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Travelling Wave Solutions in Multigroup Age-Structured Epidemic Models

Arnaut Ducrot, Pierre Magal & Shigui Ruan

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Abstract

Age-structured epidemic models have been used to describe either the age of individuals or the age of infection of certain diseases and to determine how these characteristics affect the outcomes and consequences of epidemiological processes. Most results on age-structured epidemic models focus on the existence, uniqueness, and convergence to disease equilibria of solutions. In this paper we investigate the existence of travelling wave solutions in a deterministic age-structured model describing the circulation of a disease within a population of multigroups. Individuals of each group are able to move with a random walk which is modelled by the classical Fickian diffusion and are classified into two subclasses, susceptible and infective. A susceptible individual in a given group can be crisscross infected by direct contact with infective individuals of possibly any group. This process of transmission can depend upon the age of the disease of infected individuals. The goal of this paper is to provide sufficient conditions that ensure the existence of travelling wave solutions for the age-structured epidemic model. The case of two population groups is numerically investigated which applies to the crisscross transmission of feline immunodeficiency virus (FIV) and some sexual transmission diseases.

1. Introduction

In 1956, Barlett [3] used a spatial deterministic model to predict a wave of infection moving out from the initial source of infection. Given a spatially homogenous initial condition, the spatial model predicts the eventual establishment of an epidemic front which divides the infected and uninfected regions and moves into the uninfected region with a constant velocity [5]. The velocity at which an infection wave moves is set by the rate of divergence from the disease-free state and can be determined by linear methods [17]. The existence of travelling wave

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fronts in various epidemic models described by reaction–diffusion systems has been extensively studied, we refer to the monographs of Murray [17] and Rass and Radcliffe [18], a survey paper by Ruan [19], and the references cited therein.

On the other hand, age-structured epidemic models have been used to describe either the age of individuals or the age of infection of certain diseases and to determine how these characteristics affect the outcomes and consequences of epidemiological processes. We refer to Diekmann and Heesterbeek [6], Fitzgibbon et al. [9], Gurtin and MacCamy [11], Iannelli [13], Inaba [14], Magal and Ruan [16], Thieme [21], Webb [22,23], and the references cited therein. Most results on age-structured epidemic models focus on the existence, uniqueness, and convergence to disease equilibria of solutions, there are very few results on the existence of travelling waves in age-structured epidemic models [1,7,8,20].

In this paper we investigate the existence of travelling wave solutions in a deterministic age-structured model describing the circulation of a disease within a population of n groups. Individuals of each group are able to move with a random walk which is modelled by the classical Fickian diffusion. We introduce the age structure, since the infection of the disease is a continuously structured variable. In each group the population is split into two subclasses, the susceptible individuals and the infective individuals. A susceptible individual in a given group can be crisscross infected by direct contact with infective individuals of possibly any group. By direct contact we mean that the transmission of the disease is only local in space. This process of transmission can depend upon the age of the disease of infected individuals.

We consider the following mathematical model:

$$\begin{cases} \dfrac{\partial \theta_i}{\partial t} = d_{ui}\Delta\theta_i - \theta_i(t,x)\displaystyle\sum_{j=1}^{n}\int_0^\infty K_{i,j}(a)\psi_j(t,a,x)\,\mathrm{d}a, \\ \dfrac{\partial \psi_i}{\partial t} + \dfrac{\partial \psi_i}{\partial a} = d_{vi}\Delta\psi_i - \mu_i(a)\psi_i, \\ \psi_i(t,0,x) = \theta_i(t,x)\displaystyle\sum_{j=1}^{n}\int_0^\infty K_{i,j}(a)\psi_j(t,a,x)\,\mathrm{d}a. \end{cases} \tag{1.1}$$

This system of equations is posed for $t \geqq 0$, $x \in \mathbb{R}$, $a \in (0,\infty)$ and $i = 1,\dots,n$.

