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Differential Equations and Population Dynamics II: Advanced Approaches

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Contents

Part I
Dynamical Systems in Population Dynamics

Chapter 1
Semiflows, ω-limit Sets, α-limit Sets, Attraction, and Dissipation

1.1 Introduction

Let (M, d) be a complete metric space. Typical examples in population dynamics will be

$$M = \mathbb{R}_+^n = [0, \infty)^n.$$

A more general class of subset M are the intervals in $\mathbb{R}^n$. That is a subset of the following form

$$M = [c, \infty) = \{x \in \mathbb{R}^n : c_i \leq x_i\},$$

$$M = [c, d] = \{x \in \mathbb{R}^n : c_i \leq x_i \leq d_i\},$$

or

$$M = (-\infty, d] = \{x \in \mathbb{R}^n : x_i \leq d_i\},$$

where $c, d \in \mathbb{R}^n$ and $c \leq d$.

These subsets will be endowed with usual distance (induced by the norm $\|.\|$ of $\mathbb{R}^n$)

$$d(x, y) = \|x - y\|, \forall x, y \in M.$$

Then (M, d) be a complete metric space (since the subsets M are closed subsets of $\mathbb{R}^n$).

In order to consider both discrete and continuous time dynamical systems, the time will vary either in

$$I_+ = [0, +\infty) \text{ and } I = \mathbb{R} \text{ (}\textbf{continuous time}\text{)}$$

or in

$$I_+ = \mathbb{N} \text{ and } I = \mathbb{Z} \text{ (}\textbf{discrete time}\text{)}.$$

Definition 1.1 Let $\{U(t)\}_{t \geq 0}$ be a family of continuous maps from M into itself parameterized by $t \in I_+$. We will say that U is a **continuous semiflow** on M if the following properties are satisfied

(i) $U(0)x = x, \forall x \in M$;

(ii) $U(t)U(s)x = U(t+s)x, \forall t, s \geq 0, \forall x \in M$;
(iii) The map $(t,x) \to U(t)x$ is continuous from $I_+ \times M$ into M.

Moreover we will say that U is a **continuous time semiflow** if $I_+ = [0, +\infty)$ and a **discrete time semiflow** if $I_+ = \mathbb{N}$.

Remark 1.2 Since we assumed that the map $x \to U(t)x$ continuous (for each $t \geq 0$), it follows that a discrete time semiflow is always continuous. That is to say that the property (iii) of Definition 1.1 is always satisfied for discrete time semiflow.

1.1.1 Discrete time semiflow and difference equation

Assume that a discrete semiflow $\{U(t)\}_{t\in\mathbb{N}}$ is given. Define the map $T : M \to M$

$$T(x) = U(1)(x)$$

and the sequence $\{u_n\}_{n\in\mathbb{N}}$

$$u_n := U(n)(x), \forall n \geq 0.$$

Then by using the property (ii), we obtain

$$u_n = U(n)(x) = U(1)(U(n-1)(x)) = T(u_{n-1}), \forall n \in \mathbb{N}, \text{ and } u_0 = x.$$

So we obtain a difference equation

$$u_{n+1} = T(u_n), \forall n \in \mathbb{N}, \text{ with } u_0 = x \in M.$$

We also observe that

$$\begin{aligned} u_0 &= x \\ u_1 &= T(x) \\ u_2 &= T(T(x)) = T^2(x) \end{aligned}$$

and we obtain

$$u_n = T^n(x), \forall n \in \mathbb{N},$$

where T^n is defined by

$$T^{n+1} = T \circ T^n, \forall n \in \mathbb{N} \text{ and } T^0 = I.$$

Conversely, assume that a map $T : M \to M$ is given. Then the semiflow $\{U(t)\}_{t\in\mathbb{N}}$ is defined by

$$U(n)(x) = T^n(x), \forall n \in \mathbb{N}, \forall x \in M,$$

where $T : M \to M$ is a continuous map.

1.1.2 Semiflow generated by the 1-dimensional logistic equation

Let us consider the family of maps $\{U(t)\}_{t\geq 0}$ defined on $M = \mathbb{R}_+$ as follows

$$U(t)x = \frac{e^{\lambda t}x}{1+\kappa\int_0^t e^{\lambda\sigma}xd\sigma}, \; \forall t \geq 0, \; \forall x \geq 0, \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and $\kappa \geq 0$.

Lemma 1.3 *The family* $\{U(t)\}_{t\leq 0}$ *is a semiflow on* $M = [0, \infty)$.

Remark 1.4 The semiflow $U(t)$ can not be extended (backward in time) to a flow because the solution of the logistic equation $t \to U(t)x$ is blowing up for negative time (whenever $x > \frac{\lambda}{\kappa}$). But $U(t)$ restricted to $[0, \frac{\lambda}{\kappa}]$ defines a flow (whenever $\lambda > 0$).

Proof Let us verify that $U(t)$ is a continuous semiflow on $\mathbb{R}_+$. Indeed, it is clear that

$$U(0)x = x, \forall x \in \mathbb{R}_+.$$

Let $t, s \geq 0$, we have

$$U(t)U(s)x = \frac{e^{\lambda t}\frac{e^{\lambda s}x}{1+\kappa\int_0^s e^{\lambda r}xdr}}{1+\kappa\int_0^t e^{\lambda\sigma}\frac{e^{\lambda s}x}{1+\kappa\int_0^s e^{\lambda r}xdr}d\sigma}$$

$$= \frac{e^{\lambda(t+s)}x}{1+\kappa\int_0^s e^{\lambda r}xdr + \kappa\int_0^t e^{\lambda(\sigma+s)}xd\sigma}$$

and by using a change of variable, we obtain that

$$\int_0^t e^{\lambda(\sigma+s)}xd\sigma = \int_s^{t+s} e^{\lambda r}xdr,$$

and it follows that

$$U(t)U(s)x = \frac{e^{\lambda(t+s)}x}{1+\kappa\int_0^{t+s} e^{\lambda r}xdr} = U(t+s)x.$$

Therefore U is a semiflow on $\mathbb{R}_+$. □

Remark 1.5 The map $t \to N(t) := U(t)x$ satisfies the logistic equation

$$N'(t) = \lambda N(t) - \kappa N^2(t). \tag{1.2}$$

We refer to Chapter 5 in Volume I for more results.

1.1.3 Semiflow generated by a 2-dimensional Bernoulli-Verhulst equation

Let $\theta > 0$. Consider the family of maps $V(t)$ on $\mathbb{R}^2$ defined by

$$V(t)X = \begin{cases} \left(\frac{U(t)\left(\|X\|_2^\theta\right)}{\|X\|_2^\theta}\right)^{\frac{1}{\theta}} e^{At}X, & \text{if } X = \begin{pmatrix} x \\ y \end{pmatrix} \neq 0, \\ 0, & \text{if } X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{cases} \tag{1.3}$$

where $U(t)$ is the semiflow defined by (1.1), and

$$\|X\|_2 = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 = \sqrt{x^2 + y^2}$$

is the Euclidean norm, and (see Section 3.9.3 in Volume I) we have

$$e^{At} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\omega t)x - \sin(\omega t)y \\ \sin(\omega t)x + \cos(\omega t)y \end{pmatrix},$$

with

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Lemma 1.6 *The family $\{V(t)\}_{t \leq 0}$ is a semiflow on $\mathbb{R}^2$.*

Proof We first observe that

$$\begin{aligned} \left\|e^{At}X\right\|_2^2 &= (\cos(\omega t)x)^2 - 2\cos(\omega t)x\sin(\omega t)y + (\sin(\omega t)y)^2 \\ &+ (\sin(\omega t)x)^2 + 2\sin(\omega t)x\cos(\omega t)y + (\cos(\omega t)y)^2 \\ &= x^2 + y^2 \\ &= \|X\|_2^2, \end{aligned}$$

and we deduce that e^{At} preserves the Euclidean norm of X

$$\left\|e^{At}X\right\|_2 = \|X\|_2, \forall t \in \mathbb{R}.$$

It follows that

$$V(t)\left(\mathbb{R}^2 \setminus \{0\}\right) \subset \mathbb{R}^2 \setminus \{0\}, \forall t \geq 0.$$

Let $X \neq 0$. By applying the Euclidean norm on both sides of (1.2), we deduce that

$$\|V(t)X\|_2^\theta = \frac{U(t)\left(\|X\|_2^\theta\right)}{\|X\|_2^\theta} \left\|e^{At}X\right\|_2^\theta = U(t)\left(\|X\|_2^\theta\right) \forall t \geq 0,$$

and

$$
\begin{aligned}
V(t)V(s)X &= \left(\frac{U(t)\left(\|V(s)X\|_2^\theta\right)}{\|V(s)X\|_2^\theta}\right)^{\frac{1}{\theta}} e^{At}V(s)X \\
&= \left[U(t)\left(U(s)\left(\|X\|_2^\theta\right)\right)\right]^{\frac{1}{\theta}} e^{At}\frac{V(s)X}{\|V(s)X\|_2} \\
&= \left[U(t+s)\left(\|X\|_2^\theta\right)\right]^{\frac{1}{\theta}} e^{At}\frac{e^{As}X}{\|X\|_2} \\
&= V(t+s)X,
\end{aligned}
$$

whenever $t \geq 0$ and $s \geq 0$. It follows that $V(t)$ is a continuous semiflow. □

Remark 1.7 The map $t \to X(t) = V(t)X$ satisfies the 2-dimensional Bernoulli-Verhulst equation

$$
X'(t) = \left[A + \frac{\lambda}{\theta}I\right]X(t) - \frac{\kappa}{\theta}\|X(t)\|_2^\theta\, X(t), \forall t \geq 0. \tag{1.4}
$$

We refer to chapter 5 in volume I for more results.

1.1.4 Explicit formula for the semiflow of the Poincaré normal form

In the special case $\theta = 2$, we obtain

$$
V(t)X = \begin{cases} \sqrt{U(t)\left(\|X\|_2^2\right)}\; e^{At}\left(\frac{X}{\|X\|}\right) & \text{if } X \neq 0, \\ 0 & \text{if } X = 0, \end{cases} \tag{1.5}
$$

and from the above computation we deduce that $t \to V(t)X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ satisfies the following system of ordinary differential equations

$$
\begin{cases} x'(t) = \frac{\lambda}{2}x(t) - \omega y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)x(t), \\ y'(t) = \omega x(t) + \frac{\lambda}{2}y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)y(t). \end{cases} \tag{1.6}
$$

The system (1.6) is nothing but the Poincaré normal form for Hopf bifurcation. We can say that the Poincaré normal form is a special case of the 2-dimensional Bernoulli-Verhulst equation.

1.2 Stability of an Equilibrium

Definition 1.8 We will say that $\overline{x} \in M$ is an **equilibrium** for U if

$$
U(t)\overline{x} = \overline{x}, \forall t \geq 0.
$$

Set

$$B_M(x,\varepsilon) := \{y \in M : d(x,y) \le \varepsilon\}.$$

Definition 1.9 We will say that an equilibrium $\overline{x} \in M$ is **stable** for U if for each $\varepsilon > 0$, there exists $\eta \in (0,\varepsilon]$ such that

$$U(t)B_M(\overline{x},\eta) \subset B_M(\overline{x},\varepsilon), \forall t \ge 0.$$

We will say that $\overline{x} \in M$ is **unstable** otherwise. That is to say that, there exist $\varepsilon > 0$, and a sequence $x_n \to \overline{x}$ and $t_n \to +\infty$ such that

$$\|U(t_n)x_n - \overline{x}\| > \varepsilon.$$

We will say that $\overline{x} \in M$ is **asymptotically stable**, if $\overline{x}$ is stable for U and if there exists $\eta > 0$ such that for each $x \in B_M(\overline{x},\eta)$

$$\lim_{t\to+\infty} U(t)x = \overline{x}.$$

We will say that $\overline{x} \in M$ is **exponentially asymptotically stable**, if $\overline{x}$ is stable for U and if in addition we can find three constants $\eta > 0$, $\alpha > 0$, $M \ge 1$ such that

$$d(U(t)x,\overline{x}) \le Me^{-\alpha t}d(x,\overline{x}), \forall t \ge 0, \forall x \in B_M(\overline{x},\eta).$$

Another equivalent definition of stability is the following. We will say that an equilibrium $\overline{x} \in M$ is stable if for each neighborhood V of $\overline{x}$ in M (that is to say that V contains a ball $B_M(\overline{x},\varepsilon)$), we can find a neighborhood $W \subset V$ of $\overline{x}$ in M such that

$$U(t)W \subset V, \forall t \ge 0.$$

In this case, by considering

$$\widehat{W} := \bigcup_{t\ge 0} U(t)W,$$

we have $U(0)W = W$ (since $U(0) = I$), so we deduce that

$$W \subset \widehat{W} \subset V.$$

So $\widehat{W}$ is a neighborhood of $\overline{x}$ in M, (since W is a neighborhood of $\overline{x}$ in M), and we have

$$U(t)\widehat{W} = U(t)\bigcup_{s\ge 0} U(s)W = \bigcup_{s\ge 0} U(t)U(s)W$$
$$= \bigcup_{s\ge 0} U(s+t)W = \bigcup_{l\ge t} U(l)W.$$

Thus

$$U(t)\widehat{W} \subset \widehat{W}, \forall t \ge 0.$$

Therefore we obtain the following lemma.

Lemma 1.10 *The following properties are equivalent*

(i) $\overline{x} \in M$ *is stable equilibrium for* U*;*
(ii) *For each neighborhood* V *of* $\overline{x}$ *in* M*, we can find neighborhood* $W \subset V$ *of* $\overline{x}$ *in* M*, such that*

$$U(t)W \subset V, \forall t \geq 0.$$

(iii) *For each neighborhood* V *of* $\overline{x}$ *in* M*, we can find a neighborhood* $W \subset V$ *of* $\overline{x}$ *in* M*, such that*

$$U(t)W \subset W, \forall t \geq 0.$$

We can observe that an equilibrium $\overline{x}$ is unstable for U if there exists $\varepsilon > 0$ such that for each $\eta > 0$ there exists $t = t(\eta) > 0$ such that

$$U(t)B_M(\overline{x}, \eta) \not\subset B_M(\overline{x}, \varepsilon).$$

Therefore by considering a sequence $\eta_n = 1/(n+1) \to 0$, we obtain the following lemma. For each integer $n \geq 0$ we can find $t_n \geq 0$ such that

$$U(t_n)B_M(\overline{x}, 1/(n+1)) \not\subset B_M(\overline{x}, \varepsilon).$$

Therefore we can finds $x_n \in B_M(\overline{x}, 1/(n+1))$ with

$$d(U(t_n)x_n, \overline{x}) \geq \varepsilon.$$

Moreover, we must have $t_n \to +\infty$ because U is a continuous semiflow. Otherwise, we can find a sub-sequence $t_{n_p} \to \widehat{t}$, and by continuity of U, we deduce that

$$\lim_{n\to\infty} d(U(t_n)x_n, \overline{x}) = 0,$$

which is impossible since $\varepsilon > 0$.

Lemma 1.11 *An equilibrium* $\overline{x} \in M$ *is unstable for* U *if and only if there exists* $\varepsilon > 0$ *and two sequences* $x_n \to \overline{x}$ *and* $t_n \to +\infty$ *such that*

$$d(U(t_n)x_n, \overline{x}) \geq \varepsilon, \forall n \geq 0.$$

Definition 1.12 Let A be a subset of M. We will say that A is **positively invariant** by U if

$$U(t)A \subset A, \forall t \geq 0.$$

The subset A is positively invariant if and only if for each $x \in A$

$$U(t)x \in A, \forall t \geq 0.$$

We will say that A is **negatively invariant** by U if

$$U(t)A \supset A, \forall t \geq 0.$$

The subset A is negatively invariant if and only if for each $y \in A$ and each $t \geq 0$, there exists $x \in A$ such that

$$U(t)x = y.$$

We will say that A is **invariant** by U if A is both positively and negatively invariant. That is

$$U(t)A = A, \forall t \geq 0.$$

Example 1.13 Consider the map $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = 4x(1 - x), \forall x \in [0, 1] .$$

Then T reaches its maximum on $[0, 1]$ at $x = 1/2$ and

$$T(1/2) = 1.$$

Therefore

$$T([0, 1]) = [0, 1] .$$

Clearly the map T is not one to one. We deduce that $[0, 1]$ is invariant by the discrete time semiflow

$$U(n) = T^n, \forall n \geq 0.$$

But the map $x \to U(n)x$ is not invertible.

Definition 1.14 We will say that $O_+ = \{u(t)\}_{t\geq 0} \subset M$ is a **positive orbit** of U if

$$u(t) = U(t)u(0), \forall t \geq 0 \Leftrightarrow u(t + s) = U(s)u(t), \forall t, s \geq 0.$$

We will say that $\{u(t)\}_{t\leq 0} \subset M$ is a **negative orbit** of U if

$$u(-t) = U(s)u(-t - s), \forall t, s \geq 0.$$

Finally we will say that $O = \{u(t)\}_{t\in I} \subset M$ is a **complete orbit** of U if

$$u(t) = U(s)u(t - s), \forall s \geq 0, \forall t \in I.$$

We will say that an orbit (positive, negative or complete) **passes through** $x \in M$ **at time** $t = 0$ if $u(0) = x$.

Remark 1.15 Let $x \in M$ be given. Then there exists at most one positive orbit passing through x at time $t = 0$, which is

$$u(t) := U(t)x, \forall t \geq 0.$$

But in general there is no negative orbit passing through x at time 0. Since the map $U(t)$ is not always onto for $t > 0$. Moreover when there exists a negative orbit passing through x, the negative orbit is not necessarily unique since the map $U(t)$ is not always one to one for $t > 0$. As an example of non-unique negative orbit consider Example 1.13.

Remark 1.16 If $\{u(t)\}_{t\in I} \subset M$ is a complete orbit passing through x then the set

$$O := \bigcup_{t\in I} \{u(t)\}$$

satisfies

$$U(t)O = O, \forall t \geq 0.$$

This is an example of invariant set.

1.3 ω-Limit and α-Limit Sets

Definition 1.17 Let $x \in M$. Let $\{u(t)\}_{t\geq 0} \subset M$ be a positive orbit of U passing through x at time 0. The **ω-limit set** of x is defined as

$$\omega(x) := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}}.$$

Let $\{u(t)\}_{t\leq 0}$ be a negative orbit of U passing through x at time 0. Then the **α-limit set** of x (with respect to this negative orbit) is

$$\alpha(x) := \bigcap_{t\leq 0} \overline{\bigcup_{s\leq t} \{u(s)\}}.$$

The omega limit set satisfies

$$\begin{aligned}\omega(x) &= \left\{y \in M : \exists \{t_n\}_{n\in\mathbb{N}} \subset I_+ \to +\infty \text{ such that } \lim_{n\to+\infty} u(t_n) = y\right\} \\ &= \{y \in M : \forall t \geq 0, \forall \varepsilon > 0, \exists s > t \text{ such that } d(u(s), y) \leq \varepsilon\}.\end{aligned}$$

Similarly the alpha limit set satisfies

$$\begin{aligned}\alpha(x) &= \left\{y \in M : \exists \{t_n\}_{n\in\mathbb{N}} \subset I_+ \to +\infty \text{ such that } \lim_{n\to+\infty} u(-t_n) = y\right\} \\ &= \{y \in M : \forall t \leq 0, \forall \varepsilon > 0, \exists s < t \text{ such that } d(u(s), y) \leq \varepsilon\}.\end{aligned}$$

Example 1.18 We have for example

$$\begin{aligned}u(t) &= \cos(t) \Rightarrow \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}} = [0, 1], \\ u(t) &= t\cos(t) \Rightarrow \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}} = \mathbb{R}, \\ u(t) &= t \Rightarrow \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}} = \emptyset.\end{aligned}$$

So the omega limit sets can be compact, non compact or empty.

Definition 1.19 Let (M, d) be a metric space.

(i) A subset $C \subset M$ is **compact** if and only if any sequence in C has a sub-sequence which converges in C.
(ii) A subset $C \subset M$ is **relatively compact** if and only if $\overline{C}$ (the closure of C in (M, d)) is compact.

In the general case ω-limit set is non empty only whenever the positive orbit

$$O_+ = \{u(t)\}_{t\geq 0}$$

is relatively compact (i.e. its closure is compact).

Theorem 1.20 *Let $\{u(t)\}_{t\geq 0} \subset M$ be a positive orbit passing through $x \in M$ at time $t = 0$. Assume that the closure of this positive orbit*

$$\overline{\bigcup_{t\geq 0} \{u(t)\}}$$

is compact.

Then the ω-limit set satisfies the following properties:

(i) *$\omega(x)$ is a non empty compact subset of M;*
(ii) *$\omega(x)$ is invariant by U;*
(iii) $\lim_{t\to+\infty} d\left(u(t), \omega(x)\right) = 0$*, where*

$$d\left(x, B\right) := \inf_{y\in B} d(x, y).$$

Remark 1.21 If M is a closed subset of $\mathbb{R}^n$, and the metric d is induced by a norm on $\mathbb{R}^n$ (i.e. $d(x, y) = \|x - y\|$), then $\overline{\bigcup_{t\geq 0} \{u(t)\}}$ is compact if and only if the positive orbit $\bigcup_{t\geq 0} \{u(t)\}$ is a bounded set.

Before proving Theorem 1.20 we need the following lemma.

Lemma 1.22 *Let $B \subset M$. Then the map $x \to d(x, B)$ is Lipschitz continuous. More precisely*

$$|d(x, B) - d(y, B)| \leq d\left(x, y\right), \forall x, y \in M.$$

Proof Let $x, y \in M$ and $z \in B$. We have

$$d(x, B) \leq d(x, z) \leq d(x, y) + d(y, z).$$

Thus

$$d(x, B) \leq d(x, y) + d(y, B)$$

and the result follows. □

Proof (of Theorem 1.20) Define

$$A_t := \overline{\bigcup_{s\geq t} \{u(s)\}}, \forall\, t \geq 0.$$

By assumption for each $t \geq 0$, the subset A_t is compact. Moreover the family $t \to A_t$ is decreasing, that is to say that

$$t \geq s \Rightarrow A_t \subset A_s.$$

Proof of (i). Let $\{t_n\} \to +\infty$ be an increasing sequence and $x_n \in A_{t_n}, \forall n \geq 0$. Then since A_0 is compact, and the family $t \to A_t$ is decreasing, we have

$$x_n \in A_0, \forall n \geq 0.$$

So, we can find a converging sub-sequence $\{x_n\}_{n\geq 0} \to z \in A_0$ (denoted for notational simplicity by the same index).

Moreover for each $t \geq 0$, we can find an integer $n_0 \geq 0$ such that $t_n \geq t, \forall n \geq n_0$, and since the family $t \to A_t$ is decreasing

$$x_n \in A_t, \forall n \geq n_0.$$

But the subset A_t is closed by construction, therefore

$$z \in A_t, \forall t \geq 0.$$

hence $\omega(x)$ is non-empty (since $z \in \omega(x)$).

Next, consider a sequence in the ω-limit set

$$x_n \in \omega(x), \forall n \geq 0.$$

Then

$$x_n \in A_t, \forall n \geq 0, \forall t \geq 0.$$

Since A_0 is compact, we can find a converging sub-sequence, and since this sub-sequence belongs to each subset A_t (which is closed), we deduce that the limit of this converging sub-sequence belongs to $\omega(x)$. Therefore $\omega(x)$ is compact.

Proof of (ii). Observe that for each $t, s \geq 0$

$$U(s)\left(\bigcup_{l\geq t}\{u(l)\}\right) = \bigcup_{l\geq t+s}\{u(l)\}. \tag{1.7}$$

This equality implies that

$$U(s)\left(\bigcup_{l\geq t}\{u(l)\}\right) \subset A_{t+s}$$

and since the map $x \to U(s)x$ is continuous we obtain

$$U(s)\left(A_t\right) \subset A_{t+s}. \tag{1.8}$$

From (1.7), we also have

$$U(s)\left(A_t\right) \supset \bigcup_{l\geq t+s}\{u(l)\}$$

and since by assumption A_t is compact and $x \to U(s)x$ is continuous, it follows that $U(s)\left(A_t\right)$ is compact and we obtain

$$U(s)\,(A_t) \supset A_{t+s}. \tag{1.9}$$

By combining the inclusions (1.8) and (1.9) we obtain

$$U(s)\,(A_t) = A_{t+s}, \forall t, s \geq 0.$$

The invariance of the ω-limit set follows from the following observation

$$U(s)\,(\omega(x)) = U(s)\left(\bigcap_{t\geq 0} A_t\right) = \bigcap_{t\geq 0} U(s)A_t = \bigcap_{t\geq 0} A_{t+s} = \omega(x).$$

Proof of (iii). Assume by contradiction that there exist $\varepsilon > 0$ and a sequence $t_n \longrightarrow +\infty$ such that

$$d\,(u(t_n), \omega(x)) \geq \varepsilon, \forall n \geq 0.$$

By compactness of A_0, we can find a converging sub-sequence (denoted with the same index) such that

$$\lim_{n\to+\infty} u(t_n) = z \in \omega(x).$$

By Lemma **1.22**, we deduce that

$$\varepsilon/2 \leq \lim_{n\to+\infty} d\,(u(t_n), \omega(x)) = d\,(z, \omega(x)) = 0,$$

which is a contradiction. □

The proof for alpha limit sets is similar to the proof of Theorem 1.20.

Theorem 1.23 *Let $\{u(t)\}_{t\leq 0} \subset M$ be a negative orbit passing through $x \in M$ at time $t = 0$. Assume that $\overline{\bigcup_{t\leq 0} \{u(t)\}}$ is compact. Then the α-limit set satisfies the following properties*

(i) *$\alpha(x)$ is a non empty compact subset of M;*
(ii) *$\alpha(x)$ is invariant by U.*
(iii) $\lim_{t\to-\infty} d\,(u(t), \alpha(x)) = 0.$

1.4 Heteroclinic and Homoclinic Orbits

Proposition 1.24 *Let $\{u(t)\}_{t\geq 0} \subset M$ be a positive orbit passing through $x \in M$ at time $t = 0$. Then*

$$\omega(x) = \{\overline{x}_+\} \Leftrightarrow \lim_{t\to+\infty} u(t) = \overline{x}_+.$$

Moreover in that case, $\overline{x}_+$ must be an equilibrium of U.

Let $\{u(t)\}_{t\leq 0} \subset M$ be a negative orbit passing through $x \in M$ at time $t = 0$. Then

$$\alpha(x) = \{\overline{x}_-\} \Leftrightarrow \lim_{t \to -\infty} u(t) = \overline{x}_-.$$

Moreover in that case, $\overline{x}_-$ *must be an equilibrium of* U*.*

Proof By using the definition of $\omega(x)$, we deduce that $\omega(x) = \{\overline{x}_+\}$ is equivalent to

$$\lim_{t \to +\infty} u(t) = \overline{x}_+.$$

Let $s \geq 0$. Since $t \to u(t)$ is a positive orbit, we have

$$u(t+s) = U(s)u(t), \forall t \geq 0,$$

and by taking the limit when $t \to +\infty$ on both side, we obtain

$$\overline{x}_+ = U(s)\overline{x}_+.$$

By using the definition of $\alpha(x)$, we deduce that$\alpha(x) = \{\overline{x}_-\}$ is equivalent to

$$\lim_{t \to -\infty} u(t) = \overline{x}_-.$$

Let $s \geq 0$. Since $t \to u(t)$ is a negative orbit, we have

$$u(t+s) = U(s)u(t), \forall t \leq -s,$$

and by taking the limit when $t \to -\infty$ on both side, we obtain

$$\overline{x}_- = U(s)\overline{x}_-.$$

Definition 1.25 A complete orbit $\{u(t)\}_{t \in I}$ is called a **heteroclinic orbit** if there exist $\overline{x}_{-\infty} \in M$ and $\overline{x}_{+\infty} \in M$ such that

$$\lim_{t \to -\infty} u(t) = \overline{x}_{-\infty} \text{ and } \lim_{t \to +\infty} u(t) = \overline{x}_{+\infty}.$$

That is equivalent to say that both the omega limit set and alpha limit set are reduced to a single point. That is

$$\alpha(x) = \{\overline{x}_{-\infty}\} \text{ and } \omega(x) = \{\overline{x}_{+\infty}\}.$$

A complete orbit $\{u(t)\}_{t \in I}$ is called a **homoclinic orbit** if this orbit is not constant and there exists $\overline{x} \in M$ such that

$$\lim_{t \to +\infty} u(t) = \overline{x} \text{ and } \lim_{t \to +\infty} u(t) = \overline{x}.$$

That is equivalent to say that this orbit is not constant the omega limit set and alpha limit set are reduced to the same single point. That is

$$\alpha(x) = \omega(x) = \{\overline{x}\}.$$

1.5 Attraction of Sets and Hausdorff Distance and Semi-Distance

Let (M, d) be a complete metric space. For any subsets A, $B \subset M$, we define Hausdorff's semi-distance of B to A as

$$\delta(B, A) := \sup_{x \in B} d(x, A),$$

where

$$d(x, A) := \inf_{z \in A} d(x, z).$$

For each $\varepsilon > 0$, we define an **open ε-neighborhood of** A (see also Section 2.4 in Chapter 2 for more results) as

$$N(A, \varepsilon) := \{x \in M : d(x, A) < \varepsilon\},$$

and a **closed ε-neighborhood of** A as

$$\overline{N}(A, \varepsilon) := \{x \in M : d(x, A) \leq \varepsilon\}.$$

Now by using the fact that $x \to d(x, A)$ is a continuous map, we deduce that $N(A, \varepsilon)$ is an open neighborhood of A and $\overline{N}(A, \varepsilon)$ is a closed neighborhood of A. From these observations, it becomes clear that $\delta(B, A)$ is measuring the distance of B to A (and not the converse). Therefore $\delta(B, A)$ is only a semi-distance (since $\delta(B, A) = 0$ does not imply $A = B$).

Remark 1.26 The open ball $B_M(x, \varepsilon)$ (respectively the closed ball $\overline{B}_M(x, \varepsilon)$) centered at x with radius $\varepsilon > 0$ satisfies

$$B_M(x, \varepsilon) = \{y \in M : d(y, x) < \varepsilon\} = N(\{x\}, \varepsilon),$$

and

$$\overline{B}_M(x, \varepsilon) = \{y \in M : d(y, x) \leq \varepsilon\} = \overline{N}(\{x\}, \varepsilon).$$

Definition 1.27 The distance between two subsets $A, B \subset M$ is measured by using the so called **Hausdorff's distance** which is defined by

$$d_H(A, B) = \max(\delta(B, A), \delta(A, B)).$$

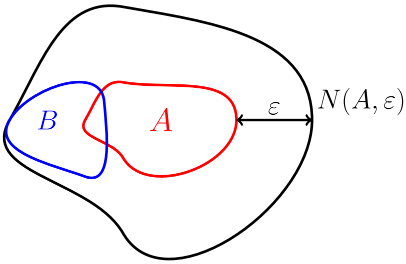

Fig. 1.1: *The figure illustrates the notion of Hausdorff's semi-distance of B to A. In the figure $\varepsilon = \delta(B, A)$. The black curve corresponds to the boundary of $N(A, \varepsilon)$ a ε-neighborhood of A.*

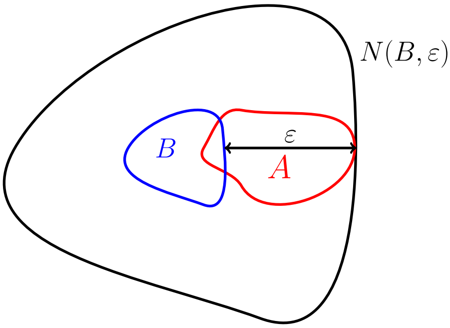

Fig. 1.2: *The figure illustrates the notion of Hausdorff's semi-distance of A to B. In the figure $\varepsilon = \delta(A, B)$. The black curve corresponds to the boundary of $N(B, \varepsilon)$ a ε-neighborhood of B.*

Definition 1.28 We say that a subset $A \subset M$ **attracts** a subset $B \subset M$ for a semiflow U on M if

$$\lim_{t \to +\infty} \delta(U(t)B, A) = 0.$$

Remark 1.29 This means that for each $\varepsilon > 0$, we can find $t_0 = t_0(\varepsilon) > 0$ (large enough) such that for each $t \geq t_0$, the subset $U(t)B = \{U(t)x : x \in B\}$ is included in $N(A, \varepsilon)$.

To illustrate the notion of attraction, we first prove that the positive orbit is attracted by the omega limit sets (whenever it is exists).

Lemma 1.30 *Let $\{u(t)\}_{t \geq 0} \subset M$ be a positive orbit passing through $x \in M$ at time $t = 0$. Assume that*

$$O_+(x) := \bigcup_{t \geq 0} \{u(t)\}$$

is relatively compact. Then $\omega(x)$ attracts $O_+(x)$ for U.

Proof We first observe that

$$U(t)O_+(x) = \bigcup_{s \geq t} \{u(s)\}, \forall t \geq 0.$$

Assume by contradiction that $\omega(x)$ does not attract $O_+(x)$ for U. Then we can find $\varepsilon > 0$ and a sequence $t_n \to +\infty$ such that

$$\delta(u(t_n), \omega(x)) \geq \varepsilon,$$

and we obtain a contradiction with Theorem **1.20**-(iii). □

Remark 1.31 If we consider the family of subsets $A_t := \bigcup_{s \geq t} \{u(s)\}$, then $\omega(x)$ attracts $O_+(x)$ for U is equivalent to say that

$$\lim_{t \to +\infty} \delta(A_t, \omega(x)) = 0.$$

Let $\{u(t)\}_{t \leq 0} \subset M$ be a negative orbit passing through $x \in M$ at time $t = 0$. We can adapt this last notion of attractivity for the alpha limit sets, by saying that if

$$O_-(x) = \bigcup_{t \leq 0} \{u(t)\}$$

is relatively compact, then

$$\lim_{t \to -\infty} \delta(B_t, \alpha(x)) = 0.$$

where

$$B_t = \bigcup_{s \leq t} \{u(s)\}.$$

The Hausdorff semi-distance measure the distance of B to A. Therefore if $B \subset A$ then $\delta(B, A) = 0$. Moreover, if $\varepsilon > 0$ then

$$\delta(B, A) = \varepsilon \Rightarrow B \subset \overline{N}(A, \varepsilon).$$

This means that we can find a sequence $x \in B$ such that

$$d(x, A) \leq \varepsilon,$$

and there exists a sequence $x_n \in B$ such that

$$\lim_{n \to \infty} d(x_n, A) = \varepsilon.$$

The Hausdorff distance can also be defined as follows

$$d_H(A, B) = \inf \{\varepsilon > 0 : A \subset N(B, \varepsilon) \text{ and } B \subset N(A, \varepsilon)\}.$$

Actually the Hausdorff distance is not a real distance, because we only have

$$d_H(A,B) = 0 \Leftrightarrow \overline{A} = \overline{B}.$$

But the Hausdorff is a real distance if we restrict to the closed subsets.

Proposition 1.32 *Let (M, d) be a metric space. Then the set of closed subsets of M is a metric space endowed with the Hausdorff distance.*

Proof It remains to prove the triangle inequality for the Hausdorff distance. Assume that

$$d_H(A,B) < \varepsilon \text{ and } d_H(B,C) < \varepsilon'.$$

From the proof of Lemma 1.22, we know that for each $x \in C$, $z \in B$,

$$d(x,A) \leq d(x,z) + d(z,A).$$

Let $x \in C$ be fixed. Since

$$d_H(B,C) < \varepsilon' \Rightarrow C \subset N(B,\varepsilon'),$$

we can choose $z \in B$ such that $d(z,x) \leq \varepsilon'$, and

$$d_H(A,B) < \varepsilon \Rightarrow B \subset N(A,\varepsilon).$$

We deduce that

$$d(x,A) \leq \varepsilon' + \varepsilon, \forall x \in C.$$

By taking the supremum in x, we obtain

$$\delta(C,A) \leq d_H(A,B) + d_H(B,C),$$

and by symmetry of the problem, we obtain

$$\delta(A,C) \leq d_H(A,B) + d_H(B,C).$$

We conclude that

$$d_H(C,A) \leq d_H(A,B) + d_H(B,C).$$

The proof is completed. □

1.6 Connectivity of ω-Limit Sets and α-Limit Sets

Internally chain transitive ω-limit sets and α-limit sets

The fundamental property of omega limit sets is the fact that each couple of points in a omega limit set can be almost connected by an orbit staying in a small neighborhood of the omega limit set. The following result is making this statement more precise.

Lemma 1.33 *Assume that the positive orbit $O_+(x)$ (respectively the negative orbit $O_-(x)$) is relatively compact. For each $a, b \in \omega(x)$ (respectively $a, b \in \alpha(x)$) and* *each* $\varepsilon > 0$ *we can find* $a_\varepsilon, b_\varepsilon \in M$ *and* $t_\varepsilon > 0$ *such that*

$$d(a, a_\varepsilon) \leq \varepsilon,\ d(b, b_\varepsilon) \leq \varepsilon,$$

$$U(t_\varepsilon)a_\varepsilon = b_\varepsilon,$$

and

$$U(t)a_\varepsilon \in N(\omega(x), \varepsilon), \forall t \in [0, t_\varepsilon],$$

which is equivalent to

$$d\left(U(t)a_\varepsilon, \omega\left(x\right)\right) \leq \varepsilon, \forall t \in [0, t_\varepsilon].$$

Proof We have $\lim_{t\to+\infty} \delta(U(t)x, \omega(x)) = 0$. So we can find $t_0 > 0$ such that

$$\delta(U(t)x, \omega(x)) \leq \varepsilon,\ \forall t \geq t_0.$$

Since a belongs to $\omega(x)$ we can find $t_1 > t_0$ such that $d(U(t_1)x, a) \leq \varepsilon$. Set $a_\varepsilon = U(t_1)x$. Then since b belongs to $\omega(x)$ we can find $t_2 > t_1 > t_0$ such that $d(U(t_2)x, b) \leq \varepsilon$. Set $b_\varepsilon = U(t_2)x$ and the result follows. □

Definition 1.34 Let A be a subset of M. We say that $a \in A$ **is chained to** $b \in A$ **in** A, if for each $t^\star > 0$, for each $\varepsilon > 0$, and each $\eta > 0$, there exist $\tau \in [t^\star, t^\star + \eta] \cap I$, and $x_1, x_2, ..., x_m \in A$ (with $m \geq 2$) such that

$$x_1 = a,\ x_m = b,\ \text{and}\ d\left(U(\tau)x_i, x_{i+1}\right) \leq \varepsilon, \forall i = 1, ..., m-1.$$

We will say that A is **internally chain transitive**, if for each $a, b \in A$, a is chained to b in A.

Theorem 1.35 *Let* $\{U(t)\}_{t\in I}$ *be a continuous semiflow on* (M, d)*. Then the omega limit set (respectively alpha limit set) of a relatively compact positive orbit (respectively negative orbit) is internally chain transitive.*

Lemma 1.36 *Let* $\{U(t)\}_{t\in I}$ *be a continuous semiflow on* (M, d)*. Let* C *be a compact subset of* M*. Then for each* $\varepsilon > 0$*, and each* $t^\star \in I$ *(with* $t^\star > 0$*), there exists* $\delta > 0$*, such that* $s \in [t^\star, t^\star + \delta]$*,* $u, v \in M$*,*

$$d\left(u, C\right) \leq \delta, d\left(v, C\right) \leq \delta,\ \textit{and}\ d\left(u, v\right) \leq \delta \Rightarrow d\left(U(s)u, U(s)v\right) \leq \varepsilon.$$

Proof Assume by contradiction that there exists $\varepsilon > 0$ and two sequences $u_n \in M$, and $v_n \in M$, and $s_n \in [t^\star, t^\star + 1/n]$, such that for each integer $n > 0$,

$$d\left(u_n, C\right) \leq 1/n, d\left(v_n, C\right) \leq 1/n,\ \text{and}\ d\left(u_n, v_n\right) \leq 1/n,$$

implies

$$d\left(U(s_n)u_n, U(s_n)v_n\right) \geq \varepsilon. \tag{1.10}$$

By definition of distance $d\left(x, C\right)$, we can find two sequences $u_n^C \in C$ and $v_n^C \in C$ such that

$$d\left(u_n, u_n^C\right) \leq 2/n \text{ and } d\left(v_n, v_n^C\right) \leq 2/n.$$

By using the triangle inequality we obtain

$$d\left(u_n^C, v_n^C\right) \leq d\left(u_n^C, u_n\right) + d(u_n, v_n) + d\left(v_n, v_n^C\right) \leq 5/n.$$

Now, by using the fact that C is compact, we can find some converging sub-sequences (denoted with the same index), such that

$$u_n^C \to w, \text{ and } v_n^C \to w, \text{ as } n \to +\infty.$$

By using the continuity of $(t, x) \to U(t)x$, we deduce from (1.10) that

$$0 = d\left(U(t^\star)w, U(t^\star)w\right) \geq \varepsilon > 0.$$

A contradiction. The proof is completed. □

Proof (of Theorem 1.35) Let us prove the result for omega limit set (the proof for alpha limit set is similar). Let $x \in M$, and assume that $\overline{\gamma^+(x)}$ is compact. Then $\omega(x)$ is nonempty, compact, invariant and

$$\lim_{t \to +\infty} d\left(U(t)x, \omega(x)\right) = 0.$$

Let $t^\star \in I$ (with $t^\star > 0$), $\varepsilon > 0$, and $\eta > 0$ be fixed. By continuity of U, and compactness of $\omega(x)$, we can find $\delta \in (0, \varepsilon/3) \cap (0, \eta)$, with the following property: If $s \in [t^\star, t^\star + \delta]$, $u, v \in M$,

$$d\left(u, \omega(x)\right) \leq \delta, d\left(v, \omega(x)\right) \leq \delta, \text{ and } d\left(u, v\right) \leq \delta \Rightarrow d\left(U(s)u, U(s)v\right) \leq \varepsilon/3.$$

Since

$$\lim_{t \to +\infty} d\left(U(t)x, \omega(x)\right) = 0,$$

we can find $t_1 \in I$, such that

$$d\left(U(t)x, \omega(x)\right) < \delta, \forall t \geq t_1.$$

Let $a, b \in \omega(x)$. Then we can find $t_a \geq t_1$, such that

$$d\left(U(t_a)x, a\right) < \delta.$$

Let $k \in \mathbb{N}$, $k \geq 2$, such that $t^\star/k \leq \delta$. Then we can find $t_b \geq t_a + kt^\star$, such that

$$d\left(U(t_b)x, b\right) < \delta.$$

However there exists $m \geq k + 1$, such that $(m-1)\,t^\star \leq t_b - t_a < mt^\star$. We set $\tau = \dfrac{t_b - t_a}{m-1}$. Then by construction of $t^\star$, we have $\tau \in I$, and

$$\tau \in \left[t^\star, \left(1 + \frac{1}{m-1}\right)t^\star\right] \subset \left[t^\star, \left(1 + \frac{1}{k}\right)t^\star\right] \subset \left[t^\star, t^\star + \delta\right].$$

We set

$$y_1 = a, y_2 = U(\tau) U(t_a)x, \dots, y_{m-1} = U((m-2)\tau) U(t_a)x, y_m = b.$$

Then

$$d(U(\tau) y_1, y_2) = d(U(\tau) a, U(\tau) U(t_a)x)$$

and since

$$a \in \omega(x), d(U(t_a)x, \omega(x)) < \delta, \text{ and } d(U(t_a)x, a) < \delta,$$

and $\tau \in [t, t+\delta]$, we deduce that

$$d(U(\tau) y_1, y_2) = d(U(\tau) a, U(\tau) U(t_a)x) \le \varepsilon/3,$$

$$U(\tau) y_j = y_{j+1}, \forall j = 2, ..., m-2,$$

and

$$d(U(\tau) y_{m-1}, y_m) = d(U(t_b)x, b) \le \delta \le \varepsilon/3.$$

Now for each $j = 2, ..., m-1$, there exists $x_j \in \omega(x)$, such that $d\left(x_j, y_j\right) \le \delta$. By setting $x_1 = a$ and $x_m = b$, we obtain for $j = 1, ..., m-1$,

$$\begin{aligned} d\left(U(\tau) x_j, x_{j+1}\right) &\le d\left(U(\tau) x_j, U(\tau) y_j\right) + d\left(U(\tau) y_j, y_{j+1}\right) + d\left(y_{j+1}, x_{j+1}\right) \\ &\le \varepsilon/3 + \varepsilon/3 + \delta \le \varepsilon. \end{aligned}$$

The proof is completed. □

Invariantly connected ω-limit and α-limit Sets

Definition 1.37 A compact invariant set A is said to be **invariantly connected** if it is not the union of two nonempty disjoint compact invariant subsets. That is to say that if $A_1 \neq \emptyset$, and $A_2 \neq \emptyset$ are non empty and compact subsets satisfying

$$A = A_1 \cup A_2, \text{ with } A_1 \cap A_2 = \emptyset.$$

Then either A_1 or A_2 are not invariant by U. That is,

$$\text{either } U\left(t^\star\right) A_1 \neq A_1, \text{ or } U(t^\star) A_2 \neq A_2.$$

for some $t^\star > 0$.

Theorem 1.38 *Let $\{U(t)\}_{t\in I}$ be a continuous semiflow on (M, d). Then the omega limit set (respectively alpha limit set) of a relatively compact positive orbit (respectively negative orbit) is invariantly connected.*

Theorem 1.38 follows from Theorem 1.35 and the following lemma.

Lemma 1.39 *Let A be a compact subset of M which is invariant by U. If A is internally chain transitive then A is also invariantly connected.*

Proof We can prove the result by contradiction. Assume that A is the union of two disjoint closed invariant sets $A = A_1 \cup A_2$. We get a contradiction because the subsets A_1 and A_2 are invariant. So if we fix $\tau > 0$ then when $d(x, A_1) \leq \varepsilon$, then $d(U(\tau)x, A_2) > 2\varepsilon$ whenever $\varepsilon > 0$ is small enough. So a point of A_1 can not be chained to a point of A_2. □

Connected ω-limit and α-limit sets

Definition 1.40 Let (M, d) be a metric space. Let $C \subset M$ be a subset of M. We will say that a pair of of non empty subsets $A \subset C$ and $B \subset C$ is a **partition of** C if

$$A \cup B = C \text{ and } A \cap B = \emptyset.$$

Definition 1.41 Let (M, d) be a complete metric space. A subset C of M is said to be **connected** if there exists no partition of C in two subsets A and B which are both open subsets for the topology of (C, d). We will say that C is **disconnected** whenever C is not connected.

Remark 1.42 Recall that a subset is closed in (C, d) if its complementary set in C is open in (C, d). Therefore, in the above definition, it is equivalent to say that both subsets A and B are also closed in (C, d).

Assume that C is not connected. Then we can there exists a partition of C in two subsets A and B which are both open subsets for the topology of (C, d). Recall that a subset A is open in (C, d), if and only if for each $x \in A$, there exists $\varepsilon_x > 0$ such that

$$B_C(x, \varepsilon_x) = \{y \in C : d(x, y) \leq \varepsilon_x)\} \subset A,$$

where $B_C(x, \varepsilon_x)$ is the ball of center x and radius ε_x (in C).

Remark that

$$B_C(x, \varepsilon_x) = B_M(x, \varepsilon_x) \cap C,$$

where

$$B_M(x, \varepsilon_x) = \{y \in M : d(x, y) \leq \varepsilon_x\}.$$

So we must have

$$B_M(x, \varepsilon_x) \cap B = \emptyset.$$

Now we can define

$$U = \bigcup_{x \in A} B_M(x, \varepsilon_x),$$

and

$$V = \bigcup_{x \in B} B_M(x, \varepsilon_x).$$

where for each $x \in B$, ε_x is chosen small enough to guaranty

$$B_M(x, \varepsilon_x) \cap A = \emptyset.$$

We observe that U and V satisfies

$$U \cap C = A \text{ and } V \cap C = B$$

and

$$U \cap V = \emptyset.$$

Therefore we obtain the following lemma.

Proposition 1.43 *Let (M, d) be a complete metric space. A subset C is disconnected if there exist two open subsets $U \subset M$ and $V \subset M$, such that*

$$(C \cap U) \neq \emptyset \text{ and } (C \cap V) \neq \emptyset$$

and

$$U \cap V = \emptyset.$$

Definition 1.44 An **interval** in $\mathbb{R}$ is a subset I such that

$$a < c < b \text{ and } a, b \in I \Rightarrow c \in I.$$

Remark 1.45 This is definition can be extended to $\mathbb{R}^n$ endowed with some partial order $\leq$.

Theorem 1.46 *A connected set of real numbers is an interval.*

Remark 1.47 The converse is also true. An interval of real numbers is a connected set.

Proof Let C be a connected set in $\mathbb{R}$. Assume by contradiction that C is not an interval. Then we can find $a, b \in C$ and $c \notin C$ with $a < c < b$. The subsets $U = (-\infty, c)$ and $V = (c, \infty)$ are both open in $\mathbb{R}$, and

$$a \in C \cap U \neq \emptyset, \text{ and } b \in C \cap V \neq \emptyset.$$

Therefore $A = C \cap U$ and $B = C \cap V$ are both open in C. We deduce that C is disconnected. □

Theorem 1.48 *Let $T : M \to \widetilde{M}$ be a continuous map from a metric space (M, d) to a metric space $(\widetilde{M}, \widetilde{d})$. Then M is connected implies that $T(M)$ is connected.*

Proof Assume by contradiction that $T(M)$ is not connected. Then there exists $\widetilde{A}$ and $\widetilde{B}$ two open subset of $(T(M), \widetilde{d})$

$$\widetilde{A} \cup \widetilde{B} = T(M) \text{ and } \widetilde{A} \cap \widetilde{B} = \emptyset.$$

By continuity of T, we deduce that $A = T^{-1}\left(\widetilde{A}\right)$ and $B = T^{-1}\left(\widetilde{B}\right)$ are open in (M, d) and

$$A \cup B = M \text{ and } A \cap B = \emptyset.$$

This contradicts the fact that M is connected. The proof is completed. □

As a consequence of the Proposition 1.43, Theorem 1.46, and Theorem 1.48, we obtain the following results that will be useful in the applications.

Theorem 1.49 *Let $M \subset X$ be a subset of Banach space $(X, \|.\|)$. Then we have the following properties:*

(i) *M is convex $\Rightarrow$ M is connected.*
(ii) *If M is connected and $x^* : X \to \mathbb{R}$ is a bounded linear map then*

$$I = \{x^*(x) : x \in M\} \subset \mathbb{R}$$

is an interval in $\mathbb{R}$.

As a consequence of the above theorem we have for example, a connected set in $\mathbb{R}^n$ becomes an interval when it is projected onto the axes.

Example of not connected ω-limit and α-limit sets: The ω-limit set (respectively the α-limit set) of a relatively compact positive orbit (respectively negative orbit) generated by a discrete time semiflow is not connected in general. Indeed, assume that $T : M \to M$ has a 2-periodic orbit

$$T(a) = b \text{ and } T(b) = a,$$

with

$$a \neq b.$$

If we define the complete orbit

$$u(n) = \begin{cases} a, \text{ if } n = 2k, \text{ for some integer } k \in \mathbb{Z}, \\ b, \text{ if } n = 2k+1, \text{ for some integer } k \in \mathbb{Z}, \end{cases}$$

Then the ω-limit set of the solution starting from a or b is

$$\omega(a) = \omega(b) = \{a, b\},$$

and

$$\alpha(a) = \alpha(b) = \{a, b\}.$$

This provides an example of disconnected ω-limit and α-limit sets.

The case of continuous time semiflow is different.

Theorem 1.50 *Let (M, d) be a complete metric space. The ω-limit set of a relatively compact orbit generated by a continuous time semiflow $\{U(t)\}_{t \in \mathbb{R}_+}$ is connected.*

Proof (of Theorem 1.50) Assume that $\omega(x)$ is disconnected. Then there would be disjoint open subsets U and V of M such that $U \cap \omega(x)$ and $V \cap \omega(x)$ are nonempty and $\omega(x) \subset U \cup V$. Let $a \in U \cap \omega(x)$ and $b \in U \cap \omega(x)$. Then we can have a sequence $t_1, t_2, \dots t_k \to \infty$ and a sequence $s_1, s_2, \dots s_k \to \infty$ (with $t_k \leq s_k$) such that $U(t_k)x \in U \to a$, and $U(s_k)x \in U \to b$. But $\{U(t)x : t \in [t_k, s_k]\}$ is a connected curve going from a point in U to a point in V. Therefore, there must be able to find $\tau_k \in (t_k, s_k)$ such that $U(\tau_k)x \in M \setminus (U \cup V)$. But the sequence

$k \to U(\tau_k)x$ is relatively compact, so up to a sub-sequence (denoted with the same index) we can assume that

$$U(\tau_k)x \to c.$$

But by construction $M \setminus (U \cup V)$ is a closed subset, and it follows that

$$c \in M \setminus (U \cup V) \text{ and } c \in \omega(x).$$

We obtain a contradiction since $\omega(x) \subset (U \cup V)$. □

1.7 Dissipation and Absorbing Sets

Definition 1.51 A continuous semiflow $\{U(t)\}_{t \in I}$ on a metric space (M, d) is said to be **point dissipative (respectively compact dissipative, bounded dissipative)** if there exists a bounded set $B_0 \subset M$ attracting the point (respectively the compact subsets, the bounded subsets).

The notion of dissipative semiflow can be expressed by using one of the two following equivalent properties:

(i) There exists $B_0 \subset M$ a bounded subset such that

$$\lim_{t \to +\infty} \delta(U(t)B, B_0) = 0,$$

whenever B is a point (respectively a compact subset, a bounded subset).

(ii) For each $\varepsilon > 0$, and each subset $B \subset M$ that is a point (respectively a compact subset, a bounded subset), there exists $t_0 = t_0(\varepsilon, B) > 0$ such that

$$U(t)B \subset \overline{N}(B_0, \varepsilon), \forall t \geq t_0,$$

where $\overline{N}(B_0, \varepsilon)$ is a closed ε-neighborhood of B_0 defined by

$$\overline{N}(B_0, \varepsilon) := \{x \in M : d(x, B_0) \leq \varepsilon\}.$$

Definition 1.52 A subset $B_0 \subset M$ is called **point absorbing, compact absorbing, bounded absorbing** if for each subset $B \subset M$ which is respectively a single point, a compact subset, a bounded subset, there exists $t_0 = t_0(B) \geq 0$ such that

$$U(t)B \subset B_0, \forall t \geq t_0.$$

A bounded absorbing subset is called **absorbing subset**.

1.8 Examples

1.8.1 Logistic equations: heteroclinic orbit

Consider the scalar logistic equation

$$N'(t) = N(t) - N(t)^2, \forall t \geq 0 \text{ and } N(0) = x. \tag{1.11}$$

The solution is explicitly given by

$$N(t) = \frac{e^t x}{1 + \int_0^t e^l x dl}, \forall t \geq 0.$$

Define the maximal backward time of existence

$$\tau^-(x) = \inf \left\{ t < 0 : 1 - \int_s^0 e^l x dl > 0, \forall s \in [t, 0] \right\}.$$

Then

$$\int_{-\infty}^0 e^l dl = 1,$$

therefore

$$\tau^-(x) = -\infty, \forall x \in [0, 1]$$

and the solution

$$N(t) = \frac{e^t x}{1 + \int_0^t e^\sigma x d\sigma}, \forall t \in \mathbb{R}. \tag{1.12}$$

It is clear that 0 and 1 are equilibrium solutions. Moreover for each $x \in (0, 1)$, the solution (1.12) is a heteroclinic orbit and

$$\lim_{t \to -\infty} N(t) = 0 \text{ and } \lim_{t \to +\infty} N(t) = 1.$$

That is equivalent to say that

$$\alpha(x) = \{0\} \text{ and } \omega(x) = \{1\}.$$

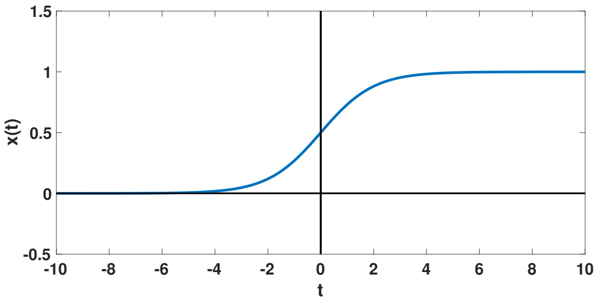

Fig. 1.3: *In this figure the blue curve represents the heteroclinic orbit* (1.6) *whenever* $x = 0.5$.

Dissipation: Consider the map

$$V(N) = (N-1)^2.$$

We have

$$V'(N) = 2(N-1)N' = 2(N-1)N(1-N) = -2N(1-N)^2 \leq 0.$$

Theorem 1.53 *The semiflow*

$$U(t)x = \frac{e^t x}{1 + \int_0^t e^{\sigma} x d\sigma}, \forall t \in \mathbb{R}$$

of scalar logistic equation (1.11) *is bounded dissipative on* $\mathbb{R}_+$. *More precisely, each such that* $B_0 = [0, N_1]$ *(with* $N_1 > 2$*) is bounded absorbing set. That is to say that for each bounded set* $B \subset [0, +\infty)$, *there exists* $t_0 = t_0(B)$, *such that*

$$U(t)B \subset B_0, \forall t \geq t_0.$$

Proof We first observe that

$$\sup_{x \in [0, N_1]} V(x) = (N_1 - 1)^2, \forall N_1 > 2,$$

therefore

$$\left\{x \geq 0 : V(x) \leq (N_1 - 1)^2\right\} = [0, N_1], \forall N_1 > 2.$$

We choose $B_0 = \left\{x \geq 0 : V(x) \leq (N_1 - 1)^2\right\}$ for some $N_1 > 2$. Next, we observe that B_0 is positively invariant by U. Indeed, we have for each $x \in B_0$

$$V(U(t)x) \leq V(x) \leq \sup_{x \in B_0} V(x) = (N_1 - 1)^2.$$

Therefore

$$\sup_{x \in B_0} V(U(t)x) \leq (N_1 - 1)^2 \Rightarrow U(t)B_0 \subset B_0, \forall t \geq 0.$$

Assume by contradiction, that there exists a bounded $B \subset [0, \infty)$, such that

$$\sup_{x \in B} V(U(t)x) \geq (N_1 - 1)^2, \forall t \geq 0.$$

Then we can construct a sequence $x_n \in B$ such that for each integer $n \geq 0$,

$$V(U(n)x_n) \geq (N_1 - 1)^2, \forall n \in \mathbb{N}.$$

Since the sequence x_n is bounded, we can find a sub-sequence (denoted with the same index) since $x_n \to x_\infty$ and by continuity of U we deduce that

$$V(U(t)x_\infty) \geq (N_1 - 1)^2, \forall t \geq 0. \tag{1.13}$$

But since $N_1 \geq 2$, we must have

$$U(t)x_\infty > N_1, \forall t \geq 0. \tag{1.14}$$

Finally observe that

$$V'(U(t)x_\infty) = -U(t)x_\infty \times (1 - U(t)x_\infty)^2 = -U(t)x_\infty \times V\left(U(t)x_\infty\right),$$

so by integrating this formula and by using (1.14), we obtain

$$V\left(U(t)x_\infty\right) = e^{-\int_0^t U(\sigma)x_\infty d\sigma} V\left(U(t)x_\infty\right) \leq e^{-tN_1} V(x) \to 0, \text{ as } t \to +\infty.$$

We obtain a contradiction with (1.13). □

Theorem 1.54 *Consider the semiflow $U(t)$ of scalar logistic equation (1.11) restricted to $(0, \infty)$. Then the subset $B_0 = [N_0, N_1]$ (with $0 < N_0 < 1 < N_1$) is a compact absorbing set, but B_0 is not a bounded absorbing set. Moreover precisely*

(i) *The subset $(0, 1]$ attracts all the bounded subsets in $(0, +\infty)$;*
(ii) *The subset $(0, 1]$ is invariant by U. That is*

$$U(t)(0, 1] = (0, 1], \forall t \geq 0;$$

(iii) *The subset $(0, 1]$ is not compact in $(0, +\infty)$;*
(iv) *The subset $(0, N_1]$ is a bounded absorbing set in $(0, +\infty)$.*

1.8.2 Poincaré normal form: periodic orbit

Consider the Poincaré normal form

$$\begin{cases} x'(t) = \frac{\lambda}{2}x(t) - \omega y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)x(t) \\ y'(t) = \omega x(t) + \frac{\lambda}{2}y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)y(t) \end{cases} \tag{1.15}$$

From Section 1.1, we know that the semiflow generated by (1.15) is defined by

$$V(t)\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{\frac{e^{\lambda t}\left(x^2+y^2\right)}{1+\kappa\int_0^t e^{\lambda s}\left(x^2+y^2\right) ds}} \times \frac{1}{\sqrt{x^2+y^2}} \times \begin{pmatrix} \cos(\omega t)x - \sin(\omega t)y \\ \sin(\omega t)x + \cos(\omega t)y \end{pmatrix}, \tag{1.16}$$

whenever $(x, y) \neq (0, 0)$ and

$$V(t)\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Exercise 1.55 Derive the above formula by using the following changes of variables. Consider

$$r(t)^2 = x^2(t) + y(t)^2$$

and prove that

$$(r(t)^2)' = \lambda r(t)^2 - \kappa r(t)^4.$$

Consider

$$X(t) = \frac{x(t)}{\sqrt{x^2(t)+y^2(t)}}$$

and

$$Y(t) = \frac{y(t)}{\sqrt{x^2(t)+y^2(t)}}$$

whenever $(x, y) \neq (0, 0)$. Prove that

$$X' = -\omega Y, \text{ and } Y' = \omega X.$$

Figures 1.4, 1.5 and 1.6 illustrate the behavior of the solutions of the Poincaré normal form.

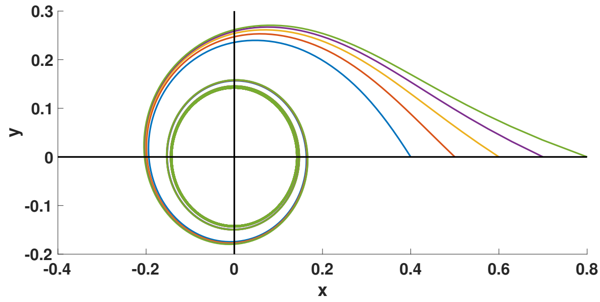

Fig. 1.4: *We plot some solutions of* (1.15) *in the phase plane* $(x(t), y(t))$ *whenever* $\lambda = 0.02$*,* $\omega = 0.1$ *and* $\kappa = 1$*. We choose several initial values where* $x = 0.2, 0.3, \dots, 0.8$ *and* $y = 0$*. One may observe that the omega limit set of these solutions is the central circle, while the alpha limit set is empty since the norm of the solutions eventually blowup when the time goes to* $-\infty$*.*

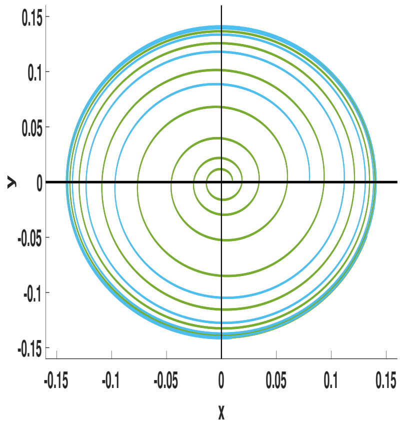

Fig. 1.5: *We plot some solutions of* (1.15) *in the phase plane* $(x(t), y(t))$ *whenever* $\lambda = 0.02$, $\omega = 0.1$ *and* $\kappa = 1$. *We choose several initial values, where* $x = 0.01, 0.08$ *and* $y = 0$*. The solutions are part of a complete orbits jointing the trivial equilibrium* 0 *to the circular periodic orbit.*

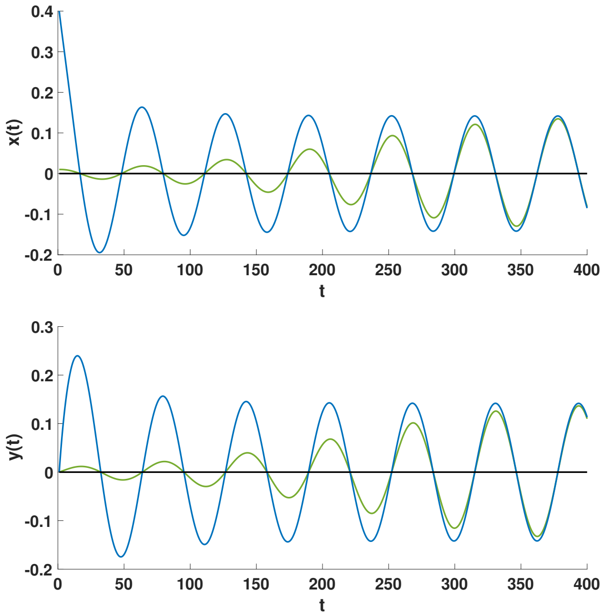

Fig. 1.6: *We plot* $(t, x(t))$ *(top) and* $(t, y(t))$ *(bottom) for two solutions of* (1.15) *whenever* $\lambda = 0.02$, $\omega = 0.1$ *and* $\kappa = 1$. *We choose several initial values where* $x = 0.01, 0.4$ *and* $y = 0$. *Both solutions coincide whenever they converge to the periodic orbit. That is because both solutions turn around* 0 *with the same rotation speed. This appears explicitly in the semiflow formula* (1.16) *(i.e. the rotation is guided by the linear term* $e^{At}X$*).*

1.8.3 Homoclinic orbit for a second order logistic equation

Consider the equation

$$x''(t) = -x(t)(1 - x(t)), \forall t \geq 0, \text{ with } x(0) = x_0 \text{ and } x'(0) = x_0'. \tag{1.17}$$

This equation can be regarded as the following system

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t)(1 - x(t)) \end{cases} \tag{1.18}$$

and the equilibria of (1.18) are

$$(x, y) = (0, 0) \text{ and } (x, y) = (1, 0).$$

The system (1.18) is an Hamiltonian which corresponds to a special case of the Bogdanov-Takens normal form (see [32] and [129]). The terminology Hamiltonian means that we have

$$x(1-x)x' = -yy'.$$

Therefore we obtain the following conservation property along the trajectories

$$\frac{x(t)^2}{2} - \frac{x(t)^3}{3} = -\frac{y(t)^2}{2} + h,$$

where the constant of integration $h \in \mathbb{R}$.

In order to study the trajectories passing through the points $y_0 = 0$ and x_0 between 0 and 1, we obtain the following condition for h

$$x_0^2\left(\frac{1}{2} - \frac{x_0}{3}\right) = h.$$

The function $x \longrightarrow \frac{x^2}{2} - \frac{x^3}{3}$ is increasing between 0 and 1. Therefore h varies between 0 (for $x = 0$) and $\frac{1}{6}$ (for $x = 1$).

We deduce that the orbit passing through the point $(x, 0)$ (with $x \in [0, 1]$) satisfies

$$y = \pm\sqrt{2\left[h - \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\right]},$$

where x takes some appropriate values in order for the quantity below the square root to be positive.

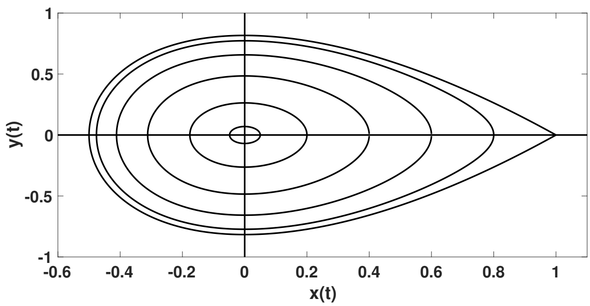

Fig. 1.7: *We plot the complete orbits* $(x(t), y(t))$ *passing respectively through* $x = 0.05, 0.2, 0.4, 0.6, 0.8$ *and* $y = 0$. *The last orbit is a homoclinic orbit passing through* $(x, y) = (-0.5, 0)$ *and the alpha limit set as well as the omega limit is the equilibrium* $(x, y) = (1, 0)$.

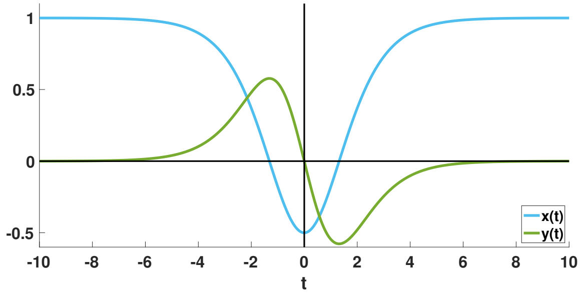

Fig. 1.8: *We plot the homoclinic orbits* $t \longrightarrow (x(t), y(t))$ *which is the solution passing through* $(x, y) = (-0.5, 0)$. *We observe numerically the convergence to* $(1, 0)$ *when the time* t *goes to* $\pm\infty$.

1.8.4 Beverton and Holt discrete time model

The model of Beverton and Holt [18] was introduced in the context of fisheries in 1957. This model is the following

$$N(t+1) = \frac{\beta N(t)}{1 + \alpha N(t)}, \forall t \geq 0, \text{ with } N(0) = N_0 \geq 0, \tag{1.19}$$

where $N(t)$ is the number of individuals, $\beta > 0$ is the growth rate of the population and the term $1/(1 + \alpha N(t))$ (with $\alpha \geq 0$) describes the competition for food, cannibalism effect (indeed the adult fish eat the larvea when they reproduce) etc... The competition occurs between individuals of the same species. Therefore this effect is usually called *intra-specific competition*.

Semiflow: Recall that the semiflow $\{U(t)\}_{t \in \mathbb{N}}$ is defined by

$$U(t)(N_0) = T^t(N_0), \forall t \geq 0, \forall N_0 \geq 0,$$

where $T(N_0) = \frac{\beta N(t)}{1+\alpha N(t)}$ and T^t is defined by induction as following

$$T^{t+1}(N_0) = T\left(T^t(N_0)\right) = T^t(T(N_0)), \forall t \geq 0,$$

and

$$T^0(N_0) = N_0.$$

Equilibria: It is readily checked that 0 is always an equilibrium of (1.19). The non-zero equilibrum satisfies

$$\frac{\beta}{1 + \alpha \overline{N}} = 1 \Leftrightarrow \overline{N} = \frac{\beta - 1}{\alpha}.$$

Dissipation property: We follow the construction of the Liapunov of function by Fisher and Goh [76]. Define

$$V(N(t)) = |N(t) - \overline{N}|.$$

Then

$$\begin{aligned} V(N(t+1)) = |N(t+1) - \overline{N}| &= |\frac{\beta N(t)}{1+\alpha N(t)} - \frac{\beta - 1}{\alpha}| \\ &= |\frac{\alpha\beta N(t) - (\beta-1)(1+\alpha N(t))}{\alpha(1+\alpha N(t))}| \\ &= |\frac{\alpha N(t) - (\beta - 1)}{\alpha(1+\alpha N(t))}|, \end{aligned}$$

and we obtain

$$V(N(t+1)) = \frac{1}{(1+\alpha N(t))} V(N(t)). \tag{1.20}$$

We deduce that if we define $\{U(t)\}_{t\in\mathbb{N}}$ the discrete time semiflow generated by (1.19) for each $t \in \mathbb{N}$,

$$\begin{cases} V(U(t)N_0) < V(N_0), & \text{if } N_0 > 0, \text{ and } N_0 \neq \overline{N}, \\ V(U(t)N_0) = V(N_0), & \text{if } N_0 = 0, \text{ or } N_0 = \overline{N}. \end{cases} \tag{1.21}$$

Theorem 1.56 *The discrete time semiflow generated by* (1.19) *is bounded dissipative on* $[0, \infty)$. *More precisely, for each* $\gamma_0 > \overline{N}$ *the subset*

$$B_{\gamma_0} = \{x \geq 0 : V(x) \leq \gamma_0\}$$

is a bounded absorbing subset.

Proof We have for each $N_0 > 0$, with $V(N_0) \geq \overline{N}$,

$$V(U(t)N_0) < V(N_0), \forall t \geq 0,$$

the result follows. □

Remark 1.57 One may observe that this model can be derived from the logistic equation (1.11) by using the following semi-implicit scheme

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = N(t)(\lambda - \chi N(t+\Delta t))$$

which is equivalent to

$$N(t+\Delta t) = \frac{(1+\Delta t\lambda)N(t)}{1+\Delta t\chi N(t)}.$$

The behavior of the system (1.19) is analogous to the behavior of the solution of the logistic equation (1.11) (see Figure 1.9).

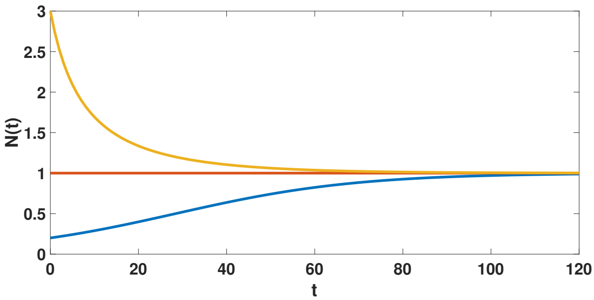

Fig. 1.9: *In this figure we plot three solutions of equation* (1.19) *with* $\alpha = 0.05$ *small and* $\lambda = 1 + \alpha$ *(similarly to the above approximation of the logistic equation). Then we choose three initial value* $N_0 = 0.2$ *(blue curve),* $N_0 = 1$ *(green curve) and* $N_0 = 3$ *(red curve). The green curve corresponds to the equilibrium* $N = 1$*. The second member of* (1.19) *being (concave) monotone increasing no other behavior can be observed whenever* $\lambda > 1$.

1.8.5 Ricker model: chaotic behavior

The first population dynamics model introduced in the context of fisheries to describe the reproduce of salmons in Canada was introduced by Ricker [179, 180] in 1954. The model is the following

$$N(t+1) = \beta N(t) \exp(-N(t)), \ \forall t \geq 0, \text{ with } N(0) = N_0 \geq 0, \tag{1.22}$$

where $\beta > 0$ is the growth rate of the population and the term $\exp(-N(t))$ (with $\alpha \geq 0$) describes the intra-specific competition.

Semiflow: Recall that the semiflow $\{U(t)\}_{t \in \mathbb{N}}$ is defined by

$$U(t)(N_0) = T^t(N_0), \forall t \geq 0, \forall N_0 \geq 0,$$

where $T(N_0) = \beta N_0 \exp(-N_0)$ and T^t is defined by induction as following

$$T^{t+1}(N_0) = T\left(T^t(N_0)\right) = T^t(T(N_0)), \forall t \geq 0,$$

and

$$T^0(N_0) = N_0.$$

Equilibria: It is readily checked that 0 is always an equilibrium of (1.22). The non-zero equilibrum satisfies

$$\beta \exp(-\overline{N}) = 1 \Leftrightarrow \overline{N} = \ln(\beta).$$

Dissipation property: We follow the construction of the Liapunov of function by Fisher and Goh [76]. Define

$$V(N(t)) = \left(N(t) - \overline{N}\right)^2.$$

Then

$$\begin{aligned}
V(N(t+1)) - V(N(t)) &= \left(N(t+1) - \overline{N}\right)^2 - \left(N(t) - \overline{N}\right)^2 \\
&= \left(\beta N(t)\exp(-N(t)) - \overline{N}\right)^2 - \left(N(t) - \overline{N}\right)^2 \\
&= (\beta N(t)\exp(-N(t)))^2 - N(t)^2 \\
&\quad -2\beta N(t)\exp(-N(t))\overline{N} + 2N(t)\overline{N} \\
&= [\beta\exp(-N(t)) - 1]\,[\beta\exp(-N(t)) + 1]\,N(t)^2 \\
&\quad -2N(t)\overline{N}\,[\beta\exp(-N(t)) - 1]
\end{aligned}$$

and we obtain

$$V(N(t+1)) - V(N(t)) = h(N(t))\,[\beta\exp(-N(t)) - 1]\,, \tag{1.23}$$

where

$$\begin{aligned}
h(N(t)) &= \left\{[\beta\exp(-N(t)) + 1]\,N(t) - 2\overline{N}\right\} N(t) \\
&= \left\{N(t)\beta\exp(-N(t)) - \overline{N} + N(t) - \overline{N}\right\} N(t) \\
&= \left\{N(t)\exp(\overline{N} - N(t)) + \left[N(t) - 2\overline{N}\right]\right\} N(t)
\end{aligned} \tag{1.24}$$

Proposition 1.58 *Assume that $\beta \in (1, e^2]$. We deduce that if we define $\{U(t)\}_{t\in\mathbb{N}}$ the discrete time semiflow generated by (1.22) for each $t \in \mathbb{N}$,*

$$V(U(t)N_0) \leq V(N_0), \forall t \in \mathbb{N},\ \forall N_0 \geq 0. \tag{1.25}$$

Proof Let us prove that

$$h(N(t)) \leq 0, \forall N \leq \overline{N}$$

and

$$h(N(t)) \geq 0, \forall N \geq \overline{N}.$$

Indeed, we have

$$h(0) = 0,\ \text{and}\ h(\overline{N}) = 0,\ \text{and}\ h(N) > 0, \forall N \geq 2\overline{N}.$$

Consider $N \in (0, 2\overline{N})$ and $N \neq \overline{N}$. Then $h(N) = 0$ if

$$N(t)\exp(\overline{N}-N(t))+\left[N(t)-2\overline{N}\right]=0$$
$$\Leftrightarrow \exp(\overline{N}-N(t))=2\frac{\overline{N}}{N(t)}-1$$
$$\Leftrightarrow \overline{N}-N(t)=\ln\left(2\frac{\overline{N}}{N(t)}-1\right)$$

which is equivalent to

$$\overline{N}=\frac{\ln\left(2\frac{\overline{N}}{N(t)}-1\right)}{1-\frac{N(t)}{\overline{N}}}. \tag{1.26}$$

First case: $N\in(0,\overline{N})$**:** Set $y=1/(1-N(t)/\overline{N})>1$. Then (1.26) gives

$$\overline{N}=y\left\{\ln(1+1/y)-\ln(1-1/y)\right\}=y\left\{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{y^{-n}}{n}+\sum_{n=1}^{\infty}\frac{y^{-n}}{n}\right\}$$
$$=2\sum_{n=0}^{\infty}\frac{y^{-2n}}{2n+1}>2.$$

Second case: $N\in(\overline{N},2\overline{N})$**:** Set $y=1/(N(t)/\overline{N}-1)>1$. Then (1.26) gives

$$\overline{N}=y\left\{\ln(1+1/y)-\ln(1-1/y)\right\}>2.$$

Finally $\beta\in(1,e^2]$ implies $\overline{N}\in(0,2]$ we conclude that

$$h(N)\neq 0 \text{ and } N\in(0,2\overline{N})\Rightarrow N=\overline{N},$$

and the proof is completed. □

Theorem 1.59 *Assume that* $\beta\in(1,e^2]$. *The discrete time semiflow generated by* (1.22) *is bounded dissipative on* $[0,\infty)$. *More precisely, for each* $\gamma_0>\overline{N}$ *the subset*

$$B_{\gamma_0}=\{x\geq 0: V(x)\leq\gamma_0\}$$

is bounded absorbing subset.

Proof Let $U(t)$ be the discrete time semiflow generated by (1.22). We have

$$V(U(t)0)=V(0)=\overline{N},$$

and

$$V(U(t)\overline{N})=V(\overline{N})=0,$$

and for each $N_0\in(0,+\infty)\setminus\left\{\overline{N}\right\}$,

$$V(U(t)N_0)<V(N_0),\forall t\in\mathbb{N},$$

the result follows. □

Remark 1.60 An Hopf bifurcation for maps occurs at $\beta = e^2$ (i.e. the derivative of the map $x \to \beta x e^{-x}$) crosses -1 when β crosses e^{-2}). It means that periodic orbit with period 2 with appear. Then see Figure 1.10 and Figure 1.11 periodic orbits of period 4, 8, etc... appear and chaos appear when periodic orbit of period 3 appears.

Assume that $\beta > e^2$. we can use the fact that $x \to \beta x \exp(-x)$ is bounded to derive the dissipation property. Indeed,

$$(\beta x \exp(-x))' = \beta\,[1 - x]\exp(-x).$$

Therefore the map $x \to \beta x \exp(-x)$ reaches a maximum at $x = 1$, and this maximum is βe^{-1}. Define

$$W(N) = \max(N, \beta e^{-1}).$$

Let $U(t)$ be the discrete time semiflow generated by (1.22). Then we have

$$W(U(t)N_0) \leq W(N_0), \forall t \geq 0, \forall N_0 \geq 0.$$

Theorem 1.61 *Assume that $\beta \geq 0$. The discrete time semiflow generated by* (1.22) *is bounded dissipative on* $[0, \infty)$.

Remark 1.62 Then by using a simple first order approximation $\exp(N(t)) = 1 + N(t) + h.o.t$, we can regard the model of Beverton and Holt as a first order approximation of the Ricker model. Nevertheless the behavior of this two systems is very different.

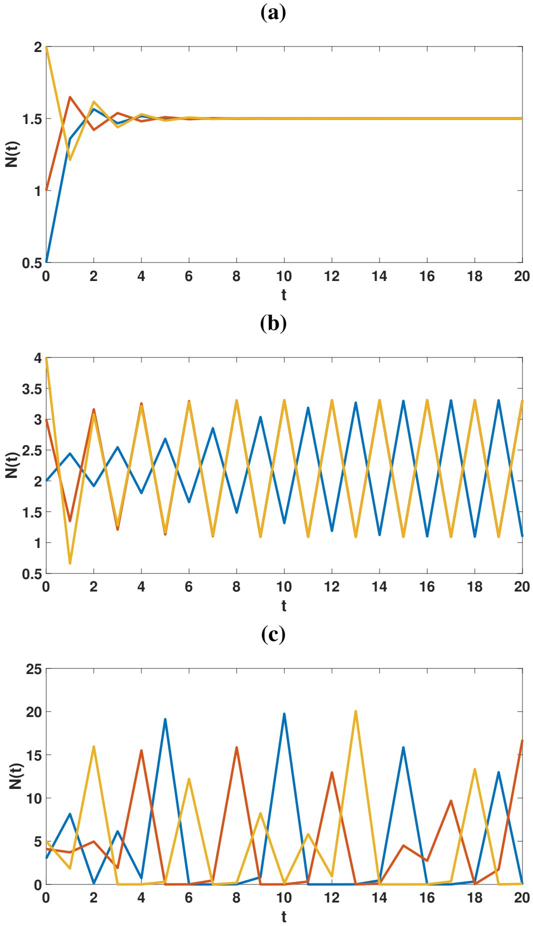

Fig. 1.10: *In Figures (a) (b) and (c) we plot three solutions of equation* (1.22). *In Figure (a), we fix* $\beta = e^{1.5}$, *we observe behavior that is similar to the behavior for the Beverton and Holt model (but oscillations around the positive equilibrium). In Figure (b) we fix* $\beta = e^{2.2}$, *the solutions converge to some periodic solutions. In Figure (c) we fix* $\beta = e^4$, *the behavior becomes chaotic (unpredictable).*

An Hopf bifurcation for maps occurs at $\beta = e^2$. That is to say that derivative of the map $x \longrightarrow \beta x e^{-x}$ crosses -1 when β crosses e^{-2}, and a branch of (non-trivial) periodic orbits of period 2 appear at $\beta = e^2$ (see Figures 1.10 and 1.11). Then periodic orbits of period 4, 8, etc... appear. That is often called a period-doubling bifurcation.

When the parameter β is large enough a periodic orbit of period 3 appears, and the system becomes chaotic (see Figures 1.10 and 1.11). In other words, period three implies chaos Li and Yorke [139].

This is the so called chaos of Sharkovskii [190, 191]. The result of Sharkovskii was rediscovered by Li and Yorke [139]. Both results of Sharkovskii [190, 191] and Li and Yorke chaos show that when the system has a periodic solution of period 3 then periodic orbit of all periods exists. Therefore the system has infinitely many periodic orbit. This could serve as first step to characterize the chaos. But Li and Yorke explored the dependency with respect to the initial condition in the following sense.

Theorem 1.63 (Li and Yorke) *Let $T : I \to I$ be a continuous map on some interval $I \subset \mathbb{R}$. Assume that there exists $N_0 \geq 0$ such that*

$$T^3(N_0) \leq N_0 < T(N_0) < T^2(N_0)$$

or

$$T^3(N_0) \geq N_0 > T(N_0) > T^2(N_0).$$

Then

(i) *For each integer $k = 1, 2, \ldots$ there exists a periodic point in I having the period k.*

(ii) *There exists an uncountable set $S \subset I$ (containing no periodic points), which satisfies the following conditions:*

(a) *For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{t \to \infty} |T^t(p) - T^t(q)| > 0,$$

and

$$\liminf_{t \to \infty} |T^t(p) - T^t(q)| = 0.$$

(b) *For every $p \in S$ and every periodic point $q \in I$,*

$$\limsup_{t \to \infty} |T^t(p) - T^t(q)| > 0.$$

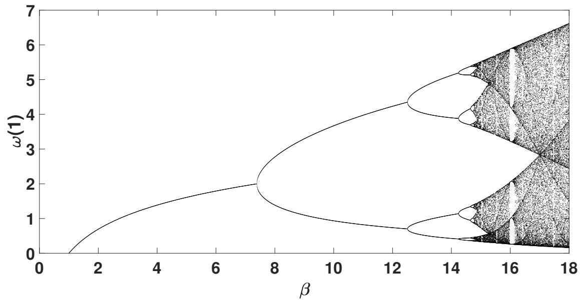

Fig. 1.11: *In this figure we plot for each β varying from 0 to 18 the omega limit set $\omega(1)$ of the solution of (1.22) starting from the initial value $N_0 = 1$.*

In Figure 1.11, we observe a cascade of bifurcations. The branch of single point corresponds to a positive equilibrium, which first bifurcate to a 2 periodic orbit. One may notice that the positive equilibrium exists for each β but becomes unstable after the bifurcation. Then we observe a period doubling bifurcation, we period orbits of period 2, 4, 8, etc. Periodic orbit of period 3 appears only when the parameter β becomes large enough, the chaos in the sense of Li and Yorke [139] appear.

1.8.6 Lorentz system: chaos and dissipation properties

The system of Lorenz [148] was introduced in his paper in 1963 as taken from Saltzman [186] 1962 as a minimalist model of thermal convection in a box. This system is the following

$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \end{cases} \tag{1.27}$$

where $\sigma > 0, \rho > 0,$ and $\beta > 0$, and with initial value

$$x(0) = x_0, y(0) = y_0, \text{ and } z(0) = z_0.$$

In Figures 1.12-1.13, we plot two solutions of system (1.27) with the parameters values

$$\sigma = 10, \rho = 28, \text{ and } \beta = 8/3.$$

In Figures 1.12-1.13, we observe that starting from two very close initial values gives some very different trajectories. In Figure 1.13, the solution starting from the green dot (respectively from the black dot) end up at the yellow dot (respectively at the red dots) at time $t = 19$. So we can visualize the fact that changing a little bit the initial value may have a large impact on the trajectory.

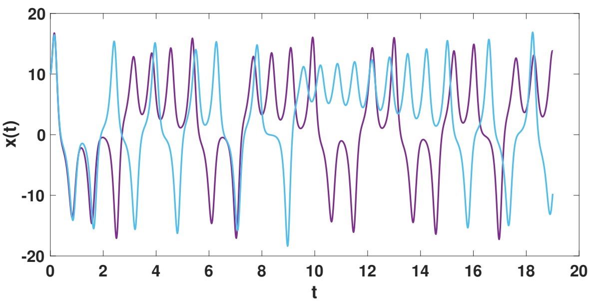

Fig. 1.12: *In this figure, we plot the first component $t \to x(t)$ of two solutions starting from the initial value from $[x_0, y_0, z_0] = [10, 10, 10]$ (light blue curve) and from $[x_0, y_0, z_0] = [10, 10, 10.7]$ (purple curve).*

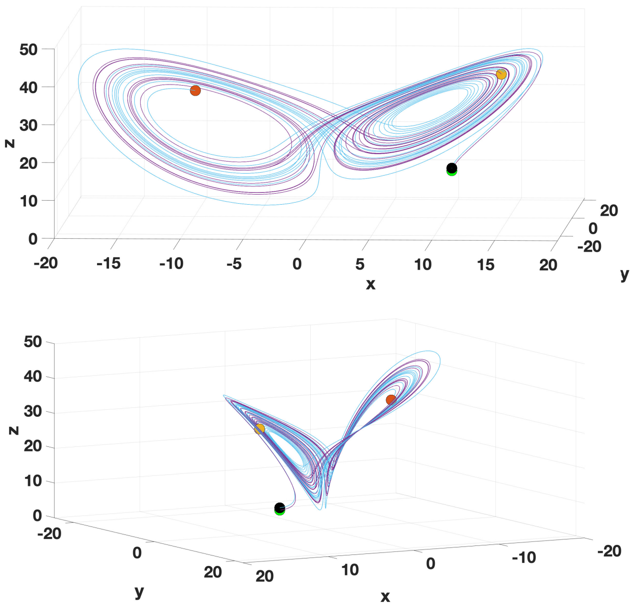

Fig. 1.13: *In this figure, the light blue curve (respectively the purple curve) corresponds to the solution* $t \to (x(t), y(t), z(t))$ *starting from the initial value* $[x_0, y_0, z_0] = [10, 10, 10]$ *(green dot) (respectively from the initial value* $[x_0, y_0, z_0] = [10, 10, 10.7]$ *(black dot)). The two solutions are plotted on the time interval* $[0, 19]$.

Lorenz looks at his system as a simplified model for weather prediction. He is assuming that one loop corresponds to the calm weather and the other loop to tornados. The orbit generated by a slight disturbance of the initial distribution of the system looks very different. The shape of the omega limit set is very similar for each solution. The chaotic nature of this system is due to the "fractal" structure of the attractor in between the loops.

The frequency at which a solution passes from one loop to the next turns to be similar for each solution. Lorentz already mentioned this phenomenon during his presentation at the American Association for the Advancement of Science in 1972. During his presentation is says that:

Over the years minuscule disturbances neither increase nor decrease the frequency of occurrences of various weather events such as tornados; the most they may do is to modify the sequences in which they occur.

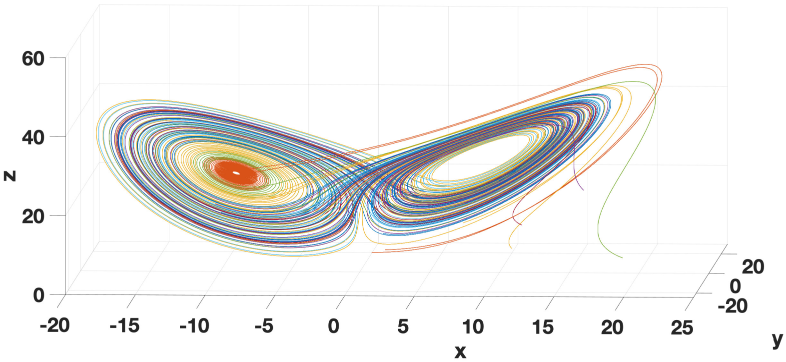

Fig. 1.14: *In this figure, we use 20 randomly chosen initial distributions.*

It turns out that the Lorenz attractor is an attractor in the sense developed in the Chapter 2. That is to say that, it is a compact invariant subset which attracts one of its neighborhood at each point of $\mathbb{R}^3$. The existence such a global attractor is a consequence of the dissipation of the system. So we now investigate the dissipation of the Lorentz system.

For the Lorenz system, the dissipativity can be study by using the results of Leonov [135]. Let us consider

$$V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2. \tag{1.28}$$

Then

$$\frac{dV(x(t), y(t), z(t))}{dt} = 2x(y - x) + 2yx(\rho - z) - 2y^2 + 2xyz - 2\beta z^2$$

and after simplification we obtain

$$\frac{dV(x(t), y(t), z(t))}{dt} = -2\left[x^2 + y^2 + \beta z^2\right] + 2(\rho + 1)xy$$

but $2xy = x^2 + y^2 - (x - y)^2$ therefore

$$\frac{dV(x(t), y(t), z(t))}{dt} = -(1 - \rho)x^2 - (1 - \rho)y^2 - 2\beta z^2 - (\rho + 1)(x - y)^2.$$

Lemma 1.64 *Assume that $\rho \in (0, 1)$. Then each solution of system (1.27) converges to $0_{\mathbb{R}^3}$. More precisely, we have the following estimation of the*

$$V(x(t), y(t), z(t)) \leq e^{-\delta t}V(x_0, y_0, z_0), \forall t \geq 0, \tag{1.29}$$

where

$$\delta := \min\{(1-\rho)\sigma, (1-\rho), 2\beta\}.$$

Proof By choosing the above constant $\delta > 0$, we have

$$\frac{dV(x(t), y(t), z(t))}{dt} \le -\delta V(x(t), y(t), z(t))$$

and the result follows from the differential form of Gronwall's lemma (see Chapter 8 in Volume I). □

Theorem 1.65 *Assume that* $\rho \in (0, 1)$. *Then the semiflow* $U(t)$ *generated by the system* (1.27) *is bounded dissipative.*

Proof Let $\gamma_0 > 0$ be fixed. Let us define

$$B_{\gamma_0} = \left\{(x, y, z) \in \mathbb{R}^3 : V(x, y, z) \le \gamma_0\right\}.$$

Then for each bounded subset $B \subset \mathbb{R}^3$ we have

$$\gamma := \sup_{(x,y,z)\in B} V(x, y, z) < \infty.$$

Let $t_0 = t_0(B) > 0$ such that

$$e^{-\delta t_0}\gamma \le \gamma_0,$$

then by using the definition of B_{γ_0}, we deduce that

$$U(t)B \subset B_{\gamma_0}, \forall t \ge t_0,$$

and the proof is completed. □

Next we consider the functional

$$W(x, y, z) = y^2 + (z-\rho)^2. \tag{1.30}$$

Then

$$\begin{aligned}\frac{dW(x(t), y(t), z(t))}{dt} &= 2yx(\rho - z) - 2y^2 + 2(z-\rho)(xy - \beta z)\\ &= 2yx(\rho - z) - 2y^2 + 2xy(z-\rho) - 2\beta z(z-\rho)\end{aligned}$$

hence

$$\frac{dW(x(t), y(t), z(t))}{dt} = -2y^2 - 2\beta z(z-\rho).$$

Lemma 1.66 *Let* $\lambda \in (0, 2\min(1, \beta))$. *Then*

$$\frac{dW(x(t), y(t), z(t))}{dt} \le -\lambda W(x(t), y(t), z(t)) + \chi, \tag{1.31}$$

where

$$\chi := [2\beta - \lambda]\left(\frac{\beta\rho}{\lambda - 2\beta}\right)^2.$$

Proof By using the fact that $\lambda < 2$ we deduce that

$$\begin{aligned}\frac{dW(x(t), y(t), z(t))}{dt} & +\lambda W(x(t), y(t), z(t)) \\ & = -2y^2 - 2\beta z(z-\rho) + \lambda y^2 + \lambda(z-\rho)^2 \\ & \leq [\lambda - 2\beta]\,(z-\rho)^2 - 2\beta\rho(z-\rho) \\ & = [\lambda - 2\beta]\left\{(z-\rho) - \frac{\beta\rho}{\lambda - 2\beta}\right\}^2 - [\lambda - 2\beta]\left(\frac{\beta\rho}{\lambda - 2\beta}\right)^2.\end{aligned}$$

and since $\lambda < 2\beta$ the inequality (1.31) follows. □

Theorem 1.67 *Let* $\lambda \in (0, 2\min(1,\beta))$. *Then, for each* $t \geq 0$,

$$W(x(t), y(t), z(t)) \leq e^{-\lambda t} W(x_0, y_0, z_0) + \int_0^t e^{-\lambda(t-s)} \chi ds,$$

and

$$\limsup_{t \to +\infty} W(x(t), y(t), z(t)) \leq \frac{\chi}{\lambda}.$$

Moreover the semiflow $U(t)$ *generated by the Lorenz system* (1.27) *is bounded dissipative.*

Proof The proof is left as an exercise. (**Hint:** Start first with the two last components y and z use the same arguments than in the proof of Theorem 1.65. Then use the comparison principle for the x-equation to derive the dissipativity of the full system). □

1.9 Notes and Remarks

Existence and invariance and compactness of ω-limit sets

In this chapter, we presented some notions about continuous semiflow, omega and alpha limit sets and we refer to the books of Lasalle [130], Hale [91, 93]. A we will see in the next chapter the existence of ω-limit sets can be extended by replacing a point $x \in M$ by subset $B \in M$ and replacing the orbit

$$t \to U(t)x.$$

by

$$t \to U(t)B.$$

we refer to Hale [93], Sell and You [189], Raugel [177] for more results.

Connectivity of ω-limit sets

The notion of invariantly connected sets was introduced by Lasalle [130]. The notion of chain transitive sets was first introduced for discrete time semiflow by Hirsch, Smith, and Zhao [104] and for continuous time semiflow by Magal [150]. We also refer to Raugel [177] for more results.

Dissipation in Beverton and Holt model and Ricker model

The Liapunov function for the Beverton and Holt model and Ricker model was introduced by Fisher and Goh [76]. We refer to their original articles for more results.

Chaos in one dimensional discrete time model

The understanding of one dimensional maps improved since Sharkovskii [190] and Li and Yorke [139]. We refer to [12] [66] [86] [141] [169] [227] for more xc xresults on the subject.

More about one dimensional discrete time model

The discrete time logistic equation was invented by Verhulst [219] in 1838. That is the following difference equation

$$N(t+1) = \beta N(t)(1 - N(t)),, \forall t \geq 0, \text{ with } N(0) = N_0 \in [0, 1],$$

where $\beta \in [0, 4]$.

This equation also generates a chaos in the sense of Li and Yorke [139] for $\beta = 4$.

Remark 1.68 The discrete logistic equation was introduced by Berkson [17] who invented the statistical logistic regression in 1944.

The connection of Ricker's model the Beverton and Holt model is the following we use a first order (Talyor expansion) approximation (which is only valid whenever $\alpha N(t)$ is small enough)

$$\frac{\beta N(t)}{\exp(\alpha N(t))} \approx \frac{\beta N(t)}{1 + \alpha N(t)}.$$

The significant difference between the Ricker's model and the Beverton and Holt model probably arise at the second order. It would be natural to consider the following extension of the above approximation

$$N(t+1) = \frac{\beta N(t)}{1+\alpha N(t) + (\alpha N(t))^2/2 + \ldots + (\alpha N(t))^m/m!},$$

as well the higher order version of this model.

The discrete time logistic equation also follows from a first order approximation of the Ricker's model

$$\beta N(t) \exp(-\alpha N(t)) \approx \beta N(t)(1-\alpha N(t)).$$

A extended version of the model of the Beverton and Holt is the cooperative model of Hill [102]

$$N(t+1) = \frac{\beta N(t)^n}{1+\alpha N(t)^n}.$$

Monod, Wyman and Changeux [167] also proposed a further extension cooperative Hill's model which is called the Allosteric model

$$N(t+1) = \frac{\beta N(t)(1+N(t))^{n-1}}{1+\alpha(1+N(t))^n}.$$

Further application to embryology of such model can be found in the framework of gradient differential system in the paper of Demongeot, Glade and Forest [51] and Cinquin and Demongeot [44].

n-dimensional Ricker model

Ricker's model shows that the salmon are first born into a river then swim to the sea. The female salmon spend several years in the ocean before returning to the river they were born to reproduce. Therefore Ricker [179] considers an age-structured model of the following form

$$\begin{cases} N_1(t+1) = [\beta_1 N_1(t) + \ldots + \beta_m N_m(t)] \times \exp(-\alpha [\beta_1 N_1(t) + \ldots + \beta_m N_m(t)]) \\ N_2(t+1) = \pi_1 N_1(t) \\ \vdots \\ N_m(t+1) = \pi_{m-1} N_{m-1}(t), \end{cases}$$

where $N_1(t), \ldots, N_m(t)$ are the number of (female) individuals in each age classes going respectively from 1 to $m \geq 1$, β_i is the birth rate of the age class i, π_i is the probability to survive from age class i to the age class $i+1$. Ricker [179] was actually interested by the following special case

$$\beta_1 = \ldots = \beta_{m-1} = 0,$$

and

$$\beta_m > 0.$$

The age structured Ricker's model actually falls down into a more general class of age structured model so called density dependent model. Such a model was considered by Guckenheimer, Oster and Ipaktchi [87] and by Liu and Cohen [143] to quote a few.

Lorenz like attractor

An alternative to the Lorenz system is the model of Shimizu-Morioka [196]

$$\begin{cases} x' = y \\ y' = x(1-z) - \alpha y \\ z' = x^2 - \beta z \end{cases} \tag{1.32}$$

where $\beta > 0$, and $\alpha > 0$, and with initial value

$$x(0) = x_0, y(0) = y_0, \text{ and } z(0) = z_0.$$

Shilnikov [194] found some Lorenz like attractor for the parameters values

$$\alpha = 0.85, \text{ and } \beta = 0.5.$$

In Figures 1.15-1.16, we illustrate this result with a simulation of system (1.32) for the above parameters values.

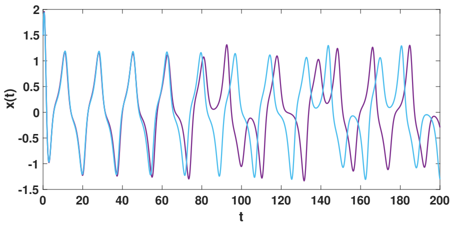

Fig. 1.15: *In this figure, we plot the first component* $t \to x(t)$ *of two solutions starting from the initial value from* $[x_0, y_0, z_0] = [1.5, 1.5, 1.5]$ *(light blue curve) and from* $[x_0, y_0, z_0] = [1.5, 1.5, 1.605]$ *(purple curve).*

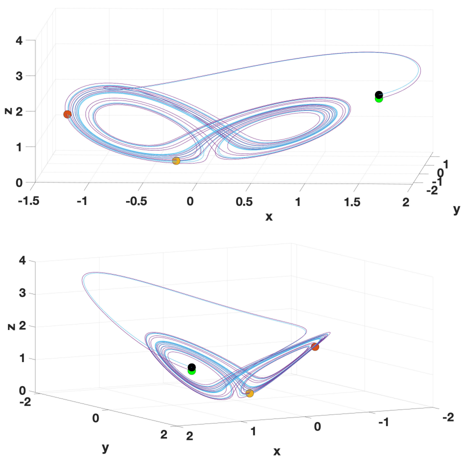

Fig. 1.16: *In this figure, the light blue curve (respectively the purple curve) corresponds to the solution* $t \to (x(t), y(t), z(t))$ *starting from the initial value* $[x_0, y_0, z_0] = [1.5, 1.5, 1.5]$ *(green dot) (respectively from the initial value* $[x_0, y_0, z_0] = [1.5, 1.5, 1.605]$ *(black dot)). The two solutions are plotted on the time interval* $[0, 200]$.

Remark 1.69 Shimizu-Morioka system can be regarded as the following a second order ordinary differential equation

$$\begin{cases} x'' = x(1-z) - \alpha y \\ z' = x^2 - \beta z. \end{cases}$$

Lorenz like attractor in a multi-strain epidemic model

In this section, we present some multi-strain model simulations motivated by the application to the epidemiology of dengue fever. This chaotic numerical simulation was discovered by Aguiar, Kooi, and Stollenwerk [2]. We also refer to Aguiar et al. [1, 3] for more results going in that direction.

Consider an epidemic with two different strains, 1 and 2. Susceptibles to both strains (S) get infected with strain 1 (I_1 or I_{21}) or strain 2 (I_2 or I_{12}), with force of infection (β_1 and $\phi_1\beta_1$ respectively) and (β_2 and $\phi_2\beta_2$) respectively. They recover from infection with strain 1 (becoming R_1) or from strain 2 (becoming R_2), with recovery rate γ. In this recovered class, people have full and life-long immunity against the strain that they were exposed to and infected, and also a period of temporary cross-immunity against the other strain. After this, with rate α, they enter again in the susceptible classes (S_1 respectively S_2), where the index represents the first infection strain. Now, S_1 can be reinfected with strain 2 (becoming I_{12}), meeting I_2 with infection rate β_2 or meeting I_{12} with infection rate $\phi_2\beta_2$, secondary infected contributing differently to the force of infection than primary infected, and S_2 can be reinfected with strain 1 (becoming I_{21}) meeting I_1 or I_{21} with infections rates β_1 and $\phi_1\beta_1$ respectively.

The system introduced by Aguiar, Kooi, and Stollenwerk [2] is the following

$$\begin{cases} S' = -\beta_1 S(I_1 + \phi_1 I_{21}) - \beta_2 S(I_2 + \phi_2 I_{12}) + \mu(N - S), \\ I_1' = \beta_1 S(I_1 + \phi_1 I_{21}) - (\gamma + \mu) I_1, \\ I_2' = \beta_2 S(I_2 + \phi_2 I_{12}) - (\gamma + \mu) I_2, \\ R_1' = \gamma I_1 - (\alpha + \mu) R_1, \\ R_2' = \gamma I_2 - (\alpha + \mu) R_2, \\ S_1' = -\beta_2 S_1 (I_2 + \phi_2 I_{12}) + \alpha R_1 - \mu S_1, \\ S_2' = -\beta_1 S_2 (I_1 + \phi_1 I_{21}) + \alpha R_2 - \mu S_2, \\ I_{12}' = \beta_2 S_1 (I_2 + \phi_2 I_{12}) - (\gamma + \mu) I_{12}, \\ I_{21}' = \beta_1 S_2 (I_1 + \phi_1 I_{21}) - (\gamma + \mu) I_{21}, \\ R' = \gamma (I_{12} + I_{21}) - \mu R, \end{cases} \tag{1.33}$$

where $\beta_1 > 0, \beta_2 > 0$, and $\alpha > 0$, and with initial value

$$S(0) = S^0, I_1(0) = I_1^0, I_2(0) = I_2^0, R_1(0) = R_1^0, R_2(0) = R_2^0,$$
$$S_1(0) = S_1^0, S_2(0) = S_2^0, I_{12}(0) = I_{12}^0, I_{21}(0) = I_{21}^0, \text{ and } R(0) = R^0.$$

In all the figures, we use the following values of the parameters

$$N = 100, \mu = 1/65, \gamma = 52, \beta_1 = \beta_2 = 2\gamma,$$
$$\alpha = 2, \text{ and } \phi_1 = \phi_2 = \phi = 0.6.$$

In Figures 1.17-1.19 we plot two solutions of the system. The system seams to be very much initial condition dependent and the key to get some chaos is to use non symmetric initial percentage for each strain.

