)dt = y(\mathcal{P}(w)).$$

Now for the equilibrium state m_F one obtains

$$m_F = \frac{1}{\int \tau(u)d\mu_{\tilde{f}}}\Big(\mu_{\tilde{f}}(w) \times l\Big)$$

since

$$\mu_{f(w)} = \mu_{\tilde{f}(w)+y(w)-y(\mathcal{P}(w))} = \mu_{\tilde{f}}(w).$$

Hear μ_q denote the equilibrium state of q which is a probability measure invariant with respect to $\mathcal{P}$ and l is the Lebesgue measure on $\mathbb{R}$. Obviously, $\tilde{f}$ depends only on $x = \pi_U(w) \in U$. Moreover, adding a constant, we preserve m_F and can arrange $\Pr(F) = 0$. Since

$$\Pr_{\mathcal{P}}(f(w) - \Pr(F)\tau) = 0,$$

this implies

$$0 = \Pr_{\mathcal{P}}(\tilde{f}(w) + y(w) - y(\mathcal{P}(w)) = \Pr_{\mathcal{P}}(\tilde{f}(w)).$$

We may express the pressure by

$$\Pr_{\mathcal{P}}(\tilde{f}) = \lim_{n\to\infty} \log \sum_{\mathcal{P}^n w = w} e^{\tilde{f}^n(w)}.$$

Since $\tilde{f}(w)$ depends only on $x = \pi_U w$ and $\mathcal{P}^n w = w$ implies $\sigma^n x = x$, one deduces $\Pr_{\mathcal{P}}(\tilde{f}(x)) = \Pr_\sigma(\tilde{f}(x)) = 0$. Therefore the Ruelle operator $\mathcal{L}_{\tilde{f}}$ has 1 as an eigenvalue with eigenfunction $h(x) > 0$. Moreover, we have $d\mu_{\tilde{f}}(x) = h(x)d\nu(x)$ and $\nu(x)$ is a σ-invariant measure on U which can be considered as a $\mathcal{P}$-invariant measure on R as we have mentioned above. For this measure we have

$$(\mathcal{L}^*_{\tilde{f}})^n \nu(x) = \nu(x),\ n \geq 1. \tag{2.13}$$

3. Representation of the function $Z(s, \omega, a)$

Let $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R}^+)$ be a fixed cut-off function. Set $q_n(t) = e^{\xi(a)t}\chi_n(t)$, where

$$\chi_n(t) = \chi\Big(\frac{t}{\epsilon_n}\Big),\ \epsilon_n = e^{-\epsilon n},\ 0 < \epsilon \leq \mu_0/8,$$

$\mu_0 > 0$ being the constant introduced in Proposition 4.2 in Section 4.

The Fourier transform of χ_n satisfies $\widehat{\chi_n}(\omega) = \epsilon_n\, \hat{\chi}(e_n\, \omega)$. Given a continuous function Q on R^τ, consider

$$Q^T(\tilde{w}) = \int_0^T Q(\varphi_t^\tau(\tilde{w}))dt,\ \tilde{w} \in R^\tau.$$

Notice that we have

$$\Pr_{\mathcal{P}}(q - \Pr(Q)\tau) = 0$$

if $q(w) = \int_0^{\tau(w)} Q(w, t)dt,\ w \in R$ (see [11]).

To establish Theorem 1.3, we will study the nonnegative function

$$\rho_n(T) := \int_{R^\tau} q_n((G^T - aT)(y))dm_F(y).$$

We have

$$\begin{aligned}\rho_n(T) = \int_{R^\tau} q_n((G-a)^T(y))\, dm_F(y) &= \int_{R^\tau} e^{\xi(a)(G-a)^T(y)}\, \chi_n((G-a)^T(y))\, dm_F(y)\\ &= \frac{1}{2\pi}\int_{R^\tau}\int_{\mathbb{R}} e^{\xi(a)(G-a)^T(y)}\, e^{\mathbf{i}\omega(G-a)^T(y)}\, \hat{\chi}_n(\omega)\, dm_F(y)\, d\omega\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\int_{R^\tau} e^{z\,(G-a)^T(y)}\, dm_F(y)\Big)\, \hat{\chi}_n(\omega)\ d\omega\ ,\end{aligned}$$

where $z = \xi(a) + \mathbf{i}\omega$.

Define $\Gamma_z(T)$ for $T \geq 0$ by

$$\Gamma_z(T) = \int_{R^\tau} e^{z\,(G-a)^T(y)}\, dm_F(y)$$

and $\Gamma_z(T) = 0$ for $T < 0$. Our purpose is to study the Laplace transform

$$Z(s, \omega, a) = \int_0^\infty e^{-sT}\, \Gamma_z(T)\, dT,\ s \in \mathbb{C},\ \omega \in \mathbb{R}. \tag{3.1}$$

Since for large $M_0 > 0$ we have $|\Gamma_z(T)| \leq Ce^{C_1 T}$, $\mathrm{Re}\, s > M_0$, with some constant $C_1 > 0$, the transformation $Z(s, \omega, a)$ exists for $\mathrm{Re}\, s > M_0$ uniformly with respect to $\omega \in \mathbb{R}$. In Section 4 we will show that $Z(s, \omega, a)$ has an analytic extension to

$$\{s \in \mathbb{C} :\ \gamma(a) - \mu_0 \leq \mathrm{Re}\, s, \omega \in \mathbb{R}\} \setminus \{(s(\omega, a) :\ |\omega| \leq \epsilon_0\},$$

with $\epsilon_0 > 0$ and $\mu_0 > 0$ sufficiently small. Here $s(\omega, a)$ is a simple pole described in Section 4. We use the notation $Z(s, \omega, a)$ since the Laplace transform depends on $s \in \mathbb{C}, \omega \in \mathbb{R}$ and a.

Set

$$f(w) = \int_0^{\tau(w)} F(w, t) dt,\ g(w) = \int_0^{\tau(w)} G(w, t) dt,\ w \in R.$$

We repeat the argument of Section 4 in [20] (see also [15]) to obtain a representation of $Z(s, \omega, a)$. For the equilibrium state m_F of F we apply the reduction in Subsection 2.4 and we obtain the measure $d\mu_{\tilde{f}}(x) = h(x)\nu(x)$, where $\tilde{f}(x)$ depends only on $x \in U$. For simplicity of the notation in the following we will denote $\tilde{f}(x)$ again by $f(x)$ and $\mu_{\tilde{f}}$ by μ_f. Moreover, in the following we assume that G **is constant on the stable foliations** in R^τ, so $g(x)$ depends only on $x \in U$.

Given a Hölder function $Q(w)$ on R^τ, we have

$$\int_{R^\tau} Q(w) dm_F(w) = \frac{\int_R \int_0^{\tau(x)} Q(x, \eta) d\eta d\mu_q(x)}{\int \tau d\mu},$$

where μ_q is the equilibrium state of $q(w) = \int_0^{\tau(w)} Q(w, t) dt$. Therefore, setting $Q = z(G - a)$, we obtain

$$Z(s, w, a) = \frac{1}{\int \tau d\mu} \int_0^\infty e^{-(s+az)T} \Big(\int_R \int_0^{\tau(x)} e^{zG^T(x,\eta)} d\eta d\mu(x) \Big) dT$$

$$= \frac{1}{\int \tau d\mu} \int_0^\infty e^{-(s+az)T} \Big(\int_U \int_0^{\tau(x)} e^{zG^{T+\eta}(x,0) - zG^\eta(x,0)} d\eta h(x) d\nu(x) \Big) dT.$$

Here we interpret the integral on R as an integral on U as we have mentioned in Remark 1(c) in Section 2. Given $T > 0, x \in U, 0 \leq \eta \leq \tau(x)$, there exists a unique choice of $n \geq 0$ and $0 \leq v < \tau(\sigma^n x)$ so that $T + \eta = v + \tau^n(x)$. Notice that when $x \in U$ changes the integer n may change but since

$$T - \tau(\sigma^n x) \leq \tau^n(x) = T + \eta - v \leq T + \tau(x),$$

we deduce that for fixed T, $T \geq T_0$, and all $x \in U$ there is only a finite number (depending on T) of possible choices for n.

For $T + \eta = \tau^n(x) + v$ one applies the formula

$$e^{G^{T+\eta}(x,0)} = \sum_{n=0}^\infty \int_0^{\tau(\sigma^n x)} e^{G^{v+\tau^n(x)}(x,0)} \delta(\eta + T - v - \tau^n(x)) dv,$$

where for fixed $x \in U$, $T \geq T_0$ only one term in the infinite sum is not vanishing (see [15], [20] for a similar argument). Then we may transform the integral

$$\frac{1}{\int \tau d\mu} \int_0^\infty e^{-(s+az)T} \Big(\int_U \int_0^{\tau(x)} e^{zG^{T+\eta}(x,0) - zG^\eta(x,0)} d\eta h(x) d\nu(x) \Big) dT$$

in the above expression for $Z(s, \omega, a)$, as in Section 4 in [20] to obtain the representation

$$Z(s,\omega,a) = \frac{1}{\int \tau d\mu} \sum_{n=0}^\infty \int_U e^{-(s+(\xi(a)+\mathbf{i}\omega)a)\tau^n(x) + (\xi(a)+\mathbf{i}\omega)g^n(x)}$$
$$\times B_1(s,\omega,a,\sigma^n(x)) B_2(s,\omega,a,x) h(x) d\nu(x),$$

where

$$B_1(s,\omega,a,x) = \int_0^{\tau(x)} \exp\Big(-(s+az)v + zG^v(x,0) \Big) dv,$$
$$B_2(s,\omega,a,x) = \int_0^{\tau(x)} \exp\Big((s+az)\eta - zG^\eta(x,0) \Big) d\eta.$$

We apply (2.13) and then use the adjoint of the Ruelle operator $\mathcal{L}_f^*$, noting that

$$\Big[\mathcal{L}_f^n \Big(e^{-(s+(\xi(a)+\mathbf{i}\omega)a)\tau^n + (\xi(a)+\mathbf{i}\omega))g^n} d \Big) \Big](y)$$
$$= \Big[\mathcal{L}^n_{f - s\tau + (\xi(a)+\mathbf{i}\omega)(g - a\tau)} d \Big](y).$$

Therefore, we conclude that

$$Z(s,\omega,a) = \frac{1}{\int \tau d\mu} \sum_{n=0}^\infty \int_U B_1(s,\omega,a,y)$$
$$\times \Big(\mathcal{L}^n_{f - (s\tau + (\xi(a)+\mathbf{i}\omega)(g - a\tau)} \Big[h(.) B_2(s,\omega,a,.) \Big] \Big)(y) d\nu(y). \tag{3.2}$$

4. Meromorphic extension of $Z(s, \omega, a)$

We assume $f(x)$ and $g(x)$, $x \in U$, fixed as in Section 3. Introduce the Ruelle operator

$$\mathcal{L}_{s,\omega,a} = \mathcal{L}_{f - s\tau + (\xi(a)+\mathbf{i}\omega)(g - a\tau)},\ s \in \mathbb{C},\ \omega \in \mathbb{R}.$$

It is easy to see that for $s = \gamma(a)$ and $\omega = 0$ we have

$$\mathrm{Pr}_{\mathcal{P}}(f + \xi(a)(g - a\tau) - \gamma(a)\tau) = 0.$$

Indeed,

$$\gamma(a) = \mathrm{Pr}\,(F + \xi(a)G) - \xi(a)a$$

and

$$\mathrm{Pr}_{\mathcal{P}}(f + \xi(a)g - \mathrm{Pr}\,(F + \xi(a)G)\tau) = 0.$$

Notice that there is an unique number t such that

$$\mathrm{Pr}_{\mathcal{P}}(f + \xi(a)(g - a\tau) - t\tau) = 0.$$

Since f, g, τ depend only on $x \in U$, as in Subsection 2.4 we deduce that

$$\mathrm{Pr}_\sigma(f + \xi(a)(g - a\tau) - \gamma(a)\tau) = 0.$$

Below we will write simply Pr instead of Pr_σ if there are no confusions.

Set $f_a = f + \xi(a)(g - a\tau)$ and consider the Ruelle operator $\mathcal{L}_{s,\omega,a}$, where $s = \gamma(a) + q + \mathbf{i}b,\ q \in \mathbb{R},\ b \in \mathbb{R},\ \omega \in \mathbb{R}$. Let

$$p(s, w, a) = f_a - s\tau + \mathbf{i}\omega(g - a\tau),\ s = \gamma(a) + q + \mathbf{i}b.$$

Since $\Pr p(\gamma(a), 0, a) = 0$, by a standard argument we may define $\Pr p(s, \omega, a)$ for (s, ω) in a small neighbourhood of $(\gamma(a), 0)$ in $\mathbb{C}^2$ (see [11]). Recall the following result proved in [20].