$\theta_i(t,x)$ is the density of susceptible individuals that belong to the ith group at time t and location $x \in \mathbb{R}$, while $\psi_i(t,a,x)$ denotes the density of infective individuals in the group i at time t location x and have entered the infected class since time $a > 0$. The parameters d_{ui} and d_{vi} are positive constants describing the diffusion rates of the susceptible and infective populations, respectively. Here we do not suppose that susceptibles and infectives spread in the same manner. The disease can influence the spreading behavior of infectives. For the sake of simplicity we suppose that the diffusion coefficients d_{vi} do not depend upon the age of the disease.

The function $K_{i,j}(a)$ denotes the age-specific contamination rate from group j to group i. These functions depend explicitly on the variable a that allows us to take into account a (possibly) latent period. A specific form for these functions reads as a characteristic function of some interval:

$$K_{i,j}(a) = \widehat{K_{i,j}} 1_{[\tau_{i,j},\infty)}(a),$$

where $\widehat{K_{i,j}} \geqq 0$ and $\tau_{i,j} \geqq 0$ are some constants. This simple form means that an individual in the jth group that enters the infected class cannot infect an individual of the species group i until it has stayed more than a time $\tau_{i,j}$ in the infected class without being removed.

Finally, the function $\mu_i(a)$ denotes the infection-age-specific removed rate for the infectives of group i. This removed rate describes the death rate of individuals due to the disease or a recovery rate due to complete immunization (or both).

Here we mention that another interesting problem consists in considering a spatial nonlocal force of infection of the form

$$\Lambda_i(x) = \sum_{j=1}^{n} \int_{\mathbb{R}} \int_0^{\infty} K_{i,j}(a, x-y)\psi_j(t,a,y)\,\mathrm{d}a\,\mathrm{d}y.$$

This kind of problem with nonlocal interaction seems to exhibit complex patterns if the support of the infection kernels is sufficiently large. We refer to Genieys et al. [10] for a study of pattern formation for some nonlocal (in space) Fisher type of equations. We would like to investigate disease propagation for this kind of nonlocal interaction in a forthcoming work.

Model (1.1) has been previously considered in the literature. The case $n = 1$ corresponds to the Kermack and McKendrick model [15]. The case $n = 2$ has also been considered by Fitzgibbon et al. [9] to model the circulation of Feline Immunodeficiency Virus within a population of domestic cats. They also mentioned in their work that this model with $n = 2$ can be applied to study some sexually transmitted diseases. In this case the two groups of population correspond to males and females, respectively. The crisscross infection kinetics mean that infected males can infect susceptible males as well as susceptible females and infected females can infect both susceptible males and susceptible females.

Note that in model (1.1), as in [9], we ignore the vital demographic dynamics of different populations. The introduction of demographic parameters could be of particular interest but complicates the mathematical analysis significantly. Indeed in a similar model for one population and without age dependency, the introduction of logistic growth of the population leads to oscillating patterns. We refer to the book of Murray [17] for more details.

The aim of this work is to study front solutions for problem (1.1). By front solutions we mean travelling wave solutions, that are solutions of the particular form

$$\theta_i(t,x) = \widetilde{\theta_i}(x-ct), \quad \psi_i(t,a,x) = \widetilde{\psi_i}(a, x-ct).$$

Here $c > 0$ is an unknown parameter, the wave speed, that should be found together with the unknown functions $\widetilde{\theta_i}$ and $\widetilde{\psi_i}$. In order to study this problem, we perform the change of unknown functions

$$u_i(x) = \widetilde{\theta_i}(x), \quad v_i(a,x) = \widetilde{\psi_i}(a,x)e^{\int_0^a \mu_i(s)\,\mathrm{d}s}$$

and set

$$k_{i,j}(a) = K_{i,j}(a)e^{-\int_0^a \mu_i(s)\,\mathrm{d}s}. \tag{1.2}$$

In the moving frame coordinate, the travelling wave problem reduces to finding positive functions u_i and v_i satisfying the following elliptic-parabolic boundary value problem:

$$\begin{cases} d_{ui}u_i'' + cu_i' - u_i(x)\displaystyle\sum_{j=1}^{n}\int_0^\infty k_{i,j}(a)v_j(a,x)\,\mathrm{d}a = 0, & x \in \mathbb{R}, \\ \dfrac{\partial v_i}{\partial a} = d_{vi}\dfrac{\partial^2 v_i}{\partial x^2} + c\dfrac{\partial v_i}{\partial x}, \quad x \in \mathbb{R}, \quad a > 0, \\ v_i(0,x) = u_i(x)\displaystyle\sum_{j=1}^{n}\int_0^\infty k_{i,j}(a)v_j(a,x)\,\mathrm{d}a. \end{cases} \tag{1.3}$$