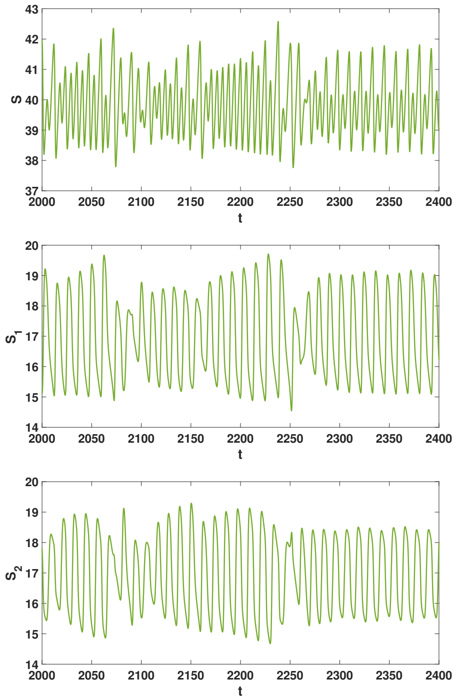

Fig. 1.17: *In this figure we use the initial value* $[S_0, I_1^0, I_2^0] = [70, 10, 20]$ *with the remaining components of the initial distribution equal to* 0*. In this figure we plot* $t \to S(t)$*,* $t \to S_1(t)$*, and* $t \to S_2(t)$ *for* $t \in [2000, 2400]$.

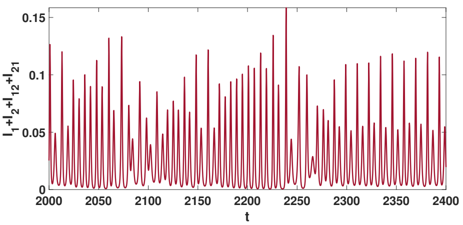

Fig. 1.18: *In this figure we use the initial value* $[S_0, I_1^0, I_2^0] = [70, 10, 20]$ *with the remaining components of the initial distribution equal to* 0*. In this figure we plot* $t \to I_1(t) + I_2(t) + I_{12}(t) + I_{21}(t)$ *for* $t \in [2000, 2400]$.

In Figure 1.19, the global attractor is comparable to the Lorenz attractor because the solution jumps randomly from a region where the first strain is domain to a region where the second strain is dominant back and forth. The changes from one dominant strain to the next are decided uncertainly along the axis $(S, 0, 0)$ where both strains are close to 0.

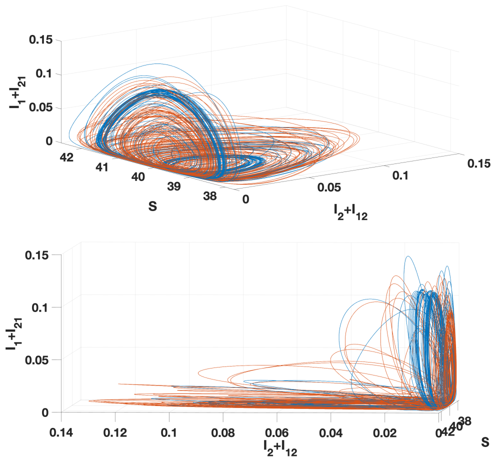

Fig. 1.19: *In this figure we use the initial value* $[S_0, I_1^0, I_2^0] = [70, 10, 20]$ *(blue curve)* $[S_0, I_1^0, I_2^0] = [70, 20, 10]$ *(orange curve) with the remaining components of the initial distribution equal to* 0*. The top and bottom figures corresponds to the same simulations but regarded from a different angle.*

Homoclinic orbits

Homoclinic trajectories are very important in the bifurcation theory and may induce chaos as in the system of Lorenz [86] [195] [226]. The existence of homoclinic orbits has be studied for Lorentz system by Leonov in [136], Leonov and Kuznetsov [137].

In the context of population dynamics, the existence of homoclinic orbits are usual induced by Bogdanov-Takens bifurcations (see Section 1.8.3 and Figure 1.8).

The existence homoclinic orbit have been obtained by Tang et al. [206] for an epidemic system of the form

$$\begin{cases} S' = b - \delta S - \kappa \dfrac{I^2 S}{1 + \alpha I^2} + \nu R \\ I' = \kappa \dfrac{I^2 S}{1 + \alpha I^2} - (\delta + \gamma) I \\ R' = \gamma I - \nu R. \end{cases}$$

A model without loss of immunity of the recovered (i.e. for $\nu = 0$ in the above model) was considered earlier by Xiao and Ruan [228].

ω-limit for non-autonomous system

Explain the skew product semi-flow and refer to the ODE book of George Sell.

We consider a non autonomous ordinary differential equation of the following

$$x'(t) = F(\lambda(t), x(t))$$

where $t \to \lambda(t)$ is a parameter which is time dependent, and which defines a continuous map from $\mathbb{R}$ to $\mathbb{R}^p$.

$$T(t)(\lambda)(x) = \lambda(t + x), \forall t \in \mathbb{R}, \forall x \in \mathbb{R},$$

where $\lambda : \mathbb{R} \to \mathbb{R}^n$ is almost periodic continuous function.

There are several equivalent definition for almost periodic function (Ref???). The following is one of them.

Definition 1.70 The function $t \to \lambda(t) \in \text{BUC}\,(\mathbb{R}, \mathbb{R}^p)$ is almost periodic, if and only if

$$O(\lambda) := \bigcup_{t \geq 0} \{u(t)\}$$

is relatively compact in $\text{BUC}\,(\mathbb{R}, \mathbb{R}^p)$. That is to say that for every sequence t_n we can find a sub-sequence t_{n_p} (denoted with the same index) such that there exists $x \to \overline{\lambda}(x) \in \text{BUC}(\mathbb{R}, \mathbb{R}^p)$ such that

$$\lim_{p \to \infty} \sup_{x \in \mathbb{R}} |\lambda(t_{n_p} + x) - \overline{\lambda}(x)| = 0.$$

Strange attractors

We refer to Lu, Wang, and Young [149] for more references and more results about strange attractor. In [149] they prove that a periodic perturbation of a system undergoing an Hopf bifurcation create sustained chaotic behavior. Specifically, strange attractors are shown to exist. The analysis is carried out for infinite dimensional systems.

Principle of competitive exclusion

McGehee, R., & Armstrong, R. A. (1977). Some mathematical problems concerning the ecological principle of competitive exclusion. Journal of Differential Equations, 23(1), 30-52.

1.10 MATLAB Codes

1.10.1 Figure 1.3

```
L=10;
t=-L:0.1:L;
x0=0.5
x=exp(t)*x0./(1+(exp(t)-1)*x0);

aux=-L:0.1:L;
aux1=ones(size(aux));
aux2=-2:0.1:2;
hold on
plot(t,x,'Color','[      0    ,    0.4470    ,    0.7410]','
    LineWidth',5)
plot(aux,0*aux,'k',0*aux2,aux2,'k','LineWidth',3)
xlabel('t');
ylabel('N(t)');



xlabel('t')
ylabel('x(t)')

set(gca,'YLim',[-0.5  1.5])
set(gca,'XLim',[-L L])

set(gca,'fontweight','bold','FontSize',30);
ax=gca;
hold off
```

1.10.2 Figure 1.4 and Figure 1.5 and Figure 1.6

```
close all
clear all;
figure(1)
dt=0.5
tspan = 1:dt:500;
hold on
y0 = [0.4;0];
[T,Y1] = ode45(@myfun,tspan,y0);

y0 = [0.5;0];
[T,Y2] = ode45(@myfun,tspan,y0);

```

```
y0 = [0.6;0];
[T,Y3] = ode45(@myfun,tspan,y0);


y0 = [0.7;0];
[T,Y4] = ode45(@myfun,tspan,y0);


y0 = [0.8;0];
[T,Y5] = ode45(@myfun,tspan,y0);

plot(Y1(:,1),Y1(:,2),Y2(:,1),Y2(:,2),Y3(:,1),Y3(:,2)
    ,Y4(:,1),Y4(:,2),Y5(:,1),Y5(:,2),'linewidth',3);
sol = ode45(@myfun,tspan,y0);

plot([-0.4,0.8],[0,0],'k','linewidth',3)

plot([0,0],[-0.2,0.3],'k','linewidth',3)

xlabel('x')
ylabel('y')

set(gca,'XLim',[-0.4,0.8])
set(gca,'YLim',[-0.2,0.3])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;
hold off

figure(2)

dt=0.1
tspan = 1:dt:500;
y0 = [0.01;0];
[T,Y1] = ode45(@myfun,tspan,y0);

tspan = 1:dt:300;
y0 = [0.08;0];
[T,Y2] = ode45(@myfun,tspan,y0);
hold on

plot(Y1(:,1),Y1(:,2),'Color',[ 0.4660 , 0.6740
    , 0.1880]','linewidth',3);

plot(Y2(:,1),Y2(:,2),'Color',[ 0.3010 , 0.7450
    , 0.9330]','linewidth',3);
```

```
plot([-0.2,0.2],[0,0],'k','linewidth',3)

plot([0,0],[-0.2,0.2],'k','linewidth',3)
xlabel('x')
ylabel('y')

set(gca,'XLim',[-0.16,0.16])
set(gca,'YLim',[-0.16,0.16])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;
hold off


figure(3)

dt=0.1
tspan = 1:dt:400;
y0 = [0.01;0];
[T,Y1] = ode45(@myfun,tspan,y0);

tspan = 1:dt:400;
y0 = [0.4;0];

[T,Y2] = ode45(@myfun,tspan,y0);
hold on

plot(T,Y1(:,1),'Color','[ 0.4660 , 0.6740 ,
    0.1880]','linewidth',3);

plot(T,Y2(:,1),'Color','[ 0 , 0.4470 ,
    0.7410]','linewidth',3);
plot(T,0*T,'k','linewidth',3)

%plot([0,0],[-0.2,0.2],'k','linewidth',3)
xlabel('t')
ylabel('x(t)')

%set(gca,'XLim',[-0.16,0.16])
%set(gca,'YLim',[-0.16,0.16])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;
hold off

figure(4)

dt=0.1
```

```
tspan = 1:dt:400;
y0 = [0.01;0];
[T,Y1] = ode45(@myfun,tspan,y0);

tspan = 1:dt:400;
y0 = [0.4;0];

[T,Y2] = ode45(@myfun,tspan,y0);
hold on

plot(T,Y1(:,2),'Color','[ 0.4660 , 0.6740 ,
    0.1880]','linewidth',3);

plot(T,Y2(:,2),'Color','[ 0 , 0.4470 ,
    0.7410]','linewidth',3);
plot(T,0*T,'k','linewidth',3)


xlabel('t')
ylabel('y(t)')


set(gca,'fontweight','bold','FontSize',30);
ax=gca;
hold off

function dy = myfun(t,y)
lambda = 0.02;
omega = 0.1;
kappa = 1;
dy = zeros(2,1);
dy(1) =(lambda/2)*y(1)-omega*y(2)-(kappa/2)*(y(1)*y
    (1)+y(2)*y(2))*y(1);
dy(2) = omega*y(1)+(lambda/2)*y(2)-(kappa/2)*(y(1)*y
    (1)+y(2)*y(2))*y(2);
dy=dy(:);
end
```

1.10.3 Figure 1.7

In this program we use the matlab solver *roots(p)* which compute the roots a polynomial $p(x) = a_n x^n + \ldots + a_0$ which is defined in Matlab as $[a_n \; a_{n-1} \; \ldots \; a_0]$.

```
clf;
aux2=0.05
h=(aux2^2)/2-(aux2^3)/3;
p = [(1/3) -(1/2) 0 h]
```

```
r = roots(p)
aux1=r(3)
x1=aux1:0.0001:aux2;
y1=2*(h- x1.^2/2+x1.^3/3).^0.5;
z1=-y1;

plot(x1,y1,'k',x1,-y1,'k','LineWidth',3)
hold on;

aux2=0.2;
h=(aux2^2)/2-(aux2^3)/3;
p = [(1/3) -(1/2) 0 h];
r = roots(p);
aux1=r(3);
x1=aux1:0.0001:aux2;
y1=2*(h- x1.^2/2+x1.^3/3).^0.5;
z1=-y1;

plot(x1,y1,'k',x1,-y1,'k','LineWidth',3)
hold on;


aux2=0.4;
h=(aux2^2)/2-(aux2^3)/3;
p = [(1/3) -(1/2) 0 h];
r = roots(p);
aux1=r(3);
x1=aux1:0.0001:aux2;
y1=2*(h- x1.^2/2+x1.^3/3).^0.5;
z1=-y1;

plot(x1,y1,'k',x1,-y1,'k','LineWidth',3);

aux2=0.6;
h=(aux2^2)/2-(aux2^3)/3;
p = [(1/3) -(1/2) 0 h];
r = roots(p);
aux1=r(3);
x1=aux1:0.0001:aux2;
y1=2*(h- x1.^2/2+x1.^3/3).^0.5;
z1=-y1;


plot(x1,y1,'k',x1,-y1,'k','LineWidth',3)

aux2=0.8;
```

```
h=(aux2^2)/2-(aux2^3)/3;
p = [(1/3) -(1/2) 0 h];
r = roots(p);
aux1=r(3);
x1=aux1:0.0001:aux2;
y1=2*(h- x1.^2/2+x1.^3/3).^0.5;
z1=-y1;


plot(x1,y1,'k',x1,-y1,'k','LineWidth',3)


aux2=1;
h=(aux2^2)/2-(aux2^3)/3;
p = [(1/3) -(1/2) 0 h];
r = roots(p);
aux1=r(3);
x1=aux1:0.0001:aux2;
y1=2*(h- x1.^2/2+x1.^3/3).^0.5;
z1=-y1;
plot(x1,y1,'k',x1,-y1,'k','LineWidth',3)


aux=-2:0.1:2;
plot(aux,0*aux,'k',0*aux,-aux,'k','LineWidth',3)
xlabel('x(t)');
ylabel('y(t)');
set(gca,'YLim',[-1 1])
set(gca,'XLim',[-0.6 1.1])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;
```

1.10.4 Figure 1.8

In this program we need to increase the precision of the MATLAB ODE solver in order to get the convergence of the solution when t goes to $\pm\infty$.

```
clf;
X0=[-0.5 0];
t0=0:0.002:10;

options=odeset('RelTol',10^(-10));
[t1,x1]=ode45(@Model1,t0,X0,options);
hold on;
```

```
h=plot(t1,x1(:,1),'Color','[    0.3010  ,   0.7450   ,
    0.9330]','LineWidth',5);
h=plot(t1,x1(:,2),'Color','[ 0.4660   ,   0.6740     ,
    0.1880]','LineWidth',5);

X0=[-0.5  0];
t0=0:0.002:10;

options=odeset('RelTol',10^(-10));
[t2,x2]=ode45(@Model2,t0,X0,options);


h=plot(-t2,x2(:,1),'Color','[    0.3010  ,   0.7450   ,
    0.9330]','LineWidth',5);
h=plot(-t2,x2(:,2), 'Color','[ 0.4660   ,   0.6740     ,
    0.1880]','LineWidth',5);


aux=-10:0.1:10;
plot(aux,0*aux,'k',0*aux,aux,'k','LineWidth',3)
h=legend('x(t)','y(t)','Location','southeast');
xlabel('t');
%ylabel('x(t)   y(t)');
set(gca,'YLim',[-0.6  1.1])
set(gca,'XLim',[-10  10])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;


function dy=Model1(t,y)
dy=zeros(2,1);
dy(1)=-y(2);
dy(2)=y(1).*(1-y(1));
end

function dy=Model2(t,y)
dy=zeros(2,1);
dy(1)=y(2);
dy(2)=-y(1).*(1-y(1));
end
```

1.10.5 Figure 1.9

```
alpha=0.05;
```

```
lambda=1+alpha;
n=120; % number of time steps
x1=zeros(n+1,1);
x2=zeros(n+1,1);
x3=zeros(n+1,1);
t=zeros(n+1,1);
x1(1)=0.2;
x2(1)=1;
x3(1)=3;
t(1)=0;
for i=1:n
    x1(i+1)=lambda*x1(i)/(1+alpha*x1(i));
    x2(i+1)=lambda*x2(i)/(1+alpha*x2(i));
    x3(i+1)=lambda*x3(i)/(1+alpha*x3(i));
    t(i+1)=i;
end


clf;
plot(t,x1,t,x2,t,x3,'LineWidth',5)
xlabel('t');
ylabel('N(t)');
set(gca,'XLim',[0 n])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;
```

1.10.6 Figure 1.10

```
close all;
clear all;
figure(1)
beta=exp(1.5);
n=20; % number of time steps
x1=zeros(n+1,1);
x2=zeros(n+1,1);
x3=zeros(n+1,1);
t=zeros(n+1,1);
x1(1)=0.5;
x2(1)=1;
x3(1)=2;
t(1)=0;
for i=1:n
    x1(i+1)=beta*x1(i)*exp(-x1(i));
    x2(i+1)=beta*x2(i)*exp(-x2(i));
    x3(i+1)=beta*x3(i)*exp(-x3(i));
    t(i+1)=i;
```

```
end
plot(t,x1,t,x2,t,x3,'LineWidth',5)
xlabel('t');
ylabel('N(t)');
set(gca,'XLim',[0 n])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;


figure(2)
beta=exp(2.2);
n=20; % number of time steps
x1=zeros(n+1,1);
x2=zeros(n+1,1);
x3=zeros(n+1,1);
t=zeros(n+1,1);
x1(1)=2;
x2(1)=3;
x3(1)=4;
t(1)=0;
for i=1:n
    x1(i+1)=beta*x1(i)*exp(-x1(i));
    x2(i+1)=beta*x2(i)*exp(-x2(i));
    x3(i+1)=beta*x3(i)*exp(-x3(i));
    t(i+1)=i;
end


clf;
plot(t,x1,t,x2,t,x3,'LineWidth',5)
xlabel('t');
ylabel('N(t)');
set(gca,'XLim',[0 n])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;

figure(3)
beta=exp(4);
n=20; % number of time steps
x1=zeros(n+1,1);
x2=zeros(n+1,1);
x3=zeros(n+1,1);
t=zeros(n+1,1);
x1(1)=3;
x2(1)=4.1;
x3(1)=5;
```

```
t(1)=0;
for i=1:n
    x1(i+1)=beta*x1(i)*exp(-x1(i));
    x2(i+1)=beta*x2(i)*exp(-x2(i));
    x3(i+1)=beta*x3(i)*exp(-x3(i));
    t(i+1)=i;
end


clf;
plot(t,x1,t,x2,t,x3,'LineWidth',5)
xlabel('t');
ylabel('N(t)');
set(gca,'XLim',[0 n])
set(gca,'fontweight','bold','FontSize',30);
ax=gca;
```

1.10.7 Figure 1.11

```
clear all;
close all;
n=10000;
p=50; % p must be smaller than n
j=1;
for beta=1:0.005:exp(log(18))
x=1;


    for i=1:n
        x=beta*x*exp(-x);
        if (i >=n-p)
        aux(1,j)=beta;
        aux(2,j)=x;
        j=j+1;
        end
    end
end

clf;
plot(aux(1,:),aux(2,:),'k.','markersize', 2)
set(gca, 'xlim', [0 18]);
xlabel('\beta');
ylabel('\omega(1)');

set(gca,'fontweight','bold','FontSize',30);
ax=gca;
```

1.10.8 Figure 1.12 and Figure 1.13

```
close all;
clear all;

sigma=10;
rho=28;
beta=8/3

eps = 1e-15;


Tmax=19;


figure(1)

T = [0 Tmax];


initV = [10 10 10];


options = odeset('RelTol',eps,'AbsTol',[eps eps eps
    /10]);
[T,X] = ode45(@(T,X) F(T, X, sigma, rho, beta), T,
    initV, options);



plot3(X(:,1),X(:,2),X(:,3),'Color','[ 0.4940 ,
    0.1840 , 0.5560]','LineWidth', 1);

view
    ([19.1287420824236,-356.3357721553517,80.00898161095051])

hold on;

plot3(X(1,1),X(1,2),X(1,3),'ok', 'MarkerSize', 20, '
    MarkerFaceColor', 'g');

plot3(X(end,1),X(end,2),X(end,3),'ok', 'MarkerSize',
     20, 'MarkerFaceColor', '[ 0.9290 , 0.6940 ,
    0.1250]');

```

```
T = [0 Tmax];

initV = [10 10 10.7];

[T,X] = ode45(@(T,X) F(T, X, sigma, rho, beta), T, initV, options);


plot3(X(:,1),X(:,2),X(:,3),'Color','[ 0.3010 , 0.7450 , 0.9330 ]','LineWidth', 1);


plot3(X(1,1),X(1,2),X(1,3),'ok', 'MarkerSize', 20, 'MarkerFaceColor', 'k');

plot3(X(end,1),X(end,2),X(end,3),'ok', 'MarkerSize', 20, 'MarkerFaceColor', '[ 0.8500 , 0.3250 , 0.0980]');


grid;
xlabel('x'); ylabel('y'); zlabel('z');
set(gca, 'FontSize', 30);
set(gca, 'FontWeight', 'bold');

figure(2)

T = [0 Tmax];

initV = [10 10 10];

[T,X] = ode45(@(T,X) F(T, X, sigma, rho, beta), T, initV, options);

plot3(X(:,1),X(:,2),X(:,3),'Color','[ 0.4940 , 0.1840 , 0.5560]','LineWidth', 1);

hold on;

plot3(X(1,1),X(1,2),X(1,3),'ok', 'MarkerSize', 20, 'MarkerFaceColor', 'g');

plot3(X(end,1),X(end,2),X(end,3),'ok', 'MarkerSize', 20, 'MarkerFaceColor', '[ 0.9290 , 0.6940 , 0.1250]');

```

```
T = [0 Tmax];

initV = [10 10 10.7];

[T,X] = ode45(@(T,X) F(T, X, sigma, rho, beta), T,
    initV, options);

plot3(X(:,1),X(:,2),X(:,3),'Color','[   0.3010  ,
    0.7450   ,   0.9330  ]','LineWidth', 1);

view
    ([183.3188733682035,331.5379873898024,112.19634248895])


plot3(X(1,1),X(1,2),X(1,3),'ok', 'MarkerSize', 20, '
    MarkerFaceColor', 'k');

plot3(X(end,1),X(end,2),X(end,3),'ok', 'MarkerSize',
     20, 'MarkerFaceColor', '[   0.8500  ,    0.3250   ,
    0.0980]');

grid;
xlabel('x'); ylabel('y'); zlabel('z');
set(gca, 'FontSize', 30);
set(gca, 'FontWeight', 'bold');


figure(3)
T = [0 Tmax];

initV = [10 10 10];


options = odeset('RelTol',eps,'AbsTol',[eps eps eps
    /10]);
[T,X] = ode45(@(T,X) F(T, X, sigma, rho, beta), T,
    initV, options);

plot(T,X(:,1),'Color','[   0.4940  ,    0.1840  ,
    0.5560]','LineWidth', 3);

hold on;
T = [0 Tmax];

initV = [10 10 10.7];

```

```
options = odeset('RelTol',eps,'AbsTol',[eps eps eps
    /10]);
[T,X] = ode45(@(T,X) F(T, X, sigma, rho, beta), T,
    initV, options);

plot(T,X(:,1),'Color','[  0.3010  ,  0.7450  ,
    0.9330 ]','LineWidth', 3);

xlabel('t'); ylabel('x(t)');
set(gca, 'FontSize', 30);
set(gca, 'FontWeight', 'bold');



function dx = F(T, X, sigma, rho, beta)
% Evaluates the right hand side of the Lorenz system
% x' = sigma*(y-x)
% y' = x*rho - x*z - y
% z' = x*y - beta*z


    dx = zeros(3,1);

    dx(1) = sigma*(X(2) - X(1));
    dx(2) = rho*X(1) - X(1)*X(3)- X(2);
    dx(3) = X(1)*X(2) - beta*X(3);

end
```

1.10.9 Figures 1.17-1.19

```
close all;
clear all;

% Aguiar, M., Ballesteros, S., Kooi, B. W., &
    Stollenwerk, N. (2011). The role of seasonality
    and import in a minimalistic
% multi-strain dengue model capturing differences
    between primary and secondary infections: complex
     dynamics and its implications
% for data analysis. Journal of theoretical biology,
     289, 181-196.



```

```
Tmax=2800;


T =[0 Tmax];
initV=zeros(1,10);

initV(1,1) =70; % S at t=0
initV(1,2) =10; % I_1 at t=0
initV(1,3) =20; % I_2 at t=0


eps = 1e-40;
options = odeset('RelTol',eps,'AbsTol',[eps eps eps
    eps eps eps eps eps eps eps]);

[T,X] = ode45(@(T,X) F(T, X), T, initV, options);

figure(1)
    plot(T,X(:,1),'Color','[  0.4660  ,  0.6740  ,
        0.1880]','LineWidth', 3);
    xlabel('t'); ylabel('S');
    set(gca, 'FontSize', 30);
    set(gca, 'FontWeight', 'bold');
    xlim([2000,2400])

figure(2)
    plot(T,X(:,6),'Color','[  0.4660  ,  0.6740  ,
        0.1880]','LineWidth', 3);
    xlabel('t'); ylabel('S_1');
    set(gca, 'FontSize', 30);
    set(gca, 'FontWeight', 'bold');
    xlim([2000,2400])

figure(3)
    plot(T,X(:,7),'Color','[  0.4660  ,  0.6740  ,
        0.1880]','LineWidth', 3);
    xlabel('t'); ylabel('S_2');
    set(gca, 'FontSize', 30);
    set(gca, 'FontWeight', 'bold');
    xlim([2000,2400])
figure(4)
    plot(T,X(:,2)+X(:,3)+X(:,8)+X(:,9),'Color','
        [0.6350 ,  0.0780  , 0.1840]','LineWidth',
        3);
    xlabel('t'); ylabel('I_1+I_2+I_{12}+I_{21}');
    set(gca, 'FontSize', 30);
```

```
    set(gca, 'FontWeight', 'bold');
    xlim([2000,2400])

figure(5)
    n=find(T>=2000);
    plot3(X(n(1):end,3)+X(n(1):end,8), X(n(1):end,1),X(n(1):end,2)+X(n(1):end,9),'Color','[ 0 , 0.4470 , 0.7410]','LineWidth', 1);


    hold on;

    initV=zeros(1,10);
    initV(1,1) =70; % S at t=0
    initV(1,2) =20; % I_1 at t=0
    initV(1,3) =10; % I_2 at t=0

    [T1,X1] = ode45(@(T,X) F(T, X), T, initV, options);
    n=find(T1>=2000);
    plot3(X1(n(1):end,3)+X1(n(1):end,8), X1(n(1):end,1),X1(n(1):end,2)+X1(n(1):end,9),'Color','[ 0.8500 , 0.3250 , 0.0980]','LineWidth', 1);
    grid;
    ylabel('S'); xlabel('I_2+I_{12}'); zlabel('I_1+I_{21}');
    set(gca, 'FontSize', 30);
    set(gca, 'FontWeight', 'bold');


figure(6)

    plot3(X(n(1):end,3)+X(n(1):end,8), X(n(1):end,1)+X(n(1):end,6)+X(n(1):end,7),X(n(1):end,2)+X(n(1):end,9),'Color','[ 0 , 0.4470 , 0.7410]','LineWidth', 1);

    hold on;


    plot3(X1(n(1):end,3)+X1(n(1):end,8), X1(n(1):end,1)+X1(n(1):end,6)+X1(n(1):end,7),X1(n(1):end,2)+X1(n(1):end,9),'Color','[ 0.8500 , 0.3250 , 0.0980]','LineWidth', 1);
    grid;
```

```
    ylabel('S+S_1+S_2'); xlabel('I_2+I_{12}'); zlabel('I_1+I_{21}');
    set(gca, 'FontSize', 30);
    set(gca, 'FontWeight', 'bold');
function dx = F(T, X)

N=100;
mu=1/65;
gamma= 52  ;
beta1= 2*gamma;
beta2= 2*gamma;
alpha= 2  ;
phi=0.6;
phi1=phi;
phi2=phi;

    dx = zeros(10,1);

    dx(1) =- beta1/N*X(1)*(X(2)+phi1*X(9)) - beta2/N*X(1)*(X(3)+phi2*X(8))+mu*(N-X(1)); % S-equation

    dx(2) = beta1/N*X(1)*(X(2)+phi1*X(9))- (gamma+mu)*X(2);                 % I_1 equation
    dx(3) = beta2/N*X(1)*(X(3)+phi2*X(8)) - (gamma+mu)*X(3);                % I_2 equation

    dx(4)=gamma*X(2)-(alpha+mu)*X(4);                                       % R_1 equation
    dx(5)=gamma*X(3)-(alpha+mu)*X(5);                                       % R_2 equation

    dx(6)= - beta2/N*X(6)*(X(3)+phi2*X(8))+alpha*X(4)-mu*X(6);              % S_1 equation
    dx(7)= - beta1/N*X(7)*(X(2)+phi1*X(9))+alpha*X(5)-mu*X(7);              % S_2 equation

    dx(8)= beta2/N*X(6)*(X(3)+phi2*X(8))- (gamma+mu)*X(8);                  % I_12 equation
```

```
        dx(9)= beta1/N*X(7)*(X(2)+phi1*X(9))- (gamma+mu
            )*X(9);                                   % I_21
            equation

        dx(10)=gamma*(X(8)+X(9))-mu*X(10);

            % R equation
    end
```

Chapter 2
Global Attractors and Uniform Persistence

2.1 Interior Global Attractor for an Elementary Example

The illustrate the idea of the chapter we consider the difference equation

$$U(t+1)x = \frac{\lambda U(t)x}{1+U(t)x}, \forall t \in \mathbb{N}, \text{ and } U(0)x = x \geq 0,$$

where $\lambda > 1$.

Then we set $M = \mathbb{R}_+$ endowed with the distance induced by the absolute value

$$d(u, v) = |u - v|.$$

Then $A = [0, \lambda - 1]$ (where $\overline{x} = \lambda - 1$ is the positive equilibrium) is compact and

$$\lim_{t \to \infty} \delta(U(t)B, A) = 0,$$

whenever $B \subset \mathbb{R}_+$.

The subset A is the global attractor for U in $\mathbb{R}_+$.

Next, we observe that

$$M_0 = (0, +\infty)$$

is positively invariant by U. But the existence of a global for U restricted (M_0, d) is more delicate because the metric space (M_0, d) is not complete.

If we consider positive interior equilibrium

$$\overline{U} = \lambda - 1 > 0.$$

We would like to define

$$A_0 = \left\{\overline{U}\right\}$$

as the global attractor for U restricted to M_0. This is delicate, because the interior attractor A_0 will attract the compact subset of M_0 (in general), and A_0 may not the bounded subsets of M_0 (in general).

2.2 Positive Orbit for a Set

Let (M,d) be a complete metric space. Let $\{U(t)\}_{t\geq 0}$ be a semiflow on a metric space (M,d). In Chapter 1, we described the omega-limit set for a point $x \in M$. Recall that the omega-limit set is defined as

$$\omega(x) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} U(s)x}.$$

The idea of global attractor is to understand the asymptotic properties of $t \to U(t)B$ when t goes to infinity. So we need to extend the notion omega-limit for the trajectory of a point x to the trajectory of a subset B of M. So we will consider

$$\omega(B) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} U(s)B}$$

for a subset B of M.

2.3 Examples of Metric Spaces

In practice the notion of metric spaces is introduced in order to keep a notion of distance between points for a subset of a Banach space which is not a vector space. So a metric space will be a subset M of a Banach space $(X, \|.\|)$ endowed with the distance induced by the norm of X. That is

$$d(x,y) = \|x-y\|, \forall x, y \in M.$$

Definition 2.1 A metric space (M,d) is **complete** , if any Cauchy sequence $\{u_n\}_{n\geq 0}$ converges in M. Recall that a Cauchy sequence $\{u_n\}_{n\geq 0}$ in M is a sequence such that for each $\varepsilon > 0$, there exists an integer $r \in \mathbb{N}$, such that

$$d(u_m, u_p) \leq \varepsilon, \forall m, p \geq r.$$

Then (M,d) is complete if for any such a Cauchy sequence $\{u_n\}_{n\geq 0}$ in M, there exists $u \in M$ such that

$$d(u_m, u) \to 0, \text{ as } m \to \infty.$$

A metric space (M,d) is **not complete** otherwise. That is, if there exists a Cauchy sequence in (M,d) that is not convergent in M.

Example 2.2 (of metric space in $\mathbb{R}$) Consider the metric space $M_0 = (0,1] \subset \mathbb{R}$ endowed with the metric $d_0(x,y) = |x-y|$. Then (M_0, d_0) is closed (because the empty set $\emptyset$ is open). If we take a sequence

$$x_n = \frac{1}{2^n}.$$

Then

$$|x_n - x_{n+q}| = x_n - x_{n+q} \leq x_n = \frac{1}{2^n}, \forall q \in \mathbb{N}.$$

Therefore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. But this sequence converges to 0, which does not belong to M_0. Hence (M_0, d_0) is not a complete metric space.

Example 2.3 (of metric space in $\mathbb{R}^2$**)** We can consider

$$M_1 = [0, 1] \times [0, 1], \text{ with } d_1(x, y) = \|x - y\|_{\mathbb{R}^2}$$

which is a closed subset. Therefore, (M_1, d_1) will give a complete metric (i.e. any Cauchy sequence in (M_1, d_1) converges to an element of M_1).

An example of non complete metric space is given by

$$M_2 = M_1 \setminus \{(0, 0)\} = \{y \in M_1 : y \neq 0\},$$

which is not closed in $\mathbb{R}^2$. Then (M_2, d_1) is not a complete metric space. Indeed, let us consider a sequence

$$(x_n, y_n) \in (0, 1] \times (0, 1] \to (0, 0) \notin M_2, \text{ as } n \to \infty.$$

Then $\{(x_n, y_n)\}_{n\geq 0}$ is a Cauchy sequence in (M_2, d_1), but $\{(x_n, y_n)\}_{n\geq 0}$ does not converge in M_2.

Exercise 2.4 Prove the following properties.

(i) The map

$$d_2(x, y) = \left|\frac{1}{\|x\|_{\mathbb{R}^2}} - \frac{1}{\|y\|_{\mathbb{R}^2}}\right| + \|x - y\|_{\mathbb{R}^2}, \forall x, y \in M_2,$$

is a distance on M_2.

(ii) M_2 endowed with the distance d_2 is a complete metric space.
(**Hint:** The proof should become clear at the end of this chapter.)

Example 2.5 (of metric space in $L^1(0, 1)$**)** We can also consider some infinite dimensional example such as

$$M_3 = \left\{u \in L^1(0, 1) : 0 \leq u \leq 1\right\},$$

endowed with

$$d_3(u, v) = \|u - v\|_{L^1} = \int_0^1 |u(\sigma) - v(\sigma)| d\sigma.$$

Then since M_3 is closed in $L^1(0, 1)$, and the metric space (M_3, d_3) is complete. Next, If we consider

$$M_4 = M_3 \setminus \{0_{L^1}\}.$$

Then (M_4, d_3) is not a complete metric space. Indeed, let us consider a sequence

$$u_n \in L^1(0,1),$$

such that

$$\|u_n - 0_{L^1(0,1)}\|_{L^1} = \int_0^1 |u_n(\sigma)| d\sigma \to 0, \text{ as } n \to \infty.$$

Then

$$u_n \to 0_{L^1(0,1)} \notin M_4, \text{ as } n \to \infty,$$

where the limit is understood in $L^1(0,1)$. Then $\{u_n\}_{n\geq 0}$ is a Cauchy sequence in (M_4, d_3), but $\{u_n\}_{n\geq 0}$ is not convergent in M_4.

Exercise 2.6 Prove the following properties.

(i) The map

$$d_4(u,v) = \left|\frac{1}{\|u\|_{L^1}} - \frac{1}{\|v\|_{L^1}}\right| + \|u-v\|_{L^1}, \forall u, v \in M_4,$$

is a distance on M_4.

(ii) M_4 endowed with the distance d_4 is a complete metric space.

(**Hint:** The proof should become clear at the end of this chapter.)

2.4 Neighborhood and ε-Neighborhood of a Subset

Let (M, d) be a metric space.

Definition 2.7 Let $A \subset M$ be a subset. Then a subset $N \subset M$ (which is not necessarily open or closed in M) is called a **neighborhood of** A , if and only if for each $x \in A$, there exists $\varepsilon = \varepsilon(x) > 0$ (depends on x in general) such that

$$B_M(x, \varepsilon) := \{y \in M : d(x,y) < \varepsilon\} \subset N.$$

We will say that N is an **open neighborhood of** A , whenever N is open, and we will say that N is an **closed neighborhood of** A, whenever N is closed.

Recall that the distance between a point x and a subset $A \subset M$ is defined by

$$d(x, A) := \inf_{y \in A} d(x,y).$$

The quantity $d(x, A)$ is measuring the distance of x to A. Therefore if $x \in A$ then $d(x, A) = 0$. Moreover, if $d(x, A) = \varepsilon \geq 0$, this means that we can find a sequence $y_n \in A$ such that

$$\lim_{n \to \infty} d(x, y_n) = \varepsilon.$$

In particular $d(x, A) = 0$ does not imply that $x \in A$. But $d(x, A) = 0$ implies that $x \in \overline{A}$.

By Lemma 1.22, we know that the map $x \to d(x, A)$ is Lipschitz continuous. It follows that for each $\varepsilon > 0$, the subset

$$N(A, \varepsilon) := \{x \in M : d(x, A) < \varepsilon\},$$

is an open neighborhood of A (since it is the inverse image of $(-\infty, \varepsilon)$ by a continuous map), and the subset

$$\overline{N}(A, \varepsilon) := \{x \in M : d(x, A) \leq \varepsilon\},$$

is a closed neighborhood of A (since it is the inverse image of $(-\infty, \varepsilon]$ by a continuous map).

Definition 2.8 The subset $N(A, \varepsilon)$ and $\overline{N}(A, \varepsilon)$ are called **ε-neighborhood of** A. The subset $N(A, \varepsilon)$ is called an **open ε-neighborhood of** A, and the subset $\overline{N}(A, \varepsilon)$ is called a **closed ε-neighborhood of** A.

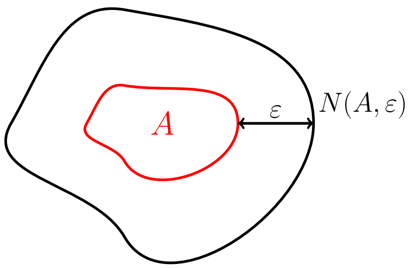

Fig. 2.1: *This figure illustrates the notion of ε-neighborhood of subset A.*

The neighborhoods $N(A, \varepsilon)$ are very specific. Indeed, for each $x \in A$, we have

$$B_M(x, \varepsilon) \subset N(A, \varepsilon),$$

where $\varepsilon > 0$ is independent of x. One can compare with Definition 2.7 where the value of ε varies with x.

2.5 Compact Subsets in (Non Complete) Metric Spaces: Definitions and Main Results

Let (M, d) be a metric space.

Definition 2.9 A subset of $\mathcal{O}$ of M is **open**, if for each $x \in \mathcal{O}$, there exists $\varepsilon > 0$ (depending on x) such that

$$B_M(x, \varepsilon) = \{y \in M : d(x, y) < \varepsilon\} \subset \mathcal{O}.$$

A subset $\mathcal{F}$ is **closed** if its complementary in M is open. That is

$$M \setminus \mathcal{F} = \{y \in M : y \notin \mathcal{F}\}$$

is an open subset.

Definition 2.10 Let C be a subset of M. We will say that

(i) A family of subsets $\{\mathcal{F}_\alpha\}_{\alpha \in E}$ is a **covering** of C if

$$C \subset \bigcup_{\alpha \in E} \mathcal{F}_\alpha.$$

(ii) A family of subsets $\{\mathcal{F}_\alpha\}_{\alpha \in E}$ is an **open covering** of C, if $\{\mathcal{F}_\alpha\}_{\alpha \in E}$ is a covering of C, and for each $\alpha \in E$, the subset $\mathcal{F}_\alpha$ is an open subset of (M, d).
(iii) A family of subset $\{\mathcal{F}_\alpha\}_{\alpha \in E}$ is an **finite covering** of C, if $\{\mathcal{F}_\alpha\}_{\alpha \in E}$ is a covering of C, and E is finite.
(iv) If $\{\mathcal{F}_\alpha\}_{\alpha \in E}$ is a covering of C, and $E' \subset E$. Then $\{\mathcal{F}_\alpha\}_{\alpha \in E'}$ is an **sub-covering** of C (extracted from $\{\mathcal{F}_\alpha\}_{\alpha \in E}$), if

$$C \subset \bigcup_{\alpha \in E'} \mathcal{F}_\alpha.$$

Definition 2.11 Let C be a subset of M. We will say that

(i) The subset C is **compact** if any open covering of C admits a finite sub-covering of C. That is for each family $\{\mathcal{F}_\alpha\}_{\alpha \in E}$ such that

$$C \subset \bigcup_{\alpha \in E} \mathcal{F}_\alpha,$$

there exists a finite subset $E' \subset E$ such that

$$C \subset \bigcup_{\alpha \in E'} \mathcal{F}_\alpha.$$

(ii) The subset C is **sequentially compact** if each sequence $\{u_n\}_{n \in \mathbb{N}}$ admits a convergent sub-sequence $\left\{u_{n_p}\right\}_{p \in \mathbb{N}}$ converging to an element $x \in C$.
(iii) The subset C is **relatively compact**, if its closure $\overline{C}$ is compact.
(iv) The subset C is **totally bounded (or precompact)**, if for each $r > 0$, C has a finite covering with balls of radius r.

Theorem 2.12 (Corollary 3.8 p.34 in [132]) *Assume that* (M, d) *is a metric space (complete or not). Then we have the following*

(i) *A subset C is compact if and only if it is sequentially compact.*

(ii) *A subset C is compact if and only if (C, d) is a complete metric space, C is totally bounded.*

Example 2.13 (of non compact subset which is totally bounded and closed) Consider the metric space $M = (0, 1] \subset \mathbb{R}$ endowed with the metric $d(x, y) = |x-y|$. If we take a sequence

$$C := \bigcup_{p\in\mathbb{N}} \{x_p\},$$

where

$$x_n = \frac{1}{2^n}, \forall n \in \mathbb{N}.$$

(i) **The subset C is not compact.**
Indeed, we have

$$x_{n+1} = \frac{x_n}{2}, \text{ and } x_{n-1} = 2x_n.$$

Define

$$r_n = \frac{x_n - x_{n+1}}{2} = \frac{1}{2^{n+2}},$$

then

$$B_M(x_n, r_n) \cap C = \{x_n\}.$$

It follows that $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ the family defined by

$$\mathcal{F}_n = B_M(x_n, r_n)$$

is an open covering for the sequence

$$C = \bigcup_{p\in\mathbb{N}} \{x_p\}.$$

But we can find no finite sub-covering of C extracted from $\mathcal{F}$ (since each element of $\mathcal{F}$ covers exactly one element of C). Therefore C is not compact.

(ii) **C is totally bounded.**
Indeed, if we consider any covering $\mathcal{G} = \{\mathcal{G}_n\}_{n\geq 0}$ the family defined by

$$\mathcal{G}_n = B(x_n, r), \forall n \in \mathbb{N},$$

with $r > 0$.

Then for each $n_0 = n_0(r) > 0$, such that $x_{n_0} < r$, we have

$$C \subset \bigcup_{n=1,\ldots,n_0} \{\mathcal{G}_n\}.$$

(iii) **C is a closed subset in (M, d).**
Indeed, the complementary subset $M \setminus C$ is open (M, d), because we can find an open ball around each point of $M \setminus C$ included in $M \setminus C$.

Therefore C is closed, totally bounded, and C is not compact.

The above example together with the following theorem explain why the notion of totally bounded sets is used only in complete metric spaces. This will be crucial (later in this chapter) when we will consider the measures of non compactness.

Theorem 2.14 (Corollary 3.9 p.35 in [132]) *Assume that* (M, d) *is a complete metric space. If* C *is totally bounded, then the subset* $\overline{C}$ *is compact.*

2.6 Cantor's Diagonal Process

In order to explain the diagonal process, we consider an example of sequence with double indexation

$$(n, m) \in \mathbb{N} \times \mathbb{N} \to u_n^m \in [0, 1].$$

This double indexation can be regarded as a sequence $m \to u^m$, where each element u^m is itself a sequence $n \to u_n^m$ with values in $[0, 1]$.

Claim 2.15 *There exists a increasing sequence* $p \to m_p \in \mathbb{N}$*, and there exists a sequence* $n \to \overline{u}_n \in [0, 1]$*, such that*

$$\lim_{p\to\infty} u_n^{m_p} = \overline{u}_n, \forall n \in \mathbb{N}.$$

Proof (of the claim) In the principle of this proof is schematically represented in Figure 2.2.

Step $n = 0$**:** Let us first consider the sequence $m \to u_0^m \in [0, 1]$. Since $[0, 1]$ is compact in $\mathbb{R}$, we can find a convergent sub-sequence. That is, we can find a strictly increasing sequence of integer $p \to m_p^0 \in \mathbb{N}$ (with $m_p^0 \to \infty$), and $\overline{u}_0 \in [0, 1]$, such that

$$\lim_{p\to\infty} u_0^{m_p^0} = \overline{u}_0.$$

Step $n = 1$**:** We consider the sub-sequence $p \to u_1^{m_p^0} \in [0, 1]$. We can extract a sub-sequence $p \to m_p^1 \in \mathbb{N}$ (with $m_p^1 \to \infty$), and $\overline{u}_1 \in [0, 1]$, such that

$$\lim_{p\to\infty} u_1^{m_p^1} = \overline{u}_1.$$

Here to extract a sub-sequence from $p \to m_p^0 \in \mathbb{N}$ means that, for each $p \in \mathbb{N}$,

$$m_p^1 \in \bigcup_{\widehat{p}\in\mathbb{N}} m_{\widehat{p}}^0,$$

and

$$\lim_{p\to\infty} m_p^1 = \infty.$$

This process is represented by the diagonal red arrows in Figure 2.2.

Due to this construction, we keep the convergence from the previous sub-sequence. That is

$$\lim_{p\to\infty} u_0^{m_p^1} = \overline{u}_0.$$

We can keep the diagonal elements from the previous column without changing the convergence properties,

$$m_0^1 = m_0^0.$$

This the process represented by the horizontal red arrows in Figure 2.2.

Step $n = k+1$**:** We consider the sub-sequence $p \to u_{k+1}^{m_p^k} \in [0,1]$. We can extract a sub-sequence $p \to m_p^{k+1} \in \mathbb{N}$ (i.e. $m_p^{k+1} \to \infty$) and $\overline{u}_{k+1} \in [0,1]$, such that

$$\lim_{p\to\infty} u_{k+1}^{m_p^{k+1}} = \overline{u}_{k+1}.$$

Here to extract a sub-sequence from $p \to m_p^{k+1} \in \mathbb{N}$ means that, for each $p \in \mathbb{N}$,

$$m_p^{k+1} \in \bigcup_{\widehat{p}\in\mathbb{N}} m_{\widehat{p}}^k,$$

and

$$\lim_{p\to\infty} m_p^{k+1} = \infty.$$

This process is represented by the diagonal red arrows in Figure 2.2.

Due to this construction, we keep the convergence from the previous sub-sequence. That is

$$\lim_{p\to\infty} u_0^{m_p^{k+1}} = \overline{u}_0, \lim_{p\to\infty} u_1^{m_p^{k+1}} = \overline{u}_1, \dots, \lim_{p\to\infty} u_k^{m_p^{k+1}} = \overline{u}_k.$$

We can keep the above diagonal elements from the previous column without changing the convergence properties,

$$m_p^{k+1} = m_p^k, \forall p = 0, 1, \dots, p-1.$$

This process is represented by the horizontal red arrows in Figure 2.2.

Final step: By induction arguments we can construct such a sequence for each integer $k \in \mathbb{N}$, and we obtain

$$\lim_{p\to\infty} u_k^{m_p^k} = \overline{u}_k, \forall k \in \mathbb{N}.$$

Now we choose the diagonal element in Figure 2.2 (in green). That is we consider the sequence

$$\widehat{m}_p = m_p^p, \forall p \in \mathbb{N}.$$

Then by construction the sequence $p \to \widehat{m}_p$ is a sub-sequence of any $p \to m_p^k$. That is for each $p \in \mathbb{N}$, and $k \in \mathbb{N}$,

$$\widehat{m}_p \in \bigcup_{\widehat{p}\in\mathbb{N}} m^k_{\widehat{p}},$$

and

$$\lim_{p\to\infty} \widehat{m}_p = \infty.$$

Therefore, we obtain

$$\lim_{p\to\infty} u_k^{\widehat{m}_p} = \overline{u}_k, \forall k \in \mathbb{N}.$$

The claim is proved. □

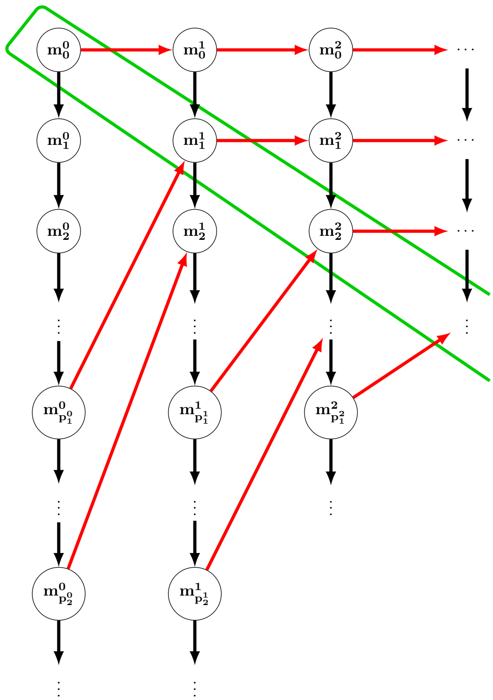

Fig. 2.2: *The figure gives a schematic representation of the diagonal process. The columns represent the index of each sub-sequence. The red arrows indicate where the values are taken in the previous column (i.e. on the left hand side). One must realize that all the indexes of a given column appear in the previous one (i.e., on the left-hand side). That is, each column is extracted from the previous one. The coefficients above the diagonal are unchanged. The diagonal process consists in choosing the diagonal elements.*

Example 2.16 (of sequence that is not converging uniformly with respect to $n \in \mathbb{N}$) The convergence in the Claim 2.15 is called **local convergence** in $n \in \mathbb{N}$.

This means that the convergence holds simultaneously only for a finite values of $n \in \mathbb{N}$.

To convince ourselves that this convergence can not be uniform, let us consider the following example

$$u_n^p = \frac{n+1}{n+1+p+1}$$

Then

$$\lim_{p\to\infty} u_n^p = 0, \forall n \geq 0.$$

But we have

$$\|u^p\|_\infty = \sup_{n\in\mathbb{N}} u_n^p = \lim_{n\to\infty} \frac{n+1}{n+1+p+1} = 1, \forall p \in \mathbb{N}.$$

Therefore

$$\lim_{p\to\infty} \|u^p\|_\infty = 1 \neq 0.$$

This example show that the convergence in the Claim 2.15 is not (in general) uniform with respect to $n \in \mathbb{N}$.

Corollary 2.17 *The space of sequences*

$$M = \{\{u_n\}_{n\in\mathbb{N}} : u_n \in [0,1], \forall n \in \mathbb{N}\},$$

endowed with the distance

$$d(u,v) = \sup_{n\in\mathbb{N}} \frac{|u_n - v_n|}{2^n},$$

is sequentially compact.

Proof Let $m \longrightarrow u^m$ be a sequence of elements of M. That is for each $m \in \mathbb{N}$, u^m is a sequence of the form

$$n \longrightarrow u_n^m \in [0,1].$$

By using the Claim 2.15, there exists a increasing sequence $p \longrightarrow m_p \in \mathbb{N}$, and the exists a sequence $n \longrightarrow \overline{u}_n \in [0,1]$, such that

$$\lim_{p\to\infty} u_n^{m_p} = \overline{u}_n, \forall n \in \mathbb{N}.$$

Let us prove that

$$\lim_{p\to\infty} \sup_{n\in\mathbb{N}} \frac{|u_n^{m_p} - \overline{u}_n|}{2^n} = 0.$$

Let $\varepsilon > 0$ be fixed. Since $u_n^m \in [0,1]$, and $\overline{u}_n \in [0,1]$, we deduce that

$$\frac{|u_n^{m_p} - \overline{u}_n|}{2^n} \leq \frac{2}{2^n}.$$

Therefore, for $n_0 \in \mathbb{N}$ large enough, we have

$$\frac{|u_n^{m_p} - \overline{u}_n|}{2^n} \leq \frac{2}{2^n} \leq \varepsilon, \forall n \geq n_0.$$

It follows that

$$\sup_{n\in\mathbb{N}} \frac{|u_n^{m_p} - \overline{u}_n|}{2^n} \leq \max\left(\max_{n=0,\ldots,n_0} \frac{|u_n^{m_p} - \overline{u}_n|}{2^n}, \varepsilon\right)$$

and since

$$\lim_{p\to\infty} |u_n^{m_p} - \overline{u}_n| = 0, \forall n = 0, \ldots, n_0,$$

we deduce that there exists an integer $p_0 \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \frac{|u_n^{m_p} - \overline{u}_n|}{2^n} \leq \varepsilon, \forall p \geq p_0.$$

The proof of the corollary is completed. □

2.7 Spaces of Continuous Functions: Compactness Properties and Diagonal Process

Theorem 2.18 (Arzelà–Ascoli theorem) *Let X be a compact subset of a metric space (M, d). Let $(F, \|.\|_F)$ be a Banach space. Let Φ be a part of $C(X, F)$ the space of continuous maps from X into F. Then Φ is relatively compact in $C(X, F)$ (endowed with the distance $d(u, v) = \sup_{x\in X} \|u(x) - v(x)\|_F$) if and only if*

(i) *Φ is equi-continuous. That is, for each $\varepsilon > 0$, there exists $\eta > 0$, such that*

$$d(x, y) \leq \eta \Rightarrow \|u(x) - u(y)\|_F \leq \varepsilon, \forall u \in \Phi.$$

(ii) *For each $x \in X$, the set $\{u(x) : u \in \Phi\}$ is relatively compact in F.*

To illustrates the above result, we will prove the following typical consequence of Arzelà–Ascoli theorem.

Lemma 2.19 *Let $I \subset \mathbb{R}$, $J_0 \subset \mathbb{R}$, and $J_1 \subset \mathbb{R}$ be three closed and bounded intervals. Then the set*

$$C := \{u \in C^1(I, J_0) : u(x) \in J_0, \text{ and } u'(x) \in J_1, \text{ for all } x \in I\}$$

is relatively compact in $C(I, J_0)$ endowed with the ***supremum topology*** *(i.e., $d(u, v) = \sup_{x\in I} |u(x) - v(x)|$). That is, for each sequence $\{u^m\}_{m\in\mathbb{N}} \subset C$, there exists a sub-sequence $p \to m_p \in \mathbb{N}$ (i.e., strictly increasing sequence of integers) and a continuous $\bar{u} \in C(I, J_0)$, such that*

$$\lim_{p\to\infty} \sup_{x\in I} |u^{m_p}(x) - \bar{u}(x)| = 0.$$

Proof (Proof of Lemma 2.19) To prove Lemma 2.19 we show that the Arzelà-Ascoli Theorem 2.18 can be applied to the set C. First we remark that the set $\{u(x) : u \in X\} = J_0$ is a bounded interval in $\mathbb{R}$, therefore condition (ii) in Theorem 2.18 is satisfied.

Next we show that the condition (i) holds. We let

$$D := \sup\{|y| : y \in J_1\},$$

so that

$$|u'(x)| \leq D \text{ for all } u \in C \text{ and } x \in I.$$

Then by the mean value theorem, we have

$$|u(x) - u(y)| \leq |x - y| \sup_{z \in I} |u'(z)| \leq D|x - y| \text{ for all } u \in C \text{ and } x, y \in I.$$

In particular for each $\varepsilon > 0$ we can find

$$\eta := \frac{\varepsilon}{D},$$

such that

$$|x - y| \leq \eta \Rightarrow |u(x) - u(y)| \leq D|x - y| \leq D\frac{\varepsilon}{D} = \varepsilon \text{ for all } u \in C.$$

Thus, by the Arzelà-Ascoli Theorem 2.18, C is relatively compact in $C(I, J_0)$. In other words for each sequence $\{u^m\}_{m \in \mathbb{N}} \subset C$, there exists a function $\bar{u} \in C(I, \mathbb{R})$ and a subsequence $p \longrightarrow m_p \in \mathbb{N}$ such that

$$\lim_{p \to +\infty} d(u^{m_p}, \bar{u}) = \lim_{p \to +\infty} \sup_{x \in I} |u^{m_p}(x) - \bar{u}(x)| = 0.$$

Lemma 2.19 is proved. □

Theorem 2.20 *Let $J_0 \subset \mathbb{R}$, and $J_1 \subset \mathbb{R}$ be two closed and bounded intervals. Consider the subset of $C(\mathbb{R}, \mathbb{R})$*

$$M := \{u \in C(\mathbb{R}, \mathbb{R}) : u(x) \in J_0, \textit{ for all } x \in \mathbb{R}\},$$

endowed with the distance associated to the ***local uniform topology****. That is*

$$d(u, v) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \sup_{x \in [-n, n]} |u(x) - v(x)|.$$

Then (M, d) is a complete metric space. Moreover, the space

$$C := \{u \in C^1(\mathbb{R}, \mathbb{R}) : u(x) \in J_0 \textit{ and } u'(x) \in J_1 \textit{ for all } x \in \mathbb{R}\},$$

is relatively compact in (M, d). That is, for each sequence $\{u^m\}_{m \in \mathbb{N}} \subset C$, there exists a sub-sequence $p \longrightarrow m_p \in \mathbb{N}$ (i.e., strictly increasing sequence of integers) and a continuous $\bar{u} \in C(\mathbb{R}, J_0)$, such that

$$\lim_{p\to+\infty}\sum_{n=1}^{+\infty}\frac{1}{2^n}\sup_{x\in[-n,n]}|u^{m_p}(x)-\bar{u}(x)|=0.$$

Proof Let $\{u^m\}_{m\geq 0}$ be a sequence in C. We use a diagonal process.

Step 1: Compactness on $[-1,1]$.
Consider the restriction of $\{u^m\}_{m\in\mathbb{N}}$. By Lemma 2.19, there exists a sub-sequence $\{u^{m_p^1}\}_{p\in\mathbb{N}}$ and a continuous function $\bar{u}_1:[-1,1]\to\mathbb{R}$ such that

$$\lim_{p\to+\infty}\sup_{x\in[-1,1]}|u^{m_p^1}(x)-\bar{u}_1(x)|=0.$$

Step 2: Compactness in $[-2,2]$.
By applying Lemma 2.19 to the sequence $\{u^{m_p^1}\}_{p\in\mathbb{N}}$ restricted to the interval $[-2,2]$, we get that there exists a subsequence $p\to m_p^2$ extracted from $p\to m_p^1$ and a continuous function $\bar{u}_2\in C([-2,2],\mathbb{R})$ such that

$$\lim_{p\to+\infty}\sup_{x\in[-2,2]}|u^{m_p^2}(x)-\bar{u}_2(x)|=0.$$

Since m_p^2 is extracted from m_p^1, the restriction of $u^{m_p^2}$ to $[-1,1]$ converges to $\bar{u}_1$. Therefore

$$\bar{u}_2(x)=\bar{u}_1(x)\text{ for all }x\in[-1,1].$$

Step 3: Construction of $\{u^{m_p^n}\}_{p\in\mathbb{N}}$.
By induction, assume that we have constructed sequences $\{m_p^k\}_{p\in\mathbb{N}}$ for $k=1,\ldots,n$, such that each $\{m_p^k\}_{p\in\mathbb{N}}$ is extracted from $\{m_p^{k-1}\}_{p\in\mathbb{N}}$, and there exists $\bar{u}_k\in C([-k,k],\mathbb{R})$ such that

$$\sup_{x\in[-k,k]}|u_p^{m_p^k}(x)-\bar{u}_k(x)|\xrightarrow[p\to+\infty]{}0\text{ for all }k=1,\ldots,n.$$

By applying the Arzelà-Ascoli theorem to the sequence $\{u^{m_p^n}\}_{p\in\mathbb{N}}$ restricted to $[-(n+1),(n+1)]$, we find a subsequence $\{m_p^{n+1}\}_{p\in\mathbb{N}}$ extracted from $\{m_p^n\}_{p\in\mathbb{N}}$ and a function $\bar{u}_{n+1}\in C([-(n+1),n+1],\mathbb{R})$ such that

$$\lim_{p\to+\infty}\sup_{x\in[-(n+1),n+1]}|u^{m_p^{n+1}}(x)-\bar{u}_{n+1}(x)|=0.$$

Since m_p^{n+1} is extracted from m_p^n, the restriction of $u^{m_p^n}$ to $[-n,n]$ converges to $\bar{u}_n$. Therefore

$$\bar{u}_{n+1}(x)=\bar{u}_n(x)\text{ for all }x\in[-n,n].$$

The sequence $\{u^{m_p^n}\}_{p\in\mathbb{N}}$ is thus constructed for all $n\in\mathbb{N}$.

Step 4: Diagonal process.
Now we choose $\widehat{m}_n:=m_n^n$ for all $n\in\mathbb{N}$ and $\bar{u}(x):=\bar{u}_n(x)$ for all $x\in[-n,n]$ and $n\geq 1$. Observe that, by construction, the sequence $\{\widehat{m}_p\}_{p\geq n}$ is extracted from

$\{m_p^n\}_{p\in\mathbb{N}}$ for all $n \in \mathbb{N}$. Therefore

$$\lim_{p\to+\infty} \sup_{x\in[-n,n]} |u^{\widehat{m}_p}(x) - \bar{u}(x)| = 0, \text{ for all } n \in \mathbb{N}.$$

Define $D := \sup_{x\in J_0} |x|$. Then, since $u^{\widehat{m}_p} \in C(\mathbb{R}, J_0)$ and $\bar{u} \in C(\mathbb{R}, J_0)$, we have

$$\sup_{x\in[-n,n]} |u^{\widehat{m}_p}(x) - \bar{u}(x)| \leq 2D.$$

Thus

$$\begin{aligned} d(u^{\widehat{m}_p}, \bar{u}) &= \sum_{n=1}^{+\infty} \frac{1}{2^n} \sup_{x\in[-n,n]} |u^{\widehat{m}_p}(x) - \bar{u}(x)| \\ &\leq \sum_{n=1}^{n_0} \frac{1}{2^n} \sup_{x\in[-n,n]} |u^{\widehat{m}_p}(x) - \bar{u}(x)| + \sum_{n=n_0+1}^{+\infty} \frac{1}{2^n} \times 2D \\ &= \sum_{n=1}^{n_0} \frac{1}{2^n} \sup_{x\in[-n,n]} |u^{\widehat{m}_p}(x) - \bar{u}(x)| + \frac{2D}{2^{n_0+1}} \frac{1}{1-\frac{1}{2}}. \end{aligned}$$

Let $\varepsilon > 0$. Then for $n_0 \geq 0$ sufficiently large we have

$$\frac{2D}{2^{n_0+1}} \frac{1}{1-\frac{1}{2}} = \frac{2D}{2^{n_0}} \leq \frac{\varepsilon}{2},$$

and we obtain for all $n \geq n_0$,

$$d(u^{\widehat{m}_p}, \bar{u}) \leq \sum_{n=1}^{n_0} \frac{1}{2^n} \sup_{x\in[-n,n]} |u^{\widehat{m}_p}(x) - \bar{u}(x)| + \frac{\varepsilon}{2}.$$

Now we can find $n_1 \geq n_0$ such that $p \geq n_1$ with n_1 sufficiently large we have

$$\sup_{x\in[-n_0,n_0]} |u^{\widehat{m}_p}(x) - \bar{u}(x)| \leq \frac{\varepsilon}{2n_0}.$$

Therefore for all $p \geq n_1$ we have

$$d(u^{\widehat{m}_p}, \bar{u}) \leq n_0 \frac{\varepsilon}{2n_0} + \frac{\varepsilon}{2} = \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, we have proved that $u^{\widehat{m}_p}$ converges to $\bar{u}$ as $p \to +\infty$ for the topology induced by d. Theorem 2.20 is proved. □

Remark 2.21 Instead of the local uniform topology, we could have used any other distance of the form

$$d^\star(u, v) = \sup_{x\in\mathbb{R}} \rho(x)|u(x) - v(x)|,$$

where $\rho : \mathbb{R} \to (0, \infty)$ is any continuous function, satisfying

$$\lim_{|x|\to\infty} \rho(x) = 0.$$

The conclusion of Theorem 2.20 holds, That is, for each sequence $\{u^m\}_{m\in\mathbb{N}} \subset C$, there exists a sub-sequence $p \longrightarrow m_p \in \mathbb{N}$ (i.e., strictly increasing sequence of integers) and a continuous $\overline{u} \in C(\mathbb{R}, J_0)$, such that

$$\lim_{p\to\infty} d^\star(u^{m_p}, \overline{u}) = 0.$$

2.8 Compactness Properties for Families of Subsets

In this section, we prepare some key ingredients for the global attractor theory. We first recall that in any metric space (M, d) (complete or not) a subset C of M is compact if and only if any sequence in C has a sub-sequence which converges in C. So for each sequence $\{x_n\}_{n\geq 0} \subset C$, we can find a sub-sequence $\left\{x_{n_p}\right\}_{p\geq 0}$ (that is we can find a strictly increasing sequence of integer $p \to n_p \in \mathbb{N}$) and there exists $x \in C$, such that

$$\lim_{p\to\infty} x_{n_p} = x.$$

Definition 2.22 Let $I \subset [0, +\infty)$ be unbounded, let $\{A_t\}_{t\in I}$ be a family of non-empty subsets of M. We say that $\{A_t\}_{t\in I}$ is **point-wise sequentially compact** if and only if for each sequence $\{t_n\}_{n\geq 0} \subset I \to +\infty$, and each sequence $\{x_n\}_{n\geq 0}$ satisfying

$$x_n \in A_{t_n}, \forall n \geq 0,$$

has a convergent sub-sequence.

Remark 2.23 Every family $\{A_t\}_{t\in I} \subset \mathbb{R}^n$ such that $A_t \subset B_{\mathbb{R}^n}(0, r), \forall t \in I$ (for some $r > 0$) is point-wise sequentially compact. So this provide an example of such a class of point-wise sequentially compact family. The same property is true in locally compact metric spaces.

Definition 2.24 Let $I \subset [0, +\infty)$ be unbounded, let $\{A_t\}_{t\in I}$ be a family of non-empty subsets of M. We will say that $\{A_t\}_{t\in I}$ is **decreasing** if and only if

$$t \geq s \Rightarrow A_t \subset A_s.$$

Recall that in any metric space (M, d) (complete or not) a subset C of M is compact if and only if every sequence has a convergent sub-sequence whose limit is in C.

Theorem 2.25 *Let $I \subset [0, +\infty)$ be unbounded, let $\{A_t\}_{t\in I}$ be a family of non-empty subsets of M. Assume that $\{A_t\}_{t\in I}$ is point-wise sequentially compact. Then*

$$A_\infty := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} A_s}$$

is compact and non-empty and

$$\delta(A_t, A_\infty) \to 0, \text{ as } t \to +\infty.$$

Moreover, if we assume in addition that A_t is a decreasing family of closed of subsets of M. Then

$$A_\infty = \bigcap_{t \ge 0} A_t.$$

Proof We set $B_t = \overline{\bigcup_{s \ge t} A_s}, \forall t \ge 0$. Let $\{t_n\}_{n \ge 0} \subset I \to \infty$, and each sequence $\{x_n\}_{n \ge 0} \subset M$, such that

$$x_n \in B_{t_n}, \forall n \ge 0.$$

We claim that $\{x_n\}_{n \ge 0}$ has a convergent sub-sequence. Indeed, there exists for each $n \ge 0$, $y_n \in \bigcup_{s \ge t_n} A_s$, such that

$$d(y_n, x_n) \le \frac{1}{n+1},$$

For each $n \ge 0$, there exists $s_n \ge t_n$ such that $y_n \in A_{s_n}$, and by assumption $\{y_n\}_{n \ge 0}$ has a convergent sub-sequence. Let $\left\{y_{n_p}\right\}_{p \ge 0} \to x \in M$, then

$$d(x, x_{n_p}) \le d(x, y_{n_p}) + d(y_{n_p}, x_{n_p}) \to 0, \text{ as } p \to +\infty.$$

Moreover, since $A_t \subset B_t$, we deduce that

$$\delta(A_t, A_\infty) \le \delta(B_t, A_\infty), \forall t \ge 0,$$

We can replace A_t by B_t, and we can always assume that A_t is a non-increasing family of closed and non-empty subsets of M. We set $A_\infty = \bigcap_{t \ge 0} A_t$. Then it is clear that A_∞ is closed since it is an intersection of closed subsets.

Let us prove that A_∞ is not empty. Indeed, by taking $t_n \to \infty$ and $x_n \in A_{t_n}$, and by taking a sub-sequence we can assume that $x_n \to x$. Let $t \in I$. We have for all $n \in \mathbb{N}$, with $t_n \ge t$, we have

$$x_n \in A_{t_n} \subset A_t$$

and since A_t is closed, we deduce that

$$x_n \to x \in A_t, \forall t \ge 0.$$

Hence

$$x \in A_\infty.$$

Moreover, if we take $\{y_n\}_{n \ge 0} \subset A_\infty$, then by induction we can find a sequence $\{t_n\}_{n \ge 0} \subset I$, and a sequence $\{x_n\}_{n \ge 0} \subset M$, such that $x_n \in A_{t_n}, \forall n \ge 0$, $t_n \to +\infty$, and $d(y_n, x_n) \to 0$, as $n \to +\infty$. By assumption we can find a sub-sequence $\left\{x_{n_p}\right\}_{p \ge 0} \to y$ as $n \to +\infty$. Now since A_∞ is closed, and by construction $y \in A_\infty$, and $y_{n_p} \to y$ as $p \to +\infty$, we deduce that $y \in A_\infty$. It follows that A_∞ is compact.

Assume by contradiction that there exist $\varepsilon > 0$, and $\{t_n\}_{n\geq 0} \to +\infty$, such that $\delta(A_{t_n}, A_\infty) \geq \varepsilon, \forall n \geq 0$. Then for each $n \geq 0$, we can find $x_n \in A_{t_n}$, such that $d\left(x_n, A_\infty\right) \geq \varepsilon/2$. By assumption we can find a sub-sequence $\left\{x_{n_p}\right\}_{p\geq 0} \to x$. But by construction $x \in A_\infty$,

$$0 = d(x, A_\infty) \geq \varepsilon/2,$$

and we obtain a contradiction. □

Definition 2.26 We say that a subset $C \subset M$ **attracts** a family $\{A_t\}_{t\in I}$, if

$$\lim_{t\to\infty} \delta(A_t, C) = 0.$$

That is equivalent to say that for each $\varepsilon > 0$, we can find $t_0 = t_0(\varepsilon) \geq 0$, such that

$$A_t \subset N(C, \varepsilon), \forall t \geq t_0.$$

We deduce the following result from Theorem 2.25.

Theorem 2.27 *The two following assertions are equivalent:*

(i) *The family $\{A_t\}_{t\in I}$ is point-wise sequentially compact.*
(ii) *The family $\{A_t\}_{t\in I}$ is attracted by some compact subset $C \subset M$.*

Moreover, if (i) *or* (ii) *is satisfied, then*

$$A_\infty = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} A_s}$$

is the smallest compact subset $C \subset M$, attracting the family $\{A_t\}_{t\in I}$.

Definition 2.28 Let $I \subset [0, +\infty)$ be unbounded, let $\{A_t\}_{t\in I}$ be a family of non-empty subsets of M. We say that $\{A_t\}_{t\in I}$ is **Hausdorff sequentially compact** if and only if for each sequence $\{t_n\}_{n\geq 0} \subset I \to +\infty$, there exist a sub-sequence $\left\{t_{n_p}\right\}_{p\geq 0}$ and a subset $\widehat{A}_\infty \subset M$ satisfying

$$d_H\left(A_{t_{n_p}}, \widehat{A}_\infty\right) \to 0, \text{ as } p \to +\infty.$$

The convergence of the family $\widehat{A}_t$ to A_∞ for the Hausdorff distance means that, for each $\varepsilon > 0$, we have both

$$A_t \subset N(\widehat{A}_\infty, \varepsilon) \tag{2.1}$$

and

$$\widehat{A}_\infty \subset N(A_t, \varepsilon) \tag{2.2}$$

whenever $t > 0$ is large enough.

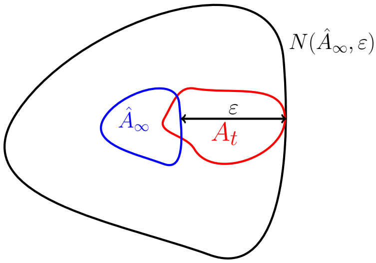

Fig. 2.3: *This figure illustrates the inclusion* (2.1). *When ε goes to* 0*, the set A_t approaches to $\widehat{A}_\infty$.*

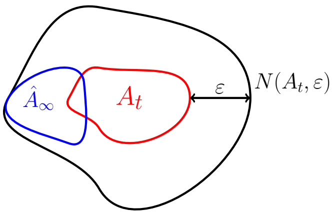

Fig. 2.4: *This figure illustrates the inclusion* (2.2). *When ε goes to* 0*, the set $\widehat{A}_\infty$ approaches to A_t.*

In this previous of convergence results, the property (2.2) was missing. This property can be interpreted by saying that every point of the limit set A_∞ is approached by some point of A_t. The following lemma says that the convergence is also true in the sense of the Hausdorff distance, but only for a sub-sequence A_{t_n} and a smaller subset $\widehat{A}_\infty \subset A_\infty$.

Theorem 2.29 *Let $I \subset [0, +\infty)$ be unbounded, let $\{A_t\}_{t\in I}$ be a family of non-empty closed subsets of M. If the family $\{A_t\}_{t\in I}$ is point-wise sequentially compact, then $\{A_t\}_{t\in I}$ is Hausdorff sequentially compact. More precisely, for each $\{t_n\}_{n\geq 0} \subset I \to +\infty$, there exists a sub-sequence $\left\{t_{n_p}\right\}_{p\geq 0}$ such that*

$$d_H\left(A_{t_{n_p}}, \widehat{A}_\infty\right) \to 0, \text{ as } p \to +\infty,$$

where

$$\widehat{A}_\infty = \bigcap_{p\geq 0} \overline{\bigcup_{q\geq p} A_{t_{n_q}}}.$$

Proof Set

$$A_\infty = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} A_s}.$$

then by Theorem 2.25 we know that A_∞ is compact, and

$$\delta(A_t, A_\infty) \to 0, \text{ as } t \to +\infty.$$

Since A_∞ is compact, for each integer $m \geq 0$, there exists $\left\{x_1^m, ..., x_{k_m}^m\right\} \subset A_\infty$, such that

$$A_\infty \subset \bigcup_{i=1}^{k_m} B\left(x_i^m, \frac{1}{2^{m+1}}\right).$$

Let $\{t_n\}_{n\geq 0} \subset I \to +\infty$. By using a diagonal process, we can find a sequence of integers $\{l_m\}_{m\geq 0}$, such that $\left\{y_1^m, ..., y_{l_m}^m\right\} \subset \left\{x_1^m, ..., x_{k_m}^m\right\}$ such that for integer p and m with $p \geq m$

$$A_{t_{n_p}} \subset \bigcup_{i=1}^{l_m} B\left(y_i^m, \frac{1}{2^m}\right), \tag{2.3}$$

and

$$A_{t_{n_p}} \cap B\left(y_i^m, \frac{1}{2^m}\right) \neq \emptyset, \forall i = 1, ..., l_m. \tag{2.4}$$

Set

$$\widehat{A}_\infty = \bigcap_{p\geq 0} \overline{\bigcup_{q\geq p} A_{t_{n_q}}},$$

by Theorem 2.25, we know that $\widehat{A}_\infty$ is a non-empty compact set, and

$$\delta(A_{t_{n_p}}, \widehat{A}_\infty) \to 0, \text{ as } p \to +\infty.$$

Moreover, by using (2.3) and (2.4), we deduce that for each integer $m \geq 0$,

$$\widehat{A}_\infty \subset \bigcup_{i=1}^{l_m} \overline{B}\left(y_i^m, \frac{1}{2^m}\right),$$

and

$$\widehat{A}_\infty \cap \overline{B}\left(y_i^m, \frac{1}{2^m}\right) \neq \emptyset, \forall i = 1, ..., l_m.$$

Let $x \in \widehat{A}_\infty$, for each $m \geq 0$, there exists $i_0 \in \{1, ..., l_m\}$, such that

$$x \in \overline{B}\left(y_{i_0}^m, \frac{1}{2^m}\right) \text{ and } A_{t_{n_p}} \cap B\left(y_{i_0}^m, \frac{1}{2^m}\right) \neq \emptyset, \forall p \geq m.$$

So

$$d\left(x, A_{t_{n_p}}\right) \leq \frac{1}{2^{m-1}}, \forall p \geq m.$$

We conclude that

$$\delta\left(\widehat{A}_\infty, A_{t_{n_p}}\right) \leq \frac{1}{2^{m-1}}, \forall p \geq m,$$

and the proof is completed. □

Example 2.30 (Discrete time example) Consider the following example $\{A_n\}_{n\in\mathbb{N}}$

$$A_{2n} = \{0\}, \forall n \in \mathbb{N},$$

and

$$A_{2n+1} = \{1\}, \forall n \in \mathbb{N}.$$

Then

$$A_\infty = \{0, 1\},$$

and

$$\delta(A_n, A_\infty) = 0, \forall n \geq 0,$$

implies

$$\lim_{n\to\infty} \delta(A_n, A_\infty) = 0.$$

Moreover we have

$$\delta(A_\infty, A_n) = \sup_{x\in\{0,1\}} d(x, A_n) = 1, \forall n \in \mathbb{N}.$$

Therefore, the Hausdorff distance satisfies

$$d_H\left(A_n, A_\infty\right) = 1, \forall n \geq 0.$$

Moreover if we consider constant sub-sequence $n \longrightarrow A_{2n}$ then

$$\widehat{A}_\infty = \{0\},$$

and if we consider constant sub-sequence $n \longrightarrow A_{2n+1}$ then

$$\widehat{A}_\infty = \{1\}.$$

Example 2.31 (Continuous time example) Consider the following example $\{A_t\}_{t\in\mathbb{R}_+}$

$$A_t = \{\cos(t)\}, \forall t \in \mathbb{R}_+.$$

Then

$$A_\infty = [-1, 1],$$

and

$$\delta(A_\infty, A_t) = \sup_{x\in[-1,1]} d(x, A_t) = \max\left(|1 - \cos(t)|, |1 + \cos(t)|\right) \geq 1, \forall t \in \mathbb{R}_+.$$

Therefore, the Hausdorff distance satisfies

$$d_H\left(A_t, A_\infty\right) \geq 1, \forall t \geq 0.$$

Moreover if we consider constant sub-sequence $n \longrightarrow A_{t+2n\pi}$ then

$$A_{t+2n\pi} = \{\cos(t)\}, \forall n \geq 0,$$

and we obtain for this sequence $\widehat{A}_\infty = \{\cos(t)\}$.

2.9 Measure of Non-Compactness

Definition 2.32 The **Kuratowski measure of non-compactness** , κ, is defined by

$$\kappa(B) = \inf\{r > 0 : \ B \text{ has a finite cover with ball of radius } r\},$$

for any bounded set B of M. We set $\kappa(B) = +\infty$, whenever B is unbounded.

Remark 2.33 From Example 2.13, we can see that the measure of non-compactness is not measuring the compactness in non complete metric spaces.

From this definition, we deduce that

$$\kappa(B) = \kappa(\overline{B}),$$

for any subset B of M.

The following lemma can be reformulated by saying that (M, d) is a complete metric space if and only if each totally bounded subset of M is relatively compact.

Theorem 2.34 *The following properties are equivalent*

(i) (M, d) *is complete metric space.*
(ii) *For each subset* C *of* M, $\kappa(C) = 0$ *implies that* $\overline{C}$ *is compact.*

Proof **Proof of (ii)⇒(i)**. Let be a Cauchy sequence $\{x_n\}_{n\geq 0}$ then for each $\varepsilon > 0$, there exists $n_0 \geq 0$, such that

$$d(x_{n_0}, x_{n_0+l}) \leq \varepsilon, \forall l \geq 0.$$

So

$$\kappa(\bigcup_{n\geq 0} \{x_n\}) = \kappa(\bigcup_{n\geq n_0} \{x_n\}) \leq \varepsilon.$$

hence

$$\kappa(\bigcup_{n\geq 0} \{x_n\}) = 0,$$

and by using (ii), we deduce that

$$\overline{\bigcup_{n\geq 0} \{x_n\}}$$

is compact. By compactness, we deduce that $\{x_n\}_{n\geq 0}$ has a sub-sequence converging in M. But, by using again the fact that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence, we conclude that the sequence $\{x_n\}_{n\geq 0}$ converges in M.

Proof of (i)⇒(ii). Let C be closed subset of M, and assume that $\kappa(C) = 0$. Let $\{x_n\}_{n\geq 0}$ be a sequence in C. For each integer $m \geq 0$, let be $\left\{z_1^m, ..., z_{k_m}^m\right\} \subset M$, such that

$$C \subset \bigcup_{i=1}^{k_m} B\left(z_i^m, \frac{1}{2^m}\right),$$

and by using a diagonal process, we can find a sub-sequence $\left\{x_{n_m}\right\}_{m\geq 0}$ and a sequence of integer $\{i_m\}_{m\geq 0}$ such that

$$x_{n_p} \in B\left(z_{i_m}^m, \frac{1}{2^m}\right), \forall p \geq m.$$

So

$$d\left(z_{i_m}^m, z_{i_{m+1}}^{m+1}\right) \leq d\left(z_{i_m}^m, x_{n_{m+1}}\right) + d\left(x_{n_{m+1}}, z_{i_{m+1}}^{m+1}\right) \leq \frac{1}{2^m} + \frac{1}{2^{m+1}} \leq \frac{1}{2^{m-1}}.$$

So we deduce that for all $m \geq 0$, and $l \geq 0$,

$$\begin{aligned} d\left(z_{i_m}^m, z_{i_{m+l}}^{m+l}\right) &\leq d\left(z_{i_m}^m, z_{i_{m+1}}^{m+1}\right) + \ldots + d\left(z_{i_{m+l-1}}^{m+l-1}, z_{i_{m+l}}^{m+l}\right) \\ &\leq \frac{1}{2^{m-1}} + \frac{1}{2^m} + ... + \frac{1}{2^{m+l-1}} \\ &\leq \frac{1}{2^{m-1}}\left[1 + \frac{1}{2} + ... + \frac{1}{2^l}\right] \leq \frac{1}{2^{m-2}}. \end{aligned}$$

So $\left\{z_{i_m}^m\right\}_{m\geq 0}$ is a Cauchy sequence. So by using (i), we deduce that $z_{i_m}^m \longrightarrow x \in C$, as $m \longrightarrow +\infty$, and since

$$d\left(x_{n_p}, z_{i_m}^m\right) \leq \frac{1}{2^m}, \forall p \geq m,$$

the result follows. □

Proposition 2.35 *The following statements are valid*

(i) *Let (M, d) be complete metric space, let $I \subset [0, +\infty)$ be unbounded, and let $\{A_t\}_{t\in I}$ be a decreasing family of non-empty closed subsets of M. Assume that*

$$\kappa(A_t) \longrightarrow 0, \text{ as } t \longrightarrow +\infty.$$

Then

$$A_\infty = \bigcap_{t\geq 0} A_t$$

is non-empty and compact, and

$$\delta(A_t, A_\infty) \longrightarrow 0, \text{ as } t \longrightarrow +\infty.$$

(ii) *For each $A \subset M$, and $B \subset M$, we have*

$$\kappa(B) \leq \kappa(A) + \delta(B, A).$$

Proof **Proof of (i)**. Assume first that $\kappa(A_t) \to 0$, as $t \to +\infty$. Let two sequences $\{t_n\}_{n\geq 0} \subset I \to \infty$ and $\{x_n\}_{n\geq 0} \subset M$, such that $x_n \in A_{t_n}, \forall n \geq 0$. Then

$$\kappa\left(\bigcup_{n\geq 0}\{x_n\}\right) = \kappa\left(\bigcup_{n\geq p}\{x_n\}\right) \leq \kappa(A_{t_p}), \forall p \geq 0,$$

hence

$$\kappa\left(\bigcup_{n\geq 0}\{x_n\}\right) = 0.$$

By Theorem 2.34, we know that $\overline{\bigcup_{n\geq 0}\{x_n\}}$ is compact. We deduce that the family $\{A_t\}_{t\in I}$ is point-wise sequentially compact, and the result follows from Theorem 2.25.

Proof of (ii). Let $\varepsilon > 0$. Then there exists $\{x_1, ..., x_N\} \subset M$, such that

$$A \subset \bigcup_{n=1,\dots,N} B\left(x_n, \kappa(A) + \varepsilon/2\right) \tag{2.5}$$

Let $x \in B$. Then $d(x, A) \leq \delta(B, A)$, so there exists $y \in A$, such that

$$d(x, y) \leq \delta(B, A) + \varepsilon/2.$$

But by (2.5), there exists $n \in \{1, ..., N\}$, such that

$$d(x_n, y) \leq \kappa(A) + \varepsilon/2,$$

so

$$d(x, x_n) \leq d(x, y) + d(y, x_n) \leq \kappa(A) + \delta(B, A) + \varepsilon.$$

So

$$B \subset \bigcup_{n=1,\dots,N} B\left(x_n, \kappa(A) + \delta(B, A) + \varepsilon\right).$$

and

$$\kappa(B) \leq \kappa(A) + \delta(B, A) + \varepsilon, \forall \varepsilon > 0,$$

and the result follows when $\varepsilon \to 0$. □

2.10 Omega-Limit Sets of a Subset

Definition 2.36 For each subset $B \subset M$, we denote by

$$\gamma^+(B) = \bigcup_{t\geq 0} U(t)(B)$$

the **positive orbit of** B **for** U, and by

$$\omega(B) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} U(s)(B)}$$

the **omega-limit set of** B for U.

Definition 2.37 We say that a subset $A \subset M$ **attracts** a subset $B \subset M$ for U, if

$$\lim_{t\to\infty} \delta(U(t)B, A) = 0.$$

Lemma 2.38 *Let U be a continuous semiflow. If $B \subset M$ is positively invariant for U so is $\overline{B}$.*

Proof Assume that $B \subset M$ is positively invariant for U, that is to say that

$$U(t)B \subset B, \forall t \geq 0.$$

Let be a convergent sequence $x_n \in B \to x \in \overline{B}$. Then since U is a continuous semiflow

$$U(t)x_n \in B \to U(t)x \in \overline{B},$$

and we deduce that $\overline{B}$ is positively invariant for U. □

The theory of attractors is based on the following fundamental result, which is related to Hale [93, Lemmas 2.1.1 and 2.1.2].

Proposition 2.39 (Omega-limit sets) *Let $\{U(t)\}_{t\in I}$ be a continuous semiflow on (M, d). Let B be a subset of M, which is attracted by compact subset $C \subset M$ for U. Then*

(i) *$\omega(B)$ is non-empty, compact, and attracts B for U.*
(ii) *$\omega(B)$ invariant for U. That is*

$$U(t)\omega(B) = \omega(B), \forall t \geq 0.$$

(iii) *$\omega(B)$ attracts B for U. That is*

$$\lim_{t\to+\infty} d\left(U(t)B, \omega(B)\right) = 0.$$

Proof **Proof of (i)**. Set

$$A_t = \overline{\bigcup_{s\geq t} U(s)B}, \forall t \geq 0.$$

It is clear that the family $\{A_t\}_{t\in I}$ is decreasing and is attracted by C. By Theorem 2.27, we deduce that $\omega(B) = \bigcap_{t\geq 0} A_t$ is non-empty, compact, and attracts $\{A_t\}_{t\in I}$. Therefore $\omega(B)$ attracts B for U.

Proof of (ii). Let $\tau \geq 0$. We have

$$U(\tau)\left(\bigcup_{s\geq t}U(s)B\right)=\bigcup_{s\geq t+\tau}U(s)B,\forall t\geq 0. \tag{2.6}$$

Therefore, we obtain

$$\bigcup_{s\geq t+\tau}U(s)B\subset\overline{U(\tau)A_t},\forall t\geq 0,$$

hence

$$A_{t+\tau}\subset\overline{U(\tau)A_t},\forall t\geq 0. \tag{2.7}$$

By using again (2.6), we have

$$U(\tau)\left(\bigcup_{s\geq t}U(s)B\right)\subset A_{t+\tau},\forall t\geq 0,$$

since $U(\tau)$ is continuous,

$$U(\tau)A_t\subset A_{t+\tau},\forall t\geq 0. \tag{2.8}$$

We deduce from (2.8) that

$$U(\tau)\omega(B)\subset U(\tau)A_t\subset A_{t+\tau},\forall t\geq 0,$$

hence

$$U(\tau)\omega(B)\subset\omega(B),\forall t\geq 0. \tag{2.9}$$

To prove the converse inclusion, let $y\in\omega(B)$. Since the family A_t is decreasing, we can find a sequence $t_n\longrightarrow\infty$ and $y_n\in A_{t_n+\tau}$ such that $y_n\longrightarrow y$ as $n\longrightarrow\infty$. By using (2.7) we deduce that we can find a sequence $x_n\in A_{t_n}$ such that

$$d(y_n,U(\tau)x_n)\leq\frac{1}{2^n}.$$

But we can also find a convergent sub-sequence

$$x_{n_p}\longrightarrow x\in\omega(B).$$

Now by using the continuity of $U(\tau)$, it follows that $U(\tau)x=y$. It follows that

$$\omega(B)\subset U(\tau)\omega(B), \tag{2.10}$$

and the results follows from (2.9) and (2.10).

Proof of (iii). Assume by contradiction that there exist $\varepsilon>0$ and a sequence $t_n\longrightarrow+\infty$ such that

$$\sup_{x\in B}d\left(U(t_n)x,\omega(B)\right)=\delta\left(U(t_n)B,\omega(B)\right)\geq\varepsilon,\forall n\geq 0.$$

Then we can find a sequence $x_n\in B$ such that

$$d\left(U(t_n)x_n, \omega(B)\right) \geq \varepsilon/2, \forall n \geq 0.$$

By compactness of C, we can find a convergent sub-sequence (denoted with the same index) such that

$$\lim_{n \to +\infty} U(t_n)x_n = z \in \omega(B).$$

By Lemma 1.22, we deduce that

$$0 < \varepsilon/2 \leq \lim_{n \to +\infty} d\left(U(t_n)x_n, \omega(B)\right) = d\left(z, \omega(B)\right) = 0,$$

which gives a contradiction. □

2.11 Global Attractors

Let (M, d) be a complete metric space. Let $U : I \times M \to M$ be a continuous semiflow on M with $I = \omega\mathbb{N}$ or $\mathbb{R}$.

Definition 2.40 The semiflow U is said to be **point** (**compact**, **bounded**) **dissipative** if there is a bounded set B_0 in M such that B_0 attracts each point (compact set, bounded set) in M;

Definition 2.41 The semiflow U is said to be **asymptotically smooth** if for any nonempty closed bounded set $B \subset M$ which is positively invariant for U (i.e. $U(t)B \subset B, \forall t \geq 0$), there is a compact set $C \subset M$ such that C attracts B for U (i.e. $\delta(U(t)B, C) \to 0$, as $t \to \infty$).

Lemma 2.42 *The semiflow U is asymptotically smooth if and only if*

$$\lim_{t \to \infty} \kappa\left(U(t)B\right) = 0,$$

for any nonempty closed bounded subset $B \subset M$ which is positively invariant for U.

Proof ($\Leftarrow$): Let nonempty bounded subset $B \subset M$ positively invariant for U. Assume that

$$\lim_{t \to \infty} \kappa\left(U(t)B\right) = 0$$

Then

$$\lim_{t \to \infty} \kappa\left(\overline{U(t)B}\right) = 0$$

by Proposition 2.35-(i) (applied to the family $A_t = \overline{U(t)B}$), we deduce that $A_\infty = \bigcap_{t \geq 0} \overline{U(t)B}$ is non-empty and compact, and

$$\delta(U(t)B, A_\infty) \leq \delta(A_t, A_\infty) \to 0, \text{ as } t \to +\infty.$$

($\Rightarrow$): Conversely assume that there exists a compact subset $C \subset M$ such that

$$\delta(U(t)B, C) \to 0, \text{ as } t \to \infty.$$

Then by Proposition 2.35-(ii), we have

$$\kappa(U(t)B) \leq \kappa(C) + \delta(U(t)B, C) = \delta(U(t)B, C) \to 0, \text{ as } t \to \infty,$$

the proof is completed. □

Definition 2.43 A positively invariant subset $A \subset M$ for U is said to be **stable** if for any neighborhood V of A, there exists a neighborhood $W \subset V$ of A such that

$$U(t)W \subset V, \forall t \geq 0.$$

We say that A is **globally asymptotically stable** for U if, in addition, A attracts points of M for U. That is

$$\lim_{t\to\infty} d\left(U(t)x, A\right) = 0, \forall x \in M.$$

Lemma 2.44 *A subset A is stable for U if and only if for each neighborhood V of A, there exists $W \subset V$ neighborhood of A satisfying*

$$U(t)W \subset W, \forall t \geq 0.$$

Proof $(\Rightarrow)$: Assume that $A \subset M$ is stable for U. Let V be neighborhood A. Then there exists a neighborhood $W \subset V$ of A such that

$$U(t)W \subset V, \forall t \geq 0.$$

Set

$$\widehat{W} = \bigcup_{s\geq 0} U(s)W.$$

Then $\widehat{W}$ is a neighborhood of A (since it contains W) and

$$U(t)\widehat{W} = \bigcup_{s\geq 0} U(s+t)W \subset \bigcup_{s\geq 0} U(s)W = \widehat{W}.$$

$(\Leftarrow)$: This implication is trivial. □

Lemma 2.45 *Assume that A is a compact subset of M and V is a neighborhood of A. Then there exists $\varepsilon > 0$ such that*

$$N\left(A, \varepsilon\right) \subset V.$$

Proof Assume by contradiction that

$$N\left(A, \frac{1}{n+1}\right) \not\subset V, \forall n \in \mathbb{N}.$$

This implies that there exists $x_n \in A$ such that

$$B_M\left(x_n, \frac{1}{n+1}\right) \not\subset V, \forall n \in \mathbb{N}.$$

But since A is compact, we can find a converging subsequence $x_{n_p} \to \overline{x} \in A$. Moreover for each $p \in \mathbb{N}$ we can find $y_{n_p} \in B_M\left(x_{n_p}, \frac{1}{n_p+1}\right)$, and

$$y_{n_p} \notin V.$$

We deduce that

$$d\left(\overline{x}, y_{n_p}\right) \le d\left(\overline{x}, x_{n_p}\right) + d\left(x_{n_p}, y_{n_p}\right),$$

so we deduce that

$$\lim_{p\to\infty} d\left(\overline{x}, y_{n_p}\right) = 0.$$

Since $\overline{x} \in A$ and $y_{n_p} \notin V$, we obtain a contradiction with the fact that V is neighborhood of A. □

The following result can be found in the book of Hale [93, Theorem 2.2.5], we have the following result.

Proposition 2.46 (Stability) *Let $A \subset M$ be a compact subset which is positively invariant for U. If A attracts the compact subsets of one of its neighborhoods, then A is stable.*

Proof Let W be a neighborhood of A and assume that A attracts every compact subset of W for U. Assume by contradiction that A is not stable. So assume that there exists a neighborhood V of A, such that for each neighborhood V' of A, with $V' \subset V$, there exists $t = t(V') > 0$, such that

$$U(t)V' \not\subset V.$$

Since A is compact and $V \cap W$ is a neighborhood of A, we can find $\varepsilon > 0$, such that

$$N(A, \varepsilon) \subset V \cap W.$$

Moreover by construction for each integer $m \ge 0$, we can find $t_m \ge 0$, such that

$$U(t_m)\left(N\left(A, \frac{\varepsilon}{m+1}\right)\right) \not\subset V.$$

So for each $m \ge 0$, we can find $x_m \in N\left(A, \frac{\varepsilon}{m+1}\right)$, such that

$$U(t_m)x_m \notin V.$$

Since A is compact, by taking a sub-sequence, we can always assume that

$$x_m \to x \in A, \text{ as } m \to +\infty.$$

Moreover since

$$C = \{x_m : m \ge 0\} \cup \{x\},$$

is compact and is a subset of W, there exists $t_0 > 0$, such that

$$U(t)C \subset N(A,\varepsilon) \subset V, \forall t \geq t_0.$$

So

$$t_m \in I \cap [0, t_0], \forall m \geq 0,$$

and we have by construction

$$d(U(t_m)x_m, A) \geq \varepsilon, \forall m \geq 0. \tag{2.11}$$

Since A is positively invariant for U, we deduce that for all $m \geq 0$,

$$d(U(t_m)x_m, A) \leq d(U(t_m)x_m, U(t_m)x) \leq \sup_{t \in I \cap [0,t_0]} d(U(t)x_m, U(t)x).$$

Since U is continuous, we deduce that U is uniformly on the compact subset $[0, t_0] \times C$. Therefore,

$$x_m \to x \in A,$$

for all $m \geq 0$ large enough, we have

$$d(U(t_m)x_m, A) \leq \varepsilon/2,$$

a contradiction with (2.11). □

As a direct consequence of Proposition 2.46, we deduce the following result.

Proposition 2.47 *Let $A \subset M$ be a compact subset which is positively invariant for U. The following properties are equivalent*

(i) *A attracts the compact subsets of M;*
(ii) *A is stable and attracts the point of M;*

Definition 2.48 A nonempty, compact and invariant subset $A \subset M$ is said to be

(i) an **attractor for** U if A attracts one of its neighborhoods;
(ii) a **global attractors for** U if A is an attractor that attracts every point in M;
(iii) a **strong global attractor for** U if A attracts every bounded subset of M.

We remark that the notion of attractor and global attractor was used in [104, 160, 189, 238]. The strong global attractor was defined as global attractor in [93, 207]. As we will see in the following the notion of strong global attractor is not applicable (in general) in the context of the uniform persistence.

A discrete time version of the following theorem was proof by Magal and Zhao [160]. The proof of the following Theorem is inspired by [93, Theorem 2.4.2, Lemmas 2.4.4 and 2.4.5]. This result is also similar to the result about existence of global attractors in [189, Theorem 23.12] for continuous-time semiflows.

Theorem 2.49 (Global attractor in M) *Let U be a continuous semiflow on a complete metric space (M, d). Assume that*

(i) *U is point dissipative;*

(ii) *U is asymptotically smooth;*
(ii) *The positive orbits for U of compact subsets of M are bounded.*

Then U has a global attractor $A \subset M$. Moreover, A attracts any subset $B \subset M$ with an eventually bounded positive orbit, that is to say that A attract any subset $B \subset M$ satisfying

$$\gamma^+(U(t_0)(B)) = \bigcup_{t \geq t_0} U(t)B,$$

is bounded for some $t_0 \geq 0$ (large enough).

Remark 2.50 As a direct consequence of Proposition 2.46, the subset A is a stable compact invariant for U in (M, d).

Proof Assume that (i) is satisfied. Since U is point dissipative, we can find a closed and bounded subset B_0 in M such that for each $x \in M$, there exists $t_0 = t_0(x) \in I$,

$$U(t)x \in B_0, \forall t \geq t_0.$$

Define

$$J(B_0) = \{y \in B_0 : U(t)y \in B_0, \forall t \geq 0\}.$$

Then,

$$U(t)J(B_0) \subset J(B_0), \forall t \geq 0,$$

and for every $x \in M$, there exists $t_0 = t_0(x) \geq 0$,

$$U(t_0)x \in J(B_0).$$

Since $J(B_0)$ is non-empty closed and bounded, and U is asymptotically smooth, Proposition 2.39 implies that $\omega(J(B_0))$ is compact invariant, and attracts the points of M.

Assume, in addition, that (ii) is satisfied. We claim that there exists an $\varepsilon > 0$ such that

$$\gamma^+(N(\omega(J(B_0)), \varepsilon))$$

is bounded. Assume, by contradiction, that

$$\gamma^+\left(N\left(\omega(J(B_0)), \frac{1}{n+1}\right)\right)$$

is unbounded for each $n > 0$.

Let $z \in M$ be fixed. Then we can find a sequence $x_n \in N\left(\omega(J(B_0)), \frac{1}{n+1}\right)$, and a sequence $t_n \geq 0$ such that

$$d(z, U(t_n)x_n) \geq n.$$

Since $\omega(J(B_0))$ is compact, we can find a convergent sub-sequence $x_{n_p} \to x \in \omega(J(B))$ (up to a sub-sequence), as $p \to +\infty$. Since

$$H := \left\{x_{n_p} : p \geq 0\right\} \cup \{x\}$$

is compact, the assumption (ii) implies that $\gamma^+(H)$ is bounded, and we obtain a contradiction.

Therefore $D = \overline{\gamma^+(N(\omega(J(B_0)), \varepsilon))}$ is bounded for $\varepsilon > 0$ small enough. Then D is closed, bounded, and positively invariant for U. Since $\omega(J(B_0))$ attracts points of M for U, and

$$\omega(J(B_0)) \subset N(\omega(J(B_0)), \varepsilon) \subset \mathring{D}.$$

So $\omega(J(B_0))$ attracts the point of M, we deduce that $N(\omega(J(B_0)), \varepsilon)$ is an absorbing set of M, it follows that for each $x \in M$, there exists $t_0 = t_0(x) \geq 0$ such that

$$U(t)x \in N(\omega(J(B_0)), \varepsilon) \subset \mathring{D}, \forall t \geq t_0.$$

Claim 2.51 *Since U is a continuous semiflow, it follows that for each compact subset C of M, there exists an integer $t_0 \geq 0$ such that*

$$U(t_0)C \subset D.$$

Proof (Proof of the claim:) Assume by contradiction that

$$U(t)C \not\subset D, \forall t \geq 0.$$

Then there exists a $t_n \longrightarrow \infty$ such that

$$U(t_n)x_n \notin D, \forall n \geq 0.$$

Moverover, the subset D is positively invariant by U, so we deduce that

$$U(t)x_n \notin D, \forall t \in [0, t_n], \forall n \in \mathbb{N}. \tag{2.12}$$

But since C is compact, we can assume that $x_n \longrightarrow x \in C$. By construction there exists $t_0 > 0$ such that

$$U(t)x \in \mathring{D}, \forall t \geq t_0.$$

But the inverse image $U(t_0)^{-1}\mathring{D}$ is open (since U is continuous) and contains x. Now since $x_n \longrightarrow x$, we deduce that there exists an integer $n_0 \geq 0$ such that

$$x_n \in U(t_0)^{-1}\mathring{D}, \forall n \geq n_0.$$

Therefore

$$U(t_0)x_b \in \mathring{D}, \forall n \geq n_0,$$

and we obtain a contradiction with (2.12). The proof of the claim is completed. □

By using the Claim 2.51, we deduce that

$$A = \omega(D)$$

attracts every compact subset of M.

Fix a bounded neighborhood V of A. Since U is continuous, we can apply Proposition 2.46, it follows that A is stable, and hence, there is a neighborhood W of A such that $U(t)(W) \subset V$, $\forall t \geq 0$. Clearly, the set

$$W' := \overline{\cup_{t \geq 0} U(t) W}$$

is a bounded neighborhood of A, and

$$U(t)W' \subset W', \forall t \geq 0.$$

Since U is asymptotically smooth, there is a compact set $J \subset W'$ such that J attracts W'. By Proposition 2.39, $\omega(W')$ is non-empty, compact, invariant for U, and attracts W'. Since A attracts $\omega(W')$, we deduce that

$$\omega(W') \subset A.$$

We conclude that A is a global attractor for U.

To prove the last part of the theorem, without loss of generality we assume that B is a bounded subset of M and $\gamma^+(B)$ is bounded. We set

$$K = \overline{\gamma^+(B)}.$$

Then

$$U(t)K \subset K, \forall t \geq 0.$$

Since K is closed and bounded and U is asymptotically smooth, so we deduce that there exists a compact C which attracts K for U. Note that

$$U(t)B \subset U(t)\gamma^+(B) \subset U(t)K, \ \forall t \geq 0.$$

Thus, C attracts B for U. By Proposition 2.39, we deduce that $\omega(B)$ is non-empty, compact, invariant for U and attracts B. Since A is attracts the compact subset of M for U, we deduce that

$$\omega(B) \subset A,$$

and A attracts B for U. □

2.12 Uniform Persistence and Global Attractors

Let (M, d) be a complete metric space, and $\rho : M \to [0, +\infty)$ a continuous function. We decompose M into the open subset

$$M_0 := \{x \in M : \rho(x) > 0\},$$

and the closed subset

$$\partial M_0 := \{x \in M : \rho(x) = 0\}.$$

Then by using the definition of M_0 and ∂M_0, we deduce that

$$M = M_0 \cup \partial M_0,$$

and

$$M_0 \cap \partial M_0 = \emptyset.$$

Throughout this section, we assume that $\{U(t)\}_{t \in I}$ is a continuous semiflow, and

$$U(t)M_0 \subset M_0, \forall t \geq 0.$$

Definition 2.52 The region M_0 is called the **interior region**, and ∂M_0 is called the **boundary region**.

Remark 2.53 The boundary region ∂M_0 does not correspond (in general) to the topological notion of boundary of M_0 in the sense of the (M, d) (i.e. we may have $\partial M_0 \not\subset \overline{M}_0$ in general).

Remark 2.54 In most of the examples, the boundary region ∂M_0 is also positively invariant by U. But this assumption is not needed to prove the existence of a global attractors in M_0.

Example 2.55 (of function $\rho(x)$) The function $\rho(x)$ measure the distance of x to the boundary region ∂M_0. A typical example of function ρ is the following

$$\rho(x) = d(x, \partial M_0) = \inf_{y \in \partial M_0} d(x, y).$$

The idea in the following definition is to consider ∂M_0 as part infinite.

Definition 2.56 A subset $B \subset M_0$ is said to be **ρ-strongly bounded** if B is bounded in (M, d), and

$$\inf_{x \in B} \rho(x) > 0.$$

Example 2.57 (of function $\rho(x)$) If

$$M = L^1_+(0, 1),$$

and

$$\partial M_0 = \{0_{L^1}\}, \text{ and } M_0 = L^1_+(0, 1) \setminus = \{0_{L^1}\}.$$

There are many ways to measure the distance of a non-negative function u to the null function 0_{L^1}.

We can take

$$\rho(u) = \int_0^1 \chi(\sigma) u(\sigma) d\sigma$$

where $\chi \in C_+([0, 1], \mathbb{R})$ and

$$\chi(x) > 0, \forall x \in (0, 1).$$

So in particular we allow $\chi(0)$ and $\chi(1)$ to be $0_{\mathbb{R}}$.

Then the distance

$$d(u, \{0_{L^1}\}) = \|u\|_{L^1} = \int_0^1 u(\sigma)d\sigma,$$

which corresponds only to the case $\chi \equiv 1$. We will discuss this more in the section devoted to the examples at the end of this chapter.

Remark 2.58 If we change the function ρ, we will change the ρ-bounded sets.