Proposition 4.1 (Proposition 4, [20])**.** *Let $G^\alpha(R^\tau)$ be a function such that G and φ_t^τ are flow independent. Assume that G is constant on stable leaves. Then*
(i) The function $Z(s, \omega, a)$ is analytic for $(s, \omega) \in \{s \in \mathbb{C} :\ \operatorname{Re} s > \gamma(a)\} \times \mathbb{R}$.
(ii) There exists an open neighbourhood W of $(\gamma(a), 0)$ in $\mathbb{C}^2$ such that for $(s, \omega) \in W$ we have

$$Z(s, \omega, a) = \frac{B_3(s, \omega, a)}{1 - \exp\Big(\Pr(p(s, \omega, a))\Big)} + J(s, \omega, a), \tag{4.1}$$

where

$$B_3(s, \omega, a) = \frac{1}{\int \tau d\mu} \int_R B_1(s, \omega, a, .) h_{p(s,\omega,a)}(x) d\nu(x) \int h B_2(s, \omega, a, .) d\nu_{p(s,\omega,a)}. \tag{4.2}$$

and $J(s, \omega, a)$ is analytic for $(s, \omega) \in W$. Here $h_{p(s,\omega,a)}(x) > 0$ is the eigenfunction corresponding to the eigenvalue $e^{\Pr(p(s,\omega,a))}$ of $\mathcal{L}_{p(s,\omega,a)}$ and similarly the measure $\nu_{p(s,\omega,a)}$ is determined by the eigenvalue $e^{\overline{\Pr(p(s,\omega,a))}}$ of the dual operator $\mathcal{L}^*_{p(s,\omega a)}$.
(iii) $Z(s, \omega, a)$ is analytic for (s, ω) in an open neighbourhood V_1 of $\{s : \operatorname{Re} s = \gamma(a),\ s \neq \gamma(a))\} \times \{0\}$.
(iv) For each $\omega \in \mathbb{R} \setminus \{0\}$, $Z(s, \omega, a)$ is analytic for (s, ω) in an open neighbourhood V_2 of $\{s : \operatorname{Re} s = \gamma(a)\} \times \{\omega\}$.

For our analysis we need to estimate the norms

$$\|B_1(s, \omega, a, x)\|_\infty,\ \|B_2(s, \omega, a, .)\|_\beta$$

The norm $\|B_2(s, \omega, a, .)\|_\infty$ is easily estimated uniformly with respect to $\omega \in \mathbb{R}$, since

$$\Big|\exp\Big((s + a(\xi(a) + \mathbf{i}\omega))\eta - (\xi(a) + \mathbf{i}\omega)G^\eta(x, 0)\Big)\Big|$$
$$\leq \exp\Big((|\operatorname{Re} s| + a|\xi(a)|)\eta + |\xi(a)||G^\eta(x, 0)|\Big)$$

and

$$\|B_2(s, \omega, a, x)\|_\infty \leq \exp\Big((|\operatorname{Re} s| + a|\xi(a)|)\kappa_1 + |\xi(a)| \max_{x \in U} G(x, 0)\kappa_1\Big),$$

where $\kappa_1 = \max_{x \in U} \tau(x)$. Similarly, one treats the norm $\|B_1(s, \omega, a, .)\|_\infty$. For the norm $|B_2(s, \omega, a, .)|_\beta$ we apply the following elementary estimate. Let

$$k(x) = \int_0^{\tau(x)} e^{(s+az)\eta} e^{K^\eta(x,0)} d\eta,\ K \in C^\beta(R^\tau).$$

Then

$$|k(x) - k(y)| = \Big|\int_0^{\tau(x)} e^{(s+az)\eta} e^{K^\eta(x,0)} d\eta - \int_0^{\tau(y)} e^{(s+az)\eta} e^{K^\eta(y,0)} d\eta\Big|$$
$$\leq \int_0^{\tau(x)} e^{|\operatorname{Re} s + a\xi(a)|\eta} \Big|e^{K^\eta(x,0)} - e^{K^\eta(y,0)}\Big| d\eta + \Big|\int_{\tau(x)}^{\tau(y)} e^{(s+az)\eta} e^{K^\eta(y,0)} d\eta\Big|.$$

The second term in the right-hand-side is estimated by

$$\exp(\max_{0\leq\eta\leq\kappa_1}|\operatorname{Re} K^\eta| + |\operatorname{Re} s + a\xi(a)|\kappa_1)|\tau(x)-\tau(y)|.$$

For the first term on the right we use the inequality

$$|e^{z_1}-e^{z_2}| = \Big|\int_{z_1}^{z_2} e^u du\Big| \leq e^{|\operatorname{Re} z_1|+|\operatorname{Re} z_2|}|z_1-z_2|,$$

and we obtain

$$\int_0^{\tau(x)} e^{|\operatorname{Re} s+a\xi(a)|\eta}\Big|e^{K^\eta(x,0)}-e^{K^\eta(y,0)}\Big|d\eta \leq \exp\Big((2\|\operatorname{Re} K\|_\infty + |\operatorname{Re} s + a\xi(a)|)\kappa_1\Big)$$

$$\times\int_0^{\tau(x)}|K^\eta(x,0)-K^\eta(y,0)|d\eta$$

$$\leq \exp\Big((2\|\operatorname{Re} K\|_\infty + |\operatorname{Re} s + a\xi(a)|)\kappa_1\Big)\int_0^{\tau(x)}\int_0^\eta\Big|K(\sigma_t^\tau(x,0))-K(\sigma_t^\tau(y,0))\Big|dtd\eta$$

which yields an estimate for $|B_2(s,\omega,a,.)|_\beta$ uniformly with respect to $\omega\in\mathbb{R}$.

For $(s,\omega)=(\gamma(a),0)$ one has a maximal real eigenvalue 1 of $\mathcal{L}_{(\gamma(a),0,a)}$ and the rest of the spectrum is contained in a disk of radius $0<r<1$. By perturbation theory there exists an unique eigenvalue with maximal modulus of $\mathcal{L}_{s,\omega,a}$ given by

$$\lambda_{s,\omega,a} = \exp\Big(\Pr(p(s,\omega,a))\Big),$$

defined for $(s,\omega)\in W$. We get

$$\frac{\partial\lambda_{s,\omega,a}}{\partial s}\Big|_{(s,\omega,a)=(\gamma(a),0,a)} = -\int\tau d\mu_{f_a} < 0,$$

where μ_{f_a} is the equilibrium state of f_a. By the implicit function theorem (see Lemma 3 in [20]) for small $\epsilon_1>0$ we may determine $s=s(\omega,a)$, $|\omega|\leq\epsilon_0$, from the equation $\lambda_{s,\omega,a}=1$ so that

$$\lambda_{(s(\omega,a),\omega,a)} = 1,\ s(0,a)=\gamma(a).$$

Therefore

$$\frac{s-s(\omega,a)}{1-\exp\Big(\Pr(p(s,\omega,a))\Big)} = \Big(\int\tau d\nu_{f_a-s(\omega,a)\tau+\mathbf{i}\omega g}\Big)^{-1} + \mathcal{O}(s-s(\omega,a)).$$

This shows that we have a pole at $s=s(\omega,a)$ and taking the residue at $s(\omega,a)$, the singular term in (4.1) becomes

$$\Big(\frac{B_3(s(\omega,a),\omega,a)}{\int\tau d\nu_{f_a-s(\omega,a)\tau+\mathbf{i}\omega g}}\Big)\frac{1}{s-s(\omega,a)}.$$

Now we will show that $Z(s,\omega,a)$ has an meromorphic continuation across the line $\operatorname{Re} s=\gamma(a)$. First note that for $a=\int_U G(y)dm_{F+\xi(a)G}$ with a Hölder function $G\geq g_0>0$ on U one has

$$g_0\leq a\leq\max_{y\in U}G(y)=m.$$

Let $0 < \eta < g_0/2$ be a fixed number. We will apply the spectral estimates for the operator $\mathcal{L}_{s,\omega,a}$ given in Theorem 2.1 in Section 2 (see [14]). It is possible to write $\mathcal{L}_{s,\omega,a}$ in two different forms

$$\mathcal{L}_{1,s,\omega,a} = \mathcal{L}_{h_a-(\operatorname{Re} s-\gamma(a))\tau-\mathbf{i}\operatorname{Im} s\tau+\mathbf{i}\omega(g-a\tau)},$$
$$\mathcal{L}_{2,s,\omega,a} = \mathcal{L}_{h_a-(\operatorname{Re} s-\gamma(a))\tau-\mathbf{i}(\operatorname{Im} s+a\omega)\tau+\mathbf{i}\omega g},$$

where $h_a = f_a - \gamma(a)\tau$ and $\Pr_\sigma(h_a) = 0$. In the operators $\mathcal{L}_{k,s,\omega,a}$, $k = 1, 2$, we have different factors $-\mathbf{i}\operatorname{Im} s$ and $-\mathbf{i}(\operatorname{Im} s + a\omega)$ in front of τ. Applying Theorem 2.1, we can find $a_0 > 0$ and constants $0 < \rho < 1,\ M > 0$ such that for $|\operatorname{Re} s - \gamma(a)| \leq a_0$ and any $\nu > 0$ we have

$$\|\mathcal{L}^m_{1,s,\omega,a} h\|_{\beta,\operatorname{Im} s} \leq C(\nu, B_1)\rho^m |\operatorname{Im} s|^\nu \|h\|_{\beta,\operatorname{Im} s} \tag{4.3}$$

for $|\operatorname{Im} s| \geq M,\ |\omega| \leq B_1 |\operatorname{Im} s|$,

$$\|\mathcal{L}^m_{2,s,\omega,a} h\|_{\beta,(\operatorname{Im} s+a\omega)} \leq D(\nu, B_2)\rho^m |\operatorname{Im} s + a\omega|^\nu \|h\|_{\beta,\operatorname{Im} s+a\omega} \tag{4.4}$$

for $|\operatorname{Im} s + a\omega| \geq M,\ |\omega| \leq B_2 |\operatorname{Im} s + a\omega|$. Let us remark that we can take the same constants a_0, ρ and M in both estimates above, since if we have constants

$$a_k > 0,\ 0 < \rho_k < 1, M_k > 0, k = 1, 2$$

for the operators $\mathcal{L}_{k,s,\omega,a}$, we can choose

$$a_0 = \min\{a_1, a_2\},\ \rho = \max\{\rho_1, \rho_2\},\ M = \max\{M_1, M_2\}.$$

On the other hand, the constants $C(\nu, B_1)$ and $D(\nu, B_2)$ depend on (ν, B_1) and (ν, B_2), respectively.

Proposition 4.2. *Assume the assumptions of Theorem* 1.3 *fulfilled. Then for any Hölder continuous functions* $F, G \in C^\alpha(\widehat{R}^\tau)$ *there exist* $\mu_0 > 0$ *and* $\epsilon_0 > 0$ *such that the function* $Z(s, \omega, a)$ *admits a meromorphic continuation for*

$$(s, \omega) \in \{(s, \omega) \in \mathbb{C}^2 :\ \operatorname{Re} s \geq \gamma(a) - \mu_0,\ \omega \in \mathbb{R}\} \tag{4.5}$$

with only one simple pole at $s(\omega, a)$, $|\omega| \leq \epsilon_0$. *The pole* $s(\omega, a)$ *is determined as the root of the equation* $\Pr(f_a - s\tau + \mathbf{i}\omega(g - a\tau)) = 0$ *with respect to* s *for* $|\omega| \leq \epsilon_0$. *Moreover, there exist constants* $\eta > 0$, $M > 0$ *such that for any* $\nu > 0$ *if* $|\operatorname{Im} s| \geq M$ *or* $|\omega| \geq \frac{1}{\eta} M$, *we have the estimate*

$$|Z(s, \omega, a)| \leq B_\nu(|\operatorname{Im} s|^\nu + |\omega|^\nu),\ \operatorname{Re} s \geq \gamma(a) - \mu_0, \tag{4.6}$$

uniformity with respect to $a \in J$ *in a compact interval* $J \Subset \Gamma_G$ *with a constant* $B_\nu > 0$ *independent on* s, ω *and* $a \in J$.

Proof. We suppose below that $|\operatorname{Re} s - \gamma(a)| \leq a_0$, since for $\operatorname{Re} s > \gamma(a) + a_0$ the result follows from Proposition 4.1, (i). Consider three cases.

Case 1. $(\operatorname{Im} z, \omega) \in D_M = \{|\operatorname{Im} z| \leq M,\ |\omega| \leq \frac{1}{\eta} M\}$.
For $(\operatorname{Im} s, \omega) \in B_{\epsilon_0} = \{|\operatorname{Im} z| < \epsilon_0, |\omega| < \epsilon_0\}$ the result follows from Proposition 4.1, (ii). So assume that $(\operatorname{Im} z, \omega) \in D_M \setminus B_{\epsilon_0}$.

In this situation we may apply the statement (iii) of Proposition 4.1. For reader's convenience we present a proof. Let $(s_0,\ w_0)$ with $(\operatorname{Im} s_0, \omega_0) \in D_M \setminus B_{\epsilon_0}$ be fixed. Assume first that $\operatorname{Im} p(s_0, \omega_0, a)$ is cohomologous to $c + 2\pi Q$ with an integer-valued function $Q \in C(U; \mathbb{Z})$ and a constant $c \in [0, 2\pi)$. Then we define the pressure $\Pr(p(s_0, \omega_0, a)) = \Pr(f_a) + c$ and we extend the pressure in a small neighbourhood of (s_0, ω_0). Since G and σ_t^τ are flow independent, the functions g and τ are σ-independent. If we have $c = 0$, from the fact that $\operatorname{Im} s_0 \tau + \omega_0 g$ is cohomologous to a function in $C(U; 2\pi\mathbb{Z})$, we deduce $\operatorname{Im} s_0 = \omega_0 = 0$ which is impossible. Thus we have $c \neq 0$. Consequently, the operator $\mathcal{L}_{s_0,\omega_0,a}$ has an eigenvalue $e^{\mathbf{i}c}$. Then there exists a neighborhood U_2 of (s_0, ω_0) such that for $(s, \omega) \in U_2$ we have $\Pr(p(s, \omega, a)) \neq 0$ and for $(s, \omega) \in U_2$ we have an analytic extension of $Z(s, \omega, a)$ given by

$$Z(s, \omega, a) = \Big[\frac{B_4(s, \omega, a)}{1 - e^{\Pr(p(s,\omega,a))}} + J_2(s, \omega, a)\Big]$$

with a function $J_2(s, \omega, a)$ analytic with respect to s for $(s, \omega) \in U_2$. Second, let $\operatorname{Im} p(s_0, \omega_0, a)$ be not cohomologous to $c + 2\pi Q$. Then the spectral radius of $\mathcal{L}_{s_0,\omega_0,a}$ is strictly less than 1 and this will be the case for (s, ω) is a small neighbourhood U_3 of (s_0, ω_0). Therefore it is easy to see that the series in (3.2) is absolutely convergent and we obtain again an analytic extension. Covering the compact set $D_M \setminus B_{\epsilon_0}$ by a finite number of neighbourhoods, we may choose $\mu_0 > 0$ small so that for $\gamma(a) - \mu_0 \leq \operatorname{Re} s \leq \gamma(a)$, $(\operatorname{Im} s, \omega) \in D_M \setminus B_{\epsilon_0}$, we have an analytic extension of $Z(s, \omega, a)$ in (4.5).