We also impose the limit behavior of the solutions

$$u_i(\infty) = 1, \quad u_i(-\infty) = u_i^+, \quad v_i(a, \pm\infty) = 0, \tag{1.4}$$

where $u_i^+ \in [0,1]$ are unknown numbers. These limit conditions simply mean that at $x = \pm\infty$ the population is entirely composed with susceptible individuals. The density of infectives is a spike like function. At $x = +\infty$ the population is not yet affected by the epidemic while at $x = -\infty$ it has ever passed and the density of susceptible is u_i^+. Therefore, the unknown parameters u_i^+, $i = 1, \ldots, n$ describe the severity of the epidemics.

The goal of this paper is to provide sufficient conditions that ensure the existence of nontrivial positive solutions for problem (1.3)–(1.4). The paper is organized as follows. Section 2 discusses the main assumptions of this work and describes the main results. Section 3 provides mathematical proofs of the main results while Sect. 4 investigates numerical simulations of the model in the case of two population groups ($n = 2$) which applies to the crisscross transmission of certain sexual transmission diseases and feline immunodeficiency virus (FIV) (see, for example, [9]).

2. Assumptions and main results

In this section we give the main assumptions of this work and state the main results of the paper. We first suppose that all individuals (susceptibles as well as infectives) can diffuse in space, which means that

$$d_{ui} > 0, \quad d_{vi} > 0, \quad \forall i = 1, \ldots, n,$$

and set

$$d = \sup_{i=1,\ldots,n} d_{vi}. \tag{2.1}$$

Next we consider system (1.3)–(1.4) and suppose that

$$k_{i,j} \in L^1_+((0,\infty),\mathbb{R}) \cap L^\infty_+((0,\infty),\mathbb{R}), \quad \forall i,j = 1,\ldots,n. \tag{2.2}$$

If we go back to the original model (1.1) and definition (1.2), then the functions $K_{i,j}$ are supposed to be positive and bounded but non-integrable. Therefore, assumption (2.2) means that the disease is lethal for each group and essentially when the age of the disease is large enough.

Next as we will see in the proofs in Section 3, the existence of solutions for (1.3)–(1.4) is related to the following eigenvalue problem: Find $c > 0$, $\lambda \in \left(0, \frac{c}{2d}\right)$ and $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{R}^n \setminus \{0\}$ with $\varphi_i \geqq 0$ such that

$$\varphi_i = \sum_{j=1}^{n} \left(\int_0^\infty k_{i,j}(a) e^{(d_{vj}\lambda^2 - c\lambda)a} \, \mathrm{d}a \right) \varphi_j \quad \text{for } i = 1, \dots, n. \tag{2.3}$$

The existence of solutions for the above eigenvalue problem can be discussed in term of the spectral radius of the matrix

$$M = \left(\int_0^\infty k_{i,j}(a) \, \mathrm{d}a \right)_{(i,j)=1,\dots,n}.$$

To ensure the existence of solutions for this eigenvalue problem we suppose for simplicity that

$$\text{Matrix } M \text{ is either irreducible or primitive.} \tag{2.4}$$

Under this assumption the existence of travelling wave solutions is related to the location of the spectral radius $\rho(M)$ of matrix M with respect to 1.

