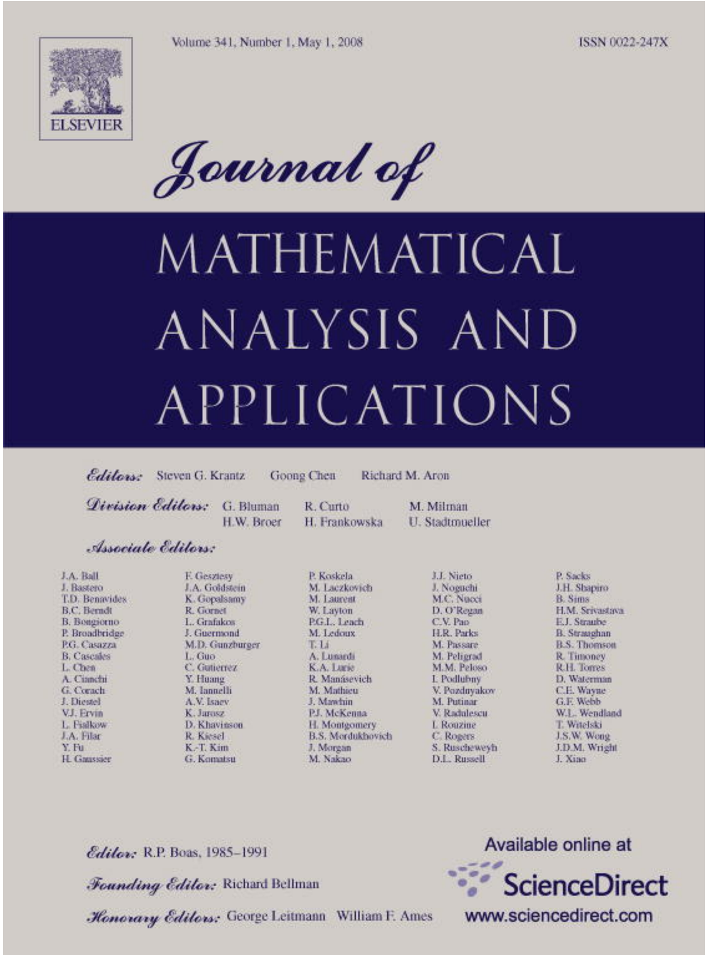
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Essential growth rate for bounded linear perturbation of non-densely defined Cauchy problems

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Abstract

This paper is devoted to the study of the essential growth rate of some class of semigroup generated by bounded perturbation of some non-densely defined problem. We extend some previous results due to Thieme [H.R. Thieme, Quasi-compact semigroups via bounded perturbation, in: Advances in Mathematical Population Dynamics—Molecules, Cells and Man, Houston, TX, 1995, in: Ser. Math. Biol. Med., vol. 6, World Sci. Publishing, River Edge, NJ, 1997, pp. 691–711] to a class of non-densely defined Cauchy problems in L^p. In particular in the context the integrated semigroup is not operator norm locally Lipschitz continuous. We overcome the lack of Lipschitz continuity of the integrated semigroup by deriving some weaker properties that are sufficient to give information on the essential growth rate.

Keywords: Semigroups; Integrated semigroups; Essential growth rate; Bounded perturbation; Non-densely defined operators

1. Introduction

The goal of this paper is to study the essential growth rate of some class of semigroup generated by bounded perturbation of some non-densely defined Cauchy problem. In order to investigate such problems, we first need to consider non-densely defined non-homogeneous Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t), \quad t \in [0, \tau_0], \qquad u(0) = x \in \overline{D(A)}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space X and $f \in L^1((0, \tau_0), X)$. When A is a Hille–Yosida operator and is densely defined (i.e., $\overline{D(A)} = X$), the problem has been extensively studied (see Pazy [15] and Yosida [26]). When A is a Hille–Yosida operator but its domain is non-densely defined, Da Prato and Sinestrari [5]

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investigated the existence of several types of solutions for (1.1). They first reformulated (1.1) as a sum of operator problems (i.e., $Bu = Au + f$ with $Bu(t) = \frac{du}{dt}$), and then obtained the existence and uniqueness of integrated solutions of (1.1) for each $x \in \overline{D(A)}$ and each $f \in L^1((0, \tau_0), X)$.

A very important and useful approach to investigate such non-densely defined Cauchy problems is to use the integrated semigroup theory, which was first introduced by Arendt [1,2]. In the context of Hille–Yosida operators, we have the following relationship between the integrated semigroup and integrated solutions of (1.1). An integrated semigroup $\{S(t)\}_{t\geqslant 0}$ is a strongly continuous family of bounded linear operators on X, which commute with the resolvent of A, such that for each $x \in X$ the map $t \to S(t)x$ is an integrated solution of the Cauchy problem

$$\frac{du}{dt} = Au(t) + x, \qquad u(0) = 0. \tag{1.2}$$

Arendt [1,2] proved that if there is a strongly continuous family of bounded linear operators $\{S(t)\}_{t\geqslant 0}$ on X, which is assumed to be exponentially bounded (see Section 2 for precise definition), and if $(\mu I - A)^{-1}x = \mu \int_0^\infty e^{-\mu t} S(t)x\,dt$ holds for all $x \in X$ and all $\mu > \omega$ large enough (where $(\omega, \infty) \subset \rho(A)$), then $\{S(t)\}_{t\geqslant 0}$ is an integrated semigroup and A is called its generator. Kellermann and Hieber [8] further developed the integrated semigroup theory and provided an easy proof of Da Prato and Sinestrari's result [5]. To be more specific, Kellermann and Hieber [8] proved that when A is a Hille–Yosida operator, the map $t \to (S * f)(t) := \int_0^t S(t-s) f(s)\,ds$ is continuously differentiable and $u(t) = \frac{d}{dt}(S * f)(t)$ is an integrated solution of (1.1). For recent studies on the integrated semigroup theory, we refer to the monographs of Arendt et al. [3], Xiao and Liang [27] and the references cited therein.

In this article, as in Magal and Ruan [10] and Thieme [22], we consider the case where the integrated solution of the Cauchy problem (1.1) only exists whenever f belongs the $L^p((0, \tau_0), X)$ for some $p \in [1, +\infty)$. The situation is motivated in particular by application to age-structured model, or application to neutral delay differential equations in L^p. We make the following assumptions.

Assumption 1.1. We assume that the resolvent set of A is non-empty, and the part A_0 of A in $X_0 = \overline{D(A)}$ is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geqslant 0}$ of bounded linear operators on X_0.

Under Assumption 1.1, it follows that $\rho(A) = \rho(A_0)$, because both the resolvent set of A and the resolvent set of A_0 are not empty, and it also follows that A generates a integrated semigroup $\{S_A(t)\}_{t\geqslant 0}$ on X.

Assumption 1.2. Let $p \in [1, +\infty)$ be fixed. We assume that there exist $\widehat{M} > 0$ and $\widehat{\omega} \in \mathbb{R}$, such that for each $\tau > 0$ and each $f \in L^p((0, \tau), X)$, there exists $u_f \in C([0, \tau], X)$ an integrated solution of the Cauchy problem (1.1) with $x = 0$, satisfying

$$\left\| u_f(t) \right\| \leqslant \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(.) \right\|_{L^p((0,t),X)}, \quad \forall t \in [0, \tau].$$

Let $L : \overline{D(A)} \to X$ be a bounded linear operator. The purpose of this paper is to obtain an estimation of $\omega_{0,\mathrm{ess}}((A + L)_0)$, the essential growth rate of $\{T_{(A+L)_0}(t)\}_{t\geqslant 0}$ (see Section 3 for a precise definition of the essential growth rate). When the domain of A is dense in X, then $\{T_{(A+L)_0}(t)\}_{t\geqslant 0}$ is obtained as the unique solution of

$$T_{(A+L)_0}(t) = T_{A_0}(t) + \int_0^t T_{A_0}(t-s) L T_{(A+L)_0}(s)\,ds,$$

and when

$$L T_{A_0}(t) \text{ is compact} \quad \text{for each } t > 0,$$

by using the approach of Webb [23] (see also [13, Theorem 3.2]), one may deduce that (because $x \to \int_0^t T_A(t-s) L T_{(A+L)}(s)x\,ds$ is compact for each $t \geqslant 0$)

$$\omega_{0,\mathrm{ess}}\big((A + L)_0\big) \leqslant \omega_{0,\mathrm{ess}}(A_0).$$

The main problem here is to obtain the same results as above whenever A is non-densely defined. In Section 2, we obtain the following theorem (see Theorem 2.7).