Case 2. $|\operatorname{Im} s| \leq M,\ |\omega| > \frac{1}{\eta} M > \frac{2}{g_0} M.$

Notice that $\frac{2M}{g_0} \geq \frac{2M}{a}$. We consider the operator $\mathcal{L}_{2,s,\omega,a}$ and observe that

$$|\operatorname{Im} s + a\omega| \geq a|\omega| - |\operatorname{Im} s| \geq M$$

and also

$$|\operatorname{Im} s + a\omega| \geq \frac{a}{2}|\omega| + \frac{a}{2}|\omega| - |\operatorname{Im} s| \geq \frac{a}{2}|\omega|.$$

Hence $|\omega| \leq \frac{2}{a}|\operatorname{Im} s + a\omega| \leq \frac{2}{g_0}|\operatorname{Im} s + a\omega|$.

To apply the estimate (4.4) with $B_2 = \frac{2}{\delta_0}$ to the series in (3.2), we must estimate the norm

$$\|hB_2(s, \omega, a, .)\|_{\beta, \operatorname{Im} s + a\omega}$$

uniformly with respect to $\omega \in \mathbb{R}$. The norm $\|B_2(s, \omega, a, .)\|_\beta$ has been estimated above. Next one gets

$$\Big|\frac{1}{\operatorname{Im} s + a\omega}\Big| \leq \frac{1}{M}$$

and we deduce the needed estimate.

Now the series in (3.2) is absolutely convergent and we obtain an analytic extension of $Z(s, w, a)$ for $|\operatorname{Re} s - \gamma(a)| \leq a_0$, $|\operatorname{Im} s| \geq M$, $|\omega| > \frac{1}{\eta} M$ as well as the estimate

$$|Z(s, \omega, a)| \leq C_{\nu, B_2} |\operatorname{Im} s + a\omega|^\nu \leq C_{\nu, B_2, M}(1 + |\omega|)^\nu. \tag{4.7}$$

Decreasing, if is necessary, μ_0 we obtain an analytic extension in (4.5).

Case 3. $|\operatorname{Im} s| > M.$

We consider two subcases:

Subcase 3a. $|\omega| \leq \frac{1}{\eta}|\operatorname{Im} s|$. We work with the operator $\mathcal{L}_{1,s,\omega,a}$. For every $\nu > 0$ with $b = \operatorname{Im} s$, $B_1 = \frac{1}{\eta}$ and $|\operatorname{Im} s| \geq M$ one obtains from (4.3) the spectral estimates

$$\|\mathcal{L}^m_{s,\omega,a} h\|_{\beta, b} \leq C(\nu, \eta)\rho^m |\operatorname{Im} s|^\nu \|h\|_{\beta, b},\ m \in \mathbb{N}. \tag{4.8}$$

The series (3.2) is absolutely convergent and we deduce

$$|Z(s,\omega,a)| \leq \tilde{C}(\nu,\eta)|\operatorname{Im} s|^{\nu}.$$

Subcase 3b. $|\omega| > \frac{1}{\eta}|\operatorname{Im} s|$. We work now with the operator $\mathcal{L}_{2,s,\omega,a}$. In this case

$$|\operatorname{Im} s + a\omega| \geq a|\omega| - |\operatorname{Im} s| \geq (a-\eta)|\omega| \geq \frac{1}{2} g_0|\omega|.$$

Hence

$$|\omega| \leq \frac{2}{g_0}|\operatorname{Im} s + a\omega|.$$

On the other hand, for $|\omega| > \frac{1}{\eta}|\operatorname{Im} s| \geq \frac{1}{\eta}M$ we have

$$|\operatorname{Im} s + a\omega| \geq \frac{g_0}{2}|\omega| \geq \frac{g_0}{2\eta}M \geq M,$$

because $2\eta < g_0$. Therefore, for any $\nu > 0$ we can use (4.4) with $B_2 = \frac{2}{g_0}$ and obtain

$$\begin{aligned}\|\mathcal{L}^m_{s,\omega,a} h\|_{\beta, \operatorname{Im} s + a\omega} &\leq D(\nu, g_0)\rho^m |\operatorname{Im} s + a\omega|^{\nu} \|h\|_{\beta,\operatorname{Im} s + a\omega}\\ &\leq D(\nu, g_0, M)\rho^m (1+|\omega|)^{\nu} \|h\|_{\beta,\operatorname{Im} s + a\omega},\ m \in \mathbb{N}. \qquad (4.9)\end{aligned}$$

These estimates lead again to (4.7) and the proof is complete. □

5. Tauberian theorem

In this section we prove a Tauberian theorem for a sequence of functions $\{g_n(t)\}_{n\in\mathbb{N}}$ similar to that in [8] (see also Proposition 6 (i) in [20]). The novelty here is that the leading terms contain a factor $A_n \geq e^{-\mu n}$ with $\mu > 0$ which can converge to 0 exponentially fast and the remainders must be smaller that this leading term. Moreover, we have two limits $n \to \infty$ and $t \to \infty$ and this creates new difficulties. Under some assumptions on the Laplace transform of $g_n(t)$, stronger that those in [8], we obtain an asymptotic for $t \geq n - q$ and $n \to \infty$.

Proposition 5.1. *Let $g_n(t)$, $n \in \mathbb{N}$, be monotonic nondecreasing functions defined for $t \in [0,\infty)$ such that $g_n(0) = 0$, $n \in \mathbb{N}$. Assume that for any $n \in \mathbb{N}$ the Laplace transform*

$$F_n(s) = \int_0^{\infty} e^{-st} g_n(t)dt$$

is analytic for $\operatorname{Re} s > 1$. Assume that there exist $\mu_0 > 0$ and $M > 0, C_0 > 0, C_1 > 0, \delta_0 > 0$ such that $F_n(s)$ has a representation

$$F_n(s) = \frac{A_n}{\sqrt{s-1}} + A_n K_n(s) + L_n(s), \qquad (5.1)$$

where $C_0 e^{-\mu n} \leq A_n \leq C_1$, $0 < \mu \leq \mu_0/4$, $\forall n \in \mathbb{N}$, $K_n(s)$ are analytic functions for $\operatorname{Re} s > 1$ and for $t \in \mathbb{R}$ and $s = 1 + \delta + \mathbf{i}t, 0 < \delta \ll 1$, $K_n(s)$ has a limit $k_n(1 + \mathbf{i}t) \in W^{1,1}_{loc}(\mathbb{R})$ (functions which are locally integrable and have locally integrable derivatives) almost everywhere on $\mathbb{R}$ as $\delta \searrow 0$ and $|k_n(1+\delta+\mathbf{i}t)| \leq k_0(t)$, where $k_0(t)$ is locally integrable, while $L_n(s)$ has an analytic continuation for $1 - \mu_0 \leq \operatorname{Re} s \leq 1 + \delta_0$. Moreover, assume that the functions $K_n(s)$ have analytic continuations to $1 - \mu_0 \leq \operatorname{Re} s \leq 1 + \delta_0$, $|\operatorname{Im} s| \geq M$, and for every compact set $D \subset \mathbb{R}$ uniformly with respect to n we have

$$\|k_n'(1+\mathbf{i}t)\|_{L^1(D)} \leq C(D),\ \forall n \in \mathbb{N}.$$

Next assume that for any $0 < \nu < 1$ with a constant $B(\nu) > 0$ independent on n we have for any $n \in \mathbb{N}$ the estimates

$$\Big|\frac{d^k}{ds^k} L_n(s)\Big| \leq B(\nu)(1 + |\operatorname{Im} s|^{\nu}),\ 1 - \mu_0 \leq \operatorname{Re} s \leq 1 + \delta_0,\ k = 0, 1, \tag{5.2}$$

$$\Big|\frac{d^k}{ds^k} K_n(s)\Big| \leq B(\nu)(1+|\operatorname{Im} s|^{\nu}), 1-\mu_0 \leq \operatorname{Re} s \leq 1+\delta_0, k = 0, 1,\ |\operatorname{Im} s| \geq M. \tag{5.3}$$

Then for fixed $q \geq 0$ and for any $0 < \eta \ll 1$ there exists $n_0(\eta) \in \mathbb{N}$ such that for $t \geq n - q$ and $n \geq n_0(\eta) + q$ we have

$$\frac{A_n e^t}{\sqrt{\pi t}}(1 - \eta) \leq g_n(t) \leq \frac{A_n e^t}{\sqrt{\pi t}}(1 + \eta). \tag{5.4}$$

Remark 5.2. Notice that if the estimates (5.2), (5.3) are satisfied for $\mu_0 > 0$ and $k = 0$, then fixing numbers $0 < \mu_1 < \mu_0$, $0 < \delta_1 < \delta_0$, we obtain the estimates (5.2) (resp.(5.3)) for $1 - \mu_1 \leq \operatorname{Re} \leq 1 + \delta_1$, (resp. for $1 - \mu_1 \leq \operatorname{Re} s \leq 1 + \delta_1$, $|\operatorname{Im} s| \geq M$) with another constants $B_1(\nu)$ by applying the Cauchy formula for the first derivative of the analytic functions $K_n(s)$ and $L_n(s)$. In the proof we use only the estimates with $\nu = 1/3$, but if we change the relation between μ, μ_0 and $t \geq \beta(n)$, we need estimates with different ν. In Section 6 we establish (5.2), (5.3).

Proof. We follow the proof in [8] with a more precise analysis concerning the dependence of n. If we replace the function $g_n(t)$ by $\tilde{g}_n(t) = g_n(t)\sqrt{t}$, one obtains

$$F_n(s) = \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-st} \tilde{g}_n(t) dt.$$

For simplicity of the notations below we will denote $\tilde{g}_n(x)$ by $g_n(x)$. Now introduce the function $H_n(x) = g_n(x)e^{-x}$ and for $s = 1 + \epsilon + \mathbf{i}t$ define

$$\mathcal{K}_{\epsilon,n}(t) = F_n(s) - \frac{A_n}{\sqrt{s-1}}.$$

After a change of variable $x = v^2$, one obtains

$$\mathcal{K}_{\epsilon,n}(t) = 2\int_0^{\infty} e^{-(s-1)v^2}\Big(H_n(v^2) - \frac{A_n}{\sqrt{\pi}}\Big) dv.$$

Consequently,

$$\mathcal{K}_{\epsilon,n}(t) = \lim_{\xi \to \infty} 2\int_0^{\xi} \Big(H_n(v^2) - \frac{A_n}{\sqrt{\pi}}\Big) e^{-(s-1)v^2} dv$$

and for fixed n and fixed ϵ this limit is uniform for $|t| \leq 2\lambda$. We multiply $\mathcal{K}_{\epsilon,n}(t)$ by $\sqrt{y}e^{\mathbf{i}ty}\Big(1 - \frac{|t|}{2\lambda}\Big)$ and integrate over t in $[-2\lambda, 2\lambda]$. Thus

$$\int_{-2\lambda}^{2\lambda} \sqrt{y}\Big(1 - \frac{|t|}{2\lambda}\Big)\mathcal{K}_{\epsilon,n}(t) e^{\mathbf{i}yt} dt$$

$$= \lim_{\xi\to\infty} 2\int_{-2\lambda}^{2\lambda} \sqrt{y} e^{\mathbf{i}ty}\Big(1 - \frac{|t|}{2\lambda}\Big)\Big(\int_0^{\xi}\Big(H_n(v^2) - \frac{A_n}{\sqrt{\pi}}\Big) e^{-\epsilon v^2 - \mathbf{i}tv^2} dv\Big) dt.$$

As in [8], we interchange the limit $\xi \to \infty$ and the integration and write the right hand side of the last term as

$$2\int_0^{\infty}\Big(H_n(v^2) - \frac{A_n}{\sqrt{\pi}}\Big) e^{-\epsilon v^2}\Big(\int_{-2}^{2} \lambda\sqrt{y} e^{\mathbf{i}\lambda(y - v^2)u}\Big(1 - \frac{|u|}{2}\Big) du\Big) dv.$$

We change the variable $v = \sqrt{y - \frac{w}{\lambda}}$ and the last term becomes

$$\int_{-\infty}^{\lambda y} \Big(H_n\Big(y - \frac{w}{\lambda}\Big) - \frac{A_n}{\sqrt{\pi}}\Big) e^{-\epsilon(y-(w/\lambda))} \sqrt{y} \Big(\int_{-2}^{2} \Big(1 - \frac{|u|}{2}\Big) e^{\mathbf{i}wu} du\Big) \frac{dw}{\sqrt{y - (w/\lambda)}}$$

$$= 2\int_{-\infty}^{\lambda y} H_n\Big(y - \frac{w}{\lambda}\Big) e^{-\epsilon(y-w/\lambda)} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} dw$$

$$- \frac{2A_n}{\sqrt{\pi}} \int_{-\infty}^{\lambda y} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} e^{-\epsilon(y-w/\lambda)} dw.$$

Now, as in [8], we take the limit $\epsilon \searrow 0$ and set $\mathcal{K}_{0,n}(t) = \lim_{\epsilon \searrow 0} \mathcal{K}_{\epsilon,n}(t)$. By the Lebesgue convergence theorem and Sub-Lemma 4.5 in [8] we obtain

$$\lim_{y\to\infty} \frac{1}{2} \int_{-2\lambda}^{2\lambda} \mathcal{K}_{0,n}(t) \Big(1 - \frac{|t|}{2\lambda}\Big) \sqrt{y} e^{\mathbf{i}ty} dy + A_n \sqrt{\pi}$$

$$= \lim_{y\to\infty} \int_{-\infty}^{\lambda y} H_n\Big(y - \frac{w}{\lambda}\Big) \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} dw. \tag{5.5}$$

By using an integration by parts and the fact that $\mathcal{K}_{0,n}(t) \in W^{1,1}_{loc}(\mathbb{R})$, we may deduce that for every fixed $\lambda > 1$ the first term on the left hand side of (5.5) has a limit 0 as $y \to \infty$. However, for every fixed $0 < \eta < 1$ and fixed $\lambda > 1$ if we wish to arrange the inequality

$$\Big|\frac{1}{2} \int_{-2\lambda}^{2\lambda} \mathcal{K}_{0,n}(t) \Big(1 - \frac{|t|}{2\lambda}\Big) \sqrt{y} e^{\mathbf{i}ty} dt\Big| \leq A_n \eta, \tag{5.6}$$

we must take $y \geq Y(\eta, \lambda, n)$ and we may have $Y(\eta, \lambda, n) \to \infty$ as $n \to \infty, \lambda \to \infty$.