More precisely, we will prove the following result:

Theorem 2.1. *Suppose that assumptions* (2.2) *and* (2.4) *hold. If* $\rho(M) > 1$, *then there exists* $c^* > 0$ *such that system* (1.3)–(1.4) *has* (*at least*) *a nontrivial solution* $(\mathbf{u}, \mathbf{v}, c)$ *for any* $c > c^*$, *where* $c^* > 0$ *is given by the resolution of the equation*:

$$\rho \left(\int_0^\infty k_{i,j}(a) e^{\frac{d_{vj} - 2d}{4d^2} c^{*2} a} \, \mathrm{d}a \right) = 1. \tag{2.5}$$

Moreover, we have the following estimate on the severity vector $\mathbf{u}^\infty = (u_1^\infty, \dots, u_n^\infty)^T$:

$$\rho \left(\begin{pmatrix} u_1^\infty & 0 & \cdots & 0 \\ 0 & \ddots & & \\ 0 & \cdots & 0 & u_n^\infty \end{pmatrix} M \right) \leqq 1. \tag{2.6}$$

Remark 2.2. If condition (2.4) does not hold, then the disease can propagate only through a subgroup of the population and the situation becomes more complicated to study.

This existence result has one immediate corollary. If we go back to the original problem (1.1) with the following infection parameters:

$$K_{i,j}(a) = \widetilde{K_{i,j}} 1_{[\tau_{i,j}, \infty)}(a), \quad \mu_i(a) = \widetilde{\mu_i},$$

where $\widetilde{K_{i,j}} \geqq 0$, $\tau_{i,j} \geqq 0$ and $\widetilde{\mu_i} > 0$. Then we obtain, by integrating model (1.1) over the age variable, the following result:

Corollary 2.3. *Suppose that the matrix*

$$M = \left(\frac{\widetilde{K_{i,j}}}{\widetilde{\mu_i}} e^{-\tau_{i,j}\widetilde{\mu_i}} \right)_{i,j}$$

satisfies assumption (2.4). *If* $\rho(M) > 1$, *then there exists* $c^* > 0$ *such that the system*

$$\begin{aligned} \frac{\partial \theta_i}{\partial t} &= d_{ui}\Delta\theta_i - \theta_i \sum_{j=1}^{n} e^{\tau_{i,j}\widetilde{\mu_i}} \widetilde{K_{i,j}} T_{d_{vj}\Delta}(\tau_{i,j})\psi_j(t-\tau_{i,j},.), \\ \frac{\partial \Psi_i}{\partial t} &= d_{vi}\Delta\psi_i - \widetilde{\mu_i}\psi_i + \theta_i \sum_{j=1}^{n} e^{\tau_{i,j}\widetilde{\mu_i}} \widetilde{K_{i,j}} T_{d_{vj}\Delta}(\tau_{i,j})\psi_j(t-\tau_{i,j},.) \end{aligned} \tag{2.7}$$

has a travelling wave solution for any wave speed $c > c^*$. *Here* $T_{d\Delta}(\tau)\varphi$ *is defined by*

$$T_{d\Delta}(t)\varphi(x) = \frac{1}{\sqrt{4\pi t d}} \int_{\mathbb{R}} \varphi(x-y) e^{-\frac{y^2}{4dt}}\, \mathrm{d}y.$$

To prove the above corollary, we first observe that travelling waves are complete orbits of system (1.1). Then by using solutions of system (1.1) integrated along the characteristics (see [8,16]) and by integrating in the age variable, we deduce that any complete orbit of system (1.1) gives a solution of the partial differential equation with delay (2.7), and Corollary 2.3 follows.

Another corollary arises when $\tau_{i,j} = 0$ for any $i, j = 1, \ldots, n$.

Corollary 2.4. *Suppose that the matrix*

$$M = \left(\frac{\widetilde{K_{i,j}}}{\widetilde{\mu_i}} \right)_{i,j}$$

satisfies assumption (2.4). *If* $\rho(M) > 1$, *then there exists* $c^* > 0$ *such that the system*

$$\begin{aligned} \frac{\partial \theta_i}{\partial t} &= d_{ui}\Delta\theta_i - \theta_i \sum_{j=1}^{n} \widetilde{K_{i,j}}\psi_j, \\ \frac{\partial \Psi_i}{\partial t} &= d_{vi}\Delta\psi_i - \widetilde{\mu_i}\psi_i + \theta_i \sum_{j=1}^{n} \widetilde{K_{i,j}}\psi_j \end{aligned} \tag{2.8}$$

has a travelling wave solution for any wave speed $c > c^*$.