Definition 2.59 (i) The semiflow $\{U(t)\}_{t\in I}$ is said to be ρ-**uniformly persistent** if there exists $\varepsilon > 0$ such that

$$\liminf_{t\to+\infty} \rho\,(U(t)x) \geq \varepsilon,\ \forall x \in M_0.$$

(ii) The semiflow $\{U(t)\}_{t\in I}$ is said to be **weakly** ρ-**uniformly persistent** if there exists $\varepsilon > 0$ such that

$$\lim_{t\to+\infty} \sup \rho\,(U(t)x) \geq \varepsilon,\ \forall x \in M_0.$$

(iii) The set ∂M_0 is said to be ρ-**ejective for** U if there exists $\varepsilon > 0$ such that for every $x \in M$ with $0 < \rho\,(x) < \varepsilon$, there is $t_0 = t_0(x) \geq 0$ such that

$$\rho\,(U(t_0)x) \geq \varepsilon.$$

For a given open subset $M_0 \subset M$, let $\partial M_0 := M \backslash M_0$. Then if $\partial M_0 \neq \emptyset$, we can use the continuous function $\rho : M \to [0, \infty)$ defined by $\rho(x) = d\,(x, \partial M_0)\,,\ \forall x \in M$, to obtain the traditional definition of persistence.

Proposition 2.60 *Assume that there is a compact subset C of M which attracts every point in M for U. Then the following statements are equivalent:*

(i) *U is weakly ρ-uniformly persistent;*
(ii) *U is ρ-uniformly persistent;*
(iii) *∂M_0 is ρ-ejective for U.*

Proof The observations (i)⇔(iii) and (ii)⇒(i) are obvious. Let us prove that (i)⇒(ii). Let $\varepsilon > 0$ be fixed such that

$$\lim_{t\to+\infty} \sup \rho\,(U(t)x) \geq \varepsilon, \forall x \in M_0. \tag{2.13}$$

Then for each $x \in M_0$, and each $t \geq 0$, there exists $r \geq 0$ such that $\rho\,(U(t+r)x) \geq \varepsilon/2$. Assume that U is not ρ-uniformly persistent. Then we can find a sequence $\{x_m\}_{m\geq 0} \subset M_0$ such that

$$\lim_{t\to+\infty} \inf \rho\,(U(t)x_m) \leq \frac{1}{m+1}, \forall m \geq 0. \tag{2.14}$$

By using the fact that C every point of M and (2.13), we deduce that there exists $t_m^\star \geq 0$ such that

$$d(U(t)x_m, C) \leq \frac{1}{m+1}, \forall t \geq t_m^\star, \text{ and } \rho\left(U(t_m^\star)x_m\right) \geq \varepsilon/2.$$

By using (2.14) we deduce that there exists $l_m > 0$ such that

$$\rho\left(U(t_m^\star + l_m)x_m\right) \leq \frac{2}{m+1}.$$

Assuming that $\varepsilon/2 > \frac{2}{m+1}$ (which is true for all m large enough), we deduce that there exists $t_m \geq t_m^\star$ such that

$$\rho\left(U(t_m)x_m\right) \geq \varepsilon/2, \text{ and } \rho\left(U(t_m + l)x_m\right) \leq \varepsilon/2, \forall l \in (0, l_m].$$

We conclude that there exist $l_m > 0$ and $t_m \geq 0$ such that

$$\begin{aligned} &d(U(t_m)x_m, C) \leq \frac{1}{m+1}, \\ &\rho\left(U(t_m)x_m\right) \geq \varepsilon/2, \\ &\rho\left(U(t_m + l)x_m\right) \leq \varepsilon/2, \forall l \in (0, l_m], \end{aligned}$$

and

$$\rho\left(U(t_m + l_m)x_m\right) \leq \frac{2}{m+1}. \tag{2.15}$$

Since C is compact, by taking a subsequence that we denote with the same index, we can always assume that $y_m = U(t_m)x_m \to y \in C$. Since ρ is continuous and U is a continuous semiflow, we deduce that

$$\rho\left(y\right) \geq \varepsilon/2, \text{ and } \rho\left(U(l)y\right) \leq \varepsilon/2, \ \forall l \in \left[0, l^\star\right),$$

where $l^\star = \lim_{m\to+\infty} \inf l_m$. Since $H = \{y_m : m \geq 0\} \cup \{y\}$ is compact, and $(t, x) \to U(t)x$ is a continuous map, we deduce that for each $\tau > 0$, $\bigcup_{t\in[0,\tau]\cap I} U(t)H$ is a compact subset which is included in M_0. Hence for each $\tau > 0$, $\bigcup_{t\in[0,\tau]\cap I} U(t)H$ is ρ-bounded. By using (2.15), we deduce that

$$l^\star = +\infty.$$

Otherwise, assume by contradiction that we can find a sub-sequence $l_{m_p} \to \widehat{l}$ as $p \to \infty$. By continuity of U, and by (2.15), we deduce that

$$\rho(U(l_{m_p})y_{m_p}) \to \rho(U(\widehat{l})y) = 0,$$

and since $\rho\left(y\right) \geq \varepsilon/2$, we obtain a contradiction with the fact that M_0 is positively invariant by U.

We conclude that

$$\lim_{t\to+\infty} \sup \rho\left(U(t)y\right) \leq \varepsilon/2 < \varepsilon,$$

which contradicts (2.13). □

We note that the concept of general ρ-persistence was used in [200, 210, 238]. It was also shown in [210] that the ρ-uniform persistence implies the weak ρ-uniform persistence for non-autonomous semiflows under appropriate conditions. The following result shows that the notion of ρ-uniform persistence is independent of the choice of continuous function ρ.

Proposition 2.61 *Let $\xi : M \to [0, +\infty)$ be a continuous function such that $\partial M_0 = \{x \in M : \xi(x) = 0\}$. Assume that there is a compact subset C of M which attracts every point in M for U. Then U is ρ-uniformly persistent if and only if U is ξ-uniformly persistent.*

Proof It suffices to prove that ρ-uniform persistence implies ξ-uniform persistence since the problem is symmetric. Let us first remark that U is ρ-uniformly persistent if and only if there exists $\varepsilon > 0$ such that

$$\inf_{x \in M_0} \inf_{y \in \omega(x)} \rho(y) \geq \varepsilon,$$

where $\omega(x)$ is the omega-limit set of the positive orbit of x. Define

$$A_\omega = \cup_{x \in M_0} \omega(x), \text{ and } V = \{y \in M : \rho(y) \geq \varepsilon\}.$$

Then

$$\inf_{x \in M_0} \inf_{y \in \omega(x)} \rho(y) = \inf_{x \in A_\omega} \rho(x) \geq \varepsilon.$$

Clearly, $A_\omega \subset C$, so $\overline{A_\omega}$ is compact. Since A_ω is include in $V \subset M_0$ which is closed, we deduce that $\overline{A_\omega} \subset V \cap C \subset M_0$. So $\overline{A_\omega} \subset M_0$ is compact, and hence, there exists $\eta > 0$ such that $\inf_{x \in \overline{A_\omega}} \xi(x) \geq \eta$, which implies that U is ξ-uniformly persistent. □

Definition 2.62 Let A be a nonempty subset of M. A is said to be **ejective** for U if there exists a neighborhood V of A such that for every $x \in V \setminus A$, there is $t_0 = t_0(x) \geq 0$ such that

$$U(t_0)x \in M \setminus V.$$

Proposition 2.63 *Assume that $\partial M_0 \neq \emptyset$ and that there is a compact subset C of M which attracts every point in M for U. Then the following statements are equivalent*

(i) *U is ρ-uniformly persistent;*
(ii) *∂M_0 is ejective for U.*

Proof **Proof of** (i) $\Rightarrow$ (ii). Assume that (i) is true. Let $\varepsilon > 0$ be fixed such that

$$\lim_{t \to +\infty} \sup \rho\left(U(t)x\right) \geq \varepsilon, \forall x \in M_0.$$

Then it is clear that ∂M_0 is ejective for U, with $V = \{x \in M : \rho\left(x\right) \leq \varepsilon/2\}$.

Proof of (ii) $\Rightarrow$ (i). Conversely, assume that ∂M_0 is ejective for U. Let V be a neighborhood of ∂M_0 such that for every $x \in M_0 \cap V$, there is $n_0 = n_0(x) \geq 0$ such that $U(t_0)x \in M \setminus V$. By Proposition 2.61, it is sufficient to prove that U is ρ-uniformly persistent when $\rho\left(x\right) = d(x, \partial M_0)$. Assume, by contradiction, that U is not ρ-uniformly persistent. Then for each $n \geq 1$, there exists $x_n \in M_0$, such that

$$\lim_{t\to+\infty}\sup\rho\left(U(t)x_n\right)\leq\frac{1}{n}.$$

By the attractivity of C, it follows that for each $n \geq 1$, there exists $t_n \geq 0$ such that each $y_n := U(t_n)x_n \in M_0$ satisfies

$$d\left(U(s)y_n, C\right)\leq\frac{2}{n},\text{ and } d\left(U(s)y_n,\partial M_0\right)\leq\frac{2}{n},\forall s\geq 0.$$

Since C is compact and V is a neighborhood of ∂M_0, there exists $\delta > 0$ such that

$$\{x\in M : d\left(x, C\right)\leq\delta,\text{ and } d\left(x,\partial M_0\right)\leq\delta\}\subset V.$$

Let $n_0 \geq 2/\delta$ be fixed. Then we have $y_{n_0} \in M_0$, and

$$d\left(U(s)y_{n_0}, C\right)\leq\delta,\text{ and } d\left(U(s)y_n,\partial M_0\right)\leq\delta,\forall s\geq 0.$$

Thus, we obtain

$$y_{n_0}\in M_0\cap V,\text{ and } U(s)y_{n_0}\in V,\forall s\geq 0,$$

a contradiction with the property (ii). □

Observe that M_0 is an open subset in (M, d). In order to make M_0 become a complete metric space, we define a new metric function d_0 on M_0 by

$$d_0(x,y)=\left|\frac{1}{\rho\left(x\right)}-\frac{1}{\rho\left(y\right)}\right|+d(x,y),\ \forall x,y\in M_0. \tag{2.16}$$

Lemma 2.64 (M_0, d_0) *is a complete metric space.*

Proof It is easy to see that d_0 is a distance. Let $\{x_n\}_{n\geq 0}$ be a Cauchy sequence in (M_0, d_0). Since $d(x, y) \leq d_0(x, y)$, $\forall x, y \in M_0$, we deduce that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (M, d), and there exists $x \in M$, such that $d(x_n, x) \to 0$ as $n \to +\infty$. To prove that $d_0(x_n, x) \to 0$ as $n \to +\infty$, it is sufficient to show that $x \in M_0$. Given $\varepsilon > 0$, since $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (M_0, d_0), there exists $n_0 \geq 0$ such that $d_0(x_n, x_p) \leq \varepsilon$, $\forall n, p \geq n_0$. In particular, we have $d_0(x_n, x_{n_0}) \leq \varepsilon, \forall n \geq n_0$. Then

$$\left|\frac{1}{\rho\left(x_n\right)}-\frac{1}{\rho\left(x_{n_0}\right)}\right|\leq\varepsilon,\forall n\geq n_0,$$

So there exists $r > 0$ such that $\inf_{n\geq 0}\rho\left(x_n\right) \geq r$. Since ρ is continuous and $d(x_n, x) \to 0$ as $n \to +\infty$, we deduce that $\rho\left(x\right) \geq r$, and hence $x \in M_0$. Thus, (M_0, d_0) is complete. □

We denote for each couple of subsets $A, B \subset M$,

$$\delta\left(B,A\right)=\sup_{x\in B}\inf_{y\in A}d(x,y),$$

and if $A, B \subset M_0$, we denote

$$\delta_0(B, A) = \sup_{x \in B} \inf_{y \in A} d_0(x, y).$$

Lemma 2.65 *The following two statements are valid:*

(i) *Let $\{B_t\}_{t \in I}$ be a family of subsets of M_0, where I is a unbounded subset of $[0, +\infty)$. If $A \subset M_0$ is compact in (M, d) and $\lim_{t \to \infty} \delta(B_t, A) = 0$, then $\lim_{t \to \infty} \delta_0(B_t, A) = 0$.*

(ii) *If U is asymptotically smooth in (M, d), then U is asymptotically smooth in (M_0, d_0).*

Proof **Proof of (i).** Let

$$\kappa := \frac{1}{2} \inf_{x \in A} \rho(x) > 0.$$

Assume, by contradiction that

$$\limsup_{t \to +\infty} \delta_0(B_t, A) > \varepsilon > 0.$$

Then we can find a sequence $\{t_p\}_{p \geq 0} \subset I$ such that $t_p \to +\infty$, $p \to +\infty$, and a sequence $\{x_{t_p}\}_{p \geq 0} \in B_{t_p} \subset M_0$ such that

$$d_0\left(x_{t_p}, A\right) \geq \varepsilon/2, \forall p \geq 0.$$

Since A attracts B_t in (M, d), we deduce that

$$d\left(x_{t_p}, A\right) \to 0, \text{ as } p \to +\infty.$$

So without loss of generality, we can assume that there exists $x \in A$ such that $d(x_{t_p}, x) \to 0$, as $p \to +\infty$. Since ρ is continuous and $\rho(x) > \kappa$, there exists $p_0 \geq 0$ such that $\rho(x_{t_p}) \geq \kappa, \forall p \geq p_0$. Thus, we have

$$0 < \varepsilon/2 \leq d_0\left(x_{t_p}, A\right) \leq d_0\left(x_{t_p}, x\right) \leq \kappa^{-2} \left|\rho(x_{t_p}) - \rho(x)\right| + d(x_{t_p}, x) \to 0 \text{ as } p \to +\infty,$$

a contradiction.

Proof of (ii). Let B be a bounded subset in (M_0, d_0) such that

$$U(t)B \subset B, \forall t \geq 0.$$

Since U is asymptotically smooth in (M, d), there exists a compact subset $C \subset M$ which attracts B for U in (M, d). That is

$$\lim_{t \to \infty} \delta(U(t)B, C) = 0.$$

The subset $C_0 = C \cap \overline{B} \subset M_0$ is compact in (M, d), and attracts B for U in (M, d). It easily follows that C_0 is also compact in (M_0, d_0). Since C_0 attracts B for U in (M, d), the property (i) implies that C_0 attracts B for U in (M_0, d_0). □

The main result of this section is the following theorem.

Theorem 2.66 (Global attractor in M_0) *Assume that U is asymptotically smooth, ρ-uniformly persistent, and U has a global attractor A in M. Then U has a global attractor A_0 in (M_0, d). Moreover, for each subset B of M_0, if there exists $t \geq 0$ such that $\gamma^+ (U(t)(B))$ is ρ-strongly bounded, then A_0 attracts B for U.*

Proof Since U is point dissipative and ρ -uniformly persistent, U is point dissipative in (M_0, d_0). Moreover, Lemma 2.87 implies that U is asymptotically smooth in (M_0, d_0). It is clear that U is continuous in (M_0, d_0). Let C be a compact subset in (M_0, d_0), and $\{x_n\}_{n\geq 0}$ a bounded sequence in $\gamma^+(C)$ in (M_0, d_0). Then $x_n = U(t_n)z_n$, $z_n \in C$, $\forall n \geq 1$, and the sequence $\{x_n\}_{n\geq 0}$ is ρ-strongly bounded in (M, d). If $\{t_n\}_{n\geq 0}$ is bounded by $T > 0$, since $\overline{\bigcup_{t\in[0,T]\cap I} U(t)C}$ is compact a compact subset of M_0, we deduce that $\{x_p\}$ has a convergent subsequence in (M_0, d_0). If $\{t_n\}_{n\geq 0}$ is an bounded, by taking a subsequence, we can assume that $t_n \to +\infty$, as $n \to +\infty$.

Since C is also compact in (M, d), we have $\lim_{t\to\infty} \delta(U(t)(C), A) = 0$. Thus, $\{x_n\}_{n\geq 0}$ has a convergent subsequence $x_{n_k} \to x$ in (M, d) as $k \to \infty$. By the continuity of ρ and the ρ-strong boundedness of $\{x_n\}_{n\geq 0}$, it follows that $\rho(x) > 0$, i.e., $x \in M_0$, and hence, $x_{p_k} \to x$ in (M_0, d_0) as $k \to \infty$. Thus, Lemma 2.83 (with $t_0 = 0$) implies that positive orbits of compact sets are bounded for U in (M_0, d_0). Then the conclusion for U in (M_0, d) follows from Theorem 2.49, as applied to U in (M_0, d_0). □

As a consequence of Proposition 2.46, we deduce that A_0 is a stable set for U with respect to the original distance d.

Corollary 2.67 (Stability of the global attractor in M_0) *Assume that A_0 is a global attractor for U in (M_0, d). Then A_0 is stable invariant subset for U in (M, d).*

Proof Since A_0 is compact, we must have

$$\eta := \inf_{x\in A_0} \rho(x) > 0.$$

The result follows from that fact that A_0 attracts the compact subsets of

$$\{x \in M : \rho(x) \geq \eta/2\}$$

which is a neighborhood of A_0. □

2.13 Coexistence Steady States

In this section, we establish the existence of coexistence steady state (i.e., the fixed point in M_0) for uniformly persistent dynamical systems.

Definition 2.68 A map $T : M \to M$ is said to be **κ-condensing** if T takes bounded sets to bounded sets and

$$\kappa(T(B)) < \kappa(B).$$

for any nonempty bounded set $B \subset M$ with $0 < \kappa(B) < +\infty$.

We assume that M is a closed and convex subset of a Banach space $(X, \|\cdot\|)$, that $\rho : M \to [0, +\infty)$ is a continuous function such that $M_0 = \{x \in M : \rho(x) > 0\}$ is nonempty and convex, and that $T : M \to M$ is a continuous map with $T(M_0) \subset M_0$. For convenience, we set $\partial M_0 := M \setminus M_0$.

Assume that $T : M_0 \to M_0$ has a global attractor A_0. It follows that for every compact set $K \subset M_0$, there exists an open neighborhood of K which is attracted by A_0. This property of A_0 is enough for the arguments in the proof of [233, Theorem 2.3] (see also [238, Theorem 1.3.6]) instead of the property that A_0 attracts ρ-strongly bounded sets in M_0. Thus, the proof of [233, Theorem 2.3] actually implies the following fixed point theorem.

Theorem 2.69 *Assume that T is κ-condensing. If $T : M_0 \to M_0$ has a global attractor A_0. Then T has a fixed point $x_0 \in A_0$.*

2.14 Two Examples

In this section, we first provide four examples of discrete and continuous-time semi-flows which admit global attractors, but no strong global attractors in the complete metric spaces (M_0, d_0) introduced in section 3. The examples show that our notion of global attractors is needed.

2.14.1 Asymptotically smooth semiflows on (M_0, d_0)

Let $C([0,1], \mathbb{R})$ be endowed with the usual norm $\|\varphi\|_\infty = \sup_{a\in[0,1]} |\varphi(a)|$. Let $M := C_+([0,1], \mathbb{R})$ be endowed with the metric $d(x, y) = \|x - y\|$, and $T : M \to M$ be defined by

$$T(\varphi) = \delta \frac{\mathcal{F}_\beta(\varphi)}{1 + \mathcal{F}_\beta(\varphi)} 1_{[0,1]},$$

where $1_{[0,1]}(a) = 1, \forall a \in [0,1]$, and $\mathcal{F}_\beta(\varphi) = \int_0^1 \beta(a)\varphi(a)da, \forall \varphi \in X$. We assume that

(A1) $\delta > 1, \beta \in C([0,1], \mathbb{R}), \int_0^1 \beta(a)da = 1, \beta(a) > 0, \forall a \in [0,1)$, *and* $\beta(1) = 0$.

Consider the following discrete time system on M:

$$u_{n+1} = T(u_n), \forall n \geq 0, \text{ and } u_0 \in M.$$

It is easy to see that the map T is continuous, and maps bounded sets into compact sets of M. Note that $T(M) \subset [0, \delta]\, 1_{[0,1]} = \left\{\alpha 1_{[0,1]} : \alpha \in [0, \delta]\right\}$ is bounded. So T compact and point dissipative, and has a strong global attractor in M. Set

$$\partial M_0 = \{0\}, \text{ and } M_0 = M \setminus \{0\}, \text{ and } \rho(x) = \|x\|_\infty .$$

Clearly, $T(M_0) \subset M_0$, $T(\partial M_0) \subset \partial M_0$, and the fixed points of T are 0 and $\overline{u} = (\delta - 1)\, 1_{[0,1]}$. Then it is easy to see that for each $\varphi \in M_0$, $T^m(\varphi) \to \overline{u}$, as $m \to +\infty$. So T is ρ-uniformly persistent. Let $\overline{\alpha} = (\delta - 1)$ and $B := \{x \in M : \|x\|_\infty = \overline{\alpha}\}$. Since $\beta(1) = 0$, we have $\mathcal{F}_\beta(B) = (0, \overline{\alpha}]$. Moreover, $T(B) = \left\{\alpha 1_{[0,1]} : \alpha \in (0, \overline{\alpha}]\right\}$, and $T^n(B) = T(B)$, $\forall n \geq 1$. Thus, there exists no compact subset in M_0 that attracts B for T. In particular, there is no strong global attractor for $T : (M_0, d_0) \to (M_0, d_0)$, where d_0 is defined as in (2.16).

2.14.2 κ-contracting maps on (M_0, d_0)

In this subsection, we construct κ-contracting maps on (M_0, d_0) such that they admits a global attractor, but no strong global attractor.

We set

$$X = L^1((0, +\infty), \mathbb{R}) \times \mathbb{R},\ X_+ = L^1_+((0, +\infty), \mathbb{R}) \times \mathbb{R}_+,$$

and endow X with the product norm $\|(\varphi, y)\| = \|\varphi\|_{L^1} + |y|$. Define $1_{[0,1]} \in X$ by $1_{[0,1]}(l) = 1$, $\forall l \in (0, 1)$, and $1_{[0,1]}(l) = 0$, $\forall l \in [1, \infty)$. Let a, b and c be three real numbers. Define $T : X_+ \to X_+$ by $T(\varphi, y) = (T_1(\varphi, y), T_2(\varphi, y))$ with

$$\begin{aligned} T_1(\varphi, y) &= a\varphi(\cdot + 1) + \left[a \int_0^1 \varphi(l)dl + c\frac{\int_0^1 \varphi(l)dl}{1+\|(\varphi,y)\|}\right] 1_{[0,1]}, \\ T_2(\varphi, y) &= ay + b\frac{\|(\varphi,y)\|}{1+\|(\varphi,y)\|}. \end{aligned}$$

We assume that

(A3) $a \in (0, 1)$, $b > 0$, $c > 0$, $\sqrt{a} < a + b < 1$, and $a + c > 1$.

Consider the discrete time system

$$x_{n+1} = T(x_n), \forall n \geq 0, \text{ and } x_0 \in X_+.$$

It is easy to see that $T^n(0, y) \to 0$, as $n \to +\infty$. Clearly, T is not uniformly persistent for $X_+ \setminus \{0\}$. We will find a closed subset M of X_+ such that it contains 0 and is positively invariant for T, and show that T is uniformly persistent for $M \setminus \{0\}$.

Lemma 2.70 *There exists a non-decreasing and rigth-continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0, f(x) > 0, \forall x > 0$, $\lim_{x \to 0} f(x) = 0$, and the set $M := \{(\varphi, y) \in X_+ : y \leq f(\|\varphi\|)\}$ is positively invariant for T.*

Proof We define $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by

$$F(x_1, x_2) = \left(ax_1, ax_2 + b\frac{x_1 + x_2}{1 + x_1 + x_2}\right),\ \forall x = (x_1, x_2) \in \mathbb{R}^2_+.$$

Then F is non-decreasing on $\mathbb{R}^2_+$. Set

$$\chi(t) = (ta + (1 - t), 1)\,,\ \forall t \in [0, 1]\,.$$

By induction, we define $\chi : \mathbb{R}_+ \to \mathbb{R}_+^2$ by

$$\chi(t) = F(\chi(t-1)),\ \forall t \in (n, n+1],\ \ \forall n \geq 1.$$

Note that $\chi(1)_1 = F(\chi(0))_1$ and $a < 1$. Then the function $t \to \chi(t)_1$ is strictly decreasing and continuous. Since $F(1,1) \leq (a,1)$, the function $t \to \chi(t)_2$ is non-increasing and left continuous. Moreover, since $a+b < 1$, we have $\lim_{t\to+\infty} \chi(t) = 0$. We further set

$$\chi(t) = (1 - t, 1)\,,\ \forall t \in (-\infty, 0]\,.$$

Since $\chi(t)_1$ is strictly decreasing in $t \in \mathbb{R}$, we can define

$$f(x) = \begin{cases} \chi(\chi(x)_1^{-1})_2, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

It is easy to see that f has the desired properties.

Let $D := \left\{(x_1, x_2) \in \mathbb{R}_+^2 : x_2 \leq f(x_1)\right\}$. Since f is non-decreasing and right-continuous, it easily follows that D is closed. Now we show that $F(D) \subset D$. Let $x = (x_1, x_2) \in D$, then $x_2 \leq f(x_1)$. If $x_1 = 0$, there is nothing to prove because $F(0) = 0$. Assume that $x_1 > 0$, then there exists $t \in \mathbb{R}$ such that $\chi(t)_1 = x_1$, and hence, $x_2 \leq f(x_1) = \chi(t)_2$. Clearly, $x = (x_1, x_2) \leq \chi(t)$, and $F(x) \leq F(\chi(t))$. In the case where $t \geq 0$, we have

$$\chi(t+1)_1 = F(\chi(t))_1 = F(x)_1\,,$$

and hence,

$$f(F(x)_1) = \chi(t+1)_2 = F(\chi(t))_2 \geq F(x)_2\,,$$

which implies that $F(x) \in D$. In the case where $t \leq 0$, we have

$$x_1 \geq F(x)_1 = F(\chi(t))_1 = a\chi(t)_1 = a(1 - t) \geq a = \chi(1)_1,$$

and hence, there exists $s \in [t, 1]$ such that $\chi(s)_1 = F(x)_1$. It then follows that

$$f(F(x)_1) = \chi(s)_2 = 1 \geq F(\chi(t))_2 \geq F(x)_2,$$

which implies that $F(x) \in D$. This proves that $F(D) \subset D$.

Finally, we prove that $T(M) \subset M$. For any $(\varphi, y) \in M$, we have $(\|\varphi\|, y) \in D$, and hence, the positive invariance of D for F implies that $F(\|\varphi\|, y)_2 \leq f(F(\|\varphi\|, y)_1)$. Note that $\|T_1(\varphi, y)\| \geq a\|\varphi\| = F(\|\varphi\|, y)_1$ and $T_2(\varphi, y) = F(\|\varphi\|, y)_2$. By the monotonicity of f, it then follows that

$$T_2(\varphi, y) = F(\|\varphi\|, y)_2 \leq f(F(\|\varphi\|, y)_1) \leq f(\|T_1(\varphi, y)\|),$$

which implies that $T(\varphi, y) \in M$. Thus, M is positively invariant for T. □

Now we consider $T : M \to M$, where M is endowed with the usual distance $d(x, \widehat{x}) = \|x - \widehat{x}\|$. We set

$$\partial M_0 = \{0\}\,,\ M_0 = M \setminus \{0\}\,,\ \text{and}\ \rho(x) = \|x\|\,.$$

Since T is the sum of a compact operator and a linear operator with norm being a, we have $\kappa\left(T(B)\right) \le a\kappa\left(B\right)$ for any bounded set $B \subset M$. Thus, T is κ-contraction. Moreover, for each $x \in M$, we have $\|T(x)\| \le a\,\|x\| + b + c$, and hence

$$\|T^n(x)\| \le a^n\,\|x\| + \left(\sum_{i=0}^{n-1} a^i\right)(b+c),\ \forall n \ge 1.$$

It then follows that $B = \left\{x \in M : \|x\| \le \frac{b+c}{1-a}\right\}$ is positively invariant for T, and attracts every bounded subset of M for T. So $T : (M,d) \to (M,d)$ has a strong global attractor.

Let $\varepsilon > 0$ be fixed such that $a + \frac{c}{1+\varepsilon} > 1$. We claim that

$$\limsup_{n\to\infty} \|T^n x\| \ge \varepsilon,\ \forall x = (\varphi, y) \in M_0.$$

Assume, by contradiction, that $\limsup_{n\to\infty} \|T^n x\| < \varepsilon$ for some $x = (\varphi, y) \in M_0$. We set $(\varphi_n, y_n) = T^n x$, $\forall n \ge 0$. By the definition of M, we have $\varphi \in L^1_+\left((0,+\infty), \mathbb{R}\right) \setminus \{0\}$. It then follows that there exists $n_0 \ge 0$ such that $\int_0^1 \varphi_{n_0}(l)dl > 0$ and

$$\int_0^1 \varphi_{n+1}(l)dl \ge \left(a + \frac{c}{1+\varepsilon}\right)\int_0^1 \varphi_n(l)dl, \forall n \ge n_0.$$

Thus, we obtain

$$\int_0^1 \varphi_n(l)dl \to +\infty,\ \text{as}\ n \to +\infty,$$

a contradiction. By Proposition 2.60, we conclude that T is ρ-uniform persistence. Since $T : (M,d) \to (M,d)$ has a global attractors, it follows from Theorem 3.12 that $T : (M_0, d_0) \to (M_0, d_0)$ has a global attractor.

To avoid possible confusion, we denote by κ_0 the Kuratowski measure of non-compactness on the complete metric (M_0, d_0). We now consider $T : (M_0, d_0) \to (M_0, d_0)$. Let $\varepsilon > 0$ be fixed such that

$$\sqrt{a} < d := a + \frac{b}{1+\varepsilon} < 1.$$

Then for each $x \in M$, we have

$$\|T(x)\| \ge a\,\|x\| + b\frac{\|x\|}{1+\|x\|} \ge d\min(\varepsilon, \|x\|).$$

Let $B \subset M_0$ be a ρ-bounded set. We set $\rho_0 = \inf_{x\in B} \rho(x)$. Then for each $x \in B$, we obtain

$$\|T(x)\| \ge a\,\|x\| + b\frac{\|x\|}{1+\|x\|} \ge d\min(\varepsilon, \|x\|).$$

By induction, it follows that

$$\rho\left(T^n(x)\right) \geq d^n \min(\varepsilon, \rho_0), \quad \forall n \geq 1,\ \forall x \in B.$$

Thus, for each $x, y \in B$, we have

$$\begin{aligned} d_0\left(T^n(x), T^n(y)\right) &= \left|\frac{1}{\rho\left(T^n(x)\right)} - \frac{1}{\rho\left(T^n(y)\right)}\right| + \left\|T^n(x) - T^n(y)\right\| \\ &\leq \left[\frac{1}{\rho\left(T^n(x)\right)\rho\left(T^n(y)\right)} + 1\right]\left\|T^n(x) - T^n(y)\right\| \\ &\leq \left[\frac{1}{d^{2n}\min(\varepsilon, \rho_0)^2} + 1\right] d(T^n(x), T^n(y)), \end{aligned}$$

and hence,

$$\begin{aligned} \kappa_0\left(T^n(B)\right) &\leq \left[\frac{1}{d^{2n}\min(\varepsilon, \rho_0)^2} + 1\right]\kappa\left(T^n(B)\right) \\ &\leq a^n\left[\frac{1}{d^{2n}\min(\varepsilon, \rho_0)^2} + 1\right]\kappa(B). \end{aligned}$$

Since $d > \sqrt{a}$, we obtain $\kappa_0\left(T^n(B)\right) \to 0$ as $n \to +\infty$. So $T : (M_0, d_0) \to (M_0, d_0)$ is κ_0-contracting.

It remains to show that $T : (M_0, d_0) \to (M_0, d_0)$ has no strong global attractor. Let $\delta > 0$ be fixed, and consider the ρ-strongly bounded set

$$B_\delta = \{x \in M : \rho(x) = \delta\}.$$

For each $m \geq 0$, we set $x^m := (\varphi^m, 0)$ with $\varphi^m = \delta 1_{[m,m+1]}$, and

$$x_n^m := \left(\varphi_n^m, y_n^m\right) = T^n(x^m), \quad \forall n \geq 0.$$

Then for each $m \geq 1$ and each $n \in \{0, ..., m-1\}$, we have $\int_0^1 \varphi_n^m(l)dl = 0$, and hence,

$$\begin{cases} \varphi_{n+1}^m(\cdot) = a\varphi_n^m(\cdot + 1) + a\int_0^1 \varphi_n^m(l)dl 1_{[0,1]}(\cdot) \\ y_{n+1}^m = a y_n^m + b\frac{\|x_n^m\|}{1+\|x_n^m\|}. \end{cases}$$

Thus, for each $m \geq 1$ and each $n \in \{0, ..., m-1\}$, we obtain

$$\left\|x_{n+1}^m\right\| \leq (a+b)\left\|x_n^m\right\| \leq (a+b)^n \delta.$$

It follows that $\inf_{x \in B_\delta} \rho\left(T^n(x)\right) \to 0$, as $n \to +\infty$. So the κ_0-contracting map $T : (M_0, d_0) \to (M_0, d_0)$ has a global attractor, but no strong global attractor.

2.15 Notes and Remarks

- Add some comment about the exclusion principle
- Continuity of attractors with respect to a parameter

The section devoted to the uniform persistence is inspried by Thieme [211], Smith and Zhao [200], and Magal and Zhao [160]. More result on uniform persistence can be founded in the books of Smith and Thieme [203] and Zhao [237].

More results about uniform persistence

Some book on global attractors

For more reading on global attractors theory we refer to the books of Temam [207], Hale [93], Sell and You [189], Raugel [177], Chueshov [43], Robinson [182], Cholewa, Dlotko, and Chafee [29], Carvalho Langa and Robinson [24], Zhao [238],

Please up date with more book on global attractors

So far, the only book similar to this chapter is the book of Zhao [238], where only the discrete-time case is considered. The novelty in this chapter is that we provide a global attractors theory in the context of uniform persistence for both discrete and continuous time dynamical systems.

The following text is taken from the book of Zhao

Theorem 1.1.1 is due to LaSalle [130]. Theorem 1.1.3 (a) with $n0 = 1$ is due to Billotti and LaSalle [36]. Theorem 1.1.4 is due to Nussbaum [257] and Hale and Lopes [95]. Theorems 1.1.2 and 1.1.3, Lemma 1.1.5, and their proofs are adapted from Magal and Zhao [160].

Section 1.2 is adapted from Hirsch, Smith and Zhao [104]. Lemma 1.2.4 and Example 1.2.2 are taken from Smith and Zhao [200].

The notion of chain recurrence was introduced by Conley [45]. Bowen [20] proved that omega limit sets of precompact orbits of continuous invertible maps are internally chain transitive. Robinson [181] proved that omega limit sets of precompact orbits of continuous maps are internally chain recurrent. Thieme [208] [209] [209] studied the long-term behavior in asymptotically autonomous differential equations, and Mischaikow, Smith, and Thieme [166] discussed chain recurrence and Liapunov functions in asymptotically autonomous semiflows. Asymptotic pseudo-orbits were introduced by Benaïm and Hirsch [16] for continuous-time semiflows. The embedding approach in the proof of Lemma 1.2.2 was used earlier by Zhao [234] [235] to prove that the omega limit set of a precompact orbit of an asymptotically autonomous process is nonempty, compact, invariant, and internally chain recurrent for the limiting map (see [234, Theorem 2.1] and [235, Theorem 1.2]). Freedman and So [79, Theorem 3.1] proved the Butler–McGehee lemma of limit sets for continuous maps. By an embedding approach and [79, Theorem 3.1], Hirsch, Smith and Zhao [105, Lemma 3.3] proved Lemma 1.2.7. Theorem 1.2.1 was proved earlier by Smith and Zhao [200, Lemma 4.1].

Uniform persistence (permanence) has received extensive investigation for both continuous- and discrete-time dynamical systems. We refer to Waltman [221], Hutson and Schmitt [122], and Hofbauer and Sigmund [106] for surveys and reviews, and to Thieme [210], Zhao [236], Smith and Thieme [202], and references therein for further developments.

Subsections 1.3.1 and 1.3.2 are adapted from Hirsch, Smith and Zhao [164] and Smith and Zhao [340]. Theorem 1.3.3 was generalized to non-autonomous semiflows by Thieme [368, 369]. The concept of a generalized distance function was motivated by ideas in Thieme [369], where uniform ρ-persistence was developed for nonautonomous semiflows. General theorems on uniform per- sistence were established earlier by Hale and Waltman [146], Thieme [365] for autonomous semiflows, and Freedman and So [122], Hofbauer and So [168] for continuous maps.

Various concepts of practical persistence were utilized by Hutson and Schmitt [186], Cantrell, Cosner and Hutson [54], Hutson and Zhao [443], Cosner [67], Cantrell and Cosner [52], Hutson and Mischaikow [184], Smith and Zhao [336, 337], and Ruan and Zhao [298]. The p-function in Exam- ple 1.3.1 was employed by Thieme [369] for a scalar functional differential equation. Two p-functions in Example 1.3.2 were used by Smith and Zhao [340] and Zhao [439], respectively, for an autonomous microbial population growth model and almost periodic predator–prey reaction–diffusion systems.

Subsection 1.3.3 is taken from Magal and Zhao [241], and Subsection 1.3.4 is adapted from Zhao [430] and Magal and Zhao [241]. For a class of continuous Kolmogorov-type maps on $\mathbb{R}_+^m$, Hutson and Moran [185] proved that the existence of a compact attracting set in $int(\mathbb{R}_+^m)$ implies that of a (com- ponentwise) positive fixed point. By applying Theorem 1.3.8 to the Poincaré map associated with a periodic semiflow, one can obtain the existence of a periodic orbit in X_0, and hence that of periodic coexistence solutions for periodic systems of differential equations. Freedman and Yang [419, Theorem 4.11] proved the existence of interior periodic solutions for periodic, dissipative, and uniformly persistent systems of ODEs. For periodic and uniformly persistent Kolmogorov systems of ODEs, Zanolin [424, Lemma 1] also proved the existence of positive periodic solutions. For autonomous Kolmogorov systems of ODEs and a class of autonomous differential equations with finite delay, Hutson [183] proved the existence of positive equilibria. Hofbauer [166] generalized an index theorem for dissipative ordinary differential systems, which implies the existence of a positive equilibrium (see also [167]). For autonomous 2-species Kolmogorov reaction–diffusion systems, Cantrell, Cosner and Hutson [54, Theorem 6.2] also proved a result on the existence of stationary coexis- tence states under appropriate assumptions.

Subsection 1.4.1 is taken from Smith and Waltman [335]. Subsection 1.4.2 is adapted from Hirsch, Smith and Zhao [164] and Smith and Zhao [340]. Subsection 1.4.3 is taken from Hirsch, Smith and Zhao [164]. Smith and Zhao [336, Theorem 4.3] proved a similar result on uniform persistence uniform in parameter. Earlier, Hutson

[182] discussed robustness of permanence for autonomous ordinary differential systems defined on Rn+ by using Liapunov function techniques. Schreiber [302] established criteria for C^r–robust permanence, $r \geq 1$, of autonomous Kolmogorov ordinary differential systems.

Book of Smith and Thieme [204]

Book of X-Q Zhao

P 5 Definition 1.1.3.

P 28 Remark 1.3.3. A result similar to Theorem 1.3.7 was already presented for discrete- and continuous-time dynamical systems in [430] and [146], respectively. The only difference, compared with the earlier results, is that we add a strong boundedness assumption for case (a). In general, this assumption is necessary for the existence of a strong global attractor in X0 for f, which can be seen from the counter example below.

P 40 Remark and notes

Juste make a remark of example and counter example and refer to our paper

Chains and uniform persistence

We now recall some results taken from Hale and Waltman [96], and Hofbauer and So [107]. Let (M, d) be a complete metric space, and $\rho : M \to [0, +\infty)$ a continuous function. We decompose M into the open subset

$$M_0 := \{x \in M : \rho(x) > 0\},$$

and the closed subset

$$\partial M_0 := \{x \in M : \rho(x) = 0\}.$$

Assumption 2.71 Let $\{U(t)\}_{t \in I}$ be a continuous semiflow. We assume that

(i) $U(t)M_0 \subset M_0$, and $U(t)\partial M_0 \subset \partial M_0, \forall t \geq 0$.

(ii) U has a global attractor A in M.

Lemma 2.72 *Let Assumption 2.71 be satisfied. Then*

$$A_\partial = A \cap \partial M_0.$$

is a global attractor for U in ∂M.

Definition 2.73 Let C be a subset of M. We recall that the *stable set (or attracting set)* of a compact invariant set C is denoted by $W^s(C)$ and is defined as

$$W^s(C) = \{x \in M : \omega(x) \neq \emptyset \text{ and } \omega(x) \subset C\},$$

or equivalently

$$W^s(C) = \left\{x \in M : \lim_{t \to +\infty} d(U(t)x, C) = 0\right\}.$$

The *unstable set (or repelling set)*, W^u is defined by

$$W^u(C) = \{x \in M : \text{ there exists a backward orbit } \gamma^-(x) \text{ such that } \alpha_\gamma(x) \neq \emptyset \text{ and } \alpha_\gamma(x) \subset C\Big\}.$$

where $\alpha_\gamma(x)$ is the alpha limit set corresponding to the specific negative orbit $\gamma^-(x)$.

That is also equivalent to say that

$$W^u(C) = \{x \in M : \text{ there exists a complete orbit } t \to u(t) \text{ passing through } x \text{ such that } \lim_{t\to+\infty} d(U(t)x, C) = 0\}.$$

Theorem 2.74 *Let Assumption 2.71 be satisfied. Assume in addition that U has a global attractor A_0 in M_0. Then*

$$A = A_0 \cup W^u(A_\partial)$$

Proof To be done (See Theorem 3.2 p. 391 in [96]) □

Definition 2.75 A nonempty invariant subset C of M is called an *isolated invariant set* if it is the maximal invariant set of a neighborhood of itself. The neighborhood is called an *isolating neighborhood*.

Definition 2.76 Let C_1, C_2 be isolated invariant sets (not necessarily distinct). The subset C_1 is said to be *chained* to C_2, written $C_1 \to C_2$, if there exists an element $x \in M$ such that

$$x \notin C_1 \cup C_2 \text{ and } x \in W^u(C_1) \cap W^s(C_2).$$

If we assume in addition that C_1 and C_2 are two compact subsets. This is also equivalent to say that there exists a complete orbit $t \to u(t)$ such that

$$u(0) = x \notin C_1 \cup C_2,$$

with

$$\lim_{t\to-\infty} d(u(t), C_1) = 0, \text{ and } \lim_{t\to+\infty} d(u(t), C_2) = 0.$$

A finite sequence $C_1, C_2, \ldots, C_k$ of isolated invariant sets will be called a *chain* if $C_1 \to C_2 \to \ldots \to C_k$ (with $C_1 \to C_1$, if $k = 1$). The chain will be called a *cycle* if $C_k = C_1$.

The particular invariant sets of interest are

$$\widehat{A}_\partial = \bigcup_{x \in A_\partial} \omega(x).$$

Definition 2.77 $\widehat{A}_\partial$ is *isolated* if there exists a covering $C = \{C_i\}_{i=1,\ldots,k}$ of $\widehat{A}_\partial$ by pairwise disjoint, compact, isolated invariant sets $C_1, \ldots, C_n$ for U_∂ (i.e. U restricted to ∂M_0) such that each C_i is also an isolated invariant set for U (i.e. U on M). C is called an *isolated covering*. $\widehat{A}_\partial$ will be called *acyclic* if there exists some isolated covering C of $\widehat{A}_\partial$ such that no subset of the C_i forms a cycle. An isolated covering C satisfying this condition will be called *Morse decomposition* or *acyclic*.

Theorem 2.78 *Let Assumption 2.71 be satisfied. Assume in addition that $\widehat{A}_\partial$ is isolated has an acyclic covering. Then U is ρ-uniform persistent (with respect to $\rho(x) = d(x, \partial M_0)$) if an only if*

$$W^s(C_i) \cap M_0 = \emptyset, \forall i = 1, \ldots, k.$$

Proof To be done (See Theorem 4.2 p. 393 in [96]) □

More properties on the measure of non-compactness

Recall that for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}$$

is the Kuratovsky measure of non-compactness.

We now list various properties of the Kuratowski's measure of non-compactness in a Banach space X. We refer to Deimling [49], Martin [161], and Sell and You [189, Lemma 22.2].

Theorem 2.79 *Let $(X, \|.\|)$ be a Banach space and $\kappa(.)$ the measure of non-compactness defined as above. Then for any bounded subsets B and $\widehat{B}$ of X, we have the following properties:*

(i) $\kappa(B) = 0$ *if and only if $\overline{B}$ is compact;*
(ii) $\kappa(B) = \kappa\left(\overline{B}\right)$*;*
(iii) $\kappa(\overline{\text{co}}(B)) = \kappa(B)$*;*
(iv) $\kappa(\overline{\text{co}}(B)) = \kappa(B)$*;*
(iv) *If $B \subset \widehat{B}$ then $\kappa(B) \leq \kappa\left(\widehat{B}\right)$;*
(v)
$$\kappa(\lambda B) \leq |\lambda| \kappa(B),$$
where
$$\lambda B = \{\lambda x : x \in B\}$$
and
$$\kappa\left(B + \widehat{B}\right) \leq \kappa(B) + \kappa\left(\widehat{B}\right),$$
where
$$B + \widehat{B} = \left\{x + y : x \in B, y \in \widehat{B}\right\}.$$

Let $L \in \mathcal{L}(X)$. Then the *essential semi-norm* $\|L\|_{\text{ess}}$ of T is defined by

$$\|L\|_{\text{ess}} = \kappa(L(B_X(0,1))),$$

where $B_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$,

Proposition 2.80 *For each pair of bounded linear operators $L, \widehat{L} \in \mathcal{L}(X)$, we have the following properties:*

(i) $\|L\|_{\text{ess}} = 0$ *if and only if L is compact (i.e. L maps bounded subsets into relatively compact subsets);*
(ii) $\|\lambda L\|_{\text{ess}} \leq |\lambda| \, \|L\|_{\text{ess}}, \forall \lambda \in \mathbb{C}$;
(iii) $\left\|L + \widehat{L}\right\|_{\text{ess}} \leq \|L\|_{\text{ess}} + \left\|\widehat{L}\right\|_{\text{ess}}$;
(iv) $\left\|L\widehat{L}\right\|_{\text{ess}} \leq \|L\|_{\text{ess}} \left\|\widehat{L}\right\|_{\text{ess}}$;
(v) $\|L\|_{\text{ess}} \leq \|L\|_{\mathcal{L}(X)}$;

Asymptotic smoothness

Assumption 2.81 Let $F : X \to X$ is a map on a Banach space X which is Lipschitz on bounded sets. Let $\left\{e^{At}\right\}_{t \geq 0} \subset \mathcal{L}(X)$ be a strongly continuous family of bounded linear operator on a Banach space X. We assume F is completely continuous, that is to say that the closure $\overline{F(B)}$ is compact for each bounded subset B of X. In other words,

$$\kappa(F(B)) = 0$$

for any bounded set $B \subset X$.

Let $U(t)$ be the semiflow obtained as fixed point of

$$U(t)x = e^{At}x + \int_0^t e^{A(t-s)} F(U(s)x) ds.$$

Let B be a bounded set such that

$$U(t)B \subset B, \forall t \geq 0.$$

Assume that F maps bounded sets into relatively compact subsets of X. Then by Mazur's theorem

$$\int_0^t e^{A(t-s)} F(U(s)x) ds \in t \times \overline{\text{co}}(\left\{e^{A(t-s)} F(U(s)x) : s \in [0, t]\right\}).$$

$$\kappa(\int_0^t e^{A(t-s)} F(U(s)B) ds) \leq t \times \overline{\text{co}}(\left\{e^{A(t-s)} F(x) : x \in B, s \in [0, t]\right\}). = 0,$$

Since the measure of compactness is additive (i.e. $\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)$)

$$\kappa(U(t)B) = \kappa(e^{At}B).$$

Now since $\kappa(L(B)) \leq \|L\|_{\mathcal{L}(X)} \kappa(B)$ we deduce the following proposition.

The following extend some idea originally proved in Webb [223].

Proposition 2.82 *Assume that $F : X \to X$ is a map on a Banach space that is Lipschitz on bounded sets and F is completely continuous, that is to say that the closure $\overline{F(B)}$ is compact for each bounded subset B of X.*

Then the semiflow $t \to U(t)$ is asymptotically smooth if we impose in addition that

$$\|e^{At}\|_{\mathcal{L}(X)} \to 0 \text{ as } t \to \infty.$$

We refer to Magal and Thieme [159] for more result going in that direction.

Gurtin-MacCamy population dynamics model

Z. Ma and P. Magal, Global asymptotic stability for Gurtin-MacCamy's population dynamics model, Proceedings of the AMS (to appear)

Bernoulli-Kermack-McKendrick epidemic model with age of infection

P. Magal, C. C. McCluskey, and G. F. Webb (2010), Liapunov functional and global asymptotic stability for an infection-age model, Applicable Analysis 89, 1109 -1140.

P. Magal and C.C. McCluskey (2013), Two group infection age model: an application to nosocomial infection, SIAM J. Appl. Math., 73(2), 1058-1095.

Existence of strong global attractors

The following lemma provides sufficient conditions for the positive orbit of a compact set to be bounded.

Lemma 2.83 *Assume that U is continuous and point dissipative. Let C be a compact subset of M. Assume that for every bounded sequence $\{x_n\}_{n\geq 0}$ in $\gamma^+(C)$, there exists $t_0 \geq 0$, and the exists a subsequence $\left\{x_{n_p}\right\}_{p\geq 0}$, such that the two following properties are satisfied:*

(i) $\left\{U\left(t_0)x_{n_p}\right)\right\}_{p\geq 0}$ *converges;*

(ii) *The subset* $\bigcup_{p\geq 0}\bigcup_{0\leq t\leq t_0}\left\{U(t)x_{n_p}\right\}$ *is bounded.*

Then $\gamma^+(C)$ is bounded in M.

Proof Since U is point dissipative, we can choose a bounded and open subset V of M such that for each $x \in M$ there exists $\tau(x) \geq 0$ such that

$$U(t)x \in V, \forall t \geq \tau(x).$$

For each $x \in C$ we can find

$$\widehat{\tau}(x) = \inf\{t > 0 : U(t)x \in V\}.$$

We claim that

$$\sup_{x \in C} \widehat{\tau}(x) < \infty. \tag{2.17}$$

Assume by contradiction that, there exists a sequence $x_n \in C$ such that

$$\widehat{\tau}(x_n) \to \infty.$$

Since C is compact we can assume (by taking a sub-sequence) that $x_n \to x_\infty \in C$, and since

$$\widehat{\tau}(x_\infty) < \infty,$$

and $x \to U(\tau(x_\infty))x$ is continuous and V is open, we deduce that for all n large enough

$$U(\tau(x_\infty))x_n \in V,$$

and we obtain a contradiction with $\widehat{\tau}(x_n) \to \infty$. So we deduce (2.17) holds.

Let $r > \sup_{x \in C} \widehat{\tau}(x)$. Let $x \in C$, by using the definition of $\widehat{\tau}(x)$, we deduce that there exists $s = s(x) \in I \cap [0, r]$, such that

$$U(s)x \in V.$$

Let $z \in M$ be fixed. Assume, by contradiction, that $\gamma^+(C)$ is unbounded. Then there exists a sequence $\{x_p\}$ in $\gamma^+(C)$ such that

$$x_p = U\left(\tau_p\right) z_p, \text{ with } z_p \in C, \text{ and } \lim_{p \to \infty} d(z, x_p) = \infty.$$

Since U is continuous and V is bounded, we can assume that

$$\lim_{p \to \infty} \tau_p = \infty, \text{ and } \tau_p > r, \, x_p \notin \overline{V}, \, \forall p \geq 1.$$

For each $z_p \in C$, there exists $s_p \leq r$ such that

$$U(s_p)z_p \in V.$$

Since $x_p = U(\tau_p)(z_p) \notin V$, there exists $\delta_p \in [s_p, \tau_p)$ (that is the last time before τ_p such that the orbit starting from x_p belongs to $\overline{V}$) such that

$$y_p = U(\delta_p)z_p \in \overline{V}, \text{ and } U(l)y_p \notin V, \, \forall l \in I \cap \left(0, l_p\right], \text{ with } l_p = \tau_p - \delta_p.$$

Clearly,

$$x_p = U(l_p)y_p, \, \forall p \geq 1,$$

and $\{y_p\}_{p \in \mathbb{N}}$ is a bounded sequence in $\gamma^+(C)$.

By using assumptions (i) and (ii), we deduce that there exists $t_0 \in I$, and taking a sub-sequence, we have

$$\left\{U\left(t_0\right) y_p\right\}_{p \geq 0} \to \widehat{x} \in M.$$

and

$$\bigcup_{p \geq 0, t \in [0, t_0] \cap I} \left\{U(t)y_p\right\}$$

is bounded.

Since $\lim_{p\to\infty} d(z, x_p) = \infty$, we deduce that for all $p \geq 0$ large enough

$$l_p > t_0.$$

By construction the subset

$$H = \left\{U(t_0)\, y_p : p \geq 0\right\} \cup \{\widehat{x}\}$$

is compact, and since U is a continuous semiflow, the subset

$$\bigcup_{s\in I\cap[0,t]} U(s)H$$

is bounded for each $t \in I$, and we deduce that $l_p \to +\infty$, as $p \to +\infty$.

Now since U is continuous, and V is open, we deduce that

$$U(t)\widehat{x} \notin V, \forall t > 0,$$

which contradicts the definition of V (the fact that any orbit starting from a single point x ultimately belongs to V). □

Definition 2.84

(i) A map $T : M \to M$ is **compact** if T maps any bounded set $B \subset M$, $\overline{T(B)}$ is compact.

(ii) A semiflow U is said to be **eventually compact** there exists $t_0 > 0$, such that $U(t_0)$ is compact.

We complete this section with a variant of [93, Theorems 2.4.6 and 2.4.7] on the existence of strong global attractors.

Theorem 2.85 (Strong global attractor) *Let U be a continuous semiflow on a metric space (M, d). Assume that U is point dissipative on M, and assume that one of the two following conditions holds:*

(i) *U is eventually compact;*
(ii) *U is asymptotically smooth, and for each bounded set $B \subset M$, there exists $t_0 = t_0(B) \geq 0$ such that $\gamma^+(U(t_0)(B))$ is bounded.*

Then U has a strong global attractor $A \subset M$.

Proof The conclusion in case (ii) is an immediate consequence of Theorem 2.49. In the case of (i), since $U(t_0)$ is compact for some $t_0 \in I \setminus \{0\}$, it is sufficient to show that for each compact subset $C \subset M$,

$$\bigcup_{t\geq 0} U(t)(C)$$

is bounded.

By applying Lemma 2.83 to the discrete time semiflow $\{V(t)\}_{t\in\mathbb{N}}$, defined by

$$V(n) = U(t_0)^n, \forall n \geq 0,$$

we deduce that for each compact subset $C \subset M$,

$$\bigcup_{n\geq 0} V(n)\,(C)$$

is bounded.

So Theorem 2.49 implies that V has a global attractor $\widetilde{A} \subset M$. We set

$$\widetilde{B} = \bigcup_{0\leq t\leq t_0} U(t)\left(\widetilde{A}\right).$$

Since U is a continuous, the subset $\widetilde{B}$ is compact, and $\widetilde{B}$ and attracts every compact subset of M for U, and hence, the result follows from Theorem 2.49. □

Connected global attractors

We are interested in connexity property of the global attractor. The goal is to give a generalized version of the connectness results in Gobbino and Sardella [83].

Definition 2.86 A subset C of M is said to be **connected** if there exists no partition of C in two subsets A and B that are both open and closed. That is to say that if for each $A, B \subset C$, such that

$$A \neq \emptyset,\ B \neq \emptyset,\ C = A \cup B,\ \text{and } A \cap B = \emptyset,$$

we have

$$\left(\overline{A} \cap B\right) \cup \left(A \cap \overline{B}\right) \neq \emptyset.$$

If C is closed and not connected, there exist A and B two non empty subset of C, such that

$$C = A \cup B,\ A \cap B = \emptyset,\ \text{and } \left(\overline{A} \cap B\right) \cup \left(A \cap \overline{B}\right) = \emptyset.$$

Assume for example that $A \neq \overline{A}$. Then $\overline{A} \subset C$, and $\left(\overline{A} \cap B\right) = \emptyset$. So $C = A \cup B \cup \left(\overline{A} \setminus A\right)$, and $C \neq A \cup B$. So $A = \overline{A}$. We conclude that if C is closed and not connected, there exist A and B two nonempty closed subset of C, such that

$$C = A \cup B,\ \text{and } A \cap B = \emptyset.$$

Lemma 2.87 *Let $\{U(t)\}_{t\in I}$ be a continuous semiflot, let A be a nonempty compact subset of M invariant for U, and if there exists C a connected subset of M, such that $A \subset C$, and A attracts C for U, then A is connected.*

Proof Assume that A is not connected then we can find A_1 and A_2 two disjoint nonempty compact subsets of A such that $A = A_1 \cup A_2$. Since A_1 and A_2 are compact, we can find $\varepsilon > 0$, such that $\overline{N}(A_1, \varepsilon) \cap \overline{N}(A_2, \varepsilon) = \emptyset$. Since A attracts C for U there exists $t > 0$, such that $U(t)C \subset \overline{N}(A_1, \varepsilon) \cup \overline{N}(A_2, \varepsilon)$, and since $A \subset C$, and A is invariant for U, we deduce that $U(t)C \cap \overline{N}(A_i, \varepsilon) \neq \emptyset, \forall i = 1, 2$. But since $x \to U(t)x$ is continuous, we deduce that $\widehat{C} = U(t)C$ is connected. But $\widehat{C} = C_1 \cup C_2$ with $C_i = U(t)C \cap \overline{N}(A_i, \varepsilon), \forall i = 1, 2$. By construction we have

$$\left(C_1 \cap \overline{C_2}\right) \cup \left(\overline{C_1} \cap C_2\right) \subset \overline{N}(A_1, \varepsilon) \cap \overline{N}(A_2, \varepsilon) = \emptyset,$$

we obtain a contradiction with the fact that $\widehat{C}$ is connected. □

Theorem 2.88 *Let $\{U(t)\}_{t \in I}$ be a continuous semiflot on M a closed and convex subset of a Banach space $(X, \|\cdot\|)$. Assume that U has a global attractor $A \subset M$. Then A is connected.*

Proof Since A is compact, $\overline{\text{co}}(A)$ (the closed convex hull of A) is also compact. Moreover $A \subset \overline{\text{co}}(A)$, and since A attracts the compact subset of M by A the result follows from Lemma 2.87. □

The following result has been proved first by Zhao [233, Lemma 2.1].

Lemma 2.89 (Zhao) *Let M be a closed subset of a Banach space $(X, \|\cdot\|)$. Assume that $\rho : M \to [0, +\infty)$ is a continuous map, and assume that*

$$M_0 = \{x \in M : \rho(x) > 0\}$$

is nonempty and convex. Let $A \subset M_0$ be a compact subset, then

$$\overline{\text{co}}(A) \subset X_0.$$

Theorem 2.90 *Let $\{U(t)\}_{t \in I}$ be a continuous semiflot on M a closed subset of a Banach space $(X, \|\cdot\|)$. Assume that $\rho : M \to [0, +\infty)$ is a continuous map, and assume that*

$$M_0 = \{x \in M : \rho(x) > 0\}$$

is nonempty and convex. Assume that U has a global attractor $A \subset M_0$. Then A is connected.

Upper semi-continuity of global attractors

We now consider investigate the depence to a parameter. Let (Λ, d_Λ) be a metric space, let $\lambda_0 \in \Lambda$ be fixed, let $\{U_\lambda(t)\}_{t \in I}$ be a familly of semiflow on M parametred by $\lambda \in \Lambda$. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a familly of compact subset of M. We will say that $\{A_\lambda\}_{\lambda \in \Lambda}$ is **upper semi-continuous at** $\lambda_0 \in \Lambda$, if and only if

$$\delta\left(A_\lambda, A_{\lambda_0}\right) \to 0 \text{ as } \lambda \to \lambda_0.$$

For each $\lambda \in \Lambda$, and each $\eta \geq 0$, we set

$$\overline{B}_{\Lambda}(\lambda, \eta) = \{\alpha \in \Lambda : d_{\Lambda}(\lambda, \alpha) \leq \eta\}.$$

We first have the following result, which will be used in section 3.

Proposition 2.91 *We assume that for each* $(t, x_0) \in I \times M$, *the map* $(\lambda, x) \to U_\lambda(t)x$ *is continuous at* (λ_0, x_0) *, for each* $\lambda \in \Lambda$, A_λ *is invariant for* U_λ, *and* A_{λ_0} *is a global attractors for* U_{λ_0}. *Then the following statement are equivalents:*

(i) $\{A_\lambda\}_{\lambda \in \Lambda}$ *is upper semi-continuous at* $\lambda_0 \in \Lambda$.
(ii) *There exists a compact subset* $C \subset M$, *such that* $\delta(A_\lambda, C) \to 0$ *as* $\lambda \to \lambda_0$.

Proof (i)⇒(ii) is obvious with $C = A_{\lambda_0}$.

(ii)⇒(i). Conversely, assume that there exists a compact subset C of M such that $\delta(A_\lambda, C) \to 0$ as $\lambda \to \lambda_0$. Assume that there exist $\varepsilon > 0$, and a sequence $\lambda_n \to \lambda_0$, as $n \to +\infty$, such that

$$\delta\left(A_{\lambda_n}, A_{\lambda_0}\right) \geq \varepsilon, \forall n \geq 0.$$

Then by using Theorem 2.29, we find a sub-sequence $\lambda_{n_p} \to \lambda_0$, and a compact subset $\widehat{A}_\infty$ such that $d_H(A_{\lambda_{n_p}}, \widehat{A}_\infty) \to 0$, as $p \to +\infty$. By construction we have $\delta\left(\widehat{A}_\infty, A_{\lambda_0}\right) \geq \varepsilon$. Since for each $p \geq 0$, $A_{\lambda_{n_p}}$ is invariant for $U_{\lambda_{n_p}}$, and since for each $t \in I$, and each $x_0 \in M$, the map $(\lambda, x) \to U_\lambda(t)x$ is continuous at (λ_0, x_0), we deduce that $\widehat{A}_\infty$ is invariant for U_{λ_0}. Finally since A_{λ_0} is a global attractor for U_{λ_0}, we deduce that $\widehat{A}_\infty \subset A_{\lambda_0}$, so $\delta\left(\widehat{A}_\infty, A_{\lambda_0}\right) = 0$, a contradiction. □

Proposition 2.92 *We assume that for each* $\lambda \in \Lambda$, A_λ *is invariant for* U_λ, *and there exists a subset* B *of* M *such that:*
a) A_{λ_0} *attracts* B *for* U_{λ_0}.
b) $A_\lambda \subset B, \forall \lambda \in \Lambda$.
c) For each $t \geq 0$, $U_\lambda(t)x \to U_{\lambda_0}(t)x$ *as* $\lambda \to \lambda_0$ *uniformly in* $x \in B$.
Then $\{A_\lambda\}_{\lambda \in \Lambda}$ *is upper semi-continuous at* $\lambda_0 \in \Lambda$.

Proof We have for each $t \geq 0$, and each $\lambda \in \Lambda$

$$\begin{aligned}
\delta\left(A_\lambda, A_{\lambda_0}\right) &= \delta\left(U_\lambda(t) A_\lambda, A_{\lambda_0}\right) \\
&= \sup_{x \in A_\lambda} \inf_{y \in A_{\lambda_0}} d\left(U_\lambda(t) x, y\right) \\
&\leq \sup_{x \in B} \inf_{y \in A_{\lambda_0}} d\left(U_\lambda(t) x, y\right) \\
&\leq \sup_{x \in B} \inf_{y \in A_{\lambda_0}} d\left(U_\lambda(t) x, U_{\lambda_0}(t) x\right) + d\left(U_{\lambda_0}(t) x, y\right),
\end{aligned}$$

so

$$\delta\left(A_\lambda, A_{\lambda_0}\right) \leq \sup_{x \in B} d\left(U_\lambda(t) x, U_{\lambda_0}(t) x\right) + \delta\left(U_{\lambda_0}(t) B, A_{\lambda_0}\right).$$

and the result follows. □

The idea of the following result is impired by Theorem 23.14 p.41 in Sell and You [189]. It seams necessary to make some additionnal continuity assumptions (see assumption c)).

Proposition 2.93 *Let (Λ, d_Λ) be a metric space, let $\lambda_0 \in \Lambda$ be fixed, and let $\{U_\lambda(t)\}_{t\in I}$ be a familly of continuous semiflow on M parametred by $\lambda \in \Lambda$. We assume that for each $\lambda \in \Lambda$, U_λ is asymptically smooth, and there exists A_{λ_0} is a global attractor for U_{λ_0}. We assume in addition that there exists B_0 a bounded subset of M, such that:*
a) A_{λ_0} attracts B_0 for U_{λ_0}.
b) For each $\lambda \in \Lambda$, and each $x \in M$, there exists $t_0 = t_0(\lambda, x) \geq 0$, such that

$$U_\lambda(t)x \in B_0, \forall t \geq t_0.$$

c) For each bounded set $B \subset M$, and each $t^* \in I$, $U_\lambda(t)x \to U_{\lambda_0}(t)x$ as $\lambda \to \lambda_0$ uniformly in $(t,x) \in ([0,t^*] \cap I) \times B$.
Then there exists $\eta > 0$, such that for all $\lambda \in \overline{B}_\Lambda(\lambda_0, \eta)$, $U_\lambda(t)$ has a global attractors $A_\lambda \subset M$, and the family $\{A_\lambda\}_{\lambda \in \overline{B}_\Lambda(\lambda_0,\eta)}$ is upper semi-continuous at $\lambda_0 \in \Lambda$.

Proof For each $\lambda \in \Lambda$, we set

$$J_\lambda(B_0) := \{y \in B_0 : U_\lambda(t)(y) \in B_0, \forall t \in I\}.$$

Then $J_\lambda(B_0) \subset B_0$ is is bounded, and positively invariant for U_λ, and for each $x \in M$, there exists $t \geq 0$, such that $U_\lambda(t)(x) \in J_\lambda(B_0)$. Since U_λ is asymptotically smooth, we deduce that $\omega_\lambda(J_\lambda(B_0)) = \bigcap_{s\geq 0} \overline{\bigcup_{t \geq s} U_\lambda(t)(J_\lambda(B_0))}$ is compact, invariant for U_λ, and attracts the point of M for U_λ. By applying Proposition 2.92 (with $A_\lambda = \omega_\lambda(J_\lambda(B_0)), \forall \lambda \in \Lambda \setminus \{\lambda_0\}, B = B_0$), we deduce that

$$\lim_{\lambda \to \lambda_0} \delta\left(\omega_\lambda(J_\lambda(B_0)), A_{\lambda_0}\right) = 0. \tag{2.1}$$

Let $\varepsilon > 0$ be fixed. Since A_{λ_0} is a global attractors for U_{λ_0}, it is follows that A_{λ_0} is stable for U_{λ_0}, and A_{λ_0} attracts one of its neigborhood for U_{λ_0}. So there exists $\widehat{\varepsilon} \in (0, \varepsilon)$, such that A_{λ_0} attracts $\overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}\right)$ for U_{λ_0}, and

$$U_{\lambda_0}(t)\overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}\right) \subset \overline{N}\left(A_{\lambda_0}, \varepsilon/4\right), \forall t \geq 0.$$

and

$$U_{\lambda_0}(t^*)\overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}\right) \subset \overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}/4\right).$$

By using assumption c), we deduce that there exists $\eta_1 > 0$, such that for each $\lambda \in \Lambda$, with $d_\Lambda(\lambda_0, \lambda) \leq \eta_1$,

$$U_\lambda(t)\overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}\right) \subset \overline{N}\left(A_{\lambda_0}, \varepsilon/2\right), \forall t \in [0, t^*] \cap I,$$

and

$$U_\lambda(t^*)\overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}\right) \subset \overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}/2\right).$$

We set for each $\lambda \in \Lambda$, with $d_\Lambda(\lambda_0, \lambda) \leq \eta_1$,

$$B_\lambda = \overline{\bigcup_{t\in[0,t^*]} U_\lambda(t)\overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}\right)},$$

then B_λ is positively invariant for U_λ, and $\overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}\right) \subset B_\lambda \subset \overline{N}\left(A_{\lambda_0}, \varepsilon/2\right)$. Moreover by using (2.1) there exists $\eta \in (0, \eta_1)$, such that $\omega_\lambda\left(J_\lambda(B_0)\right) \subset \overline{N}\left(A_{\lambda_0}, \widehat{\varepsilon}/2\right)$, for each $\lambda \in \Lambda$, with $d_\Lambda\left(\lambda_0, \lambda\right) \leq \eta$. Then for each $\lambda \in \Lambda$, with $d_\Lambda\left(\lambda_0, \lambda\right) \leq \eta$, B_λ attracts the compact subset of M, and $A_\lambda = \omega_\lambda\left(B_\lambda\right)$ is a global attractors for $U_\lambda(t)$. Finally since $B_\lambda \subset \overline{N}\left(A_{\lambda_0}, \varepsilon/2\right)$, we deduce that

$$A_\lambda \subset \overline{N}\left(A_{\lambda_0}, \varepsilon/2\right),$$

and the result follows. □

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Chapter 3
Bifurcations

This chapter is concerned with bifurcation theory, that roughly consists in the study of the topological changes of the solutions of some equations as some parameters are varying. Along this chapter we shall mainly focus on bifurcation arising when a simple eigenvalue crosses zero, typically yielding the appearance (or the disappearance) of branches of solutions. The results presented below are mostly based on the implicit function theorem, recalled in Section 3.1. This theorem is used to study local branches of the solutions and is applies to state and prove the so-called Hopf-bifurcation theorem for systems ordinary differential equations.

3.1 Implicit Function Theorem

Let $(X, \|\cdot\|)$ be a Banach space. Throughout this chapter the closed ball centered at $x \in X$ with radius $\delta \geq 0$, denoted by $\overline{B}_X(x, \delta)$, is defined by

$$\overline{B}_X(x, \delta) := \{y \in X : \ \|x - y\| \leq \delta\}$$

while the open ball centered at $x \in X$ with radius $\delta > 0$ is denoted by $B_X(x, \delta)$ and defined by

$$B_X(x, \delta) := \{y \in X : \ \|x - y\| < \delta\}.$$

We start this chapter by recalling the following result that will be used to prove the implicit function theorem below.

Theorem 3.1 *Let $(X, \|\cdot\|)$ be a Banach space. Let $0 \leq \kappa < 1$ and $\delta > 0$. Then the following properties hold*

(i) **(Global inverse)** *If $T : X \to X$ is a $\kappa-$ Lipschitz continuous map, then the map $I - T$ is a invertible and $(I - T)^{-1}$ is $(1 - \kappa)^{-1}-$Lipschitz continuous.*

(ii) **(Local inverse)** *If $T : \overline{B}_X(0, \delta) \to X$ is a $\kappa-$Lipschitz continuous map and $\|T(0)\| < \delta(1 - \kappa)$, then*

$$\overline{B}_X(0, \varrho) \subset (I - T)\left(\overline{B}_X(0, \delta)\right),$$

when

$$\varrho := (1-\kappa)\delta - \|T(0)\|.$$

Moreover there exists a unique map $S : \overline{B}_X(0,\varrho) \to \overline{B}_X(0,\delta)$ *which is* $(1-\kappa)^{-1}$*–Lipschitz continuous and that satisfies*

$$(I-T)Sy = y, \ \forall y \in \overline{B}_X(0,\varrho). \tag{3.1}$$

Proof **Proof of** (i)**:** Let $y \in X$. Then we consider the fixed point problem

$$x = Tx + y. \tag{3.2}$$

Note that the Banach fixed point theorem applied to the map κ–contraction map $\Phi : X \to X$ defined by $\Phi(x) = Tx + y$ (namely by using the iteration procedure $x_{n+1} = Tx_n + y$). Hence we deduce that (3.2) has a unique solution Sy satisfying

$$Sy = TSy + y \Leftrightarrow (I-T)\,Sy = y.$$

We conclude, $(I-T)$ is onto (i.e. $(I-T)(X) = X$), for each $y \in X$, there exists a unique $x = S(y)$ satisfying $(I-T)Sy = y$.

Next we observe that, for each $x \in X$, we can fix $y = x - Tx$, by uniqueness of the fixed point problem (3.2), we deduce

$$Sy = x.$$

So the map $S : X \to X$ is onto (i.e. $S(X) = X$).

Moreover, by applying S on both sides of $(I-T)Sy = y$, and by setting $z = Sy$, we deduce that

$$S(I-T)z = z, \forall z \in X.$$

We conclude that $I - T : X \to X$ is invertible, and its inverse is S. Next by (3.2), we observe that we have, for any $y, \hat{y} \in X$,

$$Sy - S\hat{y} = TSy + y - (TS\hat{y} + \hat{y})$$

thus

$$\|Sy - S\hat{y}\| \le \kappa\|Sy - S\hat{y}\| + \|y - \hat{y}\| \ \Leftrightarrow \ \|Sy - S\hat{y}\| \le (1-\kappa)^{-1}\|y - \hat{y}\|,$$

that completes the proof of (i).

Proof of (ii)**:** Let us fix $y \in \overline{B}_X(0,\varrho)$, and consider the map $\Phi : \overline{B}_X(0,\delta) \to X$ given by

$$\Phi(x) = Tx + y, \ \forall x \in \overline{B}_X(0,\delta).$$

Then for each $x \in \overline{B}_X(0,\delta)$, one has

$$\begin{aligned}\|\Phi(x)\| &\leq \|Tx\| + \|y\| \\ &\leq \|Tx - T(0)\| + \|T(0)\| + \varrho \\ &\leq \kappa\|x\| + \|T(0)\| + \varrho\end{aligned}$$

hence

$$\|\Phi(x)\| \leq \kappa\delta + \|T(0)\| + \varrho = \kappa\delta + \|T(0)\| + (1-\kappa)\delta - \|T(0)\| = \delta.$$

Therefore we deduce that

$$\Phi\left(\overline{B}_X(0,\delta)\right) \subset \overline{B}_X(0,\delta).$$

Moreover, since Φ is κ–Lipschitz continuous, there exists a unique $x \in \overline{B}_X(0,\delta)$ such that

$$\Phi(x) = x \;\Leftrightarrow\; x - T(x) = y.$$

Denoting by $S : \overline{B}_X(0,\varrho) \to \overline{B}_X(0,\delta)$ the map defined by the resolution of the above equation (i.e. $Sy = x$), one may observe that it satisfies (3.1). Moreover, by using the same argument as for the proof for (i), S is also $(1-\kappa)^{-1}$–Lipschitz continuous. □

We now turn to the statement of the implicit function theorem, that will be extensively used along this chapter.

Theorem 3.2 (Implicit function theorem) *Let X, Y and Z be three given Banach spaces. Let $U \subset X$ and $V \subset Y$ be two open sets. We assume that $F : U \times V \to Z$ is continuous function, which is differentiable with respect to the second variable y and the map $(x, y) \to \partial_y F(x, y) \in \mathcal{L}(Y, Z)$ is continuous. We assume that there exists $(x_0, y_0) \in U \times V$ such that*

$$F(x_0, y_0) = 0_Z.$$

We further assume that $\partial_y F(x_0, y_0) \in \mathcal{L}(Y, Z)$ is one to one and onto and its inverse is a bounded linear operator, that is

$$\partial_y F(x_0, y_0)^{-1} \in \mathcal{L}(Z, Y).$$

Then there exist $r > 0$, $\delta > 0$ with $\overline{B}_X(x_0, r) \subset U$ and $\overline{B}_Y(y_0, \delta) \subset V$ and there exists a unique continuous map $S : \overline{B}_X(x_0, r) \to \overline{B}_Y(y_0, \delta)$ with $Sx_0 = y_0$ such that

$$F(x, Sx) = 0_Z, \;\forall x \in \overline{B}_X(x_0, r).$$

Proof Without loss of generality we can assume that $x_0 = 0_X$ and $y_0 = 0_Y$. We set $L = \partial_y F(0, 0) \in \mathcal{L}(Y, Z)$. We observe that

$$F(x, y) = 0_Z \Leftrightarrow Ly - F(x, y) = Ly \Leftrightarrow y - L^{-1}F(x, y) = y \Leftrightarrow y - (y - L^{-1}F(x, y)) = 0_Y,$$

and we set

$$\Lambda(x, y) := y - L^{-1}F(x, y).$$

So we will apply the local invertibility result Theorem 3.1-(ii) to find the zeros of

$$y - \Lambda(x, y) = 0_Y.$$

So, let us prove that Λ satisfies the assumptions of (ii) in Theorem 3.1. We observe that

$$\partial_y \Lambda(x, y) = I - L^{-1}\partial_y F(x, y).$$

One deduces that

$$\partial_y \Lambda(0_X, 0_Y) = I - L^{-1}\partial_y F(0_X, 0_Y) = 0_{\mathcal{L}(Y)}.$$

Let $\kappa \in (0, 1)$ be fixed. Since the map $(x, y) \to \partial_y F(x, y)$ is continuous, we can find $\delta > 0$ and $r_0 > 0$ such that

$$\|\partial_y \Lambda(x, y)\|_{\mathcal{L}(Y)} \leq \kappa, \ \forall y \in \overline{B}_Y(0_Y, \delta), \ \forall x \in \overline{B}_X(0_X, r_0).$$

By using the formula (obtained by taking the derivative of the function $s \to \Lambda(x, y + s(\hat{y} - y))$)

$$\Lambda(x, \hat{y}) - \Lambda(x, y) = \int_0^1 \partial_y \Lambda(x, y + s(\hat{y} - y))(\hat{y} - y)\mathrm{d}s,$$

we deduce that

$$\|\Lambda(x, \hat{y}) - \Lambda(x, y)\| \leq \kappa\|\hat{y} - y\|, \ \forall y \in \overline{B}_Y(0_Y, \delta), \ \forall x \in \overline{B}_X(0_X, r_0).$$

Moreover one has

$$\Lambda(0_X, 0_Y) = 0_Y - L^{-1}F(0_X, 0_Y) = 0_Y,$$

so that due to the continuity of F, one can find $r \in (0, r_0)$ such that

$$\|\Lambda(x, 0_Y)\| \leq (1 - \kappa)\delta, \ \forall x \in \overline{B}_X(0_X, r).$$

Now let $x \in \overline{B}_X(0_X, r)$ be given. By applying Theorem 3.1-(ii) to the map $T : \overline{B}_Y(0_Y, \delta) \to Y$, given by $Ty = \Lambda(x, y)$, it follows that for each $x \in \overline{B}_X(0_X, r)$, there exists a unique $y \in B_Y(0_Y, \delta)$, such that

$$y - \Lambda(x, y) = 0_Y.$$

Set

$$Sx = y.$$

By using the uniqueness result in Theorem 3.1-(ii), we deduce that $Sx_0 = y_0$. It remains to prove the continuity of S. To that aim we fix $x \in \overline{B}_X(0_X, r)$ and we vary $\widehat{x} \in \overline{B}_X(0_X, r)$, and we observe that

$$S(\widehat{x}) - S(x) = \Lambda(\widehat{x}, S(\widehat{x})) - \Lambda(\widehat{x}, S(x)) + \Lambda(\widehat{x}, S(x)) - \Lambda(x, S(x))$$

hence

$$\|S(\widehat{x}) - S(x)\| \leq \kappa \|S(\widehat{x}) - S(x)\| + \|\Lambda(\widehat{x}, S(x)) - \Lambda(x, S(x))\|$$

therefore

$$\|S(\widehat{x}) - S(x)\| \leq (1-\kappa)^{-1} \|\Lambda(\widehat{x}, S(x)) - \Lambda(x, S(x))\|,$$

and by using the continuity of the map $\widehat{x} \to \Lambda(\widehat{x}, S(x))$, we deduce that $S(\widehat{x}) \to S(x)$ as $\widehat{x} \to x$. □

3.2 Bifurcating Branch of Equilibria: Finite Dimensional Case

In this section, we discuss some result to describe local branches of solutions for some finite dimensional equations. We start by discussing two simple examples.

Example 3.3 (Logistic equation) The equilibria of the logistic equation satisfy a quadratic equation of the form

$$u(\lambda \kappa - u) = 0$$

for some given and fixed parameter $\kappa \neq 0$ while $\lambda \in \mathbb{R}$ is a varying parameter. Then the solutions of the above equation are given by the two branches of solutions

$$u = 0 \text{ and } u = \lambda \kappa.$$

The two branches of solutions cross each other at $\lambda = 0$. This special point is called a bifurcation point for the branch of trivial solution $u = 0$ in the plane (λ, u).

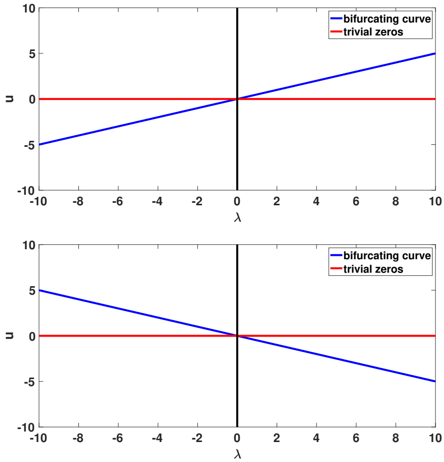

Fig. 3.1: *The figure on the top and the figure on the bottom correspond respectively to* $\kappa = 0.5$ *and* $\kappa = -0.5$.

Example 3.4 (Bernoulli-Verhulst equation) We consider now the equation of the form

$$U(\lambda\kappa - U^2) = 0$$

with $\kappa \neq 0$. Then the solution are given by

$$U = 0 \text{ and } \lambda\kappa = U^2.$$

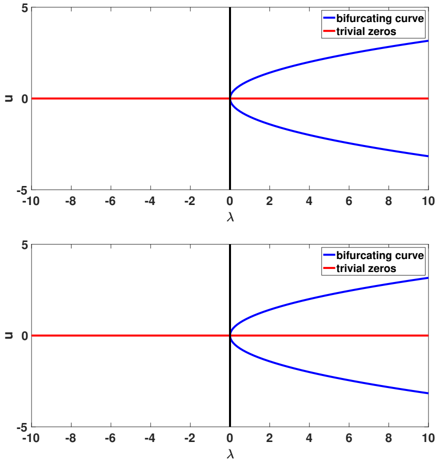

Fig. 3.2: *The figure on the top and the figure on the bottom correspond respectively to* $\kappa = 1$ *and* $\kappa = -1$.

Theorem 3.5 (Bifurcating Branch of Equilibria) *Let* $e \in \mathbb{R}^n$ *with* $e \neq 0_{\mathbb{R}^n}$. *Let* I *be an open interval in* $\mathbb{R}$ *and let* $B_{\mathbb{R}^n}(0, r)$ *be the open ball of* $\mathbb{R}^n$ *centered* $0_{\mathbb{R}^n}$ *with radius* $r > 0$. *Let* $\lambda_0 \in I$ *and* $F : I \times B_{\mathbb{R}^n}(0_{\mathbb{R}^n}, r) \to \mathbb{R}^n$ *a two times continuously differentiable map satisfying*

(i) $F(\lambda, 0_{\mathbb{R}^n}) = 0_{\mathbb{R}^n}, \ \forall \lambda \in I$;

(ii) $\mathrm{N}(\partial_x F(\lambda_0, 0_{\mathbb{R}^n})) = \mathbb{R}e$, *where* $\mathbb{R}e := \{\lambda e : \lambda \in \mathbb{R}\}$;

(iii) $\partial_\lambda \partial_x F(\lambda_0, 0_{\mathbb{R}^n}) e \notin \mathrm{R}(\partial_x F(\lambda_0, 0_{\mathbb{R}^n}))$.

Then there exist a constant $\delta > 0$ *and a continuous function* $(\lambda^*, x^*) : (-\delta, \delta) \subset \mathbb{R} \to I \times \mathbb{R}^n$ *satisfying*

$$\lambda^*(0_{\mathbb{R}}) = \lambda_0, x^*(0_{\mathbb{R}}) = 0_{\mathbb{R}^n} \text{ and } x^*(s) \neq 0_{\mathbb{R}^n}, \ \forall s \neq 0_{\mathbb{R}},$$

and

$$F(\lambda^*(s), x^*(s)) = 0_{\mathbb{R}^n}, \ \forall s \in (-\delta, \delta).$$

Moreover the curve $s \to x^*(s)$ *is tangent to* $\mathbb{R}e$ *at* $s = 0$, *that is to say that*

$$\lim_{s \to 0} \frac{x^*(s)}{s} = e.$$

Furthermore, there is a neighborhood of $(\lambda_0, 0)$ *such that any zero of* F *either belongs to this curve or has the form* $(\lambda, 0)$.

Proof Let Z denotes a complement space of $\mathbb{R}e$ in $\mathbb{R}^n$, namely

$$\mathbb{R}e \oplus Z = \mathbb{R}^n.$$

Fix $\delta > 0$ and $\varrho > 0$ such that $\delta(\|e\| + \varrho) \leq r$ and let $G : B_{\mathbb{R}}(0, \delta) \times I \times B_Z(0, \varrho) \to \mathbb{R}^n$ be the map defined by

$$G(s, \lambda, z) = \begin{cases} \dfrac{F(\lambda, s(e + z))}{s}, & \text{if } s \neq 0, \\ \partial_x F(\lambda, 0)(e + z), & \text{if } s = 0, \end{cases}$$

whenever $(s, \lambda, z) \in B_{\mathbb{R}}(0, \delta) \times I \times B_Z(0, \varrho)$.

Since F is of class C^2, it readily follows that G is of class C^1. Next note that one has $G(0, \lambda_0, 0) = 0$ and

$$\partial_{(\lambda,z)} G(0, \lambda_0, 0)(\lambda, z) = \partial_\lambda G(0, \lambda_0, 0)\lambda + \partial_z G(0, \lambda_0, 0)z,$$

hence

$$\partial_{(\lambda,z)} G(0, \lambda_0, 0)(\lambda, z) = \lambda\, \partial_\lambda \partial_x F(\lambda_0, 0)e + \partial_x F(\lambda_0, 0)z.$$

Now let us prove that the operator $L : (\lambda, z) \to \partial_{(\lambda,z)} G(0, \lambda_0, 0)(\lambda, z)$ is invertible from $\mathbb{R} \times Z$ into $\mathbb{R}^n$. Note that since $\dim(Z) = n - 1$, it is sufficient to prove that $\mathrm{N}(L) = 0$. To that aim we argue by contradiction by assuming that $\mathrm{N}(L) \neq 0$. Let $(\lambda, z) \in \mathrm{N}(L) \setminus \{0\}$ be given. Then one has

$$\lambda\, \partial_\lambda \partial_x F(\lambda_0, 0)e = -\partial_x F(\lambda_0, 0)z \in \mathrm{R}(\partial_x F(\lambda_0, 0)).$$

Due to (iii) this implies that $(\lambda, z) = (0, 0)$, a contradiction that ensures the invertibility of L.

Now since

$$G(0, \lambda_0, 0) = \partial_x F(\lambda_0, 0)e = 0,$$

the implicit function theorem applies to the function G (with $x = s$ and $y = (\lambda, z)$) and we deduce that there exist $\delta_0 > 0$ sufficiently small and a continuous map $(\lambda^*, z^*) : (-\delta_0, \delta_0) \to I \times Z$ with $\lambda^*(0) = \lambda_0$ and $z^*(0) = 0$ and satisfying

$$F(\lambda^*(s), s(e + z^*(s))) = 0, \ \forall s \in (-\delta_0, \delta_0).$$

The result follows by setting $x^*(s) = s(e + z^*(s))$. □

3.3 Application to the Scalar Generalized Logistic Equation

Consider a one dimensional logistic equation

$$U'(t) = \lambda\,\kappa\,U(t) - U(t)^m, \text{ for } t \geq 0, \; U(0) = U_0 \geq 0,$$

where $\kappa \in \mathbb{R} \setminus \{0\}$ is a fixed positive parameter, while $\lambda \in \mathbb{R}$ is a varying parameter and $m \geq 2$ is an integer. Then the equilibria correspond to the zeros of the function

$$F(\lambda, U) = \lambda\,\kappa\,U - U^m.$$

First we have

$$F(\lambda, 0) = 0, \; \forall \lambda \in \mathbb{R},$$

while

$$\partial_U F(\lambda, U) = \lambda\,\kappa - mU^{m-1},$$

hence

$$\partial_U F(\lambda, 0) = \lambda\,\kappa.$$

Set $\lambda_0 = 0$ so that we get

$$\partial_U F(0, 0) = 0.$$

If we regard the partial derivative as a linear map $\partial_U F(0,0) : U \to \partial_U F(0,0)U$ from $\mathbb{R}$ into itself, one has

$$\dim(\mathrm{N}(\partial_U F(0,0))) = \dim(\mathbb{R}) = 1 \text{ and } \mathrm{R}\,(\partial_U F(0,0)) = \{0_{\mathbb{R}}\}.$$

Furthermore one also has

$$\partial_\lambda \partial_U F(0,0) = \kappa,$$

and the linear map $e \to \partial_\lambda \partial_U F(0,0)e \in \mathcal{L}(\mathbb{R}, \mathbb{R})$ satisfies for each $e \neq 0$,

$$\partial_\lambda \partial_U F(0,0)e \notin \mathrm{R}(\partial_U F(0,0)).$$

So we can apply Theorem 3.5 and we find a branch of equilibrium. Actually, this branch of equilibrium is obtained as

$$\lambda(s) = s^{n-1}/\kappa^{-1} \text{ and } U(s) = s.$$

Remark 3.6 From this example, the condition (iii) of Theorem 3.5 corresponds which to $\kappa \neq 0$. We can see that this assumption is needed.

Remark 3.7 The cases $m = 2$ and $m = 3$ provide different type of bifurcating branches. We may observe that Theorem 3.5 is not informing us about the shape of the bifurcating branches.

3.4 Application to the $n-$Dimensional Logistic Equation

Consider the $n-$dimensional logistic equation

$$\begin{cases} U_i'(t) = \sum_{j=1}^n A_{ij}(\lambda)U_j(t) - \sum_{j=1}^n \kappa_{ij} U_j(t)U_i(t), \ t \geq 0, \\ U_i(0) = U_{0i} \geq 0. \end{cases} \tag{3.3}$$

We assume that

$$\kappa_{ij} > 0, \text{ for all } i, j = 1, \dots, n,$$

and

$$\lambda \in (\lambda_0 - \delta_0, \lambda_0 + \delta_0), \text{ for some } \delta_0 > 0 \text{ and } \lambda_0 \in \mathbb{R}.$$

Assumption 3.8 We assume that the function $\lambda \to A(\lambda) = \left(A_{ij}(\lambda)\right)_{i,j=1,\dots,n}$ is twice continuously derivable and satisfies the following properties

(i) There exists a vector $e \in \mathbb{R}^n$ such that $e \gg 0$ and $\mathrm{N}(A(\lambda_0)) = \mathbb{R}e$;

(ii) $\partial_\lambda A(\lambda_0)e \notin \mathrm{R}(A(\lambda_0))$.

Now the equilibria of the $n-$dimensional logistic equation above correspond to the zeros of the map $F : I \times \mathbb{R}^n \to \mathbb{R}^n$, with $I = (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$, given by

$$F_i(\lambda, U)_i = \sum_{j=1}^n A_{ij}(\lambda)U_j - \sum_{j=1}^n \kappa_{ij} U_j U_i, \ i = 1, \dots, n. \tag{3.4}$$

Next the following result holds.

Theorem 3.9 *Let Assumption 3.8 be satisfied. Then there exist some constant $\delta > 0$ and a continuous function $(\lambda^*, U^*) : (-\delta, \delta) \to I \times \mathbb{R}^n$ satisfying*

$$\lambda^*(0) = \lambda_0 \text{ and } U^*(0) = 0,$$
$$U^*(s) \neq 0, \forall s \in (-\delta, \delta) \setminus \{0\},$$

and

$$F(\lambda^*(s), U^*(s))) = 0, \forall s \in (-\delta, \delta).$$

Moreover for $|s|$ small enough

$$\begin{cases} U^*(s) \gg 0, \text{ if } s > 0, \\ U^*(s) \ll 0, \text{ if } s < 0. \end{cases} \tag{3.5}$$

Furthermore, there is a neighbourhood of $(\lambda_0, 0)$ such that any zero of F either belongs to this curve or belongs to the curve $(\lambda, 0)$.

Proof In order to apply the Theorem 3.5 it is sufficient to observe that

$$\partial_x F(\lambda, 0) = A(\lambda_0).$$

Moreover since the curve $s \to U^*(s)$ is tangent to $\mathbb{R}e$ at $s = 0$ and since $e \gg 0$ we deduce (3.5). □

Example 3.10 If we consider for example the matrix

$$A(\lambda) = \begin{pmatrix} -\pi + d_1\lambda & \beta \\ \pi & -\beta + d_2\lambda \end{pmatrix}$$

where $\pi > 0$, $d_1 > 0$, $d_2 > 0$ and $\beta > 0$. Then for $\lambda = 0$, we have $\mathrm{N}(A(0)) = \mathbb{R}e$ with

$$e = \begin{pmatrix} \beta \\ \pi \end{pmatrix}.$$

Moreover

$$A(\lambda)e = \lambda \begin{pmatrix} d_1\beta \\ d_2\pi \end{pmatrix} = \lambda De,$$

where $D := \mathrm{Diag}(d_1, d_2)$ and

$$\partial_\lambda A(\lambda)e = De.$$

Moreover

$$A(0)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1\pi \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \alpha_2\beta \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (\alpha_1\pi - \alpha_2\beta)\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since $d_1\beta > 0$ and $d_2\pi > 0$, we deduce that

$$De \neq \gamma \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \forall \gamma \in \mathbb{R},$$

which implies that

$$\partial_\lambda A(\lambda)e \notin \mathrm{R}(A(\lambda_0)).$$

For this two dimensional matrix example, the zeros of the corresponding two dimensional logistic system reads as the following algebraic system of equations

$$\begin{cases} 0 = (-\pi + d_1\lambda)U_1 + \beta U_2 - (\kappa_{11}U_1 + \kappa_{12}U_2)U_1 \\ 0 = \pi U_1 + (-\beta + d_2\lambda)U_2 - (\kappa_{21}U_1 + \kappa_{22}U_2)U_2 \end{cases}$$

for which we can no longer compute the equilibria explicitly. Therefore Theorem 3.9 is particularly useful when no simplification arises.

3.5 Bifurcating Branch of Equilibria: Infinite Dimensional Case

In order to prove an infinite dimensional version of Theorem 3.5 we will need the following result (see Brezis [22, Corollary 2.7 p. 35]).

Theorem 3.11 *Let X and Y be two Banach spaces and let L be a continuous linear operator from X into Y that is bijective, namely one-to-one and onto. Then L^{-1} is also continuous from Y into X.*

The infinite dimensional version of Theorem 3.5 reads as follows.

Theorem 3.12 *[Bifurcating Branch of Equilibria] Let I be an open interval in $\mathbb{R}$ and let X and Y be two Banach spaces. Let $\lambda_0 \in I$, $r > 0$ and $F : I \times B_X(0,r) \to Y$ a twice continuously differentiable map satisfying*

(i) $F(\lambda, 0_X) = 0_Y$, $\forall \lambda \in I$;

(ii) $\dim(\mathrm{N}(\partial_x F(\lambda_0, 0_X))) = 1$;

(iii) $\partial_\lambda \partial_x F(\lambda_0, 0_X)e \notin \mathrm{R}(\partial_x F(\lambda_0, 0_X))$ *for some* $e \in \mathrm{N}(\partial_x F(\lambda_0, 0_X)) \setminus \{0\}$.

(iv) $\mathrm{codim}\mathrm{R}(\partial_x F(\lambda_0, 0_X)) = 1$ *which is equivalent to say that there exists* $v \in Y$ *such that*

$$Y = \mathbb{R}v \oplus \mathrm{R}(\partial_x F(\lambda_0, 0_X))$$

whenever $v \notin \mathrm{R}(\partial_x F(\lambda_0, 0_X))$ *(which implies that* $v \neq 0$*).*

Then the conclusions of Theorem 3.5 hold.

Proof Let Z be a closed complement space of $\mathbb{R}e$ in X (such a subspace exists due to the Analytic form of the Hahn-Banach theorem exists see [22, Example p. 38])). That is to say that

$$X = \mathbb{R}e \oplus Z.$$

Consider the map $B : \mathbb{R} \times Z \to Y$ given by

$$B(\lambda, z) = \lambda \hat{e} + L_0 z,$$

wherein we have set $\hat{e} := \partial_\lambda \partial_x F(\lambda_0, 0)e$ and $L_0 = \partial_x F(\lambda_0, 0)$. Then in order to prove the theorem, it is sufficient to prove that B is a bijection from $\hat{X} := \mathbb{R} \times Z$ onto Y. To do so we first observe that since $\mathrm{codim}\, \mathrm{R}(L_0) = 1$ and $\hat{e} \notin \mathrm{R}(L_0)$ we have

$$Y = \mathbb{R}\hat{e} \oplus \mathrm{R}(L_0).$$

Now since $X = \mathbb{R}e \oplus Z$ one deduces that $\mathrm{R}(L_0) = L_0(Z)$ and B is surjective. Next by using again the fact that $\hat{e} \notin \mathrm{R}(L_0)$ and $z \in Z$ we deduce that

$$B(\lambda, z) = 0 \;\Leftrightarrow\; \lambda \hat{e} + L_0 z = 0 \;\Leftrightarrow\; \lambda = 0_{\mathbb{R}} \text{ and } z = 0_X.$$

Finally the invertiblity is B follows from Theorem 3.11. Finally the result follows by using the same arguments as in the proof of Theorem 3.5.

3.6 Example of the Poincaré normal form and a first Hopf bifurcation Theorem

In order to understand the idea of Hopf bifurcation theorem (see Hopf [111]), we first consider the so called Poincaré normal form. This normal form was introduced by Poincaré in its memoir celestial mechanics [176] in 1893. This system is the following

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \kappa \left(x(t)^2 + y(t)^2 \right) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{3.6}$$

where the bifurcation parameter α varies from negative values to positive values, while the parameters

$$\omega \neq 0 \text{ and } \kappa \neq 0$$

are fixed.

Complex number reformulation: Set

$$z(t) = x(t) + i\, y(t).$$

Then

$$z'(t) = x'(t) + i\, y'(t) = \alpha\,(x + i\, y) - \omega y + \omega\, i\, x + \kappa\, |z|^2 z = \alpha\,(x + i\, y) + i\omega\,(x + i\, y) + \kappa\, |z|^2 z$$

and we obtain the Poincaré's normal form

$$z'(t) = \lambda\, z + \kappa\, |z|^2 z, \text{ and } z(0) = z_0 \in \mathbb{C}, \tag{3.7}$$

where

$$\lambda = \alpha + i\omega.$$

The equation (3.7) is nothing but a complex value Bernoulli-Verhulst's equation. In order to derive an explicite formula for the solution we first set

$$\widehat{z}(t) = e^{-i\,\omega t} z(t).$$

We obtain

$$\widehat{z}'(t) = -i\,\omega \widehat{z} + e^{-i\,\omega t} z'(t),$$

it follows that

$$\widehat{z}'(t) = \alpha \widehat{z} + \kappa\, |\widehat{z}|^2 \widehat{z}, \text{ and } z(0) = z_0 \in \mathbb{C}. \tag{3.8}$$

We are now in position to use the explicit formula for the Bernoulli-Verhulst's equation, and we deduce that the solution of equation (3.7) is given by the explicit formula

$$z(t) = \frac{e^{\lambda t} z_0}{\left(1 + 2\kappa \int_0^t e^{2\,\mathrm{Re}(\lambda)\,\sigma} |z_0|^2 d\sigma\right)^{\frac{1}{2}}}. \tag{3.9}$$

Moreover if we set $\rho(t) = |z|^2$, then we obtain

$$\rho(t)' = (z\overline{z})' = z'\overline{z} + z\overline{z}' = 2\,\mathrm{Re}(\lambda)|z|^2 + 2\,\kappa\,|z|^4$$

which is a logistic equation

$$\rho' = 2\,\mathrm{Re}(\lambda)\,\rho + 2\,\kappa\,\rho^2. \tag{3.10}$$

The equilibrium for equation (3.10)

$$\overline{\rho} = -\frac{\mathrm{Re}(\lambda)}{\kappa}.$$

So the equilibrium $\overline{\rho}$ is strictly positive if and only if $\mathrm{Re}(\lambda)\,\kappa < 0$. Moreover in order to understand the stability of the equilibrium $\overline{\rho}$, we can make the change of variable $x = \rho - \overline{\rho}$ and we obtain

$$x' = 2\kappa(x + \overline{\rho})x.$$

Now the equilibrium 0 of the above equation is stable if and only if $\kappa < 0$ and unstable if $\kappa > 0$. We summarize this in the following theorem.

Theorem 3.13 *The system* (3.6) *has a periodic orbit if and only if*

$$\mathrm{Re}(\lambda)\,\kappa < 0. \tag{3.11}$$

Moreover the periodic orbit (when it exists) is unique. Furthermore, the periodic orbit (when it exists) is locally asymptotically stable only if $\kappa < 0$ *and unstable if* $\kappa > 0$.

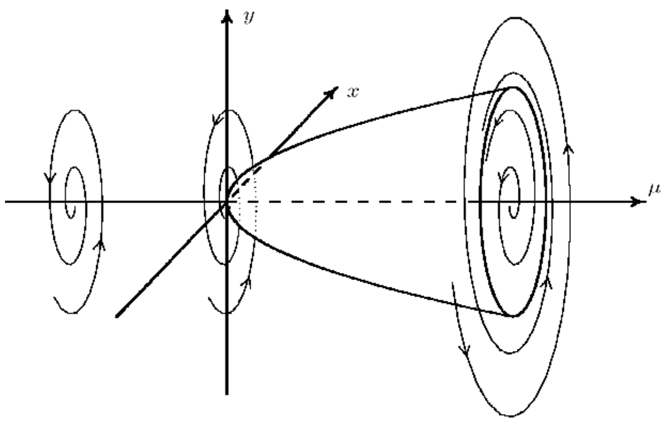

Fig. 3.3: *When* $\kappa < 0$ *we use* $\mu := \alpha = \mathrm{Re}(\lambda)$ *as a bifurcation parameter, and when* μ *passes through* 0 *a stable periodic orbit is appearing. The case* $\kappa > 0$ *can be understood from the case* $\kappa < 0$ *by going backward in time; that is, by considering* $\hat{x}(t) := x(-t)$ *and* $\hat{y}(t) := y(-t)$. *When* $\kappa > 0$ *we use* $\mu := -\alpha = -\mathrm{Re}(\lambda)$ *as a bifurcation parameter, when* μ *passes through* 0 *an unstable periodic orbit is appearing.*

3.7 Hopf Bifurcation Theorem

Let I be an open interval in $\mathbb{R}$ containing 0 and $r > 0$ be fixed. Let $F : I \times B_{\mathbb{R}^n}(0, r) \to \mathbb{R}^n$ be a continuously differential map. Consider the ordinary differential equation

$$U'(t)x = F(\mu, U(t)x),\ \forall t \geq 0,\ U(0)x = x \in \mathbb{R}^n. \tag{3.12}$$

Before stating the main result of this section, we first describe the assumptions the will be needed.

Assumption 3.14 We assume that $(\mu, x) \to F(\mu, x)$ is twice continuously differentiable on $I \times B_{\mathbb{R}^n}(0, r)$ into $\mathbb{R}^n$ and satisfies the following properties

(i) $F(\mu, 0_{\mathbb{R}^n}) = 0_{\mathbb{R}^n},\ \forall \mu \in I$;

(ii) **(Transversality condition)** For each $\mu \in I$, there exists a pair of conjugated simple eigenvalues of $\partial_x F(\mu, 0)$ denoted by $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ written as

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \text{ with } \alpha(\mu),\ \omega(\mu) \in \mathbb{R}$$

such that the map $\mu \to \lambda(\mu)$ is continuously derivable,

$$\omega(0) > 0,\ \alpha(0) = 0 \text{ and } \frac{\mathrm{d}\alpha(0)}{\mathrm{d}\mu} \neq 0.$$

(iii) The only eigenvalues of $\partial_x F(0, 0)$ which are equal to $i\omega(0)k$ or $-i\omega(0)k$ for some integer $k \in \mathbb{N}$ are $\lambda(0)$, $\overline{\lambda(0)}$. In other words, one has

$$\sigma(\partial_x F(0, 0)) \cap i\omega(0)\mathbb{Z} = \left\{\lambda(0), \overline{\lambda(0)}\right\}. \tag{3.13}$$

Remark 3.15 The condition (3.13) implies $0 \notin \sigma(\partial_x F(0, 0))$.

Theorem 3.16 (Hopf bifurcation) *Let Assumption 3.14 be satisfied. Then there exist $\varepsilon^* > 0$ and continuous maps, $\varepsilon \to \mu(\varepsilon)$ from $[0, \varepsilon^*)$ to $\mathbb{R}$, $\varepsilon \to x(\varepsilon)$ from $[0, \varepsilon^*)$ to $\mathbb{R}^n$, $\varepsilon \to \gamma(\varepsilon)$ from $[0, \varepsilon^*)$ to $\mathbb{R}$, such that for each $\varepsilon \in [0, \varepsilon^*)$ there exists a $\gamma(\varepsilon)$-periodic function u_ε of class C^1 which is a solution of (3.12) with the parameter $\mu = \mu(\varepsilon)$ and with initial condition $u_\varepsilon(0) = x(\varepsilon)$. Moreover the branch of periodic orbit is bifurcating from 0 at $\mu = 0$, that is to say that*

$$\mu(0) = 0,\ x(0) = 0 \text{ and } x(\varepsilon) \neq 0,\ \forall \varepsilon \in (0, \varepsilon^*).$$

Furthermore one also has

$$\gamma(0) = 2\pi\omega(0).$$

3.8 Proof of the Hopf Bifurcation Theorem

Throughout this section we assume that Assumption 3.14 is satisfied and we aim at proving Theorem 3.16. Up to time rescaling we assume without loose of generality that $\omega(0) = 1$. Now to prove the Hopf bifurcation theorem above, we first rewrite the system under a more convenient form. Using a change of basis we can rewrite (3.12) under the following form

$$\begin{cases} \dot{x}_1(t) = B(\mu)x_1(t) + G_1(\mu, x_1(t), x_2(t)) \in \mathbb{R}^2, \\ \dot{x}_2(t) = C(\mu)x_2(t) + G_2(\mu, x_1(t), x_2(t)) \in \mathbb{R}^{n-2}, \end{cases} \tag{3.14}$$

with some initial data

$$(x_1(0), x_2(0)) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}.$$

Moreover the following properties are satisfied:

(i) The functions $\mu \to B(\mu)$, $\mu \to C(\mu)$ are of class C^1;

(ii) The matrix $B(\mu)$ takes the following form $B(\mu) = \begin{pmatrix} \alpha(\mu) & \omega(\mu) \\ -\omega(\mu) & \alpha(\mu) \end{pmatrix}$, with $\alpha(0) = 0$, $\omega(0) = 1$ as well as the transversality condition $\alpha'(0) \neq 0$;

(iii) The matrix $I - e^{2\pi C(0)}$ is invertible from $\mathbb{R}^{n-2}$ into itself;

(iv) The functions G_i for $i = 1, 2$ are of class C^2;

(v) $G_i(\mu, 0, 0) = 0$, for $i = 1, 2$ and for all μ in a small neighboorhood of 0;

(vi) $\partial_{x_i} G_i(\mu, 0, 0) = 0$, for $i = 1, 2$ and for μ in a small neighboorhood of 0.