By using the representation (5.1) and the estimates (5.2), (5.3), we will prove a more precise result.

Lemma 5.3. *Let $q \geq 0$ be fixed and let $y \geq n - q$, $\lambda = \lambda_n = e^{\frac{1}{2}\mu_0 n}$. Then for any $\eta > 0$ there exists $n_0(\eta) \in \mathbb{N}$ such that for all $n \geq n_0(\eta) + q$ we have (5.6).*

Proof. First we treat the integral

$$J_n(y) = \int_{-2\lambda_n}^{2\lambda_n} L_n(1 + \mathbf{i}t) \Big(1 - \frac{|t|}{2\lambda_n}\Big) \sqrt{y} e^{\mathbf{i}ty} dt,$$

where $L_n(1 + \mathbf{i}t) = \lim_{\delta \searrow 0} L_n(1 + \delta + \mathbf{i}t)$. We write this integral as follows

$$J_n(y) = -\int_{\gamma_1} \mathbf{i} L_n(s) \Big(1 - \frac{s-1}{2\mathbf{i}\lambda_n}\Big) \sqrt{y} e^{(s-1)y} ds$$

$$- \int_{\gamma_2} \mathbf{i} L_n(s) \Big(1 + \frac{s-1}{2\mathbf{i}\lambda_n}\Big) \sqrt{y} e^{(s-1)y} ds,$$

where

$$\gamma_1 = \{s \in \mathbb{C} : s = 1 + \mathbf{i}t,\ 0 \leq t \leq 2\lambda_n\},\ \gamma_2 = \{s \in \mathbb{C} :\ s = 1 + \mathbf{i}t,\ -2\lambda_n \leq t \leq 0\}.$$

Since $L_n(s)$ has an analytic continuation for $1 - \mu_0 \leq \operatorname{Re} s \leq 1 + \delta_0$, we have the equality

$$\int_{\gamma_1} + \int_{\omega_{1,1}} + \int_{\omega_{1,2}} + \int_{\omega_{1,3}} = 0,$$

where the function under integration is $-\mathbf{i}L_n(s)\Big(1-\frac{s-1}{2\mathbf{i}\lambda_n}\Big)\sqrt{y}e^{(s-1)y}$ and

$$\omega_{1,1}=\{s\in\mathbb{C}:\ s=z+2\mathbf{i}\lambda_n,\ 1-\mu_0\le z\le 1\},$$

$$\omega_{1,2}=\{s\in\mathbb{C}:\ s=1-\mu_0+\mathbf{i}t,\ 0\le t\le 2\lambda_n\},\ \omega_{1,3}=\{s\in\mathbb{R}:\ 1-\mu_0\le s\le 1\}$$

with suitable orientation (see Figure 1). The integral over $\omega_{1,2}$ has the form

$$-e^{-\mu_0 y}\sqrt{y}\int_0^{2\lambda_n}L_n(1-\mu_0+\mathbf{i}t)\Big(1+\frac{\mu_0-\mathbf{i}t}{2\mathbf{i}\lambda_n}\Big)e^{\mathbf{i}ty}dt.$$

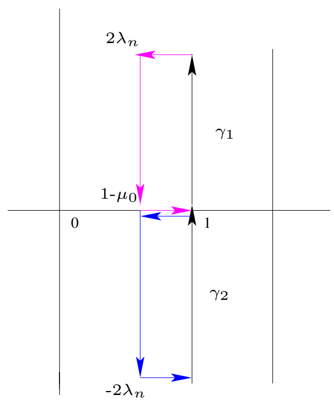

FIGURE 1. Contour of integration for $L_n(1+\mathbf{i}t)$

We integrate by parts and one obtains

$$\frac{e^{-\mu_0 y}}{\mathbf{i}\sqrt{y}}\int_0^{2\lambda_n}\frac{d}{dt}\Big[L_n(1-\mu_0+\mathbf{i}t)\Big(1+\frac{\mu_0-\mathbf{i}t}{2\mathbf{i}\lambda_n}\Big)\Big]e^{\mathbf{i}ty}dt$$

$$-\frac{e^{-\mu_0 y}}{\mathbf{i}\sqrt{y}}\Big(\frac{\mu_0}{2\mathbf{i}\lambda_n}L_n(1-\mu_0+2\mathbf{i}\lambda_n)e^{2\mathbf{i}\lambda_n y}-\Big(1+\frac{\mu_0}{2\mathbf{i}\lambda_n}\Big)L_n(1-\mu_0)\Big).$$

The function $L_n(1+\mathbf{i}t)$ has an analytic continuation $L_n(z+\mathbf{i}t)$ for $1-\mu_0\le z\le 1+\delta_0$ and for any $0<\nu<1$ and any n we have the estimate

$$\Big|\frac{d}{dt}L_n(z+\mathbf{i}t)\Big|\le B(\nu)(1+|t|^{\nu}),\ 1-\mu_0\le z\le 1+\delta_0.$$

Thus for $y\ge n-q$ and large n one gets

$$\Big|\frac{e^{-\mu_0 y}}{\mathbf{i}\sqrt{y}}\int_0^{2\lambda_n}\frac{d}{dt}\Big[L_n(1-\mu_0+\mathbf{i}t)\Big(1+\frac{\mu_0-\mathbf{i}t}{2\mathbf{i}\lambda_n}\Big)\Big]e^{\mathbf{i}ty}dt\Big|$$

$$\le C_{1,2}(\nu)\frac{e^{-\mu_0 y}}{\sqrt{y}}\lambda_n^{1+\nu}\le C_{1,2}(\nu)e^{-\mu_0(n-q)}e^{\frac{1}{2}(1+\nu)\mu_0 n}$$

$$=C_{1,2}(\nu)e^{\mu_0 q}e^{(-\frac{1}{2}+\frac{1}{2}\nu)\mu_0 n}.$$

We choose $\nu=1/3$ and for the last term of the above inequality one obtains a bound $\mathcal{O}(e^{-\frac{1}{3}\mu_0 n})$. Since $A_n\ge C_0e^{-\frac{\mu_0}{4}n}$, we obtain a term $A_n o(n)$. The boundary terms are easily estimated and we get

$$|\int_{\omega_{1,2}}|\le C_{1.2}e^{-\frac{1}{3}\mu_0 n}. \tag{5.7}$$

Passing to the integral over $\omega_{1,1}$, notice that for $s \in \omega_{1,1}$ we have

$$1 - \Big(\frac{s-1}{2\mathbf{i}\lambda_n}\Big) = \frac{1-z}{2\mathbf{i}\lambda_n},\ |e^{(s-1)y}| \leq e^{(z-1)y} \leq 1,\ \operatorname{Re} s = z.$$

We integrate by parts with respect to z and deduce

$$|\int_{\omega_{1,1}}| \leq A_{1,1}(\nu)\frac{e^{-\mu_0 y}}{\sqrt{y}}\frac{(1+\lambda_n^\nu)}{\lambda_n} + \frac{1}{2\sqrt{y}\lambda_n}\int_{1-\mu_0}^1 \Big|\frac{d}{dz}\Big[(1-z)L_n(z+2\mathbf{i}\lambda_n)\Big]\Big|dz.$$

Therefore, applying (5.2) for the second term in the right-hand-side, one obtains

$$|\int_{\omega_{1,1}}| \leq \frac{C_{1,1}(\nu)}{\sqrt{n}}e^{-\frac{1}{2}(1-\nu)\mu_0 n} \leq C_{1,1}e^{-\frac{1}{3}\mu_0 n}, \tag{5.8}$$

choosing $\nu = 1/3$. Before treating the integral over $\omega_{1,3}$, consider the equality

$$\int_{\gamma_2} + \int_{\omega_{2,1}} + \int_{\omega_{2,2}} + \int_{\omega_{2,3}} = 0,$$

where the function under integration is $-\mathbf{i}L_n(s)\Big(1+\frac{s-1}{2\mathbf{i}\lambda_n}\Big)\sqrt{y}e^{(s-1)y}$ and

$$\omega_{2,1} = \{s \in \mathbb{C} :\ s = z - 2\mathbf{i}\lambda_n,\ 1-\mu_0 \leq z \leq 1\},$$

$$\omega_{2,2} = \{s \in \mathbb{C} :\ s = 1-\mu_0 + \mathbf{i}t,\ -2\lambda_n \leq t \leq 0\},\ \omega_{2,3} = \{s \in \mathbb{R} :\ 1-\mu_0 \leq s \leq 1\}$$

with suitable orientation (see Figure 1). In particular, the curves $\omega_{1,3}$ and $\omega_{2,3}$ coincide, but they have inverse orientations. The analysis of $\int_{\omega_{2,2}}$ is completely similar and one obtains (5.7). For $s \in \omega_{2,1}$ we have

$$1 + \Big(\frac{s-1}{2\mathbf{i}\lambda_n}\Big) = \frac{z-1}{2\mathbf{i}\lambda_n},\ |e^{(s-1)y}| \leq e^{(z-1)y} \leq 1$$

and as above one has (5.8). Now we take the sum of the integrals over $\omega_{1,3}$ and $\omega_{2,3}$ and we are going to estimate the integral

$$\frac{1}{\lambda_n}\int_{1-\mu_0}^1 (z-1)\sqrt{y}L_n(z)e^{(z-1)y}dz.$$

We integrate by parts and the analysis is reduced to the integral

$$\frac{1}{\sqrt{y}\lambda_n}\int_{1-\mu_0}^1 e^{(z-1)y}\frac{d}{dz}\Big((z-1)L_n(z)\Big)dz$$

which can be estimated by $C_{1,3}e^{-\frac{1}{2}\mu_0 n}$.

Next we pass to the analysis of the integral

$$I_n(y) = \int_{-2\lambda_n}^{2\lambda} k_n(1+\mathbf{i}t)\Big(1-\frac{|t|}{2\lambda_n}\Big)\sqrt{y}e^{\mathbf{i}ty}dt,$$

where

$$k_n(1+\mathbf{i}t) = \lim_{\delta \searrow 0} K_n(1+\delta+\mathbf{i}t).$$

Our purpose is to show that for $y \geq n-q$ and $\lambda_n = e^{-\frac{1}{2}\mu_0 n}$ for any $0 < \eta \ll 1$ there exists $n_0(\eta)$ such that for $n \geq n_0(\eta)+q$ we have $|I_n(y)| < \eta$. We integrate by parts with respect to t and deduce

$$I_n(y) = -\frac{1}{\mathbf{i}\sqrt{y}}\int_{-2\lambda_n}^{2\lambda_n}\frac{d}{dt}\Big[\Big(1-\frac{|t|}{2\lambda_n}\Big)k_n(1+\mathbf{i}t)\Big]e^{\mathbf{i}ty}dt$$

$$= \frac{\mathbf{i}}{\sqrt{y}}\Big(\int_{-M-1}^{M+1} + \int_{-2\lambda_n}^{-M-1} + \int_{M+1}^{2\lambda_n}\Big).$$

The integral over $[-M-1, M+1]$ can be estimated taking n large by using the factor $\frac{1}{\sqrt{y}}$ and the fact that by hypothesis

$$\|k_n(1+\mathbf{i}t)\|_{L^1(-M-1,M+1)} + \|k_n'(1+\mathbf{i}t)\|_{L^1(-M-1,M+1)} \leq C(M),\ \forall n \in \mathbb{N}.$$