Theorem 1.1. *Let Assumptions* 1.1 *and* 1.2 *be satisfied. Let $L:\overline{D(A)}\to X$ be a bounded linear operator. Then $A+L:D(A)\subset X\to X$ also satisfies Assumptions* 1.1 *and* 1.2. *In particular $(A+L)_0$ the part of $A+L$ in X_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L)_0}(t)\}_{t\geqslant 0}$ of bounded linear operators on X_0. Moreover $\{T_{(A+L)_0}(t)\}_{t\geqslant 0}$ is the unique solution of the fixed point problem*

$$T_{(A+L)_0}(t)=T_{A_0}(t)+\frac{d}{dt}\int_0^t S_A(t-s)LT_{(A+L)_0}(s)\,ds.$$

Also inspired by the results of Thieme [19], we obtain the following theorem which is the main result of this paper.

Theorem 1.2. *Let Assumptions* 1.1 *and* 1.2 *be satisfied. Let $L:\overline{D(A)}\to X$ be a bounded linear operator. Assume that*

$$LT_{A_0}(t) \text{ is compact} \quad \text{for each } t>0.$$

Then we have the following inequality

$$\omega_{0,\mathrm{ess}}\big((A+L)_0\big)\leqslant\omega_{0,\mathrm{ess}}(A_0).$$

One may note that when A is a Hille–Yosida operator (which corresponds here to the case $p=1$ in Assumption 1.2), the above result basically summarizes the results proved by Thieme [19]. The above question has been studied by Rhandi and Schnaubelt [16] using extrapolation method, but they assume in addition that the map $t\to LT_{A_0}(t)$ is operator norm continuous which is not satisfied in general for age structured models. The above result uncompass this difficilty. We also refer to Thieme [20,21] for further result going in that direction. So finally here the point is to extend the previous results of Thieme [19] for the case $p=1$ to the case $p\in[1,+\infty)$.

The above theorem can apply to various class of examples, such as age-structured problems in L^p, functional differential equations in the space of continuous functions (see Liu, Magal and Ruan [9] for more details). In particular, the above theorem can be applied to study neutral function differential equation in L^p (see Ducrot, Liu and Magal [6] for more details).

The plan of the paper is the following. In Section 2, we recall some results about integrated semigroups. In Section 3, we recall some results about the spectral theory for linear operators. The Section 4 is devoted to the proof the main result Theorem 1.2.

2. Integrated semigroup

In this section we recall some results about integrated semigroups. We refer to Arendt [1,2], Neubrander [14], Kellermann and Hieber [8], Thieme [18], and Arendt et al. [3], and Magal and Ruan [10] for more detailed results on this subject.

Let X and Z be two Banach spaces. Denote by $\mathcal{L}(X,Z)$ the space of bounded linear operators from X into Z and by $\mathcal{L}(X)$ the space $\mathcal{L}(X,X)$. Let $A:D(A)\subset X\to X$ be a linear operator. If A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X, we denote by $\{T_A(t)\}_{t\geqslant 0}$ this semigroup. We denote by $\rho(A)=\{\lambda\in\mathbb{C}:\ \lambda I-A \text{ is invertible}\}$ the resolvent of A. From here on, we also denote by $X_0:=\overline{D(A)}$, and A_0 the part of A in X_0, which is a linear operator on X_0 defined by

$$A_0x=Ax,\quad \forall x\in D(A_0):=\big\{y\in D(A):\ Ay\in X_0\big\}.$$

If $(\omega,+\infty)\subset\rho(A)$, then it is easy to check that for each $\lambda>\omega$,

$$D(A_0)=(\lambda I-A)^{-1}X_0\quad\text{and}\quad(\lambda I-A_0)^{-1}=(\lambda I-A)^{-1}|_{X_0}.$$

In the following, we assume that operator A satisfies the following assumption.

Assumption 2.1. We assume that $A:D(A)\subset X\to X$ is a linear operator on a Banach space $(X,\|.\|)$, satisfying the following properties*:*

(a) There exist two constants $\omega_A \in R$ and $M_A \geqslant 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ and

$$\left\| (\lambda I - A)^{-k} \right\|_{\mathcal{L}(X_0)} \leqslant \frac{M_A}{(\lambda - \omega_A)^k}, \quad \forall \lambda > \omega_A,\ \forall k \geqslant 1;$$

(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0$, $\forall x \in X$.

By using Hille–Yosida theorem (see Pazy [15, Theorem 5.3 on p. 20]), Lemma 2.1 in [10], and the fact that if $\rho(A) \neq \emptyset$ then $\rho(A) = \rho(A_0)$, one obtains the following lemma.

Lemma 2.1. *Assumption* 2.1 *is satisfied if and only if* $\rho(A) \neq \emptyset$, A_0 *is the infinitesimal generator of a* C_0*-semigroup* $\{T_{A_0}(t)\}_{t \geqslant 0}$ *on* X_0, *and*

$$\left\| T_{A_0}(t) \right\| \leqslant M_A e^{\omega_A t}, \quad \forall t \geqslant 0.$$

Now we give the definition of integrated semigroups.

Definition 2.2. Let $(X, \|.\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t \geqslant 0}$ on X is called an integrated semigroup if

(i) $S(0) = 0$;
(ii) The map $t \to S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$;
(iii) $S(t)$ satisfies $S(s)S(t) = \int_0^s (S(r+t) - S(r))\, dr$, $\forall t, s \geqslant 0$.

An integrated semigroup $\{S(t)\}_{t \geqslant 0}$ is said to be *non-degenerate*, if $S(t)x = 0$, $\forall t \geqslant 0$, implies $x = 0$. According to Thieme [18], we say that a linear operator $A : D(A) \subset X \to X$ is the *generator* of a non-degenerate integrated semigroup $\{S(t)\}_{t \geqslant 0}$ on X if and only if

$$x \in D(A), \quad y = Ax \quad \Leftrightarrow \quad S(t)x - tx = \int_0^t S(s)y\, ds, \quad \forall t \geqslant 0.$$

From [18, Lemma 2.5], we know that if A generates $\{S_A(t)\}_{t \geqslant 0}$, then for each $x \in X$ and $t \geqslant 0$,

$$\int_0^t S_A(s)x\, ds \in D(A) \quad \text{and} \quad S_A(t)x = A \int_0^t S_A(s)x\, ds + tx.$$

An integrated semigroup $\{S(t)\}_{t \geqslant 0}$ is said to be *exponentially bounded* if there exist two constants $\widehat{M} > 0$ and $\widehat{\omega} > 0$, such that

$$\left\| S(t) \right\|_{\mathcal{L}(X)} \leqslant \widehat{M} e^{\widehat{\omega} t}, \quad \forall t \geqslant 0.$$

When we restrict ourself to the class of non-degenerate exponentially bounded integrated semigroups, Thieme's notion of generator is equivalent the one introduced by Arendt [2]. More precisely, combining Theorem 3.1 in Arendt [2] and Proposition 3.10 in Thieme [18], one has the following result.

Theorem 2.3. *Let* $\{S(t)\}_{t \geqslant 0}$ *be a strongly continuous exponentially bounded family of bounded linear operators on a Banach space* $(X, \|.\|)$ *and* $A : D(A) \subset X \to X$ *be a linear operator. Then* $\{S(t)\}_{t \geqslant 0}$ *is a non-degenerate integrated semigroup and A its generator if and only if there exists some* $\widehat{\omega} > 0$ *such that* $(\widehat{\omega}, +\infty) \subset \rho(A)$ *and*

$$(\lambda I - A)^{-1} x = \lambda \int_0^\infty e^{-\lambda s} S(s)x\, ds, \quad \forall \lambda > \hat{\omega}.$$

The following result is well-known in the context of integrated semigroups (see [10, Proof of Proposition 2.4]).

Proposition 2.4. *Let Assumption* 2.1 *be satisfied. Then A generates a uniquely determined non-degenerate exponentially bounded integrated semigroup $\{S_A(t)\}_{t\geqslant 0}$. Moreover, for each $x \in X$, each $t \geqslant 0$, and each $\mu > \omega_A$, $S_A(t)x$ is given by*

$$S_A(t)x = \mu \int_0^t T_{A_0}(s)(\mu I - A)^{-1} x\,ds + \left[I - T_{A_0}(t)\right](\mu I - A)^{-1} x. \tag{2.1}$$

Furthermore, the map $t \to S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS_A(t)x}{dt} = T_{A_0}(t)x, \quad \forall t \geqslant 0,\ \forall x \in X_0.$$

From now on we denote by

$$(S_A * f)(t) = \int_0^t S_A(t-s) f(s)\,ds, \quad \forall t \in [0,\tau],$$

whenever $f \in L^1((0,\tau), X)$.

We now consider the inhomogeneous Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t), \quad t \in [0,\tau_0], \qquad u(0) = x \in \overline{D(A)}, \tag{2.2}$$

and assume that f belongs to some appropriate subspace of $L^1((0,\tau_0), X)$.