Remark 3.17 The property (iii) is a consequence of the assumption (3.13). Indeed, let P an invertible matrix such that

$$J = P^{-1}C(0)P = \begin{pmatrix} J_{\lambda_1} & 0 & \dots & 0 \\ 0 & J_{\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{\lambda_m} \end{pmatrix}$$

is a Jordan decomposition of $C(0)$. We have

$$e^{2\pi J} = \begin{pmatrix} e^{2\pi J_{\lambda_1}} & 0 & \dots & 0 \\ 0 & e^{2\pi J_{\lambda_2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{2\pi J_{\lambda_m}} \end{pmatrix},$$

and

$$x - e^{2\pi C(0)}x = 0 \Leftrightarrow x = e^{2\pi C(0)}x \Leftrightarrow P^{-1}x = e^{2\pi J}P^{-1}x.$$

Therefore

$$N\left(I - e^{2\pi C(0)}\right) = \{0\} \Leftrightarrow N\left(I - e^{2\pi J_{\lambda_i}}\right) = \{0\}, \forall i = 1, \dots, m \Leftrightarrow e^{2\pi\lambda_i} \neq 1, \forall i = 1, \dots, m.$$

The system (3.14) is equivalent to

$$\frac{dx(t)}{dt} = A(\mu)x + G(\mu, x), t > 0, x \in \mathbb{R}^n,$$

with some initial value $x(0) \in \mathbb{R}^n$ and

$$A(\mu) = \begin{pmatrix} B(\mu) & 0 \\ 0 & C(\mu) \end{pmatrix}, \; G(\mu, x) = \begin{pmatrix} G_1(\mu, x_1, x_2) & 0 \\ 0 & G_2(\mu, x_1, x_2) \end{pmatrix}.$$

In order to look for nontrivial $2\pi\gamma-$periodic solutions, we consider the rescaled function

$$\hat{x}(t) := x(\gamma t),$$

so that, omitting the hat for the sake of simplicity, the system becomes

$$x'(t) = \gamma A(\mu)x + \gamma G(\mu, x), \tag{3.15}$$

and we look for $2\pi-$periodic solution of the above system. A solution $t \longrightarrow x(t)$ is on some time interval $[0, 2\pi]$ satisfies

$$x(t) = e^{\gamma A(\mu)t}x(0) + \gamma \int_0^t e^{\gamma A(\mu)(t-s)}G(\mu, x(s))\mathrm{d}s, \; \forall t \in [0, 2\pi] \,.$$

In order to investigate the existence of $2\pi-$periodic solution of the above integral integral equation, Let us define the map $F : \mathbb{R}^2 \times C_{2\pi}([0, 2\pi]; \mathbb{R}^n) \longrightarrow C_0([0, 2\pi]; \mathbb{R}^n)$ by

$$F(\gamma, \mu, x)(t) = x(t) - e^{\gamma A(\mu)t}x(0) - \gamma \int_0^t e^{\gamma A(\mu)(t-s)}G(\mu, x(s))\mathrm{d}s,$$

whenever $(\gamma, \mu, x) \in \mathbb{R}^2 \times C_{2\pi}([0, 2\pi]; \mathbb{R}^n)$.

We define $C_{2\pi}([0, 2\pi]; \mathbb{R}^n)$ the space continuous functions h satisfying $h(0) = h(2\pi)$, and we define $C_0([0, 2\pi]; \mathbb{R}^n)$ the space of continuous functions h satisfying $h(0) = 0$. Both spaces $C_{2\pi}([0, 2\pi]; \mathbb{R}^n)$ and $C_0([0, 2\pi]; \mathbb{R}^n)$ are understood as Banach spaces endowed with the supremum norm on $[0, 2\pi]$.

We now aim at investigating the zeros of the equation

$$F(\gamma, \mu, x) = 0,$$

for (γ, μ, x) close to $(1, 0, 0_{\mathbb{R}^n})$, using the implicit function theorem.

Due to the finite dimensional setting, one can calculate the following derivatives directly

$$\begin{aligned} F_x(1, 0, 0)(x)(t) &= x(t) - e^{A(0)t}x(0), \\ F_{\mu,x}(1, 0, 0)(x)(t) &= -A'(0)e^{A(0)t}x(0), \\ F_{\gamma,x}(1, 0, 0)(x)(t) &= -A(0)e^{A(0)t}x(0). \end{aligned} \tag{3.16}$$

Now using (iii), it is readily checked that $\mathrm{N}(F_x(1, 0, 0)) = \mathrm{span}\{u_0, u_1\}$, wherein the functions u_0 and u_1 are given by

$$u_0(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; u_1(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.17}$$

Define $X_1 := \text{span}\{u_0, u_1\}$ and fix X_2 the complement space of X_1 as given by

$$X_2 = \{z \in C_{2\pi}([0, 2\pi]; \mathbb{R}^n) : \int_0^{2\pi} z(t)u_i(t)\text{d}t = 0_{\mathbb{R}^n} \text{ for } i = 0, 1\},$$

where the multiplication under the above integral is understood componentwized.

By using the fact that $L^2((0, 2\pi); \mathbb{R}^n)$ is an Hilbert (see Brezis [22, Remark 5 p. 137]) we know that

$$L^2((0, 2\pi); \mathbb{R}^n) = X_1 \oplus X_1^{\perp},$$

where

$$X_1^{\perp} = \{z \in L^2((0, 2\pi); \mathbb{R}^n) : \int_0^{2\pi} z(t)u_i(t)\text{d}t = 0_{\mathbb{R}^n} \text{ for } i = 0, 1\}.$$

It follows that $C_{2\pi}([0, 2\pi]; \mathbb{R}^n)$ inherits the properties of $L^2((0, 2\pi); \mathbb{R}^n)$. Indeed, $v \in C_{2\pi}([0, 2\pi]; \mathbb{R}^n)$ implies $v \in L^2((0, 2\pi); \mathbb{R}^n)$ and therefore

$$v = v_1 + w_1$$

where $v_1 \in X_1$ and $w_1 \in X_1^{\perp}$.

Moreover $X_1 \subset C_{2\pi}([0, 2\pi]; \mathbb{R}^n)$ so we deduce that

$$w_1 = v - v_1 \in C_{2\pi}([0, 2\pi]; \mathbb{R}^n).$$

We conclude that

$$C_{2\pi}([0, 2\pi]; \mathbb{R}^n) = X_1 \oplus X_2.$$

Let us now define the map $h : \mathbb{R}^3 \times X_2 \to C_0([0, 2\pi], \mathbb{R}^n)$ by

$$h(s, \gamma, \mu, z) = \begin{cases} \dfrac{F(\gamma, \mu, s(u_0 + z))}{s}, \text{ if } s \neq 0, \\ F_x(\gamma, \mu, 0)(u_0 + z), \text{ if } s = 0. \end{cases}$$

Then one can prove that h is of class C^1. One has $h(0, 1, 0, 0) = 0$ while the partial derivative with respect to (γ, μ, z) is given by

$$\partial_{(\gamma,\mu,z)} h(0, 1, 0, 0)(\hat{\gamma}, \hat{\mu}, \hat{z}) = F_x(1, 0, 0)\hat{z} + \hat{\gamma} F_{\gamma,x}(1, 0, 0)u_0 + \hat{\mu} F_{\mu,x}(1, 0, 0)u_0,$$

that more explicitly rewrites using (3.16) as

$$\partial_{(\gamma,\mu,z)}h(0,1,0,0)(\hat{\gamma},\hat{\mu},\hat{z})(t) = \hat{z}(t)-e^{A(0)t}\hat{z}(0)-\hat{\gamma}tA(0)e^{A(0)t}u_0(0)-\hat{\mu}tA'(0)e^{A(0)t}u_0(0). \tag{3.18}$$

We now claim that

Claim 3.18 *The bounded linear operator*

$$\partial_{(\gamma,\mu,v)}h(0,1,0,0) \in \mathcal{L}\left(\mathbb{R}^2 \times X_2, C_0([0,2\pi];\mathbb{R}^n)\right)$$

is invertible.

Proof (Proof of Claim 3.18) To prove this claim, let $y \in C_0([0,2\pi];\mathbb{R}^n)$ be given. Set

$$M := \partial_{(\gamma,\mu,v)}h(0,1,0,0)$$

and let us investigate the equation

$$M\,(\hat{\gamma},\hat{\mu},\hat{z}) = y. \tag{3.19}$$

To do so, for any $y = (y_1, y_2)^T \in \mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$ we define the projection maps $\pi_1 : \mathbb{R}^n \to \mathbb{R}^2$ and $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-2}$ by $\pi_1 y = y_1$ and $\pi_2 y = y_2$.

Setting $\hat{z}_i := \pi_i \hat{z}$ and $y_i = \pi_i y$ for $i = 1, 2$.

From (3.17) it follows that $\pi_2 u(0) = 0$. Therefore by applying π_1 and π_2 on both side of (3.19), the system becomes for all $t \in [0, 2\pi]$,

$$\begin{cases} \hat{z}_1(t) - e^{B(0)t}\hat{z}_1(0) - \hat{\gamma}tB(0)e^{B(0)t}\pi_1 u_0(0) - \hat{\mu}tB'(0)e^{B(0)t}\pi_1 u_0(0) = y_1(t), \\ \hat{z}_2(t) - e^{C(0)t}\hat{z}_2(0) = y_2(t). \end{cases} \tag{3.20}$$

Since $\hat{z}_2$ is 2π−periodic and $I - e^{2\pi C(0)}$ is invertible, one obtains

$$\hat{z}_2(t) = e^{tC(0)}\left(I - e^{2\pi C(0)}\right)^{-1} y_2(2\pi) + y_2(t),$$

so that $\hat{z}_2 \in C_{2\pi}\left([0,2\pi], \mathbb{R}^{n-2}\right)$.

We now turn to the resolution of the first equation is (3.20). To do so, note that one has

$$tB(0)e^{tB(0)}\pi_1 u_0(0) = t(\cos t, -\sin t)^T,$$

while

$$tB'(0)e^{tB(0)}\pi_1 u_0(0) = t\alpha'(0)(\sin t, \cos t)^T + t\beta'(0)(\cos t, -\sin t)^T.$$

As a consequence, the first equation in (3.20) rewrites as finding $\hat{z}_1 \in \pi_1 X_2$, $(\hat{\gamma}, \hat{\mu}) \in \mathbb{R}^2$ such that

$$\hat{z}_1(t) - e^{B(0)t}\hat{z}_1(0) + c_1 t\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} + c_2 t\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} = y_1(t). \tag{3.21}$$

with

$$c_1 = -\alpha'(0)\hat{\mu}, \quad c_2 = -\hat{\gamma} - \beta'(0)\hat{\mu}.$$

Since $\tilde{z}_1$ is $2\pi-$periodic, taking $t = 2\pi$ in the above equation yields, since $e^{B(0)2\pi} = I$,

$$\tilde{z}_1(2\pi) - e^{B(0)2\pi}\tilde{z}_1(0) = \tilde{z}_1(0) - e^{B(0)2\pi}\tilde{z}_1(0) = 0.$$

and (3.21) becomes

$$c_1 2\pi \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2\pi c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = y_1(2\pi) =: \begin{pmatrix} y_1^1(2\pi) \\ y_1^2(2\pi) \end{pmatrix}.$$

Therefore we obtain $c_1 = y_1^2(2\pi)/2\pi$ and $c_2 = y_1^1(2\pi)/2\pi$. Since $\alpha'(0) \neq 0$, this allows to recover $\hat{\mu}$ and $\hat{\gamma}$ that are given by the following expressions:

$$\hat{\mu} = -\frac{y_1^2(2\pi)}{2\pi\alpha'(0)} \text{ and } \hat{\gamma} = -\frac{y_1^1(2\pi)}{2\pi} + \frac{\beta'(0)y_1^2(2\pi)}{2\pi\alpha'(0)}.$$

Next we focus on $\hat{z}_1$. To do so we set

$$Y_1(t) = y_1(t) - c_1 t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} - c_2 t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix},$$

so that $\hat{z}_1 \in \pi_1 X_2$ satisfies

$$\hat{z}_1(t) - e^{B(0)t}\hat{z}_1(0) = Y_1(t), \ \forall t \in [0, 2\pi].$$

Here note that $Y_1 \in C_0([0, 2\pi], \mathbb{R}^2) \cap C_{2\pi}([0, 2\pi], \mathbb{R}^2)$. Consider now the linear operator $Q : C_{2\pi}([0, 2\pi]; \mathbb{R}^2) \to C_{2\pi}([0, 2\pi]; \mathbb{R}^2)$ defined by

$$Q(z)(t) = e^{B(0)t}z(0), \ \forall t \in [0, 2\pi],$$

so that the above equation rewrites as

$$\hat{z}_1 \in \pi_1 X_2 \text{ and } (I - Q)\hat{z}_1 = Y_1 \in C_0([0, 2\pi], \mathbb{R}^2) \cap C_{2\pi}([0, 2\pi], \mathbb{R}^2).$$

Now note that Q is a projector (i.e. $Q^2 = Q$) and since $t \to e^{B(0)t}$ is $2\pi-$periodic, one has

$$\mathrm{R}(I - Q) = C_0([0, 2\pi], \mathbb{R}^2) \cap C_{2\pi}([0, 2\pi], \mathbb{R}^2),$$

while

$$N(I - Q) = \mathrm{R}(Q) = \mathrm{span}\{\pi_1 u_0, \pi_1 u_1\}.$$

Since $C_{2\pi}([0, 2\pi], \mathbb{R}^2) = \pi_1 X_2 \oplus \mathrm{span}\{\pi_1 u_0, \pi_1 u_1\}$, we obtain that the linear bounded operator $(I - Q)_{|\pi_1 X_2}$ is bijective from $\pi_1 X_2$ onto $C_0([0, 2\pi], \mathbb{R}^2) \cap C_{2\pi}([0, 2\pi], \mathbb{R}^2)$ and this invertible on these spaces (see Theorem 3.11). Hence we end-up with

$$\hat{z}_1 = (I - Q)^{-1}_{|\pi_1 X_2} Y_1.$$

To sum-up the above analysis, we have obtained for each $y \in C_0([0, 2\pi], \mathbb{R}^n)$ the exists a unique $(\hat{\gamma}, \hat{\mu}, \hat{z}) \in \mathbb{R}^2 \times C_{2\pi}([0, 2\pi], \mathbb{R}^n)$ satisfying (3.19) and this completes the proof of Claim 3.18. □

To conclude the proof of the Hopf bifurcation theorem, we apply the implicit function theorem to the function $h : \mathbb{R}^3 \times X_2 \to C_0([0, 2\pi], \mathbb{R}^n)$ and we deduce that there exists a C^1–mapping $(\gamma, \mu, z) : (-\delta, \delta) \to \mathbb{R}^2 \times X_2$, for some $\delta > 0$ small enough, such that

$$h(s, \gamma(s), \mu(s), z(s)) = 0, \ \forall s \in (-\delta, \delta).$$

By the definition of h, this is equivalent to say that

$$F(\gamma(s), \mu(s), s(u_0 + z(s))) = 0,$$

when $s \neq 0$ with $(\gamma(0), \mu(0), z(0)) = (1, 0, 0)$.

3.9 Remarks and Notes

Analytic Implicit function Theorem

An analytic version of the implicit function theorem can found in Theorem 15.3 p.151 in the book of Deimling [49]. Such a result provides an analytic dependency of the solution with respect to the parameters.

Implicit function Theorem in scale of Banach spaces

Theorem 15.8 p. 162 in Deimling [49].

Global Implicit function Theorem

The global solvability of $F(\lambda, x) = 0$ have been studied in Rheinboldt [178], Sandberg [187].

Bifurcations for finite dimensional systems

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Bifurcations for infinite dimensional systems

The results presented in this chapter are based on the three papers by Crandall and Rabinowitz [46, 47, 48] for infinite dimensional system.

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More about Hopf bifurcation

We need extend Theorem 3.13 in order to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions described in Theorem 3.16. The following result is taken from the book by Chow et al. [32].

Consider a C^∞ system defined in a neighborhood of the origin in $\mathbb{R}^2$

$$\begin{aligned} x'(t) &= f(x, y, \mu) \\ y'(t) &= g(x, y, \mu) \end{aligned} \tag{3.22}$$

where $x, y, \mu \in \mathbb{R}^1$. We first describe the following assumption that will be needed.

Assumption 3.19 We suppose that the origin is an equilibrium of (3.22), and, for μ near 0, there exists a pair of conjugated eigenvalues of the linear part of (3.22) around the origin denoted by $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ written as

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \text{ with } \alpha(\mu),\ \omega(\mu) \in \mathbb{R}$$

such that the map $\mu \to \lambda(\mu)$ is continuously derivable,

$$\omega(0) = \omega_0 > 0,\ \alpha(0) = 0 \text{ and } \frac{\mathrm{d}\alpha(0)}{\mathrm{d}\mu} \neq 0.$$

Before state the main result about the property of the Hopf bifurcation, we derive a normal form for (3.22) for $\mu = 0$. By making a suitable linear change of coordinates and $z = x + iy$, equation (3.22) for $\mu = 0$ becomes

$$\begin{aligned} z'(t) &= i\omega_0 z + F(z, \overline{z}) \\ \overline{z}'(t) &= -i\omega_0 \overline{z} + \overline{F(z, \overline{z})}. \end{aligned} \tag{3.23}$$

Since the eigenvalues of the linear part of (3.22) for $\mu = 0$ are $\lambda_{1,2} = \pm i\omega_0$,by a polynomial change of variables

$$z = w + \sum_{2 \le k+l \le m} b_{kl} w^k \overline{w}^l,$$

System (3.23) takes the form

$$w' = i\omega_0 w + C_1 w^2 \overline{w} + \cdots + O(|w|^{2k+3}).$$

Theorem 3.20 *Let Assumption 3.14 be satisfied and* $\mathrm{Re}(C_1) \neq 0$. *Then there are* $\sigma > 0$ *and a neighborhood* U *of* $(x, y) = (0, 0)$ *such that*

(i) *if* $|\mu| < \sigma$ *and* $\mu \, \mathrm{Re}(C_1) \dfrac{\mathrm{d}\alpha(0)}{\mathrm{d}\mu} < 0$, *the system (3.22) has exactly one limit cycle inside* U*;*

(ii) *if* $|\mu| < \sigma$ *and* $\mu \, \mathrm{Re}(C_1) \dfrac{\mathrm{d}\alpha(0)}{\mathrm{d}\mu} \geq 0$, *the system (3.22) has no periodic orbit inside* U.

Moreover, the limit cycle is stable (unstable) if $\mathrm{Re}(C_1) < 0$ $(\mathrm{Re}(C_1) > 0)$, *and it tends to the equilibrium* $(0, 0)$ *as* $\mu \to 0$.

Example of application to the partner formation to predator prey system

A nice class of problem Murray [169] [170] and [171]

H.-B. Shi and S. Ruan (2015), Spatial, temporal, and spatiotemporal patterns of diffusive predator-prey models with mutual interference, IMA Journal of Applied Mathematics 80(5), 1534-1568.

Put some result and Fig. 9. of the above paper

Chapter 4
Center Manifold and Center Unstable Manifold Theory

4.1 Introduction

In this section we will consider differential equations of the form

$$u'(t) = Au(t) + F(u(t)), \forall t \in \mathbb{R}, \text{ and } u(0) = u_0 \in \mathbb{R}^n, \tag{4.1}$$

where $A \in M_n(\mathbb{R})$ is a n by n matrix and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a of class $C^k(\mathbb{R}^n)$ with $k \geq 1$.

We assume that the system (4.1) has an equilibrium $\overline{u}$.

Assumption 4.1 We assume that there exists an equilibrium $\overline{u} \in \mathbb{R}^n$ such that

$$A\overline{u} + F(\overline{u}) = 0.$$

Replacing replacing u by $v = u - \overline{u}$ in (4.1),

$$v'(t) = Av(t) + F(v(t) + \overline{u}) + A\overline{u}, \forall t \in \mathbb{R}, \text{ and } v(0) = u_0 - \overline{u} \in \mathbb{R}^n.$$

So, we can assume (without loss of generality) that

$$\overline{u} = 0.$$

Define

$$\widehat{A} := A + DF(\overline{u}) \in M_n(\mathbb{R}) \text{ and } \widehat{F}(x) := F(x + \overline{u}) + A\overline{u} - DF(\overline{u})x. \tag{4.2}$$

So without loss of generality, under Assumption 4.1, we can assume that

Assumption 4.2 We assume that

$$F(0) = 0_{\mathbb{R}^n} \text{ and } DF(0) = 0_{M_n(\mathbb{R})}. \tag{4.3}$$

4.2 State Space Decomposition of a Linear Equation

Consider first the linear system

$$u'(t) = Au(t), \forall t \in \mathbb{R}, \text{ and } u(0) = u_0 \in \mathbb{R}^n. \tag{4.4}$$

We denote by $\sigma(A) \in \mathbb{C}$ the set of eigenvalues of A. This spectrum is the disjoint union of the **stable spectrum** $\sigma_s(A)$, the **central spectrum** $\sigma_c(A)$ and the **unstable spectrum** $\sigma_u(A)$, where

$$\begin{aligned} \sigma_s(A) &= \{\lambda \in \sigma(A) : \text{Re}(\lambda) < 0\}, \\ \sigma_c(A) &= \{\lambda \in \sigma(A) : \text{Re}(\lambda) = 0\}, \\ \sigma_u(A) &= \{\lambda \in \sigma(A) : \text{Re}(\lambda) > 0\}. \end{aligned} \tag{4.5}$$

Assumption 4.3 We assume that $\sigma_c(A) = \sigma(A) \cap i\mathbb{R} \neq \emptyset$.

The span of the generalized eigenspace of A corresponding to the above spectrum decomposition is

$$\begin{aligned} X_s^{\mathbb{C}} &= \bigoplus_{\lambda \in \sigma_s(A)} N((\lambda I - A)^n), \\ X_c^{\mathbb{C}} &= \bigoplus_{\lambda \in \sigma_c(A)} N((\lambda I - A)^n), \\ X_u^{\mathbb{C}} &= \bigoplus_{\lambda \in \sigma_u(A)} N((\lambda I - A)^n), \end{aligned} \tag{4.6}$$

where A is regarded a linear operator on $\mathbb{C}^n$.

We have a complex state space decomposition

$$\mathbb{C}^n = X_s^{\mathbb{C}} \oplus X_c^{\mathbb{C}} \oplus X_u^{\mathbb{C}}. \tag{4.7}$$

Let $k = s, c, u$. We observe that since A is a real valued matrix, we have

$$\overline{(\lambda I - A)^n x} = \left(\overline{\lambda} I - A\right)^n \overline{x}, \forall \lambda \in \mathbb{C}.$$

We deduce that

$$\lambda \in \sigma_k(A) \Leftrightarrow \overline{\lambda} \in \sigma_k(A),$$

and

$$x \in X_k^{\mathbb{C}} \Leftrightarrow \overline{x} \in X_k^{\mathbb{C}}.$$

We deduce that

$$x \in X_k^{\mathbb{C}} \Rightarrow \text{Re}(x) = \frac{x + \overline{x}}{2} \in X_k^{\mathbb{C}} \text{ and } \text{Im}(x) = \frac{x - \overline{x}}{2i} \in X_k^{\mathbb{C}}.$$

Moreover since

$$x \in X_k^{\mathbb{C}} \Leftrightarrow \pm ix \in X_k^{\mathbb{C}},$$

it follows that

$$X_k^{\mathbb{C}} = X_k \oplus i\, X_k,$$

where

$$X_k = \text{Re}\left(X_k^{\mathbb{C}}\right) = \left\{\text{Re}(x) : x \in X_k^{\mathbb{C}}\right\} \subset \mathbb{R}^n. \tag{4.8}$$

Hence the space X_k $(k = s, c, u)$ is the real part of the subspace of $X_k^{\mathbb{C}}$ spanned by the generalized eigenvectors of A corresponding to eigenvalues $\sigma_k(A)$.

Now, we obtain a **state space decomposition**

$$\mathbb{R}^n = X_s \oplus X_c \oplus X_u, \tag{4.9}$$

such that

$$A(X_s) \subset X_s, A(X_c) \subset X_c \text{ and } A(X_u) \subset X_u. \tag{4.10}$$

For $k = s, c, u$, we define $A_k : X_k \to X_k$ the bounded linear operator as

$$A_k x = Ax, \forall x \in X_k.$$

Then A_k is a bounded linear operator on X_k, which we write for short $A_k \in \mathcal{L}(X_k)$. The linear operator A_k is also **the part of** A **in** X_k which is defined as $A : D(A_k) \subset X_k \to X_k$ such that

$$D(A_k) = \{x \in X_k : Ax \in X_k\}.$$

By construction, we have the following lemma.

Lemma 4.4 *The spectrum of A_s, A_c and A_u satisfy the following*

$$\sigma(A_s) = \sigma_s(A), \sigma(A_c) = \sigma_c(A) \textit{ and } \sigma(A_u) = \sigma_u(A). \tag{4.11}$$

Define $\Pi_s, \Pi_c, \Pi_u \in \mathcal{L}(\mathbb{R}^n)$ the projectors such that

$$\Pi_s(\mathbb{R}^n) = X_s \text{ and } (I - \Pi_s)(\mathbb{R}^n) = X_c \oplus X_u,$$

$$\Pi_c(\mathbb{R}^n) = X_c \text{ and } (I - \Pi_c)(\mathbb{R}^n) = X_s \oplus X_u,$$

$$\Pi_u(\mathbb{R}^n) = X_u \text{ and } (I - \Pi_u)(\mathbb{R}^n) = X_s \oplus X_c,$$

and since Π_s, Π_c and Π_u are projectors (i.e. $\pi^2 = \pi$) this is also equivalent to

$$\text{R}(\Pi_s) = X_s \text{ and } \text{N}(\Pi_s) = X_c \oplus X_u,$$

$$\text{R}(\Pi_c) = X_c \text{ and } \text{N}(\Pi_c) = X_s \oplus X_u,$$

$$\text{R}(\Pi_u) = X_u \text{ and } \text{N}(\Pi_u) = X_s \oplus X_c.$$

Finally we define

$$\Pi_h := \Pi_s + \Pi_u$$

the projector on the **hyperbolic space** $X_h = X_s \oplus X_u$, and

$$\Pi_{cs} := \Pi_c + \Pi_s$$

the projector on the **center stable space** $X_{cs} = X_c \oplus X_s$, and

$$\Pi_{cu} := \Pi_c + \Pi_u$$

the projector on the **center unstable space** $X_{cu} = X_c \oplus X_u$.

Each of these projections commutes with A, and therefore the corresponding subspaces are invariant under the flow of (4.4). That is to say that

$$\Pi_k A = A \Pi_k, \forall k = s, c, u, h, cs, cu, \tag{4.12}$$

and by using the series formula of

$$e^{At} = I + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

we deduce that

$$\Pi_k e^{At} = e^{At} \Pi_k, \forall k = s, c, u, h, cs, cu. \tag{4.13}$$

Define

$$\beta_+ = \min \{\text{Re}(\lambda) : \lambda \in \sigma_u(A)\} > 0, \quad \beta_- = \min \{-\text{Re}(\lambda) : \lambda \in \sigma_s(A)\} > 0,$$

and

$$\beta = \min(\beta_-, \beta_+)/2. \tag{4.14}$$

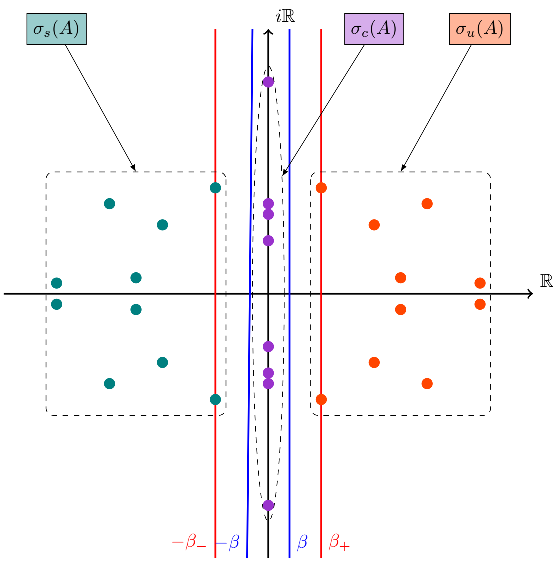


Fig. 4.1: *In this Figure, we illustrate the different parts of the spectrum $\sigma_s(A)$, $\sigma_c(A)$ and $\sigma_u(A)$ as well as β_- and β_+. The region between the blue vertical lines only contains the purely imaginary spectrum of A.*

Lemma 4.5 *There exists a non increasing map* $\varepsilon \to M_c(\varepsilon)$ *from* $(0, +\infty)$ *to itself such that*

$$\|e^{At}\Pi_c\|_{\mathcal{L}(\mathbb{R}^n)} \leq M_c(\varepsilon)e^{\varepsilon|t|}, \forall t \in \mathbb{R}, \forall \varepsilon > 0. \tag{4.15}$$

There exist two constant number $M_s > 0$ *and* $M_u > 0$ *such that*

$$\begin{aligned} \|e^{At}\Pi_u\|_{\mathcal{L}(\mathbb{R}^n)} &\leq M_u e^{(\beta-\varepsilon)t}, \forall t \leq 0, \forall \varepsilon \geq 0, \\ \|e^{At}\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)} &\leq M_s e^{(-\beta+\varepsilon)t}, \forall t \geq 0, \forall \varepsilon \geq 0. \end{aligned} \tag{4.16}$$

Remark 4.6 In general we may have $M_c(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

Proof The result is based on the stability theorem for linear system (see Theorem 3.17 in Chapter 3 of the first volume of this book [55]). For example we have for $t \geq 0$

$$e^{-\varepsilon|t|}e^{At}\Pi_c = e^{(A_c-\varepsilon I)t}\Pi_c$$

an the spectrum $\sigma(A_c - \varepsilon I)$ contains only complex number with strictly negative real part. So by using the stability theorem for linear systems we deduce that for each $\varepsilon > 0$

$$M_c^+(\varepsilon) := \sup_{t\geq 0} e^{-\varepsilon|t|}\|e^{At}\Pi_c\|_{\mathcal{L}(\mathbb{R}^n)} < +\infty,$$

$$M_c^-(\varepsilon) := \sup_{t\leq 0} e^{-\varepsilon|t|}\|e^{At}\Pi_c\|_{\mathcal{L}(\mathbb{R}^n)} < +\infty,$$

therefore we can define $M_c(\varepsilon) = \max\left(M_c^+(\varepsilon), M_c^-(\varepsilon)\right)$ and we obtain (4.15).

To prove the last part of the lemma we observe that from (4.14) we obtain

$$\beta < \beta_- \text{ and } \beta < \beta_+,$$

and by using again the stability theorem we deduce that for each $\varepsilon \geq 0$

$$\widehat{M}_u(\varepsilon) := \sup_{t\leq 0} e^{-(\beta-\varepsilon)\,t}\|e^{At}\Pi_u\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{t\geq 0} e^{-\varepsilon\,t}\|e^{(\beta I-A)t}\Pi_u\|_{\mathcal{L}(\mathbb{R}^n)} < +\infty,$$

$$\widehat{M}_s(\varepsilon) := \sup_{t\geq 0} e^{-(-\beta+\varepsilon)\,t}\|e^{At}\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{t\geq 0} e^{-\varepsilon\,t}\|e^{(\beta I+A)\,t}\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)} < +\infty,$$

and by choosing $M_s = \widehat{M}_s(0)$ and $M_u = \widehat{M}_u(0)$ the proof is completed. □

Example 4.7 The above lemma is sharp. Indeed, by consider the example of Jordan block

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R}).$$

Then 0 is the only point of the spectrum of A. That is

$$\sigma(A) = \{0\},$$

and we know that

$$e^{At} := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R}).$$

see Chapter 3 of the first volume of this book [55] for more details.

Now, by using the norm

$$\|(x, y)\|_1 = |x| + |y|.$$

We deduce that

$$1+t = \|e^{At}e_2\|_1 \leq \|e^{At}\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{\|x\|_1 \leq 1} \|e^{At}x\|_1 \leq \sup_{p \in [0,1]} \|e^{At}(pe_1 + (1-p)e_2)\|_1 \leq 2+t.$$

So, for this example we obtain

$$M_c(\varepsilon) \leq \sup_{t \in \mathbb{R}} e^{-\varepsilon|t|}(2+t) < +\infty$$

and

$$M_c(\varepsilon) \geq \sup_{t \in \mathbb{R}} e^{-\varepsilon|t|}(1+t) \to +\infty, \text{ as } \varepsilon \to 0^+.$$

The following lemma gives some characterizations of X_c in (4.18) that will be used as a definition for the center manifold.

Lemma 4.8 (Characterization of center space) *We have the following*

$$X_c = \left\{ x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} \|\Pi_h e^{At}x\| < +\infty \right\}, \tag{4.17}$$

and for each $\eta \in (0, \beta)$,

$$X_c = \left\{ x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} e^{-\eta|t|}\|e^{At}x\| < +\infty \right\}. \tag{4.18}$$

Proof If $x \in X_c$ then $\Pi_h x = 0$. We deduce that the map $t \to \Pi_h e^{At}x = e^{At}\Pi_h x = 0$ is bounded function. Next assume that there exists a constant $C \in (0, +\infty)$ such that

$$\sup_{t \in \mathbb{R}} \|\Pi_h e^{At}x\| \leq C.$$

Then by Lemma 4.5, we have

$$\|e^{At}x\| = \|e^{At}(\Pi_c + \Pi_h)x\| \leq \|e^{At}\Pi_c x\| + \|\Pi_h e^{At}x\| \leq M(\eta)e^{\eta|t|} + C$$

and we deduce that

$$\sup_{t \in \mathbb{R}} e^{-\eta|t|}\|e^{At}x\| \leq M(\eta) + C.$$

Finally assume that $\|e^{At}x\| \leq e^{\eta|t|}C, \forall t \in \mathbb{R}$ for some $C > 0$ and some $\eta \in (0, \beta)$. Then by using (4.16), we deduce that for all $t \leq 0$ and $\varepsilon > 0$

$$\|\Pi_u x\| = \|e^{At}\Pi_u e^{-At}x\| \leq Me^{(\beta-\varepsilon)t}e^{-\eta t}C,$$

and by choosing $\varepsilon < \beta - \eta \leq \beta_+ - \eta$, we deduce when t goes to $-\infty$ that $\Pi_u x = 0$. Similar arguments gives $\Pi_s x = 0$. The proof is completed. □

The following lemma gives some characterizations of X_{cu} in (4.20) that will be used as a definition for the center-unstable manifold.

Lemma 4.9 (Characterization of center unstable space) *We have the following*

$$X_{cu} = \left\{ x \in \mathbb{R}^n : \sup_{t \le 0} \|\Pi_s e^{At} x\| < +\infty \right\}, \tag{4.19}$$

and for each $\eta \in (0, \beta)$,

$$X_{cu} = \left\{ x \in \mathbb{R}^n : \sup_{t \le 0} e^{-\eta|t|} \|e^{At} x\| < +\infty \right\}. \tag{4.20}$$

Proof Assume that $\sup_{t \le 0} \|\Pi_s e^{At} x\| \le C$ (for some $C > 0$). Then by Lemma 4.5, we have

$$\|\Pi_s x\| = \|e^{-At} \Pi_s e^{At} x\| \le M e^{(-\beta+\varepsilon)(-t)} C, \forall t \le 0,$$

and when $t \to -\infty$ we obtain $\Pi_s x = 0$ which implies that $x \in X_{cu}$. Conversely if $x \in X_{cu}$ then $\Pi_s e^{At} x = e^{At} \Pi_s x = 0, \forall t \le 0$.

Next assume that $\sup_{t \le 0} e^{-\eta|t|} \|e^{At} x\| \le C$ (for some $C > 0$). Then by Lemma 4.5, we have

$$\|\Pi_s x\| = \|e^{A|t|} \Pi_s e^{At} x\| \le M e^{(-\beta_- + \varepsilon)|t|} e^{\eta|t|} C, \forall t \le 0,$$

by choosing again $\varepsilon < \beta - \eta \le \beta_- - \eta$ and let $t \to -\infty$ we deduce that $\Pi_s x = 0$. Conversely assume that $\Pi_s x = 0$, by Lemma 4.5, we have for all $t \le 0$,

$$e^{-\eta|t|} \|e^{At} x\| \le e^{-\eta|t|} \left[\|e^{At} \Pi_c x\| + \|e^{At} \Pi_u x\| \right] \le e^{-\eta|t|} \left[M e^{\varepsilon|t|} + M e^{-(\beta_+ - \varepsilon)|t|} \right]$$

so by choosing $0 < \varepsilon < \eta$ and $0 < \varepsilon < \beta_+$ we deduce

$$e^{-\eta|t|} \|e^{At} x\| \le 2M, \forall t \le 0.$$

The proof is completed. □

To conclude this section the norm

$$|x| = \|\Pi_s x\| + \|\Pi_c x\| + \|\Pi_u x\|.$$

Since $I = \Pi_s + \Pi_c + \Pi_u$, and by using the triangle inequality

$$\|x\| = \|\Pi_s x + \Pi_c x + \Pi_u x\| \le |x|$$

and by using the fact that Π_s Π_s and Π_u are continuous linear map

$$|x| \le \left(\|\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)} + \|\Pi_c\|_{\mathcal{L}(\mathbb{R}^n)} + \|\Pi_u\|_{\mathcal{L}(\mathbb{R}^n)} \right) \|x\|$$

therefore the two norms $|.|$ and $\|.\|$ (are equivalent even in infinite dimensional space).

We observe that

$$|\Pi_k x| = \|\Pi_k^2 x\| = \|\Pi_k x\| \leq |x|, \forall k = s, u, c.$$

So without loss of generality we can make the following assumption.

Assumption 4.10 We assume that

$$\|\Pi_k\|_{\mathcal{L}(\mathbb{R}^n)} \leq 1, \forall k = s, u, c,$$

where

$$\|\Pi_k\|_{\mathcal{L}(\mathbb{R}^n)} := \sup_{\|x\| \leq 1} \|\Pi_k x\|.$$

4.3 Center Manifold Theory

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous map. In this section, we investigate the existence and smoothness of the center manifold for a nonlinear semiflow $\{U(t)\}_{t \geq 0}$ on $\mathbb{R}^n$ generated by solutions of the Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(u(t)), \text{ for } t \geq 0, \text{ with } u(0) = x \in \mathbb{R}^n. \tag{4.21}$$

A solution of (4.21) is a continuous map $t \in [0, +\infty) \to U(t)x$ satisfying the fixed point problem

$$U(t)x = x + A \int_0^t U(s)x ds + \int_0^t F(U(s)x) ds, \forall t \geq 0.$$

We know (see Proposition 2.21 of the first volume of this book [55]) that the above fixed point problem is also equivalent to the variation of constant formula

$$U(t)x = e^{At}x + \int_0^t e^{A(t-s)} F(U(s)x) \mathrm{d}s, \forall t \geq 0. \tag{4.22}$$

In variation of constant formula not only is useful to problem the local stability of equilibrium (see Chapter 6 of the first volume of this book [55]). But it is also a crucial tool to understand the center manifold theory.

The variation of constant formula (4.22) can be rewritten into the following more condensed form

$$U(t)x = e^{At}x + \left(e^{A\cdot} * F(U(.)x)\right)(t), \forall t \geq 0,$$

where the convolution is defined as

$$\left(e^{A\cdot} * f(.)\right)(t) := \int_0^t e^{A(t-s)} f(s)\, \mathrm{d}s, \forall t \geq 0. \tag{4.23}$$

Since F is Lipschitz continuous, we know (see Theorem 8.11 in the first volume of this book [55]) that for each $x \in \mathbb{R}^n$, (4.22) has a unique solution $t \to U(t)x$ from

$[0, +\infty)$ into $\mathbb{R}^n$. Moreover, the family $\{U(t)\}_{t\geq 0}$ defines a continuous semiflow. That is

(i) $U(0) = I$ and $U(t)U(s) = U(t+s), \forall t, s \geq 0$;
(ii) The map $(t, x) \to U(t)x$ is continuous from $[0, +\infty) \times \mathbb{R}^n$ into $\mathbb{R}^n$.

Furthermore,

$$\|U(t)x - U(t)y\| \leq e^{\|F\|_{\text{Lip}} t} \|x - y\| \text{ for all } t \geq 0,$$

where

$$\|F\|_{\text{Lip}} := \sup_{x, y \in \mathbb{R}^n : x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|}.$$

4.3.1 Weighted spaces of exponential bounded continuous functions

Let $(Y, \|.\|_Y)$ be a Banach space. Let $\eta \in \mathbb{R}$ be a constant and $I \subset \mathbb{R}$ be an interval. Define

$$BC^\eta(I, Y) = \left\{ f \in C(I, Y) : \sup_{t \in I} e^{-\eta |t|} \|f(t)\|_Y < +\infty \right\}.$$

It is well known (see Chapter 2 of the first volume of this book [55]) that $BC^\eta(I, Y)$ is a Banach space when it is endowed with the norm

$$\|f\|_{BC^\eta(I,Y)} = \sup_{t \in I} e^{-\eta |t|} \|f(t)\|_Y .$$

We observe that

$$\|f\|_{BC^\eta(I,Y)} \leq M \Leftrightarrow \|f(t)\|_Y \leq M e^{\eta |t|}, \forall t \in I.$$

Therefore such spaces characterize the exponential growth speed of the function at plus or minus infinity.

In this chapter, the examples of set I used are the following

$$I = [0, +\infty), \; I = (-\infty, 0], \text{ and } I = (-\infty, +\infty),$$

The weighted spaces $BC^\eta(I, Y)$ are invariant by a shift.

Lemma 4.11 (Shift-invariance) *Let $\eta > 0$. Then for each $f \in BC^\eta(\mathbb{R}, Y)$ the map $t \to f(t + t^\star)$ belongs to $BC^\eta(\mathbb{R}, Y)$, and we have the following estimation*

$$e^{-\eta |t^\star|} \|f\|_{BC^\eta(\mathbb{R},Y)} \leq \left\| f(. + t^\star) \right\|_{BC^\eta(\mathbb{R},Y)} \leq e^{\eta |t^\star|} \|f\|_{BC^\eta(\mathbb{R},Y)} .$$

Proof By using the triangle inequality, we have

$$|t + t^\star| \leq |t| + |t^\star| \Leftrightarrow |t + t^\star| - |t| \leq |t^\star| \Leftrightarrow -\left[|t| - |t + t^\star|\right] \leq |t^\star|,$$

and

$$|t| = |t + t^\star - t^\star| \le |t + t^\star| + |-t^\star| \Leftrightarrow |t| - |t + t^\star| \le |t^\star|,$$

hence we obtain

$$-|t^\star| \le -\left[|t| - |t + t^\star|\right] \le |t^\star|.$$

We deduce that

$$e^{-\eta|t^\star|} \le e^{-\eta\left[|t|-|t+t^\star|\right]} \le e^{\eta|t^\star|},$$

and by using

$$\sup_{t\in\mathbb{R}} e^{-\eta|t|}\|f(t+t^\star)\| = \sup_{t\in\mathbb{R}} e^{-\eta\left[|t|-|t+t^\star|\right]} e^{-\eta|t+t^\star|}\|f(t+t^\star)\|$$

we obtain by setting $\sigma = t + t^\star$,

$$e^{-\eta|t^\star|}\sup_{\sigma\in\mathbb{R}} e^{-\eta|\sigma|}\|f(\sigma)\| \le \sup_{t\in\mathbb{R}} e^{-\eta|t|}\|f(t+t^\star)\| \le e^{\eta|t^\star|}\sup_{\sigma\in\mathbb{R}} e^{-\eta|\sigma|}\|f(\sigma)\|$$

and the proof is completed. □

The family $\left\{\left(BC^\eta(I,Y), \|.\|_{BC^\eta(I,Y)}\right)\right\}_{\eta>0}$ forms a **scale of Banach spaces**, that is, if $0 < \zeta < \eta$ then $BC^\zeta(I,Y) \subset BC^\eta(I,Y)$ and the embedding is continuous; more precisely, we have

$$\|f\|_{BC^\eta(I,Y)} \le \|f\|_{BC^\zeta(I,Y)}, \quad \forall f \in BC^\zeta(I,Y).$$

Let $(Z, \|.\|_Z)$ be a Banach spaces. From now on, we denote by $\mathrm{Lip}(Y,Z)$ (respectively $\mathrm{Lip}_B(Y,Z)$) the space of Lipschitz (respectively Lipschitz and bounded) maps from Y into Z. Define the semi norm

$$\|F\|_{\mathrm{Lip}(Y,Z)} := \sup_{x,y\in Y: x\neq y} \frac{\|F(x)-F(y)\|_Z}{\|x-y\|_Y}.$$

The above Lipschitz norm is an extended notion of the supremum norm of the first derivative. So $\|.\|_{\mathrm{Lip}(Y,Z)}$ is only a semi-norm not a norm, because

$$\|F\|_{\mathrm{Lip}(Y,Z)} = 0$$

whenever F is a constant function.

Definition 4.12 (Center manifold) Let $\eta \in (0,\beta)$. We define the **center manifold** V_η is the set of point x in $\mathbb{R}^n$ satisfying the two following properties

(i) There exists $t \in \mathbb{R} \to u(t)$ a complete orbit of the semiflow U passing through x at $t = 0$. That is

$$U(t-s)u(s) = u(t), \forall t \ge s, \text{ and } u(0) = x.$$

(ii) The exponential growth of u is bounded by η when t goes to $\pm\infty$. That is

$$u \in BC^\eta(\mathbb{R},\mathbb{R}^n) \Leftrightarrow \|u(t)\| \le Me^{\eta|t|}, \forall t \in \mathbb{R} \text{ (for some constant } M > 0).$$

For short the center manifold can be rewritten as

$$V_\eta = \left\{x \in \mathbb{R}^n : \exists u \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right) \text{ a complete orbit of } \{U(t)\}_{t \geq 0}, \text{ with } u(0) = x\right\}.$$

Lemma 4.13 (Invariance of the center manifold by the semiflow) *Let $\eta \in (0, \beta)$. Then V_η is invariant by the semiflow U. That is $U(t)V_\eta = V_\eta, \forall t \geq 0$.*

Proof Let $t^\star > 0$. Let us prove that $U(t^\star)V_\eta \subset V_\eta$. Let $x \in V_\eta$ and let $u \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right)$ a complete orbit of U and $u(0) = x$. Then by construction we have $y = u(t^\star) = U(t^\star)x$, so

$$y \in U(t^\star)V_\eta.$$

Moreover $t \to u_{t^\star}(t) = u\left(t + t^\star\right)$ is also a complete orbit, and by Lemma 4.11 we know that the shifted orbit satisfies

$$t \to u_{t^\star}(t) = u\left(t + t^\star\right) \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right).$$

So $y \in V_\eta$.

Conversely, let us prove that $V_\eta \subset U(t^\star)V_\eta$. Let $y \in V_\eta$ then there exists $u \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right)$ a complete orbit of U with $u(0) = y$. Let $x = u(-t^\star)$ and $t \to u_{-t^\star}(t) = u(t - t^\star)$. Then $U(t^\star)x = y$ because $t \to u(t)$ is a complete orbit of U, and x belongs to V_η because $t \to u_{-t^\star}(t)$ is a complete orbit of U, and by Lemma 4.16 we deduce that

$$u_{-t^\star} \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right).$$

The proof is completed. □

4.3.2 Reduced equation

In this chapter, we will see that, if $\|F\|_{\mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^n)}$ is small enough, then we can find a map $\Psi : X_c \to X_h$ such that

$$V_\eta = \{x_c + \Psi(x_c) : x_c \in X_c\}.$$

We can also reformulate the definition of the center manifold by saying that

$$\Pi_c V_\eta = X_c,$$

and

$$x \in V_\eta \Leftrightarrow \Pi_h x = \Phi\left(\Pi_c x\right).$$

As consequence Lemma 4.19, the set V_η is invariant by $U(t)$, so we deduce that

$$U(t)x \in V_\eta, \forall t \geq 0, \forall x \in V_\eta.$$

Therefore the invariance

$$\Pi_h U(t)x = \Psi(\Pi_c U(t)x), \forall t \geq 0, \forall x \in V_\eta.$$

We deduce that on the manifold V_η, we can reduce the dimension of the system (4.21), by projecting (4.21) on X_c and we obtain

$$u_c'(t) = A_c\, u_c(t) + \Pi_c F\left(u_c(t) + \Psi\left(u_c(t)\right)\right). \quad \textbf{(Reduced Equation)}$$

The advantage of the center manifold is that the dimension of reduced system has the same dimension than X_c which is much smaller than n in general. The center manifold contains all the bounded orbited (for example periodic orbits, homoclinic orbit of an equilibrium, and heteroclinic orbits) of the original system (4.21).

The disadvantage of the center manifold is that Ψ is fully implicit. Therefore we have no explicit formula for Ψ in general. In this chapter, we will see that if all the derivative

$$\Pi_h D^k F(0) = 0, \forall k = 1, \ldots, m, \tag{4.24}$$

then all the derive of the center manifold

$$D^k \Psi(0) = 0, \forall k = 1, \ldots, m.$$

In that case, by using a Taylor expansion at 0 of the center manifold, we can obtain some n^{th} order approximation locally around 0 for the reduce system. Furthermore, in the next Chapter 5, we will prove that it is always possible to make some changes of variable in order to satisfy the condition (4.24) at any order.

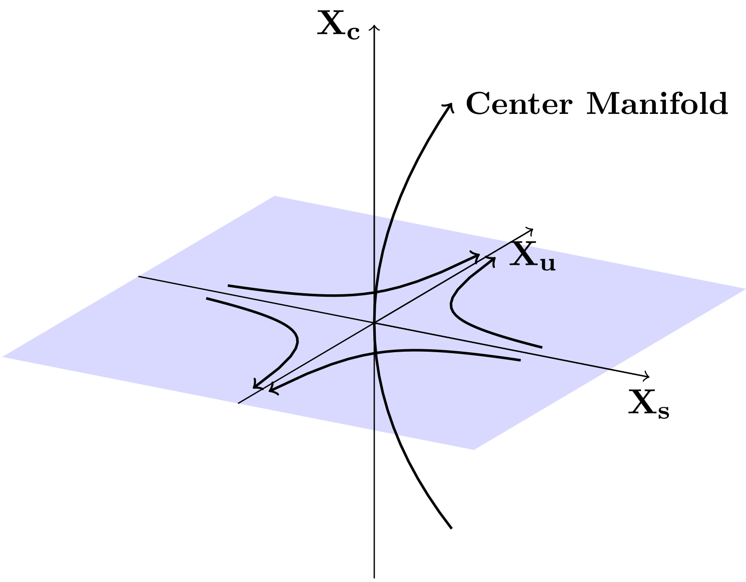


Fig. 4.2: *Schematic representation of the center manifold.*

4.3.3 Existence of the center manifold

In this subsection, we investigate the existence of the center manifold.

Definition 4.14 Let us recall that a function $u : \mathbb{R} \to \mathbb{R}$ is a **complete orbit** for the semiflow $\{U(t)\}_{t \geq 0}$ if and only if the function $t \to u(t)$ on $\mathbb{R}$ satisfies

$$u(t) = U(t-s)u(s), \forall t, s \in \mathbb{R} \text{ with } t \geq s,$$

where $\{U(t)\}_{t \geq 0}$ is a continuous semiflow generated by (4.22).

That is equivalent to say that for each $s \in \mathbb{R}$ (fixed) the function $t \to u(t)$ from $[s, \infty)$, satisfies

$$u(t) = u(s) + A \int_0^{t-s} u(s+r)dr + \int_0^{t-s} F\left(u(s+r)\right) dr, \forall t \geq s,$$

or equivalently by using the variation of constant formula

$$u(t) = e^{A(t-s)}u(s) + \int_s^t e^{A(t-\sigma)} F(u(\sigma))d\sigma, \forall t \geq s,$$

or (for short) by using the convolutions

$$u(t) = e^{A(t-s)}u(s) + \Big(e^{A.} * F\left(u(s+.)\right)\Big)(t-s), \forall t \geq s. \tag{4.25}$$

In the above intergal equation s and $u(s)$ are fixed, and the map $t \in [s, +\infty) \to u(t)$ is a fixed solution of the above integral equation.

Definition 4.15 Let $\eta \in (0, \beta)$. The η-**center manifold** of (4.21) is the set

$$V_\eta = \left\{ x \in \mathbb{R}^n : \exists u \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right), \text{ a complete orbit of } \{U(t)\}_{t\geq 0}, \text{ such that } u(0) = x \right\}. \tag{4.26}$$

That is the set of the points $x \in \mathbb{R}^n$, such that there exists $u \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right)$, a complete orbit of $\{U(t)\}_{t \geq 0}$, passing through x at $t = 0$.

Let $u \in BC^\eta\left(\mathbb{R}, \mathbb{R}^n\right)$. By Lemma 4.11 we have for each $\tau \in \mathbb{R}$,

$$e^{-\eta|\tau|} \|u\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} \leq \|u(.+\tau)\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} \leq e^{\eta|\tau|} \|u\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)}.$$

So for each $\eta > 0$, V_η is invariant under the semiflow $\{U(t)\}_{t \geq 0}$, that is,

$$U(t)V_\eta = V_\eta, \quad \forall t \geq 0.$$

Moreover, we say that $\{U(t)\}_{t \geq 0}$ is **reduced on** V_η if there exists a map $\Psi : X_c \to X_h$ such that

$$V_\eta = \text{Graph}\left(\Psi\right) = \left\{x_c + \Psi\left(x_c\right) : x_c \in X_c\right\}.$$

Before proving the main results of this section, first we need some preliminary lemmas.

Lemma 4.16 *Let Assumption 4.3 be satisfied. Let $\tau > 0$ be fixed. Then for each $f \in C([0,\tau], \mathbb{R}^n)$ and each $t \in [0,\tau]$ and $k = s, c, u$, we have*

$$\Pi_k \left(e^{A\cdot} * f(.)\right)(t) = \left(e^{A\cdot} * \Pi_k f(.)\right)(t) = \left(e^{A_k \cdot} * \Pi_k f(.)\right)(t), \forall t \geq 0. \tag{4.27}$$

Furthermore, for each $\gamma > -\beta$, there exists $\widehat{C}_{s,\gamma} := \dfrac{2M_s}{\beta+\gamma} > 0$ such that for each $f \in C([0,\tau], \mathbb{R}^n)$ and each $t \in [0,\tau]$, we have

$$e^{-\gamma t} \|\Pi_s \left(e^{A\cdot} * f\right)(t)\| \leq \widehat{C}_{s,\gamma} \sup_{s\in[0,t]} e^{-\gamma s} \|\Pi_s f(s)\| \, \mathrm{d}s. \tag{4.28}$$

Proof From Lemma 4.5 we have for each $\varepsilon > 0$

$$\|e^{At}\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)} \leq M_s e^{(-\beta+\varepsilon)t}, \forall t \geq 0.$$

Therefore

$$\begin{aligned} e^{-\gamma t} \left\|\Pi_s \left(e^{A\cdot} * f\right)(t)\right\| &\leq e^{-\gamma t} \int_0^t \|e^{A(t-s)}\Pi_s f(s)\| \, \mathrm{d}s \\ &\leq M_s \int_0^t e^{(-\beta+\varepsilon-\gamma)(t-s)} \mathrm{d}s \, \sup_{s\in[0,t]} e^{-\gamma s} \|\Pi_s f(s)\|, \end{aligned}$$

and

$$\int_0^t e^{(-\beta+\varepsilon-\gamma)(t-s)} \mathrm{d}s = \int_0^t e^{-(\beta+\gamma-\varepsilon)s} \mathrm{d}s \leq \frac{1}{\beta+\gamma-\varepsilon},$$

and the result follows by choosing $\varepsilon = (\beta+\gamma)/2$. □

Convolution's formula: By using the convolution formula (4.23) we deduce that

$$\left(e^{A\cdot} * f\right)(t) = e^{A(t-s)} \left(e^{A\cdot} * f\right)(s) + \left(e^{A\cdot} * f(s+.)\right)(t-s), \tag{4.29}$$

whenever $t, s \in [0,\tau]$ with $s \leq t$, and $f \in C([0,\tau], \mathbb{R}^n)$ (for some $\tau > 0$).

By using the above formula (4.29) and the property of $e^{A_s t}$ we obtain the following lemma.

Lemma 4.17 *Let Assumption 4.3 be satisfied. Then we have the following:*

(i) *For each $\eta \in [0,\beta)$, each $f \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$, and each $t \in \mathbb{R}$,*

$$K_s(f)(t) := \lim_{\sigma\to-\infty} \Pi_s \left(e^{A\cdot} * f(\sigma+.)\right)(t-\sigma) \text{ exists.}$$

(ii) *For each $\eta \in [0,\beta)$, K_s is a bounded linear operator from $BC^\eta(\mathbb{R}, \mathbb{R}^n)$ into itself. More precisely, for each $\nu \in (-\beta, 0)$, we have*

$$\|K_s\|_{\mathcal{L}(BC^\eta(\mathbb{R},\mathbb{R}^n), BC^\eta(\mathbb{R},X_s))} \leq \widehat{C}_{s,\nu} \, \|\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)}, \forall \eta \in [0, -\nu],$$

where $\widehat{C}_{s,\nu} = \dfrac{2M_s}{\beta+\nu} > 0$ is the constant introduced in (4.28).

(iii) *For each* $\eta \in [0, \beta)$, *each* $f \in BC^{\eta}(\mathbb{R}, \mathbb{R}^n)$, *and each* $t, s \in \mathbb{R}$ *with* $t \geq s$,

$$K_s(f)(t) - e^{A_s(t-s)}K_s(f)(s) = \Pi_s\left(e^{A\cdot} * f(s+.)\right)(t-s).$$

Remark 4.18 The definition of K_s by using a limit as above can be extended to infinite dimensional systems (see Magal and Ruan [155]). Nevertheless for ordinary differential equation we have the following

$$\begin{aligned} K_s(f)(t) &= \lim_{\sigma\to-\infty} \int_0^{t-\sigma} e^{A_s(t-\sigma-l)}\Pi_s f(\sigma+l)dl \\ &= \lim_{\sigma\to-\infty} \int_\sigma^t e^{A_s(t-s)}\Pi_s f(s)ds \\ &= \lim_{\sigma\to-\infty} \int_0^{t-\sigma} e^{A_s\theta}\Pi_s f(t-\theta)d\theta \end{aligned}$$

hence we obtain the explicit formula

$$K_s(f)(t) = \int_0^{+\infty} e^{A_s\theta}\Pi_s f(t-\theta)d\theta.$$

Proof Let start by proving (i) and (iii). Let $\eta \in (0, \beta)$ be fixed. By using (4.29), we have for each $t, s, r \in \mathbb{R}$ with $r \leq s \leq t$, and each $f \in BC^{\eta}(\mathbb{R}, \mathbb{R}^n)$ that

$$\left(e^{A\cdot} * f(r+.)\right)(t-r) = e^{A(t-s)}\left(e^{A\cdot} * f(r+.)\right)(s-r) + \left(e^{A\cdot} * f(s+.)\right)(t-s).$$

By projecting this equation on X_s, we obtain for all $t, s, r \in \mathbb{R}$ with $r \leq s \leq t$ that

$$\Pi_s\left(e^{A\cdot} * f(r+.)\right)(t-r) = e^{A_s(t-s)}\Pi_s\left(e^{A\cdot} * f(r+.)\right)(s-r) + \Pi_s\left(e^{A\cdot} * f(s+.)\right)(t-s). \tag{4.30}$$

Let $\nu \in (-\beta, -\eta)$ be fixed. Then by using (4.28) and (4.30), we have for all $t, s, r \in \mathbb{R}$ with $r \leq s \leq t$ that

$$\begin{aligned} &\left\|\Pi_s\left(e^{A\cdot} * f(r+.)\right)(t-r) - \Pi_s\left(e^{A\cdot} * f(s+.)\right)(t-s)\right\| \\ &= \left\|e^{A_s(t-s)}\Pi_s\left(e^{A\cdot} * f(r+.)\right)(s-r)\right\| \\ &\leq M_s e^{-\beta(t-s)}\widehat{C}_{s,\nu}e^{\nu(s-r)} \sup_{l\in[0,s-r]} e^{-\nu l}\|\Pi_s f(r+l)\| \\ &= M_s\widehat{C}_{s,\nu}e^{-\beta(t-s)}e^{\nu(s-r)} \sup_{\sigma\in[r,s]} e^{-\nu(\sigma-r)}\|\Pi_s f(\sigma)\| \\ &= M_s\widehat{C}_{s,\nu}e^{-\beta(t-s)}e^{\nu s} \sup_{l\in[r,s]} e^{-\nu\sigma}e^{\eta|\sigma|}e^{-\eta|\sigma|}\|\Pi_s f(\sigma)\| \\ &\leq \|\Pi_s f\|_{BC^{\eta}(\mathbb{R},\mathbb{R}^n)} M_s\widehat{C}_{s,\nu}e^{-\beta(t-s)}e^{\nu s} \sup_{\sigma\in[r,s]} e^{-\nu\sigma}e^{\eta|\sigma|}. \end{aligned}$$

Hence, for all $s, r \in \mathbb{R}_-$ with $s \geq r$, we obtain

$$\begin{aligned} &\left\|\Pi_s\left(e^{A\cdot} * f(r+.)\right)(t-r) - \Pi_s\left(e^{A\cdot} * f(s+.)\right)(t-s)\right\| \\ &\quad \leq \|\Pi_s f\|_{BC^{\eta}(\mathbb{R},\mathbb{R}^n)} M_s\widehat{C}_{s,\nu}e^{-\beta(t-s)}e^{\nu s} \sup_{\sigma\in[r,s]} e^{-(\nu+\eta)\sigma}. \end{aligned}$$

Because $-(\nu+\eta) > 0$, we have

$$\begin{aligned}&\left\|\Pi_s\left(e^{A\cdot} * f(r+.)\right)(t-r) - \Pi_s\left(e^{A\cdot} * f(s+.)\right)(t-s)\right\| \\ &\leq \|\Pi_s f\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} M_s \widehat{C}_{s,\nu} e^{-\beta(t-s)} e^{\nu s} e^{-(\nu+\eta)s} \\ &= \|\Pi_s f\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} M_s \widehat{C}_{s,\nu} e^{-\beta t} e^{(\beta-\eta)s}.\end{aligned}$$

Since $\beta - \eta > 0$, by using Cauchy sequences, we deduce that

$$K_s(f)(t) = \lim_{s\to-\infty} \Pi_s\left(e^{A\cdot} * f(s+.)\right)(t-s) \text{ exists.}$$

Taking the limit as r goes to $-\infty$ in (4.30), we obtain (iii).

Let us prove the property (ii). Let $\nu \in (-\beta, 0)$ and $\eta \in [0, -\nu]$ be fixed. For each $f \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$ and each $t \in \mathbb{R}$, we have

$$\begin{aligned}\|K_s(f)(t)\| &= \lim_{r\to-\infty}\left\|\Pi_s\left(e^{A\cdot} * f(r+.)\right)(t-r)\right\| \\ &\leq \widehat{C}_{s,\nu} \limsup_{r\to-\infty} e^{\nu(t-r)} \sup_{l\in[0,t-r]} e^{-\nu l} \|\Pi_s f(r+l)\| \\ &= \widehat{C}_{s,\nu} \limsup_{r\to-\infty} e^{\nu(t-r)} \sup_{\sigma\in[r,t]} e^{-\nu(\sigma-r)} \|\Pi_s f(\sigma)\| \\ &= \widehat{C}_{s,\nu} \limsup_{r\to-\infty} e^{\nu t} \sup_{\sigma\in[r,t]} e^{-\nu\sigma} e^{\eta|\sigma|} e^{-\eta|\sigma|} \|\Pi_s f(\sigma)\| \\ &= \widehat{C}_{s,\nu} e^{\nu t} \|\Pi_s f\|_\eta \sup_{\sigma\in(-\infty,t]} e^{-\nu\sigma} e^{\eta|\sigma|}.\end{aligned}$$

Since $(\nu+\eta) \leq 0$, we deduce that if $t \leq 0$,

$$\begin{aligned}e^{-\eta|t|}\|K_s(f)(t)\| &\leq \widehat{C}_{s,\nu} e^{(\nu+\eta)t} \|\Pi_s f\|_\eta \sup_{\sigma\in(-\infty,t]} e^{-(\nu+\eta)\sigma} = \widehat{C}_{s,\nu} e^{(\nu+\eta)t} \|\Pi_s f\|_\eta e^{-(\nu+\eta)t} \\ &= \widehat{C}_{s,\nu} \|\Pi_s f\|_\eta\end{aligned}$$

and since $(\eta - \nu) > 0$, it follows that if $t \geq 0$,

$$\begin{aligned}e^{-\eta|t|}\|K_s(f)(t)\| &\leq \widehat{C}_{s,\nu} e^{(\nu-\eta)t} \|\Pi_s f\|_\eta \sup_{\sigma\in(-\infty,t]} e^{-\nu\sigma} e^{\eta|\sigma|} \\ &\leq \widehat{C}_{s,\nu} \|\Pi_s f\|_\eta e^{(\nu-\eta)t} \max\left(\sup_{\sigma\in(-\infty,0]} e^{-(\nu+\eta)\sigma}, \sup_{\sigma\in[0,t]} e^{(\eta-\nu)\sigma}\right) \\ &= \widehat{C}_{s,\nu} \|\Pi_s f\|_\eta e^{(\nu-\eta)t} e^{(\eta-\nu)t} = \widehat{C}_{s,\nu} \|\Pi_s f\|_\eta.\end{aligned}$$

This completes the proof. □

Lemma 4.19 *Let Assumption 4.3 be satisfied. Let $\eta \in [0, \beta)$ be fixed. Then we have the following:*

(i) *For each $f \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$ and each $t \in \mathbb{R}$,*

$$K_u(f)(t) := -\int_t^{+\infty} e^{-A_u(l-t)} \Pi_u f(l) dl := -\lim_{r\to+\infty} \int_t^r e^{-A_u(l-t)} \Pi_u f(l) dl$$

exists.

(ii) K_u *is a bounded linear operator from* $BC^\eta(\mathbb{R},\mathbb{R}^n)$ *into* $BC^\eta(\mathbb{R},X_u)$ *and*

$$\|K_u\|_{\mathcal{L}(BC^\eta(\mathbb{R},\mathbb{R}^n))} \leq \frac{M_u \|\Pi_u\|_{\mathcal{L}(\mathbb{R}^n)}}{(\beta-\eta)}.$$

(iii) *For each* $f \in BC^\eta(\mathbb{R},\mathbb{R}^n)$ *and each* $t,s \in \mathbb{R}$ *with* $t \geq s$,

$$K_u(f)(t) - e^{A_u(t-s)}K_u(f)(s) = \Pi_u\left(e^{A\cdot} * f(s+.)\right)(t-s).$$

Proof By Lemma 4.5, we have

$$\left\|e^{-A_u t}\right\|_{\mathcal{L}(X_u)} \leq M_u e^{-\beta t}, \forall t \geq 0. \tag{4.31}$$

By using (4.31) and the same argument as in the proof of Lemma 4.17, we obtain (i) and (ii). Moreover, for each $s,t,r \in \mathbb{R}$ with $s \leq t \leq r$, we have

$$\begin{aligned}\int_s^r e^{A_u(s-l)}\Pi_u f(l)dl &= \int_s^t e^{A_u(s-l)}\Pi_u f(l)dl + \int_t^r e^{A_u(s-l)}\Pi_u f(l)dl \\ &= \int_s^t e^{A_u(s-l)}\Pi_u f(l)dl + e^{A_u(s-t)}\int_t^r e^{A_u(t-l)}\Pi_u f(l)dl.\end{aligned}$$

It follows that

$$e^{A_u(t-s)}\int_s^r e^{A_u(s-l)}\Pi_u f(l)dl = \int_s^t e^{A_u(t-l)}\Pi_u f(l)dl + \int_t^r e^{A_u(t-l)}\Pi_u f(l)dl.$$

When $r \longrightarrow +\infty$, we obtain for all $s,t \in \mathbb{R}$ with $s \leq t$ that

$$\begin{aligned}-e^{A_u(t-s)}K_{u,\eta}(f)(s) &= \int_0^{t-s} e^{A_u(t-s-r)}\Pi_u f(s+r)dr - K_{u,\eta}(f)(t) \\ &= \Pi_u\left(e^{A\cdot} * f(s+.)\right)(t-s) - K_{u,\eta}(f)(t).\end{aligned}$$

This gives (iii). □

Lemma 4.20 *Let Assumption 4.3 be satisfied. Let* $\eta \in (0,\beta)$ *be fixed. For each* $x_c \in X_c$, *each* $f \in BC^\eta(\mathbb{R},\mathbb{R}^n)$, *and each* $t \in \mathbb{R}$, *denote*

$$K_1(x_c)(t) := e^{A_c t}x_c, \quad K_c(f)(t) := \int_0^t e^{A_c(t-s)}\Pi_c f(s)ds.$$

Then K_1 *(respectively* K_c*) is a bounded linear operator from* X_c *into* $BC^\eta(\mathbb{R},X_c)$ *(respectively from* $BC^\eta(\mathbb{R},\mathbb{R}^n)$ *into* $BC^\eta(\mathbb{R},X_c)$, *and*

$$\|K_1\|_{\mathcal{L}(X_c,BC^\eta(\mathbb{R},\mathbb{R}^n))} \leq \max\left(\sup_{t\geq 0}\left\|e^{(A_c-\eta I)t}\right\|, \sup_{t\geq 0}\left\|e^{-(A_c+\eta I)t}\right\|\right),$$

$$\|K_c\|_{\mathcal{L}(BC^\eta(\mathbb{R},\mathbb{R}^n))} \leq \|\Pi_c\|_{\mathcal{L}(\mathbb{R}^n)} \max\left(\int_0^{+\infty}\left\|e^{(A_c-\eta I)l}\right\|dl, \int_0^{+\infty}\left\|e^{-(A_c+\eta I)l}\right\|dl\right).$$

Proof The proof is straightforward. □

Lemma 4.21 *Let Assumption 4.3 be satisfied. Let $\eta \in (0, \beta)$ be fixed. Then*

$$u \in BC^{\eta}(\mathbb{R}, \mathbb{R}^n),$$

is a complete orbit for the semiflow $\{U(t)\}_{t\geq 0}$ if and only if

$$u(t) = K_1(\Pi_c u(0))(t) + K_c(F(u(.)))(t) + K_u(F(u(.)))(t) + K_s(F(u(.)))(t), \forall t \in \mathbb{R}. \tag{4.32}$$

Proof Let $u \in BC^{\eta}(\mathbb{R}, \mathbb{R}^n)$. Assume first that u is a complete orbit of $\{U(t)\}_{t\geq 0}$. Then for $k \in \{c, u\}$ we have for all $l, r \in \mathbb{R}$ with $r \leq l$ that

$$\Pi_k u(l) = e^{A_k(l-r)}\Pi_k u(r) + \int_r^l e^{A_k(l-s)}\Pi_k F(u(s))ds.$$

Thus,

$$\frac{d\Pi_k u(t)}{dt} = A_k \Pi_k u(t) + \Pi_k F(u(t)), \quad \forall t \in \mathbb{R}.$$

From this ordinary differential equation, we first deduce that

$$\Pi_c u(t) = e^{A_c t}\Pi_c u(0) + \int_0^t e^{A_c(t-s)}\Pi_c F(u(s))ds, \forall t \in \mathbb{R}. \tag{4.33}$$

Hence, for each $t, l \in \mathbb{R}$,

$$\Pi_u u(t) = e^{A_u(t-l)}\Pi_u u(l) + \int_l^t e^{A_u(t-s)}\Pi_u F(u(s))ds, \forall t, l \in \mathbb{R}.$$

It follows that for each $l \geq 0$,

$$\left\| e^{-A_u(l-t)}\Pi_u u(l) \right\| \leq M_u \|\Pi_u\|_{\mathcal{L}(\mathbb{R}^n)} e^{-\beta(l-t)} e^{\eta l} \|u\|_{BC^{\eta}(\mathbb{R},\mathbb{R}^n)}.$$

So when l goes to $+\infty$, we obtain

$$\Pi_u u(t) = -\int_t^{+\infty} e^{A_u(t-s)}\Pi_u F(u(s))ds, \quad \forall t \in \mathbb{R}. \tag{4.34}$$

Furthermore, we have for all $t, l \in \mathbb{R}$ with $t \geq l$ that

$$\Pi_s u(t) = e^{A_s(t-l)}\Pi_s u(l) + \Pi_s \left(e^{A \cdot} * F(u(l + .)) \right)(t - l),$$

and for each $l \leq 0$ that

$$\left\| e^{A_s(t-l)}\Pi_s u(l) \right\| \leq e^{-\beta t} M_s \|\Pi_s u\|_{\eta} e^{(\beta-\eta)l}.$$

Taking $l \to -\infty$, we obtain

$$\Pi_s u(t) = K_s(F(u(.)))(t), \quad \forall t \in \mathbb{R}. \tag{4.35}$$

Finally, by summing up (4.33), (4.34), and (4.35), we obtain (4.32).

Conversely, assume that u is a solution of (4.32). Then

$$\Pi_c u(t) = e^{A_c t}\Pi_c u(0) + \int_0^t e^{A_c(t-s)}\Pi_c F(u(s))ds, \quad \forall t \in \mathbb{R}.$$

It follows that

$$\frac{d\Pi_c u(t)}{dt} = A_c \Pi_c u(t) + \Pi_c F(u(t)), \quad \forall t \in \mathbb{R}.$$

Thus, for $l, r \in \mathbb{R}_-$ with $r \leq l$,

$$\Pi_c u(l) = e^{A(t-s)}\Pi_c u(r) + \Pi_c \left(e^{A\cdot} * F(u(s+.))\right)(t-s).$$

Moreover, by using Lemma 4.17 (iii) and Lemma 4.19 (iii), we deduce that for all $t, s \in \mathbb{R}$ with $t \geq s$

$$\Pi_s u(t) - e^{A(t-s)}\Pi_s u(s) = \Pi_s \left(e^{A\cdot} * F(u(s+.))\right)(t-s),$$
$$\Pi_u u(t) - e^{A(t-s)}\Pi_u u(s) = \Pi_u \left(e^{A\cdot} * F(u(s+.))\right)(t-s).$$

Therefore, u satisfies (4.25) and is a complete orbit of $\{U(t)\}_{t\geq 0}$. □

Let $\eta \in (0, \beta)$ be fixed. We rewrite equation (4.32) as the following fixed point problem: To find $u \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$ such that

$$u = K_1(\Pi_c u(0)) + K_2 \Phi_F(u), \tag{4.36}$$

where Φ_F is a *Nemytskii operator* which is defined by

$$\Phi_F(u)(t) = F(u(t)), \quad \forall t \in \mathbb{R},$$

and since we assumed that $F \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^n)$, we deduce that

$$\Phi_F \in \text{Lip}\left(BC^\eta(\mathbb{R}, \mathbb{R}^n), BC^\eta(\mathbb{R}, \mathbb{R}^n)\right),$$

and

$$K_2 \in \mathcal{L}\left(BC^\eta(\mathbb{R}, \mathbb{R}^n), BC^\eta(\mathbb{R}, \mathbb{R}^n)\right),$$

is the linear operator defined by

$$K_2 = K_c + K_s + K_u.$$

Moreover, we have the following estimates

$$\|K_1\|_{\mathcal{L}(X_c, BC^\eta(\mathbb{R},\mathbb{R}^n))} \leq \max(\sup_{t\geq 0}\left\|e^{(A_c - \eta Id)t}\right\|, \sup_{t\geq 0}\left\|e^{-(A_c + \eta Id)t}\right\|),$$

$$\|\Phi_F\|_{\text{Lip}} \leq \|F\|_{\text{Lip}},$$

and for each $\nu \in (-\beta, 0)$, we have

$$\|K_2\|_{\mathcal{L}(BC^{\eta}(\mathbb{R},\mathbb{R}^n))} \leq \gamma(\nu,\eta), \forall \eta \in (0,-\nu],$$

where

$$\begin{aligned}\gamma(\nu,\eta) := \frac{2M_s}{\beta+\nu}\|\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)} + \frac{M_u\|\Pi_u\|_{\mathcal{L}(\mathbb{R}^n)}}{(\beta-\eta)} \\ + \|\Pi_c\|_{\mathcal{L}(\mathbb{R}^n)} \max\left(\int_0^{+\infty}\left\|e^{(A_c-\eta I)l}\right\|dl, \int_0^{+\infty}\left\|e^{-(A_c+\eta I)l}\right\|dl\right).\end{aligned} \tag{4.37}$$

Moreover, by Lemma 4.21, the η-center manifold is given by

$$V_\eta = \{x \in \mathbb{R}^n : \exists u \in BC^{\eta}(\mathbb{R},\mathbb{R}^n) \text{ a solution of (4.36) and } u(0) = x\}. \tag{4.38}$$

We are now in position to prove the existence of a center manifold for the semilinear equations with non-dense domain, which is a generalization of Vanderbauwhede and Iooss [217, Theorem 1, p.129].

Theorem 4.22 *Let Assumption 4.3 be satisfied. Let $\eta \in (0,\beta)$ be fixed. Assume that $F \in \mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)$ with*

$$\|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)}\|K_2\|_{\mathcal{L}(BC^{\eta}(\mathbb{R},\mathbb{R}^n))} < 1.$$

Then there exists a Lipschitz continuous map $\Psi : X_c \to X_h$ such that

$$V_\eta = \{x_c + \Psi(x_c) \in \mathbb{R}^n : x_c \in X_c\},$$

or equivalently

$$V_\eta = \{x \in \mathbb{R}^n : \Pi_c x = x_c, \Pi_h x = \Psi(x_c), \text{ and } x_c \in X_c\}.$$

Moreover, we have the following properties:

(i)

$$\sup_{x_c\in X_c}\|\Psi(x_c)\| \leq \|K_s + K_u\|_{\mathcal{L}(BC^{\eta}(\mathbb{R},\mathbb{R}^n))}\sup_{x\in\mathbb{R}^n}\|\Pi_h F(x)\|.$$

(ii)

$$\|\Psi\|_{\mathrm{Lip}(X_c,X_h)} \leq \frac{\|K_s+K_u\|_{\mathcal{L}(BC^{\eta}(\mathbb{R},\mathbb{R}^n))}\|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)}\|K_1\|_{\mathcal{L}(X_c,BC^{\eta}(\mathbb{R},\mathbb{R}^n))}}{1-\|K_2\|_{\mathcal{L}(BC^{\eta}(\mathbb{R},\mathbb{R}^n))}\|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)}}. \tag{4.39}$$

(iii) *For each $u \in C(\mathbb{R},\mathbb{R}^n)$, the following statement are equivalent*

(a) *$u \in BC^{\eta}(\mathbb{R},\mathbb{R}^n)$ is a complete orbit of $\{U(t)\}_{t\geq 0}$.*

(b) *The map $u_c : \mathbb{R} \to X_c$ is a solution of the reduced equation*

$$\frac{du_c(t)}{dt} = A_c u_c(t) + \Pi_c F[u_c(t) + \Psi(u_c(t))], \tag{4.40}$$

and

$$u(t) = u_c(t) + \Psi(u_c(t)), \forall t \in \mathbb{R}.$$

Proof (i) Since $\|F\|_{\mathrm{Lip}} \|K_2\|_{\mathcal{L}(BC^\eta(\mathbb{R},\mathbb{R}^n))} < 1$ we can use the global inverse theorem (see Theorem 3.1 in Chapter 3), and we deduce that the map $(Id - K_2\Phi_F)$ is invertible, the inverse $(Id - K_2\Phi_F)^{-1}$ is Lipschitz continuous, and

$$\left\|(Id - K_2\Phi_F)^{-1}\right\|_{\mathrm{Lip}(BC^\eta(\mathbb{R},\mathbb{R}^n))} \leq \frac{1}{1 - \|K_2\|_{\mathcal{L}(BC^\eta(\mathbb{R},\mathbb{R}^n))} \|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)}}. \tag{4.41}$$

Let $x \in \mathbb{R}^n$ be fixed. By Lemma 4.21, we know that $x \in V_\eta$ if and only if there exists $u_{\Pi_c x} \in BC^\eta(\mathbb{R},\mathbb{R}^n)$, such that $u_{\Pi_c x}(0) = x$ and

$$u_{\Pi_c x}(t) = K_1(\Pi_c x)(t) + K_2\Phi_F\left(u_{\Pi_c x}\right)(t), \forall t \in \mathbb{R}.$$

So

$$V_\eta = \left\{(Id - K_2\Phi_F)^{-1} K_1(x_c)(0) : x_c \in X_c\right\}.$$

We define $\Psi : X_c \to X_h$ by

$$\Psi(x_c) = \Pi_h (Id - K_2\Phi_F)^{-1} K_1(x_c)(0), \forall x_c \in X_c.$$

Then

$$V_\eta = \{x_c + \Psi(x_c) : x_c \in X_c\}.$$

For each $x_c \in X_c$, set

$$u_{x_c} = (Id - K_2\Phi_F)^{-1} K_1(x_c),$$

which is equivalent to

$$u_{x_c} = K_1(x_c) + K_2\Phi_F\left(u_{x_c}\right).$$

By projecting this last equation on X_h, we obtain

$$\Pi_h u_{x_c} = [K_s + K_u]\,\Phi_{\Pi_h F}\left(u_{x_c}\right),$$

and we obtain

$$\Psi(x_c) = [K_s + K_u]\,\Phi_{\Pi_h F}\left(u_{x_c}\right)(0) \tag{4.42}$$

and assertion (i) follows.

Assertion (ii) follows from (4.41) and (4.42).

It remains to prove assertion (iii). Assume first that $u \in BC^\eta(\mathbb{R},\mathbb{R}^n)$ is a complete orbit of $\{U(t)\}_{t\geq 0}$. By using the invariance property of the center manifold (see Lemma 4.13), we deduce that

$$u(t) \in V_\eta, \forall t \in \mathbb{R},$$

and we deduce that

$$\Pi_h u(t) = \Psi(\Pi_c u(t)), \quad \forall t \in \mathbb{R}.$$

Moreover, by projecting (4.25) on X_c, we have for each $t, s \in \mathbb{R}$ with $t \geq s$ that

$$\Pi_c u(t) = e^{A_c(t-s)}\Pi_c u(s) + \int_0^{t-s} e^{A_c(t-s-l)}\Pi_c F(u(s+l))\,dl.$$

Thus, $t \to \Pi_c u(t)$ is a solution of the reduced equation (4.40).

Conversely, assume that $u \in C(\mathbb{R}, \mathbb{R}^n)$ satisfies (iii)-(b). Then

$$\Pi_h u(t) = \Psi(\Pi_c u(t)), \quad \forall t \in \mathbb{R},$$

and $\Pi_c u(.) : \mathbb{R} \to X_c$ is a solution of (4.40). Set $x = u(0)$. We know that $x \in V_\eta$ (since $\Pi_h x = \Psi(\Pi_c x)$), therefore by using the definition of the center manifold V_η, we deduce that there exists a function $v \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$, which is a complete orbit of $\{U(t)\}_{t\geq 0}$, and $v(0) = x$. But since V_η is invariant under the semiflow, we deduce that

$$\Pi_h v(t) = \Psi(\Pi_c v(t)), \quad \forall t \in \mathbb{R},$$

and $\Pi_c v(.) : \mathbb{R} \to X_c$ is a solution of (4.40). Finally, since $\Pi_c v(0) = \Pi_c u(0)$, and since F and Ψ are Lipschitz continuous, we deduce that (4.40) has at most one solution. It follows that

$$\Pi_c v(t) = \Pi_c u(t), \forall t \in \mathbb{R},$$

and by construction

$$\Pi_h v(t) = \Psi(\Pi_c v(t)) = \Psi(\Pi_c u(t)) = \Pi_h u(t), \quad \forall t \in \mathbb{R}.$$

Thus,

$$u(t) = v(t), \ \forall t \in \mathbb{R}.$$

Therefore, $u \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$ is a complete orbit of $\{U(t)\}_{t\geq 0}$. □

We defined $\text{Lip}_B(\mathbb{R}^n, \mathbb{R}^n)$, as the space of maps $F : \mathbb{R}^n \to \mathbb{R}^n$ which are Lipschitz continuous and bounded. That is

$$\|F\|_{\text{Lip}} = \sup_{x,y\in\mathbb{R}^n : x\neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|} < +\infty.$$

and

$$\|F\|_\infty = \sup_{x\in\mathbb{R}^n} \|F(x)\| < +\infty.$$

Lemma 4.23 *Let Assumption 4.3 be satisfied. Assume that $F \in \text{Lip}_B(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$V_\eta = V_\xi, \quad \forall \eta, \xi \in (0, \beta).$$

Proof Let $\eta, \xi \in (0, \beta)$ be such that $\xi < \eta$. Let $x \in V_\xi$. By the definition of V_ξ there exists $u \in BC^\xi(\mathbb{R}, \mathbb{R}^n)$, a complete orbit of $\{U(t)\}_{t\geq 0}$, such that $u(0) = x$. But $BC^\xi(\mathbb{R}, \mathbb{R}^n) \subset BC^\eta(\mathbb{R}, \mathbb{R}^n)$, so $u \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$, and we deduce that $x \in V_\eta$.

Conversely, let $x \in V_\eta$ be fixed. By the definition of V_η there exists $u \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$, a complete orbit of $\{U(t)\}_{t\geq 0}$, such that $u(0) = x$. By Lemma 4.21 we deduce that u is a solution of

$$u = K_1(\Pi_c u(0)) + K_2 \Phi_F(u).$$

But $K_1(\Pi_c u(0)) \in BC^{\xi}(\mathbb{R},\mathbb{R}^n)$ and F is bounded, so we have $\Phi_F(u) \in BC^0(\mathbb{R},\mathbb{R}^n) \subset BC^{\xi}(\mathbb{R},\mathbb{R}^n)$ and

$$K_2\Phi_F(u) \in BC^{\xi}(\mathbb{R},\mathbb{R}^n).$$

Hence, $u \in BC^{\xi}(\mathbb{R},\mathbb{R}^n)$ and

$$u = K_1(\Pi_c u(0)) + K_2\Phi_F(u).$$

Using again Lemma 4.21 once more, we obtain that $x \in V_{\xi}$. □

4.3.4 Smoothness of the Center Manifold

In the sequel, we will use the following notations. Let $k \geq 1$ be an integer, let $Y_1, Y_2, .., Y_k$, Y and Z be Banach spaces. Denote $\mathcal{L}^{(k)}(Y_1, Y_2, .., Y_k, Z)$ (resp. $\mathcal{L}^{(k)}(Y,Z)$) the space of bounded k-linear maps (or multilinear map) from $Y_1 \times \ldots \times Y_k$ into Z (resp. from Y^k into Z). We recall that if L k-linear maps Y^k into Z its norm is defined as

$$\|L\|_{\mathcal{L}^{(k)}(Y,Z)} = \sup_{\|u_1\|\leq 1,\ldots,\|u_k\|\leq 1} \|L(u_1,\ldots,u_k)\|.$$

Let V be an open subset of Y, and let $W \in C^k(V,Z)$ be fixed. We choose the convention that if $l = 1,\ldots,k-1$ and $u, \widehat{u} \in V$ with $u \neq \widehat{u}$, the quantity

$$\sup_{u_1,\ldots,u_l \in B_Y(0,1)} \frac{\left\| \left[D^l W(u) - D^l W(\widehat{u})\right](u_1,\ldots,u_l) - D^{l+1} W(\widehat{u})(u-\widehat{u}, u_1,\ldots,u_l)\right\|}{\|u-\widehat{u}\|}$$

goes to 0 as $\|u-\widehat{u}\| \to 0$.

Set

$$C_b^k(V,Z) := \left\{ W \in C^k(V,Z) : |W|_{j,V} := \sup_{u\in V} \left\|D^j W(u)\right\|_{\mathcal{L}^{(j)}(Y,Z)} < +\infty,\ 0 \leq j \leq k \right\}.$$

For each $\eta \in [0,\beta)$, consider $K_h : BC^{\eta}(\mathbb{R},\mathbb{R}^n) \to BC^{\eta}(\mathbb{R},X_h)$, the bounded linear operator defined by

$$K_h = K_s + K_u,$$

where K_s and K_u are the bounded linear operators defined, respectively, in Lemma 4.17 and Lemma 4.19.

For each $\varrho > 0$ we define some open and closed neighborhood of X_c as

(Open neighborhood of X_c) $\quad V_{\varrho} := \{x \in \mathbb{R}^n : \|\Pi_h x\| < \varrho\}$,

and

(Closed neighborhood of X_c) $\quad \overline{V}_{\varrho} := \{x \in \mathbb{R}^n : \|\Pi_h x\| \leq \varrho\}$.

Moreover for each $\eta \geq 0$, we define

$$\overline{V}_{\varrho}^{\eta} := \left\{ u \in BC^{\eta}\left(\mathbb{R}, \mathbb{R}^{n}\right) : u(t) \in \overline{V}_{\varrho}, \forall t \in \mathbb{R} \right\}.$$

Note that since $\overline{V}_{\varrho}$ is a closed and convex subset of $\mathbb{R}^n$, the subset $\overline{V}_{\varrho}^{\eta}$ is also a closed convex subset of $BC^{\eta}\left(\mathbb{R}, \mathbb{R}^{n}\right)$ for each $\eta \geq 0$.

We make the following assumption.

Assumption 4.24 Let $k \geq 1$ be an integer and let $\eta, \widehat{\eta} \in (0, \beta)$ such that

$$k\eta < \widehat{\eta} < \beta.$$

Let $\varrho > 0$. Assume

(i)

$$F \in \operatorname{Lip}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right) \cap C_{b}^{k}\left(V_{\varrho}, \mathbb{R}^{n}\right).$$

(ii)

$$\varrho_0 := \|K_h\|_{\mathcal{L}\left(BC^{0}(\mathbb{R},\mathbb{R}^{n})\right)} \|\Pi_h F\|_{\infty} < \varrho,$$

where

$$\|F\|_{\infty} = \sup_{x \in \mathbb{R}^{n}} \|F(x)\|.$$

(iii)

$$\sup_{\theta \in [\eta, \widehat{\eta}]} \|K_2\|_{\mathcal{L}\left(BC^{\theta}(\mathbb{R},\mathbb{R}^{n})\right)} \|F\|_{\operatorname{Lip}(\mathbb{R}^{n},\mathbb{R}^{n})} < 1.$$

Note that by using (4.37) we deduce that

$$\sup_{\theta \in [\eta, \widehat{\eta}]} \|K_2\|_{\mathcal{L}\left(BC^{\theta}(\mathbb{R},\mathbb{R}^{n})\right)} < +\infty.$$

Thus, Assumption 4.24 makes sense.

Following the approach of Vanderbauwhede [216, Corollary 3.6] and Vanderbauwhede and Iooss [217, Theorem 2], we obtain the following result on the smoothness of the center manifold.

Theorem 4.25 (Smoothness of the Center Manifold) *Let Assumptions 4.3 and 4.24 be satisfied. Then the map Ψ obtained in Theorem 4.22 belongs to the space $C_b^k\left(X_c, X_h\right)$.*

Definition 4.26 Let (M, d) be a metric space and $H : M \to M$ be a map. A fixed point $\overline{x} \in M$ of H is said to be **attracting** if

$$\lim_{n \to +\infty} H^{n}(x) = \overline{x} \quad \text{for each } x \in M.$$

The following lemma is an extension of the Fibre contraction theorem (which corresponds to the case $k = 1$). This result is taken from [216, Corollary3.6].

Theorem 4.27 (Fibre contraction) *Let $k \geq 1$ be an integer and let $(M_0, d_0), \ldots, (M_k, d_k)$ be complete metric spaces, and consider a map $H : M_0 \times \ldots \times M_k \to M_0 \times \cdots \times M_k$ having the following the form*

$$H(x_0, x_1, \ldots, x_k) = (H_0(x_0), H_1(x_0, x_1), \ldots, H_k(x_0, x_1, \ldots, x_k)),$$

where each map $H_l : M_0 \times M_1 \times \ldots \times M_l \to M_l$ (for each $l = 0, \ldots, k$,) is a uniform contraction. That is, H_0 is a contraction, and for each $l = 1, \ldots, k$, there exists a constant $0 \leq \tau_l < 1$ such that

$$d_l(H_l(x_0, x_1, \ldots, x_{l-1}, x_l), H_l(x_0, x_1, \ldots, x_{l-1}, \widehat{x}_l)) \leq \tau_l \, d_l(x_l, \widehat{x}_l),$$

whenever $(x_0, x_1, \ldots, x_{l-1}) \in M_0 \times M_1 \times \ldots \times M_{l-1}$, and $x_l, \widehat{x}_l \in M_l$.

Then F has a unique fixed point $(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_k)$, satisfying

$$\begin{aligned} \overline{x}_0 &= H_0(\overline{x}_0), \\ \overline{x}_1 &= H_1(\overline{x}_0, \overline{x}_1), \\ &\vdots \\ \overline{x}_k &= H_k(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_k). \end{aligned}$$

If we assume assume in addition that each map (for each $l = 1, \ldots, k$)

$$H_l(., \overline{x}_l) : M_0 \times M_1 \times \ldots \times M_{l-1} \to M_l$$

is continuous, then $(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_k)$ is an attracting fixed point of H.

Proof We prove the lemma for $k = 1$. The proof for any integer $k \geq 1$ can be easily derived from this case. By the Banach fixed point theorem, H_0 has a unique fixed point $\overline{x}_0 \in M_0$, and the map $x_1 \to H_1(\overline{x}_0, x_1)$ also has a unique fixed point $\overline{x}_1 \in M_1$. It is clear that $(\overline{x}_0, \overline{x}_1)$ is the unique fixed point of H, so we only need to prove its attractivity.

Let $(x_0, x_1) \in M_0 \times M_1$. Consider the sequence $(x_0(n), x_1(n))$ defined by

$$(x_0(0), x_1(0)) := (x_0, x_1)$$

and

$$(x_0(n+1), x_1(n+1)) := (H_0(x_0(n)), H_1(x_0(n), x_1(n))), \ \forall n \geq 0.$$

Since H_0 is a contraction, it is clear that $\lim_{n\to+\infty} x_0(n) = \overline{x}_0$. It remains to show that $\lim_{n\to+\infty} x_1(n) = \overline{x}_1$. We observe first that

$$\begin{aligned} d(x_1(n+1), \overline{x}_1) &= d(H_1(x_0(n), x_1(n)), H_1(\overline{x}_0, \overline{x}_1)) \\ &\leq d(H_1(x_0(n), x_1(n)), H_1(x_0(n), \overline{x}_1)) + d(H_1(x_0(n), \overline{x}_1), H_1(\overline{x}_0, \overline{x}_1)) \\ &\leq \tau_1 d(x_1(n), \overline{x}_1) + \alpha_n, \end{aligned}$$

where

$$\alpha_n := d(H_1(x_0(n), \overline{x}_1), H_1(\overline{x}_0, \overline{x}_1)) \to 0 \text{ as } n \to +\infty.$$

Setting $\delta_n := d(x_1(n), \overline{x}_1)$, we obtain

$$\delta_{n+1} \le \tau_1 \delta_n + \alpha_n, \ \forall n \ge 0.$$

Since $\tau_1 < 1$, it is not difficult to prove that $\{\delta_n\}$ is bounded sequence and

$$\limsup_{n\to+\infty} \delta_n \le \tau_1 \limsup_{n\to+\infty} \delta_n.$$

Hence, $\limsup_{n\to+\infty} \delta_n = 0$, and the proof is completed. □

We recall that the map $\Psi : X_c \to X_h$ is defined by

$$\Psi(x_c) = \Pi_h (I - K_2\Phi_F)^{-1} (K_1 x_c)(0), \quad \forall x_c \in X_c.$$

We define the map $\Gamma_0 : BC^\eta(\mathbb{R}, X_c) \to BC^\eta(\mathbb{R}, \mathbb{R}^n)$ by

$$\Gamma_0(u) = (I - K_2\Phi_F)^{-1}(u), \quad \forall u \in BC^\eta(\mathbb{R}, X_c).$$

For each $\delta \ge 0$, the bounded linear operator $L : BC^\delta(\mathbb{R}, \mathbb{R}^n) \to X_h$ is defined by

$$L(u) = \Pi_h u(0), \quad \forall u \in BC^\delta(\mathbb{R}, X_c).$$

Then we have

$$\Psi(x_c) = L\Gamma_0(K_1 x_c), \quad \forall x_c \in X_c,$$

where Γ_0 is a the unique solution of the fixed problem

$$\Gamma_0(u) = u + K_2\Phi_F(\Gamma_0(u)), \quad \forall u \in BC^\eta(\mathbb{R}, X_c).$$

We deduce that $\Gamma_0 : BC^\eta(\mathbb{R}, X_c) \to BC^\eta(\mathbb{R}, \mathbb{R}^n)$ is the unique solution of the fixed problem

$$\Gamma_0 = J + K_2 \circ \Phi_F \circ (\Gamma_0), \tag{4.43}$$

where $J(u) = u$ is the continuous imbedding from $BC^\eta(\mathbb{R}, X_c)$ into $BC^\eta(\mathbb{R}, \mathbb{R}^n)$.

From (4.43), we deduce that for each $u \in BC^\eta(\mathbb{R}, X_c)$,

$$\|\Gamma_0(u) - u\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} \le \|K_2\|_{\mathcal{L}(BC^\eta(\mathbb{R},\mathbb{R}^n), BC^\eta(\mathbb{R},\mathbb{R}^n))} \|F\|_\infty,$$

$$\|\Pi_h\Gamma_0(u)(t)\|_{BC^0(\mathbb{R},\mathbb{R}^n)} \le \|K_h\|_{\mathcal{L}(BC^0(\mathbb{R},\mathbb{R}^n))} \|\Pi_h F\|_\infty = \varrho_0, \quad \forall t \in \mathbb{R}.$$

For each subset $E \subset BC^\eta(\mathbb{R}, X_c)$, denote

$$M_{0,E} = \left\{\Theta \in C\left(E, \overrightarrow{V}^0_{\varrho_0}\right) : \sup_{u\in E} \|\Theta(u) - u\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} < +\infty\right\}$$

and set

$$M_0 = M_{0,BC^\eta(\mathbb{R},X_c)}.$$

From the above remarks, it follows that Γ_0 (respectively $\Gamma_0 \mid_E$) must be an element of M_0 (respectively $M_{0,E}$). Since $\overrightarrow{V}^0_{\varrho_0}$ is a closed subset of $BC^\eta(\mathbb{R}, \mathbb{R}^n)$, we know that for each subset $E \subset BC^{\eta}(\mathbb{R}, X_c)$, $M_{0,E}$ is a complete metric space endowed with the metric

$$d_{0,E}\left(\Theta, \widetilde{\Theta}\right) = \sup_{u\in E} \left\|\Theta(u) - \widetilde{\Theta}(u)\right\|_{BC^{\eta}(\mathbb{R},\mathbb{R}^n)}.$$

Set

$$d_0 = d_{0,BC^{\eta}(\mathbb{R},X_c)}.$$

Lemma 4.28 (Regularity of the Nemytskii operator) *Let E be a Banach space and $W \in C_b^1(V_\varrho, E)$. Let $\xi \geq \delta \geq 0$ be fixed. Define $\Phi_W : V_\varrho^\eta \to BC^\xi(\mathbb{R}, E)$, $\Phi_{DW} : V_\varrho^\eta \to BC^\xi(\mathbb{R}, \mathcal{L}(\mathbb{R}^n, E))$, and $\Phi_W^{(1)} : V_\varrho^\eta \to \mathcal{L}\left(BC^\delta(\mathbb{R},\mathbb{R}^n), BC^\xi(\mathbb{R}, E)\right)$ for each $t \in \mathbb{R}$, each $u \in V_\varrho^\eta$, and each $v \in BC^\delta(\mathbb{R},\mathbb{R}^n)$ by*

$$\begin{aligned}
&\Phi_W(u)(t) := W(u(t)) \in E,\\
&\Phi_{DW}(u)(t) := DW(u(t)) \in \mathcal{L}(\mathbb{R}^n, E),\\
&\Phi_W^{(1)}(u)(v)(t) := DW(u(t))(v(t)) \in E,
\end{aligned}$$

respectively. Then we have the following:

(a) *If $\xi > 0$, then Φ_W and Φ_{DW} are continuous.*
(b) *For each $u, v \in V_\varrho^\eta$, $\Phi_W^{(1)}(u) \in \mathcal{L}\left(BC^\delta(\mathbb{R},\mathbb{R}^n), BC^\xi(\mathbb{R}, E)\right)$,*

$$\begin{aligned}
&\left\|\Phi_W^{(1)}(u) - \Phi_W^{(1)}(v)\right\|_{\mathcal{L}\left(BC^\delta(\mathbb{R},\mathbb{R}^n), BC^\xi(\mathbb{R},E)\right)}\\
&= \sup_{w\in BC^\delta(\mathbb{R},\mathbb{R}^n)} \left\|\Phi_W^{(1)}(u)(w) - \Phi_W^{(1)}(v)(w)\right\|_{BC^\xi(\mathbb{R},E)}\\
&\leq \sup_{t\in\mathbb{R}} e^{-(\xi-\delta)|t|} \|DW(u(t)) - DW(v(t))\|_{\mathcal{L}(\mathbb{R}^n,E)}\\
&= \|\Phi_{DW}(u) - \Phi_{DW}(v)\|_{BC^{\xi-\delta}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,E))}
\end{aligned}$$

and

$$\left\|\Phi_W^{(1)}(u)\right\|_{\mathcal{L}\left(BC^\delta(\mathbb{R},\mathbb{R}^n), BC^\xi(\mathbb{R},E)\right)} \leq \|\Phi_{DW}(u)\|_{BC^{\xi-\delta}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,E))} \leq |W|_{1,V_\varrho}.$$

(c) *If $\xi > \delta$, then $\Phi_W^{(1)}$ is continuous.*
(d) *If $\xi \geq \delta \geq \eta$, we have for each $u, \widehat{u} \in V_\varrho^\eta$ that*

$$\left\|\Phi_W(u) - \Phi_W(\widehat{u}) - \Phi_W^{(1)}(\widehat{u})(u - \widehat{u})\right\|_{BC^\xi(\mathbb{R},E)} \leq \|u - \widehat{u}\|_{BC^\delta(\mathbb{R},\mathbb{R}^n)} \varkappa_{\xi-\delta}(u, \widehat{u})$$

where

$$\varkappa_{\xi-\delta}(u,\widehat{u}) = \sup_{s\in[0,1]} \|\Phi_{DW}(su + (1-s)\widehat{u}) - \Phi_{DW}(\widehat{u})\|_{BC^{\xi-\delta}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,E))},$$

and if $\xi > \delta \geq \eta$, we have (by continuity of Φ_{DW})

$$\varkappa_{\xi-\delta}(u,\widehat{u}) \to 0 \quad \text{as } \|u - \widehat{u}\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} \to 0.$$

Proof We first prove that $\Phi_W \in C_b^0\left(V_\varrho^\eta, BC^\xi(\mathbb{R}, E)\right)$. For each $u, \widehat{u} \in V_\varrho^\eta$ and each $R > 0$, we have

$$\begin{aligned}\|\Phi_W(u) - \Phi_W(\widehat{u})\|_{BC^\xi(\mathbb{R},E)} &= \sup_{t\in\mathbb{R}} e^{-\xi|t|} \|W(u(t)) - W(\widehat{u}(t))\|_E \\ &= \max\left(\sup_{|t|\le R} e^{-\xi|t|} \|W(u(t)) - W(\widehat{u}(t))\|_E, 2\|W\|_0 e^{-\xi R}\right).\end{aligned} \tag{4.44}$$

Fix some arbitrary $\varepsilon > 0$. Let $R > 0$ be such that $2\|W\|_0 e^{-\xi R} < \varepsilon$ and denote $\Omega = \{\widehat{u}(t) : |t| \le R\}$. Since W is continuous and Ω is compact, we can find $\delta_1 > 0$ such that

$$\|W(x) - W(\widehat{x})\| \le \varepsilon \text{ if } \widehat{x} \in \Omega, \text{ and } \|x - \widehat{x}\| \le \delta_1.$$

Let $\delta = e^{-\eta R}\delta_1$. If $\|u - \widehat{u}\|_{BC^\eta(\mathbb{R},\mathbb{R}^n)} \le \delta$, then

$$\|u(t) - \widehat{u}(t)\| \le \delta_1, \forall t \in [-R, R],$$

and (4.44) implies

$$\|\Phi_W(u) - \Phi_W(\widehat{u})\|_{BC^\xi(\mathbb{R},E)} \le \varepsilon.$$

We now prove that $\Phi_W^{(1)} \in C\left(V_\varrho^\eta, \mathcal{L}\left(BC^\delta(\mathbb{R}, \mathbb{R}^n), BC^\xi(\mathbb{R}, E)\right)\right)$. From the first part of the proof, since E is an arbitrary Banach space, we deduce that Φ_{DW} is continuous. Moreover, for each $u, \widehat{u} \in V_\varrho^\eta$ and each $v \in BC^\delta(\mathbb{R}, \mathbb{R}^n)$,

$$\begin{aligned}\left\|\Phi_W^{(1)}(u)(v)\right\|_{BC^\xi(\mathbb{R},E)} &= \sup_{t\in\mathbb{R}} e^{-\xi|t|} \|DW(u(t))(v(t))\|_E \\ &\le \sup_{t\in\mathbb{R}} e^{-(\xi-\delta)|t|} \|DW(u(t))\|_{\mathcal{L}(\mathbb{R}^n,E)} e^{-\delta|t|}\|v(t)\|_{\mathbb{R}^n} \\ &\le \|\Phi_{DW}(u)\|_{BC^{\xi-\delta}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,E))} \|v\|_{BC^\delta(\mathbb{R},\mathbb{R}^n)},\end{aligned}$$

and

$$\begin{aligned}&\left\|\left[\Phi_W^{(1)}(u) - \Phi_W^{(1)}(\widehat{u})\right](v)\right\|_{BC^\xi(\mathbb{R},E)} \\ &= \sup_{t\in\mathbb{R}} e^{-\xi|t|} \|DW(u(t))(v(t)) - DW(\widehat{u}(t))(v(t))\|_E \\ &\le \|\Phi_{DW}(u) - \Phi_{DW}(\widehat{u})\|_{BC^{\xi-\delta}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,E))} \|v\|_{BC^\delta(\mathbb{R},\mathbb{R}^n)}.\end{aligned}$$

Thus, if $\xi \ge \delta$, we have for each $u \in V_\varrho^\eta$ that

$$\Phi_W^{(1)}(u) \in \mathcal{L}\left(BC^\delta(\mathbb{R}, \mathbb{R}^n), BC^\xi(\mathbb{R}, E)\right), \quad \forall u \in V_\varrho^\eta,$$

and if $\xi > \delta$,

$$\Phi_W^{(1)} \in C\left(V_\varrho^\eta, \mathcal{L}\left(BC^\delta(\mathbb{R}, \mathbb{R}^n), BC^\xi(\mathbb{R}, E)\right)\right), \quad \forall \mu > 0.$$

Since V_ϱ is an open and convex subset of $\mathbb{R}^n$, we have the following classical formula

$$W(x) - W(y) = \int_0^1 DW(sx + (1-s)y)(x-y)\, ds, \quad \forall x, y \in V_\varrho.$$

Therefore, for each $u, \widehat{u} \in V_\varrho^\eta$,

$$\begin{aligned}
&\left\| \Phi_W(u) - \Phi_W(\widehat{u}) - \Phi_W^{(1)}(\widehat{u})(u - \widehat{u}) \right\|_{BC^\xi(\mathbb{R},E)} \\
&= \sup_{t\in\mathbb{R}} e^{-\xi|t|} \| W(u(t)) - W(\widehat{u}(t)) - DW(\widehat{u}(t))(u(t) - \widehat{u}(t)) \| \\
&\le \sup_{t\in\mathbb{R}} \sup_{s\in[0,1]} e^{-\xi|t|} \| [DW(su(t) + (1-s)\widehat{u}(t)) - DW(\widehat{u}(t))](u(t) - \widehat{u}(t)) \| \\
&\le \|u - \widehat{u}\|_{BC^\delta(\mathbb{R},\mathbb{R}^n)} \sup_{s\in[0,1]} \|\Phi_{DW}(su + (1-s)\widehat{u}) - \Phi_{DW}(\widehat{u})\|_{BC^{\xi-\delta}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,E))}.
\end{aligned}$$

The proof is complete. □

The following Lemma is taken from Vanderbauwhede and Iooss [217, Lemma 3].