Next we have the following non-existence result:

Proposition 2.5. *Suppose that assumption* (2.2) *holds and* $\rho(M) < 1$. *If* $(\mathbf{u}, \mathbf{v}, c)$ *is a solution of* (1.3)–(1.4) *with* $\mathbf{v}(0, .) \in L^1_+(\mathbb{R}, \mathbb{R}^n)$, *then*

$$u_i \equiv 1, \quad \forall i = 1, \ldots, n, \quad \mathbf{v} \equiv 0.$$

Remark 2.6. We now give some remarks about our results. First, when $n = 1$ it is proved in [8] that wave solutions exist if and only if $\rho(M) > 1$. Here we obtained a weaker result but we expect that wave solutions for the general system (1.3)–(1.4) exist if and only if $\rho(M) > 1$. We also conjecture that the linear speed c^* provided in Theorem 2.1 corresponds to the minimal wave speed of the problem. This question will be addressed in the forthcoming work by developing Harnack inequalities for this kind of problem.

3. Proof of Theorem 2.1

The aim of this section is to provide a rigorous mathematical proof of Theorem 2.1. The proof is split into several parts. The first one considers the construction of suitable sub- and super-solution pairs. Then we consider a similar problem posed on a bounded spatial interval. This step allows us to use the compactness of the semigroup generated by the Laplace operator together with Dirichlet conditions on a bounded domain. Finally we let the length of the bounded interval tend to infinity to obtain a solution of (1.3)–(1.4) on the whole real line. This last step uses a priori estimates together with the sub- and super-solution pairs. Throughout this section we suppose that assumptions (2.2) and (2.4) hold with $\rho(M) > 1$. Then we define $c^* > 0$ by (2.5) and fix $c > c^*$.

We introduce the following continuous family of matrices

$$M(t) = \left(\int_0^\infty k_{i,j}(a) e^{(d_{vj} t^2 - ct)a}\, \mathrm{d}a\right)_{(i,j)}$$

for any $t \in [0, \frac{c}{2d}]$. First note that since the matrix $M(t)$ is positive and satisfies assumption (2.4), it also satisfies (2.4). Next, the map $t \to \rho(M(t))$ is continuous and decreasing with respect to $t \in [0, \frac{c}{2d}]$, satisfies $\rho(M(0)) = \rho(M) > 1$, and

$$\rho\left(M\left(\frac{c}{2d}\right)\right) = \rho\left(\int_0^\infty k_{i,j}(a) e^{\frac{d_{vj} - 2d}{4d^2} c^2 a}\, \mathrm{d}a\right). \tag{3.1}$$

Then since $d_{vj} \leqq d$ for any j and $c > c^*$, we have

$$\rho\left(M\left(\frac{c}{2d}\right)\right) < \rho\left(\int_0^\infty k_{i,j}(a) e^{\frac{d_{vj} - 2d}{4d^2} c^{*2} a}\, \mathrm{d}a\right) = 1. \tag{3.2}$$

Therefore, there exists $\lambda^* \in (0, \frac{c}{2d})$ such that $\rho(M(\lambda^*)) = 1$. Since the matrix $M(\lambda^*)$ is positive and is either irreducible or primitive, Perron–Frobenius Theorem applies and provides the existence of a positive eigenvector $\varphi = (\varphi_1, \ldots, \varphi_n)^T \in \mathbb{R}^n$ of $M(\lambda^*)$ which is associated to its spectral radius 1; this means that

$$M(\lambda^*)\varphi = \varphi, \quad \varphi_i > 0, \quad \forall i = 1, \ldots, n. \tag{3.3}$$

Next we have the following lemma:

Lemma 3.1. *The vector valued map*

$$\widehat{v}(a,x) = e^{-\lambda^* x} \begin{pmatrix} e^{(d_{v1}\lambda^{*2} - c\lambda^*)a}\varphi_1 \\ \vdots \\ e^{(d_{vn}\lambda^{*2} - c\lambda^*)a}\varphi_n \end{pmatrix}$$

satisfies the system of equations

$$\begin{aligned} \frac{\partial v_i}{\partial a} &= d_{vi}\frac{\partial^2 v_i}{\partial x^2} + c\frac{\partial v_i}{\partial x}, \quad \forall i = 1, \dots, n, \\ v_i(0,x) &= \sum_{j=1}^{n} \int_0^\infty k_{i,j}(a) v_j(a,x)\,\mathrm{d}a. \end{aligned} \tag{3.4}$$