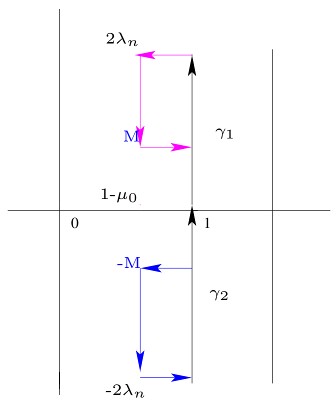

FIGURE 2. Contour of integration for $K_n(1+\mathbf{i}t)$

For the other two integrals we apply the argument used above exploiting the analytic continuation of $k_n(s)$ for $1-\mu_0 \leq \operatorname{Re} s \leq 1, |\operatorname{Im} s| \geq M$. We treat below only the integral over $[M+1, 2\lambda_n]$, the analysis of the other one is very similar. We have

$$\frac{1}{\sqrt{y}}\int_{M+1}^{2\lambda_n} \frac{d}{dt}\Big[\Big(1-\frac{t}{2\lambda_n}\Big)k_n(1+\mathbf{i}t)\Big]e^{\mathbf{i}ty}dt$$
$$= -\frac{1}{2\lambda_n\sqrt{y}}\int_{M+1}^{2\lambda_n} k_n(1+\mathbf{i}t)e^{\mathbf{i}ty}dt$$
$$+\frac{1}{\sqrt{y}}\int_{M+1}^{2\lambda_n}\Big(1-\frac{t}{2\lambda_n}\Big)\frac{d}{dt}k_n(1+\mathbf{i}t)dt.$$

We write the term on the right hand side as follows

$$\frac{\mathbf{i}}{2\lambda_n\sqrt{y}}\int_{\beta_1} k_n(s)e^{(s-1)y}ds - \frac{1}{\sqrt{y}}\int_{\beta_1}\Big(1-\frac{s-1}{2\mathbf{i}\lambda_n}\Big)\frac{d}{ds}(k_n(s))ds,$$

where

$$\beta_1 = \{s \in \mathbb{C}:\ s = 1+\mathbf{i}t,\ M+1 \leq t \leq 2\lambda_n\}.$$

The integral is equal to a sum of three integrals over the curves

$$\beta_{1,1} = \{s \in \mathbb{C}:\ s = z + 2\mathbf{i}\lambda_n,\ 1-\mu_0 \leq z \leq 1\},$$
$$\beta_{1,2} = \{s \in \mathbb{C}:\ s = 1-\mu_0+\mathbf{i}t,\ M+1 \leq t \leq 2\lambda_n\},$$
$$\beta_{1,3} = \{s \in \mathbb{C}:\ s = z + \mathbf{i}(M+1),\ 1-\mu_0 \leq z \leq 1\}$$

with suitable orientation (see Figure 2). For the integral over $\beta_{1,2}$ we obtain an estimate $\mathcal{O}\Big(\frac{e^{-\mu_0 y}\lambda_n^{1+\nu}}{\sqrt{y}}\Big) = \mathcal{O}\Big(e^{-\frac{1}{3}\mu_0 n}\Big)$, choosing $\nu = 1/3$, while for that over $\beta_{1,1}$

one deduces an estimate $\mathcal{O}(\frac{\lambda_n^{-1+\nu}}{\sqrt{y}})$ which yields the same bound. To investigate the integral over $\beta_{1.3}$, we use the factor $\frac{1}{\sqrt{y}}$ and the fact that M is fixed. This completes the proof of Lemma 5.3. □

Lemma 5.4. *For any $0 < \eta \ll 1$ there exists $n_0(\eta) \in \mathbb{N}$ such that for $y \geq 1$, $\lambda_n = e^{\frac{1}{2}\mu_0 n}$ we have*

$$\Big| \int_{-\infty}^{\lambda_n y} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} \frac{\sin^2 w}{w^2} dw - \pi \Big| < \eta,\ n \geq n_0(\eta). \tag{5.9}$$

The proof is a repetition of the proof of Sub-Lemma 4.5 in [8] and we leave the details to reader.

Combining Lemmas 5.3 and 5.4, one obtains from the equality (5.5) that for fixed $q \geq 0$ and any $0 < \eta \ll 1$ there exists $n_0(\eta) \in \mathbb{N}$ such that for $y \geq n - q$ and $n \geq n_0(\eta) + q$ we have

$$A_n \sqrt{\pi}(1-\eta) \leq \int_{-\infty}^{\lambda_n y} H_n\Big(y - \frac{w}{\lambda_n}\Big) \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} dw < A_n \sqrt{\pi}(1+\eta). \tag{5.10}$$

Passing to the function $H_n(y)$, we have the following

Lemma 5.5. *Let $q \geq 0$ be fixed. For any $0 < \eta \ll 1$ there exists $n_0(\eta) \in \mathbb{N}$ such that for $y \geq n - q$ and $n \geq n_0(\eta) + q$ we have*

$$\frac{A_n}{\sqrt{\pi}}(1-\eta) \leq H_n(y) \leq \frac{A_n}{\sqrt{\pi}}(1+\eta). \tag{5.11}$$

Proof. Since $H_n(y)$ is nonnegative, from (5.10) for $y \geq n - q$, $n \geq n_0(\eta) + q$ it follows that

$$\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n\Big(y - \frac{w}{\lambda_n}\Big) \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} dw \leq A_n \sqrt{\pi}(1+\eta). \tag{5.12}$$

By the monotonicity of $g_n(w)$ for $w \in [-\sqrt{\lambda_n}, \sqrt{\lambda_n}]$, we deduce

$$H_n\Big(y - \frac{w}{\lambda_n}\Big) \geq H_n\Big(y - \frac{1}{\sqrt{\lambda_n}}\Big) \exp\Big(-\frac{2}{\sqrt{\lambda_n}}\Big),$$

hence

$$H_n\Big(y - \frac{1}{\sqrt{\lambda_n}}\Big) \exp\Big(-\frac{2}{\sqrt{\lambda_n}}\Big) \int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} dw \leq A_n \sqrt{\pi}(1+\eta)$$

and

$$H_n\Big(y - \frac{1}{\sqrt{\lambda_n}}\Big) \leq \frac{A_n \sqrt{\pi}(1+\eta) e^{\frac{2}{\sqrt{\lambda_n}}}}{\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} dw}.$$

As in Lemma 5.4, for large n we can arrange

$$\Big| \int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} dw - \pi \Big| < \eta.$$

The above inequalities imply

$$H_n\Big(y - \frac{1}{\sqrt{\lambda_n}}\Big) \leq \frac{A_n}{\sqrt{\pi}} \Big[(1+\eta) e^{\frac{2}{\sqrt{\lambda_n}}}\Big] \Big(1 - \frac{\eta}{\pi}\Big)^{-1} \leq \frac{A_n}{\sqrt{\pi}}(1 + C_3 \eta)$$

with $C_3 > 0$ independent on n. Replacing y by $y + \frac{1}{\sqrt{\lambda_n}}$, we obtain for large n the upper bound in (5.11).

Next we pass to the analysis of the lower bound. Applying the upper bound obtained above and the monotonicity of $H_n(y)$, from the left-hand-side inequality in (5.10) we obtain

$$A_n\sqrt{\pi}(1-\eta) \le \frac{A_n}{\sqrt{\pi}}(1+\eta)\int_{-\infty}^{-\sqrt{\lambda_n}} \frac{1}{w^2}dw$$

$$+\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n\Big(y+\frac{1}{\sqrt{\lambda_n}}\Big)e^{\frac{2}{\sqrt{\lambda_n}}}\frac{\sin^2 w}{w^2}\frac{\sqrt{y}}{\sqrt{y-w/\lambda_n}}dw$$

$$+\frac{A_n}{\sqrt{\pi}}(1+\eta)\int_{\sqrt{\lambda_n}}^{\lambda_n y}\frac{\sin^2 w}{w^2}\frac{\sqrt{y}}{\sqrt{y-w/\lambda_n}}dw.$$

Clearly, for large n

$$\int_{-\infty}^{-\sqrt{\lambda_n}}\frac{1}{w^2}dw = \frac{1}{\sqrt{\lambda_n}} < \eta.$$

For the third term on the right hand side for large n one gets

$$\int_{\sqrt{\lambda_n}}^{\lambda_n y} = \int_{\sqrt{\lambda_n}}^{\lambda_n y/2} + \int_{\lambda_n y/2}^{\lambda_n y} \le \frac{1}{\sqrt{2}}\int_{\sqrt{\lambda_n}}^{\lambda_n y/2}\frac{\sin^2 w}{w^2}dw + \frac{4\sqrt{y}}{\lambda_n^2 y^2}\int_{\lambda_n y/2}^{\lambda_n y}\frac{1}{\sqrt{y-w/\lambda_n}}dw$$

$$\le \frac{1}{\sqrt{2}}\int_{\sqrt{\lambda_n}}^{\infty}\frac{\sin^2 w}{w^2}dw + \frac{4\sqrt{2}}{\lambda_n y} < \eta.$$

Consequently,

$$A_n\sqrt{\pi}(1-\eta) - 2\frac{A_n}{\sqrt{\pi}}(1+\eta)\eta \le H_n\Big(y+\frac{1}{\sqrt{\lambda_n}}\Big)e^{\frac{2}{\sqrt{\lambda_n}}}\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}}\frac{\sin^2 w}{w^2}\frac{y}{\sqrt{y-w/\lambda_n}}dw$$

and for large n with a constant $C_4 > 0$ independent on n we obtain

$$\frac{A_n\sqrt{\pi}(1-C_4\eta)e^{-\frac{2}{\sqrt{\lambda_n}}}}{\pi+\eta} \le \frac{A_n\sqrt{\pi}(1-C_4\eta)e^{-\frac{2}{\sqrt{\lambda_n}}}}{\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}}\frac{\sin^2 w}{w^2}\frac{y}{\sqrt{y-w/\lambda_n}}dw} \le H\Big(y+\frac{1}{\sqrt{\lambda_n}}\Big).$$

This estimate implies a lower bound for $y \ge n + \frac{1}{\sqrt{\lambda_n}} - q$. Changing q by $q+1$, we obtain a lower bound for $y \ge n - q$. □

Obviously Proposition 5.1 follows from Lemma 5.5.

For functions which are not monotonic we prove the following

Proposition 5.6. *Let $g_n(t) \in C^1([0,\infty[;\mathbb{R}^+)$, $n \in \mathbb{N}$, be nonnegative functions such that*

$$\max_{0\le t\le 1} g_n(t) \le B_0,\ |g_n'(t)| \le B_1\frac{e^t}{\sqrt{t}},\ t \ge 1,\ \forall n \in \mathbb{N}, \tag{5.13}$$

with constants $B_0 > 0, B_1 > 0$ independent of n. Assume that for any $n \in \mathbb{N}$ the Laplace transforms

$$F_n(s) = \int_0^\infty e^{-st}g_n(t)dt$$

are analytic for $\operatorname{Re} s > 1$ *and have the same properties as in Proposition* 5.1. *Then for any fixed* $q \geq 0$ *and any* $0 < \eta \ll 1$ *there exists* $n_0(\eta) \in \mathbb{N}$ *such that for* $t \geq n - q$ *and* $n \geq n_0(\eta) + q$ *we have*

$$\frac{A_n e^t}{\sqrt{\pi t}}(1-\eta) < g_n(t) < \frac{A_n e^t}{\sqrt{\pi t}}(1+\eta). \tag{5.14}$$

Proof. Replacing the function $g_n(t)$ by $\tilde{g}_n(t) = g_n(t)\sqrt{t}$, we can assume that

$$F_n(s) = \int_0^\infty \frac{e^{-st}}{\sqrt{t}} \tilde{g}_n(t) dt$$

has the representation (5.1). On the other hand,

$$g_n(t) = \int_1^t g_n'(\sigma) d\sigma + g_n(1) \leq B_0 + B_1 \int_1^t \frac{e^\sigma}{\sqrt{\sigma}} d\sigma < B_0 + B_1 e^t,\ t \geq 1,\ \forall n \in \mathbb{N}.$$

Therefore

$$|\tilde{g}_n'(t)| = \Big| \frac{g_n(t)}{2\sqrt{t}} + \sqrt{t} g_n'(t) \Big| \leq B_3 e^t,\ t \geq 1,\ \forall n \in \mathbb{N} \tag{5.15}$$

with a constant $B_3 > 0$ independent on n. Below we denote $\tilde{g}_n(t)$ again by $g_n(t)$ and we assume that $|g_n'(t)| \leq B_3 e^t$ for $t \geq 1$. We will repeat a part of the proof of Proposition 5.1 and for simplicity we use the notations of this proof. Set $H_n(y) = g_n(y)e^{-y}$ and for $s = 1 + \epsilon + \mathbf{i}t$ define

$$\mathcal{K}_{\epsilon,n}(t) = F_n(s) - \frac{A_n}{\sqrt{s-1}} = A_n K_n(s) + L_n(s).$$

Since for $F_n(s), K_n(s), L_n(s)$ we have the same assumptions as in Proposition 5.1, we can apply Lemma 5.3. Thus for $y \geq n - q$, $\lambda_n = e^{\frac{1}{2}\mu_0 n}$ and $n \geq n_0(\eta) + q$ we obtain the estimate (5.10).