Lemma 4.29 *Let E be a Banach space and $W \in C_b^1(V_\varrho, E)$. Let Φ_W and $\Phi_W^{(1)}$ be defined as in Lemma 4.28. Let $\Theta \in C\left(BC^\eta(\mathbb{R}, X_c), V_\varrho^\eta\right)$ be such that*

(a) *Θ is of class C^1 from $BC^\eta(\mathbb{R}, X_c)$ into $BC^{\eta+\mu}(\mathbb{R}, \mathbb{R}^n)$ for each $\mu > 0$.*
(b) *The derivative of Θ takes the form*

$$D\Theta(u)(v) = \Theta^{(1)}(u)(v), \quad \forall u, v \in BC^\eta(\mathbb{R}, X_c),$$

for some globally bounded map

$$\Theta^{(1)} : BC^\eta(\mathbb{R}, X_c) \to \mathcal{L}(BC^\eta(\mathbb{R}, X_c), BC^\eta(\mathbb{R}, \mathbb{R}^n)).$$

Then

$$\Phi_W \circ \Theta \in C_b^0(BC^\eta(\mathbb{R}, X_c), BC^\eta(\mathbb{R}, E)) \cap C^1\left(BC^\eta(\mathbb{R}, X_c), BC^{\eta+\mu}(\mathbb{R}, E)\right)$$

for each $\mu > 0$, and

$$D(\Phi_W \circ \Theta)(u)(v) = \Phi_W^{(1)}(\Theta(u))\left(\Theta^{(1)}(u)(v)\right),$$

for each $u, v \in BC^\eta(\mathbb{R}, X_c)$.

Proof By using Lemma 4.28, it follows that

$$\Phi_W \circ \Theta \in C_b^0(BC^\eta(\mathbb{R}, X_c), BC^\eta(\mathbb{R}, E))$$

and

$$u \to \Phi_W^{(1)}(\Theta(u))\Theta^{(1)}(u) \in C\left(BC^\eta(\mathbb{R}, X_c), \mathcal{L}\left(BC^\eta(\mathbb{R}, X_c), BC^{\eta+\mu}(\mathbb{R}, E)\right)\right).$$

Let $u, \widehat{u} \in BC^\eta(\mathbb{R}, X_c)$. By Lemma 4.28, we also have

$$\begin{aligned}
&\left\|\Phi_W\left(\Theta(u)\right)-\Phi_W\left(\Theta(\widehat{u})\right)-\Phi_W^{(1)}\left(\Theta(\widehat{u})\right)\left(\Theta^{(1)}(\widehat{u})\,(u-\widehat{u})\right)\right\|_{BC^{\eta+\mu}(\mathbb{R},E)}\\
&\quad\le\left\|\Phi_W\left(\Theta(u)\right)-\Phi_W\left(\Theta(\widehat{u})\right)-\Phi_W^{(1)}\left(\Theta(\widehat{u})\right)\left(\Theta(u)-\Theta(\widehat{u})\right)\right\|_{BC^{\eta+\mu}(\mathbb{R},E)}\\
&\qquad+\left\|\Phi_W^{(1)}\left(\Theta(\widehat{u})\right)\left[\Theta(u)-\Theta(\widehat{u})-\Theta^{(1)}(\widehat{u})\,(u-\widehat{u})\right]\right\|_{BC^{\eta+\mu}(\mathbb{R},E)}\\
&\quad\le\left\|\Theta(u)-\Theta(\widehat{u})\right\|_{BC^{\eta+\mu/2}(\mathbb{R},\mathbb{R}^n)}\varkappa_{\mu/2}\left(\Theta(u),\Theta(\widehat{u})\right)\\
&\qquad+\left\|\Phi_{DW}\left(\Theta(\widehat{u})\right)\right\|_{BC^{\mu/2}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,E))}\left\|\Theta(u)-\Theta(\widehat{u})-\Theta^{(1)}(\widehat{u})\,(u-\widehat{u})\right\|_{BC^{\eta+\mu/2}(\mathbb{R},\mathbb{R}^n)}
\end{aligned}$$

and the result follows. □

One may extend the previous lemma to any order $k > 1$.

Lemma 4.30 *Let E be a Banach space and let $W \in C_b^k\left(V_\varrho, E\right)$ (for some integer $k \ge 1$). Let $l \in \{1, \dots, k\}$ be an integer. Suppose $\xi \ge 0, \mu \ge 0$ are two real numbers and $\delta_1, \delta_2, \dots, \delta_l \ge 0$ such that $\xi = \mu + \delta_1 + \delta_2 + \dots + \delta_l$. Define*

$$\Phi_{D^{(l)}W}\left(u\right)\left(t\right) := D^{(l)}W\left(u\left(t\right)\right), \forall t \in \mathbb{R}, \forall u \in V_\varrho^\eta,$$

$$\Phi_W^{(l)}\left(u\right)\left(u_1, u_2, \dots, u_l\right)\left(t\right) := D^{(l)}W\left(u\left(t\right)\right)\left(u_1\left(t\right), u_2\left(t\right), \dots, u_l\left(t\right)\right),$$

$$\forall t \in \mathbb{R}, \forall u \in V_\varrho^\eta, \forall u_i \in BC^{\delta_i}\left(\mathbb{R}, \mathbb{R}^n\right), \textit{ for } i = 1, \dots, l.$$

Then we have the following:

(a) *If $\xi > 0$, then $\Phi_{D^{(l)}W} : V_\varrho^\eta \to BC^\xi\left(\mathbb{R}, \mathcal{L}^{(l)}\left(\mathbb{R}^n, E\right)\right)$ is continuous.*

(b) *For each $u, v \in V_\varrho^\eta, \Phi_W^{(l)}\left(u\right) \in \mathcal{L}^{(l)}\left(BC^{\delta_1}\left(\mathbb{R}, \mathbb{R}^n\right), \dots, BC^{\delta_l}\left(\mathbb{R}, \mathbb{R}^n\right); BC^\xi\left(\mathbb{R}, E\right)\right)$,*

$$\begin{aligned}
&\left\|\Phi_W^{(l)}\left(u\right)-\Phi_W^{(l)}\left(v\right)\right\|_{\mathcal{L}^{(l)}\left(BC^{\delta_1}(\mathbb{R},\mathbb{R}^n),\dots,BC^{\delta_l}(\mathbb{R},\mathbb{R}^n);BC^{\xi}(\mathbb{R},E)\right)}\\
&\quad\le\left\|\Phi_{D^{(l)}W}\left(u\right)-\Phi_{D^{(l)}W}\left(v\right)\right\|_{BC^{\mu}\left(\mathbb{R},\mathcal{L}^{(l)}(\mathbb{R}^n,E)\right)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\Phi_W^{(l)}\left(u\right)\right\|_{\mathcal{L}^{(l)}\left(BC^{\delta_1}(\mathbb{R},\mathbb{R}^n),\dots,BC^{\delta_l}(\mathbb{R},\mathbb{R}^n);BC^{\xi}(\mathbb{R},E)\right)}\\
&\quad\le\left\|\Phi_{D^{(l)}W}\left(u\right)\right\|_{BC^{\mu}\left(\mathbb{R},\mathcal{L}^{(l)}(\mathbb{R}^n,E)\right)}\le\left|W\right|_{l,V_\varrho}.
\end{aligned}$$

(c) *If $\mu > 0$, then $\Phi_W^{(l)}$ is continuous.*

(d) *If $\delta_1 \ge \eta$, we have for each $u, \widehat{u} \in V_\varrho^\eta$ (with the convention for the derivatives defined at the beginning of this section) that*

$$\begin{aligned}
&\left\|\Phi_W^{(l-1)}\left(u\right)-\Phi_W^{(l-1)}\left(\widehat{u}\right)-\Phi_W^{(l)}\left(\widehat{u}\right)\left(u-\widehat{u}\right)\right\|_{\mathcal{L}^{(l-1)}\left(BC^{\delta_2}(\mathbb{R},\mathbb{R}^n),\dots,BC^{\delta_l}(\mathbb{R},\mathbb{R}^n);BC^{\xi}(\mathbb{R},E)\right)}\\
&\quad\le\left\|u-\widehat{u}\right\|_{BC^{\delta_1}(\mathbb{R},\mathbb{R}^n)}\varkappa_\mu^{(l)}\left(u,\widehat{u}\right),
\end{aligned}$$

where

$$\varkappa_\mu^{(l)}\left(u,\widehat{u}\right)=\sup_{s\in[0,1]}\left\|\Phi_{D^{(l)}W}\left(su+(1-s)\widehat{u}\right)-\Phi_{D^{(l)}W}\left(\widehat{u}\right)\right\|_{BC^{\mu}\left(\mathbb{R},\mathcal{L}^{(l)}(\mathbb{R}^n,E)\right)},$$

and if $\mu > 0$, we have by continuity of $\Phi_{D^{(l)}W}$ that

$$\varkappa_{\mu}^{(l)}(u,\widehat{u}) \to 0 \; as \; \|u-\widehat{u}\|_{BC^{\eta}(\mathbb{R},\mathbb{R}^n)} \to 0.$$

Proof This proof is similar to that of Lemma 4.28. □

In the following lemma we use a formula for the k^{th} differential of the composed two maps. This formula is taken from Avez [10, p. 38]. This formula corrects the one used in Vanderbauwhede [216, Proof of Lemma 3.11].

Lemma 4.31 *Let E be a Banach space and let $W \in C_b^k(V_\varrho, E)$. Let Φ_W and $W^{(k)}$ be defined as above. Let $\Theta \in C\left(BC^\eta(\mathbb{R}, X_c), V_\varrho^\eta\right)$ be such that*

(a) *Θ is of class C^k from $BC^\eta(\mathbb{R}, X_c)$ into $BC^{k\eta+\mu}(\mathbb{R}, \mathbb{R}^n)$ for each $\mu > 0$;*
(b) *for each $l = 1, \dots, k$, its derivative takes the form*

$$D^l\Theta(u)(v_1, v_2, \dots, v_l) = \Theta^{(l)}(u)(v_1, v_2, \dots, v_l), \forall u, v_1, v_2, \dots, v_l \in BC^\eta(\mathbb{R}, X_c),$$

for some globally bounded $\Theta^{(l)} : BC^\eta(\mathbb{R}, X_c) \to \mathcal{L}^{(l)}(BC^\eta(\mathbb{R}, X_c); BC^\eta(\mathbb{R}, \mathbb{R}^n))$.

Then $\Phi_W \circ \Theta \in C_b^0(BC^\eta(\mathbb{R}, X_c), BC^\eta(\mathbb{R}, E)) \cap C^k\left(BC^\eta(\mathbb{R}, X_c), BC^{k\eta+\mu}(\mathbb{R}, E)\right)$ for each $\mu > 0$. Moreover, for each $l = 1, \dots, k$ and each $u, v_1, v_2, \dots, v_l \in BC^\eta(\mathbb{R}, X_c)$,

$$D^l(\Phi_W \circ \Theta)(u)(v) = (\Phi_W \circ \Theta)^{(l)}(u)(v_1, v_2, \dots, v_l)$$

for some globally bounded $(\Phi_W \circ \Theta)^{(l)} : BC^\eta(\mathbb{R}, X_c) \to \mathcal{L}^{(l)}(BC^\eta(\mathbb{R}, X_c); BC^\eta(\mathbb{R}, E))$. More precisely, we have for $j = 1, \dots, k$ that

(i) $(\Phi_W \circ \Theta)^{(j)}(u) = \Phi_W^{(1)}(\Theta(u)) D^{(j)}\Theta(u) + \widetilde{\Phi}_{W,j}(u)$;
(ii) $\widetilde{\Phi}_{W,1}(u) = 0$;
(iii) *for $j > 1$, the map $\widetilde{\Phi}_{W,j}(u)$ is a finite sum $\sum_{\lambda\in\Lambda_j} \widetilde{\Phi}_{W,\lambda,j}(u)$, where for each $\lambda \in \Lambda_j$ the map $\widetilde{\Phi}_{W,\lambda,j}(u) : BC^\eta(\mathbb{R}, X_c) \to \mathcal{L}^{(j)}(BC^\eta(\mathbb{R}, X_c), BC^\eta(\mathbb{R}, E))$ has the following form*

$$\widetilde{\Phi}_{W,\lambda,j}(u)\left(u_1, u_2, \dots, u_j\right) = \Phi_W^{(l)}(\Theta(u)) \begin{pmatrix} D^{(r_1)}\Theta(u)\left(u_{i_1^{r_1}}, u_{i_2^{r_1}}, \dots, u_{i_{r_1}^{r_1}}\right), \dots, \\ D^{(r_l)}\Theta(u)\left(u_{i_1^{r_l}}, \dots, u_{i_{r_l}^{r_l}}\right) \end{pmatrix}$$

with $2 \le l \le j$, $1 \le r_i \le j-1$ for $1 \le i \le l$,

$$r_1 + r_2 + \dots + r_l = j,$$

$$\left\{i_1^{rm}, \dots, i_{r_m}^{r_m}\right\} \subset \{1, \dots, j\}, \forall m = 1, \dots, l$$

$$\left\{i_1^{rm}, \dots, i_{r_m}^{r_m}\right\} \cap \left\{i_1^{rn}, \dots, i_{r_n}^{r_n}\right\} = \emptyset, \; if \; m \ne n,$$

$$i_1^{rm} \le i_2^{rm} \le \dots \le i_{r_m}^{r_m}, \forall m = 1, \dots, l,$$

and each $\lambda \in \Lambda_j$ corresponds to each particular such a choice.

Proof This proof is similar to that of Lemma 4.29. □

Proof (Proof of Theorem 4.25) :

Step 1 (Existence of a fixed point): Let k, η, and $\widehat{\eta}$ be the numbers introduced in Assumption 4.4.4. Let $\mu > 0$ be such that $k\eta + (2k-1)\mu = \widehat{\eta}$. We first apply the fibre contraction Theorem 4.27. For each $j = 1, \ldots, k$ and each subset $E \subset BC^{\eta}(\mathbb{R}, X_c)$, define $M_{j,E}$ as the Banach space of all continuous maps $\Theta_j : E \to \mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R}, X_c), BC^{j\eta+(2j-1)\mu}(\mathbb{R}, \mathbb{R}^n)\right)$ such that

$$\left|\Theta_j\right|_j = \sup_{u \in E} \left\|\Theta_j(u)\right\|_{\mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_c),BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} < +\infty.$$

For $j = 0, \ldots, k$, define the map $H_{j,E} : M_{0,E} \times M_{1,E} \times \ldots \times M_{j,E} \to M_{j,E}$ as follows: If $j = 0$, set for each $u \in E$ that

$$H_{0,E}(\Theta_0)(u) = u + K_2 \circ \Phi_F \circ \Theta_0(u).$$

If $j = 1$, set for each $u \in E$ that

$$H_{1,E}(\Theta_0, \Theta_1)(u)(.) = J^1 + K_2 \circ \Phi_F^{(1)}(\Theta_0(u)) \circ \Theta_1(u), \tag{4.45}$$

where J^1 is the continuous imbedding from $BC^{\eta}(\mathbb{R}, X_c)$ into $BC^{\eta+\mu}(\mathbb{R}, \mathbb{R}^n)$.
If $k \geq 2$, set for each $j = 2, \ldots, k$ and each $u \in E$ that

$$\begin{aligned} &H_{j,E}\left(\Theta_0, \Theta_1, \ldots, \Theta_j\right)(u) \\ &\quad = K_2 \circ \Phi_F^{(1)}(\Theta_0(u)) \circ \Theta_j(u) + \widehat{H}_{j,E}\left(\Theta_0, \Theta_1, \ldots, \Theta_{j-1}\right)(u), \end{aligned} \tag{4.46}$$

where

$$\widehat{H}_{j,E}\left(\Theta_0, \Theta_1, \ldots, \Theta_{j-1}\right)(u) = \sum_{\lambda \in \Lambda_j} \widehat{H}_{\lambda,j,E}\left(\Theta_0, \Theta_1, \ldots, \Theta_{j-1}\right)(u)$$

and

$$\begin{aligned} &\widehat{H}_{\lambda,j,E}\left(\Theta_0, \Theta_1, \ldots, \Theta_{j-1}\right)(u)\left(u_0, u_1, \ldots, u_j\right) \\ &\quad = K_2 \circ \Phi_F^{(l)}(\Theta_0(u))\left(\Theta_{r_1}(u)\left(u_{i_1^{r_1}}, u_{i_2^{r_1}}, \ldots, u_{i_{r_1}^{r_1}}\right), \ldots, \Theta_{r_l}(u)\left(u_{i_1^{r_l}}, \ldots, u_{i_{r_l}^{r_l}}\right)\right) \end{aligned}$$

with the same constraints as in Lemma 4.31 for λ, r_j, l, and $i_k^{r_j}$.

Define

$$H_j = H_{j,BC^{\eta}(\mathbb{R},X_c)} \quad \text{for each } j = 0, \ldots, k.$$

In the above definition one has to consider K_2 as a linear operator from $BC^{j\eta+(2j-1)\mu}(\mathbb{R}, \mathbb{R}^n)$ into $BC^{j\eta+(2j-1)\mu}(\mathbb{R}, \mathbb{R}^n)$, and $\Phi_F^{(l)}(\Theta_0(u))$ as an element of

$$\mathcal{L}^{(j)}\left(BC^{r_1\eta+(2r_1-1)\mu}(\mathbb{R}, \mathbb{R}^n), \ldots, BC^{r_l\eta+(2r_l-1)\mu}(\mathbb{R}, \mathbb{R}^n); BC^{j\eta+(2j-1)\mu}(\mathbb{R}, \mathbb{R}^n)\right).$$

Notice that

$$j\eta + (2j-1)\mu > \sum_{k=1}^{l} r_k\eta + (2r_k - 1)\mu \qquad (\Leftrightarrow j\eta + (2j-1)\mu > j\eta + (2j-l)\mu)$$

since $2 \leq l \leq j$ and

$$r_1 + r_2 + \ldots + r_l = j.$$

Finally, define $H : M_0 \times M_1 \times \ldots \times M_k \to M_0 \times M_1 \times \ldots \times M_k$ by

$$H(\Theta_0, \Theta_1, \ldots, \Theta_k) = (H_0(\Theta_0), H_1(\Theta_0, \Theta_1), \ldots, H_k(\Theta_0, \Theta_1, \ldots, \Theta_k)).$$

We now check that the conditions of the fibre contraction Theorem 4.27 are satisfied. We have already shown that H_0 is a contraction on $M_{0,E}$ (from the existence part of the center manifold). It follows from (4.45) and (4.46) that H_j $(1 \leq j \leq k)$ is a contraction on X_j. More precisely, from Assumption 4.24-(iii), we have for each $j = 1, \ldots, k$ that

$$\begin{aligned}
&\sup_{u \in V_\varrho^\eta} \left\| K_2 \circ \Phi_F^{(1)}(u) \right\|_{\mathcal{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \\
&\quad \leq \|K_2\|_{\mathcal{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \sup_{u \in V_\varrho^\eta} \left\| \Phi_F^{(1)}(u) \right\|_{\mathcal{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \\
&\quad \leq \sup_{\theta \in [\eta, \widehat{\eta}]} \|K_2\|_{\mathcal{L}\left(BC^\theta(\mathbb{R},\mathbb{R}^n)\right)} |F|_{1,V_\varrho} \\
&\quad \leq \sup_{\theta \in [\eta, \widehat{\eta}]} \|K_2\|_{\mathcal{L}\left(BC^\theta(\mathbb{R},\mathbb{R}^n)\right)} \|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)} < 1.
\end{aligned}$$

Thus, each H_j is a contraction with respect to Θ_j. The fixed point of H_0 is Γ_0, and we denote by $\Gamma = (\Gamma_0, \Gamma_1, \ldots, \Gamma_k)$ the fixed point of H. Moreover, for $\mu = 0$, each H_j is still a contraction so we have for each $j = 1, \ldots, k$ that

$$\sup_{u \in BC^\eta(\mathbb{R}, X_c)} \left\| \Gamma_j(u) \right\|_{\mathcal{L}^{(j)}\left(BC^\eta(\mathbb{R},\mathbb{R}^n), BC^{j\eta}(\mathbb{R},\mathbb{R}^n)\right)} < +\infty.$$

Step 2 (Attractivity of the fixed point): In this part, we apply the fibre contraction Theorem 4.27 to prove that for each compact subset C of $BC^\eta(\mathbb{R}, X_c)$ and each $\Theta \in M_0 \times M_1 \times \ldots \times M_k$,

$$\lim_{m \to +\infty} H_C^m(\Theta |_C) = \Gamma |_C . \tag{4.47}$$

Let C be a compact subset of $BC^\eta(\mathbb{R}, X_c)$. From the definition of H_C, it is clear that

$$\Gamma |_C = H_C(\Gamma |_C)$$

and from the step 1, it is not difficult to see that for each $j = 0, \ldots, k$, $H_{j,C}$ is a contraction. In order to apply the fibre contraction Theorem 4.27, it remains to prove that for each $j = 1, \ldots, k$, $H_{j,C}\left(\Theta_{0,C}, \Theta_{1,C}, \ldots, \Theta_{j-1,C}, \Gamma_j |_C\right) \in M_j$ dependents continuously on $\left(\Theta_{0,C}, \Theta_{1,C}, \ldots, \Theta_{j-1,C}\right) \in M_{0,C} \times M_{1,C} \times \ldots \times M_{j-1,C}$.

We have

$$\begin{aligned}
&H_j\left(\Theta_{0,C}, \Theta_{1,C}, \ldots, \Theta_{j-1,C}, \Gamma^{(j)} |_C\right)(u) \\
&\quad = K_2 \circ \Phi_F^{(1)}\left(\Theta_{0,C}(u)\right) \circ \Gamma^{(j)}(u) + \widehat{H}_j\left(\Theta_{0,C}, \Theta_{1,C}, \ldots, \Theta_{j-1,C}\right)(u).
\end{aligned}$$

Since $\Gamma^{(j)}(u) \in \mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},\mathbb{R}^n), BC^{j\eta}(\mathbb{R},\mathbb{R}^n)\right)$ and $\Phi(u) \in V_{\varrho}^{\eta}$, we can consider $\Phi_F^{(1)}$ as a map from V_{ϱ}^{η} into $\mathcal{L}\left(BC^{j\eta}(\mathbb{R},\mathbb{R}^n), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)$, and by Lemma 4.31 (c) this map is continuous.

Indeed, let $\Theta_0, \widehat{\Theta}_0 \in M_0$ be two maps. Then we have

$$\begin{aligned}
&\sup_{u\in C}\left\|K_2 \circ \left[\Phi_F^{(1)}(\Theta_0(u)) - \Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right)\right] \circ \Gamma^{(j)}(u)\right\|_{\mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_c), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \\
&\quad \le \|K_2\|_{\mathcal{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \\
&\qquad \cdot \sup_{u\in C}\left\|\left[\Phi_F^{(1)}(\Theta_0(u)) - \Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right)\right] \circ \Gamma^{(j)}(u)\right\|_{\mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_c), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \\
&\quad \le \|K_2\|_{\mathcal{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \sup_{u\in C}\left\|\Gamma^{(j)}(u)\right\|_{\mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_c), BC^{j\eta}(\mathbb{R},\mathbb{R}^n)\right)} \\
&\qquad \cdot \sup_{u\in C}\left\|\Phi_F^{(1)}(\Theta_0(u)) - \Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right)\right\|_{\mathcal{L}^{(j)}\left(BC^{j\eta}(\mathbb{R},\mathbb{R}^n), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}
\end{aligned}$$

and by Lemma 4.28 we have

$$\begin{aligned}
&\sup_{u\in C}\left\|\Phi_F^{(1)}(\Theta_0(u)) - \Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right)\right\|_{\mathcal{L}^{(j)}\left(BC^{j\eta}(\mathbb{R},\mathbb{R}^n), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)} \\
&\quad \le \sup_{u\in C}\left\|\Phi_{DF}(\Theta_0(u)) - \Phi_{DF}\left(\widehat{\Theta}_0(u)\right)\right\|_{BC^{(2j-1)\mu}(\mathbb{R},\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))} \\
&\quad \le \max\left(\begin{array}{l}
\sup_{|t|\ge R} e^{-(2j-1)\mu|t|}\left\|DF(\Theta_0(u)(t)) - DF\left(\widehat{\Theta}_0(u)(t)\right)\right\|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)}, \\
\sup_{|t|\le R} e^{-(2j-1)\mu|t|}\left\|DF(\Theta_0(u)(t)) - DF\left(\widehat{\Theta}_0(u)(t)\right)\right\|_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)}
\end{array}\right).
\end{aligned}$$

Since $\widehat{\Theta}_0$ is continuous, C is compact, it follows that $\widehat{\Theta}_0(C)$ is compact, and by Ascoli's theorem (see for example Lang [131]), the set $\widehat{C} = \overline{\bigcup_{|t|\le R, u\in C}\left\{\widehat{\Theta}_0(u)(t)\right\}}$ is compact. But since $DF(.)$ is continuous, we have that for each $\varepsilon > 0$, there exists $\eta > 0$, such that for each $x, y \in \mathbb{R}^n$,

$$d\left(x,\widehat{C}\right) \le \eta, \quad d\left(y,\widehat{C}\right) \le \eta, \quad \text{and } \|x-y\| \le \eta \Rightarrow \|DF(x) - DF(y)\| \le \varepsilon.$$

Hence, the map $\Theta_{0,C} \to K_2 \circ \Phi_F^{(1)}\left(\Theta_{0,C}(.)\right) \circ \Gamma^{(j)}(.)$ is continuous.

It remains to consider $1 \le r_i \le j-1$, $r_1 + r_2 + \ldots + r_l = j$. We have

$$\left\| K_2 \circ \left[\Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right] \left(\widetilde{\Theta}_{r_1}(u), \ldots, \widetilde{\Theta}_{r_l}(u) \right) \right\|_{\mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_c), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}$$
$$\leq \|K_2\|_{\mathcal{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}$$
$$\sup_{u\in C} \left\| \left[\Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right] \left(\widetilde{\Theta}_{r_1}(u), \ldots, \widetilde{\Theta}_{r_l}(u) \right) \right\|_{\mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_c), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}$$
$$\leq \|K_2\|_{\mathcal{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n), BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}$$
$$\cdot \left\| \Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right\|_{\mathcal{L}^{(l)}\left(\prod_{p=1,\ldots,l} BC^{r_p\eta+(2r_p-1)\mu}(\mathbb{R},\mathbb{R}^n); BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}$$
$$\cdot \prod_{p=1,\ldots,l} \left\| \widetilde{\Theta}_{r_p}(u) \right\|_{\mathcal{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_c), BC^{r_p\eta+(2r_p-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}$$

and by Lemma 4.30 we have

$$\sup_{u\in C} \left\| \Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right\|_{\mathcal{L}^{(l)}\left(\prod_{p=1,\ldots,l} BC^{r_p\eta+(2r_p-1)\mu}(\mathbb{R},\mathbb{R}^n); BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)\right)}$$
$$\leq \sup_{u\in C} \left\| \Phi_{D^{(l)}F}\left(\Theta_0(u)\right) - \Phi_{D^{(l)}F}\left(\widehat{\Theta}_0(u)\right) \right\|_{BC^{\delta}\left(\mathbb{R}, \mathcal{L}^{(l)}(\mathbb{R}^n,\mathbb{R}^n)\right)}$$

with $\delta = (j\eta + (2j-1)\mu) - \sum_{k=1}^{l} r_k\eta + (2r_k - 1)\mu > 0$. By using the same compactness arguments as previously, we deduce that

$$\sup_{u\in C} \left\| \Phi_{D^{(l)}F}\left(\Theta_0(u)\right) - \Phi_{D^{(l)}F}\left(\widehat{\Theta}_0(u)\right) \right\|_{BC^{\delta}\left(\mathbb{R}, \mathcal{L}^{(l)}(\mathbb{R}^n,\mathbb{R}^n)\right)} \to 0$$

as $d_{0,C}(\Theta_0, \widehat{\Theta}_0) \to 0$. We conclude that the continuity condition of the fibre contraction Theorem 4.27 is satisfied for each $H_{j,C}$ and (4.47) follows.

Step 3: In order to prove Theorem 4.25 it now remains to prove that for each $u, v \in BC^{\eta}(\mathbb{R}, X_c), \forall j = 1, \ldots, k$,

$$\Gamma_{j-1}(u) - \Gamma_{j-1}(v) = \int_0^1 \Gamma_j(s(u-v)+v)(u-v)\,ds, \tag{4.48}$$

where the last integral is a Riemann integral. As assumed that (4.48) is satisfied, we deduce that $\Gamma_0 : BC^{\eta}(\mathbb{R}, X_c) \to BC^{k\eta+(2k-1)\mu}(\mathbb{R}, \mathbb{R}^n)$ is k-times continuously differentiable, and since

$$\Psi(x_c) = L \circ \Gamma_0 \circ K_1(x_c) = \Pi_h \Gamma_0(K_1(x_c))(0),$$

and $Lu = \Pi_h u(0)$ is a bounded linear operator from $BC^{k\eta+(2k-1)\mu}(\mathbb{R}, \mathbb{R}^n)$ into X_h, it follows that $\Psi : X_c \to X_h$ is k-times continuously differentiable.

We now prove (4.48). Set

$$\Theta^0 = \left(\Theta_0^0, \Theta_1^0, \ldots, \Theta_k^0\right)$$

with

$$\Theta_0^0(u) = u, \Theta_1^0(u) = J, \text{ and } \Theta_j^0 = 0, \forall j = 2, \ldots, k$$

and set

$$\Theta^m = \left(\Theta_0^m, \Theta_1^m, \ldots, \Theta_k^m\right) = H^m\left(\Theta^0\right), \forall m \geq 1.$$

Then from Lemma 4.31, we know that $\Theta_0^m : BC^\eta(\mathbb{R}, X_c) \to BC^{k\eta+(2k-1)\mu}(\mathbb{R}, \mathbb{R}^n)$ is a C^k-map and

$$D^j\Theta_0^m(u) = \Theta_j^m(u), \quad \forall j = 1, \ldots, k, \quad \forall u \in BC^\eta(\mathbb{R}, X_c).$$

For each $u, v \in BC^\eta(\mathbb{R}, X_c)$ and each $\forall j = 1, \ldots, k, \forall m \geq 1$,

$$\Theta_{j-1}^m(u) - \Theta_{j-1}^m(v) = \int_0^1 \Theta_j^m(s(u-v)+v)(u-v)\,ds.$$

Let $u, v \in BC^\eta(\mathbb{R}, X_c)$ be fixed. Denote

$$C = \{s(u-v)+v : s \in [0,1]\}.$$

Then clearly C is a compact set, and from step 2, we have for each $j = 0, \ldots, k$ that

$$\sup_{w \in C}\left\|\Theta_j^m(w) - \Gamma_j(w)\right\|_{BC^{j\eta+(2j-1)\mu}(\mathbb{R},\mathbb{R}^n)} \to 0, \quad \text{as } m \to +\infty.$$

Thus, (4.48) follows. □

It follows from the foregoing treatment that we can obtain the derivatives of $\Gamma_0(u)$ at $u = 0$. Assume that $F(0) = 0$ and $DF(0) = 0$, we have

$$\begin{aligned}
&D\Gamma_0(0) = J,\\
&D^{(2)}\Gamma_0(0)(u_1, u_2) = K_2 \circ \Phi_F^{(2)}(0)\left(D\Gamma_0(0)(u_1), D\Gamma_0(0)(u_2)\right),\\
&D^{(3)}\Gamma_0(0)(u_1, u_2, u_3) = K_2 \circ \Phi_F^{(2)}(0)\left(D^{(2)}\Gamma_0(0)(u_1, u_3), D\Gamma_0(0)(u_2)\right)\\
&\qquad +K_2 \circ \Phi_F^{(2)}(0)\left(D\Gamma_0(0)(u_1), D^{(2)}\Gamma_0(0)(u_2, u_3)\right)\\
&\qquad +K_2 \circ \Phi_F^{(3)}(0)\left(D\Gamma_0(0)(u_1), D\Gamma_0(0)(u_2), D\Gamma_0(0)(u_3)\right),\\
&\vdots\\
&D^{(l)}\Gamma_0(0) = \sum_{\lambda \in \Lambda_j} K_2 \circ \Phi_F^{(l)}(0)\left(D^{(r_1)}\Gamma(0), \ldots, D\Gamma^{(r_l)}(0)\right).
\end{aligned} \tag{4.49}$$

We have the following Lemma.

Lemma 4.32 *Let Assumptions 4.3 and 4.24 be satisfied. Assume also that $F(0) = 0$ and $DF(0) = 0$. Then*

$$\Psi(0) = 0, \text{ and } D\Psi(0) = 0,$$

and if $k > 1$,

$$D^j\Psi(0)(x_1, \ldots, x_n) = \Pi_h D^{(l)}\Gamma_0(0)(K_1x_1, \ldots, K_1x_n)(0),$$

where $D^{(l)}\Gamma_0(0)$ is given by (4.49). In particular, if $k > 1$ and

$$\Pi_h D^j F(0)\mid_{X_c\times\ldots\times X_c} = 0, \text{ for } 2 \leq j \leq k,$$

then

$$D^j\Psi(0) = 0, \quad \textit{for } 1 \le j \le k.$$

In the context of Hopf bifurcation, we need an explicit formula for $D^2\Psi(0)$. Since $D\Gamma_0(0) = J$, we obtain from the above formula that $\forall x_1, x_2 \in X_c$,

$$D^2\Psi(0)\,(x_1, x_2) = \Pi_h K_h \left[D^{(2)}F(0)\,(K_1 x_1, K_1 x_2)\right](0),$$

where

$$K_h = K_s + K_u, \quad K_1(x_c)(t) := e^{A_c t}x_c,$$

$$K_u(f)(t) := -\int_t^{+\infty} e^{-A_u(l-t)}\Pi_u f(l)dl,$$

and

$$K_s(f)(t) := \lim_{r\to-\infty}\Pi_s\left(e^{A\cdot} * f(r+.)\right)(t-r).$$

Hence,

$$\begin{aligned}
&D^2\Psi(0)\,(x_1, x_2)\\
&= -\int_0^{+\infty} e^{-A_u l}\Pi_u D^{(2)}F(0)\left(e^{A_c l}x_1, e^{A_c l}x_2\right)dl\\
&\quad + \lim_{r\to-\infty}\Pi_s\left(e^{A\cdot} * D^{(2)}F(0)\left(e^{A_c(r+.)}x_1, e^{A_c(r+.)}x_2\right)\right)(-r).
\end{aligned}$$

In order to explicit the term of the above formula, we remark that

$$\begin{aligned}
&\lim_{r\to-\infty}\Pi_s\left(e^{A\cdot} * D^{(2)}F(0)\left(e^{A_c(r+.)}x_1, e^{A_c(r+.)}x_2\right)\right)(-r)\\
&= \lim_{r\to-\infty}\Pi_s\int_0^{-r} e^{A(-r-s)}D^{(2)}F(0)\left(e^{A_c(r+s)}x_1, e^{A_c(r+s)}x_2\right)ds\\
&= \lim_{r\to-\infty}\int_0^{-r} e^{Al}D^{(2)}F(0)\left(e^{-A_c l}x_1, e^{-A_c l}x_2\right)dl\\
&= \int_0^{+\infty} e^{Al}\Pi_s D^{(2)}F(0)\left(e^{-A_c l}x_1, e^{-A_c l}x_2\right)dl.
\end{aligned}$$

Therefore, we obtain the following formula

$$\begin{aligned}
&D^2\Psi(0)\,(x_1, x_2)\\
&= -\int_0^{+\infty} e^{-A_u l}\Pi_u D^{(2)}F(0)\left(e^{A_c l}x_1, e^{A_c l}x_2\right)dl\\
&\quad + e^{Al}\Pi_s D^{(2)}F(0)\left(e^{-A_c l}x_1, e^{-A_c l}x_2\right)dl.
\end{aligned}$$

Assume that $\mathbb{R}^n$ is a complex Banach space and F is twice continuously differentiable in $\mathbb{R}^n$ considered as a $\mathbb{C}$-Banach space. We assume in addition that A_c is diagonalisable, and denote by $\{v_1, \ldots, v_n\}$ a basis of X_c such that for each $i = 1, \ldots, n$, $A_c v_i = \lambda_i v_i$. Then by Assumption 4.3 , we must have $\lambda_i \in i\mathbb{R}, \forall i = 1, \ldots, n$. Moreover, for each $i, j = 1, \ldots, n$,

$$\begin{aligned}D^2\Psi(0)\left(v_i,v_j\right) &= -\int_0^{+\infty} e^{-(\lambda_i+\lambda_j)l}e^{-A_u l}\Pi_u D^{(2)}F(0)\left(v_i,v_j\right)dl \\ &\quad+\int_0^{+\infty} e^{Al}\Pi_s D^{(2)}F(0)\left(e^{-\lambda_i l}v_i, e^{-\lambda_j l}v_j\right)dl \\ &= -\left(\left(\lambda_i+\lambda_j\right)I-(-A_u)\right)^{-1}\Pi_u D^{(2)}F(0)\left(v_i,v_j\right) \\ &\quad+\int_0^{+\infty} e^{-(\lambda_i+\lambda_j)l}e^{A_s l}\Pi_s D^{(2)}F(0)\left(v_i,v_j\right)dl \\ &= -\left(\left(\lambda_i+\lambda_j\right)I-(-A_u)\right)^{-1}\Pi_u D^{(2)}F(0)\left(v_i,v_j\right) \\ &\quad+\left(\left(\lambda_i+\lambda_j\right)I-A_s\right)^{-1}\Pi_s D^{(2)}F(0)\left(v_i,v_j\right).\end{aligned}$$

Thus,

$$\begin{aligned}D^2\Psi(0)\left(v_i,v_j\right) &= -\left(\left(\lambda_i+\lambda_j\right)I-(-A_u)\right)^{-1}\Pi_u D^{(2)}F(0)\left(v_i,v_j\right) \\ &\quad+\left(\left(\lambda_i+\lambda_j\right)I-A_s\right)^{-1}\Pi_s D^{(2)}F(0)\left(v_i,v_j\right).\end{aligned}$$

Note that by Assumption 4.3 $i\mathbb{R}\subset\rho\left(A_s\right)$, so the above formula is well defined.

As in Vanderbauwhede and Iooss [217, Theorem 3], we have the following theorem about the existence of the local center manifold.

Theorem 4.33 *Let Assumption 4.3 be satisfied. Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a map. Assume there exists an integer $k\geq 1$ such that F is k-times continuously differentiable in some neighborhood of 0 with $F(0)=0$ and $DF(0)=0$. Then there exist a neighborhood Ω of the origin in $\mathbb{R}^n$ and a map $\Psi\in C_b^k\left(X_c,X_h\right)$, with $\Psi(0)=0$ and $D\Psi(0)=0$, such that the following properties hold:*

(i) *If I is an interval of $\mathbb{R}$ and $x_c:I\to X_c$ is a solution of*

$$\frac{dx_c(t)}{dt}=A_c x_c(t)+\Pi_c F\left[x_c(t)+\Psi\left(x_c(t)\right)\right] \tag{4.50}$$

such that

$$u(t):=x_c(t)+\Psi\left(x_c(t)\right)\in\Omega, \forall t\in I,$$

then for each $t,s\in I$ with $t\geq s$,

$$u(t)=u(s)+A\int_s^t u(l)dl+\int_s^t F\left(u(l)\right)dl.$$

(ii) *If $u:\mathbb{R}\to\mathbb{R}^n$ is a map such that for each $t,s\in\mathbb{R}$ with $t\geq s$,*

$$u(t)=u(s)+A\int_s^t u(l)dl+\int_s^t F\left(u(l)\right)dl$$

and

$$u(t)\in\Omega,\quad \forall t\in\mathbb{R},$$

then

$$\Pi_h u(t)=\Psi\left(\Pi_c u(t)\right), \forall t\in\mathbb{R},$$

and $\Pi_c u : \mathbb{R} \to X_c$ *is a solution of* (4.50).

(iii) *If* $k \geq 2$, *then for each* $x_1, x_2 \in X_c$,

$$\begin{aligned} D^2\Psi(0)\,(x_1,x_2) &= -\int_0^{+\infty} e^{-A_u l}\Pi_u D^{(2)}F(0)\left(e^{A_c l}x_1, e^{A_c l}x_2\right) dl \\ &+ \lim_{r\to-\infty} \Pi_s \left(e^{A\cdot} * D^{(2)}F(0)\left(e^{A_c(r+.)}x_1, e^{A_c(r+.)}x_2\right)\right)(-r). \end{aligned}$$

Moreover, $\mathbb{C}^n$ *is a* $\mathbb{C}$*-Banach space, and if* $\{v_1, \dots, v_n\}$ *is a basis of* X_c *such that for each* $i = 1, \dots, n$, $A_c v_i = \lambda_i v_i$, *with* $\lambda_i \in i\mathbb{R}$, *then for each* $i, j = 1, \dots, n$,

$$\begin{aligned} D^2\Psi(0)\left(v_i, v_j\right) = &-\left(\left(\lambda_i+\lambda_j\right)I - (-A_u)\right)^{-1}\Pi_u D^{(2)}F(0)\left(v_i, v_j\right) \\ &+\left(\left(\lambda_i+\lambda_j\right)I - A_s\right)^{-1}\Pi_s D^{(2)}F(0)\left(v_i, v_j\right). \end{aligned}$$

Proof Set for each $r > 0$ that

$$F_r(x) = F(x)\chi_c\left(r^{-1}\Pi_c(x)\right)\chi_h\left(r^{-1}\left\|\Pi_h(x)\right\|\right), \forall x \in \mathbb{R}^n,$$

where $\chi_c : X_c \to [0, +\infty)$ is a C^∞ map with $\chi_c(x) = 1$ if $\|x\| \leq 1$, $\chi_c(x) = 0$ if $\|x\| \geq 2$, and $\chi_h : [0, +\infty) \to [0, +\infty)$ is a C^∞ map with $\chi_h(x) = 1$ if $|x| \leq 1$, $\chi_h(x) = 0$ if $|x| \geq 4.4$. Then by using the same arguments as in the proof of Theorem 3 in [217], we deduce that there exists $r_0 > 0$, such that for each $r \in (0, r_0]$, F_r satisfies Assumption 4.24. By applying Theorem 4.25 to

$$\frac{du(t)}{dt} = Au(t) + F_r(u(t)), \quad t \geq 0, \text{ and } u(0) = x \in \overline{D(A)}$$

for $r > 0$ small enough, the result follows. □

In order to investigate the existence of an Hopf bifurcation we also need the following result.

Proposition 4.34 *Let the assumptions of Theorem 4.33 be satisfied. Assume that* $\overline{x} \in \mathbb{R}^n$ *is equilibrium of* $\{U(t)\}_{t\geq 0}$ *(i.e.* $\overline{x} \in D(A)$ *and* $A\overline{x} + F(\overline{x}) = 0$*) such that*

$$\overline{x} \in \Omega.$$

Then

$$\Pi_h \overline{x} = \Psi(\Pi_c \overline{x})$$

and $\Pi_c \overline{x}$ *is an equilibrium of the reduced equation*

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F\left[x_c(t) + \Psi(x_c(t))\right].$$

Moreover, if one considers the linearized equation at $\Pi_c \overline{x}$

$$\frac{dy_c(t)}{dt} = L(\overline{x})\, y_c(t)$$

with

$$L(\overline{x}) = [A_c + \Pi_c DF(\overline{x})[I + D\Psi(\Pi_c \overline{x})]],$$

then we have the following spectral properties

$$\sigma(L(\overline{x})) = \sigma((A + DF(\overline{x}))_0) \cap \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \in [-\eta, \eta]\}.$$

Proof Let $\overline{x} \in \mathbb{R}^n$ be an equilibrium of $\{U(t)\}_{t \geq 0}$ such that $\overline{x} \in \Omega$. We set

$$\overline{x}_c = \Pi_c \overline{x} \text{ and } \overline{u}(t) = \overline{x}, \forall t \in \mathbb{R}.$$

Then the linearized equation at $\overline{x}$ is given by

$$\frac{dw(t)}{dt} = (A + DF(\overline{x}))\, w(t), \text{ for } t \geq 0, \text{ and } w(0) = w_0 \in \mathbb{R}^n. \tag{4.51}$$

So

$$w(t) = e^{(A+DF(\overline{x}))(t)} w_0, \forall t \geq 0.$$

Moreover, we have

$$D\Psi(x_c)\, y_c = \Pi_h \left[\Gamma_0^1(\overline{u})\,(K_1 y_c)\right]$$

and

$$\Gamma_0^1(\overline{u})(v) = v + K_2 \Phi_{DF(\overline{x})}\left(\Gamma_0^1(u)(v)\right), \ \forall v \in BC^\eta(\mathbb{R}, X_c).$$

It follows that

$$\Gamma_0^1(\overline{u}) = \left(I - K_2 \Phi_{DF(\overline{x})}\right)^{-1} v.$$

Thus,

$$D\Psi(\overline{x}_c)\, y_c = \Pi_h \left[\left(I - K_2 \Phi_{DF(\overline{x})}\right)^{-1} (K_1 y_c)\right].$$

By applying Theorem 4.22 to equation (4.51), we deduce that

$$W_\eta = \{y_c + D\Psi(\overline{x}_c)\, y_c : y_c \in X_c\}$$

is invariant by $\left\{T_{(A+DF(\overline{x}))_0}(t)\right\}_{t \geq 0}$. Moreover, for each $w \in C(\mathbb{R}, \mathbb{R}^n)$ the following statements are equivalent:
(1) $w \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$ is a complete orbit of $\left\{T_{(A+DF(\overline{x}))_0}(t)\right\}_{t \geq 0}$.
(2) $\Pi_h w(t) = D\Psi(\overline{x}_c)(\Pi_c w(t)), \forall t \in \mathbb{R}$, and $\Pi_c w(.) : \mathbb{R} \to X_c$ is a solution of the ordinary differential equation

$$\frac{dw_c(t)}{dt} = A_c w_c(t) + \Pi_c DF(\overline{x})[w_c(t) + D\Psi(\overline{x}_c)(w_c(t))].$$

The result follows from the above equivalence. □

4.4 Existence and Stability of the Center Unstable Manifold

In this section we will assume that the center unstable spectrum is not empty.

Assumption 4.35 We assume that $\sigma_{cu}(A) = \{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq 0\} \neq \emptyset$.

Let

$$\beta_- = \min\{-\operatorname{Re}(\lambda) : \lambda \in \sigma_s(A)\} > 0,$$

or equivalently

$$\beta_- = -\lim_{t \to +\infty} \frac{\ln\left(\left\|e^{A_s t)}\right\|_{\mathcal{L}(X_s)}\right)}{t} > 0.$$

From now on, we fix $\beta \in (0, \beta_-)$. From the proof of Lemma 4.5, we deduce that we can find a constant number $M_s > 0$ such that

$$\|e^{At}\Pi_s\|_{\mathcal{L}(\mathbb{R}^n)} \leq M_s e^{(-\beta+\varepsilon)t}, \forall t \geq 0, \forall \varepsilon \geq 0. \tag{4.52}$$

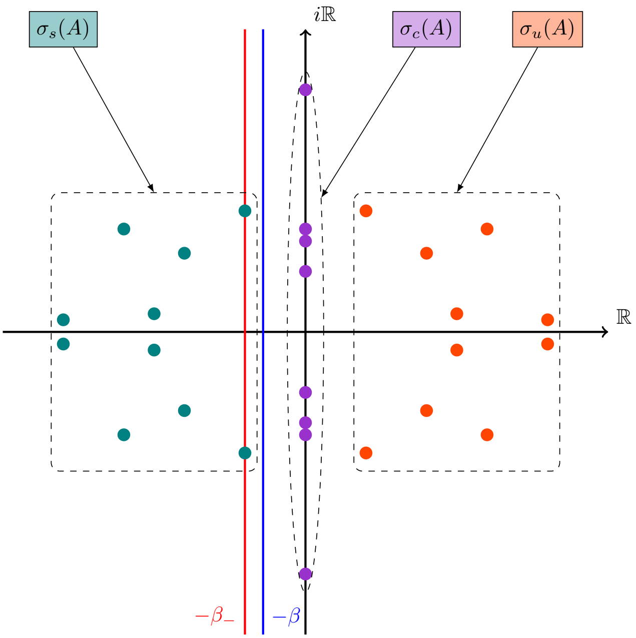


Fig. 4.3: *In this Figure, we illustrate the different parts of the spectrum $\sigma_s(A)$, $\sigma_c(A)$ and $\sigma_u(A)$ as well as β_-.*

Let us recall that a function $u : \mathbb{R} \to \mathbb{R}$ is a **negative orbit** for the semiflow $\{U(t)\}_{t\geq 0}$ if and only if the function $t \to u(t)$ on $\mathbb{R}$ satisfies

$$u(t) = U(t-s)u(s), \forall t, s \in (-\infty, 0] \text{ with } t \geq s,$$

where $\{U(t)\}_{t\geq 0}$ is a continuous semiflow generated by (4.21).

That is equivalent to say that for each $s \in (-\infty, 0]$ (fixed) the function $t \to u(t)$ from $[s, 0]$ satisfies

$$u(t) = u(s) + A\int_0^{t-s} u(s+r)dr + \int_0^{t-s} F\left(u(s+r)\right)dr, \forall t \in [s,0],$$

or equivalently by using the variation of constant formula

$$u(t) = e^{A(t-s)}u(s) + \int_s^t e^{A(t-\sigma)}F(u(\sigma))d\sigma, \forall t \in [s,0],$$

or (for short) by using the convolutions

$$u(t) = e^{A(t-s)}u(s) + \left(e^{A.} * F\left(u(s+.)\right)\right)(t-s), \forall t \in [s,0]. \tag{4.53}$$

Definition 4.36 Let $\eta \in (0,\beta_-)$. The η-**center unstable manifold** of (4.21), denoted by V_η^{cu}, is the set of all points $x \in \mathbb{R}^n$ such that there exists $u \in BC^\eta\left(\mathbb{R}_-,\mathbb{R}^n\right)$, a negative orbit of $\{U(t)\}_{t\geq 0}$, such that $u(0)=x$. That is to say that

$$V_\eta^{cu} = \left\{x \in \mathbb{R}^n : \exists u \in BC^\eta\left(\mathbb{R}_-,\mathbb{R}^n\right), \text{ a negative orbit of } \{U(t)\}_{t\geq 0}, \text{ such that } u(0)=x\right\}. \tag{4.54}$$

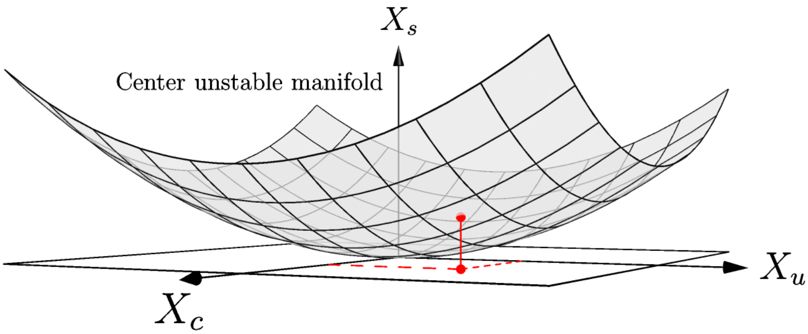


Fig. 4.4: *Schematic representation of the center unstable manifold. In this figure we plot the linear center unstable manifold and the center unstable manifold which corresponds to a surface tangent at* 0 *to the linear center unstable manifold. The surface representing center unstable manifold is delimited by the dashed curves.*

For each $\eta > 0$, V_η^{cu} is invariant under the semiflow $\{U(t)\}_{t\geq 0}$, that is,

$$U(t)\mathrm{V}_\eta^{\mathrm{cu}} = \mathrm{V}_\eta^{\mathrm{cu}}, \quad \forall t \geq 0.$$

Moreover, we say that $\{U(t)\}_{t\geq 0}$ is *reduced on* V_η^{cu} if there exists a map $\Psi_{cu} : X_{cu} \to X_s$ such that

$$\mathrm{V}_\eta^{\mathrm{cu}} = \mathrm{Graph}\left(\Psi_{cu}\right) = \left\{x_{cu} + \Psi_{cu}\left(x_{cu}\right) : x_{cu} \in X_{cu}\right\}.$$

The proof of the following lemma is similar to the proof of Lemma 4.17.

Lemma 4.37 *Let Assumption 4.35 be satisfied. Then*

(i) *For each* $\eta \in [0, \beta_-)$, *each* $f \in BC^\eta(\mathbb{R}_-, \mathbb{R}^n)$, *and each* $t \in \mathbb{R}_-$,

$$K_s(f)(t) := \lim_{r \to -\infty} \Pi_s \left(e^{A\cdot} * f(r + .) \right)(t - r) \text{ exists.}$$

(ii) *For each* $\eta \in [0, \beta_-)$, K_s *is a bounded linear operator from* $BC^\eta(\mathbb{R}_-, \mathbb{R}^n)$ *into* $BC^\eta(\mathbb{R}_-, X_s)$. *More precisely, for each* $\nu \in (-\beta_-, 0)$, *there exists a constant* $\widehat{C}_{s,\nu} > 0$ *such that*

$$\|K_s\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-,\mathbb{R}^n), BC^\eta(\mathbb{R}_-,X_s))} \le \widehat{C}_{s,\nu}, \forall \eta \in [0, -\nu].$$

(iii) *For each* $\eta \in [0, \beta_-)$, *each* $f \in BC^\eta(\mathbb{R}_-, \mathbb{R}^n)$, *and each* $t, s \in \mathbb{R}$ *with* $t \ge s$,

$$K_s(f)(t) - e^{A(t-s)} K_s(f)(s) = \Pi_s \left(e^{A\cdot} * f(s + .) \right)(t - s).$$

Remark 4.38 As explained before in the chapter we can expressed $K_s(f)(t)$ as

$$K_s(f)(t) = \int_0^{+\infty} e^{A_s \theta} \Pi_s f(t - \theta) d\theta, \forall t \le 0.$$

The following lemmas can be proved by using similar argument as for Lemma 4.20.

Lemma 4.39 *Let Assumption 4.35 be satisfied. Let* $\eta \in (0, \beta_-)$ *be fixed. For each* $x_{cu} \in X_{cu}$, *each* $f \in BC^\eta(\mathbb{R}_-, \mathbb{R}^n)$, *and each* $t \in (-\infty, 0]$, *denote*

$$K_1(x_{cu})(t) := e^{A_{cu} t} x_{cu}, \quad K_{cu}(f)(t) := \int_0^t e^{A_{cu}(t-s)} \Pi_{cu} f(s) ds,$$

where $\Pi_{cu} = \Pi_c + \Pi_u$. *Then* K_1 *is a bounded linear operator from* X_{cu} *into* $BC^\eta(\mathbb{R}_-, X_{cu})$ *and*

$$\|K_1\|_{\mathcal{L}(X_{cu}, BC^\eta(\mathbb{R}_-,\mathbb{R}^n))} \le \sup_{t \ge 0} \left\| e^{-(A_{cu} + \eta I)t} \right\| < +\infty,$$

$$\|K_{cu}\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-,\mathbb{R}^n))} \le \|\Pi_{cu}\|_{\mathcal{L}(\mathbb{R}^n)} \int_0^{+\infty} \left\| e^{-(A_{cu} + \eta I)l} \right\| dl < +\infty.$$

Lemma 4.40 *Let Assumption 4.35 be satisfied. Let* $\eta \in (0, \beta_-)$ *and* $u \in BC^\eta(\mathbb{R}_-, \mathbb{R}^n)$ *be fixed. Then* u *is a negative complete orbit of* $\{U(t)\}_{t \ge 0}$ *if and only if for each* $t \in \mathbb{R}_-$,

$$u(t) = K_1(\Pi_{cu} u(0))(t) + K_{cu}(F(u(.)))(t) + K_s(F(u(.)))(t), \tag{4.55}$$

where $\Pi_{cu} = \Pi_c + \Pi_u$.

Proof This proof is similar to the proof of Lemma 4.21. □

Let $\eta \in (0, \beta_-)$ be fixed. Rewrite equation (4.55) as the following fixed point problem: To find $u \in BC^\eta(\mathbb{R}_-, \mathbb{R}^n)$ such that

$$u = K_1(\Pi_{cu}u(0)) + K_2\Phi_F(u), \tag{4.56}$$

where the nonlinear operator $\Phi_F \in \text{Lip}\left(BC^\eta\left(\mathbb{R}_-, \mathbb{R}^n\right), BC^\eta\left(\mathbb{R}_-, \mathbb{R}^n\right)\right)$ is defined by

$$\Phi_F(u)(t) = F(u(t)), \quad \forall t \in \mathbb{R}_-,$$

and the linear operator $K_2 \in \mathcal{L}\left(BC^\eta\left(\mathbb{R}_-, \mathbb{R}^n\right), BC^\eta\left(\mathbb{R}_-, \mathbb{R}^n\right)\right)$ is defined by

$$K_2 = K_{cu} + K_s.$$

Moreover, we have the following estimates

$$\|K_1\|_{\mathcal{L}(X_{cu}, BC^\eta(\mathbb{R}_-, \mathbb{R}^n))} \leq \sup_{t\geq 0} \left\| e^{-(A_{cu}+\eta I)t} \right\|,$$

$$\|\Phi_F\|_{\text{Lip}} \leq \|F\|_{\text{Lip}},$$

and for each $\nu \in (-\beta_-, 0)$, we have

$$\|K_2\|_{\mathcal{L}(BC^\eta(\mathbb{R}, \mathbb{R}^n))} \leq \gamma(\nu, \eta), \forall \eta \in (0, -\nu],$$

where

$$\gamma(\nu, \eta) := \widehat{C}_{s,\nu} + \|\Pi_{cu}\|_{\mathcal{L}(\mathbb{R}^n)} \int_0^{+\infty} \left\| e^{-(A_{cu}+\eta I)l} \right\| dl. \tag{4.57}$$

Furthermore, by Lemma 4.40, the η-center-unstable manifold is given by

$$\mathrm{V}_\eta^{\mathrm{cu}} = \left\{x \in \mathbb{R}^n : \exists u \in BC^\eta\left(\mathbb{R}_-, \mathbb{R}^n\right) \text{ a solution of (4.56) and } u(0) = x\right\}. \tag{4.58}$$

We state the existence of center-unstable manifolds for the abstract semilinear Cauchy problem (4.21) with non-dense domain which can be proved similarly as Theorem 4.10 in Magal and Ruan [153].

Theorem 4.41 (Global center unstable manifold) *Let Assumption 4.35 be satisfied. Let $\eta \in (0, \beta_-)$ be fixed and $\delta_0 = \delta_0(\eta) > 0$ be such that*

$$\delta_0 \|K_2\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-, \mathbb{R}^n))} < 1.$$

Then for each $F \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^n)$ with $\|F\|_{\text{Lip}(\mathbb{R}^n, \mathbb{R}^n)} \leq \delta_0$, there exists a Lipschitz continuous map $\Psi_{cu} : X_{cu} \to X_s$ such that

$$\mathrm{V}_\eta^{\mathrm{cu}} = \left\{x_{cu} + \Psi_{cu}(x_{cu}) : x_{cu} \in X_{cu}\right\}.$$

Moreover, we have the following properties:

(i)

$$\sup_{x_{cu}\in X_{cu}} \|\Psi_{cu}(x_{cu})\| \leq \|K_s\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-, \mathbb{R}^n))} \sup_{x\in\mathbb{R}^n} \|\Pi_s F(x)\|.$$

(ii) *We have*

$$\|\Psi_{cu}\|_{\mathrm{Lip}(X_{cu},X_s)} \leq \frac{\|K_s\|_{\mathcal{L}(BC^{\eta}(\mathbb{R}_-,\mathbb{R}^n))}\,\|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)}\,\|K_1\|_{\mathcal{L}(X_{cu},BC^{\eta}(\mathbb{R}_-,\mathbb{R}^n))}}{1-\|K_2\|_{\mathcal{L}(BC^{\eta}(\mathbb{R}_-,\mathbb{R}^n))}\,\|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)}}. \tag{4.59}$$

We now state and prove the existence of local center-unstable manifolds.

Theorem 4.42 (Local center-unstable manifold) *Let Assumption 4.35 be satisfied. Let $r > 0$ and $F : B_{\mathbb{R}^n}(0,r) \to \mathbb{R}^n$ be a map. Assume that there exists an integer $k \geq 1$ such that F is k-time continuously differentiable in $B_{\mathbb{R}^n}(0,r)$ with*

$$F(0) = 0 \text{ and } DF(0) = 0.$$

Then there exists a neighborhood Ω of the origin in $\mathbb{R}^n$ and a map $\Psi_{cu} \in C_b^k(X_{cu}, X_s)$ with

$$\Psi_{cu}(0) = 0 \text{ and } D\Psi_{cu}(0) = 0,$$

such that

$$M_{cu} = \{x_{cu} + \Psi_{cu}(x_{cu}) : x_{cu} \in X_{cu}\}$$

is a locally invariant manifold by the semiflow generated by (4.21) around 0.

More precisely, the following properties hold:

(i) *If I is an interval of $\mathbb{R}$ and $x_{cu} : I \to X_{cu}$ is a solution of*

$$\frac{dx_{cu}(t)}{dt} = A_{0cu}x_{cu}(t) + \Pi_{cu}F(x_{cu}(t) + \Psi_{cu}(x_{cu}(t))) \text{ \textbf{(reduced equation)}} \tag{4.60}$$

such that

$$u(t) := x_{uc}(t) + \Psi_{cu}(x_{uc}(t)) \in \Omega, \forall t \in I,$$

then for each $t, s \in I$ with $t \geq s$,

$$u(t) = u(s) + A\int_s^t u(l)dl + \int_s^t F(u(l))\,dl.$$

(ii) *If $u : (-\infty, 0] \to \mathbb{R}^n$ is a map such that for each $t, s \in (-\infty, 0]$ with $t \geq s$,*

$$u(t) = u(s) + A\int_s^t u(l)dl + \int_s^t F(u(l))\,dl$$

and

$$u(t) \in \Omega, \quad \forall t \in (-\infty, 0],$$

then

$$\Pi_s u(t) = \Psi_{cu}(\Pi_{cu}u(t)), \forall t \in (-\infty, 0],$$

and $\Pi_{cu}u : (-\infty, 0] \to X_{cu}$ is a solution of (4.60).

Proof In order to prove the local center-unstable manifold theorem, we apply Theorem 4.41 to the Cauchy problem

$$\frac{du}{dt} = Au(t) + F_r(u(t)), t \geq 0, u(0) = x \in \mathbb{R}^n,$$

where $F_r : \mathbb{R}^n \to \mathbb{R}^n$ is the following truncated function

$$F_r(x) = F(x)\chi_{cu}\left(r^{-1}\Pi_{cu}(x)\right)\chi_s\left(r^{-1}\|\Pi_s(x)\|\right), \ \forall x \in \mathbb{R}^n,$$

$\chi_{cu} : X_{cu} \to [0, +\infty)$ is a C^∞ map with $\chi_{cu}(x) \leq 1$ and

$$\chi_{cu}(x) = \begin{cases} 1, & \text{if } \|x\| \leq 1, \\ 0, & \text{if } \|x\| \geq 2, \end{cases}$$

and $\chi_s : [0, +\infty) \to [0, +\infty)$ is a C^∞ map with $\chi_s(y) \leq 1, \forall y \geq 0$, and

$$\chi_s(y) = \begin{cases} 1, & \text{if } |y| \leq 1, \\ 0, & \text{if } |y| \geq 2. \end{cases}$$

The smoothness of Ψ_{cu} is obtained by applying the same arguments as in Magal and Ruan [153] to the above truncated system, and the result follows. □

The following theorem is the main result of this section. This result is proved for discrete time systems with bounded Lipschitz map F in Vanderbauwhede [215] and for ordinary differential equations in Vanderbauwhede [216] and Chow, Li and Wang [32].

Theorem 4.43 (Stability of the center unstable manifold) *Let Assumption 4.35 be satisfied. Let $\eta \in (0, \beta_-)$ be fixed. Then there exists $\delta_1(\eta) \in (0, \delta_0)$ (where $\delta_0 > 0$ is the constant introduced in Theorem 4.41), such that for each $F \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^n)$ with $\|F\|_{\mathrm{Lip}(\mathbb{R}^n,\mathbb{R}^n)} \leq \delta_1(\eta)$, there exists a continuous map $H_{cu} : \mathbb{R}^n \to \mathrm{V}_\eta^{\mathrm{cu}}$ such that for each $x \in \mathbb{R}^n$,*

$$\mathrm{V}_\eta^{\mathrm{cu}} \cap \widetilde{V}_\eta(x) = \{H_{cu}(x)\},$$

where

$$\widetilde{V}_\eta(x) = \left\{ y \in \mathbb{R}^n : \sup_{t \geq 0} e^{\eta t} \|U(t)y - U(t)x\| < +\infty \right\}.$$

More precisely, for each $x \in \mathbb{R}^n$, there is a constant $M_\eta = M_\eta(x) > 0$ such that

$$\|U(t)H_{cu}(x) - U(t)x\| \leq e^{-\eta t} M_\eta \|H_{cu}(x) - x\|, \forall t \geq 0.$$

Before proving the theorem we give some preliminary lemmas. Recall that

$$BC^{-\eta}(\mathbb{R}_+, \mathbb{R}^n) = \left\{ w \in C(\mathbb{R}_+, \mathbb{R}^n) : \|w\|_\eta = \sup_{t \in \mathbb{R}_+} e^{\eta t} \|w(t)\| < +\infty \right\}.$$

In order to determine $\widetilde{V}_\eta(x)$, we have to find all $w \in BC^{-\eta}(\mathbb{R}_+, \mathbb{R}^n)$ such that $t \to U(t)x + w(t)$ is a solution of

$$u(t) = e^{At}x + \left(e^{A\cdot} * F(u(.))\right)(t), \ \forall t \in [0, \tau]. \tag{4.61}$$

Lemma 4.44 *Let Assumption 4.35 be satisfied. Let $\eta \in (0, \beta_-)$ be fixed and $w \in BC^{-\eta}(\mathbb{R}_+, \mathbb{R}^n)$. Then the map $t \to U(t)x + w(t)$ is a solution of (4.61) if and only if for each $t \geq 0$,*

$$\begin{aligned} w(t) =& e^{A_s t}\Pi_s w(0) + \left(e^{A_s} * \Pi_s \left[F(U(.)x + w(.)) - F(U(.)x)\right]\right)(t) \\ & - \int_t^{+\infty} e^{A_{uc}(t-s)}\Pi_{cu}\left[F(U(s)x + w(s)) - F(U(s)x)\right] ds. \end{aligned} \tag{4.62}$$

Proof Let $w \in BC^{-\eta}(\mathbb{R}_+, \mathbb{R}^n)$ be fixed. Assume first that $t \to U(t)x + w(t)$ is a solution of (4.61). Then we have for each $t, s \in [0, +\infty)$ with $t \geq s$ that

$$\begin{aligned} U(t)x + w(t) =& e^{A(t-s)}\left(U(s)x + w(s)\right) \\ & + \left(e^{A \cdot} * F(U(s+.)x + w(s+.))\right)(t-s) \end{aligned}$$

and

$$U(t)x = e^{A(t-s)}U(s)x + \left(e^{A \cdot} * F(U(s+.)x)\right)(t-s).$$

Then

$$\begin{aligned} w(t) =& e^{A(t-s)}w(s) \\ & + \left(e^{A \cdot} * \left[F(U(s+.)x + w(s+.)) - F(U(s+.)x)\right]\right)(t-s). \end{aligned} \tag{4.63}$$

By projecting the above equation on X_{cu}, we obtain for each $t, s \in [0, +\infty)$ with $t \geq s$ that

$$\Pi_{cu}w(t) = e^{A_{cu}(t-s)}\Pi_{cu}w(s) + \int_s^t e^{A_{cu}(t-l)}\Pi_{cu}\left[F(U(l)x + w(l)) - F(U(l)x)\right] dl.$$

Then

$$\Pi_{cu}w(s) = e^{-A_{cu}(t-s)}\Pi_{cu}w(t) - \int_s^t e^{A_{cu}(s-l)}\Pi_{cu}\left[F(U(l)x + w(l)) - F(U(l)x)\right] dl.$$

We have $\left\| e^{-A_{cu}(t-s)} \right\|_{\mathcal{L}(X_{cu})} \leq \min\left\{ e^{\frac{\eta}{2}|t-s|}M_{c,\frac{\eta}{2}},\ e^{-\eta_1(t-s)}M_u \right\}$, $\eta_1 > 0$, $\forall t \geq s$, here η, $M_{c,\frac{\eta}{2}}$ and M_u are constants (see Magal and Ruan [153] for details). Since $w \in BC^{-\eta}(\mathbb{R}_+, \mathbb{R}^n)$, we obtain for each $t, s \in [0, +\infty)$ with $t \geq s$ that

$$\left\| e^{-A_{cu}(t-s)}\Pi_{cu}w(t) \right\| \leq \min\left\{ e^{\frac{\eta}{2}|t-s|}M_{c,\frac{\eta}{2}},\ e^{-\eta_1(t-s)}M_u \right\} \left\|\Pi_{cu}\right\|_{\mathcal{L}(\mathbb{R}^n)} \left\|w\right\|_\eta e^{-\eta t}.$$

Then

$$\left\| e^{-A_{cu}(t-s)}\Pi_{cu}w(t) \right\| \to 0 \text{ as } t \to +\infty.$$

Thus

$$\Pi_{cu}w(t) = -\int_t^{+\infty} e^{A_{cu}(t-l)}\Pi_{cu}\left[F(U(l)x + w(l)) - F(U(l)x)\right] dl, \forall t \geq 0. \tag{4.64}$$

By projecting (4.63) on X_s we obtain for each $t \geq 0$ that

$$\Pi_s w(t) = e^{A_s t}\Pi_s w(0) + \left(e^{A_s} * \Pi_s\left[F(U(.)x + w(.)) - F(U(.)x)\right]\right)(t). \tag{4.65}$$

So by summing (4.64) and (4.65), we obtain (4.62). Conversely, assume that w satisfies (4.62). Then by projecting (4.62) on X_s we obtain for each $t \geq 0$ that

$$\Pi_s w(t) = \Pi_s e^{At} w(0) + \Pi_s \left(e^{A.} * \left[F(U(.)x + w(.)) - F(U(.)x) \right] \right)(t).$$

Then

$$\Pi_s (U(t)x + w(t)) = \Pi_s e^{At} \left(w(0) + x \right) + \Pi_s \left(e^{A.} * F(U(.)x + w(.)) \right)(t). \quad (4.66)$$

Furthermore, by projecting (4.62) on X_{cu} we obtain for each $t \geq 0$ that

$$\Pi_{cu} w(t) = -\int_t^{+\infty} e^{A(t-s)} \Pi_{cu} \left[F(U(s)x + w(s)) - F(U(s)x) \right] ds.$$

Thus

$$\begin{aligned} \Pi_{cu} w(t) - e^{A_{cu} t} \Pi_{cu} w(0) &= -\int_t^{+\infty} e^{A(t-s)} \Pi_{cu} \left[F(U(s)x + w(s)) - F(U(s)x) \right] ds \\ &\quad + \int_0^{+\infty} e^{A(t-s)} \Pi_{cu} \left[F(U(s)x + w(s)) - F(U(s)x) \right] ds \\ &= \int_0^{t} e^{A(t-s)} \Pi_{cu} \left[F(U(s)x + w(s)) - F(U(s)x) \right] ds. \end{aligned}$$

Hence

$$\Pi_{cu} (U(t)x + w(t)) = e^{A_{cu} t} \Pi_{cu} (x + w(0)) + \int_0^t e^{A(t-s)} \Pi_{cu} F(U(s)x + w(s)) ds. \quad (4.67)$$

By summing up (4.66) and (4.67), we deduce that $t \to U(t)x + w(t)$ is a solution of (4.61). □

Rewrite (4.62) in the following abstract form

$$w = \widetilde{K}_1(w_s) + \widetilde{K}_2 \widetilde{\Phi}(x, w),$$

where $\widetilde{K}_1 : X_s \to BC^{-\eta}\left(\mathbb{R}_+, X_s\right)$, $\widetilde{K}_2 : BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right) \to BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right)$, and $\widetilde{\Phi} : \mathbb{R}^n \times BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right) \to BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right)$ are defined as follows

$$\begin{aligned} &\widetilde{K}_1(x_s)(t) = e^{A_s t} x_s, t \in \mathbb{R}_+, \\ &\widetilde{K}_2(f)(t) = \left(e^{A_s \cdot} * \Pi_s f \right)(t) - \int_t^{+\infty} e^{A_{cu}(t-s)} \Pi_{cu} f(s) ds, \forall t \in \mathbb{R}_+, \\ &\widetilde{\Phi}(x, f)(t) = F(U(t)x + f(t)) - F(U(t)x), \forall t \in \mathbb{R}_+. \end{aligned}$$

One has

$$\left\| \widetilde{\Phi}(x, f)(t) \right\| = \| F(U(t)x + f(t)) - F(U(t)x) \| \leq e^{-\eta t} \|F\|_{\text{Lip}} \|f\|_\eta . \quad (4.68)$$

Lemma 4.45 *Let Assumption 4.35 be satisfied. Let $\eta \in (0, \beta_-)$ be fixed. Then*

$$\widetilde{K}_1 \in \mathcal{L}(X_s, BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right)) \text{ and } \widetilde{K}_2 \in \mathcal{L}(BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right), BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right))$$

with

$$\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n),BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n))} \leq \gamma(\eta) := \widehat{C}_{s,-\eta} + \|\Pi_{cu}\|_{\mathcal{L}(\mathbb{R}^n)} \int_0^{+\infty} \left\|e^{-(A_{cu}+\eta I)l}\right\| dl$$

where $\widehat{C}_{s,-\eta} > 0$ *is a constant, and*

$$\widetilde{\Phi}(x,0) = 0,$$

$$\widetilde{\Phi}(x,.) \in \text{Lip}\left(BC^{-\eta}\left(\mathbb{R}_+,\mathbb{R}^n\right), BC^{-\eta}\left(\mathbb{R}_+,\mathbb{R}^n\right)\right), \forall x \in \mathbb{R}^n,$$

with

$$\left\|\widetilde{\Phi}(x,.)\right\|_{\text{Lip}} \leq \|F\|_{\text{Lip}}.$$

Proof This proof is straightforward. □

Proof (Proof of Theorem 4.45) Let $\eta \in (0, \beta_-)$ and $x \in \mathbb{R}^n$ be fixed. Let $\delta_0 > 0$ be the constant introduced in Theorem 4.41. Let $\delta_1^* \in (0, \delta_0)$ be such that

$$\delta_1^* \gamma(\eta) < 1. \tag{4.69}$$

Then for each $F \in \text{Lip}\left(\mathbb{R}^n, \mathbb{R}^n\right)$ with $\|F\|_{\text{Lip}} \leq \delta_1^*$, we obtain that for each $(x, w_s) \in \mathbb{R}^n \times X_s$, there exists a unique solution $w = \widetilde{w}(x, w_s) \in BC^{-\eta}\left(\mathbb{R}_+, \mathbb{R}^n\right)$ such that

$$w = \widetilde{K}_1(w_s) + \widetilde{K}_2\widetilde{\Phi}(x, w)$$

and

$$w = (Id - \widetilde{K}_2\widetilde{\Phi}(x,.))^{-1}\widetilde{K}_1(w_s).$$

We have

$$\|\widetilde{w}(x, w_s) - \widetilde{w}(x, \widetilde{w}_s)\|_\eta \leq l \left\|w_s - \widetilde{w}_s\right\|, \forall x \in \mathbb{R}^n, \forall w_s, \widetilde{w}_s \in X_s,$$

where l depends on η and $\|F\|_{\text{Lip}}$ but stays bounded as $\|F\|_{\text{Lip}} \to 0$. To see the continuous dependence of $\widetilde{w}(x, w_s)$ on $x \in \mathbb{R}^n$, we remark that (4.69) and the continuity of $\gamma(\eta)$ imply that $\gamma(\zeta)\delta_1^* < 1$ for some $\zeta \in (\eta, \beta)$. Replacing η by ζ in the above argument, we conclude that $\widetilde{w}(x, w_s)$ belongs in fact to the space $BC^{-\zeta}(\mathbb{R}_+, \mathbb{R}^n)$, which is continuously imbedded in $BC^{-\eta}(\mathbb{R}_+, \mathbb{R}^n)$. More precisely, we have

$$\begin{aligned}\|\widetilde{w}(x, w_s)\|_\zeta &\leq \left\|\widetilde{K}_1\right\|_{\mathcal{L}(X_s, BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n))} \|w_s\| + \left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n))} \left\|\widetilde{\Phi}(x + x_0, w)\right\|_\zeta \\ &\leq \left\|\widetilde{K}_1\right\|_{\mathcal{L}(X_s, BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n))} \|w_s\| + \left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n))} \|F\|_{\text{Lip}} \|\widetilde{w}(x, w_s)\|_\zeta.\end{aligned}$$

Therefore, we obtain an estimate independent of x,

$$\|\widetilde{w}(x,w_s)\|_\zeta \leq \frac{\left\|\widetilde{K}_1\right\|_{\mathcal{L}(X_s,BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n))}\|w_s\|}{1-\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n))}\|F\|_{\text{Lip}}} < +\infty.$$

Moreover, we have

$$\begin{aligned}
&\widetilde{w}(x+x_0,w_s)-\widetilde{w}(x_0,w_s)\\
&\quad=\widetilde{K}_1(w_s)+\widetilde{K}_2\widetilde{\Phi}(x+x_0,\widetilde{w}(x+x_0,w_s))-\left[\widetilde{K}_1(w_s)+\widetilde{K}_2\widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s))\right]\\
&\quad=\widetilde{K}_2\left[\widetilde{\Phi}(x+x_0,\widetilde{w}(x+x_0,w_s))-\widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s))\right]\\
&\quad=\widetilde{K}_2\left[\widetilde{\Phi}(x+x_0,\widetilde{w}(x+x_0,w_s))-\widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s))\right]\\
&\qquad+\widetilde{K}_2\left[\widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s))-\widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s))\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\|\widetilde{w}(x+x_0,w_s)-\widetilde{w}(x_0,w_s)\|_\eta\\
&\quad\leq\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n))}\|F\|_{\text{Lip}}\|\widetilde{w}(x+x_0,w_s)-\widetilde{w}(x_0,w_s)\|_\eta\\
&\qquad+\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n))}\left\|\widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s))-\widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s))\right\|_\eta.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|\widetilde{w}(x+x_0,w_s)-\widetilde{w}(x_0,w_s)\|_\eta\\
&\leq\frac{\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n))}}{1-\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n))}\|F\|_{\text{Lip}}}\left\|\widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s))-\widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s))\right\|_\eta.
\end{aligned}$$

For fixed $w\in BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n)$ we claim that the mapping $x\to\widetilde{\Phi}(x,w)$ is continuous from $\mathbb{R}^n$ into $BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n)$. In fact, by using (*4.69*), we have

$$\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta=\sup_{t\in\mathbb{R}_+}e^{\eta t}\|\mathcal{H}(t)\|,$$

where

$$\begin{aligned}
\mathcal{H}(t):=&\left[F(U(t)(x+x_0)+w(t))-F(U(t)x_0+w(t))\right]\\
&-\left[F(U(t)(x+x_0))-F(U(t)x_0)\right].
\end{aligned}$$

Thus

$$\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta=\max\left(\sup_{0\leq t\leq T}e^{\eta T}\|\mathcal{H}(t)\|,\ 2e^{(\eta-\zeta)T}\|F\|_{\text{Lip}}\|w\|_\zeta\right).$$

By the continuity of $x\to U(t)(x)$ uniformly with respect to $t\in[0,T]$, we obtain

$$\limsup_{x\to 0}\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta \leq 2e^{(\eta-\zeta)T}\ \|F\|_{\text{Lip}}\ \|w\|_\zeta\ ,\ T\geq 0.$$

So when T goes to $+\infty$, we obtain

$$\lim_{x\to 0}\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta = 0.$$

From this and the fact that $\widetilde{w}(x,w_s)\in BC^{-\zeta}(\mathbb{R}_+,\mathbb{R}^n)$, it follows that $\widetilde{w}:\mathbb{R}^n\times X_s\to BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n)$ is continuous.

Define a map $\Gamma:\mathbb{R}^n\times X_s\to X_{cu}$ by

$$\Gamma(x,w_s)=\Pi_{cu}(Id-\widetilde{K}_2\widetilde{\Phi}(x,.))^{-1}\widetilde{K}_1(w_s)(0),\ \forall x\in\mathbb{R}^n,\ w_s\in X_s.$$

Notice that $\Gamma:\mathbb{R}^n\times X_s\to X_{cu}$ is continuous and Γ is Lipschitz continuous with respect to w_s with

$$\|\Gamma(x,.)\|_{\text{Lip}}\leq\|\Pi_{0cu}\|_{\mathcal{L}(\mathbb{R}^n)}\frac{\left\|\widetilde{K}_1\right\|_{\mathcal{L}(X_s,BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n))}}{1-\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,\mathbb{R}^n))}\|F\|_{\text{Lip}}}.$$

We have by construction that

$$y\in\widetilde{V}_\eta(x)\Leftrightarrow y=x+w \text{ with } \Pi_{cu}w=\Gamma(x,\Pi_s w).$$

Then

$$\begin{aligned}\widetilde{V}_\eta(x)&=\{x_s+w_s+x_{cu}+\Gamma(x,w_s):w_s\in X_s\}\\&=\{z+x_{cu}+\Gamma(x,z-x_s):z\in X_s\}.\end{aligned}$$

Consider the map $\Theta:\mathbb{R}^n\times X_s\to X_{cu}$ defined by

$$\Theta(x,z)=x_c+\Gamma(x,z-x_s),\forall x\in\mathbb{R}^n,\ z\in X_s,$$

we have $\widetilde{V}_\eta(x)=\{z+\Theta(x,z):z\in X_s\}$. Since $\Gamma:\mathbb{R}^n\times X_s\to X_{cu}$ is continuous and $\Gamma(x,w_s)$ is Lipschitz continuous with respect to w_s, so is Θ, and $\|\Theta\|_{\text{Lip}}\leq\|\Gamma\|_{\text{Lip}}$. Finally, we look for $y\in\mathbb{R}^n$, such that

$$\Pi_s y=\Psi_{cu}(\Pi_{cu}y)\ \text{ and } \Pi_{cu}y=:\Theta(x,\Pi_s y).$$

But by (4.59), we deduce that $\|\Psi_{cu}\|_{\text{Lip}}\to 0$ as $\|F\|_{\text{Lip}}\to 0$. So (4.59) and (4.69) imply that there exists $\delta_1\in\left(0,\delta_1^*\right)$ such that for each $F\in\text{Lip}(\mathbb{R}^n,\mathbb{R}^n)$ with $\|F\|_{\text{Lip}}\leq\delta_1$,

$$\|\Theta(x,.)\|_{\text{Lip}}\|\Psi_{cu}\|_{\text{Lip}}<1.$$

Thus, there exists for each $x\in\mathbb{R}^n$ a unique $\widetilde{y}_{cu}(x)\in X_{cu}$ such that

$$\Theta(x,\Psi_{cu}(\widetilde{y}_{cu}(x)))=\widetilde{y}_{cu}(x)$$

and the map $\widetilde{y}_c : \mathbb{R}^n \to X_{cu}$ is continuous. By setting $H_{cu}(x) = \widetilde{y}_{cu}(x) + \Psi_{cu}(\widetilde{y}_{cu}(x))$ the result follows. □

Theorem 4.46 (Local uniform convergence) *Let Assumption 4.35 be satisfied. Let $\eta \in (0, \beta_-)$ be fixed. Then there exists $\delta_1(\eta) \in (0, \delta_0)$ (where $\delta_0 > 0$ is the constant introduced in Theorem 4.41), such that for each $F \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^n)$ with $\|F\|_{\text{Lip}(\mathbb{R}^n,\mathbb{R}^n)} \leq \delta_1(\eta)$, the following holds: for each $\widetilde{x} \in V_\eta^{cu}$ and for each $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$\|U(t)x - U(t)H_{cu}(x)\| \leq \varepsilon e^{-\eta t}, \ \forall t \geq 0, \tag{4.70}$$

for all $x \in \mathbb{R}^n$ with $\|x - \widetilde{x}\| < \delta$.

Proof Let $\widetilde{x} \in V_\eta^{cu}$ be fixed. The proof of Theorem 4.45 implies that

$$\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_{0s}x)(t) = U(t)(H_{cu}(x)) - U(t)(x), \forall t \geq 0, \ \ \forall x \in \mathbb{R}^n,$$

where

$$H_{cu}(x) = \widetilde{y}_c(x) + \Psi_{cu}(\widetilde{y}_c(x)) \in V_\eta^{cu}.$$

It is clear that $H_{cu}(\widetilde{x}) = \widetilde{x}$ if $\widetilde{x} \in V_\eta^{cu}$ and hence

$$\widetilde{w}(\widetilde{x}, \Psi_{cu}(\widetilde{y}_c(\widetilde{x})) - \Pi_s\widetilde{x}) = U(t)(H_{cu}(\widetilde{x})) - U(t)(\widetilde{x}) = 0, \forall t \geq 0, \forall \widetilde{x} \in V_\eta^{cu}.$$

Let $\widetilde{x} \in V_\eta^{cu}$ and $\varepsilon > 0$. By the continuity of $\widetilde{w} : \mathbb{R}^n \times X_s \to BC^{-\eta}(\mathbb{R}_+, \mathbb{R}^n)$ and $\widetilde{y}_c : \mathbb{R}^n \to X_{cu}$, we can find some $\delta > 0$ such that

$$||\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_{0s}x) - \widetilde{w}(\widetilde{x}, \Psi_{cu}(\widetilde{y}_c(\widetilde{x})) - \Pi_s\widetilde{x})||_\eta \leq \varepsilon$$

whenever $x \in \mathbb{R}^n$ and $||x - \widetilde{x}|| < \delta$. Therefore,

$$\begin{aligned}
&\sup_{t \in \mathbb{R}_+} e^{\eta t}||U(t)(H_{cu}(x)) - U(t)(x)|| \\
&= \sup_{t \in \mathbb{R}_+} e^{\eta t}||\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_s x)(t)|| \\
&= \sup_{t \in \mathbb{R}_+} e^{\eta t}||\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_s x)(t) - \widetilde{w}(\widetilde{x}, \Psi_{cu}(\widetilde{y}_c(\widetilde{x})) - \Pi_s\widetilde{x})(t)|| \\
&\leq \varepsilon \text{ if } x \in \mathbb{R}^n \text{ and } ||x - \widetilde{x}|| < \delta.
\end{aligned}$$

The proof is complete. □

Remark 4.47 Our presentations focused on center-unstable manifolds. However, similar results can be established for center-stable manifolds. In fact, we will use a center-stable result to discuss the stability of Hopf bifurcation next section.

4.5 Existence and the Uniqueness of Traveling Waves for Fisher-KPP Equation

In this section we illustrate the consequence of the global attractors theory and center unstable manifold to give a short proof for the existence and uniqueness of the travelling waves for the Fisher-KPP equation. This section is taken from Ducrot, Langlais and Magal [58].

4.5.1 Fisher-Kolmogorov-Petrovski-Piskunov's traveling waves problem

Let us consider a logistic reaction diffusion equation

$$\partial_t N = \partial_{xx} N + \lambda N \left[1 - \frac{N}{\kappa}\right]. \tag{4.71}$$

When $\lambda = \beta - \mu > 0$, the above equation corresponds a logistic or Fisher-Kolmogorov-Petrovski-Piskunov (or for short Fisher-KPP) equation [77, 126]. Classically, $\beta > 0$ denotes the birth rate, $\mu > 0$ corresponds to the death rate while $\kappa > 0$ denotes the carrying capacity of the environment. Then if we look for special solution of the form

$$N(t, x) = U(x - ct)$$

Recall that for each $c \geq c^* = 2\sqrt{\lambda}$ this logistic equation has a unique (up to translation) travelling wave solution connecting $N = 0$ to $N = \kappa$. This means that for each $c \geq c^*$, there exists a non-increasing function $U \equiv U_c(x)$ such that

$$\begin{cases} U''(x) + cU'(x) + \lambda U(x) \left[1 - \dfrac{U(x)}{\kappa}\right] = 0, \quad x \in \mathbb{R}, \\ U(-\infty) = \kappa, \quad U(\infty) = 0. \end{cases} \tag{4.72}$$

The dynamics of such a logistic equation is well known and, in many cases, strongly related to travelling wave solutions. The literature about this topic is very wide. We only quote some of them, see for instance [8, 21, 97, 133, 134, 184, 214, 220] as well as references therein.

In this section, we discuss the existence and uniqueness of solutions for (4.72) by using invariant manifold techniques. Let us notice that (4.72) can be re-written as

$$\left(\frac{d}{dx} + \frac{c}{2}\right)^2 U - \frac{c^2}{4} U + U(1 - U) = 0.$$

Next setting

$$\begin{cases} U_1 := U \\ U_2 := \left(\dfrac{d}{dx} + \dfrac{c}{2}\right) U. \end{cases}$$

we obtain the following first order system of ordinary differential equations

$$\begin{cases} \left(\frac{d}{dx} + \frac{c}{2}\right) U_1 = U_2 \\ \left(\frac{d}{dx} + \frac{c}{2}\right) U_2 = \frac{c^2}{4} U_1 - U_1(1-U_1). \end{cases}$$

Set

$$\alpha := \frac{c}{2}$$

we obtain the system

$$\begin{cases} \frac{dU_1}{dx} = -\alpha U_1 + U_2 \\ \frac{dU_2}{dx} = -\alpha U_2 + \left(\alpha^2 - 1\right) U_1 + U_1^2 \end{cases} \tag{4.73}$$

Note that this system is monotone increasing on $[0, +\infty)^2$ whenever

$$\alpha \geq 1. \tag{4.74}$$

Moreover one has

$$\frac{d\left(\alpha U_1 + U_2\right)}{dt} = -U_1(x) + U_1(x)^2 = -U_1(x)(1 - U_1(x)) \tag{4.75}$$

and the points

$$\overline{U}^0 := (0, 0) \text{ and } \overline{U}^1 := (1, \alpha)$$

are the only equilibria of the system in $[0, +\infty)^2$.

4.5.2 Existence of travelling waves

Since $[0, 1] \times [0, \alpha]$ is invariant by the semiflow $\{T(t)\}_{t \geq 0}$ generated by the system (4.73). There exists a connected subset $A \subset [0, 1] \times [0, \alpha]$, which is the global attractor of the semiflow T on $[0, 1] \times [0, \alpha]$. The global attractor is connected, because the global attractor A attracts the connected set $[0, 1] \times [0, \alpha]$). The global attractor A contains both equilibria $(0, 0)$ and $(1, \alpha)$, by considering the linear functional $P : \mathbb{R}^2 \to \mathbb{R}$

$$P(U_1, U_2) = U_1$$

we deduce that $P(A)$ is compact and connected and contains $P(0, 0) = 0$ and $P(1, \alpha) = 1$. Hence one concludes that

$$P(A) = [0, 1].$$

Moreover

$$T(t) A = A, \forall t \geq 0.$$

Therefore $\{T(t)\}_{t\in\mathbb{R}}$ is a flow on A, and it follows that there exists a complete orbit $(U_1, U_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$ of system (4.73) such that

$$(U_1(t), U_2(t)) \in A, \forall t \geq 0,$$

and passing $t = 0$ through

$$(U_1, U_2) \text{ with } U_1 = 1/2 \text{ and } U_2 \in [0, \alpha].$$

By using (4.75) we deduce that

$$\lim_{t\to+\infty} (U_1(t), U_2(t)) = (0, 0) \text{ and } \lim_{t\to-\infty} (U_1(t), U_2(t)) = (1, \alpha).$$

Therefore A contains the equilibria, and all the travelling waves going from $(1, \alpha)$ to $(0, 0)$.

4.5.3 Uniqueness of the travelling waves

In order to prove the uniqueness of the heteroclinic orbit going from $(1, \alpha)$ to $(0, 0)$, we will study the center-unstable manifold around the equilibrium $(1, \alpha)$.
Linearized equation at $\overline{U}^1 = (1, \alpha)$: The matrix of the linearized equation of system (4.73) at $(1, \alpha)$ is

$$L_U = \begin{bmatrix} -\alpha & 1 \\ \alpha^2 + 1 & -\alpha \end{bmatrix},$$

the characteristic equation is given by

$$\begin{aligned}(\alpha + \lambda)^2 - \alpha^2 - 1 = 0 &\Leftrightarrow \lambda^2 + 2\alpha\lambda + \alpha^2 - \alpha^2 - 1 = 0 \\ &\Leftrightarrow \lambda^2 + 2\alpha\lambda - 1 = 0,\end{aligned}$$

hence the spectrum of L_U is given by

$$\sigma(L_U) = \{\lambda_U^-, \lambda_U^+\},$$

with

$$\lambda_U^- := -\alpha - \sqrt{\alpha^2 + 1} < 0 < \lambda_U^+ := -\alpha + \sqrt{\alpha^2 + 1}.$$

It follows that the center-unstable manifold at $(1, \alpha)$ is a one dimensional locally invariant manifold. Since the center-unstable manifold at $(1, \alpha)$ contains the point of any negative orbit staying in some neighborhood (small enough) of $(1, \alpha)$. It follows that the traveling wave (or the complete orbit) going from $(1, \alpha)$ (at $t = -\infty$) to $(0, 0)$ (to $t = +\infty$) is unique (we refer for instance to [214] for an other proof).

The precise result proven is the following:

Theorem 4.48 *Assume that $\alpha \geq 1$ (that reads $c \geq 2$). Then there exists at most one travelling wave going from $(1, \alpha)$ to $(0, 0)$ for (4.72). More precisely, there exists a unique solution $U^*(x) = (U_1^*(x), U_2^*(x))$ of system (4.73) satisfying*

$$\lim_{x\to-\infty} U^*(x) = (1,\alpha) \ \textit{and} \ \lim_{x\to+\infty} U^*(x) = (0,0).$$

Remark 4.49 The profiles of traveling waves are not always unique. We refer to Ducrot, Langlais and Magal [58] for an example of non-unique traveling wave profile.

4.6 Remarks and Notes

The chapter is devoted to the center manifold theory that was presented by Vanderbauwhede in [216] and Vanderbauwhede and Iooss [217]. Our presentation is inspired by Magal and Ruan [154] where Vanderbauwhede's center manifold's results was extended an abstract class of Cauchy problems on Banach spaces. The second part of this chapter about the center-unstable manifold is inspired by Liu, Magal and Ruan [146]. More results and references on the center manifold theory will given at then end this chapter.

Center manifold

The classical center manifold theory was first established by Pliss [175] and Kelley [124] and was developed and completed in Carr [23], Sijbrand [199], Vanderbauwhede [216], etc. For the case of a single equilibrium, the center manifold theorem states that if a finite dimensional system has a non hyperbolic equilibrium, then there exists a center manifold in a neighborhood of the non hyperbolic equilibrium which is tangent to the generalized eigenspace associated to the corresponding eigenvalues with zero real parts, and the study of the general system near the non hyperbolic equilibrium reduces to that of an ordinary differential equation restricted on the lower dimensional invariant center manifold. This usually means a considerable reduction of the dimension which leads to simple calculations and a better geometric insight. The center manifold theory has significant applications in studying other problems in dynamical systems, such as bifurcation, stability, perturbation, etc. It has also been used to study various applied problems in biology, engineering, physics, etc. We refer to, for example, Carr [23] and Hassard et al. [99]. There are two classical methods to prove the existence of center manifolds. The Hadamard (Hadamard [88]) method (the graph transformation method) is a geometric approach which bases on the construction of graphs over linearized spaces,see Hirsch et al. [103] and Chow et al. [34, 36]. The Liapunov-Perron (Liapunov [140], Perron [174]) method (the variation of constants method) is more analytic in nature, which obtains the manifold as a fixed point of a certain integral equation. The technique originated in Krylov and Bogoliubov [127] and was furthered developed by Hale [90, 92], see also Ball [11], Chow and Lu [39], Yi [229], etc. The smoothness of center manifolds can be proved by using the contraction mapping in a scale of Banach spaces (Vanderbauwhede and van Gils [218]), the Fiber contraction mapping technique (Hirsch et al. [103]), the Henry lemma (Henry [101], Chow and Lu [38]), among other methods (Chow et al. [33]). For further results and references on center manifolds, we refer to the monographs of Carr [23], Chow and Hale [30], Chow et al. [32], Sell and You [189], Wiggins [224] and the survey papers of Bates and Jones [13], Vanderbauwhede [215] and Vanderbauwhede and Iooss [217].

Recently, great attention has been paid to the study of center manifolds in infinite dimensional systems and researchers have developed the center manifold theory for various infinite dimensional systems such as partial differential equations(Bates and Jones [8], Da Prato and Lunardi [30], Henry [54], Scheel [93]), semiflows in Banach spaces (Bates et al. [9], Chow and Lu [21], Gallay [45], Scarpellini [91],Vanderbauwhede [103], Vanderbauwhede and van Gils [105]), delay differential equations (Hale [50], Hale and Verduyn Lunel [51], Diekmann and van Gils [34,35], Diekmann et al. [36], Hupkes and Verduyn Lunel [58]), infinite dimensional non autonomous differential equations (Mielke [81, 82], Chicone and Latushkin[15]), and partial functional differential equations (Lin et al. [73], Faria et al.[43], Krisztin [68], Nguyen and Wu [83], Wu [111]). Infinite dimensional systemsusually do not have some of the nice properties the finite dimensional systems have. For example, the initial value problem may not be well posed, the solutions may not be extended backward, the solutions may not be regular, the domain of operators may not be dense in the state space, etc. Therefore, the center manifold reduction of the infinite dimensional systems plays a very important role in the theory of infinite dimensional systems since it allows us to study ordinary differential equations reduced on the finite dimensional center manifolds. Vanderbauwhede and Iooss [106] described some minimal conditions which allow to generalize the approach of Vanderbauwhede [104] to infinite dimensional systems.

In Magal and Ruan [154, 155] we consider a center manifold for non densely Cauchy problems

$$u'(t) = Au(t) + F(u(t)), \text{ for } t \geq 0, \text{ and } u(0) = x \in \overline{D(A)},$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space and $F : \overline{D(A)} \to X$ is Lipschitz continuous local arounds 0.

More recently, Ducrot and Magal [60] extended such an idea to a class of abstract second order semi-linear differential equations on the real line. In [60] we obtain the existence of wave train arising from a Hopf bifurcation.

Persistence of a normally hyperbolic manifold

The principle of the center manifold theorem is to show that the linear center manifold persists for small perturbation of the linear system. There also exists some nonlinear version of such results. That is the so called normally hyperbolic manifold. There have been several important extensions of the classical center manifold theory for invariant sets. For higher dimensional invariant sets, it is known that center manifolds exist for an invariant torus with special structure (Chow and Lu [40]), for an invariant set consisting of equilibria (Fenichel [74]), for some homoclinic orbits (Homburg

[110], Lin [142] and Sandstede [188]), for skew-product flows (Chow and Yi [42]), for any piece of trajectory of maps (Hirsch et al. [103]), and for smooth invariant manifolds and compact invariant sets (Chow et al. [35, 37]).

We also refer to Ducrot Magal and Seydi [61, 62] and Magal and Seydi [157] for more references and some extensions to abstract non densely densely defined Cauchy problems.

Center unstable manifold

Given a non hyperbolic equilibrium, the center-unstable manifold is a locally invariant manifold by the semiflow and is tangent to the generalized eigenspace associated to the corresponding eigenvalues with non-negative real parts (Kelley [124]).The local center-unstable manifold plays an important role in applications since it has some nice stability properties. Compared to center manifold, it is also easier to use in practice,since a point (locally around the equilibrium) belongs to the center-unstable manifold if only if there exists a negative orbit (staying in some small neighborhood of the equilibrium) passing through the point at time $t = 0$, while for the center manifold a complete orbitis needed. Center-unstable manifolds in in nite dynamical systems have been studied by many researchers. For example, Armbruster et al. [6] investigated center-unstable manifolds in Kuramoto-Sivashinsky equation. Chow and Lu [38] discussed the existence and smoothness of global center-unstable manifolds for semilinear and fully nonlinear differential evolution equations. Dell'Antonio and D'Onofrio [50] studied center-unstable manifolds for the Navier-Stokes equation. Nakanishi and Schlag [172] established center-unstable and center-stable manifolds around soliton manifolds for the nonlinear Klein-Gordon equation. Turyn [213] obtained a center-unstable manifold theorem for parametrically excited surface waves. Stumpf [205] discussed center-unstable manifolds for differential equations with state-dependent delay.

In Liu, Magal and Ruan [145] we consider a center unstable manifold for non densely Cauchy problems

$$u'(t) = Au(t) + F(u(t)), \text{ for } t \geq 0, \text{ and } u(0) = x \in \overline{D(A)},$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space and $F : \overline{D(A)} \to X$ is Lipschitz continuous local arounds 0. In [145], we take advantage of the stability property of the center-unstable manifold to prove a stability theorem for the periodic orbits arsing from a Hopf bifurcation.

Chapter 5
Normal Forms

This chapter treats the normal form theory for ordinary differential equations. The first part of this chapter addresses the computation of the normal form near a equilibrium solution. In order to consider the behavior near an equilibrium solution for the nonlinear system for which the linearized system has several eigenvalues on the imaginary axis, one usually focuses on the flow on the center manifold. We concentrate our efforts to introduce a method to compute the normal form associated with the flow on the center manifold in the second part. Our presentation in the first part is inspired by Chow and Li and Wang [32], Chow and Hale [30], Bibikov [19], Vanderbauwhede [216] and Wiggins [225], and in the second part is inspired by Liu, Magal and Ruan [147] where the normal form theory for an abstract class of Cauchy problems on Banach spaces is presented.

5.1 Introduction

Consider

$$u' = f(u), \tag{5.1}$$

where $u \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^k for some $k \geq 2$. Recall that the equilibrium solutions $u = \overline{u}$ of system (5.1) must satisfy

$$f(\overline{u}) = 0 \text{ with } \overline{u} \in \mathbb{R}^n.$$

The change of variables $v = u - \overline{u}$ transforms (5.1) to the form

$$v' = f(v + \overline{u}), \tag{5.2}$$

and $v = 0$ is an equilibrium solution of system (5.2). Therefore, without loss of generality, we consider

$$u'(t) = f(u) = Au(t) + F(u(t)), t \geq 0, \tag{5.3}$$

where $u \in \mathbb{R}^n$, $f(0) = 0$, $A = Df(0) \in \mathcal{L}(\mathbb{R}^n)$, and $F := f - A$ satisfies $F(0) = 0$ and $DF(0) = 0$. It is well known that there exists a $n \times n$ nondegenerate matrix T which can transform A into Jordan canonical form. Then, under the transformation

$$u = T\overline{u}$$

and after dropping the hat, (5.3) becomes

$$u'(t) = T^{-1}ATu + T^{-1}F(Tu),$$

where $u \in \mathbb{R}^n$ (or $\mathbb{C}^n$) and $T^{-1}AT$ is in Jordan canonical form. We remark that the transformation $u = T\overline{u}$ has simplified the linear part of (5.3) as much as possible. The goal of the norm form theory is to simplify the nonlinear part of (5.3), that is, to find a suitable (nonlinear) change of coordinates which will transform the nonlinear part of (5.3) to the simplest possible form.

5.2 Normal Forms for Differential Equations Near a Equilibrium Solution

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. In this section we consider

$$u'(t) = Au(t) + F(u), \tag{5.4}$$

where $u \in \mathbb{K}^n$, $A \in M_n(\mathbb{K})$, $F(0) = 0$ and $DF(0) = 0$. We write $F(u)$ around the origin $u = 0$ in (5.4) as a formal power series and then

$$u'(t) = Au(t) + F_2(u) + F_3(u) + \cdots + F_j(u) + O(|u|^{j+1}), \tag{5.5}$$

where $F_m(u) := \frac{1}{m!}D^m F(0)(u, \cdots, u)$, $2 \le m \le j$.

We introduce linear vector spaces $H_m^n(\mathbb{K}^n)$ with $F_m(u) \in H_m^n(\mathbb{K}^n)$, $2 \le m \le j$. Let $\Psi = \{\psi_1, \cdots, \psi_n\}$ denote a basis of $\mathbb{K}^n$, and let $u = (u_1, \cdots, u_n)$ be coordinates with respect to this basis. We refer to $u^d\varphi$, where $\varphi \in \mathbb{K}^n$, $d = (d_1, \cdots, d_n)$, $\sum_{j=1}^{n} d_j = m$, $d_j \ge 0$ are integers and $u^d = u_1^{d_1}u_2^{d_2}...u_n^{d_n}$, as a vector-valued homogeneous polynomial of degree m in n variables $u = (u_1, \cdots, u_n)$, and use the notation $H_m^n(\mathbb{K}^n)$ to denote the linear vector space formed by all the vector-valued homogeneous polynomials of degree m in n variables $u = (u_1, \cdots, u_n)$ with coefficients in $\mathbb{K}^n$. Then an obvious basis denoted by Φ_m for $H_m^n(\mathbb{K}^n)$ consists of all possible vector-valued homogeneous polynomials of degree m in n variables $u = (u_1, \cdots, u_n)$ with coefficients in Ψ, that is

$$\Phi_m = \left\{ u^d\psi_j \middle| \psi_j \in \Psi,\ j = 1, \cdots, n,\ \sum_{j=1}^{n} d_j = m, d_j \ge 0 \text{ are integers} \right\}.$$

The dimension of $H_m^n(\mathbb{K}^n)$ is

$$d_m := \dim H^n_m(\mathbb{K}^n) = \frac{n\,(n+m-1)!}{m!\,(n-1)!}.$$

We take the reverse lexicographic ordering of the elements in the basis Φ_m of $H^n_m(\mathbb{K}^n)$, that is, $u^\xi\psi_i \in \Phi_m$ with $\xi = (\xi_1, ..., \xi_n)$ precedes $u^\chi\psi_j \in \Phi_m$ with $\chi = (\chi_1, ..., \chi_n)$ if and only if the first non-zero difference $i - j, \xi_1 - \chi_1, ..., \xi_n - \chi_n$ is positive. We denote $u^d\psi_j$ by φ_k if $u^d\psi_j$ is the kth basis element with respect to the reverse lexicographic ordering and then the basis Φ_m with the elements in the reverse lexicographic ordering is written as

$$\Phi_m = \left\{\varphi_1, \varphi_2, \ldots, \varphi_{d_m}\right\}.$$

We will not verify these statements about the space $H^n_m(\mathbb{K}^n)$ and refer the reader to the books [32], [225]. Let us consider a specific example to explain these statements.