Definition 2.5. A continuous map $u \in C([0,\tau_0], X)$ is called an *integrated solution* of (2.2) if and only if

$$\int_0^t u(s)\,ds \in D(A), \quad \forall t \in [0,\tau_0],$$

and

$$u(t) = x + A\int_0^t u(s)\,ds + \int_0^t f(s)\,ds, \quad \forall t \in [0,\tau_0].$$

We consider the case where the map f belongs to $L^p((0,\tau_0), X)$ for some $p \in [1,+\infty)$ and we make the following assumption.

Assumption 2.2. Let be $p \in [1,+\infty)$. Assume that there exist $\widehat{M} > 0$ and $\widehat{\omega} \in \mathbb{R}$ such that for each $\tau_0 > 0$ and each $f \in C^1([0,\tau_0], X)$,

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leqslant \widehat{M} \left(\int_0^t \left(e^{\widehat{\omega}(t-s)} \| f(s) \|\right)^p ds \right)^{1/p}, \quad \forall t \in [0,\tau_0].$$

Next from Theorem 2.11 in Magal and Ruan [10], we have the following result.

Theorem 2.6. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Then for each $\tau > 0$ and each $f \in L^p((0,\tau), X)$, the map $t \to (S_A * f)(t)$ is continuously differentiable on $[0,\tau]$. The map $t \to u(t)$ defined by*

$$u(t) = T_{A_0}(t)x + \frac{d}{dt}(S_A * f)(t), \quad \forall t \in [0,\tau],$$

is an integrated solution of (2.2). *Moreover, for each $f \in L^p((0,\tau), X)$, we have the following estimate*

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leqslant \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(.) \right\|_{L^p((0,t),X)}, \quad \forall t \in [0,\tau]. \tag{2.3}$$

From now on for each $\tau > 0$ and each $f \in L^p((0,\tau), X)$, we set

$$(S_A \diamond f)(t) = \frac{d}{dt}(S_A * f)(t), \quad \forall t \in [0,\tau].$$

We now recall some properties that we will be used in the sequel. The proof of these relations can be found for instance in Magal and Ruan [10]. First we have for each $\tau > 0$ and $f \in L^p([0,\tau], X)$:

$$(S_A \diamond f)(t) = \lim_{\mu \to +\infty} \int_0^t T_{A_0}(t-l)\mu(\mu I - A)^{-1} f(l)\, dl, \quad \forall t \in [0,\tau]. \tag{2.4}$$

This approximation formula was already observed by Thieme [17] in the classical context of integrated semigroups generated by a Hille–Yosida operator. From this approximation formulation, we then deduce that for each pair $t, s \in [0,\tau]$ with $s \leqslant t$, and $f \in C([0,\tau], X)$,

$$(S_A \diamond f)(t) = T_{A_0}(t-s)(S_A \diamond f)(s) + \big(S_A \diamond f(s+.)\big)(t-s). \tag{2.5}$$

We also observe that

$$S_A(t) = \frac{d}{dt}\int_0^t S_A(t-s)x\, ds.$$

So as an immediate consequence of Assumption 2.2, we have

$$\|S_A(t)\|_{\mathcal{L}(X)} \leqslant \delta(t), \quad \forall t \geqslant 0, \tag{2.6}$$

where

$$\delta(t) := \widehat{M}\left(\int_0^t e^{p\widehat{\omega}l}\, dl\right)^{1/p}, \quad \forall t \geqslant 0. \tag{2.7}$$

The following result is a consequence of Theorem 3.1 in [10].

Theorem 2.7. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Let $L : \overline{D(A)} \to X$ be a bounded linear operator. Then $A + L : D(A) \subset X \to X$ also satisfies Assumptions* 2.1 *and* 2.2*. In particular $(A+L)_0$ the part of $A+L$ in X_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L)_0}(t)\}_{t\geqslant 0}$ of bounded linear operators on X_0. Moreover $\{T_{(A+L)_0}(t)\}_{t\geqslant 0}$ is the unique solution of the fixed point problem*

$$T_{(A+L)_0}(t) = T_{A_0}(t) + \frac{d}{dt}\int_0^t S_A(t-s)LT_{(A+L)_0}(s)\, ds.$$

Proof. Theorem 3.1 in [10] trivially applies to this situation and it remains to prove that there exists $\widetilde{\omega} \in \mathbb{R}$, and $\widetilde{M} > 0$, such that

$$\|(S_{A+L} \diamond f)(t)\| \leqslant \widetilde{M}\|e^{\widetilde{\omega}(t-.)}f(.)\|_{L^p((0,t),X)}, \quad \forall t \in [0,\tau],$$

whenever $f \in L^p((0,\tau), X)$. In order to obtain this estimation, we apply the last part of Theorem 3.1 in [10] to $A + L - \gamma I$ for some $\gamma > 0$.

Let $\gamma > 0$ be fixed. We have

$$\begin{aligned}(S_{A-\gamma I} \diamond f)(t) &= \lim_{\mu \to +\infty} \int_0^t T_{A_0-\gamma I}(t-l)\mu(\mu I - A)^{-1} f(l)\, dl \\ &= e^{-\gamma t}\lim_{\mu \to +\infty} \int_0^t T_{A_0}(t-l)\mu(\mu I - A)^{-1} e^{\gamma l} f(l)\, dl \\ &= e^{-\gamma t}\big(S_A \diamond e^{\gamma \cdot} f(.)\big)(t).\end{aligned}$$

So it follows that

$$\left\|(S_{A-\gamma I}\diamond f)(t)\right\| \leqslant \widehat{M}\left\|e^{(\widehat{\omega}-\gamma)(t-.)}f(.)\right\|_{L^p((0,t),X)}, \quad \forall t\in[0,\tau],$$

and

$$\widehat{M}\left\|e^{(\widehat{\omega}-\gamma)(t-.)}f(.)\right\|_{L^p((0,t),X)} \leqslant \delta_\gamma(t)\sup_{s\in[0,t]}\left\|f(s)\right\|,$$

where

$$\delta_\gamma(t) := \widehat{M}\left(\int_0^t e^{p(\widehat{\omega}-\gamma)l}\,dl\right)^{1/p}, \quad \forall t\geqslant 0.$$

Moreover, for $\gamma>0$ large enough, we have

$$\delta_\gamma(t)\|L\|<1, \quad \forall t\geqslant 0,$$

and it follows from the last part of Theorem 3.1 in [10] that

$$\left\|(S_{A+L-\gamma I}\diamond f)(t)\right\| \leqslant \widehat{M}'\left\|e^{(\widehat{\omega}-\gamma)(t-.)}f(.)\right\|_{L^p((0,t),X)}, \quad \forall t\in[0,\tau],$$

and by using the same argument as above the result follows. □

The following result is proved in Magal and Ruan [11, Proposition 2.15]. This result provides an exponential estimation of $\|(S_A\diamond f)(t)\|$ expressed in function of the growth rate of $\{T_{A_0}(t)\}_{t\geqslant 0}$.

Proposition 2.8. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Let $\varepsilon>0$ be fixed. Then for each $\tau_\varepsilon>0$ satisfying $M_A\delta(\tau_\varepsilon)\leqslant\varepsilon$, we have*

$$e^{-\gamma t}\left\|(S_A\diamond f)(t)\right\| \leqslant \frac{2\varepsilon\max(1,e^{-\gamma\tau_\varepsilon})}{(1-e^{(\omega_A-\gamma)\tau_\varepsilon})}\sup_{s\in[0,t]}e^{-\gamma s}\left\|f(s)\right\|, \quad \forall t\geqslant 0,$$

whenever $\gamma\in(\omega_A,+\infty)$ and $f\in C(\mathbb{R}_+,X)$.

To conclude this section we give several equivalent conditions which are necessary and sufficient conditions to verify Assumption 2.2. For that purpose let us recall some notions.

Definition 2.9. Let $(Y,\|.\|_Y)$ be a Banach space. Let E be a subspace of Y^*. Then E is called a *norming space* of Y if the map $|.|_E : T\to\mathbb{R}^+$ defined by

$$|y|_E = \sup_{y^*\in E,\ \|y^*\|_{Y^*}\leqslant 1} y^*(y), \quad \forall y\in Y,$$

is a norm equivalent to $\|.\|_Y$.

Let $(Y,\|.\|_Y)$ be a Banach space. Let $a\leqslant b$ be two given real numbers and let $g:[a,b]\to Y$ be a map. If $q\in[1,+\infty)$ we set

$$\mathrm{VL}^q(g,a,b) = \sup\left(\sum_{i=1}^n \frac{\|g(t_i)-g(t_{i-1})\|_Y^q}{|t_i-t_{i-1}|^{q-1}}\right)^{1/q},$$

where the supremum is taken over all partitions $a=t_0<\cdots<t_n=b$ of the interval $[a,b]$. We also set

$$\mathrm{VL}^\infty(g,a,b) = \sup_{a<t<s<b}\left(\frac{\|g(t)-g(s)\|_Y}{|t-s|}\right).$$

Definition 2.10. For each $q\in[1,+\infty]$, the map $g:[a,b]\to Y$ is called of q-bounded variation if $\mathrm{VL}^q(g,a,b)$ is a finite quantity.