Because $H_n(y) \geq 0$ to get an upper bound we use the inequality

$$\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n\Big(y - \frac{w}{\lambda_n}\Big) \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}} dw < A_n \sqrt{\pi}(1+\eta). \tag{5.16}$$

Clearly, for $-\sqrt{\lambda_n} \leq w \leq \sqrt{\lambda_n}$ one has

$$H_n\Big(y - \frac{w}{\lambda_n}\Big) \geq g_n(y - \frac{w}{\lambda_n}) e^{-y - \frac{1}{\sqrt{\lambda_n}}}$$
$$= H_n(y) e^{-\frac{1}{\sqrt{\lambda_n}}} + \Big(g_n(y - \frac{w}{\lambda_n}) - g_n(y)\Big) e^{-y-\frac{1}{\sqrt{\lambda_n}}}.$$

By Taylor expansion write

$$g_n\Big(y - \frac{w}{\lambda_n}\Big) - g_n(y) = -\frac{w}{\lambda_n} g_n'\Big(y - \frac{\theta w}{\lambda_n}\Big),\ 0 < \theta < 1,$$

hence for $-\sqrt{\lambda_n} \leq w \leq \sqrt{\lambda_n}$ we deduce

$$|g_n\Big(y - \frac{w}{\lambda_n}\Big) - g_n(y)| \leq \frac{B_3}{\sqrt{\lambda_n}} e^{y + \frac{1}{\sqrt{\lambda_n}}},$$

where we have used the estimate (5.15). Thus we obtain

$$H_n\Big(y - \frac{w}{\lambda_n}\Big) \geq H_n(y) e^{-\frac{1}{\sqrt{\lambda_n}}} - \frac{B_3}{\sqrt{\lambda_n}}$$

and for $y \geq n - q$ and large n we conclude that

$$H_n(y) \leq \frac{A_n\sqrt{\pi}(1+\eta)e^{\frac{1}{\sqrt{\lambda_n}}}}{\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y-\frac{w}{\lambda_n}}}dw} + \frac{B_3}{\sqrt{\lambda_n}}e^{\frac{1}{\sqrt{\lambda_n}}}$$
$$\leq \frac{A_n\sqrt{\pi}(1+\eta)^2}{\pi - \eta} + A_n\eta \leq \frac{A_n}{\sqrt{\pi}}(1 + C_5\eta)$$

with $C_5 > 0$ independent of n. This implies the upper bound in (5.14).

Passing to the lower bound, we repeat the argument of Lemma 5.5 and we are going to find an upper bound for

$$\int_{-\sqrt{\lambda_n}}^{\sqrt{\lambda_n}} H_n\Big(y - \frac{w}{\sqrt{\lambda_n}}\Big)\frac{\sin^2 w}{w^2}\frac{\sqrt{y}}{\sqrt{y - w/\lambda_n}}dw.$$

As above, by Taylor expansion for $-\sqrt{\lambda_n} \leq w \leq \sqrt{\lambda_n}$ and large n one deduces

$$H_n\Big(y - \frac{w}{\sqrt{\lambda_n}}\Big) \leq g_n(y - \frac{w}{\sqrt{\lambda_n}})e^{-y+\frac{1}{\sqrt{\lambda_n}}}$$
$$\leq H_n(y)e^{\frac{1}{\sqrt{\lambda_n}}} + \frac{B_3}{\sqrt{\lambda_n}}e^{\frac{2}{\sqrt{\lambda_n}}},$$

and we obtain easily the lower bound in (5.14). This completes the proof. □

As a preparation for the proof of the estimate (5.13) we establish the following

Lemma 5.7. *Let $\omega(t) \in C^1([0,\infty[)$ be a nonnegative function such that for a fixed $0 < \nu_0 \leq \mu_0/4$ we have the estimate*

$$|\omega'(t)| \leq D_1\frac{e^{(1+\nu_0)t}}{\sqrt{t}},\ \forall t \geq t_0 > 1. \tag{5.17}$$

Assume that the Laplace transform

$$\Omega(s) = \int_0^\infty e^{-st}\omega(t)dt$$

is analytic for $\mathrm{Re}\, s > 1$. *Assume that there exist $A > 0, \delta_0 > 0, M > 0$ such that $\Omega(s)$ has a representation*

$$\Omega(s) = \frac{A}{\sqrt{s-1}} + K(s), \tag{5.18}$$

where $K(s)$ is a function which is analytic for $\mathrm{Re}\, s > 1$ *and $K(s)$ satisfies the same assumptions as $K_1(s)$ in Proposition* 5.1. *Then there exists a constant $D_0 > 0$ such that*

$$\omega(t) \leq D_0\frac{e^t}{\sqrt{t}},\ \forall t \geq t_0. \tag{5.19}$$

Proof. We follow the proof of Proposition 5.6. First we replace the function $\omega(t)$ by $\tilde{\omega}(t) = \sqrt{t}\omega(t)$ and for the Laplace transform

$$\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\tilde{\omega}(t)dt$$

we have the representation (5.1) with $K(s)$ instead of $K_n(s)$, $L_n(s) = 0$ and $|\tilde{\omega}'(t)| \leq C_2e^{(1+\nu_0)t}$. We denote below $\tilde{\omega}(t)$ by $\omega(t)$. We set $H(y) = \frac{\omega(y)}{e^y}$, $\mathcal{K}_0(t) = \lim_{\epsilon\searrow 0} K(1+\epsilon+\mathbf{i}t)$ and prove the following

Lemma 5.8. *There exist $y_0 > 1$ and $A_1 > 0$ independent on y such that for $y \geq y_0, \lambda(y) = \exp(\frac{1}{2}\mu_0 y)$ we have*

$$\Big| \int_{-2\lambda(y)}^{2\lambda(y)} \mathcal{K}_0(t)\Big(1 - \frac{|t|}{2\lambda(y)}\Big)\sqrt{y}e^{\mathbf{i}ty}dt\Big| \leq A_1.$$

The proof is a repetition of that of Lemma 5.3 and we omit the details.

We return to the proof of Lemma 5.7. For $y \geq y_0$ and $\lambda(y) = \exp(\frac{1}{2}\mu_0 y)$ we deduce

$$\int_{-\sqrt{\lambda(y)}}^{\sqrt{\lambda(y)}} H\Big(y - \frac{w}{\lambda(y)}\Big)\frac{\sin^2 w}{w^2}\frac{\sqrt{y}}{\sqrt{y - w/\lambda(y)}}dw < A\sqrt{\pi} + A_1.$$

It is important to estimate for $-\sqrt{\lambda(y)} \leq w \leq \sqrt{\lambda(y)}$ and $0 < \theta < 1$ the term

$$\Big|\omega\Big(y - \frac{w}{\lambda(y)}\Big) - \omega(y)\Big| = \Big|\frac{w}{\lambda(y)}\omega'\Big(y - \frac{\theta w}{\lambda(y)}\Big)\Big| \leq D_1 e^{(y + \frac{1+\nu_0}{\sqrt{\lambda(y)}})}e^{(\nu_0 - \frac{1}{2}\mu_0)y}.$$

Since $0 < \nu_0 \leq \mu_0/4$, for large y we may bound the right hand side of the last inequality by ce^y with a small constant $c > 0$ independent on y and D_1. Consequently, as in the proof of Proposition 5.6, one obtains

$$H\Big(y - \frac{w}{\lambda(y)}\Big) \geq H(y)e^{-\frac{1}{\sqrt{\lambda(y)}}} - B_3$$

with a constant $B_3 > 0$ independent on y and D_1 and we complete the proof as in Proposition 5.6. □

Remark 5.9. Notice that our proof shows that the constant $D_0 > 0$ can be chosen independently of D_1 by taking $t \geq t_1 > t_0$ and t_1 sufficiently large in (5.19).

6. Asymptotic of $\rho_n(T)$

In the section we use the notations of the previous sections. It is more convenient to study the function

$$g_n(T) = \epsilon_n e^{(-\gamma(a)+1)T}\rho_n(T) = \epsilon_n e^{(-\gamma(a)+1)T}\int_{R^\tau} q_n(G^T - aT)(y)dm_F(y)$$

$$= \frac{\epsilon_n^2}{2\pi}e^{(-\gamma(a)+1)T}\int_{\mathbb{R}}\Big(\int_{R^\tau} e^{z\,(G-a)^T(y)}\,dm_F(y)\Big)\,\hat{\chi}(\epsilon_n\omega)\,d\omega.$$

Recall that $\epsilon_n = e^{-\epsilon n}$, $z = \xi(a) + \mathbf{i}\omega$, $a \in J \Subset \Gamma_G$, $q_n(t) = e^{\xi(a)t}\chi(t/\epsilon_n)$. We take $0 < \epsilon \leq \mu_0/8$, μ_0 being the constant in Proposition 4.2. We extend $g_n(T)$ as 0 for $T < 0$ and we wish to apply Proposition 5.6 for $g_n(T)$. We are going to check the assumptions of Proposition 5.6, where

$$A_n = \frac{C(a)\hat{\chi}(0)}{\sqrt{2\beta''(\xi(a))}}\epsilon_n^2,\ \forall n \in \mathbb{N},$$

with $C(a) > 0$ determined below. Clearly, $g_n(T)$ is a nonnegative function. The Laplace transform of $g_n(T)$ becomes

$$F_n(s) = \frac{\epsilon_n^2}{2\pi}\int_0^\infty e^{-(s+\gamma(a)-1)T}\int_{\mathbb{R}}\Big(\int_{R^\tau} e^{z\,(G-a)^T(y)}\,dm_F(y)\Big)\,\hat{\chi}(\epsilon_n\omega)\,d\omega dT$$

and by Proposition 4.1 (i) for $\operatorname{Re} s > 1$ the function $F_n(s)$ is analytic. Next we write the integral with respect to ω as a sum

$$\int_{\mathbb{R}} (...) = \int_{|\omega| \leq \epsilon_0} + \int_{\epsilon_0 < |\omega| \leq M} + \int_{|\omega| \geq M},$$

where $\epsilon_0 > 0$ and $M > 0$ are the constants introduced in the proof of Proposition 4.2. The corresponding decomposition of $F_n(s)$ will be a sum $F_n(s) = F_{1,n}(s) + F_{2,n}(s) + F_{3,n}(s)$. Notice that the factor $\hat{\chi}(\epsilon_n \omega)$ is not involved in the integration with respect to y and T and in the analysis of $F_{k,n}(s)$, $k = 1, 2$ we will have a coefficient ϵ_n^2 implying the factor $A_n = \mathcal{O}(\epsilon_n^2)$. Moreover, in the Laplace transform $Z(s, \omega, a)$ we must replace s by $s + \gamma(a) - 1$. According to Proposition 4.2, the functions $F_{2,n}(s)$ are analytic for $1 - \mu_0 \leq \operatorname{Re} s$, $\mu_0 > 0$ and

$$\lim_{\delta \searrow 0} F_{2,n}(1 + \delta + \mathbf{i}t) = \epsilon_n^2 f_{2,n}(1 + \mathbf{i}t)$$

with $f_{2,n}(1 + \mathbf{i}t) \in W^{1,1}_{loc}(\mathbb{R})$. For $|t| \leq M$ the function $|f_{2,n}(1 + \mathbf{i}t)|$, $|f'_{2,n}(1 + \mathbf{i}t)|$ are clearly uniformly bounded with respect to n, while for $|t| \geq M$ we have an analytic continuation $f_2(s)$ for $1 - \mu_0 \leq \operatorname{Re} s \leq 1 + \delta_0$. In the latter case we apply the estimate (4.6) with $0 < \nu < 1$ uniformly with respect to n (see the case 2 in the proof of Proposition 4.2). Since

$$\int_{\epsilon_0 \leq |\omega| \leq M} |\hat{\chi}(\epsilon_n \omega)| d\omega \leq M \sup_{\xi \in \mathbb{R}} |\hat{\chi}(\xi)|, \ \forall n \in \mathbb{N},$$

with a constant $C_2(\nu) > 0$ independent of n, we obtain

$$|f_{2,n}(s)| \leq C_2(\nu)(1 + |\operatorname{Im} s|^{\nu}). \tag{6.1}$$

Thus the term $F_{2,n}(s)$ contributes to $A_n K_n(s)$ in (5.1).

Passing to the analysis of $F_{3,n}(s)$, we apply the same argument based on the estimate (4.6). According to Proposition 4.2, $F_{3,n}(s)$ is analytic for $1 - \mu_0 \leq \operatorname{Re} s \leq 1 + \delta_0$. In this case we have an infinite integral with respect to ω and we will exploit the factor ϵ_n^2 to estimate it. By using (4.6) with $0 < \nu < 1$, we must treat

$$\epsilon_n^2 \int_{|\omega| \geq M} \Big(1 + |\operatorname{Im} s|^{\nu} + |\omega|^{\nu}\Big) |\hat{\chi}(\epsilon_n \omega)| d\omega.$$

The Fourier transform of $\chi \in C_0^{\infty}(\mathbb{R})$ satisfies

$$|\hat{\chi}(\epsilon_n \omega)| \leq D_2 (1 + |\epsilon_n \omega|)^{-2}$$

with a constant $D_2 > 0$ independent on ϵ_n and ω. By a change of variable $\epsilon_n \omega = \xi$ we get a convergent integral with respect to ξ and with a constant $C_3(\nu) > 0$ independent on n we deduce the estimate

$$|F_{3,n}(t)| \leq C_3(\nu) \epsilon_n^{1-\nu} (1 + |\operatorname{Im} s|^{\nu}) \leq C_3(\nu)(1 + |\operatorname{Im} s|^{\nu}). \tag{6.2}$$

Therefore, $F_{3,n}(s)$ contributes to the term $L_n(s)$ in (5.1) and we cannot get a coefficient A_n.