Example 5.1 Let $\Psi = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$, that is, $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$, and let $u = (u_1, u_2)$ be coordinates with respect to this basis. Then

$$H^2_2(\mathbb{R}^2) = \left\{u_1^{d_1}u_2^{d_2}\varphi \,\middle|\, \varphi \in \mathbb{R}^2,\ d_1 + d_2 = 2,\ d_j \geq 0,\ \ j = 1, 2,\ \text{are integers}\right\}$$

and

$$H^2_3(\mathbb{R}^2) = \left\{u_1^{d_1}u_2^{d_2}\varphi \,\middle|\, \varphi \in \mathbb{R}^2, d_1 + d_2 = 3,\ d_j \geq 0,\ \ j = 1, 2,\ \text{are integers}\right\}.$$

The bases Φ_2 and Φ_3 with the elements in the reverse lexicographic ordering are

$$\Phi_2 = \left\{\begin{pmatrix}0\\u_1^2\end{pmatrix}, \begin{pmatrix}0\\u_1u_2\end{pmatrix}, \begin{pmatrix}0\\u_2^2\end{pmatrix}, \begin{pmatrix}u_1^2\\0\end{pmatrix}, \begin{pmatrix}u_1u_2\\0\end{pmatrix}, \begin{pmatrix}u_2^2\\0\end{pmatrix}\right\},$$

and

$$\Phi_3 = \left\{\begin{pmatrix}0\\u_1^3\end{pmatrix}, \begin{pmatrix}0\\u_1^2u_2\end{pmatrix}, \begin{pmatrix}0\\u_1u_2^2\end{pmatrix}, \begin{pmatrix}0\\u_2^3\end{pmatrix}, \begin{pmatrix}u_1^3\\0\end{pmatrix}, \begin{pmatrix}u_1^2u_2\\0\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}u_1u_2^2\\0\end{pmatrix}, \begin{pmatrix}u_2^3\\0\end{pmatrix}\right\}$$

respectively.

Our aim is to find a change of coordinates

$$u = G(\overline{u}) \tag{5.6}$$

to bring the equation (5.3) in the simplest possible form up to terms of a specified order, called normal form, where G is a C^j transformation in a neighborhood $B(0, r)$ of the origin and $G(0) = 0$. By substituting (5.6) into (5.3) and dropping the hat for simplisity of notation, we obtain

$$u' = DG(u)^{-1}AG(u) + DG(u)^{-1}F(G(u)). \tag{5.7}$$

The linear part of (5.7) is $DG(0)^{-1}ADG(0)u$. If A is already in the Jordan canonical form, thus the diffeomorphism G may take the form

$$G(\overline{u}) = \overline{u} + O(|\overline{u}|^2) \text{ as } \overline{u} \to 0$$

and (5.7) could be written as

$$u' = Au + H(u), u \in B(0, r),$$

with $H(u) = O(|u|^2)$ as $|u| \to 0$ which is in the simplest possible form up to terms of the specified order j. The desired simplification of (5.3) will be obtained by performing inductively a sequence of near identity change of coordinates on system (5.5).

5.2.1 Computation of Normal Form and Normal Form Theorem

In the following we will obtain the simplest possible form of (5.5) up to terms j by performing inductively a sequence of change of coordinates of the form

$$u = \xi_m(\overline{u}) = \overline{u} + G_m(\overline{u}), \overline{u} \in B^m(0, r), \tag{5.8}$$

where $G_m(\overline{u}) \in H^n_m(\mathbb{K}^n)$ and $B^m(0, r)$ is a small neighborhood of the origin, $2 \le m \le j$. Notice that the map $\xi_m(\overline{u})$ is a diffeomorphism in some neighborhood of the origin and we take $B^m(0, r)$ small enough such that $D\xi_m(\overline{u}) = I + DG_m(\overline{u})$ is invertible on it and

$$D\xi_m(\overline{u})^{-1} = I - DG_m(\overline{u}) + O(|\overline{u}|^{2m-2}), \overline{u} \in B^m(0, r),$$

here I is the identity matrix. Substituting (5.8) into (5.5) and dropping the hats for simplicity of notation, we get

$$\begin{aligned} u' = & Au(t) + F_2(u) + F_3(u) + \cdots + F_{m-1}(u) \\ & + \{F_m(u) - [DG_m(u)Au - AG_m(u)]\} \\ & + \overline{F}_{m+1}(u) + \cdots + \overline{F}_j(u) \\ & + O(|u|^{j+1}), \\ & u \in B^m(0, r). \end{aligned} \tag{5.9}$$

Now we introduce the **Lie bracket** $[\cdot, \cdot]$ operation

$$[A, G_m](u) \equiv DG_m(u)Au - AG_m(u).$$

Then (5.9) can be written as

$$\begin{aligned} u' = &Au(t) + F_2(u) + F_3(u) + \cdots + F_{m-1}(u) \\ &+ \{F_m(u) - [A, G_m]\} \\ &+ \overline{F}_{m+1}(u) + \cdots + \overline{F}_j(u) \\ &+ O(|u|^{j+1}), \\ &u \in B^m(0, r). \end{aligned} \tag{5.10}$$

Notice that after the change of coordinates (5.8), only terms of order higher than $m - 1$ are modified.

Define a linear operator $\Theta_m : H_m^n(\mathbb{K}^n) \to H_m^n(\mathbb{K}^n)$ by

$$(\Theta_m G_m)(u) := [A, G_m](u) \equiv DG_m(u)Au - AG_m(u), G_m(u) \in H_m^n(\mathbb{K}^n). \tag{5.11}$$

From elementary linear algebra, we know that $H_m^n(\mathbb{K}^n)$ can be (non uniquely) represented as the direct sum

$$H_m^n(\mathbb{K}^n) = \mathcal{R}_m \oplus \mathcal{C}_m, \tag{5.12}$$

where

$$\mathcal{R}_m := R(\Theta_m)$$

is the range of Θ_m, and $\mathcal{C}_m$ is some complementary space of $\mathcal{R}_m$ into $H_m^n(\mathbb{K}^n)$.

The range of the operators Θ_m, defined in the spaces $H_m^n(\mathbb{K}^n)$, contains exactly the terms that can be taken away from the equation in the computation of the normal form. It is of interest to know when these ranges are the whole spaces $H_m^n(\mathbb{K}^n)$, since that corresponds to the cases where the terms of order m, $2 \le m \le j$, can be completely eliminated from the equation (5.5).

Theorem 5.2 (Normal Form Theorem) *Let the decomposition (5.12) of $H_m^n(\mathbb{K}^n)$ be given for $m = 2, \ldots, j$. Then the appropriate transformations $u = \xi_m(\overline{u})$, where $\overline{u} \in B^m(0, r)$, $B^m(0, r)$ is a neighborhood of the origin and $B^{m+1}(0, r) \sqsubseteq B^m(0, r)$, $m = 2, \cdots, j$, can be chosen so that the system (5.5) is transformed into*

$$\begin{aligned} u' &= Au + g_2(u) + g_3(u) + \cdots + g_j(u) + O(|u|^{j+1}), \\ u &\in B^j(0, r), \end{aligned} \tag{5.13}$$

with $g_m(u) \in \mathcal{C}_m$ for $m = 2, \cdots, j$. System (5.13) is said to be in normal form through order j.

Proof We start from $m = 2$. Substituting (5.8) with $m = 2$ into (5.5) and dropping the hats, (5.9) becomes

$$\begin{aligned} u' = &Au(t) + \{F_2(u) - [DG_2(u)Au - AG_2(u)]\} \\ &+ \overline{F}_3(u) + \cdots + \overline{F}_j(u) \\ &+ O(|u|^{j+1}), \\ &u \in B^2(0, r). \end{aligned} \tag{5.14}$$

Since $F_2(u) \in H_m^n(\mathbb{K}^n)$, $F_2(u) = f_2(u) + g_2(u)$ with $f_2 \in \mathcal{R}_2$ and $g_2 \in \mathcal{C}_2$. We can find a $G_2(u) \in H_m^n(\mathbb{K}^n)$ such that $\Theta_2(G_2(u)) = f_2(u)$ and then (5.14) becomes

$$u' = Au(t) + g_2(u) + \overline{F}_3(u) + \cdots + \overline{F}_j(u) + O(|u|^{j+1}), \quad u \in B^2(0, r). \tag{5.15}$$

We assume that after computing the normal form up to terms of order $m-1$, $2 \le m \le j$, the equation becomes

$$u'(t) = Au(t)+g_2(u)+g_3(u)+\cdots+g_{m-1}(u)+\overline{F}_m(u)+\cdots+\overline{F}_j(u)+O(|u|^{j+1}). \tag{5.16}$$

Substituting (5.8) into (5.16) and dropping the hats, we get

$$\begin{aligned} u' = {} & Au(t) + g_2(u) + g_3(u) + \cdots + g_{m-1}(u) \\ & + \left\{\overline{F}_m(u) - [A, G_m](u)\right\} \\ & + \overline{\overline{F}}_{m+1}(u) + \cdots + \overline{\overline{F}}_j(u) \\ & + O(|u|^{j+1}), \ u \in B^m_{\mathbb{R}^n}(0, r). \end{aligned} \tag{5.17}$$

Since $\overline{F}_m(u) \in H^n_m(\mathbb{K}^n)$, $\overline{F}_m(u) = f_m(u) + g_m(u)$ with $f_m(u) \in \mathcal{R}_m$ and $g_m(u) \in C_m$. We can find a $G_m(u) \in H^n_m(\mathbb{K}^n)$ such that $\Theta_m(G_m(u)) = f_m(u)$ and then (5.17) becomes

$$\begin{aligned} u' = {} & Au(t) + g_2(u) + g_3(u) + \cdots + g_m(u) \\ & + \overline{\overline{F}}_{m+1}(u) + \cdots + \overline{\overline{F}}_j(u) \\ & + O(|u|^{j+1}), \ u \in B^m_{\mathbb{R}^n}(0, r). \end{aligned}$$

The proof is completed by induction. □

It is obvious that the simplified system (5.13) is strongly depending on the specific choice of the complementary spaces C_m and then the normal form of (5.5) is not unique. The key to compute the normal form of the vector field is to find C_m.

5.2.2 Resonance Conditions and Resonant Monomial

Let $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{p\times p}$ be matrices and set $\sigma(A) = \{\lambda_1, ..., \lambda_n\}$, $\sigma(B) = \{\mu_1, ..., \mu_p\}$, $\lambda = (\lambda_1, \cdots, \lambda_n)$, $d = (d_1, \cdots, d_n)$, $\lambda \cdot d = \lambda_1 d_1 + \cdots + \lambda_n d_n$ and $d_i \ge 0, i = 1, \ldots, n$, are integers. Consider a linear operator $\Gamma : H^n_m(\mathbb{K}^p) \to H^n_m(\mathbb{K}^p)$ defined by

$$\Gamma(G)(u) := DG(u)\,Au - BG(u), \forall G(u) \in H^n_m(\mathbb{K}^p). \tag{5.18}$$

We refer to [19] for the proofs of the following results.

Lemma 5.3 *The spectrum of the linear operator* Γ *is the following set*

$$\sigma(\Gamma) = \left\{\lambda \cdot d - \mu_j : j = 1, \cdots, p, \ \sum_{j=1}^{n} d_j = m\right\}.$$

In the following, we assume that the linear operator A in (5.5) is a Jordan canonical matrix for convenience. Let $u = (u_1, u_2, \ldots, u_n)$ be coordinates with respect to the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{K}^n$. Now we present the resonance conditions and resonant monomials.

Definition 5.4 (Resonance conditions and resonant monomial) Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A. The following relations are called resonant conditions:

$$\lambda \cdot d - \lambda_j = 0, \tag{5.19}$$

where $1 \leq j \leq n$ and $\sum_{j=1}^{n} d_j \geq 2$. A monomial $u^d e_j$ ($\sum_{j=1}^{n} d_j = m \geq 2$ and $1 \leq j \leq n$) is called a resonant monomial of order m if and only if (5.19) holds for d and j. We say that equation (5.5) satisfies the nonresonance conditions relative to $\sigma(A)$ of order m if

$$\lambda \cdot d - \lambda_j \neq 0, \text{ for all } \lambda_j \in \sigma(A), \text{ all } d = (d_1, \cdots, d_n) \text{ with} \sum_{j=1}^{n} d_j = m.$$

From Lemma 5.3, we get the following result.

Theorem 5.5 *Let $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}$. The spectrum of the linear operator Θ_m defined in (5.11) is the following set:*

$$\sigma(\Theta_m) = \left\{ \lambda \cdot d - \lambda_j : j = 1, \cdots, n, \ \sum_{j=1}^{n} d_j = m \right\}. \tag{5.20}$$

Theorem 5.6 *Let A is a Jordan canonical matrix. Then the appropriate transformations $u = \xi_m(\overline{u})$, $\overline{u} \in B^m(0, r)$, where $B^m(0, r)$ is a neighborhood of the origin and $B^{m+1}(0, r) \sqsubseteq B^m(0, r), m = 2, \cdots, j$, can be chosen so that $g_m(u)$ in the right side of (5.13) consists of resonant monomials of order m, $m = 2, \cdots, j$.*

Remark 5.7 Since the change of coordinates (5.8) are local diffeomorphisms near the origin, the normal form theorem presented in the above gives only local results of (5.5) near the origin. Furthermore the normal form theorem above is about the normal form up to certain finite order j. If $F(u)$ is analytic in u, then it is clear that we can formally transform (5.5) to a normal form with $j = \infty$, that is,

$$u' = Au + \sum_{m \geq 2} g_m(u).$$

However, the issue of convergence of the power series for the normal forms and the associated transformations of variables should be considered. The answer for the issue of convergence is positive under some conditions for ordinary differential equations, and we refer to Chow et al. [32]. In fact in the applications of normal forms, usually the flow is completely determined by the terms of a normal form up to a certain finite order, for example, we only need compute the normal form up to third order for Hopf bifurcations.

5.2.3 The Matrix Representation Method

One method to find the complementary subspace C_m or normal form is to use the matrix representation $\mathbf{M}_m$ of the linear operator Θ_m with respect to the given basis Φ_m of $H_m^n(\mathbb{K}^n)$. Then $\mathbf{M}_m$ is a $d_m \times d_m$ matrix and $\mathbb{K}^{d_m} = \widetilde{C}_m \oplus \widetilde{\mathcal{R}}_m$, where $\widetilde{\mathcal{R}}_m$ is the range of $\mathbf{M}_m$ in $\mathbb{K}^{d_m}$ and $\widetilde{C}_m$ any complementary subspace. Our choice of $\widetilde{C}_m$ is certainly not unique. According to Fredholm Theorem, $\widetilde{C}_m = Ker((\mathbf{M}_m)^T)$ is a complementary subspace, where $Ker((\mathbf{M}_m)^T)$ is the null-space of the transpose of $\mathbf{M}_m$. Other complementary subspaces to $\widetilde{\mathcal{R}}_m$ can be obtained from $Ker((\mathbf{M}_m)^T)$ by performing elementary algebraic calculations. It is obvious that

$$H_m^n(\mathbb{K}^n) = C_m \oplus \mathcal{R}_m$$

where

$$C_m := \left\{ \kappa = \sum_{i=1}^{d_m} \xi_i \varphi_i \in H_m^n(\mathbb{K}^n) \,\middle|\, \varphi_i \in \Phi_m,\ (\xi_1, ..., \xi_{d_m}) \in \widetilde{C}_m \right\}, \quad (5.21)$$

$$\mathcal{R}_m := \left\{ \kappa = \sum_{i=1}^{d_m} \xi_i \varphi_i \in H_m^n(\mathbb{K}^n) \,\middle|\, \varphi_i \in \Phi_m,\ (\xi_1, ..., \xi_{d_m}) \in \widetilde{\mathcal{R}}_m \right\},$$

$\mathcal{R}_m$ is the range of Θ_m, and C_m is a complementary space of $\mathcal{R}_m$ into $H_m^n(\mathbb{K}^n)$. We refer to [32], [216] and [222] for more details and also for other methods to get the complementary subspace C_m or normal form. Here we will show an example to expain this method.

Example 5.8 (The Takens-Bogdanov Normal Form) We will compute the normal form up to terms of order 2 for the following vector field on $\mathbb{R}^2$ in the neighborhood of the origin

$$u'(t) = Au(t) + \begin{pmatrix} c_1 u_1^2 + c_2 u_1 u_2 + c_3 u_2^2 \\ d_1 u_1^2 + d_2 u_1 u_2 + d_3 u_2^2 \end{pmatrix} + O(|u|^3),$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$, and let $u = (u_1, u_2)$ be coordinates with respect to this basis. The basis Φ_2 of $H_2^2(\mathbb{R}^2)$ with the elements in the reverse lexicographic ordering is

$$\Phi_2 = \left(\begin{pmatrix} 0 \\ u_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2^2 \end{pmatrix}, \begin{pmatrix} u_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2^2 \\ 0 \end{pmatrix} \right).$$

We first obtain the matrix representation $\mathbf{M}_2$ of the linear operator Θ_2 with respect to the given basis Φ_2:

$$\mathbf{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

Then $\mathbb{R}^6 = \widetilde{\mathcal{C}}_2 \oplus \widetilde{\mathcal{R}}_2$, where $\widetilde{\mathcal{R}}_2$ is the range of $\mathbf{M}_2$ in $\mathbb{R}^6$ and $\widetilde{\mathcal{C}}_2$ any complementary subspace. Let $(e_1, e_2, \ldots, e_6)$ be the standard basis of $\mathbb{R}^6$ and then $Ker((\mathbf{M}_2)^T) = span\left\{e_{1,}e_2 + 2e_4\right\}$ is a complementary subspace of $\widetilde{\mathcal{R}}_2$. Therefore $H_2^2(\mathbb{R}^2) = \mathcal{C}_2 \oplus \mathcal{R}_2$, where

$$\mathcal{C}_2 = span\left\{\begin{pmatrix} 0 \\ u_1^2 \end{pmatrix}, \begin{pmatrix} 2u_1^2 \\ u_1 u_2 \end{pmatrix}\right\}.$$

Thus we obtain the normal form up to terms of order 2

$$u'(t) = Au(t) + d_1 \begin{pmatrix} 0 \\ u_1^2 \end{pmatrix} + \frac{2c_1 + d_2}{5} \begin{pmatrix} 2u_1^2 \\ u_1 u_2 \end{pmatrix} + O(|u|^3).$$

5.3 Normal Forms for Reduced Differential Equations on the Center Manifold

In studying nonlinear dynamical problems, the centre manifold theory of chapter 14 gives a reduction in the dimension of the system. We usually restrict our analysis to the flow on the centre manifold near a equilibrium solution which is useful for the study of bifurcation problems. In this section we will consider the computation of normal form associated with the flow on the center manifold for the differential equation of the form

$$u'(t) = Au(t) + F(u(t)), \forall t \in \mathbb{R}, \tag{5.22}$$

where $A \in M_n(\mathbb{R})$ is a n by n matrix and $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^k with $F(0) = 0$ and $DF(0) = 0$, $k > 1$.The notations X_i, Π_i, A_i with $i = c, s, u, h$, which we will use in the following, are defined in chapter 14.

Assumption 5.9 We assume that $\sigma_c(A) = \sigma(A) \cap i\mathbb{R} \neq \emptyset$.

The center manifold theorem of Chapter 14 tells us that there exists a C^k-smooth function $\Psi : X_c \to X_h$ with $\Psi(0) = 0$, $D\Psi(0) = 0$, such that

$$M = \{x_c + \Psi(x_c) : x_c \in X_c\} \quad \textbf{(center manifold)}$$

is a locally invariant center manifold of (5.22). Moreover, as Π_c is defined on $\mathbb{R}^n$, we can project (5.22) on X_c and obtain the following **reduced system**

$$\frac{du_c(t)}{dt} = A_c u_c(t) + \Pi_c F\left[u_c(t) + \Psi(u_c(t))\right]. \tag{5.23}$$

The qualitative behaviour of the flow of (5.22) near the equilibrium solution $u = 0$ is determined by its behaviour on the center manifold.

5.3.0.1 An Outline on the Methods

This chapter is devoted to the computation of the normal form of the reduced system (5.23). First, one needs to realize that the center manifold Ψ is known only through an implicit fixed point procedure and when

$$D^l\Psi(0) = 0 \text{ for each } l = 1, ..., k,$$

the Taylor's expansion of the reduced system (5.23) is simply given by

$$\frac{du_c(t)}{dt} = A_c u_c(t) + \sum_{l=1}^{k} \frac{1}{l!}\Pi_c D^l F(0)\,(u_c(t), ..., u_c(t)) + h.o.t.$$

We will show two ways to compute the normal form of the reduced system (5.23). One is to compute the Taylor's expansion of the reduced system first and then we can compute the normal form of the reduced system (5.23) using the procedure in section 2 of this chapter. The other way is to compute the normal form of the reduced system (5.23) directly.

In general, the only information available to compute the Taylor's expansion and normal form of the reduced system is the following result in Chapter 14.

Lemma 5.10 *Let Assumptions 5.9 be satisfied. Let $r > 0$ and $F : B_{\mathbb{R}^n}(0, r) \to \mathbb{R}^n$ be a k-time continuously differentiable map ($k > 1$) with $F(0) = 0$, $DF(0) = 0$ and*

$$\Pi_h D^j F(0)\,|_{X_c \times X_c \times ... \times X_c} = 0 \textit{ for each } j = 2, ..., k.$$

Then

$$D^j\Psi(0) = 0 \textit{ for each } j = 1, ..., k.$$

Description of the method for computing the normal form up to terms of order 3: Assume first that

$$\Pi_h D^2 F(0)\,|_{X_c \times X_c} \neq 0.$$

Let $G_2 \in V^2(X_c, X_h)$ be an vector-valued homogeneous polynomials of degree 2 (see next subsection for a precise definition). Consider the following global change of variable

$$v := u - G_2(\Pi_c u) \Leftrightarrow \begin{cases} \Pi_c v = \Pi_c u \\ \Pi_h v = \Pi_h u - G_2(\Pi_c u) \end{cases} \Leftrightarrow u = v + G_2(\Pi_c v). \quad (5.24)$$

Then we obtain

$$\begin{aligned}
\frac{dv(t)}{dt} &= \frac{du(t)}{dt} - DG_2(\Pi_c u)(\Pi_c \frac{du(t)}{dt}) \\
&= Au + F(u) - DG_2(\Pi_c u)(\Pi_c [Au + F(u)]) \\
&= A[v + G_2(\Pi_c v)] + F(v + G_2(\Pi_c v)) \\
&\quad - DG_2(\Pi_c v)(\Pi_c [Av + F(v + G_2(\Pi_c v))]).
\end{aligned}$$

Then we obtain the following system after the change of variable

$$\frac{dv(t)}{dt} = Av(t) + H(v(t)) \text{ for } t \geq 0 \text{ and } v(0) = x \in \overline{D(A)}, \tag{5.25}$$

where

$$H(v) = F(v + G_2(\Pi_c v)) - [A, G_2](\Pi_c v) - DG_2(\Pi_c v)(\Pi_c F(v + G_2(\Pi_c v)))$$

and $[\cdot, \cdot]$ is the Lie bracket

$$[A, G_2](v_c) = DG_2(v_c)(A_c v_c) - AG_2(v_c), \ \forall v_c \in X_c.$$

We can rewrite H as

$$\begin{aligned}
H(v) &= F(v) - [A, G_2](\Pi_c v) + [F(v + G_2(\Pi_c v)) - F(v)] \\
&\quad - DG_2(\Pi_c v)(\Pi_c F(v + G_2(\Pi_c v))).
\end{aligned}$$

Since $DF(0) = 0$, we obtain

$$\frac{1}{2!}\Pi_h D^2 H(0)(v_c, v_c) = \frac{1}{2!}\Pi_h D^2 F(0)(v_c, v_c) - [A, G_2](v_c).$$

Therefore, in order to cancel out the second order term we need to solve

$$[A, G_2](v_c) = \frac{1}{2!}\Pi_h D^2 F(0)(v_c, v_c) \text{ with } G_2 \in V^2(X_c, X_h) \tag{5.26}$$

which is the key point to compute the normal form for reduced system and we solve it in next subsection.

If we solved (5.26), then by applying center manifold theorem of chapter 14 and Lemma 5.10 to system (5.25), we deduce that the reduced system of (5.25) has the following form

$$\frac{dv_c}{dt} = A_c v_c + \Pi_c F(v_c + G_2(v_c)) + R(v_c), \tag{5.27}$$

where

$$R(v_c) = \Pi_c F(v_c + G_2(v_c) + \Psi(v_c)) - \Pi_c F(v_c + G_2(v_c))$$

and $\Psi : X_c \to X_h$ is a local center manifold of the new system (5.25) satisfying

$$\Psi(0) = 0, \ D\Psi(0) = 0, \ \text{and } D^2\Psi(0) = 0.$$

Now assuming the F is C^4-smooth locally around 0 (so is Ψ). Then we see that $R(v_c)$ is of order 4 and the Taylor's expansion of the reduced system (5.27) at the order 3 is given by

$$\begin{aligned}\frac{dv_c}{dt} &= A_c v_c + \frac{1}{2!}\Pi_c D^2 F(0)\left(v_c, v_c\right) \\ &+\frac{1}{2!}\left\{\Pi_c D^2 F(0)\left(G_2\left(v_c\right), v_c\right) + \Pi_c D^2 F(0)\left(v_c, G_2\left(v_c\right)\right)\right\} \\ &+\frac{1}{3!}\Pi_c D^3 F(0)\left[v_c, v_c, v_c\right] + h.o.t.\end{aligned}$$

Therefore, in order to compute the Taylor's expansion of the reduced system at the order 3, we only need to compute G_2. Then we can apply the normal form theory of section 2 to the Taylor's expansion of the reduced system.

An alternative approach, to compute both the Taylor's expansion and the normal form of the reduced system would be to use the following change of variables

$$u := v + G_2\left(\Pi_c v\right)$$

wherein

$$G_2 \in V^2(X_c, \mathbb{R}^n).$$

In this case

$$u := v + G_2\left(\Pi_c v\right) \Leftrightarrow \begin{cases} \Pi_c u = \Pi_c v + \Pi_c G_2\left(\Pi_c v\right) \\ \Pi_h u = \Pi_h v + \Pi_h G_2\left(\Pi_c v\right). \end{cases}$$

Then the map $\xi_c\left(v_c\right) = v_c + \Pi_c G_2\left(v_c\right)$ from X_c into itself is only locally invertible around 0. This type of change of variables leads to the normal form theory for reduced system on the center manifold directly.

5.3.1 Normal Form Theory - Nonresonant Type Results

Let $m \geq 1$. Let Y be a closed subspace of $\mathbb{R}^n$. Let $\mathcal{L}_s\left(\left(\mathbb{R}^n\right)^m, Y\right)$ be the space of bounded m-linear symmetric maps from $\left(\mathbb{R}^n\right)^m = \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ into Y and $\mathcal{L}_s\left(X_c^m, \mathbb{R}^n\right)$ be the space of bounded $m-$linear symmetric maps from $X_c^m = X_c \times X_c \times \ldots \times X_c$ into $\mathbb{R}^n$. That is, for each $L \in \mathcal{L}_s\left(X_c^m, \mathbb{R}^n\right)$,

$$L\left(x_1, \ldots, x_m\right) \in \mathbb{R}^n, \quad \forall \left(x_1, \ldots, x_m\right) \in X_c^m,$$

and the maps $\left(x_1, .., x_m\right) \to L\left(x_1, \ldots, x_m\right)$ and $\left(x_1, .., x_m\right) \to AL\left(x_1, \ldots, x_m\right)$ are $m-$linear bounded from X_c^m into $\mathbb{R}^n$. Let $\mathcal{L}_s\left(X_c^m, X_h\right)$ be the space of bounded $m-$linear symmetric maps from $X_c^m = X_c \times X_c \times \ldots \times X_c$ into X_h which belongs to $\mathcal{L}_s\left(X_c^m, \mathbb{R}^n\right)$.

Let $l = \dim(X_c)$ and Y be a subspace of X. We define $V^m(X_c, Y)$ the linear space of vector homogeneous polynomials of degree m. More precisely, let $\Lambda = \left\{b_j\right\}_{j=1,\ldots,l}$ be a basis of X_c and $x = \left(x_1, \ldots, x_l\right)$ be the coordinates with respect to this basis,

$V^m(X_c, Y)$ is the space of finite linear combinations of maps of the form

$$x_c = \sum_{j=1}^{l} x_j b_j \in X_c \to x_1^{n_1} x_2^{n_2} \ldots x_l^{n_l} V$$

where $n_1 + n_2 + \ldots + n_l = m$, $n_j \geq 0$ are integers and $V \in Y$.

Define a map $\mathcal{G} : \mathcal{L}_s\left(X_c^m, Y\right) \to V^m(X_c, Y)$ by

$$\mathcal{G}(L)(x_c) = L(x_c, \cdots, x_c),\ \forall L \in \mathcal{L}_s\left(X_c^m, Y\right).$$

Let $G \in V^m(X_c, Y)$, we have $G(x_c) = \frac{1}{m!} D^m G(0)(x_c, \cdots, x_c)$. So

$$\mathcal{G}^{-1}(G) = \frac{1}{m!} D^m G(0).$$

In other words, we have

$$L = \frac{1}{m!} D^m G(0) \Leftrightarrow G(x_c) = L\left(x_c, \ldots, x_c\right),\ \forall x_c \in X_c.$$

It follows that $\mathcal{G}$ is a bijection from $\mathcal{L}_s\left(X_c^m, Y\right)$ into $V^m(X_c, Y)$. So we can also define $V^m(X_c, \mathbb{R}^n)$ as

$$V^m(X_c, \mathbb{R}^n) := \mathcal{G}(\mathcal{L}_s\left(X_c^m, \mathbb{R}^n\right)).$$

We refer to $x_1^{n_1} x_2^{n_2} \ldots x_l^{n_l} V$ with $n_1 + n_2 + \ldots + n_l = m$ and $V \in Y$ as vector-valued homogeneous polynomials of degree m in l variables $x_1, \ldots x_l$. All the vector-valued homogeneous polynomials of degree m in l variables $x_1, \ldots x_l$ with coefficients in Y form a linear vector space, which we denote by $H_m^l(Y)$. If $\varphi = (\varphi_1, \ldots, \varphi_p)$ is a basis of Y, then an obvious basis denoted by Φ_m for $H_m^l(Y)$ consists of all possible vector-valued homogeneous polynomials of degree m in l variables $x_1, \cdots, x_l$ with coefficients in φ. Let $p = \dim(X_h)$ and $\phi = \left\{s_j\right\}_{j=1,\ldots,p}$ be a basis of X_h. Notice that $l + p = n$. An obvious basis for $H_m^l(X_h)$ is

$$\Phi_m^h = \left\{ x_1^{n_1} x_2^{n_2} \ldots x_l^{n_l} s_j \middle|\, s_j \in \phi,\ j = 1, \cdots, p,\ \sum_{j=1}^{l} n_j = m,\ n_j \geq 0 \text{ are integers} \right\}$$

and for $H_m^l(X_c)$ is

$$\Phi_m^c = \left\{ x_1^{n_1} x_2^{n_2} \ldots x_l^{n_l} b_j \middle|\ b_j \in \Lambda,\ j = 1, \cdots, l,\ \sum_{j=1}^{l} n_j = m,\ n_j \geq 0 \text{ are integers} \right\}.$$

It is obvious that under the basis $\left\{b_j\right\}_{j=1,\ldots,l}$ of X_c, if $G_m \in V^m(X_c, Y)$, then $G_m(x_c) \in H_m^l(Y)$ for $x_c \in X_c$.

In order to use the usual formalism in the context of normal form theory, we now define the Lie bracket.

Definition 5.11 For each $G_m \in V^m(X_c, Y)$, we define the *Lie bracket*

$$[A, G_m](x_c) := DG_m(x_c)\,(Ax_c) - AG_m(x_c),\ \forall x_c \in X_c.$$

Let $A_c \in \mathcal{L}\,(X_c)$ be the part of A in X_c, then we obtain

$$[A, G_m](x_c) = DG_m(x_c)\,(A_c x_c) - AG_m(x_c), \forall x_c \in X_c.$$

Setting $L := \frac{1}{m!} D^m G(0) \in \mathcal{L}_s\left(X_c^m, X_h\right)$, we have

$$DG(x_c)(y) = mL\,(y, x_c, ..., x_c)\,,\quad DG(x_c)A_c x_c = mL\,(A_c x_c, x_c, ..., x_c)\,,$$

and

$$[A, G](x_c) = \frac{d}{dt}\left[L(e^{A_c t}x_c, \ldots, e^{A_c t}x_c)\right](0) - AL(x_c, \ldots, x_c). \tag{5.28}$$

We consider two types of change of variables, namely, $G_m \in V^m(X_c, X_h)$ and $G_m \in V^m(X_c, \mathbb{R}^n)$, respectively in the following.

5.3.1.1 $G_m \in V^m(X_c, X_h)$

We consider the following change of variables

$$u = v + G_m\,(\Pi_c v)\,,\ G_m \in V^m(X_c, X_h). \tag{5.29}$$

Then $G_m(\Pi_c v) \in H^l_m(X_h)$ and

$$G_m(v_c) := L_m\,(v_c, v_c, ..., v_c)\,, \forall\, \Pi_c v := v_c \in X_c.$$

The map $v_c \to AG_m(v_c)$ is differentiable and

$$D\,(AG_m)\,(v_c)(y) = ADG_m(v_c)(y) = mAL_m\,(y, v_c, ..., v_c)\,.$$

Define a map $\xi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\xi\,(v) := v + G_m\,(\Pi_c v)\,, \forall v \in \mathbb{R}^n.$$

Since the range of G_m is included in X_h, we obtain the following equivalence

$$u = \xi\,(v) \Leftrightarrow v = \xi^{-1}\,(u)\,,$$

where

$$\xi^{-1}\,(u) := u - G_m\,(\Pi_c u)\,, \forall u \in \mathbb{R}^n,$$

and

$$\Pi_c \xi^{-1}\,(u) = \Pi_c u, \forall u \in \mathbb{R}^n.$$

The following result justifies the change of variables (5.29) .

Lemma 5.12 *Let Assumptions 5.9 be satisfied and let $G_m \in V^m(X_c, X_h)$. Assume that $u \in C([0,\tau], \mathbb{R}^n)$ is an integrated solution of the Cauchy problem*

$$\frac{du(t)}{dt} = Au(t) + F(u(t)),\ t \in [0,\tau],\ u(0) = x \in \mathbb{R}^n. \tag{5.30}$$

Then $v(t) = \xi^{-1}(u(t))$ is the integrated solution of the Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + H(v(t)),\ t \in [0,\tau],\ v(0) = \xi^{-1}(x) \in \mathbb{R}^n, \tag{5.31}$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$ is the map defined by

$$H(v) = F(\xi(v)) - [A, G_m](\Pi_c v) - DG_m(\Pi_c v)[\Pi_c F(\xi(v))].$$

Conversely, if $v \in C([0,\tau], \mathbb{R}^n)$ is an integrated solution of (5.31), then $u(t) = \xi(v(t))$ is the integrated solution of (5.30).

Proof Assume that $u \in C([0,\tau], \mathbb{R}^n)$ is the integrated solution of the system (5.30), that is,

$$u(t) = x + A\int_0^t u(l)dl + \int_0^t F(u(l))dl, \forall t \in [0,\tau].$$

Set

$$v(t) = \xi^{-1}(u(t)) = u - G_m(\Pi_c u), \forall t \in [0,\tau].$$

We have

$$\begin{aligned} A\int_0^t v(l)dl &= A\int_0^t u(l)dl - \int_0^t AG_m(\Pi_c u(l))\,dl \\ &= u(t) - x - \int_0^t F(u(l))dl - \int_0^t AG_m(\Pi_c u(l))\,dl \\ &= u(t) - G_m(\Pi_c u(t)) - (x - G_m(\Pi_c x)) \\ &\quad + (G_m(\Pi_c u(t)) - G_m(\Pi_c x)) \\ &\quad - \int_0^t F(u(l))dl - \int_0^t AG_m(\Pi_c u(l))\,dl \\ &= v(t) - \xi^{-1}(x) + (G_m(\Pi_c u(t)) - G_m(\Pi_c x)) \\ &\quad - \int_0^t F(u(l))dl - \int_0^t AG_m(\Pi_c u(l))\,dl. \end{aligned}$$

Since $\dim(X_c) < +\infty$, $t \to \Pi_c u(t)$ satisfies the following ordinary differential equations

$$\frac{d\Pi_c u(t)}{dt} = A_c \Pi_c u(t) + \Pi_c F(u(t)).$$

By integrating both sides of the above ordinary differential equations, we obtain

$$G_m(\Pi_c u(t)) - G_m(\Pi_c x) = \int_0^t DG_m(\Pi_c u(l)) \left(\frac{d\Pi_c u(l)}{dl} \right) dl$$
$$= \int_0^t DG_m(\Pi_c u(l)) \left[A_c \Pi_c u(l) + \Pi_c F(u(l)) \right] dl.$$

It follows that

$$A \int_0^t v(l) dl = v(t) - \xi^{-1}(x)$$
$$+ \int_0^t DG_m(\Pi_c u(l)) \left[A_c \Pi_c u(l) + \Pi_c F(u(l)) \right] dl$$
$$- \int_0^t F(u(l)) dl - \int_0^t A G_m(\Pi_c u(l))\, dl.$$

Thus

$$v(t) = \xi^{-1}(x) + A \int_0^t v(l) dl + \int_0^t H(v(l)) dl,$$

in which

$$H(v) = F(\xi(v)) + A G_m(\Pi_c \xi(v))$$
$$- DG_m(\Pi_c \xi(v)) \left[A_c \Pi_c \xi(v) + \Pi_c F(\xi(v)) \right].$$

Since $\Pi_c \xi = \Pi_c$, the first implication follows. The converse follows from the first implication by replacing F by H and ξ^{-1} by ξ. □

Define $\Theta_m : V^m(X_c, X_h) \to V^m(X_c, X_h)$ by

$$\Theta_m(G_m) := [A, G_m], \forall G_m \in V^m(X_c, X_h).$$

We decompose $V^m(X_c, X_h)$ into the direct sum

$$V^m(X_c, X_h) = \mathcal{R}_m \oplus C_m,$$

where

$$\mathcal{R}_m := R(\Theta_m)$$

is the range of Θ_m, and C_m is some complementary space of $\mathcal{R}_m$ into $V^m(X_c, X_h)$.

Set for each $\eta > 0$,

$$BC^\eta(\mathbb{R}, \mathbb{R}^n) := \left\{ f \in C(\mathbb{R}, \mathbb{R}^n) : \sup_{t \in \mathbb{R}} e^{-\eta |t|} \|f(t)\| < +\infty \right\}.$$

By using lemmas **??** and **??** of chapter 14, we now prove the following lemma.

Lemma 5.13 *If*

$$f(t) = t^k e^{\lambda t} x$$

for some $k \in \mathbb{N}$, $\lambda \in i\mathbb{R}$, *and* $x \in \mathbb{R}^n$, *then*

$$(K_u + K_s)(\Pi_h f)(0) = (-1)^k\, k!\,(\lambda I - A_h)^{-(k+1)}\,\Pi_h x \in D(A_h) \subset \mathbb{R}^n,$$

where K_u and K_s are defined in lemmas **??** *and* **??** *of chapter 14.*

Proof We have

$$\begin{aligned}
K_u(f)(0) &= -\int_0^{+\infty} e^{\lambda l} l^k e^{-A_u l}\Pi_u x dl \\
&= -\frac{d^k}{d\lambda^k}\int_0^{+\infty} e^{\lambda l} e^{-A_u l}\Pi_u x dl \\
&= -\frac{d^k}{d\lambda^k}(-\lambda I + A_u)^{-1}\Pi_u x \\
&= \frac{d^k}{d\lambda^k}(\lambda I - A_u)^{-1}\Pi_u x \\
&= (-1)^k k!\,(\lambda I - A_u)^{-(k+1)}\,\Pi_u x.
\end{aligned}$$

Similarly, we have for $\mu > \omega_A$ that

$$\begin{aligned}
(\mu I - A)^{-1} K_s(f)(0) &= \lim_{\tau\to-\infty}(\mu I - A)^{-1}\Pi_s\left(e^{A\cdot} * f(\tau + .)\right)(-\tau) \\
&= \lim_{\tau\to-\infty}\int_0^{-\tau} e^{A_s(-\tau-s)}(\mu I - A)^{-1}\Pi_s f(s+\tau)ds \\
&= \lim_{r\to+\infty}\int_0^{r} e^{A_s(r-s)}(\mu I - A)^{-1}\Pi_s f(s-r)ds \\
&= \int_0^{+\infty} e^{A_s l}(\mu I - A)^{-1}\Pi_s f(-l)dl.
\end{aligned}$$

So we obtain that

$$\begin{aligned}
(\mu I - A)^{-1} K_s(f)(0) &= \int_0^{+\infty}(-l)^k e^{-\lambda l} e^{Al}(\mu I - A)^{-1}\Pi_s x dl \\
&= \frac{d^k}{d\lambda^k}(\lambda I - A)^{-1}(\mu I - A)^{-1}\Pi_s x \\
&= (-1)^k k!\,(\lambda I - A)^{-(k+1)}(\mu I - A)^{-1}\Pi_s x \\
&= (\mu I - A)^{-1}(-1)^k k!\,(\lambda I - A_s)^{-(k+1)}\Pi_s x.
\end{aligned}$$

Since $(\mu I - A)^{-1}$ is one to one, we deduce that

$$K_s(f)(0) = (-1)^k k!\,(\lambda I - A_s)^{-(k+1)}\,\Pi_s x$$

and the result follows. □

The following proposition is related to nonresonant conditions (see Guckenheimer and Holmes [86], Chow and Hale [30], and Chow et al. [32]).

Proposition 5.14 *Let Assumptions 5.9 be satisfied. For each $R \in V^m(X_c, X_h)$, there exists a unique map $G_m \in V^m(X_c, X_h)$ such that*

$$[A, G_m](x_c) = R(x_c), \forall x_c \in X_c. \tag{5.32}$$

Moreover, (5.32) is equivalent to

$$G_m(x_c) = (K_u + K_s)(R(e^{A_c \cdot} x_c))(0),$$

or

$$L_m(x_1, \ldots, x_m) = (K_u + K_s)(H\left(e^{A_c \cdot} x_1, \ldots, e^{A_c \cdot} x_m\right))(0),$$

with $L_m := \frac{1}{m!} D^m G_m(0)$ *and* $H := \frac{1}{m!} D^m R(0)$.

Proof Let $G_m \in V^m(X_c, X_h)$. Then $L_m = \frac{1}{m!} D^m G_m(0) \in \mathcal{L}_s\left(X_c^m, X_h\right)$ and

$$[A, G_m](x_c) = \frac{d}{dt}\left[L_m(e^{A_c t} x_c, \ldots, e^{A_c t} x_c)\right](0) - A_h L_m(x_c, \ldots, x_c).$$

Assume that $G_m \in V^m(X_c, X_h)$ satisfies (5.32). Then $L_m = \frac{1}{m!} D^m G_m(0)$ satisfies

$$\frac{d}{dt}\left[L_m(e^{A_c t} x_1, \ldots, e^{A_c t} x_m)\right](0) = A_h L_m(x_1, \ldots, x_m) + H(x_1, \ldots, x_m),$$

where $H = \frac{1}{m!} D^m R(0) \in \mathcal{L}_s\left(X_c^m, X_h\right)$. Then (5.32) is satisfied if and only if for each $(x_1, \ldots, x_m) \in X_c^m$ and each $t \in \mathbb{R}$,

$$\begin{aligned}\frac{d}{dt}\left[L_m(e^{A_c t} x_1, \ldots, e^{A_c t} x_m)\right](t) = A_h L_m(e^{A_c t} x_1, \ldots, e^{A_c t} x_m) \\ + H(e^{A_c t} x_1, \ldots, e^{A_c t} x_m).\end{aligned} \tag{5.33}$$

Set

$$v(t) := L_m(e^{A_c t} x_1, \ldots, e^{A_c t} x_m), \ \forall t \in \mathbb{R}$$

and

$$w(t) := H(e^{A_c t} x_1, \ldots, e^{A_c t} x_m), \ \forall t \in \mathbb{R}.$$

The system (5.33) can be rewritten as

$$\frac{dv(t)}{dt} = A_h v(t) + w(t), \quad \forall t \in \mathbb{R}. \tag{5.34}$$

Since L_m and H are bounded multilinear maps and $\sigma(A_c) \subset i\mathbb{R}$, it follows that for each $\eta > 0$,

$$v \in BC^\eta(\mathbb{R}, \mathbb{R}^n) \text{ and } w \in BC^\eta(\mathbb{R}, \mathbb{R}^n).$$

Let $\eta \in \left(0, \min\left(\min_{\lambda \in \sigma(A_s)} -\operatorname{Re}(\lambda), \min_{\lambda \in \sigma(A_u)} \operatorname{Re}(\lambda)\right)\right)$. By projecting (5.34) on X_u, we have

$$\frac{d\Pi_u v(t)}{dt} = A_u \Pi_u v(t) + \Pi_u w(t),$$

or equivalently, $\forall t, s \in \mathbb{R}$ with $t \geq s$,

$$\Pi_u v(t) = e^{A_u(t-s)}\Pi_u v(s) + \int_s^t e^{A_u(t-l)}\Pi_u w(l)dl,$$
$$\Pi_u v(s) = e^{-A_u(t-s)}\Pi_u v(t) - \int_s^t e^{-A_u(l-s)}\Pi_u w(l)dl.$$

By using the fact that $v \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$, we obtain when t goes to $+\infty$ that

$$\Pi_u v(s) = K_u(\Pi_u w)(s),\ \forall s \in \mathbb{R}.$$

Thus, for $s = 0$ we have

$$\Pi_u L_m(x_1, ..., x_m) = K_u(\Pi_u H(e^{A_c \cdot}x_1, ..., e^{A_c \cdot}x_m))(0). \tag{5.35}$$

By projecting (5.34) on X_s, we obtain

$$\frac{d\Pi_s v(t)}{dt} = A_s\Pi_s v(t) + \Pi_s w(t),$$

or equivalently, $\forall t, s \in \mathbb{R}$ with $t \geq s$,

$$\Pi_s v(t) = e^{A_s(t-s)}\Pi_s v(s) + \left(e^{A_s \cdot} * \Pi_s w(. + s)\right)(t - s).$$

By using the fact that $v \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$, we have when s goes to $-\infty$ that

$$\Pi_s v(t) = K_s(\Pi_s w)(t),\ \forall t \in \mathbb{R}.$$

Thus, for $t = 0$ it follows that

$$\Pi_s L_m(x_1, ..., x_m) = K_s(\Pi_s H(e^{A_c \cdot}x_1, ..., e^{A_c \cdot}x_m))(0). \tag{5.36}$$

Summing up (5.35) and (5.36), we deduce that

$$L_m(x_1, ..., x_m) = (K_u + K_s)\,(H\left(e^{A_c \cdot}x_1, ..., e^{A_c \cdot}x_m\right))(0). \tag{5.37}$$

Conversely, assume that $L_m(x_1, ..., x_m)$ is defined by (5.37) and set

$$v(t) := (K_u + K_s)\left(H\left(e^{A_c \cdot}x_1, ..., e^{A_c \cdot}x_m\right)\right)(t),\ \forall t \in \mathbb{R}.$$

Then we have

$$v(t) = L_m(e^{A_c t}x_1, ..., e^{A_c t}x_m),\ \forall t \in \mathbb{R}.$$

Moreover, using Lemma **??**-(iii) and Lemma **??**-(iii), we deduce that for each $t, s \in \mathbb{R}$ with $t \geq s$,

$$v(t) = e^{A(t-s)}v(s) + \left(e^{A \cdot} * w(. + s)\right)(t - s),$$

or equivalently,

$$v(t) = v(s) + A\int_s^t v(l)dl + \int_s^t w(l)dl$$

and then

$$\frac{dv(t)}{dt} = Av(t) + w(t), \ \forall t \in \mathbb{R}.$$

The result follows. □

Remark 5.15 (An explicit formula for L_m in Proposition 5.14) Since $l := \dim(X_c) < +\infty$, we can find a basis $\{e_1, ..., e_l\}$ of X_c such that the matrix of A_c (with respect to this basis) is reduced to the Jordan's form. Then for each $x_c \in X_c$, $e^{A_c t}x_c$ is a linear combination of elements of the form

$$t^k e^{\lambda t} x_j$$

for some $k \in \{1, ..., l\}$, some $\lambda \in \sigma(A_c) \subset i\mathbb{R}$, and some $x_j \in \{e_1, ..., e_l\}$. Let $\lambda_1, ..., \lambda_m \in \sigma(A_c) \subset i\mathbb{R}, x_1, ..., x_m \in \{e_1, ..., e_l\}, k_1, ..., k_m \in \{1, ..., l\}$. Define

$$f(t) := H\left(t^{k_1} e^{\lambda_1 t} x_1, ..., t^{k_m} e^{\lambda_m t} x_m\right), \forall t \in \mathbb{R},$$

Since $H \in \mathcal{L}_s\left(X_c^m, X_h\right)$ is m-linear, we obtain

$$f(t) = t^k e^{\lambda t} y$$

with

$$k = k_1 + k_2 + ... + k_m, \quad \lambda = \lambda_1 + ... + \lambda_m,$$

and

$$y = H(x_1, ..., x_m).$$

Now by using Lemma 5.13, we obtain the explicit formula

$$(K_u + K_s)\left(H\left((.)^{k_1} e^{\lambda_1 \cdot} x_1, ..., (.)^{k_m} e^{\lambda_m \cdot} x_m\right)\right)(0) = (-1)^k k! (\lambda I - A_h)^{-(k+1)} \Pi_h y \in \mathbb{R}^n.$$

Remark 5.16 Let $\sigma(A_c) = \{\lambda_1, ..., \lambda_l\}, \sigma(A_h) = \{\mu_1, ..., \mu_p\}, d = (d_1, \cdots, d_l)$, $d_i \geq 0$ are integers. From the proof of proposition 5.14, we get that the spectra of Θ_m is

$$\sigma(\Theta_m) = \left\{\lambda \cdot d - \mu_j : j = 1, \cdots, p, \ \sum_{j=1}^{l} d_j = m\right\},$$

where $\lambda = (\lambda_1, \cdots, \lambda_l)$, $\lambda \cdot d = \lambda_1 d_1 + \cdots + \lambda_l d_l$. The result in Proposition 5.14 is equivalent to say that equation (5.22) satisfies the following nonresonance conditions relative to $\sigma_c(A) \subset \sigma(A)$ of order m:

$$\lambda \cdot d - \mu_j \neq 0, \text{ for all } \mu_j \in \sigma_h(A) \subset \sigma(A), \text{ all } d = (d_1, \cdots, d_l) \text{ with } \sum_{j=1}^{l} d_j = m.$$

Thus the range of Θ_m must be the whole space $V^m(X_c, X_h)$.

5.3.1.2 $G_m \in V^m(X_c, \mathbb{R}^n)$

We consider the following change of variables

$$u(t) = w(t) + G_m(\Pi_c w(t)),\ G_m \in V^m(X_c, \mathbb{R}^n). \tag{5.38}$$

Then $G_m(\Pi_c v) \in H^l_m(\mathbb{R}^n)$ and the map $I + G_m \circ \Pi_c : \mathbb{R}^n \to \mathbb{R}^n$ is locally invertible around 0. From (5.28), for each $H \in V^m(X_c, \mathbb{R}^n)$, to find $G_m \in V^m(X_c, \mathbb{R}^n)$ satisfying

$$[A, G_m](x_c) = H(x_c), \tag{5.39}$$

is equivalent to find $L_m \in \mathcal{L}_s\left(X_c^m, \mathbb{R}^n\right)$ satisfying

$$\frac{d}{dt}\left[L_m(e^{A_c t}x_1, \ldots, e^{A_c t}x_m)\right]_{t=0} = AL_m(x_1, \ldots, x_m) + \widehat{H}(x_1, \ldots, x_m) \tag{5.40}$$

for each $(x_1, \ldots, x_m) \in X_c^m$ with

$$\mathcal{G}(\widehat{H}) = H.$$

By projecting on X_c and X_h, it follows that solving system (5.39) is equivalent to find $G_m^c \in V^m(X_c, X_c)$ and $G_m^h \in V^m(X_c, X_h)$ satisfying

$$[A_c, G_m^c] = \Pi_c H \tag{5.41}$$

and

$$[A, G_m^h] = \Pi_h H. \tag{5.42}$$

Define $\Theta_m^c : V^m(X_c, X_c) \to V^m(X_c, X_c)$ by

$$\Theta_m^c\left(G_m^c\right) := [A_c, G_m^c], \forall G_m^c \in V^m(X_c, X_c) \tag{5.43}$$

and $\Theta_m^h : V^m(X_c, X_h) \to V^m(X_c, X_h)$ by

$$\Theta_m^h\left(G_m^h\right) := [A, G_m^h], \forall G_m^h \in V^m(X_c, X_h).$$

We decompose $V^m(X_c, X_c)$ into the direct sum

$$V^m(X_c, X_c) = \mathcal{R}_m^c \oplus \mathcal{C}_m^c\ ,$$

where

$$\mathcal{R}_m^c := R(\Theta_m^c)$$

is the range of Θ_m^c, and $\mathcal{C}_m^c$ is some complementary space of $\mathcal{R}_m^c$ into $V^m(X_c, X_c)$.

The range of the linear operator Θ_m^c can be characterized by using the so called non-resonance theorem.

Proposition 5.17 *Let Assumptions 5.9 be satisfied. Let $H \in \mathcal{R}_m^c \oplus V^m(X_c, X_h)$. Then there exists $G_m \in V^m(X_c, \mathbb{R}^n)$ (non-unique in general) satisfying*

$$[A, G_m](x_c) = H(x_c). \tag{5.44}$$

Furthermore, if $N(\Theta_m^c) = \{0\}$ (the null space of Θ_m^c), then G_m is uniquely determined.

Proof It is clear that we can solve (5.41). Moreover, we can apply Proposition 5.14 and deduce that (5.42) can be solved. □

Remark 5.18 In practice, we often have

$$N(\Theta_m^c) \cap R(\Theta_m^c) = \{0\},$$

In this case, a natural splitting of $V^m(X_c, X_c)$ will be

$$V^m(X_c, X_c) = R(\Theta_m^c) \oplus N(\Theta_m^c).$$

Define $P_m : V^m(X_c, \mathbb{R}^n) \to V^m(X_c, \mathbb{R}^n)$ the bounded linear projector satisfying

$$\mathcal{P}_m\left(V^m(X_c, \mathbb{R}^n)\right) = \mathcal{R}_m^c \oplus V^m(X_c, X_h), \text{ and } (I - \mathcal{P}_m)\left(V^m(X_c, \mathbb{R}^n)\right) = C_m^c.$$

Again consider the system (5.30). Assume that $DF(0) = 0$. Without loss of generality we also assume that for some $m \in \{2, \ldots, k\}$,

$$\Pi_h D^j F(0) \mid_{X_c \times X_c \times \ldots \times X_c} = 0,\ \mathcal{G}\left(\Pi_c D^j F(0) \mid_{X_c \times X_c \times \ldots \times X_c}\right) \in C_j^c, \qquad ((C_{m-1}))$$

for each $j = 1, \ldots, m-1$.

We will show that we can find $G_m \in V^m(X_c, \mathbb{R}^n)$ such that after the change of variables (5.38) we can rewrite the system (5.30) as

$$\frac{dw(t)}{dt} = Aw(t) + H(w(t)), \text{ for } t \geq 0, \text{ and } w(0) = (I + G_m \circ \Pi_c)^{-1} x \in \mathbb{R}^n, \quad (5.45)$$

where H satisfies the condition (C_m). This will provide a normal form method.

Lemma 5.19 *Let Assumptions 5.9 be satisfied. Let $G_m \in V^m(X_c, \mathbb{R}^n)$. Assume that $u \in C([0, \tau], \mathbb{R}^n)$ is an integrated solution of the Cauchy problem (5.30). Then $w(t) = (I + G_m \circ \Pi_c)^{-1}(u(t))$ is an integrated solution of the system (5.45), where $H : \overline{D(A)} \to \mathbb{R}^n$ is the map defined by*

$$H(w(t)) = F(w(t)) - [A, G_m](\Pi_c w(t)) + O(\|w(t)\|^{m+1}).$$

Conversely, if $w \in C([0, \tau], \mathbb{R}^n)$ is an integrated solution of (5.45), then $u(t) = (I + G_m \circ \Pi_c) w(t)$ is an integrated solution of (5.30).

Lemma 5.19 can be proved similarly as Lemma 5.12, here we omit it.

Proposition 5.20 *Let Assumptions 5.9 be satisfied. Let $r > 0$ and let $F : B_{\mathbb{R}^n}(0, r) \to \mathbb{R}^n$ be a map. Assume that there exists an integer $k \geq 1$ such that F is k-time continuously differentiable in $B_{\mathbb{R}^n}(0, r)$ with $F(0) = 0$ and $DF(0) = 0$. Let $m \in \{2, \ldots, k\}$ be such that F satisfies the condition (C_{m-1}). Then there exists a map $G_m \in V^m(X_c, \mathbb{R}^n)$ such that after the change of variables*

$$u(t) = w(t) + G_m(\Pi_c w(t)),$$

we can rewrite system (5.30) as (5.45) and H satisfies the condition (C_m), *where*

$$H(w(t)) = F(w(t)) - [A, G_m](\Pi_c w(t)) + O(\|w(t)\|^{m+1}).$$

Proof Let $x_c \in X_c$. We have

$$H(x_c) = F(x_c) - [A, G_m](\Pi_c x_c) + O(\|x_c\|^{m+1}).$$

It follows that

$$\begin{aligned} H(x_c) = &\frac{1}{2!}D^2F(0)(x_c, x_c) + ... + \frac{1}{(m-1)!}D^{m-1}F(0)(x_c, \ldots, x_c) \\ &+\mathcal{P}_m\left[\frac{1}{m!}D^mF(0)(x_c, \ldots, x_c)\right] + (I - \mathcal{P}_m)\left[\frac{1}{m!}D^mF(0)(x_c, \ldots, x_c)\right] \\ &- [A, G_m](x_c) + O(\|x_c\|^{m+1}) \end{aligned}$$

since $DF(0) = 0$. Moreover, by using Proposition 5.17 we obtain that there exists a map $G_m \in V^m(X_c, \mathbb{R}^n)$ such that

$$[A, G_m](x_c) = \mathcal{P}_m\left[\frac{1}{m!}D^mF(0)(x_c, \ldots, x_c)\right].$$

Hence,

$$\begin{aligned} H(x_c) = &\frac{1}{2!}D^2F(0)(x_c, x_c) + ... + \frac{1}{(m-1)!}D^{m-1}F(0)(x_c, \ldots, x_c) \\ &+(I - \mathcal{P}_m)\left[\tfrac{1}{m!}D^mF(0)(x_c, \ldots, x_c)\right] + O(\|x_c\|^{m+1}). \end{aligned} \tag{5.46}$$

By the assumption, we have for all $j = 1, ..., m-1$ that

$$\Pi_h D^j H(0) \mid_{X_c \times X_c \times \ldots \times X_c} = \Pi_h D^j F(0) \mid_{X_c \times X_c \times \ldots \times X_c} = 0$$

and

$$\mathcal{G}\left(\Pi_c D^j H(0) \mid_{X_c \times X_c \times \ldots \times X_c}\right) = \mathcal{G}\left(\Pi_c D^j F(0) \mid_{X_c \times X_c \times \ldots \times X_c}\right) \in C_j^c.$$

Now by using (**??**), we have

$$\frac{1}{m!}\Pi_h D^m H(0) \mid_{X_c \times X_c \times \ldots \times X_c} = \Pi_h \mathcal{G}^{-1}\left[(I - \mathcal{P}_m)\left(\frac{1}{m!}D^mF(0)(x_c, \ldots, x_c)\right)\right] = 0$$

and

$$\mathcal{G}\left(\Pi_c D^m H(0) \mid_{X_c \times X_c \times \ldots \times X_c}\right) = \mathcal{G}\left\{\Pi_c \mathcal{G}^{-1}\left[(I - \mathcal{P}_m)(D^mF(0)(x_c, \ldots, x_c))\right]\right\} \in C_m^c.$$

The result follows. □

5.3.2 Normal Form Computation

In this section we provide the methods to compute the Taylor's expansion and normal form of the reduced system at any order of a system topologically equivalent to the original system:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F(u(t)), \quad t \geq 0, \\ u(0) = x \in \mathbb{R}^n. \end{cases} \tag{5.47}$$

Assumption 5.21 Assume that $F \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ for some integer $k \geq 2$ with

$$F(0) = 0 \text{ and } DF(0) = 0.$$

Set

$$F_1 := F.$$

Once again we consider two cases, namely, $G_m \in V^m(X_c, X_h)$ and $G_m \in V^m(X_c, \mathbb{R}^n)$, respectively.

5.3.2.1 $G_m \in V^m(X_c, X_h)$

For $j = 2, ..., k$, we apply Proposition 5.14. Then there exists a unique function $G_j \in V^j(X_c, X_h)$ satisfying

$$\left[A, G_j\right](x_c) = \frac{1}{j!}\Pi_h D^j F_{j-1}(0)(x_c, ..., x_c), \forall x_c \in X_c. \tag{5.48}$$

Define $\xi_j : \mathbb{R}^n \to \mathbb{R}^n$ and $\xi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\xi_j(x) := x + G_j(\Pi_c x) \text{ and } \xi_j^{-1}(x) := x - G_j(\Pi_c x), \forall x \in \mathbb{R}^n.$$

Then

$$F_j(x) := F_{j-1}\left(\xi_j(x)\right) - \left[A, G_j\right](\Pi_c x) - DG_j(\Pi_c x)\left[\Pi_c F_{j-1}\left(\xi_j(x)\right)\right].$$

Moreover, we have for $x \in \mathbb{R}^n$ that

$$\Pi_c F_j(x) = \Pi_c F_{j-1}\left(\xi_j(x)\right) = \Pi_c F_{j-1}\left(x + G_j(\Pi_c x)\right).$$

Since the range of G_j is included in X_h, by induction we have

$$\Pi_c F_j(x) = \Pi_c F\left(x + G_2(\Pi_c x) + G_3(\Pi_c x) + ... + G_j(\Pi_c x)\right).$$

Now, we obtain

$$\Pi_h D^j F_k(0)\,|_{X_c \times X_c \times ... \times X_c} = 0 \text{ for all } j = 1, ..., k.$$

Setting

$$u_k(t) = \xi_k^{-1} \circ \xi_{k-1}^{-1} \circ ... \xi_2^{-1}(u(t)) = u(t) - G_2(\Pi_c u(t)) - G_3(\Pi_c u(t)) - ... - G_k(\Pi_c u(t)),$$

we deduce that $u_k(t)$ is an integrated solution of the system

$$\begin{cases} \dfrac{du_k(t)}{dt} = Au_k(t) + F_k(u_k(t)), \quad t \geq 0, \\ u_k(0) = x_k \in \overline{D(A)}. \end{cases} \tag{5.49}$$

Applying the center manifold theorem in Chapter 14 and Lemma 5.10 to system (5.49), we obtain the following result which is one of the main results of this Chapter.

Theorem 5.22 *Let Assumptions 5.9, 5.21 be satisfied. Then by using the change of variables*

$$\begin{cases} u_k(t) = u(t) - G_2(\Pi_c u(t)) - G_3(\Pi_c u(t)) - ... - G_k(\Pi_c u(t)) \\ \Leftrightarrow \\ u(t) = u_k(t) + G_2(\Pi_c u_k(t)) + G_3(\Pi_c u_k(t)) + ... + G_k(\Pi_c u_k(t)), \end{cases}$$

the map $t \to u(t)$ is an integrated solution of the Cauchy problem (5.47) if and only if $t \to u_k(t)$ is an integrated solution of the Cauchy problem (5.49). Moreover, the reduced system of Cauchy problem (5.49) is given by the ordinary differential equations on X_c :

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F\left[\begin{array}{c} x_c(t) + G_2(x_c(t)) + \\ G_3(x_c(t)) + ... + G_k(x_c(t)) \end{array}\right] + R_c(x_c(t)), \tag{5.50}$$

where the remainder term $R_c \in C^k(X_c, X_c)$ satisfies

$$D^j R_c(0) = 0 \text{ for each } j = 1, ..., k,$$

or in other words $R_c(x_c(t))$ is a remainder term of order k. If we assume in addition that $F \in C^{k+2}(\mathbb{R}^n, X)$, then the map $R_c \in C^{k+2}(X_c, X_c)$ and $R_c(x_c(t))$ is a remainder term of order $k+2$, that is

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c), \tag{5.51}$$

where $O(x_c)$ is a function of x_c which remains bounded when x_c goes to 0, or equivalently,

$$D^j R_c(0) = 0 \text{ for each } j = 1, ..., k+1.$$

Proof By applying the center manifold theorem in Chapter 14 and Lemma 5.10 to system (5.49), there exists $\Psi_k \in C^k(X_c, X_h)$ such that the reduced system of (5.49) is given by

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F[x_c(t) + G_2(x_c(t)) + G_3(x_c(t)) + ... + G_k(x_c(t)) + \Psi_k(x_c(t))]$$

and

$$D^j \Psi_k(0) = 0 \text{ for } j = 1, ..., k.$$

By setting

$$R_c(x_c) = \Pi_c F[x_c + G_2(x_c) + G_3(x_c) + ... + G_k(x_c) + \Psi_k(x_c)]$$
$$-\Pi_c F[x_c + G_2(x_c) + G_3(x_c) + ... + G_k(x_c)],$$

we obtain the first part of the theorem. If we assume in addition that $F \in C^{k+2}(\mathbb{R}^n, \mathbb{R}^n)$, then $\Psi_k \in C^{k+2}(X_c, X_h)$. Thus,

$$R_c \in C^{k+2}(X_c, X_c).$$

Set

$$h(x_c) := x_c + G_2(x_c) + G_3(x_c) + ... + G_k(x_c).$$

We have

$$\begin{aligned} R_c(x_c) &= \Pi_c \{F[h(x_c) + \Psi_k(x_c)] - F[h(x_c)]\} \\ &= \Pi_c \int_0^1 DF(h(x_c) + s\Psi_k(x_c))(\Psi_k(x_c))\, ds. \end{aligned}$$

Define

$$\widehat{h}(x_c) := h(x_c) + s\Psi_k(x_c).$$

Since $DF(0) = 0$, we have

$$\begin{aligned} DF\left(\widehat{h}(x_c)\right)(\Psi_k(x_c)) &= DF(0)(\Psi_k(x_c)) + \int_0^1 D^2F\left(l\widehat{h}(x_c)\right)\left(\widehat{h}(x_c), \Psi_k(x_c)\right) dl \\ &= \int_0^1 D^2F\left(l\widehat{h}(x_c)\right)\left(\widehat{h}(x_c), \Psi_k(x_c)\right) dl. \end{aligned}$$

Hence,

$$R_c(x_c) = \Pi_c \int_0^1 \int_0^1 D^2F(l(h(x_c) + s\Psi_k(x_c)))(h(x_c) + s\Psi_k(x_c), \Psi_k(x_c))\, dlds \tag{5.52}$$

and $h(x_c)$ is a term of order 1, $\Psi_k(x_c)$ is a term of order $k+1$, it follows that (5.52) holds. This completes the proof. □

Remark 5.23 In order to apply the above approach, we first need to compute Π_c and A_c, then $\Pi_h := I - \Pi_c$ can be derived. The point to apply the above procedure is to solve system (5.48). To do this, one may compute

$$(\lambda I - A_h)^{-k} \frac{1}{j!} \Pi_h D^j F(0) \tag{5.53}$$

for each $\lambda \in i\mathbb{R}$ and each $k \geq 1$ by using Remark 5.15, or one may directly solve system (5.48) by computing $\Pi_h \frac{1}{j!} D^j F_{j-1}$. This last approach will involve the computation of (5.53) for some specific values of $\lambda \in i\mathbb{R}$ and some specific values of $k \geq 1$. This turns out to be the main difficulty in applying the above method.

In the application of chapter 17, we will use the last part of Theorem 5.22 to avoid some unnecessary computations. We will apply this theorem for $k = 2$, F in C^4, and the remainder term $R_c(x_c)$ of order 4. This means that if we want to compute the Taylor's expansion of the reduced system to the order 3 (which is very common in such a context), we only need to compute G_2. So in application the last part of Theorem 5.22 will help to avoid a lot of computations.

5.3.2.2 $G_m \in V^m(X_c, \mathbb{R}^n)$

Now we apply Proposition 5.20 recursively to (5.47). Set

$$u_1 := u.$$

For $m = 2, ..., k$, let $G_m \in V^m(X_c, \mathbb{R}^n)$ be defined such that

$$[A, G_m](x_c) = \mathcal{P}_m \left[\frac{1}{m!} D^m F_{m-1}(0)(x_c, \ldots, x_c)\right] \text{ for each } x_c \in X_c.$$

We use the change of variables

$$u_{m-1} = u_m + G_m(\Pi_c u_m).$$

Then we consider F_m given by Proposition 5.20 and satisfying

$$F_m(u_m) = F_{m-1}(u_m) - [A, G_m](\Pi_c u_m) + O(\|u_m\|^{m+1}).$$

By applying Proposition 5.20, we have

$$\Pi_h D^j F_m(0) \mid_{X_c \times X_c \times \ldots \times X_c} = 0, \text{ for all } j = 1, ..., m,$$

and

$$\mathcal{G}\left(\Pi_c D^j F_m(0) \mid_{X_c \times X_c \times \ldots \times X_c}\right) \in C_j^c, \text{ for all } j = 1, ..., m.$$

Thus by using the change of variables locally around 0

$$u_k(t) = (I + G_k \Pi_c)^{-1} \ldots (I + G_3 \Pi_c)^{-1} (I + G_2 \Pi_c)^{-1} u(t),$$

we deduce that $u_k(t)$ is an integrated solution of system (5.49). By using the center manifold theorem in Chapter 14 and Lemma Lemma 5.10 to (5.49), we obtain the following result which is the main result of this Chapter and indicates that systems (5.49) and (5.47) are locally topologically equivalent around 0.

Theorem 5.24 *Let Assumptions 5.9 and 5.21 be satisfied. Then by using the change of variables locally around* 0

$$\begin{cases} u_k(t) = (I + G_k \Pi_c)^{-1} \ldots (I + G_3 \Pi_c)^{-1} (I + G_2 \Pi_c)^{-1} u(t) \\ \Leftrightarrow \\ u(t) = (I + G_2 \Pi_c)(I + G_3 \Pi_c) \ldots (I + G_k \Pi_c) u_k(t), \end{cases}$$

the map $t \to u(t)$ is an integrated solution of the Cauchy problem (5.47) if and only if $t \to u_k(t)$ is an integrated solution of the Cauchy problem (5.49). Moreover, the reduced equation of Cauchy problem (5.49) is given by the ordinary differential *equations on* X_c :

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \sum_{m=2}^{k} \frac{1}{m!} \Pi_c D^m F_k(0)(x_c(t), ..., x_c(t)) + R_c(x_c(t)),$$

where

$$\mathcal{G}\left(\frac{1}{m!}\Pi_c D^m F_k(0)\mid_{X_c \times X_c \times ... \times X_c}\right) \in C_m^c, \text{ for all } m = 1, ..., k,$$

and the remainder term $R_c \in C^k(X_c, X_c)$ *satisfies*

$$D^j R_c(0) = 0 \text{ for each } j = 1, ..., k,$$

or in other words $R_c(x_c(t))$ *is a remainder term of order* k. *If we assume in addition that* $F \in C^{k+2}(\mathbb{R}^n, \mathbb{R}^n)$. *Then the reduced equation of Cauchy problem (5.49) is given by the ordinary differential equations on* X_c :

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \sum_{m=2}^{k+1} \frac{1}{m!} \Pi_c D^m F_k(0)(x_c(t), ..., x_c(t)) + R_c(x_c(t)),$$

the map $R_c \in C^{k+2}(X_c, X_c)$, *and* $R_c(x_c(t))$ *is a remainder term of order* $k+2$, *that is*

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c),$$

where $O(x_c)$ *is a function of* x_c *which remains bounded when* x_c *goes to* 0, *or equivalently,*

$$D^j R_c(0) = 0 \text{ for each } j = 1, ..., k+1.$$

Proof By the center manifold theorem in Chapter 14 and 5.10 to (5.49), there exists $\Psi_k \in C^k(X_c, X_h)$ such that the reduced system of (5.49) is given by

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F_k[x_c(t) + \Psi_k(x_c(t))]$$

and

$$D^j \Psi_k(0) = 0 \text{ for } j = 1, ..., k.$$

By setting

$$R_c(x_c) = \Pi_c F_k[x_c + \Psi_k(x_c)] - \Pi_c F_k(x_c),$$

we obtain the first part of the Theorem. If we assume in addition that $F \in C^{k+2}(\mathbb{R}^n, \mathbb{R}^n)$, then $\Psi_k \in C^{k+2}(X_c, X_h)$. Thus, $R_c \in C^{k+2}(X_c, X_c)$ and

$$\begin{aligned} R_c(x_c) &= \Pi_c \{F_k[x_c + \Psi_k(x_c)] - F_k(x_c)\} \\ &= \Pi_c \int_0^1 DF_k(x_c + s\Psi_k(x_c))(\Psi_k(x_c))\, ds. \end{aligned}$$

Set

$$h(x_c) := x_c + s\Psi_k(x_c).$$

Since $DF(0) = 0$, we have

$$DF_k(h(x_c))(\Psi_k(x_c)) = DF_k(0)(\Psi_k(x_c)) + \int_0^1 D^2F_k(lh(x_c))(h(x_c), \Psi_k(x_c))\, dl$$
$$= \int_0^1 D^2F_k(lh(x_c))(h(x_c), \Psi_k(x_c))\, dl.$$

Hence,

$$R_c(x_c) = \Pi_c \int_0^1 \int_0^1 D^2F_k(l(x_c + s\Psi_k(x_c)))(x_c + s\Psi_k(x_c), \Psi_k(x_c))\, dlds$$

and $\Psi_k(x_c)$ is a term of order $k+1$, it follows that

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c).$$

The result follows. □

5.4 Remarks and notes

A normal form theorem was obtained first by Poincaré [176] and later by Siegel [197] for analytic differential equations. Simpler proofs of Poincaré's theorem and Siegel's theorem were given in Arnold [7], Meyer [165], Moser [168], and Zehnder [231]. Normal form theory has been extended to various classes of differential equations. In the context of functional differential equations we refer to Faria [72, 73]. In the context of autonomous partial differential equations we refer to Ashwin and Mei [9] (PDEs on the square), Eckmann et al. [65] (abstract parabolic equations), Faou et al. [68, 69] (Hamiltonian PDEs), Hassard, Kazarinoff and Wan [99] (Functional Differential Equations), Faria [70, 71] (PDEs with delay), Foias et al. [78] (Navier-Stokes equation), Kokubu [125] (reaction-diffusion equations), McKean and Shatah [162] (Schrödinger equation and heat equations), Nikolenko [173] (abstract semi-linear equations), Shatah [192] (Klein-Gordon equation), Zehnder [232] (abstract parabolic equations), Chow et al. [41] (and references therein) for a normal form theory in quasiperiodic partial differential equations. Liu et al. [147] present a normal form theory for an abstract non-densely defined Cauchy problem on Banach space recently.

For the computation of normal forms near a equilibrium solution presented in the first part of this chapter, we refer to the books [32], [216], [19] [30], [225], [86] by Arnold [7], Chow and Hale [30], Guckenheimer and Holmes [86], Meyer and Hall [164], Siegel and Moser [198], Chow et al. [32], Kuznetsov [129], Bibikov [19], Wiggins [225] and others for more details.

In the study of nonlinear dynamical systems, the center manifold theory is very important in reducing the dimension of equations, while the normal form theory is very useful in simplifying the forms of equations restricted on the center manifolds. Both of them are useful for the study of bifurcation problems. The second part of this chapter is devoted to show methods to compute the normal form associated with the flow on center manifold which is inspired by Liu, Magal and Ruan [147]. In this part we present two approaches to compute the normal form associated with the flow on center manifold. One method is to compute the Taylor's expansion of the reduced system associated with the flow on center manifold by performing inductively a sequence of change of coordinates on the nonlinear hyperbolic part of the system and then we can compute the normal form of the reduced system using the procedure in section 2 of this chapter. We can also use a sequence of change of coordinates to get the Taylor's expansion of the reduced system associated with the flow on center manifold and, simultaneously, to eliminate the non-resonant terms of the reduced system.

Part II
Applications to Predator Prey Systems

Chapter 6
A Holling's predator-prey model with handling and searching predators

This chapter provide an example of monotone ordinary differential equation in the context of predator prey system. This chapter is based on the paper of Hsu, Liu and Magal [118].

6.1 Introduction

The article is devoted to the following predator prey system with handling and searching predators

$$\begin{cases} N' &= \underbrace{\beta_N N - \mu_N N - \delta N^2}_{\text{Logistic growth}} \quad \underbrace{-N\,\kappa\, P_S}_{\text{Consumption of prey by predators}} \\ P_S' &= \underbrace{-(\mu_P + \eta) P_S}_{\text{Mortality}} \quad \underbrace{-P_S\,\rho\,\kappa\, N}_{\text{Searching becoming handling}} \quad + \gamma P_H \\ P_H' &= \underbrace{-\mu_P P_H}_{\text{Mortality}} \quad + P_S\,\rho\,\kappa\, N \quad \underbrace{-\gamma P_H}_{\text{Handling becoming searching}} \quad + \underbrace{\beta_P\,(P_S + P_H)}_{\text{New born predator}} \end{cases} \tag{6.1}$$

where $N(t)$ is the number of prey at time t, $P_S(t)$ is the number of predators searching for prey at time t, and $P_H(t)$ is the number of predators handling the prey at time t.

Here the terminology "handling and searching predators" refers to Holling himself [109]. We should mention that Metz and Diekmann had similar ideas of searching and handling predator in their edited book [163, pages 6-7]. In the model (6.1), the term $\beta_P\,(P_S(t) + P_H(t))$ is the flux of new born predators. Here we assume that all the new born predators are handlers. The parameter ρ should be interpreted as a conversion rate. The term $P_S(t)\,\rho\,\kappa\, N(t)$ (in the P_S-equation or the P_H-equation) is a flux of searching predators becoming handling predators. The term $\gamma P_H(t)$ (in the P_S-equation or the P_H-equation) is the flux of handling predators becoming searching predators. The term μ_P is the natural mortality of the predators and η is an extra mortality term for the searching predators only. The term $N(t)\,\kappa\, P_S(t)$ in the

N-equation corresponds to the consumption of the prey by the predators. The part $\beta_N N(t) - \mu_N N(t) - \delta N(t)^2$ in the N-equation is the standard logistic equation.

The main idea about this model is to distinguish the vital dynamics (birth and death process) of the predators and their survival due to the consumption of prey. In the model the survival of predators will depend on the status searching or handling. The handling predators are satisfied with their consumption of prey and they don't need to find more prey to survive. At the opposite the searching predators are unsatisfied with their consumption of prey and they need to find some prey to survive. Once a searching predator finds a prey (or enough prey) he becomes a handling and after some time the handling predator becomes a searching predator again.

This process only influences the survival of predators which depends on their ability to find a prey. In our model, a predator can reproduce at time t because he found enough prey to survive from its birth until the time t. In section 6.2 we will first make some basic assumptions in order for the predators to extinct in absence of prey. Then based on these setting we will analyze the dynamical properties of the system (6.1). The main advantage with the model (6.1) is that we can separate the vital dynamic and consumption of prey to describe the behavior of the predators. This will be especially very convenient if we want to add an age or size structure to the predator population. This kind of question is left for future work.

In section 6.6 we will see that our model is also comparable to the standard predator prey model whenever $\rho = \dfrac{\chi}{\varepsilon}$ and $\gamma = \dfrac{1}{\varepsilon}$ for $\varepsilon > 0$ small which means that predators are going back and forth from handling to searching very rapidly. In that case (as a singular limit) we obtain a convergence result to the standard Rosenweig-MacArthur model [183]

$$\begin{cases} N' = rN\left(1 - \dfrac{N}{K}\right) - P\,\dfrac{mN}{a+N}, \\ P' = P\left(\dfrac{mN}{a+N} - d\right), \end{cases} \tag{6.2}$$

which is the most popular predator-prey system discussed in the literature.

Let us recall that the derivation of Holling type II functional response $\frac{mN}{a+N}$ can be found in Holling [108, 109] and Hsu, Hubbell and Waltman [115]. There are two mathematical problems for the system (6.2), namely, the global asymptotic stability of the locally asymptotically stable interior equilibrium (when it exists) and the uniqueness of the limit cycles when the interior equilibrium is unstable. For the global asymptotic stability of this equilibrium we may apply the Dulac's criterion Hsu, Hubbell and Waltman [117], weak negative Bendixson Lemma Cheng, Hsu and Lin [26] or construction Lyapunov function Ardito and Ricciardi [5]. For uniqueness of limit cycle of Rosenzweig-MacArthur model (6.2), Cheng [25] employed the symmetry of the prey isocline to prove the exponential asymptotic stability of each limit cycle. Kuang and Freedman [128] reduced (6.2) to a generalized Lienard equation which has the uniqueness of limit cycle Zhang [239]. We refer to Murray [169], Hastings [100], Turchin [212] for more results about predator prey models.

The plan of the paper is the following. In section 6.2 we set some basic assumptions in order for the predators to extinct in absence of prey. In section 6.3 we prove that the system is dissipative. In section 6.4 we study the uniform persistence and extinction properties of the predators. We study the system in the interior region which corresponds the region of co-existence of prey and predators in section 6.5. We should mention that we can obtain a rather complete description of the asymptotic behavior thanks to the fact the system is competitive (for a new partial order). In section 6.6 we prove the convergence of our model to the Rosenweig-MacArthur model. In section 6.7 we apply the model to the Canadian snowshoe Hares and the Lynx.

6.2 Basic assumptions

In this section, we set some basic assumptions in order for the predators to extinct in absence of prey. Consider the total number of predators

$$P = P_H + P_S.$$

Then

$$P' = (\beta_P - \mu_P - \eta)\, P_S + (\beta_P - \mu_P)\, P_H.$$

The following assumptions mean that when $\frac{P_S}{P_H} > -\frac{\beta_P - \mu_P}{\beta_P - \mu_P - \eta}$, the total population of predators decreases. The total population of predators increases otherwise.

Assumption 6.1 We assume that all the parameters of the model (6.1) are strictly positive and

$$\beta_N - \mu_N > 0,\ \beta_P - \mu_P > 0 \text{ and } \beta_P - \mu_P - \eta < 0.$$

In absence of prey the dynamics of predator population is described by

$$\begin{cases} P_S' = -(\mu_P + \eta) P_S + \gamma P_H \\ P_H' = \beta_P P_S + (\beta_P - \mu_P - \gamma)\, P_H. \end{cases}$$

Define

$$M = \begin{bmatrix} -\mu_P - \eta & \gamma \\ \beta_P & \beta_P - \mu_P - \gamma \end{bmatrix}. \tag{6.3}$$

By using Assumption 6.1 we have

$$\mathrm{tr}\,(M) = (\beta_P - \mu_P - \gamma) - (\mu_P + \eta) < 0.$$

Therefore in absence of prey the population of predators goes to extinct if and only if

$$\det\,(M) = -\,(\mu_P + \eta)\,(\beta_P - \mu_P - \gamma) - \beta_P \gamma > 0.$$

This last inequality can be equivalently reformulated in the following assumption.

Assumption 6.2 (Extinction of the predators) We assume that

$$(\beta_P - \mu_P - \gamma) < -\frac{\beta_P \gamma}{\mu_P + \eta} \Leftrightarrow (\beta_P - \mu_P) < -\frac{\gamma}{\mu_P + \eta} (\beta_P - \mu_P - \eta) . \tag{6.4}$$

Remark 6.3 The first inequality in (6.4) implies that $(\beta_P - \mu_P - \gamma) < 0$. Moreover the second inequality in (6.4) and $(\beta_P - \mu_P) > 0$ imply that $(\beta_P - \mu_P - \eta) < 0$.

Lemma 6.4 *Let Assumptions 6.1 and 6.1 be satisfied. Then in absence of prey the population of predators goes to extinct.*

6.3 Dissipativity

In this section, we will prove that the system (6.1) is dissipative. We look for a positive left eigen-vector $(\widetilde{P}_S, \widetilde{P}_H) \in (0, +\infty)^2$ and an eigenvalue $\lambda > 0$ such that

$$(\widetilde{P}_S, \widetilde{P}_H) \begin{bmatrix} -\mu_P - \eta & \gamma \\ \beta_P & \beta_P - \mu_P - \gamma \end{bmatrix} = -\lambda (\widetilde{P}_S, \widetilde{P}_H)$$

that is equivalent to

$$\begin{cases} -(\mu_P + \eta) \widetilde{P}_S + \beta_P \widetilde{P}_H = -\lambda \widetilde{P}_S \\ \gamma \widetilde{P}_S + (\beta_P - \mu_P - \gamma) \widetilde{P}_H = -\lambda \widetilde{P}_H \end{cases} \Leftrightarrow \begin{cases} \beta_P \widetilde{P}_H = [(\mu_P + \eta) - \lambda] \widetilde{P}_S \\ \gamma \widetilde{P}_S = [-(\beta_P - \mu_P - \gamma) - \lambda] \widetilde{P}_H . \end{cases}$$

Thus the sign $\widetilde{P}_S$ and $\widetilde{P}_H$ are the same if we impose

$$\lambda \in (0, \min((\mu_P + \eta), -(\beta_P - \mu_P - \gamma)))$$

and λ must satisfy the following equation

$$1 = \frac{[(\mu_P + \eta) - \lambda]}{\beta_P} \frac{[-(\beta_P - \mu_P - \gamma) - \lambda]}{\gamma} =: \Psi(\lambda) .$$

The function $\lambda \to \Psi(\lambda)$ decreases between 0 and $\min((\mu_P + \eta), -(\beta_P - \mu_P - \gamma))$ and by using (6.4) we have $\Psi(0) > 1$. It follows that there exists a unique $\lambda^* \in (0, \min((\mu_P + \eta), -(\beta_P - \mu_P - \gamma)))$ such that

$$1 = \frac{[(\mu_P + \eta) - \lambda^*]}{\beta_P} \frac{[-(\beta_P - \mu_P - \gamma) - \lambda^*]}{\gamma} . \tag{6.5}$$

Note that

$$\frac{[-(\beta_P - \mu_P - \gamma) - \lambda^*]}{\gamma} < 1 \Leftrightarrow -(\beta_P - \mu_P) < \lambda^* .$$

By assumption $(\beta_P - \mu_P) > 0$ it follows from (6.5) that

$$\frac{[(\mu_P + \eta) - \lambda^*]}{\beta_P} > 1.$$

Since

$$\gamma \widetilde{P}_S = [-(\beta_P - \mu_P - \gamma) - \lambda^*] \widetilde{P}_H ,$$

it follows that

$$\widetilde{P}_H > \widetilde{P}_S > 0. \tag{6.6}$$

By using P_S-equation and P_H-equation of system (6.1) we obtain

$$\widetilde{P}_S P_S' + \widetilde{P}_H P_H' = -\lambda^* \left[\widetilde{P}_S P_S + \widetilde{P}_H P_H\right] - \left(\widetilde{P}_S - \widetilde{P}_H\right) P_S \, \rho \, \kappa \, N. \tag{6.7}$$

By using the N-equation and comparison principle it is clear that we can find some $N^* = \max\left(N(0), (\beta_N - \mu_N)/\delta\right)$ such that

$$N(t) \le N^*, \forall t \ge 0,$$

where $N(0)$ is the initial value. Then it follows that

$$\rho\left(\widetilde{P}_H - \widetilde{P}_S\right) N' + \widetilde{P}_S P_S' + \widetilde{P}_H P_H' \le -\rho\left(\widetilde{P}_H - \widetilde{P}_S\right)\mu_N N - \lambda^*\left[\widetilde{P}_S P_S + \widetilde{P}_H P_H\right] + \rho\left(\widetilde{P}_H - \widetilde{P}_S\right)\beta_N$$

and the dissipativity follows.

Set

$$M = \frac{\rho\left(\widetilde{P}_H - \widetilde{P}_S\right)\beta_N N^*}{\min\left(\mu_N, \lambda^*\right)} > 0.$$

As a consequence of the last inequality, we obtain the following results.

Proposition 6.5 *Let Assumptions 6.1 and 6.2 be satisfied. The system (6.1) generates a unique continuous semiflow* $\{U(t)\}_{t \ge 0}$ *on* $[0, \infty)^3$. *Moreover the domain*

$$D = \left\{(N, P_S, P_H) \in [0,\infty)^3 : \rho\left(\widetilde{P}_H - \widetilde{P}_S\right) N + \widetilde{P}_S P_S + \widetilde{P}_H P_H \le M\right\}$$

is positively invariant by the semiflow generated by U. *That is to say that*

$$U(t)D \subset D, \forall t \ge 0.$$

Furthermore D *attracts every point of* $[0,\infty)^3$ *for* U. *That is to say that*

$$\lim_{t \to \infty} \delta(U(t)x, D) = 0, \forall x \in [0,\infty)^3,$$

where $\delta(x, D) := \inf_{y \in D} \|x - y\|$ *is the Hausdorff's semi-distance. As a consequence the semiflow of* U *has a compact global attractor* $\mathcal{A} \subset [0,\infty)^3$.

6.4 Uniform persitence and extinction of predators

In this section, we study the uniform persistence and extinction of the predators. Firstly we consider the existence of the equilibrium. The equilibrium $\left(\overline{N}, \overline{P}_S, \overline{P}_H\right) \in [0,\infty)^3$ satisfies the following system

$$\begin{cases} 0 = \overline{N}\left[\beta_N - \mu_N - \delta\overline{N} - \kappa\overline{P}_S\right], \\ 0 = -(\mu_P + \eta)\overline{P}_S - \overline{P}_S\,\rho\,\kappa\,\overline{N} + \gamma\overline{P}_H, \\ 0 = -\mu_P\overline{P}_H + \overline{P}_S\,\rho\,\kappa\,\overline{N} - \gamma\overline{P}_H + \beta_P\left(\overline{P}_S + \overline{P}_H\right). \end{cases}$$

By using Assumptions 6.1 and 6.2, we deduce that the only equilibrium satisfying $\overline{N} = 0$ is $E_1 = (0, 0, 0)$. If we assume next that $\overline{N} > 0$, we obtain the system

$$\begin{cases} 0 = \beta_N - \mu_N - \delta\overline{N} - \kappa\overline{P}_S, \\ 0 = -(\mu_P + \eta)\overline{P}_S - \overline{P}_S\,\rho\,\kappa\,\overline{N} + \gamma\overline{P}_H, \\ 0 = -\mu_P\overline{P}_H + \overline{P}_S\,\rho\,\kappa\,\overline{N} - \gamma\overline{P}_H + \beta_P\left(\overline{P}_S + \overline{P}_H\right). \end{cases}$$

From the first equation we have

$$\overline{N} = \widehat{N} - \frac{\kappa}{\delta}\overline{P}_S$$

with

$$\widehat{N} = \frac{\beta_N - \mu_N}{\delta}.$$

By adding the last two equations, we have

$$\overline{P}_H = \frac{(\mu_P + \eta - \beta_P)}{(\beta_P - \mu_P)}\overline{P}_S.$$

Combining the above two equations with

$$-(\mu_P + \eta)\overline{P}_S - \overline{P}_S\,\rho\,\kappa\,\overline{N} + \gamma\overline{P}_H = 0,$$

we have

$$\left(-(\mu_P + \eta) - \frac{(\beta_N - \mu_N)\,\rho\,\kappa}{\delta} + \gamma\frac{(\mu_P + \eta - \beta_P)}{(\beta_P - \mu_P)}\right)\overline{P}_S + \frac{\kappa^2\,\rho}{\delta}\overline{P}_S^2 = 0$$

and then

$$\overline{P}_S = 0 \text{ or } \overline{P}_S = \left((\mu_P + \eta) - \frac{\gamma(\mu_P + \eta - \beta_P)}{(\beta_P - \mu_P)}\right)\frac{\delta}{\kappa^2\,\rho} + \frac{(\beta_N - \mu_N)}{\kappa}.$$

Thus we get the following lemma.

Lemma 6.6 *Let Assumptions 6.1 and 6.2 be satisfied. System (6.1) always has the following two boundary equilibria*

$$E_1 = (0, 0, 0), \quad E_2 = \left(\widehat{N}, 0, 0\right).$$

Moreover there exists a unique interior equilibrium $E^* = \left(N^*, P_S^*, P_H^*\right)$ *if and only if*

$$(\beta_N - \mu_N)\,(\beta_P - \mu_P)\,\kappa\,\rho + \delta\,(\beta_P - \mu_P)\,(\mu_P + \eta) > -\delta\gamma(\beta_P - \mu_P - \eta). \quad (6.8)$$

Furthermore, we have

$$
\begin{aligned}
N^* &= \frac{-(\beta_P - \mu_P)(\mu_P + \eta) - \gamma(\beta_P - \mu_P - \eta)}{(\beta_P - \mu_P)\,\kappa\rho} > 0,\\
P_S^* &= \frac{\delta\,(\beta_P - \mu_P)(\mu_P + \eta) + \delta\gamma(\beta_P - \mu_P - \eta) + (\beta_N - \mu_N)(\beta_P - \mu_P)\,\kappa\rho}{(\beta_P - \mu_P)\,\kappa^2\rho} > 0,\\
P_H^* &= -\frac{(\beta_P - \mu_P - \eta)}{\beta_P - \mu_P} P_S^* > 0.
\end{aligned}
$$

6.4.1 Stability of the equilibrium E_1

The Jacobian matrix at the equilibrium E_1 is

$$
\begin{bmatrix}
\beta_N - \mu_N & 0 & 0\\
0 & -\mu_P - \eta & \gamma\\
0 & \beta_P & \beta_P - \mu_P - \gamma
\end{bmatrix}
$$

and the characteristic equation is

$$[(\lambda + \mu_P + \eta)(\lambda - (\beta_P - \mu_P - \gamma)) - \beta_P\gamma]\,[\lambda - (\beta_N - \mu_N)] = 0.$$

So one of the eigenvalues is $\lambda_{1,E_1} = \beta_N - \mu_N > 0$. Thus we can get that the equilibrium E_1 is unstable. The rest of the spectrum coincides with the spectrum of the matrix M defined in (6.3). Thus we obtain the following lemma.

Lemma 6.7 *Let Assumptions 6.1 and 6.2 be satisfied. The equilibrium E_1 is hyperbolic and the unstable manifold is one dimensional.*

6.4.2 Stability of the equilibrium E_2

The Jacobian matrix at the equilibrium E_2 is

$$
\begin{bmatrix}
-(\beta_N - \mu_N) & -\kappa\widehat{N} & 0\\
0 & -\left((\mu_P + \eta) + \rho\kappa\widehat{N}\right) & \gamma\\
0 & \rho\,\kappa\,\widehat{N} + \beta_P & \beta_P - \mu_P - \gamma
\end{bmatrix}
$$

and the characteristic equation is

$$\left[\left(\lambda + \mu_P + \eta + \rho\kappa\widehat{N}\right)(\lambda - (\beta_P - \mu_P - \gamma)) - \gamma\left(\rho\kappa\widehat{N} + \beta_P\right)\right][\lambda + (\beta_N - \mu_N)] = 0.$$

So one of the eigenvalues is $\lambda_{1,E_2} = -(\beta_N - \mu_N) < 0$ and the remaining part of the characteristic equation is

$$\lambda^2 + a\lambda + b = 0$$

with

$$a = \left(\mu_P + \eta + \rho\,\kappa\,\widehat{N}\right) - (\beta_P - \mu_P - \gamma)$$

and

$$b = (\mu_P + \gamma - \beta_P)\left(\mu_P + \eta + \rho\,\kappa\,\widehat{N}\right) - \gamma\left(\rho\,\kappa\,\widehat{N} + \beta_P\right).$$

By using Assumptions 6.1 and 6.2 we have $a > 0$. Moreover by using the Routh-Hurwitz criterion E_2 is stable if and only if $b > 0$ which corresponds to

$$\begin{aligned}&(\mu_P + \gamma - \beta_P)\left(\mu_P + \eta + \rho\kappa\widehat{N}\right) - \gamma\left(\rho\,\kappa\,\widehat{N} + \beta_P\right) > 0\\ &\Leftrightarrow (\beta_N - \mu_N)(\beta_P - \mu_P)\,\kappa\,\rho + \delta\,(\beta_P - \mu_P)(\mu_P + \eta) < -\delta\gamma(\beta_P - \mu_P - \eta)\end{aligned}$$

Now we obtain the following result.

Lemma 6.8 *Let Assumptions 6.1 and 6.2 be satisfied. E_2 is unstable if the interior equilibrium exits (i.e. the condition 6.8 is satisfied) and the unstable manifold is one dimensional and the stable manifold is two dimensional.*

6.4.3 Extinction of the predators and the global stability of E_2

We decompose the positive cone $M = \mathbb{R}^3_+$ into the interior region

$$\overset{\circ}{M} = \{(N, P_S, P_H) \in M : N > 0 \text{ and } P_S + P_H > 0\},$$

the boundary region with predators only

$$\partial M_P := \{(N, P_S, P_H) \in M : N = 0\}, \tag{6.9}$$

and the boundary region with prey only

$$\partial M_N := \{(N, P_S, P_H) \in M : P_S + P_H = 0\}. \tag{6.10}$$

Each sub domain $\overset{\circ}{M}$, ∂M_P and ∂M_N is positively invariant by the semiflow generated by (6.1).

Theorem 6.9 *Let Assumptions 6.1 and 6.2 be satisfied. Assume that E_2 is locally asymptotically stable (i.e. $\left(\mu_P + \eta + \rho\kappa\widehat{N}\right)(\mu_P + \gamma - \beta_P) > \gamma\left(\beta_P + \rho\kappa\widehat{N}\right)$). Then the predator goes to extinction. More precisely for each initial value in $M = (N(0), P_S(0), P_H(0)) \in [0, \infty)^3$,*

$$\lim_{t\to\infty} P_S(t) + P_H(t) = 0.$$

and

$$\lim_{t\to\infty} N(t) = \begin{cases} \widehat{N}, & \text{if } N(0) > 0,\\ 0, & \text{if } N(0) = 0.\end{cases}$$

Proof The boundary region with predator only ∂M_P is positively invariant by the semiflow generated by (6.1) and by Assumption 6.2 any solution starting from ∂M_P exponentially converges to E_1.

So it remains to investigate the limit of a solution starting from $\overset{\circ}{M} \cup \partial M_N \backslash \{E_1\}$. We consider the Liapunov function

$$V(N, P_S, P_H) = \int_{\widehat{N}}^{N} \frac{\xi - \widehat{N}}{\xi} d\xi + c_1 P_S + c_2 P_H \tag{6.11}$$

where $c_1 > 0$ and $c_2 > 0$ to be determined. We have

$$\begin{aligned}
\dot{V} &= \left(N - \widehat{N}\right)(\beta_N - \mu_N - \delta N - \kappa P_S) \\
&+ c_1 \left(-(\mu_P + \eta) P_S - \rho\kappa P_S N + \gamma P_H\right) \\
&+ c_2 \left(-\mu_P P_H + \rho\kappa P_S N - \gamma P_H + \beta_P (P_S + P_H)\right) \\
&= \left(N - \widehat{N}\right)\left(-\delta\left(N - \widehat{N}\right) - \kappa P_S\right) \\
&+ c_1 \left(-(\mu_P + \eta) P_S - \rho\kappa P_S \left(N - \widehat{N}\right) - \rho\kappa P_S \widehat{N} + \gamma P_H\right) \\
&+ c_2 \left(-\mu_P P_H + \rho\kappa P_S \left(N - \widehat{N}\right) + \rho\kappa P_S \widehat{N} - \gamma P_H + \beta_P (P_S + P_H)\right).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\dot{V} &= -\delta\left(N - \widehat{N}\right)^2 + \kappa P_S \left(N - \widehat{N}\right)(-1 - c_1\rho + c_2\rho) \\
&+ P_S \left(-c_1(\mu_P + \eta) - c_1\rho\kappa\widehat{N} + c_2\rho\kappa\widehat{N} + c_2\beta_P\right) \\
&+ P_H \left(-c_2\mu_P + c_1\gamma - c_2\gamma + c_2\beta_P\right).
\end{aligned}$$

We claim that we can choose $c_1 > 0$ and $c_2 > 0$ such that $c_2 = c_1 + \dfrac{1}{\rho}$ and the following inequalities are satisfied

$$-c_1(\mu_P+\eta)-c_1\rho\kappa\widehat{N}+c_2\rho\kappa\widehat{N}+c_2\beta_P < 0 \text{ and } -c_2\mu_P+c_1\gamma-c_2\gamma+c_2\beta_P < 0. \tag{6.12}$$

In fact the inequalities in (6.12) lead to consider the lines

$$c_2 = c_1 \frac{\gamma}{\mu_P + \gamma - \beta_P} \quad (L_1)$$

and

$$c_2 = c_1 \frac{(\mu_P + \eta) + \rho\kappa\widehat{N}}{\beta_P + \rho\kappa\widehat{N}} \quad (L_2).$$

By Assumption 6.2 (see Remark 6.3) we have $\mu_P + \gamma - \beta_P > 0$ and by Assumption 6.1 we have $\mu_P + \eta > \beta_P$ and then

$$\frac{(\mu_P + \eta) + \rho\kappa\widehat{N}}{\beta_P + \rho\kappa\widehat{N}} > 1.$$

Note that

$$\frac{(\mu_P + \eta) + \rho\kappa\widehat{N}}{\beta_P + \rho\kappa\widehat{N}} > \frac{\gamma}{\mu_P + \gamma - \beta_P}$$
$$\Leftrightarrow \left(\mu_P + \eta + \rho\kappa\widehat{N}\right)(\mu_P + \gamma - \beta_P) > \gamma\left(\beta_P + \rho\kappa\widehat{N}\right)$$

and thus we obtain that the slope of L_2 is greater than the slope of L_1. Finally we have

$$\lim_{N\to 0^+} V(N, P_S, P_N) = (N - \widehat{N}) - \widehat{N}\ln\left(\frac{N}{\widehat{N}}\right) + c_1 P_S + c_2 P_N = +\infty.$$

By LaSalle's invariance principle we obtain that E_2 is globally asymptotically stable for the system restricted to $\overset{\circ}{M} \cup \partial M_N \backslash \{E_1\}$. □

6.4.4 Uniform persistence of the predators

We decompose the positive cone into

$$\mathbb{R}^3_+ = \partial M \cup \overset{\circ}{M}$$

where the boundary region is defined as

$$\partial M := \partial M_P \cup \partial M_N.$$

It is clear that both regions $\overset{\circ}{M}$ and ∂M are positively invariant by the semiflow generated by the system. Moreover we have the following result.

Theorem 6.10 *Let Assumptions 6.1 and 6.2 be satisfied. If the interior equilibrium exits then the predators uniformly persist with respect to the domain decomposition* $\left(\partial M, \overset{\circ}{M}\right)$. *That is to say that there exists* $\nu > 0$ *such that for each initial value* $N(0) > 0$ *and* $P_S(0) + P_H(0) > 0$

$$\liminf_{t\to\infty} N(t) > \nu \text{ and } \liminf_{t\to\infty} P_S(t) + P_H(t) > \nu.$$

Proof The equilibrium $E_1 = \{(0, 0, 0)\}$ is clearly chained to $E_2 = \{(\widehat{N}, 0, 0)\}$. By using Theorem 4.1 in [96], we only need to prove that

$$W^s(E_i) \cap \overset{\circ}{M} = \varnothing,$$

where $i = 1, 2$ and

$$W^s(E_i) = \{(N, P_S, P_H) \in M : \omega((N, P_S, P_H)) \neq \varnothing \text{ and } \omega((N, P_S, P_H)) \subset E_i\},$$

where ω means the ω-limit set. Assume that there exists $E^0 = \left(N^0, P^0_S, P^0_H\right) \in \overset{\circ}{M}$ (which means $N^0 > 0$ and $P^0_S + P^0_H > 0$) such that $\omega(E^0) \subset E_1$. Then for any $\varepsilon > 0$, there exists $t_0 \geq 0$, such that

$$N(t) + P_S(t) + P_H(t) \leq \varepsilon, \forall t \geq t_0$$

where $(N(t), P_S(t), P_H(t)) = U(t)E^0$. By using the first equation of model (6.1)

$$N' = \beta_N N - \mu_N N - \delta N^2 - N \kappa P_S,$$

we have

$$N' \geq N (\beta_N - \mu_N - \delta\varepsilon - \kappa \varepsilon).$$

Therefore for $\varepsilon > 0$ small enough, we have $\beta_N - \mu_N - \delta\varepsilon - \kappa \varepsilon > 0$ and then

$$\lim_{t\to\infty} N(t) = \infty$$

which is in contradiction to the dissipativity of the model. Assume that there exists $E^0 = \left(N^0, P_S^0, P_H^0\right) \in \overset{\circ}{M}$ such that $\omega(E^0) \subset E_2$. Then for any $\varepsilon > 0$, there exists $t_0 \geq 0$, such that

$$\left|N(t) - \widehat{N}\right| + P_S(t) + P_H(t) \leq \varepsilon, \forall t \geq t_0$$

where $(N(t), P_S(t), P_H(t)) = U(t)E^0$. By using the two last equation of system (6.1), we obtain

$$\begin{aligned} P_S' &\geq -(\mu_P + \eta)P_S - P_S\rho\kappa\left(\widehat{N} + \varepsilon\right) + \gamma P_H \\ P_H' &\geq -\mu_P P_H + P_S \rho \kappa \left(\widehat{N} - \varepsilon\right) - \gamma P_H + \beta_P (P_S + P_H) \end{aligned} \tag{6.13}$$

By using the fact that for $\varepsilon > 0$ small enough the right hand side of (6.13) is a cooperative system together with Lemma 6.8 we deduce that

$$\lim_{t\to\infty} P_S(t) + P_H(t) = \infty.$$

This gives a contradiction with the dissipativity of the system. Therefore the uniform persistence follows. □

As a consequence of the dissipativity as well as the uniform peristence (see Magal and Zhao [160]) we deduce the following result.