Then we consider a family of linear bounded operators $T(t) \in \mathcal{L}(X, Y)$ for $t \geqslant 0$, where X and Y are Banach spaces. As in Thieme [22], we introduce for each $q \in [1, \infty)$

$$V^q(T, a, b) = \sup\left\{\left\|\sum_{i=1}^{n}\bigl(T(t_i) - T(t_{i-1})\bigr)x_i\right\|\right\},$$

where the supremum is taken over all partitions $a = t_0 < \cdots < t_n = b$ of the interval $[a, b]$ and over any $(x_1, \ldots, x_n) \in X^n$ with $\sum_{i=1}^{n}(t_i - t_{i-1})\|x_i\|_X^q \leqslant 1$. Finally we consider

$$V^\infty(T, a, b) = \sup\left\{\left\|\sum_{i=1}^{n}\bigl(T(t_i) - T(t_{i-1})\bigr)x_i\right\|\right\},$$

where the supremum is taken over all partitions $a = t_0 < \cdots < t_n = b$ of the interval $[a, b]$ and over any $(x_1, \ldots, x_n) \in X^n$ with $\|x_i\|_X \leqslant 1$ $\forall i = 1, \ldots, n$.

Definition 2.11. For each $q \in [1, +\infty]$, the map $t \to T(t)$ is called of q-bounded semi-variation if $V^q(T, a, b)$ is a finite quantity.

To conclude this section, we give various equivalent conditions to verify Assumption 2.2. Combining the result of Section 4 in Magal and Ruan [10] and the results of Section 3 in Thieme [22] one has the following theorem.

Theorem 2.12. *Let Assumption* 2.1 *be satisfied. Let $p, q \in [1, \infty]$ be given such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\widehat{\omega} \in \mathbb{R}$ be given. Then the following statements are equivalent*:

(i) *There exists some constant $\widehat{M} > 0$ such that for each $\tau > 0$ and any $f \in C^1([0, \tau], X)$,*

$$\bigl\|(S_A \diamond f)(t)\bigr\| \leqslant \widehat{M}\bigl\|e^{\widehat{\omega}(t-.)}f(.)\bigr\|_{L^p((0,t),X)}, \quad \forall t \in [0, \tau].$$

(ii) *There exists a norming space E of X_0, such that for each $x^* \in E$ the map $t \to x^* \circ S_{A-\widehat{\omega}I}(t)$ is of q-bounded variation from $[0, a]$ into X^* for any $a > 0$ and*

$$\sup_{x^* \in E,\, \|x^*\|_{X_0^*} \leqslant 1} \lim_{t\to\infty} \mathrm{VL}^q\bigl(x^* \circ S_{A-\widehat{\omega}I}(.), 0, t\bigr) < +\infty.$$

(iii) *There exists a norming space E of X_0, such that for each $x^* \in E$ there exists $\chi_{x^*} \in L_+^q((0, \infty), \mathbb{R})$ with*

$$\bigl\|x^* \circ S_{A-\widehat{\omega}I}(t+h) - x^* \circ S_{A-\widehat{\omega}I}(t)\bigr\| \leqslant \int_t^{t+h} \chi_{x^*}(s)\,ds, \quad \forall t, h \geqslant 0,$$

and

$$\sup_{x^* \in E:\, \|x^*\| \leqslant 1} \|\chi_{x^*}\|_{L^q(0,+\infty)} < +\infty.$$

(iv) *There exists a norming space E of X_0, such that for each $x^* \in E$ there exists $\chi_{x^*} \in L_+^q((0, \infty), \mathbb{R})$,*

$$\bigl\|x^* \circ \bigl(\lambda I - (A - \widehat{\omega}I)\bigr)^{-n}\bigr\|_{X^*} \leqslant \frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-\lambda s}\chi_{x^*}(s)\,ds$$

for each $n \geqslant 1$ and for each λ sufficiently large, and

$$\sup_{x^* \in E:\, \|x^*\| \leqslant 1} \|\chi_{x^*}\|_{L^q(0,+\infty)} < +\infty.$$

(v) *The map $t \to S_{A-\widehat{\omega}I}(t)$ is of p-bounded semi-variation from $[0, \tau]$ into $\mathcal{L}(X, X_0)$ for each $\tau > 0$, and*

$$\sup_{\tau > 0} V^p(S_{A-\widehat{\omega}I}, 0, \tau) < +\infty.$$

3. Spectral theory

In this section we recall some known results of spectral theory. We first introduce some notations. Let $L : D(L) \subset X \to X$ be a linear operator on a complex Banach space X. Denote by $\rho(L)$ the resolvent set of L, $N(L)$ the null space of L, and $R(L)$ the range of L. The spectrum of L is $\sigma(L) = \mathbb{C}\backslash\rho(L)$. The *point spectrum* of L is the set

$$\sigma_P(L) := \{\lambda \in \mathbb{C}:\ N(\lambda - L) \neq \{0\}\}.$$

The *essential spectrum* (in the sense of Browder [4]) of L is denoted by $\sigma_{\text{ess}}(L)$. That is the set of $\lambda \in \sigma(L)$ such that at least one of the following conditions holds: (i) $R(\lambda I - L)$ is not closed; (ii) λ is a limit point of $\sigma(L)$; (iii) $N_\lambda(L) := \bigcup_{k=1}^{+\infty} N((\lambda I - L)^k)$ is infinite dimensional.

Let Y be a subspace of X. Then we denote by $L_Y : D(L_Y) \subset Y \to Y$ the part of L on Y, which is defined by

$$L_Y x = Lx, \quad \forall x \in D(L_Y) := \{x \in D(L) \cap Y:\ Lx \in Y\}.$$

Definition 3.1. Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C^0-semigroup $\{T_L(t)\}_{t\geqslant 0}$ on a Banach space X. We define $\omega_0(L) \in [-\infty, +\infty)$ the *growth bound* of L by

$$\omega_0(L) := \lim_{t\to+\infty} \frac{\ln(\|T_L(t)\|_{\mathcal{L}(X)})}{t}.$$

The *essential growth bound* $\omega_{0,\text{ess}}(L) \in [-\infty, +\infty)$ of L is defined by

$$\omega_{0,\text{ess}}(L) := \lim_{t\to+\infty} \frac{\ln(\|T_L(t)\|_{\text{ess}})}{t},$$

where $\|T_L(t)\|_{\text{ess}}$ is the essential norm of $T_L(t)$ defined by

$$\|T_L(t)\|_{\text{ess}} = \kappa\big(T_L(t)B_X(0,1)\big),$$

here $B_X(0,1) = \{x \in X:\ \|x\|_X \leqslant 1\}$, and for each bounded set $B \subset X$, $\kappa(B) = \inf\{\varepsilon > 0:\ B$ can be covered by a finite number of balls of radius $\leqslant \varepsilon\}$ is the Kuratovsky measure of non-compactness.

Then we have the following result:

Theorem 3.2. *Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C^0-semigroup $\{T_L(t)\}$ on a Banach space X. Then*

$$\omega_0(L) = \max\Big(\omega_{0,\text{ess}}(L), \max_{\lambda\in\sigma(L)\backslash\sigma_{\text{ess}}(L)} \text{Re}(\lambda)\Big).$$

Assume in addition that $\omega_{0,\text{ess}}(L) < \omega_0(L)$. Then for each $\gamma \in (\omega_{0,\text{ess}}(L), \omega_0(L)]$, $\{\lambda \in \sigma(L):\ \text{Re}(\lambda) \geqslant \gamma\} \subset \sigma_p(L)$ is non-empty, finite and contains only poles of the resolvent of L. Moreover, there exists a finite rank bounded linear operator of projection $\Pi : X \to X$ satisfying the following properties:

(a) $\Pi(\lambda I - L)^{-1} = (\lambda - L)^{-1}\Pi$, $\forall\lambda \in \rho(L)$;
(b) $\sigma(L_{\Pi(X)}) = \{\lambda \in \sigma(L):\ \text{Re}(\lambda) \geqslant \gamma\}$;
(c) $\sigma(L_{(I-\Pi)(X)}) = \sigma(L) \setminus \sigma(L_{\Pi(X)})$.

In Theorem 3.2, the existence of the projector Π was first proved by Webb [24,25] which is the projection on the direct sum the generalized eigenspaces of L associated to all points $\lambda \in \sigma(L)$ with $\text{Re}(\lambda) \geqslant \gamma$, and the fact that we have a finite number of point of the spectrum with real part $\geqslant \gamma$ is proved by Engel and Nagel [7].

The following result is due to Magal and Ruan [12, see Lemma 2.1 and Proposition 3.6].