We also have the following lemma:

Lemma 3.2. *For each $\gamma^* > 0$ sufficiently small, and $\beta > 1$ large enough, the vector valued map $\widehat{u}$ defined by*

$$\widehat{u}(x) = (1 - \beta e^{-\gamma^* x})E \text{ with } E = (1, \dots, 1)^T \in \mathbb{R}^n \tag{3.5}$$

satisfies the following system of differential inequalities

$$d_{ui}u_i'' + cu_i' - e^{-\lambda^* x}\varphi_i u_i \geqq 0, \quad \forall i = 1, \dots, n, \tag{3.6}$$

on the set $\{x \in \mathbb{R} : \ 1 - \beta e^{-\gamma^* x} \geqq 0\}$. Here λ^* and φ are defined in (3.3).

Finally we have the following lemma:

Lemma 3.3. *Let $\eta > 0$ be small enough with $\eta < \gamma^*$, $\lambda^* + \eta < \frac{c}{d}$ and $\psi \in \mathbb{R}^n$ such that*

$$M(\lambda^* + \eta)\psi = \zeta(\eta)\psi, \quad \zeta(\eta) = \rho(M(\lambda^* + \eta)) \in (0, 1),$$

with $\psi_i > 0 \ \forall i = 1, \dots, n$. Then for $k > 0$ sufficiently large the map $\widetilde{v}$ defined by

$$\widetilde{v}(a,x) = \widehat{v}(a,x) - ke^{-(\lambda^*+\eta)x} \begin{pmatrix} e^{(d_{v1}(\lambda+\eta)^2 - c(\lambda+\eta))a}\psi_1 \\ \vdots \\ e^{(d_{vn}(\lambda+\eta)^2 - c(\lambda+\eta))a}\psi_n \end{pmatrix}$$

satisfies the inequality

$$\begin{aligned} \frac{\partial v_i}{\partial a} &= d_{vi}\frac{\partial^2 v_i}{\partial x^2} + c\frac{\partial v_i}{\partial x}, \quad i = 1, \dots, n, \\ v_i(0,x) &\leqq (1 - \beta e^{-\gamma^* x})^+ \sum_{j=1}^{n} \int_0^\infty k_{i,j}(a) v_j(a,x)\,\mathrm{d}a. \end{aligned} \tag{3.7}$$

Proof. For each $i = 1, \dots, n$ the map

$$(a, x) \to e^{-\lambda^* x} e^{(d_{vi}\lambda^2 - c\lambda)a} \varphi_i - k e^{-(\lambda^*+\eta)x} e^{(d_{vi}(\lambda+\eta)^2 - c(\lambda+\eta))a} \psi_i$$

satisfies the equation

$$\frac{\partial v}{\partial a} = d_{vi} \frac{\partial^2 v}{\partial x^2} + c \frac{\partial v}{\partial x}.$$

For any $i = 1, \dots, n$ rewrite the boundary inequality as

$$\varphi_i - k\psi_i e^{-\eta x} \leqq (1 - \beta e^{-\gamma^* x})^+ \left(\varphi_i - k e^{-\eta x} (M(\lambda^* + \eta)\psi)_i\right). \tag{3.8}$$

First consider the case $x \leqq \frac{1}{\gamma^*} \ln \beta$. Then the inequality becomes

$$\varphi_i - k\psi_i e^{-\eta x} \leqq 0, \quad \forall x \leqq \frac{1}{\gamma^*} \ln \beta.$$

Thus it is sufficient that

$$k \geqq \beta^{\eta/\gamma^*} \frac{\varphi_i}{\psi_i}, \quad \forall i = 1, \dots, n.$$

We now consider the case $x \geqq \frac{1}{\gamma^*} \ln \beta$. Recall that $M(\lambda^* + \eta)\psi = \zeta(\eta)\psi$, (3.8) becomes

$$k \left(1 - \zeta(\eta) + \beta e^{-\gamma^* x}\right) \psi_i \geqq \beta \varphi_i e^{(\eta - \gamma^*)x}, \quad \forall x \geqq \frac{1}{\gamma^*} \ln \beta.$$