Theorem 6.11 *Let Assumptions 6.1 and 6.2 be satisfied. Assume in addition that the interior equilibrium exits. Then the system (6.1) has a global attractor A_0 in the interior region $\overset{\circ}{M}$. Namely A_0 is a compact invariant set by the semiflow generated by (6.1) on $\overset{\circ}{M}$ and A_0 is locally stable and attracts the compact subsets of $\overset{\circ}{M}$.*

6.5 Interior region

In this section, we will study the system in the interior region which corresponds to the region of co-existence of prey and predators.

6.5.1 Local stability of E^*

The Jacobian matrix at the equilibrium E^* is

$$\begin{bmatrix} \left(\beta_N - \mu_N - 2\delta N^* - \kappa P_S^*\right) & -N^* \kappa & 0 \\ -P_S^* \rho \kappa & -\left(\mu_P + \eta + \rho \kappa N^*\right) & \gamma \\ P_S^* \rho \kappa & \rho \kappa N^* + \beta_P & \beta_P - \mu_P - \gamma \end{bmatrix}.$$

and the characteristic equation is

$$\lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3 = 0$$

with

$$\begin{aligned} p_1 &= -\left(\beta_N - \mu_N - 2\delta N^* - \kappa P_S^*\right) + \left(\mu_P + \eta + \rho \kappa N^*\right) - \left(\beta_P - \mu_P - \gamma\right), \\ p_2 &= -\left(\mu_P + \eta + \rho \kappa N^*\right)\left(\beta_N - \mu_N - 2\delta N^* - \kappa P_S^*\right) - N^* \kappa P_S^* \rho \kappa \\ &\quad + \left(\beta_N - \mu_N - 2\delta N^* - \kappa P_S^*\right)\left(\beta_P - \mu_P - \gamma\right) \\ &\quad - \left(\mu_P + \eta + \rho \kappa N^*\right)\left(\beta_P - \mu_P - \gamma\right) - \left(\rho \kappa N^* + \beta_P\right)\gamma, \\ p_3 &= \left(\beta_N - \mu_N - 2\delta N^* - \kappa P_S^*\right)\left(\mu_P + \eta + \rho \kappa N^*\right)\left(\beta_P - \mu_P - \gamma\right) \\ &\quad + N^* \kappa \gamma P_S^* \rho \kappa + \gamma \left(\beta_N - \mu_N - 2\delta N^* - \kappa P_S^*\right)\left(\rho \kappa N^* + \beta_P\right) + \\ &\quad P_S^* \rho \kappa N^* \kappa \left(\beta_P - \mu_P - \gamma\right). \end{aligned}$$

By using Routh-Hurwitz criterion, we get that the equilibrium E^* is stable if and only if

$$p_1 > 0, p_1 p_2 - p_3 > 0 \text{ and } p_3 > 0.$$

By computing, we have

$$\begin{aligned} p_1 &= \frac{-\kappa \rho \left(\beta_P - \mu_P - \gamma\right)\left(\beta_P - \mu_P\right) - \gamma\left(\delta + \kappa \rho\right)\left(\beta_P - \mu_P - \eta\right) - \delta\left(\beta_P - \mu_P\right)\left(\mu_P + \eta\right)}{\kappa \rho \left(\beta_P - \mu_P\right)}, \\ p_2 &= \frac{\left[\left(\beta_P - \mu_P - \gamma\right)\left(\mu_P + \eta\right) + \gamma \beta_P\right]\left\{\left(\beta_P - \mu_P\right)\left[\delta\left(\beta_P + \eta + \gamma\right) + \kappa \rho\left(\beta_N - \mu_N\right)\right] - 2\delta \gamma \eta\right\}}{\kappa \rho \left(\beta_P - \mu_P\right)^2}, \\ p_3 &= \frac{\left[\left(\beta_P - \mu_P - \gamma\right)\left(\mu_P + \eta\right) + \gamma \beta_P\right]\left\{\begin{array}{c} -\delta\left(\beta_P - \mu_P - \gamma\right)\left(\mu_P + \eta\right) - \gamma \delta \beta_P - \\ \kappa \rho \left(\beta_P - \mu_P\right)\left(\beta_N - \mu_N\right) \end{array}\right\}}{\kappa \rho \left(\beta_P - \mu_P\right)}. \end{aligned}$$

Thus we have the following result.

Lemma 6.12 *Let Assumptions 6.1, 6.2 and inequality (6.8) be satisfied. The equilibrium E^* is stable if and only if $(\beta_P - \mu_P)[\kappa\rho(\beta_N - \mu_N) + \delta(\eta + \gamma)] < \delta[2\gamma\eta - \beta_P(\beta_P - \mu_P)]$.*

6.5.2 Three dimensional $\mathbb{K}$-competitive system

In this section we use a Poincaré-Bendixson theorem for three dimensional $\mathbb{K}$-competitive system.

Theorem 6.13 *([201, Theorem 4.2 p. 43]) Let the autonomous system of ordinary differential equations $x' = f(x)$ be a competitive system, where f is continuously differentiable on an open subset $D \subset \mathbb{R}^3$ and suppose that D contains a unique equilibrium point p which is hyperbolic. Suppose further that $W^s(p)$ is one-dimensional and tangent at p to a vector $v \gg 0$. If $q \in D \setminus W^s(p)$ and $\gamma^+(q)$ has compact closure in D then $\omega(q)$ is a nontrival periodic orbit.*

By applying this theorem to the system (6.1) restricted to the interior global attractor A_0 we obtain the following result.

Theorem 6.14 *Suppose that $E^* = \left(N^*, P_S^*, P_H^*\right)$ exists and is hyperbolic and unstable for (6.1). Then the stable manifold $W^s(E^*)$ of E^* is one dimensional and the omega limit set $\omega\,(N(0), P_S(0), P_H(0))$ is a nontrivial periodic orbit in $\mathbb{R}^3_+$ for every $(N(0), P_S(0), P_H(0)) \in \mathbb{R}^3_+ \setminus W^s(E^*)$.*

Proof The Jacobian matrix of the vector field (6.1) at the point $(N, P_S, P_H) \in (0,\infty)^3$ is given by

$$J = \begin{pmatrix} (\beta_N - \mu_N) - 2\delta N - \kappa P_S & -\kappa N & 0 \\ -\rho\kappa P_S & -(\mu_P + \eta) - \rho\kappa N & \gamma \\ \rho\kappa P_S & \rho\kappa N + \beta_P & -\mu_P + \beta_P - \gamma \end{pmatrix}. \tag{6.14}$$

The off-diagonal entries of J are sign-stable and sign symmetric in $\mathbb{R}^3_+$.

Let

$$\mathbb{K} = \left\{(N, P_S, P_H) \in \mathbb{R}^3 : N \geq 0, P_S \geq 0, P_H \leq 0\right\}.$$

The system is $\mathbb{K}$-competitive, since the matrix of the time-reversed linearized system $-J$ is cooperative with respect to the cone $\mathbb{K}$. □

6.6 Convergence to the Rosenzweig-MacArthur model

The time scale for the life expectancy (as well as the time scale needed for the reproduction) is the year, while the time needed for the lynx to handle the rabbit is measured by days (no more than one week). Therefore there is a huge difference between the time scales for the vital dynamic and the consumption dynamic.

The consumption of prey by the predator is a fast process compared to the vital dynamic which is slow. In the model γ^{-1} is the average time spent by the predators to handle prey. γ^{-1} should be very small in comparison with the other parameters. Then it makes sense to make the following assumption.

Assumption 6.15 Assume that

$$\rho = \frac{\chi}{\varepsilon} \text{ and } \gamma = \frac{1}{\varepsilon}$$

with $\varepsilon \ll 1$ is small.

Under the above assumption the system (6.1) becomes

$$\begin{cases} \dot{N^\varepsilon} = (\beta_N - \mu_N)\, N^\varepsilon - \delta(N^\varepsilon)^2 - \kappa N^\varepsilon\, P_S^\varepsilon \\ \dot{P_S^\varepsilon} = -(\mu_P + \eta) P_S^\varepsilon - \frac{\chi}{\varepsilon}\kappa N^\varepsilon P_S^\varepsilon + \frac{1}{\varepsilon} P_H^\varepsilon \\ \dot{P_H^\varepsilon} = -\mu_P P_H^\varepsilon \qquad + \frac{\chi}{\varepsilon}\kappa N^\varepsilon P_S^\varepsilon - \frac{1}{\varepsilon} P_H^\varepsilon + \beta_P \left(P_S^\varepsilon + P_H^\varepsilon\right) \end{cases} \tag{6.15}$$

and we fix the initial value

$$N^\varepsilon(0) = N_0 \geq 0,\ P_S^\varepsilon(0) = P_{S0} \geq 0 \text{ and } P_H^\varepsilon(0) = P_{H0} \geq 0.$$

The first equation of (6.15) is

$$\dot{N^\varepsilon} = (\beta_N - \mu_N)\, N^\varepsilon - \delta(N^\varepsilon)^2 - \kappa N^\varepsilon\, P_S^\varepsilon. \tag{6.16}$$

Hence

$$\dot{N^\varepsilon} \leq (\beta_N - \mu_N)\, N^\varepsilon. \tag{6.17}$$

By summing the two last equations of (6.15) we obtain

$$\dot{P^\varepsilon} = (\beta_P - \mu_P)\, P^\varepsilon - \eta P_S^\varepsilon \tag{6.18}$$

and $P_S^\varepsilon \geq 0$ implies that

$$\dot{P^\varepsilon} \leq (\beta_P - \mu_P)\, P^\varepsilon. \tag{6.19}$$

Therefore by using (6.17) and (6.19) we obtain the following finite time estimation uniform in ε.

Lemma 6.16 *For each $\tau > 0$ we can find a constant $M = M(\tau, N_0, P_0) > 0$ (independent of $\varepsilon > 0$) such that*

$$0 \leq N^\varepsilon(t) \leq M \textit{ and } 0 \leq P^\varepsilon(t) \leq M, \forall t \in [0, \tau]. \tag{6.20}$$

and

$$\sup_{t\in[0,\tau]} |\dot{N^\varepsilon}(t)| \leq M \textit{ and } \sup_{t\in[0,\tau]} |\dot{P^\varepsilon}(t)| \leq M. \tag{6.21}$$

Proof We first deduce (6.20) by using the inequalities (6.17) and (6.19). By using the fact $P_S \geq 0$ and $P_H \geq 0$ we have

$$0 \leq P_S^\varepsilon(t) \leq M, \text{ and } 0 \leq P_H^\varepsilon(t) \leq M, \forall t \in [0, \tau]. \tag{6.22}$$

Therefore by injecting these estimations into (6.16) and (6.18) we deduce (6.21). □

By using Lemma 6.6, and the Arzela-Ascoli theorem we deduce that we can find a sequence $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty} N^{\varepsilon_n} = N \text{ and } \lim_{n\to\infty} P^{\varepsilon_n} = P$$

where the convergence is taking place in $C([0,\tau],\mathbb{R})$ for the uniform convergence topology.

Moreover by using the fact that $P_H^\varepsilon = P^\varepsilon - P_S^\varepsilon$, the P_S^ε-equation can be rewritten as

$$\dot{P}_S^\varepsilon = -\left((\mu_P + \eta) + \frac{\chi}{\varepsilon}\kappa N^\varepsilon\right) P_S^\varepsilon + \frac{1}{\varepsilon}\left(P^\varepsilon - P_S^\varepsilon\right). \tag{6.23}$$

By using (6.22), the map $t \to P_S^\varepsilon(t)$ is bounded uniformly in ε. So the family $\varepsilon_n \to P_S^{\varepsilon_n}$ is bounded in $L^\infty((0,\tau),\mathbb{R})$ which is the dual space of $L^1((0,\tau),\mathbb{R})$. Therefore by using the Banach-Alaoglu-Bourbaki's theorem, we can find a subsequence (denoted with the same index) such that $\varepsilon_n \to P_S^{\varepsilon_n}$ convergences to $P_S \in L^\infty((0,\tau),\mathbb{R})$ for the weak star topology of $\sigma\left(L^\infty((0,\tau),\mathbb{R}), L^1((0,\tau),\mathbb{R})\right)$. That is to say that for each $\chi \in L^1((0,\tau),\mathbb{R})$

$$\lim_{n\to\infty} \int_0^\tau \chi(t)\left(P_S^{\varepsilon_n}(t) - P_S(t)\right) dt = 0.$$

By multiplying (6.23) by $\chi \in C_c^1((0,\tau),\mathbb{R})$ (the space C^1 functions with compact support in $(0,\tau)$) and by integrating over $[0,\tau]$ we obtain

$$-\int_0^\tau \dot{\chi}(t) P_S^{\varepsilon_n}(t) dt = \int_0^\tau \chi(t)\left[-\left((\mu_P+\eta) + \frac{\chi}{\varepsilon_n}\kappa N^{\varepsilon_n}(t)\right) P_S^{\varepsilon_n}(t) + \frac{1}{\varepsilon_n}\left(P^{\varepsilon_n}(t) - P_S^{\varepsilon_n}(t)\right)\right] dt$$

Hence by multiplying both sides by ε_n and by taking the limit when n goes to infinity we obtain

$$0 = \int_0^\tau \chi(t)\left[-\left(\chi\kappa N(t)\right) P_S(t) + \left(P(t) - P_S(t)\right)\right] dt$$

and since $C_c^1((0,\tau),\mathbb{R})$ is dense in $L^1((0,\tau),\mathbb{R})$ we deduce that

$$P_S^{\varepsilon_n}(t) \overset{*}{\rightharpoonup} \frac{1}{1+\chi\kappa N(t)} P(t) \text{ as } n \to \infty.$$

By using the first equation of (6.15) and (6.18), we have

$$N^{\varepsilon_n}(t) = \frac{e^{\int_0^t \beta_N - \mu_N - \kappa P_S^{\varepsilon_n}(\sigma) d\sigma} N_0}{1 + \delta \int_0^t e^{\int_0^l \beta_N - \mu_N - \kappa P_S^{\varepsilon_n}(\sigma) d\sigma} N_0 dl},$$
$$P^{\varepsilon_n}(t) = e^{(\beta_P - \mu_P)t} P_0 - \int_0^t e^{(\beta_P - \mu_P)(t-s)} \eta P_S^{\varepsilon}(\sigma) d\sigma.$$

By taking the limit on both sides we deduce that

$$\begin{cases} \dot{N} = (\beta_N - \mu_N) N(t) - \delta N(t)^2 - \dfrac{\kappa N(t)}{1 + \chi\kappa N(t)} P(t), \\ \dot{P} = (\beta_P - \mu_P) P - \eta \dfrac{1}{1 + \chi\kappa N(t)} P. \end{cases}$$

Therefore we obtain the following theorem.

Theorem 6.17 *Let $\tau > 0$ be fixed. For each fixed initial values $N_0 \geq 0$, $P_{S0} \geq 0$ and $P_{H0} \geq 0$, we have the following results:*

$$\lim_{\varepsilon \to 0} N^{\varepsilon}(t) = N(t) \text{ and } \lim_{\varepsilon \to 0} P_S^{\varepsilon}(t) + P_H^{\varepsilon}(t) = P(t)$$

where the limit is uniform on $[0, \tau]$, $(N^{\varepsilon}(t), P_S^{\varepsilon}(t), P_H^{\varepsilon}(t))$ is the solution of (6.15) with the initial conditions

$$N^{\varepsilon}(0) = N_0 \geq 0,\ P_S^{\varepsilon}(0) = P_{S0} \geq 0 \text{ and } P_H^{\varepsilon}(0) = P_{H0} \geq 0$$

and $(N(t), P(t))$ is the solution of the Rosenzweig-MacArthur model

$$\begin{cases} \dot{N} = (\beta_N - \mu_N) N(t) - \delta N(t)^2 - \dfrac{\kappa N(t)}{1 + \chi\kappa N(t)} P(t), \\ \dot{P} = (\beta_P - \mu_P - \eta) P + \eta \dfrac{\chi\kappa N(t)}{1 + \chi\kappa N(t)} P \end{cases} \tag{6.24}$$

with the initial conditions

$$N(0) = N_0 \text{ and } P(0) = P_{S0} + P_{H0}.$$

Remark 6.18 If instead of the model (6.1) we consider the following model

$$\begin{cases} \dot{N} = (\beta_N - \mu_N) N - \delta N^2 - \kappa N^l P_S \\ \dot{P}_S = -(\mu_P + \eta) P_S - \rho\kappa N^m P_S + \gamma P_H, \\ \dot{P}_H = \beta_P (P_S + P_H) - \mu_P P_H + \rho\kappa N^m P_S - \gamma P_H \end{cases} \tag{6.25}$$

Then by using the same procedure above we obtain a convergence result to the most classical predator prey model

$$\begin{cases} \dot{N} = (\beta_N - \mu_N) N(t) - \delta N(t)^2 - \dfrac{\kappa N(t)^l}{1 + \chi\kappa N(t)^m} P(t), \\ \dot{P} = (\beta_P - \mu_P - \eta) P + \eta \dfrac{\chi\kappa N(t)^m}{1 + \chi\kappa N(t)^m} P. \end{cases} \tag{6.26}$$

By choosing $l = m$ we obtain the classical Holling's type functional response.

6.7 Application to the Snowshoe Hares and the Lynx

In this section we reconsider predator-prey system form by the hares (prey) and lynxes (predator) in the years 1900-1920 recorded by the Hudson Bay Company. The data are available for example in [52].

Year	**Hares** (in thousands)	**Lynx** (in thousands)
1900	30	4
1901	47.2	6.1
1902	70.2	9.8
1903	77.4	35.2
1904	36.3	59.4
1905	20.6	41.7
1906	18.1	19
1907	21.4	13
1908	22	8.3
1909	25.4	9.1
1910	27.1	7.4
1911	40.3	8
1912	57	12.3
1913	76.6	19.5
1914	52.3	45.7
1915	19.5	51.1
1916	11.2	29.7
1917	7.6	15.8
1918	14.6	9.7
1919	16.2	10.1
1920	24.7	8.6

Table 6.1: *Numbers of hares (prey) and lynxes (predator) in the years 1900-1920 recorded by the Hudson Bay Company*

The limit model obtain for ε small enough is given by

$$\begin{cases} \dot{N} = (\beta_N - \mu_N)\, N \left(1 - \dfrac{\delta N}{\beta_N - \mu_N}\right) - \dfrac{\kappa P N}{1 + \chi \kappa N}, \\ \dot{P} = (\beta_P - \mu_P - \eta)\, P + \eta \dfrac{\chi \kappa P N}{1 + \chi \kappa N} \end{cases} \tag{6.27}$$

with initial value

$$N(0) = N_0 = 30 \times 10^3 \text{ and } P(0) = P_0 = 4 \times 10^3.$$

Symbol	Interpretation	Value	Unit	Method
$1/\mu_N$	Life expectancy of hares	1	year	fixed
β_N	Birth rate of hares	1.6567	number of new born/year	fitted
δ	Carrying capacity of hares	303000	year	fitted
κ		3.2×10^{-5}		fitted
χ		0.11		fitted
$1/\mu_P$	Life expectancy of Lynx	7	year	fixed
β_P	Birth rate of Lynx	8.5127	number of new born/year	fitted
η	Extra mortality of searching Lynx	9.24	year^{-1}	fitted
$\beta_P - \mu_P - \eta$	Growth of searching lynx	−0.8702		fitted
$\eta\chi$	Convertion rate	1.0164		fitted

Table 6.2: *List parameters for the model* (6.27)*, their interpretations, values and symbols. In this table we have fixed* μ_N *and* μ_P *and we have obtain all the remaining parameters by using a least square method between the data in Table* 6.1 *the solution of the model* (6.27)*. The life expectancy of Snowshoe Hares is not known [28, 82]. Here we fix the life expectancy of hares to be* 1 *year (similarly to [230]). In the wild a Canadian Lynx can live up to* 14 *years. Here we fix the life expectancy to be* 7 *years (see [67] for more result). A Canadian lynx can have between 1 and 8 new babies [123]. So the estimation obtained for the brith rate of lynx is still reasonable.*

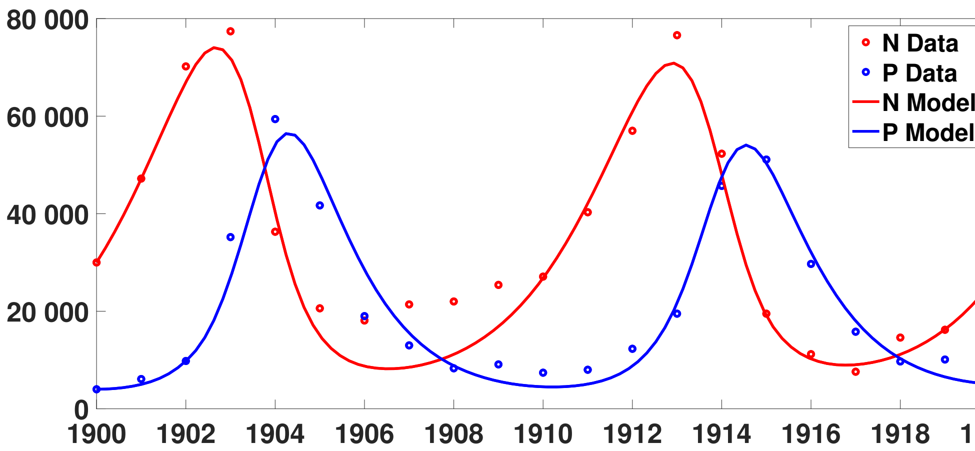


Fig. 6.1: *In this figure we run a simulation of the model* (6.27) *(solide lines) compared with the data (circles).*

In section 6.6, we proved that the model (6.27) can be obtained as singular limit (when $\varepsilon \to 0$) of the following model

$$\begin{cases} \dot{N^{\varepsilon}} = (\beta_N - \mu_N)\, N^{\varepsilon} \left(1 - \dfrac{\delta N^{\varepsilon}}{\beta_N - \mu_N}\right) - \kappa N^{\varepsilon} P_S^{\varepsilon} \\ \dot{P_S^{\varepsilon}} = -(\mu_P + \eta) P_S^{\varepsilon} - \dfrac{\chi}{\varepsilon} \kappa N^{\varepsilon} P_S^{\varepsilon} + \dfrac{1}{\varepsilon} P_H^{\varepsilon} \\ \dot{P_H^{\varepsilon}} = -\mu_P P_H^{\varepsilon} \qquad + \dfrac{\chi}{\varepsilon} \kappa N^{\varepsilon} P_S^{\varepsilon} - \dfrac{1}{\varepsilon} P_H^{\varepsilon} + \beta_P \left(P_S^{\varepsilon} + P_H^{\varepsilon}\right) \end{cases} \tag{6.28}$$

and we fix the initial value

$$N^{\varepsilon}(0) = N_0 = 30 \times 10^3 \geq 0,\ P_S^{\varepsilon}(0) = P_{S0} \geq 0 \text{ and } P_H^{\varepsilon}(0) = P_{H0} \geq 0.$$

In Theorem 6.17 we proved that for ε small enough

$$P_S^{\varepsilon}(t) \simeq \frac{1}{1 + \chi\kappa N(t)} P(t) \text{ and } P_R^{\varepsilon}(t) \simeq \left(1 - \frac{1}{1 + \chi\kappa N(t)}\right) P(t) = \frac{\chi\kappa N(t)}{1 + \chi\kappa N(t)} P(t). \tag{6.29}$$

By using the value for $\chi\kappa$ estimated in Table 6.2, we obtain the following initial values for the model (6.28)

$$P_{S0}^{\varepsilon} = \frac{P_0}{1 + \chi\kappa N_0} = \frac{4 \times 10^3}{1 + 1.0164 \times 30 \times 10^3} \text{ and } P_{R0}^{\varepsilon} = \frac{\chi\kappa N_0}{1 + \chi\kappa N_0} P_0 = \frac{1.0164 \times 30 \times 10^3}{1 + 1.0164 \times 30 \times 10^3} 4\times 1 \tag{6.30}$$

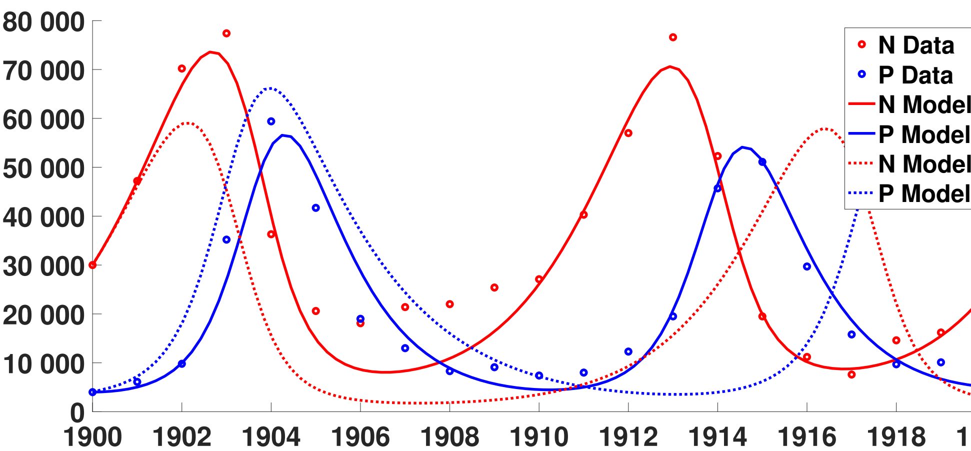


Fig. 6.2: *In this figure we run a simulation of the model* (6.28) *(solide and dotted lines) compared with the data (circles). The solide lines correspond to* $\varepsilon = 10^{-4}$ *and the dotted lines correspond to* $\varepsilon = 5.10^{-3}$.

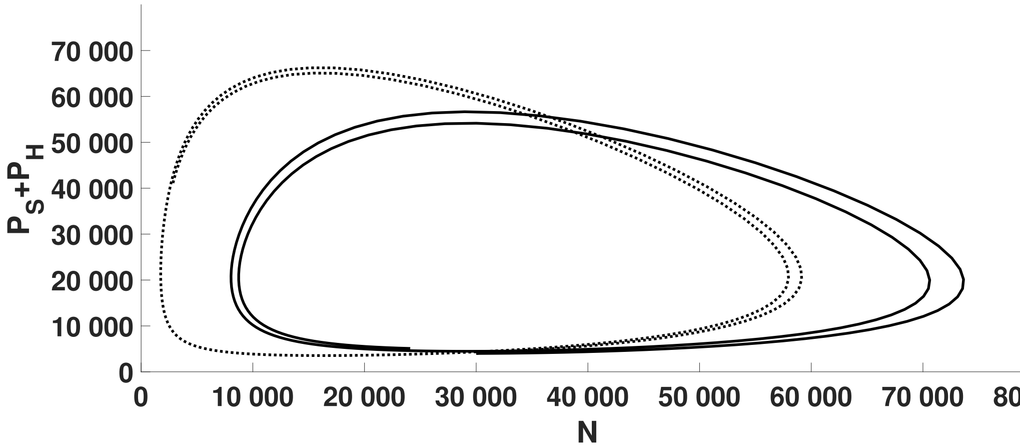


Fig. 6.3: *In this figure we run a simulation of the model* (6.28) *with* $\varepsilon = 10^{-4}$ *for solide line and with* $\varepsilon = 5.10^{-3}$ *for dotted line.*

From Figures 6.1 and 6.2 , we can see that ε does not need to be very small ($\varepsilon = 10^{-4}$) to get an almost perfect match of our model (6.28) with the Rosenzweig-MacArthur model (6.27). Our simulations for hares and lynxes fit the data reported by the Hudson Bay Company. As we mentioned the main advantage with the model (6.28) is that we can separate the vital dynamic and consumption of prey (hares) to describe the behavior of the predators (lynxes). From our model (6.28), people can study the interaction between predator and prey in detail and get more information.

6.8 Remarks and notes

6.9 MATLAB codes

6.9.1 Figure 6.1

```
function example
tspan = [1900 1920];
x=[1900 1901 1902 1903 1904 1905 1906 1907 1908 1909
    1910 1911 1912 1913 1914 1915 1916 1917 1918
   1919 1920]; % years
y=[3*1e4 4.72*1e4 70.2*1e3 77.4*1e3 36.3*1e3 20.6*1
   e3 18.1*1e3 21.4*1e3 22*1e3 25.4*1e3 27.1*1e3
   40.3*1e3 57*1e3 76.6*1e3 52.3*1e3 19.5*1e3 11.2*1
   e3 7.6*1e3 14.6*1e3 16.2*1e3 24.7*1e3];
```

```
z=[4*1e3 6.1*1e3 9.8*1e3 35.2*1e3 59.4*1e3 41.7*1e3
    19*1e3 13*1e3 8.3*1e3 9.1*1e3 7.4*1e3 8*1e3
    12.3*1e3 19.5*1e3 45.7*1e3 51.1*1e3 29.7*1e3
    15.8*1e3 9.7*1e3 10.1*1e3 8.6*1e3];
z0 = [30*1e3;(4*1e3)/(1+1.0164*30*1e3)+((1.0164*30*1
    e3)*(4*1e3))/(1+1.0164*30*1e3)];


sol = ode45(@myfun,tspan,z0);

t = linspace(tspan(1),tspan(2),100);

z_new = deval(sol,t);

% we plot the data
plot(x,y,'ro','LineWidth',3);
hold on
plot(x,z,'bo','LineWidth',3);
hold on
% we plot the solution of the model
plot(t,z_new(1,:),'r','linewidth',3);
hold on
plot(t,z_new(2,:),'b','linewidth',3);
hold on
%xlabel('year')
%ylabel('year')
legend('N Data', 'P Data','N Model','P Model')

set(gca, 'FontSize', 30);
set(gca, 'FontWeight', 'bold');

function dz = myfun(t,z)
muN = 1;
betaN = 1.6567;
delta = 0.6567/303000;
k = 3.2*1e-5;
X = 0.11;
muP = 1/7;
betaP = 8.5127;
eta = 9.24;

dz = zeros(2,1);
dz(1) = (betaN-muN)*z(1)*(1-(delta*z(1))/(betaN-muN)
    )-(k*z(1)*z(2))/(1+X*k*z(1));
dz(2) = (betaP-muP-eta)*z(2)+(eta*k*X*z(1)*z(2))/(1+
    X*k*z(1));
```

```
dz=dz(:);
```

6.9.2 Figure 6.2

```
function  example
tspan = [1900 1920];
x=[1900 1901 1902 1903 1904 1905 1906 1907 1908 1909
    1910 1911 1912 1913 1914 1915 1916 1917 1918
   1919 1920];
y=[3*1e4 4.72*1e4 70.2*1e3 77.4*1e3 36.3*1e3 20.6*1
   e3 18.1*1e3 21.4*1e3 22*1e3 25.4*1e3 27.1*1e3
   40.3*1e3 57*1e3 76.6*1e3 52.3*1e3 19.5*1e3 11.2*1
   e3 7.6*1e3 14.6*1e3 16.2*1e3 24.7*1e3];
z=[4*1e3 6.1*1e3 9.8*1e3 35.2*1e3 59.4*1e3 41.7*1e3
   19*1e3 13*1e3 8.3*1e3 9.1*1e3 7.4*1e3 8*1e3
   12.3*1e3 19.5*1e3 45.7*1e3 51.1*1e3 29.7*1e3
   15.8*1e3 9.7*1e3 10.1*1e3 8.6*1e3];
y0 = [30*1e3;(4*1e3)/(1+1.0164*30*1e3);((1.0164*30*1
   e3)*(4*1e3))/(1+1.0164*30*1e3)];
z0 = [30*1e3;(4*1e3)/(1+1.0164*30*1e3);((1.0164*30*1
   e3)*(4*1e3))/(1+1.0164*30*1e3)];
sol1 = ode45(@myfun,tspan,y0);
sol2 = ode45(@myfun2,tspan,z0);
t = linspace(tspan(1),tspan(2),100);
y_new = deval(sol1,t);
z_new = deval(sol2,t);

plot(x,y,'ro','LineWidth',3);
hold on
plot(x,z,'bo','LineWidth',3);
hold on
plot(t,z_new(1,:),'r','linewidth',3);
hold on
plot(t,z_new(2,:)+z_new(3,:),'b','linewidth',3);
hold on
plot(t,y_new(1,:),'r:','linewidth',3);
hold on
plot(t,y_new(2,:)+y_new(3,:),'b:','linewidth',3);
hold on

legend('N Data', 'P Data','N Model','P Model','N
   Model', 'P Model')

set(gca, 'FontSize', 30);
set(gca, 'FontWeight', 'bold');

```

```
function dy = myfun(t,y)
muN = 1;
betaN = 1.6567;
delta = 0.6567/303000;
k = 3.2e-5;
X = 0.11;
muP = 1/7;
betaP = 8.5127;
eta = 9.24;
epison =5*1e-3;
rho=X/epison;
r=1/epison;
dy = zeros(3,1);
dy(1) = (betaN-muN)*y(1)*(1-(delta*y(1))/(betaN-muN)
    )-k*y(1)*y(2);
dy(2) = -(muP+eta)*y(2)-rho*k*y(1)*y(2)+y(3)/epison;
dy(3) = -muP*y(3)+X/epison*k*y(1)*y(2)-y(3)*r+betaP
    *(y(2)+y(3));
hold on

function dz = myfun2(t,z)
muN = 1;
betaN = 1.6567;
delta = 0.6567/303000;
k = 3.2e-5;
X = 0.11;
muP = 1/7;
betaP = 8.5127;
eta = 9.24;
epison =1e-4;
rho=X/epison;
r=1/epison;
dz = zeros(3,1);
dz(1) = (betaN-muN)*z(1)*(1-(delta*z(1))/(betaN-muN)
    )-k*z(1)*z(2);
dz(2) = -(muP+eta)*z(2)-rho*k*z(1)*z(2)+z(3)/epison;
dz(3) = -muP*z(3)+X/epison*k*z(1)*z(2)-z(3)*r+betaP
    *(z(2)+z(3));
```

6.9.3 Figure 6.3

```
function example
tspan = [1900 1920];
x=[1900 1901 1902 1903 1904 1905 1906 1907 1908 1909
    1910 1911 1912 1913 1914 1915 1916 1917 1918
    1919 1920];
```

```
y=[3*1e4 4.72*1e4 70.2*1e3 77.4*1e3 36.3*1e3 20.6*1
    e3 18.1*1e3 21.4*1e3 22*1e3 25.4*1e3 27.1*1e3
    40.3*1e3 57*1e3 76.6*1e3 52.3*1e3 19.5*1e3 11.2*1
    e3 7.6*1e3 14.6*1e3 16.2*1e3 24.7*1e3];
z=[4*1e3 6.1*1e3 9.8*1e3 35.2*1e3 59.4*1e3 41.7*1e3
    19*1e3 13*1e3 8.3*1e3 9.1*1e3 7.4*1e3 8*1e3
    12.3*1e3 19.5*1e3 45.7*1e3 51.1*1e3 29.7*1e3
    15.8*1e3 9.7*1e3 10.1*1e3 8.6*1e3];
y0 = [30*1e3;(4*1e3)/(1+1.0164*30*1e3);((1.0164*30*1
    e3)*(4*1e3))/(1+1.0164*30*1e3)];
z0 = [30*1e3;(4*1e3)/(1+1.0164*30*1e3);((1.0164*30*1
    e3)*(4*1e3))/(1+1.0164*30*1e3)];
sol1 = ode45(@myfun,tspan,y0);
sol2 = ode45(@myfun2,tspan,z0);
t = linspace(tspan(1),tspan(2),200);
y_new = deval(sol1,t);
z_new = deval(sol2,t);


plot(y_new(1,:),y_new(2,:)+y_new(3,:),'k:','
    linewidth',3);
hold on
plot(z_new(1,:),z_new(2,:)+z_new(3,:),'k','linewidth
    ',3);
hold on
xlabel('N');
ylabel('P_S+P_H')


set(gca, 'FontSize', 30);
set(gca, 'FontWeight', 'bold');



function dy = myfun(t,y)
muN = 1;
betaN = 1.6567;
delta = 0.6567/303000;;
k = 3.2e-5;
X = 0.11;
muP = 1/7;
betaP = 8.5127;
eta = 9.24;
epison =5*1e-3;
rho=X/epison;
r=1/epison;
```

```
dy = zeros(3,1);
dy(1) = (betaN-muN)*y(1)*(1-(delta*y(1))/(betaN-muN)
    )-k*y(1)*y(2);
dy(2) = -(muP+eta)*y(2)-rho*k*y(1)*y(2)+y(3)/epison;
dy(3) = -muP*y(3)+X/epison*k*y(1)*y(2)-y(3)*r+betaP
    *(y(2)+y(3));
hold on

function dz = myfun2(t,z)
muN = 1;
betaN = 1.6567;
delta = 0.6567/303000;
k = 3.2e-5;
X = 0.11;
muP = 1/7;
betaP = 8.5127;
eta = 9.24;
epison =1e-4;
rho=X/epison;
r=1/epison;
dz = zeros(3,1);
dz(1) = (betaN-muN)*z(1)*(1-(delta*z(1))/(betaN-muN)
    )-k*z(1)*z(2);
dz(2) = -(muP+eta)*z(2)-rho*k*z(1)*z(2)+z(3)/epison;
dz(3) = -muP*z(3)+X/epison*k*z(1)*z(2)-z(3)*r+betaP
    *(z(2)+z(3));
```

Chapter 7
Hopf bifurcation for a Holling's predator-prey model with handling and searching predators

This chapter provides an application of Hopf bifurcation, center manifold and normal form theories in the context of predator prey system.

7.1 Introduction

We continue to consider the system (6.1) of Chapter 16 and assume that the system (6.1) satisfies Assumptions 6.1 and 6.2. For model (6.1), we get that there exists a unique interior equilibrium $E^* = \left(N^*, P_S^*, P_H^*\right)$ if and only if (6.8) holds in Chapter 16. Note that (6.8) is equivalent to the following inequality

$$\beta_N - \mu_N > \frac{\delta\left[\gamma\eta - (\beta_P - \mu_P)(\mu_P + \eta + \gamma)\right]}{\kappa\rho\,(\beta_P - \mu_P)}. \tag{7.1}$$

The goal of this chapter is to apply the bifurcation theory, center manifold theory and normal form method to investigate the existence and properties of the Hopf bifurcation around the unique interior equilibrium E^* for the model (6.1). For convenience, we set the following assumption in order for the unique interior equilibrium E^* to exist.

Assumption 7.1 We assume that the inequality (6.8) or the inequality (7.1) holds.

We choose the parameter $\alpha := \beta_N - \mu_N$ as the Hopf bifurcation parameter and start with the linearization of (6.1) around the positive equilibrium E^* in the following.

7.2 Linearized System

First of all, we translate the unique interior equilibrium E^* of system (6.1) to the origin. Let $\overline{N} = N - N^*$, $\overline{P}_S = P_S - P_S^*$, $\overline{P}_H = P_H - P_H^*$. Then after dropping the hat, system (6.1) becomes

$$\frac{du(t)}{dt} = f(u(t)), \quad t \geq 0 \tag{7.2}$$

with $u = \begin{pmatrix} N \\ P_S \\ P_H \end{pmatrix} \in \mathbb{R}^3$, and

$$f(u(t)) = \begin{pmatrix} (\beta_N - \mu_N)(N + N^*) - \delta (N + N^*)^2 - \kappa (N + N^*) \left(P_S + P_S^*\right) \\ -(\mu_P + \eta)\left(P_S + P_S^*\right) - \rho\kappa \left(P_S + P_S^*\right)(N + N^*) + \gamma \left(P_H + P_H^*\right) \\ -(\mu_P + \gamma)\left(P_H + P_H^*\right) + \rho\kappa \left(P_S + P_S^*\right)(N + N^*) + \beta_P \left(P_S + P_S^* + P_H + P_H^*\right) \end{pmatrix}.$$

Linearization at the zero equilibrium of (7.2) yields

$$\frac{du(t)}{dt} = Au(t), \quad t \geq 0 \tag{7.3}$$

where A is the Jacobian matrix at the equilibrium E^* defined in section 16.5.1 of Chapter 16 and $u = \begin{pmatrix} N \\ P_S \\ P_H \end{pmatrix} \in \mathbb{R}^3$, and (7.2) can be written as

$$\frac{du(t)}{dt} = Au(t) + F(u(t)), \quad t \geq 0 \tag{7.4}$$

with

$$F(u(t)) = f(u(t)) - Au(t) \tag{7.5}$$

satisfying $F(0) = 0$ and $DF(0) = 0$. The characteristic equation is

$$\Delta(\lambda)_\alpha := \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \tag{7.6}$$

which is defined in section 16.5.1 of Chapter 16 and the stability of the equilibrium E^* is also obtained in lemma 6.12 of Chapter 16.

7.3 Existence of Hopf bifurcation

Under Assumptions 6.1, 6.2 and 7.1, we get $p_3 > 0$ and then $\lambda = 0$ is not a eigenvalue of (7.6). Let $\lambda = i\omega, \omega > 0$ be a pure imaginary root of (7.6). Then we have

$$-i\omega^3 - p_1\omega^2 + ip_2\omega + p_3 = 0.$$

Separating the real part and the imaginary part of the above equation, we get

$$p_2 = \omega^2, \ \frac{p_3}{p_1} = \omega^2.$$

Thus (7.6) has a pair of pure imaginary roots $\pm i\omega$ with $\omega = \sqrt{p_2}$ if and only if

$$p_2 > 0 \text{ and } p_3 = p_1 p_2.$$

By computing, we obtain that under the following Assumption 7.2, (7.6) has a pair of pure imaginary roots $\pm i\omega$ with $\omega = \sqrt{p_2}$ if and only if $\alpha = \alpha^\star$ with

$$\alpha^\star = \frac{\delta\left[2\gamma\eta - (\beta_P - \mu_P)(\beta_P + \eta + \gamma)\right]}{\kappa\rho\,(\beta_P - \mu_P)} + \frac{\delta\,(\beta_P - \mu_P)\left[(\beta_P - \mu_P)^2 - \gamma\eta\right]}{\gamma\kappa\rho\eta - \delta\left\{(\beta_P - \mu_P - \gamma)\ (\mu_P + \eta) + \gamma\beta_P\right\}}. \tag{7.7}$$

Assumption 7.2 Assume that $\gamma\eta - (\beta_P - \mu_P)^2 > 0$.

Let $\lambda_\pm(\alpha) := \sigma(\alpha) \pm i\nu(\alpha)$ with $\sigma(\alpha^\star) = 0, \nu(\alpha^\star) = \omega > 0$ are the eigenvalues of (7.6) in a neighborhood of $\alpha = \alpha^\star$. By computation, we get

$$\left.\frac{d\mathrm{Re}\,(\lambda_\pm(\alpha))}{d\alpha}\right|_{\alpha=\alpha^\star} = -\left.\frac{\left[\,(\beta_P - \mu_P - \gamma)(\mu_P + \eta) + \gamma\beta_P\right]\left(\frac{2p_1p_2}{(\beta_P - \mu_P)} + 2p_2\right)}{\left[p_2 - 3\omega^2\right]^2 + \left[2p_1\omega\right]^2}\right|_{\alpha=\alpha^\star} \neq 0$$

and then we derive the following Theorem.

Theorem 7.3 *Let Assumptions 6.1, 6.2, 7.1 and 7.2 be satisfied. Then the model (6.1) undergoes a Hopf bifurcation at the positive equilibrium E^* when $\alpha = \alpha^\star$, where $\alpha := \beta_N - \mu_N$ and $\alpha^\star$ is defined in (7.7).*

7.4 Computation of the Normal Form

In this section, we study the direction and stability of the Hopf bifurcation by applying the normal form theory developed in Chapter 15 to system (7.4). In order to apply the center manifold theory and normal form theory, we include the parameter α into the state variable of system (7.4) and consider the system

$$\begin{cases} \dfrac{d\alpha}{dt} = 0, \\ \dfrac{du}{dt} = A(\alpha)u + F(\alpha, u), \end{cases}$$

where

$$A(\alpha) = \begin{bmatrix} \alpha - 2\delta N^* - \kappa P_S^* & -N^*\kappa & 0 \\ -P_S^*\rho\kappa & -(\mu_P + \eta + \rho\kappa N^*) & \gamma \\ P_S^*\rho\kappa & \rho\kappa N^* + \beta_P & \beta_P - \mu_P - \gamma \end{bmatrix}, \tag{7.8}$$

and $F(\alpha, u) = f(\alpha, u) - A(\alpha)u$ with

$$f(\alpha,u) = \begin{pmatrix} \alpha\left(N+N^*\right)-\delta\left(N+N^*\right)^2-\kappa\left(N+N^*\right)\left(P_S+P_S^*\right) \\ -(\mu_P+\eta)\left(P_S+P_S^*\right)-\rho\kappa\left(P_S+P_S^*\right)\left(N+N^*\right)+\gamma\left(P_H+P_H^*\right) \\ -\left(\mu_P+\gamma\right)\left(P_H+P_H^*\right)+\rho\kappa\left(P_S+P_S^*\right)\left(N+N^*\right)+\beta_P\left(P_S+P_S^*+P_H+P_H^*\right) \end{pmatrix}.$$

Making the change of variables

$$\alpha=\overline{\alpha}+\alpha^\star,$$

we obtain the system after dropping the hat

$$\begin{cases} \dfrac{d\alpha}{dt}=0, \\ \dfrac{du}{dt}=A(\alpha+\alpha^\star)u+F(\alpha+\alpha^\star,u). \end{cases}$$

Separating the linear and nonlinear part, we have

$$\begin{cases} \dfrac{d\alpha}{dt}=0, \\ \dfrac{du}{dt}=A(\alpha^\star)u+W(\alpha,u), \end{cases} \tag{7.9}$$

with $W(\alpha,u)=F(\alpha+\alpha^\star,u)+A(\alpha+\alpha^\star)u-A(\alpha^\star)u$.

Consider the linear operator $\mathcal{A}:\mathbb{R}^4\longrightarrow\mathbb{R}^4$ defined by

$$\mathcal{A}\begin{pmatrix}\alpha\\u\end{pmatrix}=\begin{pmatrix}0\\A(\alpha^\star)u\end{pmatrix}$$

and the map $\mathcal{H}:\mathbb{R}^4\longrightarrow\mathbb{R}^4$ defined by

$$\mathcal{H}\begin{pmatrix}\alpha\\u\end{pmatrix}=\begin{pmatrix}0\\W(\alpha,u)\end{pmatrix}.$$

Then we have

$$\mathcal{H}\begin{pmatrix}0\\0\end{pmatrix}=0 \text{ and } D\mathcal{H}\begin{pmatrix}0\\0\end{pmatrix}=0.$$

Now set $w=\begin{pmatrix}\alpha\\u\end{pmatrix}$ and we can reformulate system (7.9) as the following system

$$\frac{dw(t)}{dt}=\mathcal{A}w(t)+\mathcal{H}\left(w(t)\right). \tag{7.10}$$

7.4.1 Projectors on the Eigenspaces

In order to use the center manifold theory and change of variables to compute the Taylor's expansion of the reduced system of a system topologically equivalent to the original system (7.10), we need to compute the projectors on the generalized eigenspaces associated with eigenvalues of $\mathcal{A}$. Note that

$$(\lambda I - A(\alpha^\star))^{-1} = \frac{\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}_{\alpha^\star,\,\lambda}}{|\lambda I - A(\alpha^\star)|} = \frac{\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}_{\alpha^\star,\,\lambda}}{\Delta(\lambda)_{\alpha^\star}}, \tag{7.11}$$

where $\Delta(\lambda)_{\alpha^\star}$ is defined in (7.6), and $\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}_{\alpha^\star,\,\lambda}$ is the adjoint matrix of $(\lambda I - A(\alpha^\star))$ and A_{ij} is the cofactor of the element in the ith row and jth column of $(\lambda I - A(\alpha^\star))$, $i, j = 1, 2, 3$. Furthermore the resolvent of $A(\alpha^\star)$ has the following Laurent's expansion around $\widehat{\lambda} \in \sigma(A(\alpha^\star))$ with $\widehat{\lambda} = i\omega$ or $-i\omega$:

$$(\lambda I - A(\alpha^\star))^{-1} = \sum_{n=-1}^{+\infty} (\lambda - \widehat{\lambda})^n B_n^{\widehat{\lambda}}.$$

The projector $\Pi_{\widehat{\lambda}}^{A(\alpha^\star)}$ on the generalized eigenspace associated with $\widehat{\lambda} = i\omega$ or $-i\omega$ is $B_{-1}^{\widehat{\lambda}}$. Since

$$(\lambda - \widehat{\lambda})(\lambda I - A(\alpha^\star))^{-1} = \sum_{m=0}^{+\infty} (\lambda - \widehat{\lambda})^m B_{m-1}^{\widehat{\lambda}}$$

and

$$\lim_{\lambda \to \widehat{\lambda}} \frac{(\lambda - \widehat{\lambda})}{\Delta(\lambda)_{\alpha^\star}} = \lim_{\lambda \to \widehat{\lambda}} \frac{1}{\Delta'(\lambda)_{\alpha^\star}} = \frac{1}{\Delta'(\widehat{\lambda})_{\alpha^\star}},$$

$$\begin{aligned} \Pi_{\widehat{\lambda}}^{A(\alpha^\star)} &= B_{-1}^{\widehat{\lambda}} = \lim_{\lambda \to \widehat{\lambda}} (\lambda - \widehat{\lambda})(\lambda I - A(\alpha^\star))^{-1} \\ &= \lim_{\lambda \to \widehat{\lambda}} \left(\frac{(\lambda - \widehat{\lambda})}{\Delta(\lambda)_{\alpha^\star}} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}_{\alpha^\star,\,\lambda} \right) = \frac{\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}_{\alpha^\star,\,\widehat{\lambda}}}{\Delta'(\widehat{\lambda})_{\alpha^\star}}. \end{aligned}$$

Then

$$\Pi_{i\omega}^{A(\alpha^\star)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} A_{31} \\ A_{32} \\ A_{33} \end{pmatrix}_{\alpha^\star,\, i\omega}}{\Delta'(i\omega)_{\alpha^\star}}.$$

Define $\Pi_c^{A(\alpha^\star)} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Pi_c^{A(\alpha^\star)}\begin{pmatrix}N\\P_S\\P_H\end{pmatrix}=\Pi_{i\omega}^{A(\alpha^\star)}\begin{pmatrix}N\\P_S\\P_H\end{pmatrix}+\Pi_{-i\omega}^{A(\alpha^\star)}\begin{pmatrix}N\\P_S\\P_H\end{pmatrix},\ \ \forall\begin{pmatrix}N\\P_S\\P_H\end{pmatrix}\in\mathbb{R}^3.$$

Set

$$\Pi_h^{A(\alpha^\star)}:=I-\Pi_c^{A(\alpha^\star)}.$$

Denote

$$X_c:=\Pi_c^{A(\alpha^\star)}\left(\mathbb{R}^3\right),\ \ X_h:=\Pi_h^{A(\alpha^\star)}\left(\mathbb{R}^3\right),$$

and

$$A_c:=A(\alpha^\star)\,|_{X_c},\ A_h:=A(\alpha^\star)\,|_{X_h}\ .$$

Let

$$b_1=\begin{pmatrix}A_{31}\\A_{32}\\A_{33}\end{pmatrix}_{\alpha^\star,\,i\omega},\ b_2=\overline{b_1}=\begin{pmatrix}A_{31}\\A_{32}\\A_{33}\end{pmatrix}_{\alpha^\star,\,-i\omega}. \tag{7.12}$$

It is obvious that (b_1,b_2) is a basis of X_c. In the following, we set

$$e_1=\begin{pmatrix}1\\0\\0\end{pmatrix},e_2=\begin{pmatrix}0\\1\\0\end{pmatrix},e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Lemma 7.4 *Let Assumptions 6.1, 6.2, 7.1 and 7.2 be satisfied. Then for each $\lambda\in i\mathbb{R}\setminus\{-i\omega,i\omega\}$, we have*

$$\left(\lambda I-A(\alpha^\star)\big|_{\Pi_h^{A(\alpha^\star)}\left(\mathbb{R}^3\right)}\right)^{-1}\Pi_h^{A(\alpha^\star)}e_k=\Phi_{k,\lambda}$$

with

$$\Phi_{k,\lambda}=\begin{pmatrix}\Phi_{k,\lambda}^1\\\Phi_{k,\lambda}^2\\\Phi_{k,\lambda}^3\end{pmatrix}=\frac{\begin{pmatrix}A_{k1}\\A_{k2}\\A_{k3}\end{pmatrix}_{\alpha^\star,\,\lambda}}{\Delta(\lambda)_{\alpha^\star}}-\frac{\begin{pmatrix}A_{k1}\\A_{k2}\\A_{k3}\end{pmatrix}_{\alpha^\star,\,i\omega}}{(\lambda-i\omega)\,\Delta'(i\omega)_{\alpha^\star}}-\frac{\begin{pmatrix}A_{k1}\\A_{k2}\\A_{k3}\end{pmatrix}_{\alpha^\star,\,-i\omega}}{(\lambda+i\omega)\,\Delta'(-i\omega)_{\alpha^\star}},$$
$$k=1,2,3.$$

For $\lambda=i\omega$,

$$\left(i\omega I-A(\alpha^\star)\big|_{\Pi_h^{A(\alpha^\star)}\left(\mathbb{R}^3\right)}\right)^{-1}\Pi_h^{A(\alpha^\star)}e_k=\Phi_{k,i\omega}$$

with

$$\Phi_{k,i\omega} = \begin{pmatrix} \Phi^1_{k,i\omega} \\ \Phi^2_{k,i\omega} \\ \Phi^3_{k,i\omega} \end{pmatrix} = \frac{1}{2i\omega\left(\Delta'(i\omega)_{\alpha^\star}\right)^2 \Delta'(-i\omega)_{\alpha^\star}} \times$$
$$\left(2i\omega\Delta'(i\omega)_{\alpha^\star}\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} \frac{dA_{k1}}{d\lambda} \\ \frac{dA_{k2}}{d\lambda} \\ \frac{dA_{k3}}{d\lambda} \end{pmatrix}_{\alpha^\star,\, i\omega} - i\omega\Delta''(i\omega)_{\alpha^\star}\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} A_{k1} \\ A_{k2} \\ A_{k3} \end{pmatrix}_{\alpha^\star,\, i\omega} \right.$$
$$\left. - \left(\Delta'(i\omega)_{\alpha^\star}\right)^2 \begin{pmatrix} A_{k1} \\ A_{k2} \\ A_{k3} \end{pmatrix}_{\alpha^\star,\, -i\omega} \right),$$
$$k = 1, 2, 3.$$

For $\lambda = -i\omega$,

$$\left(-i\omega I - A(\alpha^\star)\big|_{\Pi_h^{A(\alpha^\star)}(\mathbb{R}^3)}\right)^{-1} \Pi_h^{A(\alpha^\star)} e_k = \Phi_{k,-i\omega}$$

with

$$\Phi_{k,-i\omega} = \begin{pmatrix} \Phi^1_{k,-i\omega} \\ \Phi^2_{k,-i\omega} \\ \Phi^3_{k,-i\omega} \end{pmatrix} = \frac{1}{-2i\omega\left(\Delta'(-i\omega)\right)^2_{\alpha^\star} \Delta'(i\omega)_{\alpha^\star}} \times$$
$$\left(-2i\omega\Delta'(i\omega)_{\alpha^\star}\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} \frac{dA_{k1}}{d\lambda} \\ \frac{dA_{k2}}{d\lambda} \\ \frac{dA_{k3}}{d\lambda} \end{pmatrix}_{\alpha^\star,\, -i\omega} \right.$$
$$+ i\omega\Delta''(-i\omega)_{\alpha^\star}\Delta'(i\omega)_{\alpha^\star} \begin{pmatrix} A_{k1} \\ A_{k2} \\ A_{k3} \end{pmatrix}_{\alpha^\star,\, -i\omega}$$
$$\left. - \left(\Delta'(-i\omega)_{\alpha^\star}\right)^2 \begin{pmatrix} A_{k1} \\ A_{k2} \\ A_{k3} \end{pmatrix}_{\alpha^\star,\, i\omega} \right), k = 1, 2, 3.$$

Proof Since

$$\left(\lambda I - A(\alpha^\star)\right)^{-1} \left(\frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\, i\omega}}{\Delta'(i\omega)_{\alpha^\star}} \right) = \frac{1}{\lambda - i\omega} \left(\frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\, i\omega}}{\Delta'(i\omega)_{\alpha^\star}} \right)$$

and

$$\left(\lambda I - A(\alpha^\star)\right)^{-1} \left(\frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\, -i\omega}}{\Delta'(-i\omega)_{\alpha^\star}} \right) = \frac{1}{\lambda + i\omega} \left(\frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\, -i\omega}}{\Delta'(-i\omega)_{\alpha^\star}} \right),$$

for each $\lambda \in i\mathbb{R} \setminus \{-i\omega, i\omega\}$, we have

$$\begin{aligned} &\left(\lambda I - A(\alpha^\star)\big|_{\Pi_h^{A(\alpha^\star)}(\mathbb{R}^3)}\right)^{-1} \Pi_h^{A(\alpha^\star)} e_1 \\ &= \left(\lambda I - A(\alpha^\star)\right)^{-1} \left(e_1 - \Pi_c^{A(\alpha^\star)} e_1\right) \\ &= \frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\, \lambda}}{\Delta(\lambda)_{\alpha^\star}} - \frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\, i\omega}}{(\lambda - i\omega)\, \Delta'(i\omega)_{\alpha^\star}} - \frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\, -i\omega}}{(\lambda + i\omega)\, \Delta'(-i\omega)_{\alpha^\star}}. \end{aligned}$$

For $\lambda = i\omega$,

$$\left(i\omega I - A(\alpha^\star)\big|_{\Pi_h^{A(\alpha^\star)}(\mathbb{R}^3)}\right)^{-1} \Pi_h^{A(\alpha^\star)} e_1$$

$$= \lim_{\substack{\lambda\to i\omega \\ \lambda\in\rho(A(\alpha^\star))}} \left(\lambda I - A(\alpha^\star)\big|_{\Pi_h^{A(\alpha^\star)}(\mathbb{R}^3)}\right)^{-1} \Pi_h^{A(\alpha^\star)} e_1$$

$$= \lim_{\substack{\lambda\to i\omega \\ \lambda\in i\mathbb{R}\setminus\{-i\omega,i\omega\}}} \left(\frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,\lambda}}{\Delta(\lambda)} - \frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,i\omega}}{(\lambda - i\omega)\,\Delta'(i\omega)_{\alpha^\star}} - \frac{\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,-i\omega}}{(\lambda + i\omega)\,\Delta'(-i\omega)_{\alpha^\star}} \right)$$

$$= \lim_{\substack{\lambda\to i\omega \\ \lambda\in i\mathbb{R}\setminus\{-i\omega,i\omega\}}} \left\{ \frac{1}{\Delta(\lambda)\,(\lambda - i\omega)\,(\lambda + i\omega)\,\Delta'(-i\omega)_{\alpha^\star}\Delta'(i\omega)_{\alpha^\star}} \times \right.$$

$$\left((\lambda - i\omega)\,(\lambda + i\omega)\,\Delta'(i\omega)_{\alpha^\star}\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,\lambda} \right.$$

$$\left. \left. -\Delta(\lambda)\,(\lambda + i\omega)\,\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,i\omega} - \Delta(\lambda)\,(\lambda - i\omega)\,\Delta'(i\omega)_{\alpha^\star} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,-i\omega} \right) \right\}$$

$$= \lim_{\substack{\lambda\to i\omega \\ \lambda\in i\mathbb{R}\setminus\{-i\omega,i\omega\}}} \frac{(\lambda - i\omega)^2}{\Delta(\lambda)\,(\lambda - i\omega)\,(\lambda + i\omega)\,\Delta'(-i\omega)_{\alpha^\star}\Delta'(i\omega)_{\alpha^\star}} \times$$

$$\lim_{\substack{\lambda\to i\omega \\ \lambda\in i\mathbb{R}\setminus\{-i\omega,i\omega\}}} \left\{ \frac{1}{(\lambda - i\omega)^2} \left((\lambda - i\omega)\,(\lambda + i\omega)\,\Delta'(i\omega)_{\alpha^\star}\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,\lambda} \right. \right.$$

$$\left. \left. -\Delta(\lambda)\,(\lambda + i\omega)\,\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,i\omega} \right) \right\}$$

$$- \lim_{\substack{\lambda\to i\omega \\ \lambda\in i\mathbb{R}\setminus\{-i\omega,i\omega\}}} \left(\Delta(\lambda)\,(\lambda - i\omega)\,\Delta'(i\omega)_{\alpha^\star} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,-i\omega} \right)$$

$$= \frac{1}{2i\omega\,(\Delta'(i\omega)_{\alpha^\star})^2\,\Delta'(-i\omega)_{\alpha^\star}} \times$$

$$\left(2i\omega\Delta'(i\omega)_{\alpha^\star}\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} \frac{dA_{11}}{d\lambda} \\ \frac{dA_{12}}{d\lambda} \\ \frac{dA_{13}}{d\lambda} \end{pmatrix}_{\alpha^\star,\,i\omega} - i\omega\Delta''(i\omega)_{\alpha^\star}\Delta'(-i\omega)_{\alpha^\star} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,i\omega} \right.$$

$$\left. - (\Delta'(i\omega)_{\alpha^\star})^2 \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix}_{\alpha^\star,\,-i\omega} \right).$$

Similarly we can prove the other results. □

It is easy to prove the following results.

Lemma 7.5 *Let Assumptions 6.1, 6.2, 7.1 and 7.2 be satisfied. We have*

$$\sigma(\mathcal{A}) = \sigma\left(A(\alpha^{\star})\right) \cup \{0\}.$$

Moreover, for $\lambda \in \rho(\mathcal{A}) = \mathbb{C} \setminus (\sigma(A(\alpha^{\star})) \cup \{0\})$, *we have*

$$(\lambda I - \mathcal{A})^{-1}\begin{pmatrix}\alpha\\ u\end{pmatrix} = \begin{pmatrix}\dfrac{\alpha}{\lambda}\\ (\lambda I - A(\alpha^{\star}))^{-1} u\end{pmatrix}$$

and the eigenvalues 0 *and* $\pm i\omega$ *of* $\mathcal{A}$ *are simple. The corresponding projectors* $\Pi_0, \Pi_{\pm i\omega} : \mathbb{R}^4 \to \mathbb{R}^4$ *on the generalized eigenspace of* $\mathcal{A}$ *associated to* $0, \pm i\omega$, *respectively, are given by*

$$\Pi_0\begin{pmatrix}r\\ u\end{pmatrix} = \begin{pmatrix}r\\ 0\end{pmatrix},$$
$$\Pi_{\pm i\omega}\begin{pmatrix}r\\ u\end{pmatrix} = \begin{pmatrix}0\\ \Pi_{\pm i\omega}^{A(\alpha^{\star})} u\end{pmatrix}.$$

In this context, the projectors $\Pi_c : \mathbb{R}^4 \to \mathbb{R}^4$ and $\Pi_h : \mathbb{R}^4 \to \mathbb{R}^4$ are defined by

$$\Pi_c(y) = (\Pi_0 + \Pi_{i\omega} + \Pi_{-i\omega})(y), \quad \forall y \in \mathbb{R}^4,$$
$$\Pi_h(y) = (I - \Pi_c)(y), \quad \forall y \in \mathbb{R}^4.$$

Denote

$$Y_c := \Pi_c\left(\mathbb{R}^4\right), \quad Y_h := \Pi_h\left(\mathbb{R}^4\right),$$

and

$$\mathcal{A}_c := \mathcal{A}\mid_{Y_c}, \ \mathcal{A}_h := \mathcal{A}\mid_{Y_h}.$$

Then we have

$$\Pi_c\begin{pmatrix}0_{\mathbb{R}}\\ e_k\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}}\\ \Pi_c^{A(\alpha^{\star})} e_k\end{pmatrix}, k = 1, 2, 3.$$

Define the basis of $Y_c = \Pi_c\left(\mathbb{R}^4\right)$ by

$$\widehat{b}_1 = \begin{pmatrix}1\\ 0_{\mathbb{R}^3}\end{pmatrix}, \ \widehat{b}_2 = \begin{pmatrix}0_{\mathbb{R}}\\ b_1\end{pmatrix}, \ \widehat{b}_3 = \begin{pmatrix}0_{\mathbb{R}}\\ b_2\end{pmatrix}.$$

We have the following lemma.

Lemma 7.6 *Let Assumptions 6.1, 6.2, 7.1 and 7.2 be satisfied. For* $\lambda \in i\mathbb{R}$ *we have*

$$(\lambda I - \mathcal{A}_h)^{-1} \Pi_h \begin{pmatrix} 0 \\ e_k \end{pmatrix} = \begin{pmatrix} 0 \\ \left(\lambda I - A(\alpha^\star)|_{\Pi_h^{A(\alpha^\star)}(\mathbb{R}^3)}\right)^{-1} \Pi_h^{A(\alpha^\star)} e_k \end{pmatrix}, \ k = 1, 2, 3.$$

7.4.2 Change of Variables

In the following, we will compute the Taylor expansion of the reduced system of a system topologically equivalent to the original system (7.10). We apply the procedure described in section 15.3.2 of Chapter 15 and apply the method with $k = 2$ in Theorem 5.22 of Chapter 15. Therefore we must find $L_2 \in \mathcal{L}_s\left(Y_c^2, Y_h\right)$ by solving the following equation for each $(w_1, w_2) \in Y_c^2$:

$$\frac{d}{dt}\left[L_2(e^{\mathcal{A}_c t} w_1, e^{\mathcal{A}_c t} w_2)\right](0) = \mathcal{A}_h L_2(w_1, w_2) + \frac{1}{2!}\Pi_h D^2 \mathcal{H}(0)(w_1, w_2). \quad (7.13)$$

Then define $G_2 \in V^2(Y_c, Y_h)$ by

$$G_2(\Pi_c w) := L_2(\Pi_c w, \Pi_c w), \quad \forall w \in \mathbb{R}^4,$$

and the change of variables $\varsigma_2 : \mathbb{R}^4 \to \mathbb{R}^4$ and $\varsigma_2^{-1} : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\varsigma_2(w) := w + G_2(\Pi_c w) \text{ and } \varsigma_2^{-1}(w) := w - G_2(\Pi_c w), \quad \forall w \in \mathbb{R}^4.$$

By applying Theorem 5.22 to (7.10) with $k = 2$, we obtain the following theorem.

Theorem 7.7 *Let Assumptions 6.1, 6.2, 7.1 and 7.2 be satisfied. By using the change of variables*

$$w_2(t) = \varsigma_2^{-1}(w(t)) = w(t) - G_2(\Pi_c w(t)) \Leftrightarrow w(t) = \varsigma_2(w_2(t)) = w_2(t) + G_2(\Pi_c w_2(t)),$$

the map $t \to w(t)$ is an integrated solution of the system (7.10) if and only if $t \to w_2(t)$ is an integrated solution of the system

$$\frac{dw_2(t)}{dt} = \mathcal{A} w_2(t) + \mathcal{H}_2(w_2(t)), \quad t \geq 0, \quad (7.14)$$

where $\mathcal{H}_2 : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$\begin{aligned} \mathcal{H}_2(w) = \mathcal{H}(\varsigma_2(w)) + \mathcal{A} G_2(\Pi_c w) - D G_2(\Pi_c w) \mathcal{A}_c \Pi_c w \\ - D G_2(\Pi_c w) \Pi_c \mathcal{H}(\varsigma_2(w)). \end{aligned}$$

Moreover, the reduced system of the system (7.14) is given by the ordinary differential equations on $\mathbb{R} \times X_c$:

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = 0, \\ \dfrac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c^{A(\alpha^\star)} W (I + G_2) \begin{pmatrix} \alpha(t) \\ x_c(t) \end{pmatrix} + R_c \begin{pmatrix} \alpha(t) \\ x_c(t) \end{pmatrix}, \end{cases} \quad (7.15)$$

where $R_c \in C^4(\mathbb{R} \times X_c, X_c)$, *and* $R_c\begin{pmatrix} \alpha(t) \\ x_c(t) \end{pmatrix}$ *is a remainder of order* 4, *that is,*

$$R_c\begin{pmatrix} \alpha \\ x_c \end{pmatrix} = \|(\alpha, x_c)\|^4 O(\alpha, x_c),$$

where $O(\alpha, x_c)$ *is a function of* (α, x_c) *which remains bounded when* (α, x_c) *goes to* 0, *or equivalently,*

$$D^j R_c(0) = 0 \textit{ for each } j = 1, \dots, 3.$$

Furthermore,

$$\frac{\partial^j R_c(0)}{\partial^j \alpha} = 0, \quad \forall j = 1, \dots, 4,$$

which implies that

$$R_c\begin{pmatrix} \alpha \\ x_c \end{pmatrix} = O\left(\alpha^3 \|x_c\| + \alpha^2 \|x_c\|^2 + \alpha \|x_c\|^3 + \|x_c\|^4\right).$$

Proof We apply Theorem 5.22 to (7.10) and deduce that there exists $\Psi_2 \in C^2(Y_c, Y_h)$ such that the reduced system of (7.14) consists of ordinary differential equations on Y_c of the form

$$\frac{dw_c(t)}{dt} = \mathcal{A}_c w_c(t) + \Pi_c \mathcal{H}[w_c(t) + G_2(w_c(t))] + \widehat{R}_c(w_c(t)),$$

where $\widehat{R}_c \in C^4(Y_c, Y_c)$ is the remainder term of order 4 and

$$\widehat{R}_c(w_c) := \Pi_c\{\mathcal{H}[w_c + G_2(w_c) + \Psi_2(w_c)] - \mathcal{H}[w_c + G_2(w_c)]\},$$

with $D^j \widehat{R}_c(0) = 0, \forall j = 1, 2, 3$. Since the first component of $\mathcal{H}$ is 0, by using the formula for Π_c we deduce that $\widehat{R}_c(w_c)$ has the following form

$$\widehat{R}_c\begin{pmatrix} \alpha \\ x_c \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ R_c\begin{pmatrix} \alpha \\ x_c \end{pmatrix} \end{pmatrix}.$$

Moreover, since for each $\alpha \in \mathbb{R}$ small enough, $\begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix} \in Y_c$ is an equilibrium solution of (7.10) and belongs to the center manifold. It follows that

$$\begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix} = \Pi_c\begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix} + \Psi_2\left(\Pi_c\begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix}\right).$$

Thus $\Pi_c\begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix} = \begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix}$ and

$$\Psi_2\begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix} = 0, \quad \forall \alpha \in \mathbb{R} \text{ small enough.}$$

So we must have

$$\widehat{R}_c \begin{pmatrix} \alpha \\ 0_{\mathbb{R}_3} \end{pmatrix} = 0, \quad \forall \alpha \in \mathbb{R} \text{ small enough.}$$

We deduce that

$$\frac{\partial^j \widehat{R}_c\,(0)}{\partial^j \alpha} = 0, \forall j = 1, ..., 4.$$

This completes the proof. □

In order to apply the above theorem to compute the Taylor expansion of the reduced system it only remains to compute L_2.

Set

$$w_1 := \begin{pmatrix} \alpha_1 \\ u_1 \end{pmatrix}, \; w_2 := \begin{pmatrix} \alpha_2 \\ u_2 \end{pmatrix}, \; w_3 := \begin{pmatrix} \alpha_3 \\ u_3 \end{pmatrix} \in \mathbb{R}^4$$

with $u_i = \begin{pmatrix} N_i \\ P_{S,i} \\ P_{H,i} \end{pmatrix}, i = 1, 2, 3$. We have

$$D^2\mathcal{H}\,(0)\,(w_1, w_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ D^2W\,(0)\,(w_1, w_2) \end{pmatrix}$$

and

$$D^3\mathcal{H}\,(0)\,(w_1, w_2, w_3) = \begin{pmatrix} 0_{\mathbb{R}} \\ D^3W\,(0)\,(w_1, w_2, w_3) \end{pmatrix},$$

where

$$\begin{aligned} &D^2W\,(0)\,(w_1, w_2) \\ &= \begin{pmatrix} \alpha_1 N_2 + \alpha_2 N_1 - 2\delta N_2 N_1 - \kappa N_1 P_{S,2} - \kappa N_2 P_{S,1} \\ -\rho\,\kappa P_{S,1} N_2 - \rho\,\kappa P_{S,2} N_1 \\ \rho\,\kappa P_{S,1} N_2 + \rho\,\kappa P_{S,2} N_1 \end{pmatrix} \end{aligned}$$

and

$$D^3W\,(0)\,(w_1, w_2, w_3) = 0.$$

Next recall that

$$\frac{d}{dt}\left[L_2(e^{\mathcal{A}_c t} w_1, e^{\mathcal{A}_c t} w_2)\right](0) = L_2\,(\mathcal{A}_c w_1, w_2) + L_2\,(w_1, \mathcal{A}_c w_2)\,.$$

So system (7.13) can be rewritten as

$$L_2\,(\mathcal{A}_c w_1, w_2) + L_2\,(w_1, \mathcal{A}_c w_2) = \mathcal{A}_h L_2(w_1, w_2) + \frac{1}{2!}\Pi_h D^2\mathcal{H}\,(0)\,(w_1, w_2).$$

Note that $L_2(\widehat{b}_1, \widehat{b}_2) = L_2(\widehat{b}_2, \widehat{b}_1)$, $L_2(\widehat{b}_1, \widehat{b}_3) = L_2(\widehat{b}_3, \widehat{b}_1)$, $L_2(\widehat{b}_2, \widehat{b}_3) = L_2(\widehat{b}_3, \widehat{b}_2)$.

(i) Computation of $L_2(\widehat{b}_1, \widehat{b}_1)$. We have

$$\Pi_h D^2\mathcal{H}\,(0)\,(\widehat{b}_1, \widehat{b}_1) = 0, \mathcal{A}_c \widehat{b}_1 = 0.$$

So the equation

$$L_2\left(\mathcal{A}_c\widehat{b}_1,\widehat{b}_1\right)+L_2\left(\widehat{b}_1,\mathcal{A}_c\widehat{b}_1\right)=\mathcal{A}_hL_2(\widehat{b}_1,\widehat{b}_1)+\frac{1}{2!}\Pi_hD^2\mathcal{H}\left(0\right)(\widehat{b}_1,\widehat{b}_1)$$

is equivalent to

$$0=\mathcal{A}_hL_2(\widehat{b}_1,\widehat{b}_1).$$

Since 0 belongs to the resolvent set of $\mathcal{A}_h$, we obtain that

$$L_2(\widehat{b}_1,\widehat{b}_1)=0. \tag{7.16}$$

(ii) Computation of $L_2(\widehat{b}_1,\widehat{b}_2)$. We have

$$\frac{1}{2!}D^2\mathcal{H}\left(0\right)(\widehat{b}_1,\widehat{b}_2)=\frac{1}{2}\begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega}\\0\\0\end{pmatrix}\end{pmatrix}$$

and

$$\mathcal{A}_c\widehat{b}_1=0,\mathcal{A}_c\widehat{b}_2=i\omega\widehat{b}_2.$$

So in this case system (7.13) becomes

$$i\omega L_2\left(\widehat{b}_1,\widehat{b}_2\right)=\mathcal{A}_hL_2(\widehat{b}_1,\widehat{b}_2)+\frac{1}{2!}\Pi_hD^2\mathcal{H}\left(0\right)(\widehat{b}_1,\widehat{b}_2).$$

Now by using Lemmas 7.4 and 7.6, we obtain

$$\begin{aligned}L_2(\widehat{b}_1,\widehat{b}_2)&=\frac{1}{2}\left(i\omega I-\mathcal{A}_h\right)^{-1}\Pi_hD^2\mathcal{H}\left(0\right)(\widehat{b}_1,\widehat{b}_2)\\&=\frac{1}{2}\left(i\omega I-\mathcal{A}_h\right)^{-1}\Pi_h\left(\left(\begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega}\\0\\0\end{pmatrix}\end{pmatrix}\right)\right)\\&=\begin{pmatrix}0\\ \frac{1}{2}\left(A_{31}\right)_{\alpha^\star,\,i\omega}\Phi_{1,i\omega}\end{pmatrix},\end{aligned}$$

where $\Phi_{1,i\omega}$ is defined in Lemma 7.4.

(iii) Computation of $L_2(\widehat{b}_1,\widehat{b}_3)$. We have

$$\frac{1}{2!}D^2\mathcal{H}\left(0\right)(\widehat{b}_1,\widehat{b}_3)=\frac{1}{2!}\begin{pmatrix}0\\ \begin{pmatrix}(A_{31})_{\alpha^\star,\,-i\omega}\\0\\0\end{pmatrix}\end{pmatrix}$$

and

$$\mathcal{A}_c\widehat{b}_1=0,\mathcal{A}_c\widehat{b}_3=-i\omega\widehat{b}_3.$$

So in this case system (7.13) becomes

$$-i\omega L_2\left(\widehat{b}_1, \widehat{b}_3\right) = \mathcal{A}_h L_2(\widehat{b}_1, \widehat{b}_3) + \frac{1}{2!}\Pi_h D^2\mathcal{H}(0)(\widehat{b}_1, \widehat{b}_3).$$

Now by using Lemmas 7.4 and 7.6, we obtain

$$L_2(\widehat{b}_1, \widehat{b}_3) = \frac{1}{2!}(-i\omega I - \mathcal{A}_h)^{-1}\Pi_h D^2\mathcal{H}(0)(\widehat{b}_1, \widehat{b}_3) = \begin{pmatrix} 0 \\ \frac{1}{2}(A_{31})_{\alpha^\star,-i\omega}\Phi_{1,-i\omega} \end{pmatrix},$$

where $\Phi_{1,-i\omega}$ is defined in Lemma 7.4.

(iv) Computation of $L_2(\widehat{b}_2, \widehat{b}_2)$. We have

$$\frac{1}{2}D^2\mathcal{H}(0)(\widehat{b}_2, \widehat{b}_2) = \begin{pmatrix} 0 \\ \begin{pmatrix} -\delta(A_{31})^2_{\alpha^\star,i\omega} - \kappa(A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,i\omega} \\ -\rho\kappa(A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,i\omega} \\ \rho\kappa(A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,i\omega} \end{pmatrix} \end{pmatrix}$$

and

$$\mathcal{A}_c\widehat{b}_2 = i\omega\widehat{b}_2.$$

In this case system (7.13) becomes

$$2i\omega L_2(\widehat{b}_2, \widehat{b}_2) = \mathcal{A}_h L_2(\widehat{b}_2, \widehat{b}_2) + \frac{1}{2!}\Pi_h D^2\mathcal{H}(0)(\widehat{b}_2, \widehat{b}_2).$$

and Lemmas 7.4 and 7.6 imply

$$L_2(\widehat{b}_2, \widehat{b}_2) = \frac{1}{2}(2i\omega I - \mathcal{A}_h)^{-1}\Pi_h D^2\mathcal{H}(0)(\widehat{b}_2, \widehat{b}_2) = \begin{pmatrix} 0 \\ \frac{1}{2}F_1 \end{pmatrix}, \tag{7.17}$$

with

$$F_1 = \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \end{pmatrix} = 2\left(-\delta(A_{31})^2_{\alpha^\star,i\omega} - \kappa(A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,i\omega}\right)\Phi_{1,2i\omega}$$
$$+2\rho\kappa(A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,i\omega}\left(\Phi_{3,2i\omega} - \Phi_{2,2i\omega}\right),$$

where $\Phi_{j,2i\omega}$, $j = 1, 2, 3$, are defined in Lemma 7.4.

(v) Computation of $L_2(\widehat{b}_2, \widehat{b}_3)$. We have

$$D^2\mathcal{H}(0)(\widehat{b}_2, \widehat{b}_3)$$
$$= \begin{pmatrix} 0 \\ \begin{pmatrix} -2\delta(A_{31})_{\alpha^\star,i\omega}(A_{31})_{\alpha^\star,-i\omega} - \kappa(A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,-i\omega} - \kappa(A_{32})_{\alpha^\star,i\omega}(A_{31})_{\alpha^\star,-i\omega} \\ -\rho\kappa\left((A_{32})_{\alpha^\star,i\omega}(A_{31})_{\alpha^\star,-i\omega} + (A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,-i\omega}\right) \\ \rho\kappa\left((A_{32})_{\alpha^\star,i\omega}(A_{31})_{\alpha^\star,-i\omega} + (A_{31})_{\alpha^\star,i\omega}(A_{32})_{\alpha^\star,-i\omega}\right) \end{pmatrix} \end{pmatrix}$$

and

$$\mathcal{A}_c\widehat{b}_2 = i\omega\widehat{b}_2, \ \mathcal{A}_c\widehat{b}_3 = -i\omega\widehat{b}_3.$$

In this case system (7.13) reduces to

$$L_2\left(\mathcal{A}_c\widehat{b}_2,\widehat{b}_3\right)+L_2\left(\widehat{b}_2,\mathcal{A}_c\widehat{b}_3\right)=\mathcal{A}_hL_2(\widehat{b}_2,\widehat{b}_3)+\frac{1}{2!}\Pi_hD^2\mathcal{H}\left(0\right)\left(\widehat{b}_2,\widehat{b}_3\right)$$

and then

$$0=\mathcal{A}_hL_2(\widehat{b}_2,\widehat{b}_3)+\frac{1}{2!}\Pi_hD^2\mathcal{H}\left(0\right)\left(\widehat{b}_2,\widehat{b}_3\right).$$

Thus

$$-\mathcal{A}_hL_2(\widehat{b}_2,\widehat{b}_3)=\frac{1}{2!}\Pi_hD^2\mathcal{H}\left(0\right)\left(\widehat{b}_2,\widehat{b}_3\right),$$

and by Lemmas 7.4 and 7.6, we have

$$L_2(\widehat{b}_2,\widehat{b}_3)=\frac{1}{2}\left(-\mathcal{A}_h\right)^{-1}\Pi_hD^2\mathcal{H}\left(0\right)\left(\widehat{b}_2,\widehat{b}_3\right)=\begin{pmatrix}0\\ \frac{1}{2}F_2\end{pmatrix}, \tag{7.18}$$

with

$$\begin{aligned}F_2&=\begin{pmatrix}F_{21}\\F_{22}\\F_{23}\end{pmatrix}\\&=\left(-2\delta\left(A_{31}\right)_{\alpha^\star,\,i\omega}\left(A_{31}\right)_{\alpha^\star,\,-i\omega}-\kappa\left(A_{31}\right)_{\alpha^\star,\,i\omega}\left(A_{32}\right)_{\alpha^\star,\,-i\omega}-\kappa\left(A_{32}\right)_{\alpha^\star,\,i\omega}\left(A_{31}\right)_{\alpha^\star,\,-i\omega}\right)\Phi\\&\quad+\rho\,\kappa\left(\left(A_{32}\right)_{\alpha^\star,\,i\omega}\left(A_{31}\right)_{\alpha^\star,\,-i\omega}+\left(A_{31}\right)_{\alpha^\star,\,i\omega}\left(A_{32}\right)_{\alpha^\star,\,-i\omega}\right)\left(\Phi_{3,0}-\Phi_{2,0}\right),\end{aligned}$$

where $\Phi_{j,0}$, $j=1,2,3$, are defined in Lemma 7.4.
(vi) Computation of $L_2(\widehat{b}_3,\widehat{b}_3)$. We have

$$D^2\mathcal{H}\left(0\right)\left(\widehat{b}_3,\widehat{b}_3\right)=\begin{pmatrix}0\\ \begin{pmatrix}-2\delta\left(A_{31}\right)^2_{\alpha^\star,\,-i\omega}-2\kappa\left(A_{31}\right)_{\alpha^\star,\,-i\omega}\left(A_{32}\right)_{\alpha^\star,\,-i\omega}\\-2\rho\,\kappa\left(A_{31}\right)_{\alpha^\star,\,-i\omega}\left(A_{32}\right)_{\alpha^\star,\,-i\omega}\\2\rho\,\kappa\left(A_{31}\right)_{\alpha^\star,\,-i\omega}\left(A_{32}\right)_{\alpha^\star,\,-i\omega}\end{pmatrix}\end{pmatrix}$$

and

$$\mathcal{A}_c\widehat{b}_3=-i\omega\widehat{b}_3.$$

In this case system (7.13) becomes

$$-2i\omega L_2(\widehat{b}_3,\widehat{b}_3)=\mathcal{A}_hL_2(\widehat{b}_3,\widehat{b}_3)+\frac{1}{2!}\Pi_hD^2\mathcal{H}\left(0\right)\left(\widehat{b}_3,\widehat{b}_3\right).$$

It follows from Lemmas 7.4 and 7.6 that

$$L_2(\widehat{b}_3,\widehat{b}_3)=\frac{1}{2}\left(-2i\omega I-\mathcal{A}_h\right)^{-1}\Pi_hD^2\mathcal{H}\left(0\right)\left(\widehat{b}_3,\widehat{b}_3\right)=\begin{pmatrix}0\\ \frac{1}{2}F_3\end{pmatrix}$$

with

$$F_3 = \begin{pmatrix} F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = \left(-2\delta\,(A_{31})^2_{\alpha^\star,\,-i\omega} - 2\kappa\,(A_{31})_{\alpha^\star,\,-i\omega}\,(A_{32})_{\alpha^\star,\,-i\omega}\right)\Phi_{1,-2i\omega} \tag{7.19}$$
$$+2\rho\,\kappa\,(A_{31})_{\alpha^\star,\,-i\omega}\,(A_{32})_{\alpha^\star,\,-i\omega}\left(\Phi_{3,-2i\omega} - \Phi_{2,-2i\omega}\right),$$

where $\Phi_{j,-2i\omega}$, $j = 1, 2, 3$, are defined in Lemma 7.4.

7.4.3 Computation of the Taylor's Expansion of the Reduced System

By using the formula obtained for L_2 and G_2, we get the Taylor's expansion of the reduced system (7.15) up to terms of order 3.

Theorem 7.8 *Let Assumptions 6.1, 6.2, 7.1 and 7.2 be satisfied. The reduced system (7.15) expressed in terms of the basis* $\left\{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3\right\}$ *has the following form*

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = 0, \\ \dfrac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M_c \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \left(H_2 + H_3 + \widetilde{R}_c\right)\left(\begin{pmatrix} \alpha(t) \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\end{pmatrix}\right), \end{cases} \tag{7.20}$$

where

$$M_c = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix},$$

the maps H_2 and $H_3 : \mathbb{C}^3 \to \mathbb{C}^2$ are defined by

$$H_2\left(\begin{pmatrix} \alpha \\ \begin{pmatrix} x \\ y \end{pmatrix}\end{pmatrix}\right) = \left(\alpha\xi_{11} - \delta\xi_{11}^2\right)\Upsilon_1 - \kappa\xi_{21}\,\xi_{11}\Upsilon_2$$

and

$$H_3\left(\begin{pmatrix} \alpha \\ \begin{pmatrix} x \\ y \end{pmatrix}\end{pmatrix}\right) = (\alpha\xi_{12} - 2\delta\xi_{11}\xi_{12})\,\Upsilon_1 - \kappa\,(\xi_{11}\xi_{22} + \xi_{12}\xi_{21})\,\Upsilon_2,$$

the remainder term $\widetilde{R}_c \in C^4\left(\mathbb{R}^3, \mathbb{R}^2\right)$ is given by

$$\widetilde{R}_c\left(\begin{pmatrix} \alpha(t) \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\end{pmatrix}\right) = R_c\left(\begin{pmatrix} \alpha(t) \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\end{pmatrix}\right) - \delta\xi_{12}^2\Upsilon_1 - \kappa\xi_{22}\,\xi_{12}\Upsilon_2$$

and thus

$$\widetilde{R}_c\left(\begin{pmatrix} \alpha \\ \begin{pmatrix} x \\ y \end{pmatrix}\end{pmatrix}\right) = O\left(\alpha^3\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| + \alpha^2\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^2 + \alpha\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^3 + \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^4\right),$$

here

$$\Upsilon_1 = \begin{pmatrix} \Upsilon_{11} \\ \Upsilon_{12} \end{pmatrix} = \begin{pmatrix} \frac{\vartheta_1}{\Delta'(i\omega)_{\alpha^\star}} + \frac{\overline{\vartheta}_2}{\Delta'(-i\omega)_{\alpha^\star}} \\ \frac{\overline{\vartheta}_1}{\Delta'(-i\omega)_{\alpha^\star}} + \frac{\vartheta_2}{\Delta'(i\omega)_{\alpha^\star}} \end{pmatrix}, \tag{7.21}$$

$$\Upsilon_2 = \begin{pmatrix} \Upsilon_{21} \\ \Upsilon_{22} \end{pmatrix} = \rho \begin{pmatrix} \frac{\vartheta_3}{\Delta'(i\omega)_{\alpha^\star}} + \frac{\overline{\vartheta}_4}{\Delta'(-i\omega)_{\alpha^\star}} - \frac{1}{\Delta'(i\omega)_{\alpha^\star}} \\ \frac{\overline{\vartheta}_3}{\Delta'(-i\omega)_{\alpha^\star}} + \frac{\vartheta_4}{\Delta'(i\omega)_{\alpha^\star}} - \frac{1}{\Delta'(-i\omega)_{\alpha^\star}} \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{\vartheta_1}{\Delta'(i\omega)_{\alpha^\star}} + \frac{\overline{\vartheta}_2}{\Delta'(-i\omega)_{\alpha^\star}} \\ \frac{\overline{\vartheta}_1}{\Delta'(-i\omega)_{\alpha^\star}} + \frac{\vartheta_2}{\Delta'(i\omega)_{\alpha^\star}} \end{pmatrix},$$

and $\xi_{ij}, i, j = 1, 2$ *and* $\vartheta_i, i = 1, 2, 3, 4$ *are defined in (7.22) and (7.23), respectively.*

Proof Since

$$D^j W(0)(w_1, w_2, \ldots w_j) = 0, \;\; j \geq 3, \;\; w_i := \begin{pmatrix} \alpha_i \\ \begin{pmatrix} N_i \\ P_{S,i} \\ P_{H,i} \end{pmatrix} \end{pmatrix} \in \mathbb{R}^4,$$

the reduced system (7.15) can be rewritten as follows by using the Taylor's expansion of W around 0:

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = 0, \\ \dfrac{dx_c(t)}{dt} = A_c x_c(t) + \frac{1}{2}\Pi_c^{A(\alpha^\star)} D^2 W(0) \left((I + G_2) \begin{pmatrix} \alpha(t) \\ x_c(t) \end{pmatrix} \right)^2 + R_c \begin{pmatrix} \alpha(t) \\ x_c(t) \end{pmatrix}, \end{cases}$$

where $R_c \in C^4(\mathbb{R} \times X_c, X_c)$ is defined in (7.15) and is a remainder of order 4. Set

$$x_c = x b_1 + y b_2, (x, y) \in \mathbb{C}^2.$$

Since $\left\{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3\right\}$ is used as the basis for $Y_c = \Pi_c(Y)$, i.e., $\{b_1, b_2\}$ is a basis of $X_c := \Pi_c^{A(\alpha^\star)}(\mathbb{R}^3)$, we obtain that

$$M_c = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix}$$

and

$$\begin{aligned} (I + G_2)\begin{pmatrix} \alpha \\ x_c \end{pmatrix} &= \alpha\widehat{b}_1 + x\widehat{b}_2 + y\widehat{b}_3 + L_2\left(\alpha\widehat{b}_1 + x\widehat{b}_2 + y\widehat{b}_3, \alpha\widehat{b}_1 + x\widehat{b}_2 + y\widehat{b}_3\right) \\ &= \alpha\widehat{b}_1 + x\widehat{b}_2 + y\widehat{b}_3 + x^2 L_2\left(\widehat{b}_2, \widehat{b}_2\right) + y^2 L_2\left(\widehat{b}_3, \widehat{b}_3\right) \\ &\quad + 2\alpha x L_2\left(\widehat{b}_1, \widehat{b}_2\right) + 2\alpha y L_2\left(\widehat{b}_1, \widehat{b}_3\right) + 2xy L_2\left(\widehat{b}_2, \widehat{b}_3\right). \end{aligned}$$

Then it follows that

$$(I + G_2)\begin{pmatrix} \alpha \\ x_c \end{pmatrix} = \begin{pmatrix} \alpha \\ \xi \end{pmatrix}, \tag{7.22}$$

where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ with

$$\xi_1 = \xi_{11} + \xi_{12}, \xi_{11} = x\,(A_{31})_{\alpha^\star,\,i\omega} + y\,(A_{31})_{\alpha^\star,\,-i\omega}\,,$$

$$\xi_{12} = \alpha x\,(A_{31})_{\alpha^\star,\,i\omega}\,\Phi^1_{1,i\omega} + \alpha y\,(A_{31})_{\alpha^\star,\,-i\omega}\,\Phi^1_{1,-i\omega} + xyF_{21} + \frac{1}{2}x^2F_{11} + \frac{1}{2}y^2F_{31},$$

$$\xi_2 = \xi_{21} + \xi_{22}, \xi_{21} = x\,(A_{32})_{\alpha^\star,\,i\omega} + y\,(A_{32})_{\alpha^\star,\,-i\omega}\,,$$

$$\xi_{22} = \alpha x\,(A_{31})_{\alpha^\star,\,i\omega}\,\Phi^2_{1,i\omega} + \alpha y\,(A_{31})_{\alpha^\star,\,-i\omega}\,\Phi^2_{1,-i\omega} + xyF_{22} + \frac{1}{2}x^2F_{12} + \frac{1}{2}y^2F_{32}.$$

$$\xi_3 = \xi_{31} + \xi_{32}, \xi_{31} = x\,(A_{33})_{\alpha^\star,\,i\omega} + y\,(A_{33})_{\alpha^\star,\,-i\omega}\,,$$

$$\xi_{32} = \alpha x\,(A_{31})_{\alpha^\star,\,i\omega}\,\Phi^3_{1,i\omega} + \alpha y\,(A_{31})_{\alpha^\star,\,-i\omega}\,\Phi^3_{1,-i\omega} + xyF_{23} + \frac{1}{2}x^2F_{13} + \frac{1}{2}y^2F_{33}.$$

Then we deduce that

$$\frac{1}{2!}D^2W\,(0)\left(\begin{pmatrix} \alpha \\ \xi \end{pmatrix}\right)^2 = \begin{pmatrix} \alpha\xi_1 - \delta\xi_1^2 - \kappa\,\xi_1\xi_2 \\ -\rho\,\kappa\xi_2\,\xi_1 \\ \rho\,\kappa\xi_2\xi_1 \end{pmatrix}.$$

Note that

$$\det\begin{pmatrix} (A_{31})_{\alpha^\star,\,i\omega} & (A_{31})_{\alpha^\star,\,-i\omega} \\ (A_{32})_{\alpha^\star,\,i\omega} & (A_{32})_{\alpha^\star,\,-i\omega} \end{pmatrix} \neq 0$$

and

$$\Pi_c^{A(\alpha^\star)}\begin{pmatrix}1\\0\\0\end{pmatrix}=\frac{\begin{pmatrix}A_{11}\\A_{12}\\A_{13}\end{pmatrix}_{\alpha^\star,\,i\omega}}{\Delta'(i\omega)_{\alpha^\star}}+\frac{\begin{pmatrix}A_{11}\\A_{12}\\A_{13}\end{pmatrix}_{\alpha^\star,\,-i\omega}}{\Delta'(-i\omega)_{\alpha^\star}}$$

$$=\left(\frac{\vartheta_1}{\Delta'(i\omega)_{\alpha^\star}}+\frac{\overline{\vartheta}_2}{\Delta'(-i\omega)_{\alpha^\star}}\right)b_1+\left(\frac{\overline{\vartheta}_1}{\Delta'(-i\omega)_{\alpha^\star}}+\frac{\vartheta_2}{\Delta'(i\omega)_{\alpha^\star}}\right)b_2,$$

$$\Pi_c^{A(\alpha^\star)}\begin{pmatrix}0\\1\\0\end{pmatrix}=\frac{\begin{pmatrix}A_{21}\\A_{22}\\A_{23}\end{pmatrix}_{\alpha^\star,\,i\omega}}{\Delta'(i\omega)_{\alpha^\star}}+\frac{\begin{pmatrix}A_{21}\\A_{22}\\A_{23}\end{pmatrix}_{\alpha^\star,\,-i\omega}}{\Delta'(-i\omega)_{\alpha^\star}}$$

$$=\left(\frac{\vartheta_3}{\Delta'(i\omega)_{\alpha=\alpha^\star}}+\frac{\overline{\vartheta}_4}{\Delta'(-i\omega)_{\alpha=\alpha^\star}}\right)b_1+\left(\frac{\overline{\vartheta}_3}{\Delta'(-i\omega)_{\alpha=\alpha^\star}}+\frac{\vartheta_4}{\Delta'(i\omega)_{\alpha=\alpha^\star}}\right)b_2,$$

$$\Pi_c^{A(\alpha^\star)}\begin{pmatrix}0\\0\\1\end{pmatrix}=\frac{\begin{pmatrix}A_{31}\\A_{32}\\A_{33}\end{pmatrix}_{\alpha^\star,\,i\omega}}{\Delta'(i\omega)_{\alpha^\star}}+\frac{\begin{pmatrix}A_{31}\\A_{32}\\A_{33}\end{pmatrix}_{\alpha^\star,\,-i\omega}}{\Delta'(-i\omega)_{\alpha^\star}}$$

$$=\frac{1}{\Delta'(i\omega)_{\alpha^\star}}b_1+\frac{1}{\Delta'(-i\omega)_{\alpha^\star}}b_2$$

with

$$\vartheta_1=\frac{\det\begin{pmatrix}(A_{11})_{\alpha^\star,\,i\omega} & (A_{31})_{\alpha^\star,\,-i\omega}\\ (A_{12})_{\alpha^\star,\,i\omega} & (A_{32})_{\alpha^\star,\,-i\omega}\end{pmatrix}}{\det\begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega} & (A_{31})_{\alpha^\star,\,-i\omega}\\ (A_{32})_{\alpha^\star,\,i\omega} & (A_{32})_{\alpha^\star,\,-i\omega}\end{pmatrix}},\ \vartheta_2=\frac{\det\begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega} & (A_{11})_{\alpha^\star,\,i\omega}\\ (A_{32})_{\alpha^\star,\,i\omega} & (A_{12})_{\alpha^\star,\,i\omega}\end{pmatrix}}{\det\begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega} & (A_{31})_{\alpha^\star,\,-i\omega}\\ (A_{32})_{\alpha^\star,\,i\omega} & (A_{32})_{\alpha^\star,\,-i\omega}\end{pmatrix}} \tag{7.23}$$

$$\vartheta_3=\frac{\det\begin{pmatrix}(A_{21})_{\alpha^\star,\,i\omega} & (A_{31})_{\alpha^\star,\,-i\omega}\\ (A_{22})_{\alpha^\star,\,i\omega} & (A_{32})_{\alpha^\star,\,-i\omega}\end{pmatrix}}{\det\begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega} & (A_{31})_{\alpha^\star,\,-i\omega}\\ (A_{32})_{\alpha^\star,\,i\omega} & (A_{32})_{\alpha^\star,\,-i\omega}\end{pmatrix}},\ \vartheta_4=\frac{\det\begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega} & (A_{21})_{\alpha^\star,\,i\omega}\\ (A_{32})_{\alpha^\star,\,i\omega} & (A_{22})_{\alpha^\star,\,i\omega}\end{pmatrix}}{\det\begin{pmatrix}(A_{31})_{\alpha^\star,\,i\omega} & (A_{31})_{\alpha^\star,\,-i\omega}\\ (A_{32})_{\alpha^\star,\,i\omega} & (A_{32})_{\alpha^\star,\,-i\omega}\end{pmatrix}}.$$

By projecting on X_c, we obtain that

$$\frac{1}{2!}\Pi_c^{A(\alpha^\star)}D^2W(0)\left((I+G_2)\left(\begin{pmatrix}\alpha\\x_c\end{pmatrix}\right)\right)^2$$

$$=\Pi_c^{A(\alpha^\star)}\begin{pmatrix}\alpha\xi_1-\delta\xi_1^2-\kappa\,\xi_1\xi_2\\-\rho\,\kappa\xi_2\,\xi_1\\\rho\,\kappa\xi_2\xi_1\end{pmatrix}$$

$$=\left(\alpha\xi_1-\delta\xi_1^2-\kappa\,\xi_1\xi_2\right)\Pi_c^{A(\alpha^\star)}\begin{pmatrix}1\\0\\0\end{pmatrix}-\rho\,\kappa\xi_2\,\xi_1\left[\Pi_c^{A(\alpha^\star)}\begin{pmatrix}0\\1\\0\end{pmatrix}-\Pi_c^{A(\alpha^\star)}\begin{pmatrix}0\\0\\1\end{pmatrix}\right].$$

Thus we obtain H_2, H_3 and $\widetilde{R}_c$. □

7.4.4 Computation of the Normal Form of the Reduced System

We will compute the normal form of the reduced system (7.20) up to terms of order 3 by performing inductively a sequence of change of coordinates of the form

$$u = \varsigma_{m,c}(\overline{u}) = \overline{u} + G_{m,c}(\overline{u}), \overline{u} \in B^m(0,r), \tag{7.24}$$

where $G_{m,c}(\overline{u}) \in H_m^3(\mathbb{C}^3)$ and $B^m(0,r)$ is a small neighborhood of the origin, $2 \le m \le 3$. Consider the linear operator $\Theta_{m,c} : H_m^3(\mathbb{C}^3) \to H_m^3(\mathbb{C}^3)$ by

$$\left(\Theta_{m,c}G_{m,c}\right)(u) = [B, G_{m,c}](u) = D_u G_{m,c}(u)Bu - BG_{m,c}(u), \tag{7.25}$$
$$u = \begin{pmatrix} \alpha \\ x \\ y \end{pmatrix},\ G_{m,c} = \begin{pmatrix} G^1_{m,c} \\ G^2_{m,c} \\ G^3_{m,c} \end{pmatrix} \in H_m^3(\mathbb{C}^3),\ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & -i\omega \end{bmatrix}.$$

A basis denoted by Φ_m for $H_m^3(\mathbb{C}^3), 2 \le m \le 3$, consists of all possible vector-valued homogeneous polynomials of degree m in 3 variables α, x, y with coefficients in $\{e_1, e_2, e_3\}$, that is

$$\Phi_m = \left\{ \begin{array}{c} u^d e_j \Big|,\ u^d = \alpha^{d_1} x^{d_2} y^{d_3},\ d = (d_1, d_2, d_3),\ \sum_{i=1}^{3} d_i = m,\ d_i \ge 0 \text{ are integers}, \\ j = 1, \cdots, 3. \end{array} \right\}.$$

The dimension of $H_m^3(\mathbb{C}^3)$ is

$$d_m := \dim H_m^3(\mathbb{C}^3) = \frac{3\,(m+2)!}{2!m!}.$$

A direct calculation shows that for any $u^d e_j \in \Phi_m$,

$$\Theta_{m,c}\left(u^d e_j\right) = \left(i\omega d_2 - i\omega d_3 - \lambda_j\right) u^d e_j, \lambda_j \in \sigma(B),\ \text{where } \lambda_1 = 0, \lambda_2 = i\omega, \lambda_3 = -i\omega.$$

Hence we know that $H_m^3(\mathbb{C}^3)$ can be represented as the direct sum

$$H_m^3(\mathbb{C}^3) = R(\Theta_{m,c}) \oplus Ker(\Theta_{m,c}),$$

where $R(\Theta_{m,c})$ is the range of $\Theta_{m,c}$, and $Ker(\Theta_{m,c})$ is the null-space of $\Theta_{m,c}$ and spanned by all resonant monomials of order m, $2 \le m \le 3$, that is,

$$Ker(\Theta_{m,c}) = span\{u^d e_k : \lambda \cdot d = \lambda_k,\ k = 1,2,3,\ u^d = \alpha^{d_1} x^{d_2} y^{d_3}, d = (d_1, d_2, d_3),$$
$$\sum_{j=1}^{3} d_j = m,\ d_j \ge 0 \text{ are integers}, \lambda \cdot d = \sum_{j=1}^{3} d_j \lambda_j\},$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{C}^3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 = 0$, $\lambda_2 = i\omega$, $\lambda_3 = -i\omega$. Then it follows that

$$Ker(\Theta_{2,c}) = \text{span}\left\{\begin{pmatrix}\alpha^2\\0\\0\end{pmatrix},\begin{pmatrix}xy\\0\\0\end{pmatrix},\begin{pmatrix}0\\\alpha x\\0\end{pmatrix},\begin{pmatrix}0\\0\\\alpha y\end{pmatrix}\right\},$$

$$Ker(\Theta_{3,c}) = \text{span}\left\{\begin{pmatrix}\alpha^3\\0\\0\end{pmatrix},\begin{pmatrix}\alpha xy\\0\\0\end{pmatrix},\begin{pmatrix}0\\\alpha^2 x\\0\end{pmatrix},\begin{pmatrix}0\\x^2 y\\0\end{pmatrix},\begin{pmatrix}0\\0\\\alpha^2 y\end{pmatrix},\begin{pmatrix}0\\0\\xy^2\end{pmatrix}\right\}.$$

By using the change of coordinates

$$u = \varsigma_{2,c}(\overline{u}) = \overline{u} + G_{2,c}(\overline{u}), \overline{u} \in B^2(0,r) \tag{7.26}$$

and dropping the hat and the auxiliary equation introduced for handling the parameter, we get

$$\frac{d}{dt}\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = M_c\begin{pmatrix}x(t)\\y(t)\end{pmatrix} + \begin{pmatrix}C_1 x\alpha\\\overline{C}_1 y\alpha\end{pmatrix} + \left(\widetilde{H}_3 + \widetilde{R}_{1c}\right)\left(\begin{matrix}\alpha(t)\\\begin{pmatrix}x(t)\\y(t)\end{pmatrix}\end{matrix}\right),$$

with

$$C_1 = (A_{31})_{\alpha^\star, i\omega}\left(\frac{\vartheta_1}{\Delta'(i\omega)_{\alpha^\star}} + \frac{\overline{\vartheta}_2}{\Delta'(-i\omega)_{\alpha^\star}}\right). \tag{7.27}$$

In order to get the third order terms of the normal form, we must compute $G_{2,c}$ and the third order terms

$$\widetilde{H}_3 = H_3(u) + D_u H_2(u)\, G_2(u) - D_u G_2(u)\begin{pmatrix}C_1 x\alpha\\\overline{C}_1 y\alpha\end{pmatrix}$$

with $u = \left(\begin{matrix}\alpha\\\begin{pmatrix}x\\y\end{pmatrix}\end{matrix}\right)$. It is sufficient to get the coefficients of $\begin{pmatrix}x^2 y\\0\end{pmatrix}$ and $\begin{pmatrix}0\\xy^2\end{pmatrix}$ for studying the Hopf bifurcation. Since

$$H_2\left(\begin{matrix}0\\\begin{pmatrix}x\\y\end{pmatrix}\end{matrix}\right) = \varpi_1 x^2 + \varpi_2 xy + \varpi_3 y^2,$$

with

$$\varpi_1 = \begin{pmatrix} \varpi_{11} \\ \varpi_{12} \end{pmatrix} \tag{7.28}$$

$$= -\delta\,(A_{31})^2_{\alpha^\star,\,i\omega} \begin{pmatrix} \Upsilon_{11} \\ \Upsilon_{12} \end{pmatrix} - \kappa\,(A_{31})_{\alpha^\star,\,i\omega}\,(A_{32})_{\alpha^\star,\,i\omega} \begin{pmatrix} \Upsilon_{21} \\ \Upsilon_{22} \end{pmatrix},$$

$$\varpi_2 = \begin{pmatrix} \varpi_{21} \\ \varpi_{22} \end{pmatrix}$$

$$= -2\delta\,(A_{31})_{\alpha^\star,\,i\omega}\,(A_{31})_{\alpha^\star,\,-i\omega} \begin{pmatrix} \Upsilon_{11} \\ \Upsilon_{12} \end{pmatrix}$$

$$-\kappa\left[(A_{32})_{\alpha^\star,\,i\omega}\,(A_{31})_{\alpha^\star,\,-i\omega} + (A_{32})_{\alpha^\star,\,-i\omega}\,(A_{31})_{\alpha^\star,\,i\omega}\right] \begin{pmatrix} \Upsilon_{21} \\ \Upsilon_{22} \end{pmatrix},$$

$$\varpi_3 = \begin{pmatrix} \varpi_{31} \\ \varpi_{32} \end{pmatrix}$$

$$= -\delta\,(A_{31})^2_{\alpha^\star,\,-i\omega} \begin{pmatrix} \Upsilon_{11} \\ \Upsilon_{12} \end{pmatrix} - \kappa\,(A_{32})_{\alpha^\star,\,-i\omega}\,(A_{31})_{\alpha^\star,\,-i\omega} \begin{pmatrix} \Upsilon_{21} \\ \Upsilon_{22} \end{pmatrix} \tag{7.29}$$

and

$$\begin{pmatrix} G^2_{2,c}\left(\begin{pmatrix} 0 \\ x \\ y \end{pmatrix}\right) \\ G^3_{2,c}\left(\begin{pmatrix} 0 \\ x \\ y \end{pmatrix}\right) \end{pmatrix} = \frac{1}{i\omega} \begin{pmatrix} \varpi_{11}x^2 - \varpi_{21}xy - \frac{1}{3}\varpi_{31}y^2 \\ \frac{1}{3}\varpi_{12}x^2 + \varpi_{22}xy - \varpi_{32}y^2 \end{pmatrix},$$

we get that the normal form has the following form after dropping the auxiliary equation introduced for handling the parameter,

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M_c \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} C_1 x\alpha \\ \overline{C}_1 y\alpha \end{pmatrix} + \begin{pmatrix} C_2 x^2 y \\ \overline{C}_2 x y^2 \end{pmatrix} \tag{7.30}$$
$$+O(|\,(x,y)\,|\alpha^2 + |(\alpha,(x,y))|^4),$$

with

$$C_2 = -\left[2\delta F_{21}\,(A_{31})_{\alpha^\star,\,i\omega} + F_{11}\delta\,(A_{31})_{\alpha^\star,\,-i\omega}\right]\Upsilon_{11}$$
$$-\kappa\left[F_{22}\,(A_{31})_{\alpha^\star,\,i\omega} + \frac{1}{2}F_{12}\,(A_{31})_{\alpha^\star,\,-i\omega} + F_{21}\,(A_{32})_{\alpha^\star,\,i\omega} + \frac{1}{2}F_{11}\,(A_{32})_{\alpha^\star,\,-i\omega}\right]\Upsilon_{21}$$
$$+\frac{1}{i\omega}\left(\frac{2}{3}\varpi_{31}\varpi_{12} - \varpi_{11}\varpi_{21} + \varpi_{22}\varpi_{21}\right).$$

The normal form (7.30) can be written in real coordinates (w_1, w_2) through the change of variables $x = v_1 - iv_2,\ y = v_1 + iv_2$. Setting $v_1 = \rho\cos\theta,\ v_2 = \rho\sin\theta$, this normal form becomes

$$\begin{cases} \dot{\rho} = l_1\alpha\rho + l_2\rho^3 + O(\alpha^2\rho + |(\rho,\alpha)|^4), \\ \dot{\theta} = -\omega + O(|(\rho,\alpha)|), \end{cases} \tag{7.31}$$

where

$$l_1 = \text{Re}\,(C_1) = 2p_2 \left(\frac{\left(\gamma\eta - (\beta_P - \mu_P)^2\right) p_1}{\beta_P - \mu_P} + p_2 \right) > 0, \;\; l_2 = \text{Re}\,(C_2)\,.$$

From Chapter 13 or [32], we know that the sign of l_2 determines the direction of the bifurcation and the stability of the nontrivial periodic orbits. In summary we have the following theorem.