It remains to study the behaviour of $F_{1,n}(s)$. Here there are no problems with the convergence of the integral with respect to ω and we gain the factor ϵ_n^2. For $\epsilon_0 \leq |\operatorname{Im} s|$ the Laplace transform $F_{1,n}(s)$ has no singularities and as above we obtain a contribution to $A_n K_n(s)$ in (5.1). Let $U_2 = \{s \in \mathbb{C}, \omega \in \mathbb{R} : \ |s - \gamma(a)| \leq$

$\mu_0, |\omega| \leq \epsilon_0\}$. Recall that for $(s, \omega) \in U_2$ the function $Z(s, \omega, a)$ (independent on n) has a pole $s(\omega, a)$ and

$$Z(s+\gamma(a)-1, \omega, a) = \Big(\frac{B_3(s(\omega, a), \omega, a)}{\int \tau d\nu_{f_a - s(\omega,a)\tau + \mathbf{i}\omega g}}\Big)\frac{1}{s+\gamma(a)-1-s(\omega, a)} + J_5(s, \omega, a)$$

with a function $J_5(s, \omega, a)$ analytic in s and real analytic in ω. Here we may repeat without any change the argument in Section 5 in [20]. Applying the Morse lemma to the function $\operatorname{Re} s(\omega, a)$, there exists a function $y = y(\omega, a)$ defined for $|\omega| \leq \epsilon_0$ such that

$$\operatorname{Re} s(\omega, a) = \gamma(a) - y^2.$$

Therefore the analysis is reduced to the integral

$$\frac{\epsilon_n^2}{2\pi}\int_{-\epsilon_0}^{\epsilon_0} \frac{B_3\Big(\gamma(a) - y^2(\omega, a) + \mathbf{i}q(\omega, a), \omega, a\Big)}{s - 1 + y^2(\omega, a) + \mathbf{i}q(\omega, a)}\hat{\chi}(\epsilon_n\omega)d\omega,$$

where (see Lemma 3 in [20]) $q(\omega, a) = \operatorname{Im} s(\omega, a)$ is such that $q(0, a) = \frac{\partial q}{\partial \omega}(0, a) = \frac{\partial^2 q}{\partial \omega^2}(0, a) = 0$, and

$$\frac{\partial^2}{\partial \omega^2}\operatorname{Re} s(0, a) = -\sigma^2(m_{F+\xi(a)G}) = -\beta''(\xi(a)) < 0$$

with $\beta(\xi)$ and $\xi(a)$ introduced in Section 1.

Next the analysis follows that in Section 3 in [8] without any change. After a change of variable $\omega = \omega(y, a)$ the integral has the representation

$$\frac{\epsilon_n^2 C(a)}{\pi}\frac{\hat{\chi}(0)}{\sqrt{2\beta''(\xi(a))}}\Big[\int_{-y(\epsilon_0,a)}^{y(\epsilon_0,a)} \frac{1}{s-1+y^2}dy$$
$$- \int_{-y(\epsilon_0,a)}^{y(\epsilon_0,a)} \frac{\mathbf{i}Q(y,a)}{(s-1+y^2+\mathbf{i}Q(y,a))(s-1+y^2)}dy\Big]$$
$$+\frac{\epsilon_n^2}{2\pi}\frac{\sqrt{2}}{\sqrt{\beta''(\xi(a))}}\int_{-y(\epsilon_0,a)}^{y(\epsilon_0,a)} \frac{P(y)}{s-1+y^2+\mathbf{i}Q(y,a)}dy,$$

where

$$C(a) = \frac{1}{(\int \tau d\mu)^2}B_3(\gamma(a), 0, a),$$

$P(y)$ is a complex valued function such that $P(0) = 0$ and $Q(y, a)$ is real valued odd function with respect to y such that $Q(0, a) = Q'_y(0, a) = Q''_{yy}(0, a) = 0$. Here $B_3(\gamma(a), 0, a)$ is given by (4.1) and we have used that

$$\frac{\partial y}{\partial \omega}(0, a) = \Big(\sqrt{-\frac{1}{2}\frac{\partial^2}{\partial \omega^2}\operatorname{Re} s(\omega, a)}\Big)\Big|_{\omega=0} = \frac{\beta''(\xi(a))}{\sqrt{2}}$$

combined with the Taylor expansions for the functions $\operatorname{Re} s(\omega, a)$, $\operatorname{Im} s(\omega, a)$ and $\hat{\chi}(\epsilon_n\omega)$ around $\omega = 0$. In particular,

$$\hat{\chi}(\epsilon_n\omega) = \hat{\chi}(0) + \epsilon_n\omega(y, a)\hat{\chi}'(0) + \mathcal{O}(\epsilon_n^2\omega^2(y, a)).$$

The first term in the above representation yields the singularity $\frac{A_n}{\sqrt{s-1}}$ plus more regular terms, where

$$A_n = \frac{C(a)\hat{\chi}(0)}{\sqrt{2\beta''(\xi(a)}}\epsilon_n^2.$$

The dependence on n is caused by the coefficient ϵ_n^2 involved in A_n. Finally, we obtain

$$F_{1,n}(s) = \frac{A_n}{\sqrt{s-1}} + A_n \omega_{1,n}(s)$$

and for $|\operatorname{Im} s| \leq \epsilon_0$ we have uniform with respect to n bounds for the $L^1(-\epsilon_0, \epsilon_0)$ norms of $f_{1,n}(1+\mathbf{i}t)$ and $f'_{1,n}(1+\mathbf{i}t)$. Summing the analysis of $F_{k,n}(s)$, $k=1,2,3$, we get the representation (5.1).

Our purpose is to apply Proposition 5.6 for the functions $g_n(t)$. To do this, we need to estimate the derivative

$$\begin{aligned} g_n'(t) = \epsilon_n(-\gamma(a)+1)e^{(-\gamma(a)+1)t} \int_{R^\tau} q_n(G^t - at)(y) dm_F(y) \\ + \epsilon_n e^{(-\gamma(a)+1)t} \xi(a) \int_{R^\tau} q_n(G^t - at)(y)(G(\sigma_t^\tau(y)) - a) dm_F(y) \\ + e^{(-\gamma(a)+1)t} \int_{R^\tau} e^{\xi(a)(G^t(y)-at)} \chi'\Big(\frac{G^t - at}{\epsilon_n}\Big)(y)(G(\sigma_t^\tau(y) - a) dm_F(y). \end{aligned} \tag{6.3}$$

We can choose a function $0 \leq \psi(t) \in C_0^\infty(\mathbb{R})$ such that $\psi(t) = M_1 > 0$ for $t \in \operatorname{supp} \chi(\frac{t}{\epsilon_n})$, where the constant M_1 (independent of n) is chosen so that

$$\max_{t\in\mathbb{R}} \chi(t) + \max_{t \in \mathbb{R}} |\chi'(t)| < M_1.$$

Then for every fixed compact $J \Subset \Gamma_G$ and $a \in J$ there exists a constant $C(J) > 0$ independent of n and a such that

$$|g_n'(t)| \leq C(J) e^{(-\gamma(a)+1)t} \int_{R^\tau} e^{\xi(a)(G^t - at)} \psi(G^t - at)(y) dm_F(y) = C(J)\Psi(t).$$

Notice that $C(J)$ depends on $\xi(a), a$ and the maximum of $G(w)$, but $C(J)$ is independent of M_1. The problem is reduced to an estimate of $\Psi(t)$. For the nonnegative function $\Psi(t)$ we wish to apply Lemma 5.7. Consider the Laplace transform

$$Y(s) = \int_0^\infty e^{-st} \Psi(t) dt$$

for $\operatorname{Re} s > 1$ and its limit as $\operatorname{Re} s \searrow 1$. The analysis is completely the same as that of $F_n(s)$, where the function $\hat{\chi}_n(\omega)$ must be replaced by $\hat{\psi}(\omega)$ which is independent on n. Therefore with some constant $A > 0$ one deduces the representation

$$Y(s) = \frac{A}{\sqrt{s-1}} + P(s)$$

with function $P(s)$ having the properties of $K(s)$ mentioned in Lemma 5.7. To satisfy the condition (5.17), first as above we obtain an upper bound

$$|\Psi'(t)| \leq C_1(J) e^{(-\gamma(a)+1)t} \int_{R^\tau} e^{\xi(a)(G^t - at)} \psi_1(G^t - at)(y) dm_F(y) = C_1(J)\Psi_1(t), \tag{6.4}$$

where $0 \leq \psi_1(t) \in C_0^\infty(\mathbb{R})$ is such that for $t \in \operatorname{supp} \psi(t)$ we have

$$\psi_1(t) \geq \max_{t \in \mathbb{R}} \psi(t) + \max_{t\in\mathbb{R}} |\psi'(t)|.$$

We repeat this procedure once more and with a function $0 \leq \psi_2(t) \in C_0^\infty(\mathbb{R})$ and constant $C_2(J)$ one arranges the bound

$$|\Psi_1'(t)| \leq C_2(J)e^{(-\gamma(a)+1)t} \int_{R^\tau} e^{\xi(a)(G^t-at)}\psi_2(G^t - at)(y)dm_F(y) = C_2(J)\Psi_2(t).$$

Now the analysis in Section 4 shows that the Laplace transforms

$$\int_0^\infty e^{-st}\Psi_k(t)dt,\ k = 1, 2,$$

are analytic for $\operatorname{Re} s > 1$. Therefore the integral

$$\int_0^\infty e^{-st}\Psi_1'(t)dt$$

is absolutely convergent for $\operatorname{Re} s > 1$, and for $s = 1 + \delta > 1$ we have

$$\int_0^\infty e^{-st}\Psi_1'(t)dt = \Big[e^{-st}\Psi_1(t)\Big]_0^\infty + s\int_0^\infty e^{-st}\Psi_1(t)dt,$$

hence for every $\delta > 0$ we have $\lim_{t\to\infty} e^{-(1+\delta)t}\Psi_1(t) = 0$. This estimate combined with (6.4) implies a bound

$$|\Psi'(t)| \leq C_\delta C_1(J)\frac{e^{(1+\delta)t}}{\sqrt{t}},\ t \geq 1$$

and we are in position to apply Lemma 5.7 for the function $\Psi(t)$. The statement of Lemma 5.7 yields the estimate (5.19) for $\Psi(t)$ with a constant $D_0 > 0$ independent on n, C_δ, $C_1(J)$, $C_2(J)$ and we obtain

$$|g_n'(t)| \leq D_0C(J)\frac{e^t}{\sqrt{t}},\ \forall t \geq t_0,\ \forall n \in \mathbb{N}. \tag{6.5}$$

Following Remark 5.9, we may take $t_0 > 1$ large to guarantee the independence of D_0.

On the other hand, it is clear that we may estimate $\max_{0\leq t\leq 1} g_n(t)$ uniformly with respect to n and $a \in J$. Thus we can apply Proposition 5.6 for $g_n(t)$. We cancel the coefficients ϵ_n and e^t in the estimates for $g_n(t)$ and one concludes that for fixed $q \geq 0$, $T \geq n - q$ and any $0 < \eta \ll 1$ there exists $n_0(\eta) \in \mathbb{N}$ such that for $n \geq n_0(\eta) + q$ we have

$$\frac{\epsilon_n C(a)\int\chi(t)dt}{\sqrt{2\pi T\beta''(\xi(a)}}e^{\gamma(a)T}(1-\eta) \leq \rho_n(T) \leq \frac{\epsilon_n C(a)\int\chi(t)dt}{\sqrt{2\pi T\beta''(\xi(a)}}e^{\gamma(a)T}(1+\eta). \tag{6.6}$$

It is easy to see that if one examines the function

$$\tilde{\rho}_n(T) = \int_U \chi\Big(\frac{(G^T - aT)(y)}{\epsilon_n}\Big)dm_F(y),$$

then for fixed $q \geq 0$ and any $0 < \eta \ll 1$ there exists $n_0(y) \in \mathbb{N}$ such that for $n \geq n_0(\eta) + q$ we get

$$\frac{\epsilon_n C(a)\int\chi(t)dt}{\sqrt{2\pi T\beta''(\xi(a)}}e^{\gamma(a)T}(1-\eta) \leq \tilde{\rho}_n(T) \leq \frac{\epsilon_n C(a)\int\chi(t)dt}{\sqrt{2\pi T\beta''(\xi(a)}}e^{\gamma(a)T}(1+\eta). \tag{6.7}$$

since the Fourier transform $\epsilon_n\hat{\chi}(\epsilon_n\omega)$ must be replaced by the Fourier transform $\epsilon_n\hat{\chi}(\epsilon_n(\omega - \mathbf{i}\xi(a))$ and

$$\int e^{-\epsilon_n\xi(a)t}\chi(t)dt = \int\chi(t)dt + \mathcal{O}(\epsilon_n).$$

Approximating the characteristic function $\mathbf{1}_{[-1,1]}(t)$ by cut-off functions $\chi_{\pm}(t) \in C_0^{\infty}(\mathbb{R}; [0,1])$ such that

$$\chi_-(t) \leq \mathbf{1}_{[-1,1]}(t) \leq \chi_+(t),$$

we obtain for $T \geq n - q$, $n \geq n_0(\eta) + q$ the estimates (1.1) and this proves Theorem 1.3.

It is worth noting that the derivatives $\chi'_+(t)$ may increase for $-1 - \delta \leq t \leq -1$ and $1 \leq t \leq 1 + \delta$, but this is not important for the estimate (6.5) since we may arrange D_0 to be independent of the derivatives $\chi'_+(t)$ choosing $t \geq t_1$. This reflects in the choice of $n_0(\eta)$ in (6.5). The same observation holds for the derivative $\chi'_-(t)$ in $-1 \leq t \leq 1 - \delta$ and $1 - \delta \leq t \leq 1$.