Theorem 3.3. *Let $(X, \|.\|)$ be a Banach space and $L : D(L) \subset X \to X$ be a linear operator. Assume that $\rho(L) \neq \emptyset$ and L_0, the part of L in $\overline{D(L)}$, is the infinitesimal generator of a linear C^0-semigroup $\{T_{L_0}(t)\}_{t\geqslant 0}$ on a Banach space $\overline{D(L)}$. Then $\sigma(L) = \sigma(L_0)$. Let $\Pi_0 : \overline{D(L)} \to \overline{D(L)}$ be a bounded linear operator of projection. Assume that*

$$\Pi_0(\lambda I - L_0)^{-1} = (\lambda I - L_0)^{-1}\Pi_0, \quad \forall\lambda > \omega, \omega \in \mathbb{R},$$

and

$$\Pi_0\big(\overline{D(L)}\big) \subset D(L_0) \quad \text{and} \quad L_0|_{\Pi_0(\overline{D(L)})} \text{ is bounded.}$$

Then there exists a unique bounded linear operator of projection Π on X satisfying the following properties:

(i) $\Pi|_{\overline{D(L)}} = \Pi_0$;
(ii) $\Pi(X) \subset \overline{D(L)}$;
(iii) $\Pi(\lambda I - L)^{-1} = (\lambda I - L)^{-1}\Pi$, $\forall \lambda > \omega$.
Moreover, for each $x \in X$ we have the following approximation formula

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda(\lambda I - L)^{-1} x.$$

4. Essential growth rate and bounded perturbation

This section is devoted to the proof of Theorem 1.2.

4.1. Preliminary results

The main result of this section is the following theorem.

Theorem 4.1. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Let $K_1 : X_0 \to Y$ be a compact linear operator for X_0 into a Banach space Y. Let $K_2 : X_1 \to X$ be a compact linear operator from a Banach space X_1 into X. Let be $\tau > 0$. Then the map*

$$t \to K_1(S_A \diamond K_2 f)(t)$$

is uniformly continuous from $[0, \tau]$ *into Y uniformly with respect to f in bounded subsets of* $C([0, \tau], X_1)$.

Proof. From Assumption 2.2, there exists $\widetilde{M} > 0$, such that for each $f \in C([0, \tau], X)$, with $\|f\|_\infty \leqslant 1$, we have

$$\big\|(S_A \diamond f)(t)\big\| \leqslant \widetilde{M}, \quad \forall t \in [0, \tau].$$

Since K_1 is compact, it follows from Schauder's theorem that $K_1^* : Y^* \to X_0^*$ is compact. Let $\varepsilon > 0$ be fixed. Then since $K_1^* B_{Y^*}(0, 1)$ is relatively compact in X_0^*, we can find $x_1^*, \ldots, x_n^* \in X_0^*$, with $\|x_i^*\| \leqslant \|K_1\| + 1$, $\forall i = 1, \ldots, n$, such that

$$K_1^* B_{Y^*}(0, 1) \subset \bigcup_{i=1}^n B_{Y^*}\big(x_i^*, \varepsilon\big).$$

Since K_2 is compact, we can find $x_1, \ldots, x_m \in X$, such that

$$K_2 B_{X_1}(0, 1) \subset \bigcup_{i=1}^m B_X(x_i, \varepsilon).$$

Let $\eta > 0$ be fixed. Then for each $f \in C([0, \tau], X)$, with $\|f\|_\infty \leqslant 1$, and any $t \in [0, \tau - \eta]$, by the Hahn–Banach theorem, there exists $y^* \in B_{Y^*}(0, 1)$, such that

$$\begin{aligned}\big\|K_1(S_A \diamond K_2 f)(t+\eta) - K_1(S_A \diamond K_2 f)(t)\big\|_Y &= \big\langle y^*, K_1\big((S_A \diamond K_2 f)(t+\eta) - (S_A \diamond K_2 f)(t)\big)\big\rangle \\ &= \big\langle K_1^* y^*, (S_A \diamond K_2 f)(t+\eta) - K_1(S_A \diamond K_2 f)(t)\big\rangle.\end{aligned}$$

But there exists $i_0 \in \{1, \ldots, n\}$, such that $\|x_{i_0}^* - K_1^* y^*\| \leqslant \varepsilon$, so

$$\begin{aligned}&\big\|K_1(S_A \diamond K_2 f)(t+\eta) - K_1(S_A \diamond K_2 f)(t)\big\|_Y \\ &\quad \leqslant \varepsilon \widehat{M} \|K_2\|_{\mathcal{L}(X_1, X)} + \big|\big\langle x_{i_0}^*, (S_A \diamond K_2 f)(t+\eta) - (S_A \diamond K_2 f)(t)\big\rangle\big|.\end{aligned}$$

Now since the map $t \to f(t)$ is continuous on $[0,\tau]$, this map is uniformly continuous. Therefore we can find $n_0 \geqslant 1$, such that

$$|t-s| \leqslant \frac{\tau}{n_0+1} \quad \Rightarrow \quad \|f(t)-f(s)\| \leqslant \varepsilon.$$

We denote by

$$t_i^{n_0} = \frac{i\tau}{n_0+1}, \quad \forall i = 0,\dots,n_0+1,$$

and we define $f^{n_0} : [0,\tau] \to X$ by

$$f^{n_0}(t) = \frac{(t-t_i^{n_0})}{(t_{i+1}^{n_0}-t_i^{n_0})} f\big(t_{i+1}^{n_0}\big) + \frac{(t_{i+1}^{n_0}-t)}{(t_{i+1}^{n_0}-t_i^{n_0})} f\big(t_i^{n_0}\big), \quad \forall t \in \big[t_i^{n_0}, t_{i+1}^{n_0}\big],\ \forall i = 0,\dots,n_0.$$

Then we have for each $i = 0,\dots,n_0$, and each $t \in [t_i^{n_0}, t_{i+1}^{n_0}]$

$$\big\|f^{n_0}(t)-f(t)\big\| \leqslant \frac{(t-t_i^{n_0})}{(t_{i+1}^{n_0}-t_i^{n_0})}\big\|f\big(t_{i+1}^{n_0}\big)-f(t)\big\| + \frac{(t_{i+1}^{n_0}-t)}{(t_{i+1}^{n_0}-t_i^{n_0})}\big\|f\big(t_i^{n_0}\big)-f(t)\big\| \leqslant \varepsilon.$$

But for each $i \in \{0,\dots,n_0+1\}$, we can find $j_i \in \{0,\dots,m\}$, such that

$$\big\|x_{j_i} - K_2 f\big(t_i^{n_0}\big)\big\| \leqslant \varepsilon.$$

So if we set

$$g(t) = \frac{(t-t_i^{n_0})}{(t_{i+1}^{n_0}-t_i^{n_0})} x_{j_{i+1}} + \frac{(t_{i+1}^{n_0}-t)}{(t_{i+1}^{n_0}-t_i^{n_0})} x_{j_i}, \quad \forall t \in \big[t_i^{n_0}, t_{i+1}^{n_0}\big],\ \forall i = 0,\dots,n_0.$$

Then we have

$$\|g-K_2 f\|_\infty \leqslant \big\|g-K_2 f^{n_0}\big\|_\infty + \big\|K_2 f^{n_0} - K_2 f\big\|_\infty \leqslant \big(1+\|K_2\|_{\mathcal{L}(X_1,X)}\big)\varepsilon.$$

But since g can be rewritten as

$$g(t) = \sum_{i=1}^{m} \gamma_i(t) x_i,$$

where the functions $\gamma_i(t)$ are the sum of function of the form

$$\gamma_i(t) = \frac{(t-t_i^{n_0})}{(t_{i+1}^{n_0}-t_i^{n_0})} 1_{[t_i^{n_0}, t_{i+1}^{n_0}]} + \frac{(t_{i+2}^{n_0}-t)}{(t_{i+2}^{n_0}-t_{i+1}^{n_0})} 1_{[t_{i+1}^{n_0}, t_{i+2}^{n_0}]},$$

for distinct i, the function $\gamma_i(t)$ is continuous, positive, and is bounded by 1. So to achieve the proof of Theorem 4.1, it is sufficient to apply the following lemma. □

Lemma 4.2. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Let be $x \in X$ and $x^* \in X_0^*$, then the map*

$$t \to x^*\big(\big(S_A \diamond h(.)x\big)(t)\big)$$

is uniformly continuous on $[0,\tau]$, *uniformly with respect to h in bounded subsets of* $C([0,\tau],\mathbb{R})$.