Theorem 7.9 *Let Assumptions 6.1, 6.2, 7.1 and 7.2 be satisfied. The direction of the Hopf bifurcation described in Theorem 7.3 and the stability of the bifurcating periodic solutions are determined by the sign of l_2 : Hopf bifurcation is supercritical and the bifurcating periodic solutions are stable if $l_2 < 0$, and subcritical and unstable if $l_2 > 0$.*

7.5 Numerical Simulations and Conclusions

By carrying out bifurcation analysis and normal form computation of the model (6.1), we discussed the existence and properties of Hopf bifurcation. In this section, we provide some numerical simulations to illustrate the results. We firstly choose parameters $\delta = 8$, $\kappa = 1/2$, $\rho = 2$, $\mu_P = 1/7, \beta_P = 8.5127$, $\eta = 9.24$, $\gamma = 91$. In order to get the properties of Hopf bifurcation using Theorem 7.9, we need compute l_1 and l_2. By Matlab we get $\alpha^\star = 676.347, l_1 = \text{Re}\,(C_1) = 0.496645, l_2 = \text{Re}\,(C_2) = -1.61528$. Then we choose $\alpha = \beta_N - \mu_N = 1.6567 - 1 = 0.6567 < \alpha^\star, N(0) = 0.7$, $P_S(0) = 1.40684$, $P_H(0) = 0.0071$ in Figure 7.1 and $\alpha = \beta_N - \mu_N = 700 - 1 = 699 > \alpha^\star$, $N(0) = 2.70 \times 10^{-81}$, $P_S(0) = 1467, P_H(0) = 145.7$ in Figure 7.2, respectively, and obtain graphs of $N(t)$, $P_S(t)$ and $P_H(t)$ by using Matlab.

Figure 7.1 and Figure 7.2, demonstrate that the positive equilibrium E^* of system (6.1) is asymptotically stable when the intrinsic growth rate α of the prey species is less than its critical value $\alpha^\star$ and system (6.1) undergoes a Hopf bifurcation and a stable non-trivial periodic solution bifurcates from the positive equilibrium when the intrinsic growth rate α of the prey species passes through the critical value $\alpha^\star$ which consists with Theorem 7.9.

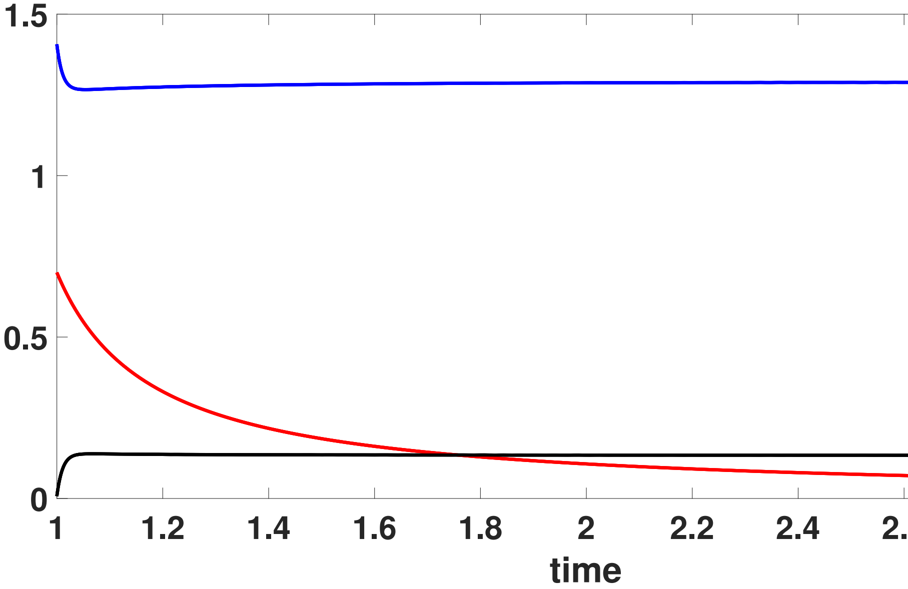


Fig. 7.1: *In this figure we run a simulation of the model* (7.1) *with stable positive equilibrium* E^*.

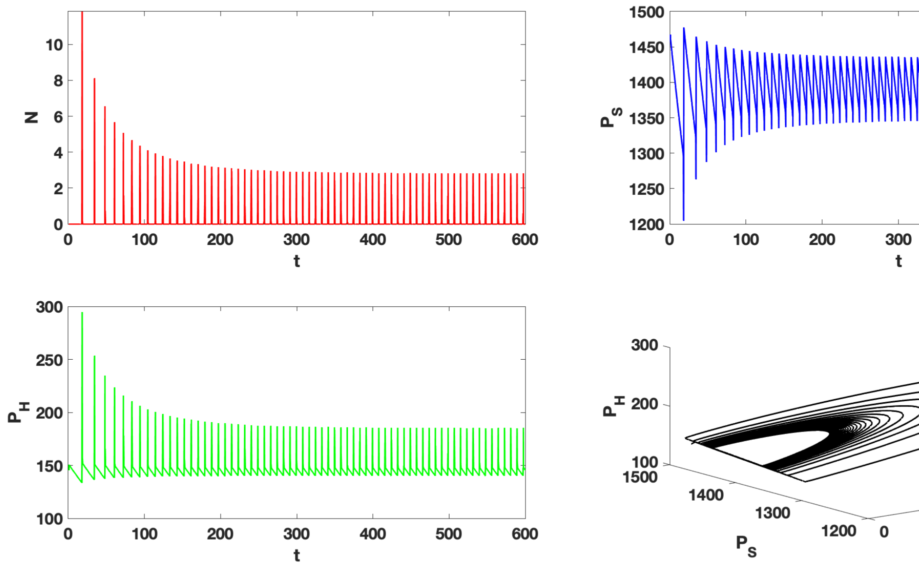


Fig. 7.2: *In this figure we run a simulation of the model* (7.1) *with stable periodic solution around the positive equilibrium* E^*.

7.6 Remarks and notes

By Chapter 15, we have two ways to compute the normal form of the reduced system of a system topologically equivalent to the original system (7.10). One is to compute the Taylor's expansion of the reduced system first and then we can compute the normal form of the reduced system. The other way is to compute the normal form of the reduced system directly. We use the first way to compute the normal form for Hopf bifurcation in this Chapter. We refer to the books [99] by Hassard, Kazarinoff and Wan, [30] by Chow and Hale, [32] by Chow et al., [129] by Kuznetsov, [225] by Wiggins and others for more results on the existence and properties of Hopf bifurcation.

7.7 MATLAB codes

7.7.1 Figure 7.1

```

tspan = [1 600];
```

```
y0 = [2.70*1e-81;1467;145.7];
[T,Y] = ode45(@myfun,tspan,y0);
subplot(2,2,1)
plot(T,Y(:,1),'r','linewidth',1.5);
xlabel('t')
ylabel('N')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');
subplot(2,2,2)
plot(T,Y(:,2),'b','linewidth',1.5);
xlabel('t')
ylabel('P_S')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');
subplot(2,2,3)
plot(T,Y(:,3),'g','linewidth',1.5);
xlabel('t')
ylabel('P_H')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');
subplot(2,2,4)
plot3(Y(:,1),Y(:,2),Y(:,3),'k','linewidth',1.5);

xlabel('N')
ylabel('P_S')
zlabel('P_H')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');

function dy = myfun(t,y)
muN = 1;
betaN = 700;
delta = 8;
k=1/2;
rho=2;
muP = 1/7;
betaP = 8.5127;
eta = 9.24;
r=91;
dy = zeros(3,1);
dy(1) = (betaN-muN)*y(1)-delta*y(1)*y(1)-k*y(1)*y(2)
    ;
dy(2) = -(muP+eta)*y(2)-rho*k*y(1)*y(2)+y(3)*r;
dy(3) = -muP*y(3)+k*y(1)*y(2)*rho-y(3)*r+betaP*(y(2)
    +y(3));
```

```
dy=dy(:);
end
```

7.7.2 Figure 7.2

```

tspan = [1 600];

y0 = [2.70*1e-81;1467;145.7];
[T,Y] = ode45(@myfun,tspan,y0);
subplot(2,2,1)
plot(T,Y(:,1),'r','linewidth',1.5);
xlabel('t')
ylabel('N')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');
subplot(2,2,2)
plot(T,Y(:,2),'b','linewidth',1.5);
xlabel('t')
ylabel('P_S')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');
subplot(2,2,3)
plot(T,Y(:,3),'g','linewidth',1.5);
xlabel('t')
ylabel('P_H')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');
subplot(2,2,4)
plot3(Y(:,1),Y(:,2),Y(:,3),'k','linewidth',1.5);

xlabel('N')
ylabel('P_S')
zlabel('P_H')
set(gca, 'FontSize', 15);
set(gca, 'FontWeight', 'bold');

function dy = myfun(t,y)
muN = 1;
betaN = 700;
delta = 8;
k=1/2;
rho=2;
muP = 1/7;
betaP = 8.5127;
eta = 9.24;
```

```
r=91;
dy = zeros(3,1);
dy(1) = (betaN-muN)*y(1)-delta*y(1)*y(1)-k*y(1)*y(2)
    ;
dy(2) = -(muP+eta)*y(2)-rho*k*y(1)*y(2)+y(3)*r;
dy(3) = -muP*y(3)+k*y(1)*y(2)*rho-y(3)*r+betaP*(y(2)
    +y(3));
dy=dy(:);
end
```

Chapter 8
Large Speed Traveling Waves for the Rosenzweig-MacArthur predator-prey Model with Spatial Diffusion

This chapter focuses on traveling wave solutions for the so-called Rosenzweig-MacArthur predator-prey model with spatial diffusion. The main results of this note are concerned with the existence and uniqueness of traveling wave solution as well as periodic wave train solution in the large wave speed asymptotic. Depending on the model parameters we more particularly study the existence and uniqueness of a traveling wave connecting two equilibria or connecting an equilibrium point and a periodic wave train. We also discuss the existence and uniqueness of such a periodic wave train. Our analysis is based on ordinary differential equation techniques by coupling the theories of invariant manifolds together with those of global attractors.

8.1 Introduction

In this work we study the traveling solutions for the so-called diffusive Rosenzweig-MacArthur predator-prey system that reads as follows

$$\begin{cases} u_t = \delta_1 u_{xx} + Au\left(1 - \dfrac{u}{K}\right) - B\dfrac{uv}{1+Eu}, \\ v_t = \delta_2 v_{xx} - Cv + D\dfrac{uv}{1+Eu}. \end{cases} \tag{8.1}$$

This system is posed for the one-dimensional spatial variable $x \in \mathbb{R}$ while t denotes time.

In the above system of equations $u = u(t,x)$ denotes the density of the prey population while $v = v(t,x)$ corresponds to those of the predator, at time $t > 0$ and spatial location $x \in \mathbb{R}$. The positive parameters δ_1 and δ_2 represent the diffusion coefficients for the prey and the predator, respectively. The underlying kinetic system describes the dynamics of the populations as well as their interactions and reads as the following ordinary differential equations (ODE for short)

$$\begin{cases} u'(t) = Au\left(1 - \dfrac{u}{K}\right) - B\dfrac{uv}{1+Eu}, \\ v'(t) = -Cv + D\dfrac{uv}{1+Eu}, \end{cases} \tag{8.2}$$

wherein A, B, C, D, E and K are given positive constants. More precisely A stands for the growth factor for the prey species, K denotes its carrying capacity, B and D are the interaction rates for the two species while C corresponds to the natural death rate for the predator. Finally the parameter E measures the "satiation" effect of the predator population. We refer the reader to Holling [**?**] and Rosenzweig [**? ?**] for more details on this model.

The aim of this work is to discuss the existence and qualitative properties of the traveling wave and the periodic wave train solutions for (8.1) . To discuss this issue, we first rescale the system by introducing

$$U = Eu,\ V = Bv/C,\ t' = Ct,\ x' = (C/\delta_2)^{1/2}x,$$
$$d = \frac{\delta_1}{\delta_2},\ \alpha = A/(ECK),\ \gamma = EK,\ \beta = D/(EC).$$

With these new variables and normalized parameters, (8.1) rewrites, omitting the prime for notational simplicity, as the following reaction-diffusion system

$$\begin{cases} U_t = dU_{xx} + \alpha U(\gamma - U) - \dfrac{UV}{1+U}, \\ V_t = V_{xx} - V + \beta\dfrac{UV}{1+U}, \end{cases} \tag{8.3}$$

while the underlying kinetic system, namely (8.2), becomes

$$\begin{cases} U' = \alpha U(\gamma - U) - \dfrac{UV}{1+U}, \\ V' = -V + \beta\dfrac{UV}{1+U}. \end{cases} \tag{8.4}$$

As mentioned above, the goal of this work is to discuss some properties of the traveling wave and periodic wave train solutions for the reaction-diffusion system (8.3). Here recall that a traveling wave solution corresponds to an entire solution of (8.3) (that is a solution defined for all time $t \in \mathbb{R}$) of the form

$$U(t,x) = u(s),\quad V(t,x) = v(s) \text{ with } s = x + ct,$$

where $c \in \mathbb{R}$ is some constant that stands for the wave speed. When the profile $s \mapsto (u(s), v(s))$ is periodic we speak about periodic wave train with speed c. Plugging this specific form into (8.3) yields the following ODE system for the wave profiles $(u,v) = (u(s), v(s))$ for $s \in \mathbb{R}$

$$\begin{cases} cu' = du'' + \alpha u(\gamma - u) - \dfrac{uv}{1+u}, \\ cv' = v'' - v + \beta\dfrac{uv}{1+u}. \end{cases} \tag{8.5}$$

Traveling wave solutions for the above system or more generally for predator-prey systems have been widely investigated in the last decades. One may refer the reader to the works of Dunbar [63, 64**?**] who proposed ODE methods coupled with topological arguments to prove the existence of such special solutions. One may also refer to Gardner [81] who developed topological arguments based on the Conley index to obtain the existence of solutions with suitable behaviour at $s = \pm\infty$. We also refer to Huang, Lu and Ruan [119] for more general results also based on a coupling between ODE methods and topological arguments. We refer to Ruan [185] a result of existence of periodic wave train by using using Hopf bifurcation method. We refer the reader to the work of Hosono [112] and the references cited therein for results about the Lotka-Volterra predator-prey system as well as to the recent work of Li and Xiao [138] (see also the references therein) for results about the existence of traveling waves for more general functional responses and also for a nice review on this topic. The connexion between wave solutions and the asymptotic behaviour of the Cauchy problem (8.3) (when equipped with suitable initial data) has been scarcely studied. One may refer the reader to Gardner [80] who studied the local stability of wave solutions and to Ducrot, Giletti and Matano [54] (and the references therein) for results related to the so-called asymptotic speed of spread. We also refer to [4, 53, 113, 121] for other recent results.

One important difficulty when studying traveling wave solutions for predator-prey interactions relies on the ability of the underlying kinetic to develop sustained oscillations, typically through Hopf bifurcation. Hence the behaviour of the solutions of the corresponding reaction-diffusion system are expected to exhibit somehow complex spatio-temporal oscillations. Therefore the traveling wave solutions describing for instance the spatial invasion of a predator is also expected to exhibit oscillating patterns, connecting a predator-free equilibrium and some oscillating state, such as a periodic wave train (see [119, 185] for results about the existence of such periodic solutions using bifurcation methods). According to our knowledge, this question related to the shape and the behaviour of traveling waves remains largely open. It has been addressed by Dunbar in [**?**] and further developed by Huang [120]. In this aforementioned work, the author developed refined singular perturbation analysis based on the hyperbolicity of the periodic solutions of the kinetic system to construct oscillating traveling wave in the large speed asymptotic. In this work we revisit this issue by developing a dynamical system approach to obtain a complete picture of the traveling wave solutions for system (8.5), in the large wave speed asymptotic, $c \gg 1$. Our methodology also allows us to provide uniqueness results, on the one hand for traveling waves and, on the other hand, also for periodic wave trains with large wave speed.

In this paper, we describe in particular sharp conditions on the parameters of the system that ensure the existence of a unique traveling wave solution for (8.5) connecting the predator free equilibrium to the interior equilibrium or to a unique periodic wave train. To reach such a refined description, we develop a methodology based on dynamical system arguments. Here we will more precisely couple center manifold and more generally invariant manifold reduction together with the global attractor theory and qualitative analysis for ODE. In fact, the method presented here is rather general and can be used to work on the traveling waves with other and more general nonlinearities. It can also be extended to handle infinite dimensional problems such as reaction-diffusion systems with time delay.

To perform our analysis, we make use of successive rescaling arguments to restrict our analysis to a system of two ordinary differential equations. Firstly let us set

$$\widehat{u}(s) = u(-cs), \widehat{v}(s) = v(-cs).$$

Then, dropping the hats on u and v for notational convenience, (8.5) becomes

$$\begin{cases} -u' = \dfrac{d}{c^2}u'' + \alpha u(\gamma - u) - \dfrac{uv}{1+u}, \\ -v' = \dfrac{1}{c^2}v'' - v + \beta\dfrac{uv}{1+u}. \end{cases} \tag{8.6}$$

Next let us set $\varepsilon = \frac{1}{c^2}$, $u_1 = u, u_2 = u', v_1 = v$ and $v_2 = v'$, so that the above problem (8.6) rewrites as

$$\begin{cases} u_1' &= u_2, \\ d\varepsilon u_2' &= -u_2 - \alpha u_1(\gamma - u_1) + \dfrac{u_1 v_1}{1+u_1}, \\ v_1' &= v_2, \\ \varepsilon v_2' &= -v_2 + v_1 - \beta\dfrac{u_1 v_1}{1+u_1}. \end{cases} \tag{8.7}$$

Set

$$\widehat{u}_1(s) = u_1(\varepsilon s), \widehat{u}_2(s) = u_2(\varepsilon s), \widehat{v}_1(s) = v_1(\varepsilon s), \widehat{v}_2(s) = v_2(\varepsilon s),$$

and (8.7) becomes (dropping the hats for notational convenience)

$$\begin{cases} u_1' &= \varepsilon u_2, \\ du_2' &= -u_2 - \alpha u_1(\gamma - u_1) + \dfrac{u_1 v_1}{1+u_1}, \\ v_1' &= \varepsilon v_2, \\ v_2' &= -v_2 + v_1 - \beta\dfrac{u_1 v_1}{1+u_1}, \end{cases} \tag{8.8}$$

where all the parameters d, α, γ, β and ε are strictly positive.

As mentioned above, in this paper we will investigate traveling waves and periodic wave trains for (8.5), that correspond to heteroclinic connexions and periodic orbits, respectively, for system (8.7) or equivalently (8.8). Here we focus our study on the large speed asymptotic, namely $c \gg 1$, that is $0 < \varepsilon = \frac{1}{c^2} \ll 1$. To study this problem we will use center manifold reduction arguments to rewrite (8.8) on a suitable invariant set as a small perturbation of the kinetic system (8.4). The description of the heteroclinic and periodic orbits of the perturbed problem are then investigated using global attractor theory.

The organization of the paper is as follows. Section 2 is devoted to the description of the global attractor for the Rosenzweig-MacArthur model (8.4) with a particular attention paid on the heteroclinic orbits and their uniqueness. Section 3 is concerned with the study of some complete orbit of (8.8), in the regime $0 < \varepsilon \ll 1$. We first reformulate this problem as a small perturbation of (8.4). We then study its global attractor and derive existence and uniqueness results for the traveling waves and periodic wave trains for (8.5) whenever c is large enough. In the last section we present some numerical simulations for the system in order illustrate our results.

8.2 Global attractors for the Rosenzweig-MacArthur model

In this section we propose a refine description of the global attractor of the Rosenzweig-MacArthur model

$$\begin{cases} U'(t) = \alpha U(\gamma - U) - \dfrac{UV}{1+U}, \\ V'(t) = -V + \beta \dfrac{UV}{1+U}. \end{cases} \tag{8.9}$$

The results presented in this section are mainly due to Hsu [116][Theorem 3.3], Hsu, Hubbell and Waltman [114, Lemma 4] where the global stability of the interior equilibrium is obtained by using the 18.13 criteria, and to Cheng [25] who proved the uniqueness of the periodic orbit. In this section, we reformulate these results using the theory of the global attractor and as mentioned above we propose a refine description of this object by studying the existence and uniqueness of heteroclinic orbit starting from the no predator region ($V = 0$) to the interior global attractor (where $U > 0$ and $V > 0$). The results presented in the next main section, about (8.8), will make use of the refined description presented in this section.

To study (8.9) let us first observe that this system admits the following equilibrium points. The boundary equilibria are given by

$$(\overline{U}_0, \overline{V}_0) = (0, 0) \text{ and } (\overline{U}_1, \overline{V}_1) = (\gamma, 0). \tag{8.10}$$

and the unique *interior equilibrium* whenever $\gamma\,(\beta - 1) > 1$, is given by

$$(\overline{U}_2, \overline{V}_2) = \left(\frac{1}{\beta - 1}, \frac{\alpha\beta\,[\gamma\,(\beta - 1) - 1]}{(\beta - 1)^2}\right). \tag{8.11}$$

Next define the functions

$$F(U, V) = \alpha U(\gamma - U) - \frac{UV}{1+U} = \frac{U}{1+U}[f(U) - V]$$

and

$$G(U, V) = V\left(\frac{\beta U}{1+U} - 1\right) = (\beta - 1)\frac{V\left(U - \overline{U}_2\right)}{1+U}$$

with the nullclines

$$f(U) = \alpha(\gamma - U)\,(1+U) \text{ and } U = \overline{U}_2$$

for U-equation and V-equation, respectively. Note that the map $f(U)$ is symmetric with respect to the vertical line $U = \frac{\gamma-1}{2}$.

We make the following parameter regimes for convenience and will refer these conditions throughout the paper:
(H1): $\gamma\,(\beta - 1) > 1$;
(H2): $\gamma\,(\beta - 1) > \beta + 1$;

(H3): $1 < \gamma(\beta - 1) < \beta + 1$.

We now discuss the existence of global attractor for (8.9) those existence is ensured by the next lemma.

Lemma 8.1 (Global attractor) *Let (H1) be satisfied. There exists $R > 0$ such that the triangle*

$$\mathbb{T} = \left\{(U, V) \in [0, \infty)^2 : \beta U + V \leq R\right\}$$

is positively invariant by the semiflow generated by (8.9)*. The (positive) semiflow generated by* (8.9) *in $\mathbb{R}^2_+$ admits a global attractor, denoted by $\mathcal{A}_{\mathbb{R}^2_+}$, that is contained in $\mathbb{T}$. Furthermore the triangle $\mathbb{T}$ contains all the non negative equilibria of* (8.9).

Note that the invariance property follows from the fact that for all $R > 0$ large enough $\beta F(U, V) + G(U, V) < 0$ whenever $\beta U + V = R$ and $U \geq 0$ and $V \geq 0$.

We now focus on the precise description $\mathcal{A}_{\mathbb{R}^2_+}$. To that aim we first discuss the existence of an interior attractor by considering the regions

$$\partial_U \mathbb{R}^2_+ = \left\{(U, V) \in \mathbb{R}^2_+ : V = 0\right\} \text{ and } \partial_V \mathbb{R}^2_+ = \left\{(U, V) \in \mathbb{R}^2_+ : U = 0\right\},$$

and leading to the state space (disjoint) decomposition $\mathbb{R}^2_+ = \text{Int}\left(\mathbb{R}^2_+\right) \cup \left(\partial_U \mathbb{R}^2_+ \cup \partial_V \mathbb{R}^2_+\right)$. In the following lemma we are using the notion of global attractor considered first by Hale [93, 94]. We refer to Magal and Zhao [160] and Magal [151] for more results and examples about global attractors only attracting compact subsets.

Lemma 8.2 *Let (H1) be satisfied. The semiflow generated by* (8.9) *restricted to $\mathbb{R}^2_+$ (respectively $\partial_U \mathbb{R}^2_+$, $\partial_V \mathbb{R}^2_+$ and $\text{Int}\left(\mathbb{R}^2_+\right)$) has a global attractor $\mathcal{A}_{\mathbb{R}^2_+}$ (respectively $\mathcal{A}_{\partial_U \mathbb{R}^2_+}$, $\mathcal{A}_{\partial_V \mathbb{R}^2_+}$ and $\mathcal{A}_{\text{Int}\left(\mathbb{R}^2_+\right)}$) which is a compact and connected subset which attracts all the compact subsets of $\mathbb{R}^2_+$ (respectively $\partial_U \mathbb{R}^2_+$, $\partial_V \mathbb{R}^2_+$ and $\text{Int}\left(\mathbb{R}^2_+\right)$).*

Remark 8.3 The global attractor $\mathcal{A}_{\text{Int}\left(\mathbb{R}^2_+\right)}$ only attracts the compact subsets of $\text{Int}\left(\mathbb{R}^2_+\right)$. That is to say that $\mathcal{A}_{\text{Int}\left(\mathbb{R}^2_+\right)}$ does not attract the bounded subsets of the interior region $\text{Int}\left(\mathbb{R}^2_+\right)$ (see Magal and Zhao [160] for more examples).

It is readily checked that the global attractor in $\partial_V \mathbb{R}^2_+$ is $\mathcal{A}_{\partial_V \mathbb{R}^2_+} = \{(0, 0)\}$ while the global attractor in $\partial_U \mathbb{R}^2_+$ is

$$\mathcal{A}_{\partial_U \mathbb{R}^2_+} = \left\{(U, V) \in \mathbb{R}^2_+ : U \in [0, \gamma] \text{ and } V = 0\right\}.$$

Indeed $\mathcal{A}_{\partial_U \mathbb{R}^2_+}$ contains the two equilibria in $\partial_U \mathbb{R}^2_+$ as well as the heteroclinic orbit joining these two equilibria.

Proof (Proof of Lemma 8.2) Note that the existence of the boundary attractors follows from the invariance of these sets together with the dissipativity stated in Lemma 8.1. Next to prove the existence of the interior attractor $\mathcal{A}_{\text{Int}\left(\mathbb{R}^2_+\right)}$, it is sufficient (see for instance Hale and Waltman [96]) to show that the state space decomposition $\left(\partial_U \mathbb{R}^2_+ \cup \partial_V \mathbb{R}^2_+; \text{Int}\left(\mathbb{R}^2_+\right)\right)$ is uniformly persistent, namely there exists some constant $\Theta > 0$, such that for each $(U_0, V_0) \in [0, \infty)^2$ with $U_0 > 0$ and $V_0 > 0$

$$\exists \Theta > 0, \quad \forall (U_0, V_0) \in \text{Int}\left(\mathbb{R}^2_+\right), \quad \liminf_{t \to \infty} \min\left(U(t), V(t)\right) \geq \Theta. \tag{8.12}$$

To prove this property, since the two equilibria, $M_1 = (0,0)$ and $M_2 = (\gamma, 0)$, on the boundary $\partial_U \mathbb{R}^2_+ \cup \partial_V \mathbb{R}^2_+$ are chained in the sense of Hale and Waltman's [96]. Therefore it is sufficient to prove the local repulsivity of each of these equilibria with respect to the interior region Int $\left(\mathbb{R}^2_+\right)$. Assume by contradiction that for some $\varepsilon > 0$ small enough one has

$$U_0 > 0 \text{ and } V_0 > 0 \text{ and } U(t) + V(t) \leq \varepsilon, \forall t \geq 0.$$

Then by using the U-equation of (8.9) we obtain

$$U' \geq [\alpha(\gamma - \varepsilon) - \varepsilon]\, U.$$

Therefore by choosing $\varepsilon > 0$ small enough (so that $[\alpha(\gamma - \varepsilon) - \varepsilon] > 0$) we deduce that $U(t) \to \infty$ as $t \to \infty$. which is impossible since the system is dissipative. Similarly assume that

$$V_0 > 0 \text{ and } |U(t) - \gamma| + V(t) \leq \varepsilon, \forall t \geq 0.$$

Then by using the V-equation of (8.9) we obtain

$$V' \geq -V + \beta \frac{(\gamma - \varepsilon)V}{1 + (\gamma - \varepsilon)}.$$

Choosing $\varepsilon > 0$ small enough (so that $\beta \dfrac{(\gamma - \varepsilon)}{1 + (\gamma - \varepsilon)} > 1 \Leftrightarrow (\beta - 1)(\gamma - \varepsilon) > 1$) one deduces that $V(t) \to \infty$ as $t \to \infty$, which is a contradiction, that completes the proof of (8.12). □

Before going to the description of the global attractor, let us first describe the interior attractor. To do so we summarized in the next theorem some important known results about (8.9).

Theorem 8.4 *System* (8.9) *enjoys the following properties.*

*(i) If **(H3)** is satisfied, then the interior equilibrium is globally asymptotically stable for system* (8.9) *restricted to* Int$(\mathbb{R}^2_+)$.
*(ii) If **(H2)** is satisfied, then* (8.9) *admits a unique stable periodic orbit surrounding the interior equilibrium and the system has no other periodic orbit.*

Note that (i) has been proved by Hsu, Hubbell and Waltman [114, Lemma 4] using 18.13's criterion which proved that the system has no periodic orbit whenever **(H3)** holds. More precisely, setting $\varphi(U, V) = \frac{1+U}{U} V^{\xi + 1}$ for some constant $\xi > 0$ such that $\frac{(\gamma-1)\alpha}{\beta-1} < \xi < \left(\frac{1}{\beta-1} - \frac{\gamma-1}{4}\right) \frac{4\alpha}{\beta-1}$, then, the aforementioned works proved that for each $0 < \eta < 1$ there exists $m_\eta > 0$ such that

$$\partial_U\left(\varphi F\right) + \partial_U\left(\varphi G\right) \leq -m_\eta,\ \forall (U, V) \in \left[\eta, \eta^{-1}\right]^2. \tag{8.13}$$

As far as the second point (ii) which is concerned the uniqueness of the periodic orbit was proved by Cheng [**?**] while its stability was proved in [114]. Note also that (8.9) undergoes an Hopf bifurcation around the interior equilibrium whenever we choose the bifurcation parameter γ and this Hopf bifurcation occurs at $\gamma = \gamma^\star$ with $\gamma^\star(\beta - 1) = \beta + 1$.

These results are related to the interior attractor and reformulated as follows.

Corollary 8.5 (Interior attractor) *The following holds.*

(i) *Assume that* ***(H3)*** *holds. Then the interior attractor $\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}$ reduces to the interior equilibrium.*

(ii) *Assume that* ***(H2)*** *holds. Then the interior attractor $\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}$ consists of the unique interior equilibrium, the unique interior periodic orbit and an infinite number of heteroclinic orbits joining the unique interior equilibrium and the unique periodic orbit.*

To complete this section, we are able to describe the global attractor $\mathcal{A}_{\mathbb{R}^2_+}$. Our result reads as follows.

Theorem 8.6 (Global attractor) *Let* ***(H1)*** *be satisfied. Then system* (8.9) *admits a unique heteroclinic orbit (U, V) joining $(\gamma, 0)$ to the boundary of the interior attractor $\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}$. The global attractor $\mathcal{A}_{\mathbb{R}^2_+}$ is composed of three disjoint parts*

$$\mathcal{A}_{\mathbb{R}^2_+} = [0, \gamma] \times \{0\} \bigcup \{(U(t), V(t)),\ t \in \mathbb{R}\} \bigcup \mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}.$$

Proof The proof of this result requires three steps. We firstly derive the existence of heteroclinic orbits using a connectedness argument for the global attractor. Then we show that heteroclinic orbits starts from the stationary point $(\gamma, 0)$ and finally we conclude to the uniqueness of such heroclinic orbit by using a center unstable manifold argument (see [59] where a rather similar argument was used to derive a uniqueness property for traveling wave solutions arising in some epidemic problem).
Connectedness arguments: The largest global attractor $\mathcal{A}_{\mathbb{R}^2_+}$ is connected since it attracts the convex subset $\mathbb{T}$. It follows that the projection of $\mathcal{A}_{\mathbb{R}^2_+}$ on the horizontal and vertical axis is a compact interval.

The global attractor $\mathcal{A}_{\mathbb{R}^2_+}$ contains the interior global attractor $\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}$ which is compact, connected and locally stable. The global attractor $\mathcal{A}_{\mathbb{R}^2_+}$ also contains the boundary attractor $\mathcal{A}_{\partial_U \mathbb{R}^2_+}$. The connectedness of $\mathcal{A}_{\mathbb{R}^2_+}$ and compactness of $\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}$ and $\mathcal{A}_{\partial_U \mathbb{R}^2_+}$ imply

$$\mathcal{A}_{\mathbb{R}^2_+} - \left(\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)} \bigcup \mathcal{A}_{\partial_U \mathbb{R}^2_+}\right) \neq \emptyset.$$

Moreover by using Theorem 3.2 due to Hale and Waltman [96] we deduce that for each point $(U, V) \in \mathcal{A}_{\mathbb{R}^2_+} - \left(\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)} \cup \mathcal{A}_{\partial_U \mathbb{R}^2_+}\right)$ the alpha and limit sets satisfy the following

$$\alpha(U, V) \in \mathcal{A}_{\partial_U \mathbb{R}^2_+} \text{ and } \omega(U, V) \in \mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}.$$

Finally since the boundary attractor has a Morse decomposition $M_1 = \{(0, 0)\}$ and $M_2 = \{(\gamma, 0)\}$ we have either

$$\alpha(U, V) = M_1 \text{ or } \alpha(U, V) = M_2, \forall (U, V) \in \mathcal{A}_{\mathbb{R}^2_+} - \left(\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)} \bigcup \mathcal{A}_{\partial_U \mathbb{R}^2_+}\right).$$

No existence of heteroclinic orbit starting from $(0,0)$**:** Assume by contradiction that there exists one. Next by looking the V-equation we deduce that

$$V(t) = \exp\left(\int_{t_0}^{t}\left(-1+\frac{U(s)}{1+U(s)}\right)ds\right)V(t_0),\ \forall t,t_0\in\mathbb{R}.$$

Since $V(t_0)>0$ and there exists $T<0$ such that $U(t)$ remains sufficiently small for all negative times $t<T$, so that $V(t)\to\infty$ as $t\to-\infty$ that which contradicts the boundedness of the global attractor.

Existence and uniqueness of an heteroclinic orbit starting from $(\gamma,0)$**:** We only need to prove the uniqueness. The linearized equation around $(\gamma,0)$ has two eigenvalues: $\lambda_1=-\alpha\gamma<0$ and $\lambda_2=\frac{\beta\gamma}{1+\gamma}-1>0$ with eigenspaces

$$E_{\lambda_1}=\left\{(U,V)\in\mathbb{R}^2:V=0\right\}\text{ and }E_{\lambda_2}=\left\{(U,V)\in\mathbb{R}^2:U-\gamma=-\frac{\gamma}{\gamma(\beta-1)-1+\alpha\gamma(1+\gamma)}V\right\}$$

with $\frac{\gamma}{\gamma(\beta-1)-1+\alpha\gamma(1+\gamma)}>0$ Note that $\mathbb{R}^2=E_{\lambda_1}\bigoplus E_{\lambda_2}$.
The center-unstable manifold at $(\gamma,0)$ is one dimensional. Let $\psi_{cu}:E_{\lambda_2}\to E_{\lambda_1}$ be a C^1 center-unstable manifold and consider the one dimensional manifold defined by

$$M_{cu}:=\{x_{cu}+\psi_{cu}(x_{cu}):x_{cu}\in E_{\lambda_2}\}.$$

It is locally invariant under the semiflow generated by (8.9) around $(\gamma,0)$. Since $D_{x_{cu}}\psi_{cu}(0)=0$, the manifold M_{cu} is tangent to E_{λ_2} at $(\gamma,0)$. Moreover we know that there exists $\varepsilon>0$, such that M_{cu} contains all negative orbits of the semiflow generated by (8.9) staying in the ball $B_{\mathbb{R}^2}((\gamma,0),\varepsilon)$ for all negative times.

In order to prove the uniqueness, we assume that there exists two heteroclinic orbits

$$O_1=(U_1(t),V_1(t))_{t\in\mathbb{R}}\subset\text{Int}\left(\mathbb{R}^2_+\right)\text{ and }O_2=(U_2(t),V_2(t))_{t\in\mathbb{R}}\subset\text{Int}\left(\mathbb{R}^2_+\right)$$

going from $(\gamma,0)$ to the interior attractor $\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}$. Since

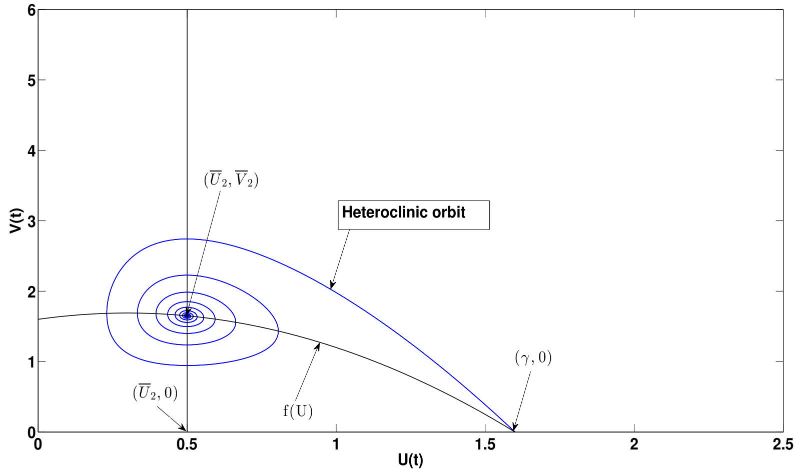
$$\lim_{t\to-\infty}(U_1(t),V_1(t))=(\gamma,0)\text{ and }\lim_{t\to-\infty}(U_2(t),V_2(t))=(\gamma,0),$$

without loss of generality, one may assume that

$$(U_1(t),V_1(t))_{t\le0}\subset B_{\mathbb{R}^2}((\gamma,0),\varepsilon)\text{ and }(U_2(t),V_2(t))_{t\le0}\subset B_{\mathbb{R}^2}((\gamma,0),\varepsilon)$$

which imply that $(U_1(t),V_1(t))_{t\le0}\subset M_{cu}$ and $(U_2(t),V_2(t))_{t\le0}\subset M_{cu}$.
Let Π_{λ_1} and Π_{λ_2} be the linear projectors from $\mathbb{R}^2$ to E_{λ_1} and E_{λ_2}, respectively. We can find $t_1<0$ and $t_2<0$ such that $\Pi_{\lambda_2}(U_1(t_1),V_1(t_1))=\Pi_{\lambda_2}(U_2(t_2),V_2(t_2))$ and then $\psi_{cu}(\Pi_{\lambda_2}(U_1(t_1),V_1(t_1)))=\psi_{cu}(\Pi_{\lambda_2}(U_2(t_2),V_2(t_2)))$. Thus $(U_1(t_1),V_1(t_1))=(U_2(t_2),V_2(t_2))$. By the uniqueness of the solutions for system (8.9), we get $(U_1(t_1+\cdot),V_1(t_1+\cdot))=(U_2(t_2+\cdot),V_2(t_2+\cdot))$ and thus $O_1=O_2$. The uniqueness of the heteroclinic orbit starting from $(\gamma,0)$ follows and this completes the proof of the theorem.

(a)



(b)

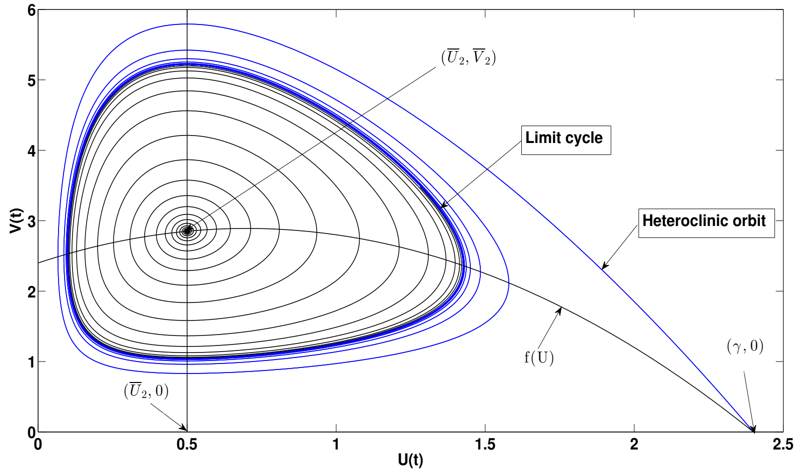


Fig. 8.1: *In this figure we run a simulation of the Rosenzweig-MacArthur model with* $\alpha = 1$, $\beta = 3$ *and* $\gamma = 18.7$ *(in Figure (a)) and* $\gamma = 2.4$ *(in Figure (b)). In both figures we plot the heteroclinic orbit joining the boundary equilibrium and the interior equilibrium (in Figure (a)) and the interior limit cycle which is a stable periodic orbit (in Figure (b)). In this figure we also plot the nullclines* $f(U) = \alpha(\gamma - U)(1 + U)$ *and* $U = \overline{U}_2$.

8.3 Application of a center manifold theorem to the traveling wave problem

This section is devoted to the study of traveling wave profile system of equations (8.8) for $\varepsilon \ll 1$. We will firstly apply a center manifold reduction on a suitable invariant region. The reduced system will be analysed. In the same spirit as in the previous section we will describe its global and interior attractor to obtain various results about the existence and uniqueness of traveling wave solutions as well as refined information about periodic wave trains.

8.3.1 Reduction of the traveling wave problem

Transformed system: In order to work with a subspace of equilibria for $\varepsilon = 0$ we use the following change of variable

$$\begin{cases} U_1 = u_1, \\ U_2 = u_2 + F(u_1, v_1), \\ V_1 = v_1, \\ V_2 = v_2 + G(u_1, v_1) \end{cases} \Leftrightarrow \begin{cases} u_1 = U_1, \\ u_2 = U_2 - F(U_1, V_1), \\ v_1 = V_1, \\ v_2 = V_2 - G(U_1, V_1). \end{cases} \tag{8.14}$$

By using this change of variable the system (8.8) becomes

$$\begin{cases} U_1' = \varepsilon u_2 = \varepsilon \left[U_2 - F(U_1, V_1) \right], \\ dU_2' = du_2' + dF(U_1, V_1)' = -U_2 + d\partial_u F(U_1, V_1) U_1' + d\partial_v F(U_1, V_1) V_1', \\ V_1' = \varepsilon v_2 = \varepsilon \left[V_2 - G(U_1, V_1) \right], \\ V_2' = v_2' + G(U_1, V_1)' = -V_2 + \partial_u G(U_1, V_1) U_1' + \partial_v G(U_1, V_1) V_1' \end{cases}$$

and therefore we obtain

$$\begin{cases} U_1' = \varepsilon \left[U_2 - F(U_1, V_1) \right], \\ dU_2' = -U_2 + \varepsilon d P(U_1, U_2, V_1, V_2), \\ V_1' = \varepsilon \left[V_2 - G(U_1, V_1) \right], \\ V_2' = -V_2 + \varepsilon Q(U_1, U_2, V_1, V_2), \end{cases} \tag{8.15}$$

wherein P and Q are given by

$$P(U_1, U_2, V_1, V_2) = \partial_u F(U_1, V_1) \left[U_2 - F(U_1, V_1) \right] + \partial_v F(U_1, V_1) \left[V_2 - G(U_1, V_1) \right]$$

and

$$Q(U_1, U_2, V_1, V_2) = \partial_u G(U_1, V_1) \left[U_2 - F(U_1, V_1) \right] + \partial_v G(U_1, V_1) \left[V_2 - G(U_1, V_1) \right].$$

Truncated system: Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that

$$\rho(x) = \begin{cases} 1, \text{ if } x \geq 0, \\ \in [0,1], \text{ if } x \in [-1/2, 0], \\ 0, \text{ if } x \leq -1/2. \end{cases}$$

Define the set

$$\mathbb{E} = \left\{(U_1, U_2, V_1, V_2) \in \mathbb{R}^4 : (U_1, V_1) \in \mathbb{T} \text{ and } |U_2 - F(U_1, V_1)| \leq 1, |V_2 - G(U_1, V_1)| \leq 1\right\}.$$

Let $L > 0$ be given large enough such that

$$\begin{aligned} L \geq 2 + &\max_{(U_1,V_1)\in\mathbb{T}} |F(U_1, V_1)| + \max_{(U_1,V_1)\in\mathbb{T}} |G(U_1, V_1)| \\ &+ \max_{(U_1,U_2,V_1,V_2)\in\mathbb{E}} |P(U_1, U_2, V_1, V_2)| + \max_{(U_1,U_2,V_1,V_2)\in\mathbb{E}} |Q(U_1, U_2, V_1, V_2)|. \end{aligned}$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that

$$\chi(x) = \begin{cases} 1, \text{ if } x \in [-L, L], \\ \in [0,1], \text{ if } x \in [-(L+1), -L] \cup [L, (L+1)], \\ 0, \text{ if } x \notin [-(L+1), (L+1)]. \end{cases}$$

Then we have

$$\begin{cases} U_1' &= \varepsilon \left[U_2 - F(U_1, V_1)\right] \chi(U_2 - F(U_1, V_1))\rho(U_1), \\ dU_2' &= -U_2 + \varepsilon dP(U_1, U_2, V_1, V_2)\chi(P(U_1, U_2, V_1, V_2))\rho(U_1), \\ V_1' &= \varepsilon \left[V_2 - G(U_1, V_1)\right] \chi(V_2 - G(U_1, V_1))\rho(U_1), \\ V_2' &= -V_2 + \varepsilon Q(U_1, U_2, V_1, V_2)\chi(Q(U_1, U_2, V_1, V_2))\rho(U_1). \end{cases} \tag{8.16}$$

Define

$$h(x) = x\chi(x), \ x \in \mathbb{R}.$$

Then system (8.16) can be rewritten as

$$\begin{cases} U_1' &= \varepsilon\, h\left(U_2 - F(U_1, V_1)\right) \rho(U_1), \\ dU_2' &= -U_2 + \varepsilon\, d\, h\left(P(U_1, U_2, V_1, V_2)\right) \rho(U_1), \\ V_1' &= \varepsilon\, h\left(V_2 - G(U_1, V_1)\right) \rho(U_1), \\ V_2' &= -V_2 + \varepsilon\, h\left(Q(U_1, U_2, V_1, V_2)\right) \rho(U_1). \end{cases} \tag{8.17}$$

Remark 8.7 In this truncation procedure the function $\rho(U_1)$ serves to avoid the singularity at $U_1 = -1$ in F and G. The function $h(.)$ is used to obtain a bounded Lipschitz perturbation of the system with $\varepsilon = 0$.

By setting $X(t) = (U_1(t), V_1(t))$ and $Y(t) = (U_2(t), V_2(t))$, system (8.17) takes the following form

$$\begin{cases} X'(t) = \varepsilon \widetilde{F}(X(t), Y(t)), \\ Y'(t) = -DY(t) + \varepsilon \widetilde{G}(X(t), Y(t)), \end{cases} \tag{8.18}$$

where $\widetilde{F}, \widetilde{G} \in C^\infty\left(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2\right)$ are bounded and Lipschitz continuous functions and where we have set $D = \text{diag}(d^{-1}, 1)$. Therefore the central space is given by

$$X_c = \left\{(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2 : Y = 0\right\},$$

while the stable space reads as

$$X_s = \left\{(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2 : X = 0\right\}.$$

Remark 8.8 Due to the choice of the constant $L > 0$ the truncated system (8.17) coincides with the original system (8.15) whenever $(U_1, U_2, V_1, V_2) \in \mathbb{E}$. Moreover the equilibria of system (8.15) belong to $\mathbb{E}$ (since $U_1 = u_1$ and $V_1 = v_1$ and the equilibria of (8.8) satisfy $u_2 = v_2 = 0$ and (u_1, v_1) must be an equilibrium of (8.9)). Conversely the equilibria of (8.17) in $\mathbb{E}$ must satisfy

$$\widetilde{U}_2 = F(\widetilde{U}_1, \widetilde{V}_1) = 0 \text{ and } \widetilde{V}_2 = G(\widetilde{U}_1, \widetilde{V}_1) = 0.$$

Now by using Lemma 8.1 we have $(\widetilde{U}_1, \widetilde{V}_1) \in \mathbb{T}$.

For $\eta > 0$ and $p \in \mathbb{N} \setminus \{0\}$ we define the weighted spaces

$$BC^\eta(\mathbb{R}; \mathbb{R}^p) = \left\{u \in C(\mathbb{R}, \mathbb{R}^p) : \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|u(t)\| < \infty\right\}.$$

Moreover for $\varepsilon > 0$ small enough we can apply the smooth center manifold theorem proved by Vanderbauwhede [216, Theorem 3.1] and Vanderbauwhede and Iooss [217, Theorem 1]. This yields the following reduction result.

Theorem 8.9 *Let $\eta \in (0, \min(1, 1/d))$ be given and fixed. Then there exists $\widetilde{\varepsilon}_0 > 0$ such that for each $\varepsilon \in [0, \widetilde{\varepsilon}_0]$ we can find a map $\Phi_\varepsilon = \left(\Phi_\varepsilon^1, \Phi_\varepsilon^2\right) \in C^k(\mathbb{R}^2, \mathbb{R}^2)$, for each integer $k > 0$, satisfying the following properties*

$$\Phi_\varepsilon(0_\mathbb{R}) = 0_{\mathbb{R}^2} \text{ and } D\Phi_\varepsilon(0_\mathbb{R}) = 0_{\mathcal{L}(\mathbb{R}^2)},$$

and Φ_ε is bounded as well as its derivatives up to the order k and

$$\lim_{\varepsilon \to 0} \|\Phi_\varepsilon\|_\infty = 0 \text{ and } \lim_{\varepsilon \to 0} \|\Phi_\varepsilon\|_{\text{Lip}} = 0.$$

Moreover we have the following properties:

(i) *The **global center manifold** $M_\varepsilon = \{(X, Y) : Y = \Phi_\varepsilon(X)\}$ is invariant by the semiflow generated by (8.17) (forward and backward in time). Namely if $t \to X(t)$ is a solution of the **reduced system** on some interval $I \subset \mathbb{R}$*

$$X'(t) = \varepsilon \widetilde{F}(X(t), \Phi_\varepsilon(X(t))), \forall t \in I, \tag{8.19}$$

then $t \to (X(t), \Phi_\varepsilon(X(t)))$ is a solution of (8.17) on I.

(ii) *If $t \to (X(t), Y(t))$ is a solution of (8.17) on $\mathbb{R}$ which belongs to $BC^\eta(\mathbb{R}; \mathbb{R}^4)$, then*

$$(X(t), Y(t)) \in M_\varepsilon, \forall t \in \mathbb{R} \Leftrightarrow Y(t) = \Phi_\varepsilon(X(t)), \forall t \in \mathbb{R}.$$

Now let us prove the following invariance result.

Proposition 8.10 *There exists $\varepsilon_0 \in (0, \widetilde{\varepsilon}_0]$ such that triangle $\mathbb{T}$ is negatively invariant by the flow generated by the reduced system (8.19). That is to say that*

$X'(t) = \varepsilon \widetilde{F}(X(t), \Phi_\varepsilon(X(t))), \forall t \in \mathbb{R}$ *and* $X(0) = X_0 \in \mathbb{T} \;\Rightarrow\; X(t) \in \mathbb{T}$ *for all* $t \leq 0$.

Furthermore the following subsets are both negatively invariant by the flow generated by the reduced system (8.19).

$$\partial_u \mathbb{T} = \{(U,V) \in \mathbb{T} : V = 0\} \text{ and } \partial_v \mathbb{T} = \{(U,V) \in \mathbb{T} : U = 0\}.$$

Proof In the first step, we investigate the invariance for the boundary regions $\partial_u \mathbb{T}$ and $\partial_v \mathbb{T}$. To that aim we claim that

$$\begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \Phi_\varepsilon \begin{pmatrix} 0 \\ V_1 \end{pmatrix} \Rightarrow U_2 = 0. \tag{8.20}$$

Indeed, assume that $U_1 = U_2 = 0$ in system, then

$$U_1 = U_2 = 0 \Rightarrow U_2 - F(U_1, V_1) = 0 \text{ and } P(U_1, U_2, V_1, V_2) = 0.$$

Therefore the two last components of the truncated system (8.16) become

$$\begin{cases} V_1' = \varepsilon\, h\left(V_2 - G(0, V_1)\right), \\ V_2' = -V_2 + \varepsilon\, h\left(Q(0, 0, V_1, V_2)\right). \end{cases} \tag{8.21}$$

Now by applying the center manifold theorem to (8.21) (which applies for the value of $\varepsilon \in (0, \widetilde{\varepsilon}_0)$ since the estimations for systems (8.17) and (8.21) remain unchanged in the proof of the center manifold theorem), we deduce that we can find a map $\Psi_\varepsilon \in C^k(\mathbb{R}, \mathbb{R})$ such that the center manifold of the two dimensional system (8.21)

$$V_2 = \Psi_\varepsilon(V_1)$$

and the solution $t \to (V_1^\star(t), V_2^\star(t))$ of (8.21) starting from an initial value $(V_1, \Psi_\varepsilon(V_1))$ satisfies

$$(V_1^\star, V_2^\star) \in BC^\eta(\mathbb{R}; \mathbb{R}^2).$$

We conclude that

$$(U_1, U_2, V_1, V_2) = (0, 0, V_1^\star, V_2^\star) \in BC^\eta(\mathbb{R}; \mathbb{R}^4)$$

is a solution of the truncated system (8.17). This completes the proof of the claim.

By using similar argument one deduces that

$$\begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \Phi_\varepsilon \begin{pmatrix} U_1 \\ 0 \end{pmatrix} \text{ and } U_1 \geq 0 \Rightarrow V_2 = 0. \tag{8.22}$$

We now turn to the invariance of the triangle $\mathbb{T}$. By using the fact that

$$\begin{cases} U_1' = \varepsilon\, h\left(\Phi_\varepsilon^1(U_1, V_1) - F(U_1, V_1)\right) \rho(U_1), \\ V_1' = \varepsilon\, h\left(\Phi_\varepsilon^2(U_1, V_1) - G(U_1, V_1)\right) \rho(U_1). \end{cases} \tag{8.23}$$

Whenever $\beta U_1 + V_1 = R$ and $U_1 \geq 0$ and $V_1 \geq 0$ in system (8.22), then $\rho(U_1) = 1$ and for $\varepsilon > 0$ small enough (h coincides with identity)

$$\beta U_1' + V_1' = \varepsilon \left(\Phi_\varepsilon^1(U_1, V_1) + \Phi_\varepsilon^2(U_1, V_1) - F(U_1, V_1) - G(U_1, V_1) \right) > 0.$$

Therefore by combining this fact together with (8.19) and (8.21), we deduce that the triangle $\mathbb{T}$ is negatively invariant by the reduced system. □

8.3.2 Global attractors

We investigate preliminary properties of the perturbed two-dimensional (reduced) system (8.19). Recall that **(H1)** is satisfied along this paper and $\mathbb{T}$, $\partial_u \mathbb{T}$, $\partial_v \mathbb{T}$ are negatively invariant with respect to this system for all $\varepsilon \in (0, \varepsilon_0]$. Before going further, by setting $t = -\varepsilon s$ and $\left(\tilde{U}, \tilde{V}\right)(s) = (U_1, U_2)(t)$ the above system (8.19) becomes, dropping the tilde for notational simplicity

$$\begin{cases} U' = \left[-\Phi_\varepsilon^1(U, V) + F(U, V) \right] := F_\varepsilon(U, V), \\ V' = \left[-\Phi_\varepsilon^2(U, V) + G(U, V) \right] := G_\varepsilon(U, V). \end{cases} \tag{8.24}$$

Notice that $\mathbb{T}$, $\partial_u \mathbb{T}$ and $\partial_v \mathbb{T}$ become positively invariant with respect to the above system. Then, for each such $\varepsilon \in [0, \varepsilon_0]$, we denote by $\{T_\varepsilon(t)\}_{t \geq 0}$ the strongly continuous semiflow on the triangle $\mathbb{T}$ generated by (8.24). One may also observe it continuously depends on ε, namely the map $(\varepsilon, t, X) \longrightarrow T_\varepsilon(t)X$ is continuous from $[0, \varepsilon_0] \times [0, \infty) \times \mathbb{T}$ into $\mathbb{T}$. Our first result reads as follows:

Lemma 8.11 *Let $\varepsilon \in [0, \varepsilon_0]$ be given. Then the semiflow $\{T_\varepsilon(t)\}_{t \geq 0}$ possesses a compact and connected global attractor $\mathcal{A}_\varepsilon \subset \mathbb{T}$ attracting $\mathbb{T}$ in the sense that*

$$\text{dist}\ (T_\varepsilon(t)X, \mathcal{A}_\varepsilon) \longrightarrow 0 \textit{ as } t \longrightarrow \infty \textit{ uniformly for } X \in \mathbb{T},$$

wherein $\text{dist}\,(X, \mathcal{A}_\varepsilon) = \inf_{Y \in \mathcal{A}_\varepsilon} \|X - Y\|$ *denotes the Euclidean distance from $X \in \mathbb{R}^2$ to $\mathcal{A}_\varepsilon$.*

Proof Fix $\varepsilon \in (0, \varepsilon_0]$. Since, for each $t \geq 0$, $T_\varepsilon(t) : \mathbb{T} \longrightarrow \mathbb{T}$ is completely continuous and bounded dissipative ($\mathbb{T}$ is compact), Theorem 3.4.8 in [94] ensures the existence of a global attractor for the semiflow T_ε. In addition, since $\mathbb{T}$ is connected, the result of Gobbino and Sardella [84] applies and ensures that $\mathcal{A}_\varepsilon$ is connected. □

Lemma 8.12 *The family $(\mathcal{A}_\varepsilon)_{\varepsilon \in [0, \varepsilon_0]}$ is upper semi-continuous, in the sense that for each $\widehat{\varepsilon} \in [0, \varepsilon_0]$ one has*

$$\lim_{\varepsilon \to \widehat{\varepsilon}} \delta\left(\mathcal{A}_\varepsilon, \mathcal{A}_{\widehat{\varepsilon}}\right) = 0,$$

wherein $\delta\left(\mathcal{A}_\varepsilon, \mathcal{A}_{\widehat{\varepsilon}}\right)$ is given by

$$\delta\left(\mathcal{A}_\varepsilon, \mathcal{A}_{\widehat{\varepsilon}}\right) = \sup_{y \in \mathcal{A}_\varepsilon} \text{dist}\,\left(y, \mathcal{A}_{\widehat{\varepsilon}}\right).$$

Proof Since the map $(\varepsilon, t, X) \mapsto T_\varepsilon(t)X$ is continuous from $[0, \varepsilon_0] \times [0, \infty) \times \mathbb{T}$ into the compact set $\mathbb{T}$, Theorem 3.5.2 in [94] ensures that the family $\{A_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ is upper semi-continuous. □

We continue this section by further studying some properties of the global attractor $\mathcal{A}_\varepsilon$. To that aim, we define

$$\partial\mathbb{T}^0 = \partial_u\mathbb{T} \cup \partial_v\mathbb{T} \text{ and } \mathbb{T}^0 = \mathbb{T} \setminus \partial\mathbb{T}^0 = \{(U, V) \in \mathbb{T} : \ U > 0 \text{ and } V > 0\}.$$

Here let us recall that, for all $\varepsilon \in [0, \varepsilon_0]$ and $t \geq 0$, one has

$$T_\varepsilon(t)\mathbb{T}^0 \subset \mathbb{T}^0 \text{ and } T_\varepsilon(t)\partial\mathbb{T}^0 \subset \partial\mathbb{T}^0. \tag{8.25}$$

We prove the following uniform persistence result for T_ε.

Lemma 8.13 *There exists $\varepsilon_1 \in (0, \varepsilon_0]$ and $\Theta > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$ and $X \in \mathbb{T}^0$ one has*

$$\liminf_{t\to\infty} \text{dist}\left(T_\varepsilon(t)X, \partial\mathbb{T}^0\right) \geq \Theta.$$

The proof of this lemma relies on the application of the results of Hale and Waltman in [96].

Proof Firstly recall that

$$(F_\varepsilon, G_\varepsilon) \to (F, G) \ \text{ as } \varepsilon \to 0 \text{ in } C^1(\mathbb{T}).$$

Next fix $\varepsilon_1 \in (0, \varepsilon_0]$ such that

$$\begin{aligned} &\partial_U F_\varepsilon(0,0) > \frac{1}{2}\partial_U F(0,0) > 0, \quad \partial_U F_\varepsilon(\gamma,0) < \frac{1}{2}\partial_U F(\gamma,0) < 0,\\ &\partial_V G_\varepsilon(0,0) < \frac{1}{2}\partial_V G(0,0) < 0, \quad \partial_V G_\varepsilon(\gamma,0) > \frac{1}{2}\partial_V G(\gamma,0) > 0. \end{aligned} \tag{8.26}$$

Now, in order to apply the result of Hale and Waltman, consider the extended semiflow $U(t) : \mathbb{T} \times [0, \varepsilon_1] \to \mathbb{T} \times [0, \varepsilon_1]$ given by

$$U(t)\begin{pmatrix} X \\ \varepsilon \end{pmatrix} := \begin{pmatrix} T_\varepsilon(t)X \\ \varepsilon \end{pmatrix}, \ \forall \begin{pmatrix} X \\ \varepsilon \end{pmatrix} \in \mathbb{T} \times [0, \varepsilon_1].$$

Then U becomes a strongly continuous semiflow on the compact set $X := \mathbb{T}\times[0, \varepsilon_1]$. Next consider the two positively invariant sets (see (8.24))

$$X^0 := \mathbb{T}^0 \times [0, \varepsilon_1] \text{ and } \partial X^0 = \partial\mathbb{T}^0 \times [0, \varepsilon_1].$$

Now in order to prove the lemma, we will show that the pair $\left(\partial X^0, X^0\right)$ is uniformly persistent with respect to the extended semiflow U. To that aim, observe that U possesses a compact global attractor, denoted by A. Then $U|_{\partial X^0}$ also admits a global attractor $A_\partial = ([0, \gamma] \times \{0\})\times[0, \varepsilon_1]$ while $\tilde{A}_\partial := \bigcup_{Z\in A_\partial} \omega(X)$ can be decomposed as the follows

$$\tilde{A}_\partial = M_1 \bigcup M_2 \text{ with } M_1 := \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \times [0, \varepsilon_1] \text{ and } M_2 := \left\{\begin{pmatrix} \gamma \\ 0 \end{pmatrix}\right\} \times [0, \varepsilon_1],$$

that corresponds to a covering of $\tilde{A}_\partial$ by disjoint compact isolated invariant sets M_1 and M_2 for $U|_{\partial X^0}$. Furthermore M_1 is chained to M_2 and this covering is acyclic (see [96]), since $\partial_U F_\varepsilon(0,0) > 0$ and $\partial_U F_\varepsilon(\gamma, 0) < 0$.

Next since $\{U(t)\}_{t\geq 0}$ is bounded dissipative and completely continuous on X for each $t \geq 0$, in view of Theorem 4.1 in [96] to prove that the pair $\left(\partial X^0, X^0\right)$ is uniformly persistent, it is sufficient to check that

$$W^s(M_i) \cap X^0 = \emptyset, \ \forall i = 1, 2.$$

This latter property follows from the same repulsiveness arguments as the ones developed in Lemma 8.2 using the inequalities in (8.25). □

Using the above lemma one obtains the following decomposition result.

Proposition 8.14 *For each $\varepsilon \in [0, \varepsilon_1]$, there exist a global attractor $\mathcal{A}_{0,\varepsilon} \subset \mathbb{T}^0$ and a global attractor $\mathcal{A}_{\partial,\varepsilon}$ in $\partial\mathbb{T}^0$ for T_ε and the following decomposition for the global attractor $\mathcal{A}_\varepsilon$ (provided by Lemma 8.12) holds true*

$$\mathcal{A}_\varepsilon = \mathcal{A}_{0,\varepsilon} \bigcup W^u\left(\mathcal{A}_{\partial,\varepsilon}\right), \tag{8.27}$$

where $W^u\left(\mathcal{A}_{\partial,\varepsilon}\right) = \left\{X \in \mathcal{A}_\varepsilon : \alpha(X) \subset \mathcal{A}_{\partial,\varepsilon}\right\}$. Furthermore the family $\left(\mathcal{A}_{0,\varepsilon}\right)_{\varepsilon\in[0,\varepsilon_0]}$ is upper semi-continuous.

Proof The proof of the above result relies on the application of Theorem 3.2 in [96] and Theorem 18.1 in [151]. To see this, let us first observe that the result in Lemma 8.13 can be reformulated as follows:

$$\liminf_{t\to\infty} \operatorname{dist}\left(T_\varepsilon(t)X, \partial\mathbb{T}^0\right) \geq \Theta,$$

for all $X \in \mathbb{T}^0$ and all $\varepsilon \in [0, \varepsilon_1]$. Hence, since for each $\varepsilon \in [0, \varepsilon_1]$, T_ε is completely continuous and bounded dissipative and satisfies (8.24), the existence $\mathcal{A}_{0,\varepsilon}$ $\mathcal{A}_{\partial,\varepsilon}$ together with the decomposition (8.26) follows from the results in [96]. Next, using Lemma 8.12 and 8.13, the results of Magal in [151] applies and ensures the upper semi-continuity for the family of interior attractors $\left\{\mathcal{A}_{0,\varepsilon}\right\}_{\varepsilon\in[0,\varepsilon_1]}$. This completes the proof of the proposition. □

Remark 8.15 One may notice that, for all $\varepsilon \in [0, \varepsilon_1]$ one has $\mathcal{A}_{\partial,\varepsilon} = [0, \gamma] \times \{0\}$. This point has – implicitly – already been used in the proof of Lemma 8.13.

In the following, we discuss some properties of the interior attractor $\mathcal{A}_{0,\varepsilon}$ for $\varepsilon \in (0, \varepsilon_1]$. Our first result consists in the perturbation of Corollary 8.5 (i) and it reads as follows.

Theorem 8.16 *Assume that **(H3)** holds. Then there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that*

$$\mathcal{A}_{0,\varepsilon} = \left\{\begin{pmatrix}\overline{U}_2\\ \overline{V}_2\end{pmatrix}\right\}, \ \forall \varepsilon \in [0, \varepsilon_2].$$

In other words, the interior attractor reduces to the interior equilibrium for all $\varepsilon > 0$ small enough.

Proof The proof of this result relies on the application of 18.13's criterion. Note that due to Lemma 8.13, one has

$$\inf_{X\in\partial\mathbb{T}^0} \text{dist}\left(X, \mathcal{A}_{0,\varepsilon}\right) \geq \Theta,\ \forall\varepsilon \in [0, \varepsilon_0].$$

Let $K \subset \mathbb{T}$ be compact such that

$$\inf_{X\in\partial\mathbb{T}^0} \text{dist}\ (X, K) \geq \frac{\Theta}{2} \text{ and } A_{0,\varepsilon} \subset K\ \ \forall\varepsilon \in [0, \varepsilon_0].$$

As for the proof of Theorem 8.4 (i) given in [114], we consider the function $\varphi(U, V) = \dfrac{1+U}{U}V^{\xi+1}$ for some suitable ξ. Then, since $(F_\varepsilon, G_\varepsilon) \to (F, G)$ as $\varepsilon \to 0$ for the topology of $C^1(\mathbb{T})$, one has

$$[\partial_U(\varphi F_\varepsilon) + \partial_V(\varphi G_\varepsilon)] \to [\partial_U(\varphi F) + \partial_V(\varphi G)],$$

uniformly for $(U, V) \in K$ as $\varepsilon \to 0$. According to the computations (8.13) recalled in Theorem 8.4 one has

$$\max_{(U,V)\in K} [\partial_U(\varphi F) + \partial_V(\varphi G)] < 0.$$

As a consequence, there exists $\varepsilon_2 \in (0, \varepsilon_1]$ small enough and $\delta > 0$ such that, for all $\varepsilon \in [0, \varepsilon_2]$ one has

$$[\partial_U(\varphi F_\varepsilon) + \partial_V(\varphi G_\varepsilon)] \leq -\delta,\ \forall(U, V) \in K.$$

Since $\mathcal{A}_{0,\varepsilon} \subset K$ for all ε small enough, the result follows using 18.13's criterion. □

Lemma 8.17 *Assume that **(H2)** holds. Then there exists $\varepsilon_3 \in (0, \varepsilon_1]$ such that the interior equilibrium $\left(\overline{U}_2, \overline{V}_2\right)$ is an unstable spiral points for the semiflow T_ε, for all $\varepsilon \in [0, \varepsilon_3]$. More precisely, the linearized equation of system (8.24) around the interior equilibrium has two complex conjugated eigenvalues with strictly positive real parts, that is a two dimensional unstable manifold.*

Proof Consider the Jacobian matrix, denoted by J_ε, associated to (8.24) at $\left(\overline{U}_2, \overline{V}_2\right)$. Since $(F_\varepsilon, G_\varepsilon)$ is $C^1(\mathbb{T})$−close to (F, G) as $\varepsilon \to 0$, one has

$$J_\varepsilon = J + o(1) \text{ as } \varepsilon \to 0.$$

Herein J is the Jacobian matrix at $\left(\overline{U}_2, \overline{V}_2\right)$ of (8.24) with $\varepsilon = 0$ (that corresponds to system (8.9)). It is readily checked that the eigenvalues $\lambda_\pm$ of J are simple so that the eigenvalues of J_ε, $\lambda_{\pm,\varepsilon}$ are simple and continuous with respect to ε. Hence $\lambda_{\pm,\varepsilon} = \lambda_\pm + o(1)$. This completes the proof of the result since $\lambda_\pm$ are conjugated complex numbers with positive real parts. □

Note that the system (8.24) has the same equilibria as system (8.9) and the system (8.24) has the boundary equilibria $(\overline{U}_0, \overline{V}_0)$ and $(\overline{U}_1, \overline{V}_1)$ given in (8.10) and the unique *interior equilibrium* $(\overline{U}_2, \overline{V}_2)$ defined in (8.11), whenever **(H1)** holds.

As a consequence of the Poincaré-Bendixon theorem, one obtains the following corollary.

Corollary 8.18 *Assume that* ***(H2)*** *holds. Then there exists* $\varepsilon_4 \in (0, \varepsilon_3]$ *such that for all* $\varepsilon \in [0, \varepsilon_4]$*, the interior attractor* $\mathcal{A}_{0,\varepsilon}$ *contains a (non-trivial) periodic orbit surrounding the interior equilibrium* $\left(\overline{U}_2, \overline{V}_2\right)$.

8.3.3 Uniqueness of the periodic orbit and interior attractor

In this section we discuss the uniqueness of the periodic orbit for system (8.24) and its relationship with the global interior attractor when the parameters satisfy the condition

$$\textbf{(H2)} \;\Leftrightarrow\; \overline{U}_2 < \frac{\gamma - 1}{2}. \tag{8.28}$$

The aim of this section is to prove the following uniqueness result.

Theorem 8.19 (Unique stable periodic orbit) *Under condition* (8.28)*, for all* $\varepsilon > 0$ *small enough, there exists a unique stable periodic orbit surrounding the interior equilibrium and the system has no other periodic orbit.*

According to Corollary 8.18, for each $\varepsilon > 0$ small enough, let $(U_\varepsilon(t), V_\varepsilon(t))$ denotes any non constant periodic orbit of (8.24) and $T_\varepsilon > 0$ its period. The associated closed curve is denoted by Γ_ε, that is

$$\Gamma_\varepsilon = \{(U_\varepsilon(t), V_\varepsilon(t)) \,,\; t \in [0, T_\varepsilon]\} \,.$$

Recall that Γ_ε encloses the interior equilibrium $(\overline{U}_2, \overline{V}_2)$. Note also that $\Gamma_\varepsilon \subset \mathcal{A}_{0,\varepsilon}$. Hence Proposition 8.14 ensures that there exists $\theta > 0$ such that for all $\varepsilon > 0$ small enough

$$U_\varepsilon(t) \geq \theta, \; V_\varepsilon(t) \geq \theta, \; \forall t \in \mathbb{R}.$$

Throughout this section we also denote by Γ_0 the unique non constant periodic orbit of (8.9), that corresponds to (8.24) with $\varepsilon = 0$ (see Theorem 8.4 (*ii*)). The corresponding periodic solution of (8.9) is denoted by $(U_0(t), V_0(t))$ while $T_0 > 0$ is its period.

The idea of this proof is to show that Γ_ε becomes close to Γ_0 as $\varepsilon \to 0$. Then, as in [25] for the unperturbed system, we will prove that for all $\varepsilon > 0$ small enough,

$$\int_0^{T_\varepsilon} \left[\partial_U F_\varepsilon(U_\varepsilon(t), V_\varepsilon(t)) + \partial_V G_\varepsilon(U_\varepsilon(t), V_\varepsilon(t))\right] dt < 0.$$

According to Hale [91], the latter condition means that Γ_ε is locally asymptotically stable and then it follows that Γ_ε is unique when $\varepsilon > 0$ is small enough.

To prove Theorem 8.19, let us firstly prove the following lemma.

Lemma 8.20 *Let condition* (8.28) *be satisfied. Then, for each* $\delta \in \left(\overline{U}_2, \frac{\gamma-1}{2}\right)$*, there exists* $\varepsilon(\delta) > 0$ *small enough such for all* $\varepsilon \in (0, \varepsilon(\delta)]$ *the curve* Γ_ε *intersects* *the line* $U = \delta$. *In other words, one has* $\max\{U_\varepsilon(t):\ t \in [0, T_\varepsilon]\} \geq \delta$ *for all* $\varepsilon \in (0, \varepsilon(\delta)]$.

Proof Consider the function

$$\mathcal{F}(U,V) = \int_{\overline{U}_2}^{U} \frac{\left(\frac{\beta\xi}{1+\xi} - 1 + \Psi_\varepsilon(\xi, V)\right)}{\frac{\beta\xi}{1+\xi}} d\xi + \frac{1}{\beta}\int_{\overline{V}_2}^{V} \frac{\eta - \overline{V}_2}{\eta} d\eta,$$

wherein we have set $\Psi_\varepsilon(U,V) = -V^{-1}\Phi_\varepsilon^2(U,V)$. Note that since $\Phi_\varepsilon(U,V)$ is C^1-small uniformly on the compact set $(U,V) \in \mathbb{T}$ with $V \geq \theta > 0$ and $U \geq \theta > 0$ then Ψ_ε is also C^1-small on the same compact set.

Next let us compute the derivative of function $\mathcal{F}(U_\varepsilon, V_\varepsilon)$ with respect to t along the periodic orbit Γ_ε, that yields

$$\frac{d\mathcal{F}(U_\varepsilon(t), V_\varepsilon(t))}{dt} = \frac{\left(\frac{\beta U_\varepsilon}{1+U_\varepsilon} - 1 + \Psi_\varepsilon(U_\varepsilon, V_\varepsilon)\right)}{\frac{\beta U_\varepsilon}{1+U_\varepsilon}}\left[-\Phi_\varepsilon^1(U_\varepsilon, V_\varepsilon) + F(U_\varepsilon, V_\varepsilon)\right] + \frac{1}{\beta}\frac{V_\varepsilon - \overline{V}_2}{V_\varepsilon}V'_\varepsilon$$
$$+ V'_\varepsilon \int_{\overline{U}_2}^{U_\varepsilon} \frac{(1+\xi)\partial_V \Psi_\varepsilon(\xi, V_\varepsilon)}{\beta\xi} d\xi.$$

This rewrites as

$$\frac{d\mathcal{F}(U_\varepsilon, V_\varepsilon)}{dt} = \frac{V'_\varepsilon}{V_\varepsilon}\frac{1+U_\varepsilon}{\beta U_\varepsilon}\left[-\Phi_\varepsilon^1(U_\varepsilon, V_\varepsilon) + F(U_\varepsilon, V_\varepsilon) + \frac{U_\varepsilon}{1+U_\varepsilon}(V_\varepsilon - \overline{V}_2)\right] + V' \int_{\overline{U}_2}^{U_\varepsilon} \frac{1+\xi}{\beta\xi}\partial_V$$

and denoting by $\tilde{\Psi}_\varepsilon(U,V) = -\frac{1+U}{\beta U}\Phi_\varepsilon^1(U,V)$, this yields

$$\frac{d\mathcal{F}(U_\varepsilon, V_\varepsilon)}{dt} = \frac{V'_\varepsilon}{\beta V_\varepsilon}\left[\tilde{\Psi}_\varepsilon((U_\varepsilon, V_\varepsilon) + f(U_\varepsilon) - \overline{V}_2\right] + V'_\varepsilon \int_{\overline{U}_2}^{U_\varepsilon} \frac{(\partial_V \Psi_\varepsilon(\xi, V_\varepsilon))}{\frac{\beta\xi}{1+\xi}} d\xi.$$

Integrating the above equality on $[0, T_\varepsilon]$ leads

$$0 = \int_0^{T_\varepsilon} \frac{d\mathcal{F}(U_\varepsilon(s), V_\varepsilon(s))}{dt} ds = \oint_{\Gamma_\varepsilon} \left\{\frac{1}{\beta V}\left(\tilde{\Psi}_\varepsilon(U,V) + f(U) - \overline{V}_2\right) + \int_{\overline{U}_2}^{U} \frac{1+\xi}{\beta\xi}\partial_V \Psi_\varepsilon(\xi, V) d$$

Now denoting by Ω_ε the interior of the periodic curve Γ_ε and using the Green-Riemann formula, we infer the following identity

$$0 = \int_{\Omega_\varepsilon} \left\{\frac{1}{\beta V}\left(\partial_U \tilde{\Psi}_\varepsilon(U,V) + f'(U)\right) + \frac{1+U}{\beta U}\partial_V \Psi_\varepsilon(U,V)\right\} dU dV. \tag{8.29}$$

Now fix $\delta \in \left(\overline{U}_2, \frac{\gamma-1}{2}\right)$ and recall that $\inf_{U \leq \delta} f'(U) = f'(\delta) > 0$. Set $K = \{(U,V) \in \mathbb{T}:\ U \in [\theta, \delta],\ V \geq \theta\}$ and observe that $\partial_V \Psi_\varepsilon$ and $\tilde{\Psi}_\varepsilon$ tend to 0 as $\varepsilon \to 0$, uniformly for $(U,V) \in K$. Hence since K is bounded by some constant $R > 0$, we obtain uniformly for $(U,V) \in K$ and for all $0 < \varepsilon \ll 1$

$$\frac{1}{V}\left(\partial_U \tilde{\Psi}_\varepsilon(U,V) + f'(U)\right) + \frac{1+U}{\beta U}\partial_V \Psi_\varepsilon(U,V) \geq f'(\delta) + o(1).$$

Hence there exists $\varepsilon(\delta) > 0$ such that for all $\varepsilon \in (0, \varepsilon(\delta)]$ one has

$$\sup_{(U,V)\in K} \frac{1}{V}\left(\partial_U \tilde{\Psi}_\varepsilon(U,V) + f'(U)\right) + \frac{1+U}{\beta U}\partial_V \Psi_\varepsilon(U,V) > 0.$$

As a consequence, since $\Gamma_\varepsilon \cup \Omega_\varepsilon \subset K$ and Γ_ε encloses the equilibrium, if the curve Γ_ε does not intersect the line $U = \delta$ for all $\varepsilon > 0$ small enough then the integral on the right hand side of (8.29) would be positive which is a contradiction and we complete the proof of the lemma. □

We continue the proof of Theorem 8.19 by showing the following lemma.

Lemma 8.21 *Let condition* (8.28) *be satisfied. Let* Γ_0 *denote the unique non-constant periodic orbit of system* (8.9). *Then the following convergence holds*

$$\lim_{\varepsilon\to 0} \mathrm{d}(\Gamma_\varepsilon, \Gamma_0) = 0$$

where $\mathrm{d}(\Gamma_\varepsilon, \Gamma_0)$ *denotes the Hausdorff's semi-distance given by*

$$\mathrm{d}(\Gamma_\varepsilon, \Gamma_0) := \sup_{x\in\Gamma_\varepsilon} \delta(x, \Gamma_0) \text{ with } \delta(x, \Gamma_0) = \inf_{y\in\Gamma_0} \|x - y\|.$$

In other words, for each neighborhood V *of* Γ_0 *there exists* $\varepsilon_V > 0$ *such that*

$$\Gamma_\varepsilon \subset V, \forall \varepsilon \in (0, \varepsilon_V].$$

Furthermore the period $T_\varepsilon > 0$ *of* Γ_ε *converges to* T_0*, the period of* Γ_0*, as* $\varepsilon \to 0$.

Proof Fix $U^\star = \frac{1}{2}\left[\overline{U}_2 + \frac{\gamma-1}{2}\right) \subset \left(\overline{U}_2, \frac{\gamma-1}{2}\right)$.
Step 1: From Lemma 8.20 for all $\varepsilon > 0$ small enough, there exists $t_\varepsilon \in \mathbb{R}$ such that

$$U_\varepsilon(t_\varepsilon) > U^\star.$$

Step 2: Using Arzela-Ascoli's theorem we can find a sequence $\varepsilon_n \to 0$ and $t \to (U(t), V(t))$ a complete orbit of the unperturbed system (8.9) such that

$$\left(U_{\varepsilon_n}(t + t_\varepsilon), V_{\varepsilon_n}(t + t_\varepsilon)\right) \to (U(t), V(t)) \tag{8.30}$$

for the topology of the local uniform convergence for $t \in \mathbb{R}$. The definition of t_ε above ensures that

$$U(0) > U^\star. \tag{8.31}$$

Moreover, since $(U_\varepsilon(t), V_\varepsilon(t)) \in \mathbb{T}$ and $U_\varepsilon(t) \geq \theta$ and $V_\varepsilon(t) \geq \theta$ for all ε small enough and $\forall t \in \mathbb{R}$, one obtains that

$$(U,V)(t) \in \mathbb{T}, \quad \forall t \in \mathbb{R} \text{ and } U(t) \geq \theta, \ V(t) \geq \theta, \ \forall t \in \mathbb{R}.$$

Hence the limit orbit (U,V) lies in the interior attractor $\mathcal{A}_{\text{Int}(\mathbb{R}^2_+)}$ of (8.9) while (8.31) implies that the complete orbit is not reduced to the interior equilibrium, therefore

$$\lim_{t\to\infty} \delta\left((U(t),V(t)),\Gamma_0\right) = 0.$$

Step 3: Let us fix $M^0 = \left(U^0,V^0\right) \in \Gamma_0$ such that

$$F(M^0) > 0 \text{ and } G(M^0) > 0.$$

In order to simplify the rest of the proof, we fix the norm $\|\cdot\|_1$ in $\mathbb{R}^2$ given by

$$\|(U,V)\|_1 = |U| + |V|, \ \forall (U,V) \in \mathbb{R}^2.$$

Let $\eta > 0$ be small enough and let $\varepsilon_0 = \varepsilon_0(\eta) > 0$ be small enough (depending on η) such that

$$F_\varepsilon(M) > F(M^0)/2 \text{ and } G_\varepsilon(M) > G(M^0)/2,$$

whenever $\|M - M^0\|_1 \leq \eta$ and $\varepsilon \in (0,\varepsilon_0)$.

By using the sign of F and G around M^0, we can find $M^1 = (U^1,V^1)$ a point on Γ_0 such that

$$M^0 < M^1 (\text{that is, } U^0 < U^1 \text{ and } V^0 < V^1) \text{ and } \|M^1 - M^0\|_1 < \eta.$$

Let $\delta \in (0,\eta)$ be such that

$$\left\{M \in \mathbb{R}^2 : \|M - M^1\|_1 \leq \delta\right\} \subset \left\{M \in \mathbb{R}^2 : M \geq M^0 \text{ and } \|M - M^0\|_1 \leq \eta\right\}.$$

Step 4: By using the continuous dependency of the semiflow generated (8.24) with respect to the initial condition and with respect to the parameter ε we deduce that we can find $\widehat{\delta} \in (0,\delta)$ and $\varepsilon_1 \in (0,\varepsilon_0)$ such that every solution of (8.24) starting in the ball

$$B(M^1,\widehat{\delta}) := \left\{M \in \mathbb{R}^2 : \|M - M^1\|_1 \leq \widehat{\delta}\right\}$$

will belong to the larger ball

$$B(M^1,\delta) := \left\{M \in \mathbb{R}^2 : \|M - M^1\|_1 \leq \delta\right\}$$

at time $t = T_0$.

Step 5: By using the Step 2, for all n large enough, we find $M_{\varepsilon_n} \in \Gamma_{\varepsilon_n}$ belonging in the ball $B(M^1,\widehat{\delta})$ and the solution of the approximated system (8.24) starting from M_{ε_n} belongs to the ball $B(M^1,\delta)$ at $t = T_0$.

Assume by contradiction that this solution leaves the triangle

$$T = \left\{M \in \mathbb{R}^2 : M \geq M^0 \text{ and } \|M - M^0\|_1 \leq \eta\right\}$$

without intersecting the point M_{ε_n}. By using Jordan's theorem, we obtain a contradiction since the closed curve Γ_{ε_n} cannot return back through the triangle T from the "exit segment"

$$S = \left\{M \in \mathbb{R}^2 : M \geq M^0 \text{ and } \|M - M^0\|_1 = \eta\right\}.$$

This completes the proof of the lemma. □

We now complete the proof of Theorem 8.19 by proving, announced above that for all $\varepsilon > 0$ small enough

$$\int_0^{T_\varepsilon} \left[\partial_U F_\varepsilon(U_\varepsilon(t), V_\varepsilon(t)) + \partial_V G_\varepsilon(U_\varepsilon(t), V_\varepsilon(t))\right] dt < 0.$$

However this estimate follows from some properties of the unique periodic orbit (U_0, V_0) of (8.9) together with the convergence result stated in Lemma 8.21. Indeed, note that Cheng [25] proved that, the unique unperturbed periodic orbit Γ_0 satisfies

$$\int_0^{T_0} \left[\partial_U F(U_0(t), V_0(t)) + \partial_V G(U_0(t), V_0(t))\right] dt < 0,$$

while Lemma 8.21 ensures that

$$\lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \left[\partial_U F_\varepsilon(U_\varepsilon(t), V_\varepsilon(t)) + \partial_V G_\varepsilon(U_\varepsilon(t), V_\varepsilon(t))\right] dt = \int_0^{T_0} \left[\partial_U F(U_0(t), V_0(t)) + \partial_V G(U$$

This completes the proof of the estimate and thus the one of Theorem 8.19.

As a consequence of the above result, we now can state the following properties of the interior attractor $\mathcal{A}_{0,\varepsilon}$ for all $0 < \varepsilon \ll 1$.

Theorem 8.22 *Assume that **(H2)** holds. Then for all $\varepsilon > 0$ small enough, the interior global attractor $\mathcal{A}_{0,\varepsilon}$ consists of the unique interior equilibrium $\left(\overline{U}_2, \overline{V}_2\right)$ and the interior of the unique periodic orbit surrounding the interior equilibrium, and an infinite number of heteroclinic orbits joining the unique interior equilibrium and the unique periodic orbit.*

8.3.4 Existence and uniqueness of a traveling wave joining $(\gamma, 0)$ and the interior global attractor

In this section, we use the previous results to provide a description of the heteroclinic orbits for (8.24) as well as their uniqueness.

Lemma 8.23 *Assume that **(H1)** holds. Then, for all $\varepsilon > 0$ small enough the equilibria $(0,0)$ and $(\gamma, 0)$ are saddle points for the semiflow T_ε. More precisely, the linearized equation of system (8.24) around the equilibrium $(0,0)$ (or $(\gamma, 0)$) has one eigenvalue with positive real part and one with negative real part.*

Proof Let us denote by J_ε the Jacobian matrix associated to (8.24) at $(0,0)$. Since $(F_\varepsilon, G_\varepsilon)$ is $C^1(\mathbb{T})$−close to (F, G) as $\varepsilon \to 0$, one has

$$J_\varepsilon = J + o(1) \text{ as } \varepsilon \to 0,$$

where J denotes the Jacobian matrix at $(0,0)$ of (8.24) with $\varepsilon = 0$ (that corresponds to system (8.9)). It is easy to check that the eigenvalues of J are the following: $\lambda_{+,J} = \alpha\gamma > 0$ and $\lambda_{-,J} = -1 < 0$. The eigenvalues $\lambda_{\pm,J_\varepsilon}$ of J_ε are continuous with respect to ε. Hence $\lambda_{\pm,J_\varepsilon} = \lambda_{\pm,J} + o(1)$. This completes the proof of the result. □

Proposition 8.24 *Assume that **(H1)** holds. Then system* (8.24) *admits a unique heteroclinic orbit going from* $(0,0)$ *to* $(\gamma,0)$*, for all* $0 < \varepsilon \ll 1$ *small enough.*

Proof Since $\partial_u\mathbb{T}$ is positively invariant with respect to the system (8.24) and

$$F(U,V) = \alpha U(\gamma - U) - \frac{UV}{1+U} = \frac{U}{1+U}[f(U) - V],$$

by using the following fact

$$\lim_{\varepsilon\to 0} \|\Phi_\varepsilon\|_\infty = 0,$$

we can deduce that there exists a unique heteroclinic orbit of system (8.24) going from $(0,0)$ to $(\gamma,0)$. □

We now discuss the existence and uniqueness of heteroclinc orbits for (8.24) joining the boundary to the interior attractor. As in the previous section, we make use of the connectedness of the global attractor to derive the existence of such connections. We then discuss further properties.

Connectedness arguments: The largest global attractor $\mathcal{A}_\varepsilon$ is connected. Since any continuous map maps a connected set into a connected set, it follows that the projection of $\mathcal{A}_\varepsilon$ on the horizontal and vertical axis is a compact interval.

The global attractor $\mathcal{A}_\varepsilon$ contains the interior global attractor $\mathcal{A}_{0,\varepsilon}$ which is compacts connected and locally stable and also contains the boundary attractor $\mathcal{A}_{\partial,\varepsilon}$. The connectedness of $\mathcal{A}_\varepsilon$ and compactness of $\mathcal{A}_{0,\varepsilon}$ and $\mathcal{A}_{\partial,\varepsilon}$ imply

$$\mathcal{A}_\varepsilon - \mathcal{A}_{0,\varepsilon} \bigcup \mathcal{A}_{\partial,\varepsilon} \neq \emptyset.$$

We deduce that for each point $(U,V) \in \mathcal{A}_\varepsilon - \mathcal{A}_{0,\varepsilon} \bigcup \mathcal{A}_{\partial,\varepsilon}$ the α and ω limit sets satisfy the following

$$\alpha(U,V) \in \mathcal{A}_{\partial,\varepsilon} \text{ and } \omega(U,V) \in \mathcal{A}_{0,\varepsilon}.$$

Finally since the boundary attractor has a Morse decomposition $M_1 = \{(0,0)\}$ and $M_2 = \{(\gamma,0)\}$ we have either

$$\alpha(U,V) = M_1 \text{ or } \alpha(U,V) = M_2, \forall (U,V) \in \mathcal{A}_\varepsilon - \mathcal{A}_{0,\varepsilon} \bigcup \mathcal{A}_{\partial,\varepsilon}.$$

Proposition 8.25 *Assume that **(H1)** holds. There for all* ε *small enough, system* (8.24) *does not admit any heteroclinic orbit going from* $(0,0)$ *to the interior global attractor.*

Proof Assume by contradiction that there exists one. Note that

$$V' = -\Phi_\varepsilon^2(U,V) + \left(-1 + \frac{\beta U}{1+U}\right)V.$$

We deduce that

$$V(t) = \exp\left(\int_{t_0}^{t} -1 + \frac{\beta U(s)}{1+U(s)} ds\right)\left(V(t_0) + \int_{t_0}^{t} -\Phi_\varepsilon^2(U(t),V(t))\exp\left(\int_{t_0}^{s} 1 - \frac{\beta U(t)}{1+U(t)} dt\right) ds\right)$$

Since there exists $T < 0$ such that $U(t)$ remains sufficiently small for all negative times $t < T$ and $V(t_0) > 0$, we deduce that

$$\lim_{t\to-\infty} V(t) = +\infty$$

which contradicts the fact that the solution belongs to the global attractor and is therefore bounded. □

We complete this section by proving the uniqueness of the traveling wave solution connecting $(\gamma, 0)$ to the interior global attractor. The arguments of this proof extend those used in [59].

Proposition 8.26 *Assume that **(H1)** holds. Then for all* $\varepsilon > 0$ *small enough, system* (8.24) *admits a unique heteroclinic orbit going from* $(\gamma, 0)$ *to the interior global attractor.*

Proof We only need to prove the uniqueness. The center-unstable manifold at $(\gamma, 0)$ is a one dimensional locally invariant manifold. By using the same arguments as in section 2 for the uniqueness of the heteroclinic orbit starting from $(\gamma, 0)$ for system (18.9), we can prove the uniqueness of the heteroclinic orbit going from $(\gamma, 0)$ to the interior global attractor for system (8.24). □

8.4 Numerical simulations

In this section we intend to observe the previous results numerically. We run some numerical simulations for the system

$$\begin{cases} U_t = dU_{xx} + \alpha U(\gamma - U) - \dfrac{UV}{1+U}, \text{ for } x \in [0, 1000] \\ V_t = V_{xx} - V + \beta \dfrac{UV}{1+U}, \text{ for } x \in [0, 1000] \end{cases} \tag{8.32}$$

with Neumann boundary conditions

$$U_x(t,x) = V_x(t,x) = 0, \text{ for } x = 0 \text{ and } x = 1000,$$

and the initial values

$$U(0,x) = \gamma \text{ and } V(0,x) = 0.1 * \exp(-\delta x).$$

Throughout the simulations the parameters will be unchanged and fixed as follows

$$d = 1,\ \alpha = 1/4,\ \gamma = 4,\ \beta = 2.$$

In order to capture large speed traveling waves, we impose exponentially decaying initial distribution for the predator while the initial distribution for the prey is set to its carrying capacity γ. As for the usual Fisher-KPP equation, we expect that reducing the exponential decay rate δ increases the wave speed of the predator invasion. Using formal computations around the unstable predator free equilibrium $(\gamma, 0)$ and using the usual antsatz $V(t,x) = e^{-\delta(x-ct)}$, we obtain the following formula for the wave speed depending on δ

$$c = \begin{cases} 2\sqrt{K} \text{ if } \delta > \sqrt{K}, \\ \delta + \frac{K}{\delta} \text{ if } \delta \in (0, \sqrt{K}), \end{cases} \qquad \text{with } K = \frac{\gamma(\beta-1)-1}{\gamma+1} > 0.$$

With the above parameter set, we have $\sqrt{K} \approx 0.77$. Below we perform numerical simulations of the model with two different decay rates $\delta = 0.1$ and $\delta = 1$. According to our formal wave speed computations, we expect to obtain a predator invasion with a small speed (in fact the minimal wave speed) for $\delta = 1$ and with a larger wave speed with $\delta = 0.1$.

In Figure 8.2, we observe the traveling wave joining $(\gamma, 0)$ and periodic wave train when we start from a $V(0,x)$ with $\delta = 0.1$.

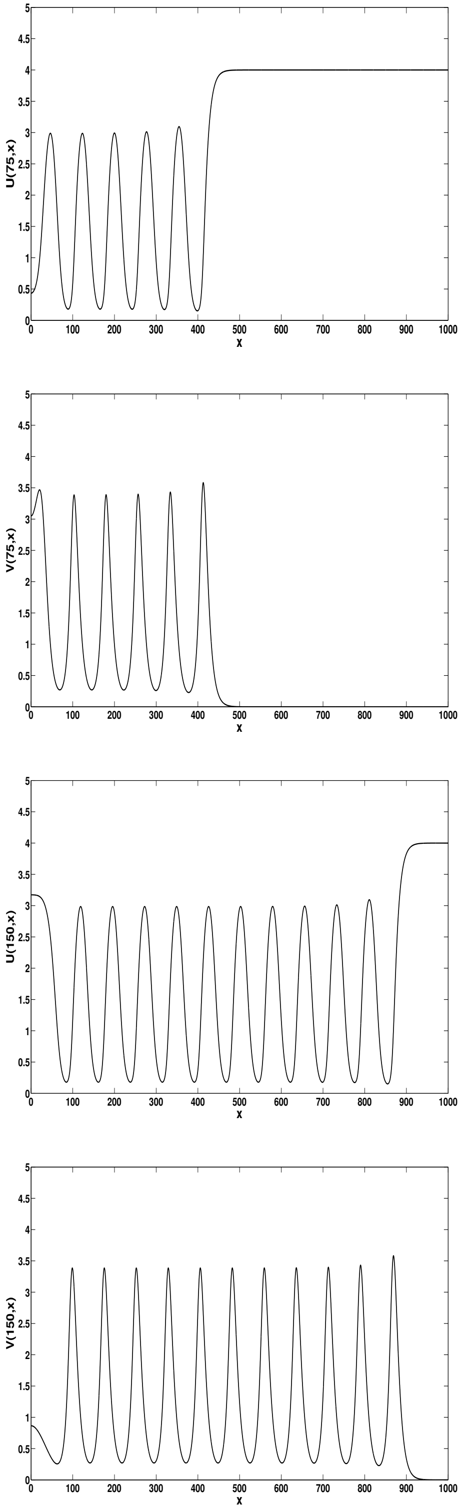


Fig. 8.2: *In this figure we plot* $U(t,x)$ *(left handside) and* $V(t,x)$ *(right handside) whenever the parameter* $\delta = 0.1$ *and* $t = 75$ *(above) and* $t = 150$ *(below). The*

In Figure 8.3, we observe slower predator invasion followed by some more complex behaviours whenever we start from a $V(0,x)$ with $\delta = 1$. The complexity in such a problem was already observed by Sherratt, Smith and Rademacher [193] in the multi-dimensional case.

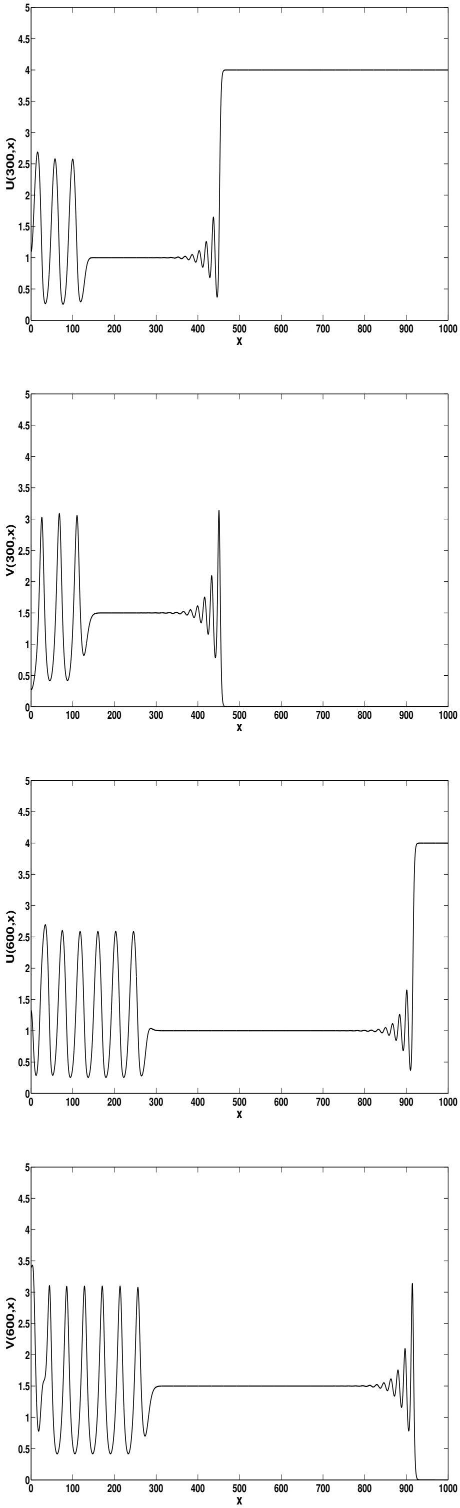


Fig. 8.3: *In this figure we plot* $U(t,x)$ *(left handside) and* $V(t,x)$ *(right handside) whenever the parameter* $\delta = 1$ *and* $t = 300$ *(above) and* $t = 600$ *(below). The initial*

The Figure 8.2 corresponds to our results when the speed of the traveling wave (with $\delta = 0.1$) is larger than in Figure 8.3 (with $\delta = 1$). The description of the small speed traveling waves is difficult question and left for further studies.

8.5 Discussion

In this paper we have studied all the large speed traveling solutions for the one-dimensional diffusive Rosenzweig-MacArthur predator-prey system. The methodology developed in this manuscript is based on center manifold reduction coupled with global attractor theory and its topological properties to understand some properties of the entire orbits (traveling waves, periodic wave train), for which the above mentioned analytical tools are particularly well adapted. Note that global attractor allows to obtain the existence of complete orbits, since every point of global attractor belongs to such a bounded complete orbit. Furthermore, the connectedness of global attractor allows in particular to prove the existence of complete orbits joining the boundary region and the interior equilibrium or periodic orbit. These complete orbits are nothing but traveling waves we are interested in.

As far as the reduction procedure is concerned, Fenichel's results about the persistence of normally hyperbolic invariant manifold may also be used. The two reduction techniques, based on Fenichel's theorem and center manifold theorem are comparable. Both methodologies provides the existence of manifold that is only locally invariant by the system (see [75, 216]).

In order to describe the system restricted to such a locally invariant manifold, one needs to carefully consider the truncation of the original system. It is important to observe that we consider the complete orbits that remain into the un-truncated region of the state space, where the two systems (with and without truncation) coincide. These tools permit to reduce the dimension as long as some bounded complete orbits remain in the un-truncated region of the state space.

Let us mention that such reduction arguments can be extended to infinite dimensional systems. We refer the reader to the books of Haragus and Iooss [98] and of Magal and Ruan [155] for infinite dimensional extension of the center manifold theorem. We also refer to Bates, Lu and Zeng [14, 15] and to Magal and Seydi [158] for results about the perturbation of normally hyperbolic manifold in infinite dimensional dynamical systems. More specifically the recent work of Magal and Seydi [158] deals with small perturbation of normally hyperbolic manifold, extended in particular the center manifold by allowing a nonlinear dynamic in the unperturbed central part of the system, that is roughly speaking for rather general – infinite dimensional – systems before perturbation with the form

$$\begin{cases} U'(t) = F(U(t)), \\ V'(t) = B(U(t))V(t), \end{cases}$$

wherein U corresponds to the central part and V corresponds to the hyperbolic part of the system.

This latest form is not only well adapted for the traveling wave problem considered in this manuscript, but similar method can be applied to fast traveling wave for general reaction-diffusion systems (see [98, 155] for more results). As mentioned in the introduction this method can be developed in particular to handle more general nonlinearities. On the other hand, as mention above, since the center manifold is a flexible analytical tool with infinite dimensional extensions, the methodology developed in this work can also be used to study large speed traveling wave solutions for some infinite dimensional problems, including as a special case reaction-diffusion systems with time delay.

To conclude we provide in the article a new combination of reduction techniques and global analysis based on global attractors theory. Such a method can be employed in many other contexts to obtain the existence of large speed traveling waves.

8.6 Remarks and Notes

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