Now it is easy to pass to the analysis of the intervals $\Big(-\frac{e^{-\epsilon T}}{T}, \frac{e^{-\epsilon T}}{T}\Big)$. In fact, let $q = 1$, $0 < \eta \ll 1$ and $n_0(\eta) \in \mathbb{N}$ be as above. Let $T \geq n_0(\eta) + 1$ and let $N(\eta) \geq n_0(\eta) + 1$ be chosen so that $N(\eta) \leq T \leq N(\eta) + 1$. Obviously, we have $\Big(-e^{-\epsilon(N(\eta)+1)}, e^{-\epsilon(N(\eta)+1)}\Big) \subset \Big(-e^{-\epsilon T}, e^{-\epsilon T}\Big) \subset \Big(-e^{-\epsilon N(\eta)}, e^{-\epsilon N(\eta)}\Big)$. Now we may examine

$$\zeta(T; a) = m_F\Big\{w \in R^{\tau} : \int_0^T G(\sigma_t^{\tau}(w) dt - aT \in \Big(-e^{-\epsilon T}, e^{-\epsilon T}\Big)\Big\}.$$

For $T \geq n_0(\eta) + 1$ we deduce the estimates

$$\frac{2e^{-\epsilon T} e^{-\epsilon} C(a)}{\sqrt{2\pi T \beta''(\xi(a)}} e^{\gamma(a)T}(1 - \eta) \leq \frac{2e^{-\epsilon(N(\eta)+1)} C(a)}{\sqrt{2\pi T \beta''(\xi(a)}} e^{\gamma(a)T}(1 - \eta) \leq \zeta(T; a), \tag{6.8}$$

$$\zeta(T; a) \leq \frac{2e^{-\epsilon N(\eta)} C(a)}{\sqrt{2\pi T \beta''(\xi(a)}} e^{\gamma(a)T}(1 + \eta) \leq \frac{2e^{\epsilon} e^{-\epsilon T} C(a)}{\sqrt{2\pi T \beta''(\xi(a)}} e^{\gamma(a)T}(1 + \eta). \tag{6.9}$$

To obtain (6.8), we exploit $T \geq [N(\eta) + 1] - 1$, while for (6.9) we use $T \geq N(\eta)$. These estimates prove the statement of Theorem 1.4.

Appendix

Proof of Proposition 2.2. In what follows we will denote global generic constants by $C > 0$ and $c > 0$.

(a) We will first show that $\tilde{G}$ is constant on stable leaves of R^{τ}. Let $\xi, \eta \in R^{\tau}$ be on the same stable leaf of R^{τ}. Thus, $\xi = \pi(x,t)$, $\eta = \pi(y,t)$ for some $x, y \in R_i$ with $\pi_U(x) = \pi_U(y) = z \in U_i$ and some $t \in [0, \tau(x))$. Then $\tau(y) = \tau(x) = \tau(z)$. Moreover, $s = \tau(\mathcal{P}^{n+1}(x)) = \tau(\mathcal{P}^{n+1}(y))$ and also $s = \tau(\mathcal{P}^n(\pi_U(\mathcal{P}(y)))$. Finally, $\pi_U(\mathcal{P}(x)) = \pi_U(\mathcal{P}(y))$. Using all these in the definition of $\tilde{G}$, gives $\tilde{G}(x,t) = \tilde{G}(y,t)$.

To prove (2.6), write

$$\begin{aligned} h(x) - h(\mathcal{P}(x)) &= \sum_{n=0}^{\infty} [g(\mathcal{P}^n(x)) - g(\mathcal{P}^n(\pi_U(x)))] \\ &\qquad - \sum_{n=0}^{\infty} [g(\mathcal{P}^n(\mathcal{P}(x))) - g(\mathcal{P}^n(\pi_U(\mathcal{P}(x))))] \\ &= g(x) - g(\pi_U(x)) + \sum_{n=1}^{\infty} [g(\mathcal{P}^n(x)) - g(\mathcal{P}^n(\pi_U(x)))] \\ &\qquad - \sum_{n=0}^{\infty} [g(\mathcal{P}^n(\mathcal{P}(x))) - g(\mathcal{P}^n(\pi_U(\mathcal{P}(x))))] \qquad \text{(A.1)} \\ &= g(x) - g(\pi_U(x)) + \sum_{n=0}^{\infty} \left[g(\mathcal{P}^{n+1}(x)) - g(\mathcal{P}^{n+1}(\pi_U(x)))\right] \\ &\qquad - \sum_{n=0}^{\infty} [g(\mathcal{P}^n(\mathcal{P}(x))) - g(\mathcal{P}^n(\pi_U(\mathcal{P}(x))))] \\ &= g(x) - \left[g(\pi_U(x)) + \sum_{n=0}^{\infty} \left(g(\mathcal{P}^{n+1}(\pi_U(x))) - g(\mathcal{P}^n(\pi_U(\mathcal{P}(x)))) \right) \right]. \end{aligned}$$

Next, for every $x \in \widehat{R}$, using the change of variable $t \mapsto s = t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x)$ in some of the integrals below, we get

$$\begin{aligned} \tilde{g}(x) &= \int_0^{\tau(x)} \tilde{G}(x,t)\, dt = \int_0^{\tau(x)} G(\pi_U(x), t)\, dt \\ &\quad + \sum_{n=0}^{\infty} [\int_0^{\tau(x)} G(\mathcal{P}^{n+1}(\pi_U(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\, dt \\ &\quad - \int_0^{\tau(x)} G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\, dt] \\ &= g(\pi_U(x)) + \sum_{n=0}^{\infty} [\int_0^{\tau(\mathcal{P}^{n+1}(x))} G(\mathcal{P}^{n+1}(\pi_U(x))), s)\, ds \\ &\quad - \int_0^{\tau(\mathcal{P}^{n+1}(x))} G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s)\, ds] \\ &= g(\pi_U(x)) + \sum_{n=0}^{\infty} \left(g(\mathcal{P}^{n+1}(\pi_U(x))) - g(\mathcal{P}^n(\pi_U(\mathcal{P}(x)))) \right). \end{aligned}$$

This and (A.1) imply $h(x) - h(\mathcal{P}(x)) = g(x) - \tilde{g}(x)$, thus proving (2.6).

It remains to prove that $\tilde{G}$ and h are β-Hölder on $\widehat{R}^\tau$ and $\widehat{R}$ respectively, for some $\beta > 0$.

Let $x \neq y$ belong to some $R_i \cap \widehat{R}$ and let $0 \leq t < \tau(x)$ and $0 \leq t' < \tau(y)$. We may assume $\tau(x) \leq \tau(y)$. We have to estimate $|\tilde{G}(x,t) - \tilde{G}(y,t')|$. We will first estimate $|\tilde{G}(x,t) - \tilde{G}(y,t)|$.

Set $\tilde{x} = \pi_U(x)$ and $\tilde{y} = \pi_U(y)$ and $s = t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x)$. Then $d(\tilde{x},\tilde{y}) \leq C(d(x,y))^\alpha$ for some global constant $C > 0$.

Let $2m$ (or $2m+1$) be the maximal positive integer so that $\mathcal{P}^j(x), \mathcal{P}^j(y)$ belong to the same rectangle R_{i_j} for $j = 0, 1, \ldots, 2m-1$. Then by (2.2),

$$c \le d(\mathcal{P}^{2m}(x), \mathcal{P}^{2m}(y)) \le (c/c_0)\gamma_1^{2m}\,(d(x,y))^{\alpha},$$

so $d(x,y) \ge \frac{c}{\gamma_1^{2m/\alpha}}$, and therefore

$$(d(x,y))^{\alpha\alpha'/2} \ge \frac{c}{\gamma_1^{\alpha' m}} = \frac{c}{(\rho\gamma)^m} > \frac{c}{\gamma^m}. \tag{A.2}$$

For any integer $n = 0, 1, \ldots, m-1$, using (2.2) and the latter, we get

$$\begin{aligned} d(\mathcal{P}^{n+1}(\tilde{x}), \mathcal{P}^{n+1}(\tilde{y})) &\le \frac{d(\mathcal{P}^{2m}(\tilde{x}), \mathcal{P}^{2m}(\tilde{y}))}{c_0\gamma^{2m-n}} \le \frac{C}{\gamma^{2m-n}} \\ &\le \frac{C}{\gamma^{m-n}} \cdot \frac{1}{\gamma^m} \le \frac{C}{\gamma^{m-n}} \cdot (d(x,y))^{\alpha\alpha'/2}. \end{aligned} \tag{A.3}$$

We will now estimate $|\tilde{G}(x,t) - \tilde{G}(y,t)|$. We have

$$\begin{aligned} |\tilde{G}(x,t) - \tilde{G}(y,t)| &\le |G(\tilde{x},t) - G(\tilde{y},t)| + \sum_{n=0}^{m-1} \left|G(\mathcal{P}^{n+1}(\tilde{x})), s) - G(\mathcal{P}^{n+1}(\tilde{y})), s)\right| \\ &\quad + \sum_{n=0}^{m-1} |G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s)| \\ &\quad + \sum_{n=m}^{\infty} \left|G(\mathcal{P}^{n+1}(\pi_U(x))), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s)\right| \\ &\quad + \sum_{n=m}^{\infty} \left|G(\mathcal{P}^{n+1}(\pi_U(y))), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(y))), s)\right| \\ &= I + II + III + IV + V. \end{aligned}$$

Clearly, $I \le |G|_\alpha (d(\tilde{x},\tilde{y}))^\alpha \le C\,|G|_\alpha\,(d(x,y))^{\alpha^2}$.

Next,

$$\begin{aligned} II &= \sum_{n=0}^{m-1} \left|G(\mathcal{P}^{n+1}(\tilde{x})), s) - G(\mathcal{P}^{n+1}(\tilde{y}), s)\right| \\ &\le \sum_{n=0}^{m-1} |G|_\alpha \left(\frac{C}{\gamma^{m-n}} \cdot (d(x,y))^{\alpha\alpha'/2}\right)^{\alpha} \le C\,|G|_\alpha\,(d(x,y))^{\beta}. \end{aligned}$$

Similarly, $III \le C\,|G|_\alpha\,(d(x,y))^\beta$.

Since $\mathcal{P}(\pi_U(x))$ and $\pi_U(\mathcal{P}(x))$ are on the same stable leaf of some rectangle, it follows from (2.3) that

$$\begin{aligned} &\left|G(\mathcal{P}^{n+1}(\pi_U(x))), s) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), s)\right| \\ &\le |G|_\alpha\,(d(\mathcal{P}^{n+1}(\pi_U(x))), \mathcal{P}^n(\pi_U(\mathcal{P}(x)))^\alpha \le C\,|G|_\alpha \frac{1}{\gamma^{\alpha n}}. \end{aligned}$$

This and (A.2) yield

$$IV \le C\,|G|_\alpha \sum_{n=m}^{\infty} \frac{1}{\gamma^{\alpha n}} \le C\,|G|_\alpha \frac{1}{\gamma^{\alpha m}} \le C\,|G|_\alpha\,(d(x,y))^{\alpha^2\alpha'/2} C\,|G|_\alpha\,(d(x,y))^\beta.$$

In a similar way we obtain $V \leq C\,|G|_\alpha\,(d(x,y))^\beta$. Thus,

$$|\tilde{G}(x,t) - \tilde{G}(y,t)| \leq C\,|G|_\alpha\,(d(x,y))^\beta .$$

An estimate of the form $|\tilde{G}(y,t) - \tilde{G}(y,t')| \leq C|G|_\alpha\,|t-t'|^\beta$ can be obtained rather easily, and we leave the details to the reader. This proves that $\tilde{G}$ is β-Hölder and $|\tilde{G}|_\beta \leq |G|_\alpha$.

The proof that h is also β-Hölder is very similar to the above, in fact it is easier. We leave the details to the reader.

(b) The proof that H is β-Hölder is very similar to the proof above that $\tilde{G}$ is β-Hölder. We leave the details to the reader.

To establish (2.9), replace $H(x,t)$ in the integral in (2.9) by the right-hand-side of (2.8) and use the change of variable $t \mapsto s = t\,\tau(\mathcal{P}^n(x))/\tau(x)$. What we obtain in this way is the right-hand-side of (2.5). This proves (2.9).

Finally, to establish (2.10), write:

$$\begin{aligned}
& H(x,t) - H(\mathcal{P}(x), t\,\tau(\mathcal{P}(x))/\tau(x)) \\
= & \sum_{n=0}^{\infty} [G(\mathcal{P}^n(x), t\,\tau(\mathcal{P}^n(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(x)), t\,\tau(\mathcal{P}^n(x))/\tau(x))] \\
& - \sum_{n=0}^{\infty} \left[G(\mathcal{P}^{n+1}(x), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\right] \\
= & G(x,t) - G(\pi_U(x),t) \\
& + \sum_{n=1}^{\infty} [G(\mathcal{P}^n(x), t\,\tau(\mathcal{P}^n(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(x)), t\,\tau(\mathcal{P}^n(x))/\tau(x))] \\
& - \sum_{n=0}^{\infty} \left[G(\mathcal{P}^{n+1}(x), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\right] \\
= & G(x,t) - G(\pi_U(x),t) \\
& + \sum_{n=0}^{\infty} \left[G(\mathcal{P}^{n+1}(x), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x)) - G(\mathcal{P}^{n+1}(\pi_U(x)), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\right] \\
& - \sum_{n=0}^{\infty} \left[G(\mathcal{P}^{n+1}(x), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x)) - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))\right] \\
= & G(x,t) - (\ G(\pi_U(x),t) + \sum_{n=0}^{\infty} [G(\mathcal{P}^{n+1}(\pi_U(x)), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x)) \\
& - G(\mathcal{P}^n(\pi_U(\mathcal{P}(x))), t\,\tau(\mathcal{P}^{n+1}(x))/\tau(x))]\ ) \\
= & G(x,t) - \tilde{G}(x,t).
\end{aligned}$$

This proves (2.10).

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