Proof. Let $h \in C_c^1((0,\tau),\mathbb{R})$ be given. Then we have

$$\begin{aligned}\frac{d}{dt}\int_0^t x^*\big(S_A(t-s)x\big)h(s)\,ds &= \frac{d}{dt}\int_0^t x^*\big(S_A(s)x\big)h(t-s)\,ds = S_A(t)h(0) + \int_0^t x^*\big(S_A(s)x\big)h'(t-s)\,ds \\ &= \int_0^t x^*\big(S_A(s)x\big)h'(t-s)\,ds.\end{aligned}$$

Due to Assumption 2.2 we have

$$\left|x^*\big(\big(S_A \diamond h(.)x\big)(\tau)\big)\right| \leqslant \widehat{C}\|h\|_{L^p((0,\tau),\mathbb{R})}$$

for some constant $\widehat{C}$ independent of function h. This implies that for any function $h \in C_c^1((0,\tau),\mathbb{R})$ we have

$$\left|\int_0^\tau x^*\big(S_A(s)x\big)h'(\tau - s)\,ds\right| \leqslant \widehat{C}\|h\|_{L^p((0,\tau),\mathbb{R})}. \tag{4.1}$$

Now if $\varphi \in C_c^1((0,\tau),\mathbb{R})$, then setting $h(s) = \varphi(\tau - s)$, we obtain $h \in C_c^1((0,\tau),\mathbb{R})$ and due to (4.1), we conclude that

$$\left|\int_0^\tau x^*\big(S_A(s)x\big)\varphi'(s)\,ds\right| \leqslant C\|\varphi\|_{L^p((0,\tau),\mathbb{R})}, \quad \forall\varphi \in C_c^1\big((0,\tau),\mathbb{R}\big).$$

Since $p \in [1,+\infty)$, by the Riesz's representation theorem, we know that there exists $g \in L^q((0,\tau),\mathbb{R})$ with $\frac{1}{q} + \frac{1}{p} = 1$ such that

$$\int_0^\tau x^*\big(S_A(s)x\big)\varphi'(s)\,ds = \int_0^\tau g(s)\varphi(s)\,ds, \quad \forall\varphi \in C_c^1\big((0,\tau),\mathbb{R}\big).$$

Therefore function $x^*(S_A(.)x)|_{[0,\tau]}$ belongs to $W^{1,q}((0,\tau),\mathbb{R})$ with $q \in (1,+\infty]$. Next since $x^*(S_A(0)x) = 0$, we obtain the following integral representation:

$$x^*\big(S_A(t)x\big) = \int_0^t g(l)\,dl, \quad \forall t \in [0,\tau],$$

and

$$x^*\big(S_A \diamond h(.)x\big)(t) = \int_0^t g(t-s)h(s)\,ds.$$

Finally we obtain that for any $(s,t) \in [0,\tau]^2$, with $t \geqslant s$,

$$\left|x^*\big(S_A \diamond h(.)x\big)(t) - x^*\big(S_A \diamond h(.)x\big)(s)\right| \leqslant \|h\|_\infty\left[\left|\int_0^{t-s}\left|g(l)\right|dl\right| + \int_0^s\left|g(t-s+l) - g(l)\right|dl\right].$$

This completes the proof of Lemma 4.2. □

4.2. *Proof of Theorem 1.2*

In this section we investigate the essential spectral growth rate of a bounded perturbation of A. Inspired by the work of Thieme [19, Theorem 3] we will make the following assumption.

Assumption 4.1. Let $L : X_0 \to X$ be a bounded linear operator such that $LT_{A_0}(t) : X_0 \to X$ is compact for every $t > 0$.

Let Z be a Banach space, and let I be an interval in $\mathbb{R}$. From now on we denote by

$$C_s\big(I, \mathcal{L}(X_0, Z)\big)$$

the space of strongly continuous map from I into $\mathcal{L}(X_0, Z)$. Then for each $V \in C_s([0,\tau], \mathcal{L}(X_0, X))$, we denote by $(S_A \diamond V(.))(t)$ the bounded linear operator from X_0 into itself, defined by

$$\big(S_A \diamond V(.)\big)(t)(x) := \big(S_A \diamond V(.)x\big)(t), \quad \forall t \in [0,\tau],\ \forall x \in X_0.$$

Next we need some preliminary lemmas.

Lemma 4.3. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Let* $\tau > 0$ *be given. Then for each* $V \in C([0,\tau], \mathcal{L}(X_0, X))$*, the map* $t \to (S_A \diamond V(.))(t)$ *is continuous from* $[0,\tau]$ *into* $\mathcal{L}(X_0)$.

Proof. Let $t, s \in [0,\tau]$ with $t \geqslant s$. From (2.5) we have

$$\big(S_A \diamond V(.)\big)(t) = T_{A_0}(s)\big(S_A \diamond V(.)\big)(t-s) + \big(S_A \diamond V(t-s+.)\big)(s).$$

Thus we obtain

$$\big(S_A \diamond V(.)\big)(t) - \big(S_A \diamond V(.)\big)(s) = T_{A_0}(s)\big(S_A \diamond V(.)\big)(t-s) + \big(S_A \diamond \big(V(.) - V(t-s+.)\big)\big)(s).$$

Next using Assumption 2.2, one has for any $x \in X$,

$$\big\|\big(S_A \diamond V(.)\big)(t) - \big(S_A \diamond V(.)\big)(s)\big\| \leqslant \delta(t-s)\big\|T_{A_0}(s)\big\| \sup_{l\in[0,t-s]} \big\|V(l)\big\| + \delta(s) \sup_{l\in[0,s]} \big\|V(l) - V(t-s+l)\big\|,$$

where $\delta(t)$ is defined by (2.7). Since $V : [0,\tau] \to \mathcal{L}(X_0, X)$ is continuous, it is also uniformly continuous and the result follows. □

By using Lemma 4.3 we obtain the following result.

Lemma 4.4. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Let be* $\tau > 0$. *Then we have the following*:

(i) *For each* $W \in C([0,\tau], \mathcal{L}(X_0))$, *there exists a unique* $V \in C([0,\tau], \mathcal{L}(X_0))$ *solution of*

$$V(t) = \big(S_A \diamond LV(.)\big)(t) + W(t), \quad \forall t \in [0,\tau].$$

(ii) *For each* $\widehat{W} \in C([0,\tau], \mathcal{L}(X_0, X))$, *there exists a unique* $\widehat{V} \in C([0,\tau], \mathcal{L}(X_0, X))$ *solution of*

$$\widehat{V}(t) = L\big(S_A \diamond \widehat{V}(.)\big)(t) + \widehat{W}(t), \quad \forall t \in [0,\tau].$$

Proof. The follows by using standart fixed point argument, and Lemma 4.3. □

Lemma 4.5. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Let* $\tau > 0$ *be fixed. Let* $\mathcal{F} \subset C([0,\tau], X)$ *be a set of equicontinuous maps, and assume that there exists* $\lambda^* > \omega$, *such that for each* $\eta \in (0,\tau]$,

$$\big\{\big(\lambda^* I - A\big)^{-1} f(t)\colon\ t \in [\eta,\tau],\ f \in \mathcal{F}\big\} \tag{4.2}$$

is relatively compact. Then for each $\tau_1 \in (0,\tau)$, *the set*

$$\big\{(S_A \diamond f)(t)\colon\ t \in [0,\tau_1],\ f \in \mathcal{F}\big\}$$

is relatively compact.

Proof. The proof is similar to the proof of Lemma 3.5 in Magal and Thieme [13]. □

The first main result of this section is the following:

Proposition 4.6. *Let Assumptions* 2.1, 2.2 *and* 4.1 *be satisfied. Let* $\tau > 0$ *be given and let* $W : [0,\tau] \to \mathcal{L}(X_0, X)$ *be strongly continuous. Then we have the following*:

(i) *If* $t \to L(S_A \diamond W(.))(t)$ *is operator norm continuous then the set*

$$\overline{\big\{L\big(S_A \diamond W(.)x\big)(t)\colon\ t \in [0,\tau_1],\ x \in B_{X_0}(0,1)\big\}}$$

is compact for each $\tau_1 \in (0,\tau)$.

(ii) *If $t \to W(t)$ is operator norm continuous, and the set*

$$\overline{\{W(t)x\colon\ t \in [\eta, \tau],\ x \in B_{X_0}(0,1)\}} \tag{4.3}$$

is compact for each $\eta \in (0, \tau]$, then the set

$$\overline{\{(S_A \diamond W(.)x)(t)\colon\ t \in [0, \tau],\ x \in B_{X_0}(0,1)\}}$$

is compact.