Since $\eta < \gamma^*$ and $\zeta(\eta) \in [0, 1)$, the maps

$$x \to \frac{\beta \varphi_i e^{(\eta - \gamma^*)x}}{(1 - \zeta(\eta) + \beta e^{-\gamma^* x})\psi_i}, \quad i = 1, \dots, n,$$

are bounded for $x \geqq \frac{1}{\gamma^*} \ln \beta$. Therefore, it is sufficient to choose $k > 0$ large enough to complete the proof of the lemma. □

From now on we set

$$\overline{u} \equiv e, \quad \underline{u} = \widehat{u}^+, \quad \overline{v} = \widehat{v}, \quad \underline{v} = \widetilde{v}^+. \tag{3.9}$$

Consider a similar problem on a bounded interval $(-z, z)$ with $z > 0$ given:

$$\begin{cases} d_{ui} u_i'' + c u_i' - u_i(x) \displaystyle\sum_{j=1}^n \int_0^\infty k_{i,j}(a) v_j(a, x)\, \mathrm{d}a = 0, \quad i = 1, \dots, n, \\ \dfrac{\partial v_i}{\partial a} = d_{vi} \dfrac{\partial^2 v_i}{\partial x^2} + c \dfrac{\partial v_i}{\partial x}, \\ v_i(0, x) = u_i(x) \displaystyle\sum_{j=1}^n \int_0^\infty k_{i,j}(a) v_j(a, x)\, \mathrm{d}a, \\ u(\pm z) = \underline{u}(\pm z), \quad v(a, \pm z) = \underline{v}(a, \pm z). \end{cases} \tag{3.10}$$

Then we have the following proposition:

Proposition 3.4. *Let $z_0 > 0$ be given such that $\underline{u}(-z) = 0$ for any $z > z_0$. Then for any $z > z_0$ system (3.10) has a solution $(\mathbf{u}, \mathbf{v})$ such that*

$$\underline{u} \leqq \mathbf{u} \leqq \overline{u}, \quad \underline{v} \leqq \mathbf{v} \leqq \overline{v}. \tag{3.11}$$

Moreover, the function $\mathbf{u}$ is increasing with respect to x.

Proof. The proof relies on the application of Schauder fixed point theorem. We start by investigating the existence of the solution. We first re-formulate problem (3.10) as a fixed point problem. For that purpose let us introduce the following parabolic initial data problem

$$\begin{aligned} &\frac{\partial v}{\partial a} = \mathbf{d_v} \frac{\partial^2 v}{\partial x^2} + c \frac{\partial v}{\partial x}, \\ &v(a, \pm z) = \underline{v}(a, \pm z), \quad v(0, x) = v_0(x). \end{aligned} \tag{3.12}$$

Here we have set $v = (v_1, \dots, v_n)^T$ and $\mathbf{d_v} = \operatorname{diag}(d_{v1}, \dots, d_{vn})$. Let v denote the solution of the above linear problem. Next we consider $u = (u_1, \dots, u_n)^T$ a solution of the linear elliptic problem:

$$d_{ui} u_i'' + c u_i' = u_i \Lambda_i(x), \quad u_i(\pm X) = \underline{u_i}(\pm z), \tag{3.13}$$

where we have set

$$\Lambda_i(x) = \sum_{j=1}^{n} \int_0^\infty k_{i,j}(a) v_j(a, x)\, \mathrm{d}a. \tag{3.14}$$

Finally problem (3.10) is equivalent to the following one:

$$v_{0,i}(x) = u_i(x) \Lambda_i(x), \quad i = 1, \dots, n.$$

In order to solve this problem, we introduce the following closed and convex subset $E \subset C([-z, z], \mathbb{R}^n)$ of the n-dimensional continuous functions on the compact set $[-z, z]$:

$$E = \{v_0 \in C([-X, X], \mathbb{R}^n), \quad \underline{v}(0, x) \leqq v_0(x) \leqq \bar{v}(0, x)\}.$$