Proof. Proof of (i). Since the map $t \to L(S_A \diamond W(.))(t)$ is continuous from $[0, \tau]$ into $\mathcal{L}(X_0, X)$, this map is also uniformly continuous. Let $\tau_1 \in (0, \tau)$ be fixed, we deduce that

$$\lim_{h \searrow 0} \sup_{t \in [0,\tau_1]} \left\| L\big(S_A \diamond W(.)\big)(t) - \frac{1}{h} \int_t^{t+h} L\big(S_A \diamond W(.)\big)(s)\, ds \right\|_{\mathcal{L}(X_0,X)} = 0.$$

But for each $t \in [0, \tau_1]$, and each $h \in (0, \tau - \tau_1)$, we have

$$\begin{aligned} \frac{1}{h} \int_t^{t+h} L\big(S_A \diamond W(.)\big)(s)\, ds &= \frac{1}{h} L\left[\int_0^{t+h} S_A(t+h-s)W(s)\, ds - \int_0^t S_A(t-s)W(s)\, ds \right] \\ &= \frac{1}{h} L \int_t^{t+h} S_A(t+h-s)W(s)\, ds + \frac{1}{h} L \int_0^t \big[S_A(t+h-s) - S_A(t-s)\big] W(s)\, ds. \end{aligned}$$

On the one hand, recalling that we have

$$S_A(t+r) = T_{A_0}(r)S_A(t) + S_A(r), \quad \forall t, r \geqslant 0,$$

we obtain

$$\frac{1}{h} L \int_0^t \big[S_A(t+h-s) - S_A(t-s)\big] W(s)\, ds = \frac{1}{h} L \int_0^t T_{A_0}(t-s) S_A(h) W(s)\, ds, \quad \forall t \in [0, \tau_1].$$

On the other hand one has

$$\left\| \frac{1}{h} L \int_t^{t+h} S_A(t+h-s)W(s)\, ds \right\|_{\mathcal{L}(X_0,X)} \leqslant \frac{1}{h} \int_0^h \big\|S_A(h-s)\big\|_{\mathcal{L}(X_0,X)}\, ds \|L\|_{\mathcal{L}(X_0,X)} \sup_{t \in [0,\tau]} \big\|W(t)\big\|_{\mathcal{L}(X_0,X)}.$$

Here we can notice that, since W is strongly continuous, the uniform boundedness principle implies that the above supremum is finite. Next, Assumption 2.2 implies $\|S_A(t)\|_{\mathcal{L}(X_0,X)} \leqslant \delta(t)$ for each $t \geqslant 0$. Therefore we deduce that $\|S_A(t)\|_{\mathcal{L}(X_0,X)} \to 0$ when $t \to 0^+$. Thus when $h \to 0^+$ we have

$$\left\| \frac{1}{h} L \int_t^{t+h} S_A(t+h-s)W(s)\, ds \right\|_{\mathcal{L}(X_0,X)} \to 0 \quad \text{uniformly with respect to } t \in [0, \tau_1].$$

It follows that

$$\lim_{h \searrow 0} \sup_{t \in [0,\tau_1]} \left\| L\big(S_A \diamond W(.)\big)(t) - \frac{1}{h} L \int_0^t T_{A_0}(t-s) S_A(h) W(s)\, ds \right\|_{\mathcal{L}(X_0,X)} = 0.$$

From Assumption 4.1, the operator $LT_{A_0}(t-s)$ is compact for any $0 \leqslant s < t < \tau_1$. Thus using the same argument as in the proof of Theorem 3.2 in [13] completes the proof of assertion (i). Finally assertion (ii) directly follows from Lemma 4.5. Indeed if we set $\mathcal{F} = \{t \to W(t)x,\ x \in B_{X_0}(0,1)\} \subset C([0,\tau], X)$. Then the map $t \to W(t)$ is operator norm continuous from $[0, \tau]$ into $\mathcal{L}(X_0, X)$, it is uniformly continuous. Thus $\mathcal{F}$ is equicontinuous subset of

$C([0,\tau],X)$. Finally for any $\lambda > \omega_A$, the map $(\lambda - A)^{-1}$ is a bounded operator of X and due to assumption (4.3), for each $\eta \in (0,\tau)$ the set

$$\left\{(\lambda - A)^{-1} W(t)x,\ t \in [\eta,\tau],\ x \in B_{X_0}(0,1)\right\}$$

is relatively compact. Then Lemma 4.5 applies and completes the proof of assertion (ii). □

We now use the above result to obtain an approximated expression for the semigroup $\{T_{(A+L)_0}(t)\}_{t\geqslant 0}$. For that purpose we consider the map $\mathcal{B}: C_s([0,+\infty), \mathcal{L}(X_0)) \to C_s([0,+\infty), \mathcal{L}(X_0))$ defined by

$$\mathcal{B}(V)(t) = \big(S_A \diamond LV(.)\big)(t),$$

and $\widetilde{\mathcal{B}}: C_s([0,+\infty), \mathcal{L}(X_0,X)) \to C_s([0,+\infty), \mathcal{L}(X_0,X))$ defined by

$$\widetilde{\mathcal{B}}(W)(t) = L\big(S_A \diamond W(.)\big)(t).$$

Then we have the following result.

Proposition 4.7. *Let Assumptions* 2.1, 2.2 *and* 4.1 *be satisfied. Assume that for some integer* $n \geqslant 0$, *the map*

$$t \to L\mathcal{B}^n(T_{A_0})(t) = \widetilde{\mathcal{B}}^n(LT_{A_0})(t)$$

is operator norm continuous on $[0,+\infty)$. *Then we have the following expression*

$$T_{(A+L)_0}(t)x = \sum_{k=0}^{n} \mathcal{B}^k\big(T_{A_0}(.)x\big)(t) + C(t)x,$$

where operator $C(t) \in \mathcal{L}(X_0)$ *is compact for each* $t \geqslant 0$.

Proof. We first recall that the semigroup $T_{(A+L)_0}(t)$ satisfies the following fixed point formulation

$$T_{(A+L)_0}(t) = T_{A_0}(t) + \big(S_A \diamond LT_{(A+L)_0}(.)\big)(t), \quad \forall t \geqslant 0.$$

This rewrites using the map $\mathcal{B}$ as follows:

$$T_{(A+L)_0} = T_{A_0} + \mathcal{B}(T_{(A+L)_0}).$$

Next multiplying this equality by L leads us to

$$LT_{(A+L)_0} = LT_{A_0} + \widetilde{\mathcal{B}}(LT_{(A+L)_0}).$$

By induction, we obtain

$$\widetilde{\mathcal{B}}^n(LT_{(A+L)_0}) = \widetilde{\mathcal{B}}^n(LT_{A_0}) + L\big(S_A \diamond \widetilde{\mathcal{B}}^n(LT_{(A+L)_0})\big). \tag{4.4}$$

Now by assumption $t \to \widetilde{\mathcal{B}}^n(LT_{A_0})(t)$ is operator norm continuous. Thus by using Lemma 4.4, we obtain that the map $t \to \widetilde{\mathcal{B}}^n(LT_{(A+L)_0})(t) = L\mathcal{B}^n(T_{(A+L)_0})(t)$ is operator norm continuous. Then from (4.4), the operator $t \to \widetilde{\mathcal{B}}^{n+1}(LT_{(A+L)_0})(t)$ is operator norm continuous. So by Proposition 4.6(i), we deduce that

$$\overline{\left\{\widetilde{\mathcal{B}}^{n+1}\big(LT_{(A+L)_0}(.)x\big)(t):\ t \in [0,\tau],\ x \in B_{X_0}(0,1)\right\}} \tag{4.5}$$

is compact for any $\tau > 0$.

Next we claim that the set

$$\overline{\left\{\widetilde{\mathcal{B}}^{n}\big(LT_{A_0}(.)x\big)(t):\ t \in [\eta,\tau],\ x \in B_{X_0}(0,1)\right\}} \tag{4.6}$$

is compact for any $0 < \eta \leqslant \tau$. Indeed for $n = 0$, this directly follows from Assumption 4.1, and if $n \geqslant 1$ since the map $t \to \widetilde{\mathcal{B}}^n(LT_{A_0})(t)$ is operator norm continuous, this follows from Proposition 4.6(ii).

Finally, using (4.4)–(4.6), we conclude that the set

$$\overline{\left\{\widetilde{\mathcal{B}}^{n}\big(LT_{(A+L)_0}(.)x\big)(t):\ t \in [\eta,\tau],\ x \in B_{X_0}(0,1)\right\}}$$

is compact for each $\eta \in (0,\tau]$.

Recalling now that

$$\mathcal{B}^{n+1}(T_{(A+L)_0}) = \big(S_A \diamond \widetilde{\mathcal{B}}^n(LT_{(A+L)_0})\big),$$

and that the map $t \to \widetilde{\mathcal{B}}^n(LT_{(A+L)_0})(t)$ is operator norm continuous, we deduce by using Proposition 4.6(ii) with this map that the set

$$\overline{\big\{\mathcal{B}^{n+1}(T_{(A+L)_0}x)(t)\colon\ t \in [0,\tau],\ x \in B_{X_0}(0,1)\big\}}$$

is compact. Finally the result follows from the following expression:

$$T_{(A+L)_0} = T_{A_0} + \mathcal{B}(T_{A_0}) + \mathcal{B}^2(T_{A_0}) + \cdots + \mathcal{B}^n(T_{A_0}) + \mathcal{B}^{n+1}(T_{(A+L)_0}).$$

This completes the proof of Proposition 4.7. □

We now prove the following proposition that will be essential in the proof of our main result.

Proposition 4.8. *Let Assumptions* 2.1, 2.2 *and* 4.1 *be satisfied. Then the map* $t \to L(S_A \diamond LT_{A_0}(.))(t)$ *is operator norm continuous from* $[0,+\infty)$ *into* $\mathcal{L}(X_0, X)$.

Proof. Let $\varepsilon > 0$ be fixed. By using formula (2.5), for $t \geqslant 2\varepsilon$,

$$L\big(S_A \diamond LT_{A_0}(.)\big)(t) = LT_{A_0}(\varepsilon)\big(S_A \diamond LT_{A_0}(.)\big)(t-\varepsilon) + \big(S_A \diamond LT_{A_0}(.+t-\varepsilon)\big)(\varepsilon)$$

and

$$\big(S_A \diamond LT_{A_0}(.)\big)(t-\varepsilon) = T_{A_0}(t-2\varepsilon)\big(S_A \diamond LT_{A_0}(.)\big)(\varepsilon) + \big(S_A \diamond LT_{A_0}(.+\varepsilon)\big)(t-2\varepsilon).$$

So

$$L\big(S_A \diamond LT_{A_0}(.)\big)(t) = K_\varepsilon\big(S_A \diamond K_\varepsilon T_{A_0}(.)\big)(t-2\varepsilon) + R(\varepsilon,t),$$

where

$$K_\varepsilon = LT_{A_0}(\varepsilon)$$

and

$$R(\varepsilon,t) = LT_{A_0}(\varepsilon)T_{A_0}(t-2\varepsilon)\big(S_A \diamond LT_{A_0}(.)\big)(\varepsilon) + \big(S_A \diamond LT_{A_0}(.+t-\varepsilon)\big)(\varepsilon)$$

is of order ε by Assumption 2.2. By Assumption 4.1 the linear operator K_ε is compact. So by Theorem 4.1 the map $t \to K_\varepsilon(S_A \diamond K_\varepsilon T_{A_0}(.))(t-2\varepsilon)$ is operator norm continuous on $[2\varepsilon,+\infty)$, and the result follows. □

Proof of Theorem 1.2. Let us first note that by Proposition 4.8, the map $t \to L(S_A \diamond LT_{A_0}(.))(t)$ is norm continuous, so by applying Proposition 4.7 for $n = 1$, we deduce that

$$T_{(A+L)_0}(t) = T_{A_0}(t) + \big(S_A \diamond LT_{A_0}(.)\big)(t) + C(t), \tag{4.7}$$

where $C(t)$ is a compact operator for each $t \geqslant 0$.

Assume first that $\omega_{0,\mathrm{ess}}(A_0) = \omega_0(A_0)$. By construction for each $\gamma > \omega_0(A_0)$, there exists $M_\gamma > 0$

$$\big\|T_{A_0}(t)\big\| \leqslant M_\gamma e^{\gamma t}, \quad \forall t \geqslant 0,$$

and it follows that Assumption 2.1 is also satisfied, whenever we replace M_A and ω_A by M_γ and γ, respectively. So by applying Proposition 2.8, it follows that for each $\gamma > \omega_0(A_0)$ there exists some constant $M_\gamma^1 > 0$ such that

$$e^{-\gamma t}\big\|\big(S_A \diamond LT_{A_0}(.)\big)(t)\big\| \leqslant M_\gamma^1\|L\| \sup_{s\in[0,t]} e^{-\gamma s}\big\|T_{A_0}(s)\big\| \leqslant M_\gamma^1\|L\|M_\gamma, \quad \forall t \geqslant 0.$$

From (4.7) we obtain

$$\big\|T_{(A+L)_0}(t)\big\|_{\mathrm{ess}} \leqslant \big\|T_{A_0}(t)\big\| + \big\|\big(S_A \diamond LT_{A_0}(.)\big)(t)\big\| \leqslant C_\gamma e^{\gamma t}, \quad \forall t \geqslant 0,$$

for some constant C_γ. Therefore we conclude that $\omega_{0,\mathrm{ess}}((A+L)_0) \leqslant \gamma$ for all $\gamma > \omega_0(A_0)$, which implies

$$\omega_{0,\mathrm{ess}}\big((A+L)_0\big) \leqslant \omega_{0,\mathrm{ess}}(A_0).$$

We now consider the case $\omega_{0,\mathrm{ess}}(A_0) < \omega_0(A_0)$. Let $\gamma \in (\omega_{0,\mathrm{ess}}(A_0), \omega_0(A_0)]$ be fixed. By using Theorem 3.2 we consider $\Pi_0 : X_0 \to X_0$ the finite rank linear bounded projector satisfying:

(a) $\Pi_0(\lambda I - A_0)^{-1} = (\lambda - A_0)^{-1}\Pi_0$, $\forall \lambda \in \rho(A_0)$;
(b) $\sigma(A_0|_{\Pi_0(X)}) = \{\lambda \in \sigma(A_0)\colon \operatorname{Re}(\lambda) \geqslant \gamma\}$;
(c) $\sigma(A_0|_{(I-\Pi_0)(X)}) = \sigma(A_0) \setminus \sigma(A_0|_{\Pi_0(X)})$.

Since Π_0 corresponds to the projection on the direct sum of some generalized eigenspaces of finite dimension, we have $\Pi_0(X_0) \subset D(A_0)$. Moreover since Π_0 is a finite rank operator, the restriction $A_0|_{\Pi_0(X_0)}$ is bounded. Then using Theorem 3.3 we extend this projector in $\Pi : X \to X$ with the following properties:

(i) $\Pi|_{X_0} = \Pi_0$;
(ii) $\Pi(X) \subset X_0$;
(iii) $\Pi(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi$, $\forall \lambda > \omega_A$.

Since Π_0 is a finite rank operator, $\Pi_0 T_{(A+L)_0}(t)$ is compact for each $t \geqslant 0$ therefore

$$\big\|T_{(A+L)_0}(t)\big\|_{\mathrm{ess}} = \big\|(I-\Pi_0)T_{(A+L)_0}(t)\big\|_{\mathrm{ess}}.$$

On the other hand one has

$$\begin{aligned}(I-\Pi_0)T_{(A+L)_0}(t) &= (I-\Pi_0)T_{A_0}(t) + (I-\Pi_0)\big(S_A \diamond LT_{A_0}(.)\big)(t) + C_0(t)\\ &= (I-\Pi_0)T_{A_0}(t) + (I-\Pi_0)\big(S_A \diamond L(I-\Pi_0)T_{A_0}(.)\big)(t)\\ &\quad + (I-\Pi_0)\big(S_A \diamond L\Pi_0 T_{A_0}(.)\big)(t)\Pi_0 + C_0(t),\end{aligned}$$

where $(I-\Pi_0)(S_A \diamond L\Pi_0T_{A_0}(.))(t)\Pi_0$ and $C_0(t)$ are compact for any $t \geqslant 0$.

It follows that

$$\big\|T_{(A+L)_0}(t)\big\|_{\mathrm{ess}} \leqslant \big\|\big[(I-\Pi_0)T_{A_0}(t) + (I-\Pi_0)\big(S_A \diamond L(I-\Pi_0)T_{A_0}(.)\big)(t)\big](I-\Pi_0)\big\|.$$

We set $Y := (I-\Pi)X$ endowed with the norm of X. Then since $(I-\Pi)$ commutes with the resolvent of A, $B : D(B) \subset Y \to Y$ the part of A in Y, and B_0 the part of B in $\overline{D(B)}$, satisfy Assumptions 2.1 and 2.2 in Y, and we have the following:

$$\big[(I-\Pi_0)T_{A_0}(t) + (I-\Pi_0)\big(S_A \diamond LT_{A_0}(.)\big)(t)\big](I-\Pi_0) = \big[T_{B_0}(t) + \big(S_B \diamond (I-\Pi)LT_{B_0}(t)\big)\big](I-\Pi_0),$$

and by construction (see (c) above) we have $\sigma(A_0|_{(I-\Pi_0)(X)}) = \sigma(B_0) \subset \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) < \gamma\}$, and $\omega_{0,\mathrm{ess}}(A_0) = \omega_{0,\mathrm{ess}}(B_0) < \gamma$. So

$$\omega_0(B_0) < \gamma$$

and

$$\big\|T_{(A+L)_0}(t)\big\|_{\mathrm{ess}} \leqslant \big\|\big[T_{B_0}(t) + \big(S_B \diamond (I-\Pi)LT_{B_0}(t)\big)\big]\big\|_{\mathcal{L}(Y)}\big\|(I-\Pi_0)\big\|.$$

So by applying the same argument as in the first part of the proof (i.e. the case $\omega_{0,\mathrm{ess}}(A_0) = \omega_0(A_0)$), we deduce that

$$\omega_{0,\mathrm{ess}}\big((A+L)_0\big) < \gamma, \quad \forall \gamma > \omega_{0,\mathrm{ess}}(A_0).$$

The proof is completed. □

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