Now we consider this fixed point problem. Consider the operator

$$\Phi : v_0 \in E \to (u_i \Lambda_i)_{i=1,\dots,n},$$

where Λ_i is given in (3.14), v is given by the resolution of

$$\begin{aligned} &\frac{\partial v}{\partial a} = \mathbf{d_v} \frac{\partial^2 v}{\partial x^2} + c \frac{\partial v}{\partial x}, \\ &v(a, \pm z) = \underline{v}(a, \pm z), \quad v(0, x) = v_0(x), \end{aligned} \tag{3.15}$$

and u is the solution of (3.13). Let us first show that the operator Φ is a compact operator from E into $C([-X, X], \mathbb{R}^n)$. We first note that the function v can be re-written as

$$v(a) = T_{\mathbf{d_v} \Delta + c \partial_x}(a)\, (v_0) + \widehat{v_0}(a),$$

where $\widehat{v_0}(a, .)$ satisfies

$$\frac{\partial v}{\partial a} = \mathbf{d_v} v'' + cv', \quad v(a, \pm z) = \underline{v}(a, \pm z), \quad v(0, x) = 0$$

and the semigroup $\left\{\mathbf{d_v} T_{\Delta + c\partial_x}(t)\right\}_{t \geqq 0}$ is generated by $\mathbf{d_v}\Delta + c\partial_x$ with Dirichlet boundary conditions. It follows that for each $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} \int_0^\infty k_{i,j}(a) v_i(a, .)\, \mathrm{d}a = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \langle \mathbf{k_i}(a), T_{\mathbf{d_v}\Delta + c\partial_x}(a)\,(v_0)\rangle\, \mathrm{d}a + \sum_{j=1}^{n} \int_0^\infty \langle \mathbf{k_i}(a), \widehat{v_0}(a, .)\rangle\, \mathrm{d}a.$$

Here we have set $\mathbf{k_i} = (k_{i,1}, \ldots, k_{i,n})^T$ and $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^n$. Due to assumption (2.2), we have $\mathbf{k_i} \in L^\infty((0, \infty), \mathbb{R}^n)$ and the limit converges uniformly with respect to v_0 in bounded sets. But the linear operator

$$\int_\varepsilon^\infty \langle \mathbf{k_i}(a), T_{\mathbf{d_v}\Delta + c\partial_x}(a)\,(v_0)\rangle\, \mathrm{d}a$$

is compact since $T_{\mathbf{d_v}\Delta + c\partial_x}(\varepsilon)$ is a compact operator. It follows that for each $i = 1, \ldots, n$ the map

$$v_0 \to \Lambda_i$$

is a compact operator from $C([-z, z], \mathbb{R}^n)$ into $C([-z, z], \mathbb{R})$.

Moreover, from standard elliptic estimates, we also obtain that the operator $v_0 \in E \to u_i$ is bounded from $C([-z, z], \mathbb{R}^n)$ into $C^1([-z, z], \mathbb{R})$ for each $i = 1, \ldots, n$. Finally, we obtain that Φ is completely continuous from E into $C([-z, z], \mathbb{R}^n)$.

Next we prove that $\Phi(E) \subset E$. This result follows from successive applications of the comparison principle. Indeed let $v_0 \in E$ be given. Since $v_0 \geqq 0$ and $v(a, \pm z) \geqq 0$, we obtain from the comparison principle that $v(a, x) \geqq 0$ for any (a, x). Therefore, $\Lambda_i(x) \geqq 0$ for each $i = 1, \ldots, n$ and the maximum principle applies to Equation (3.13). Since $0 \leqq u_i(\pm z) \leqq 1$ for each $i = 1, \ldots, n$, we obtain that $0 \leqq u_i(x) \leqq 1$ for any $x \in [-z, z]$ and any $i = 1, \ldots, n$.

Since the function $\overline{v}$ satisfies Equation (3.4), $v_0(x) = v(0, x) \leqq \overline{v}(0, x)$ and $v(a, \pm z) \leqq \overline{v}(a, \pm z)$, the comparison principle implies that

$$v(a, x) \leqq \bar{v}(a, x) \quad \text{for any } (a, x) \in [0, \infty) \times [-z, z]. \tag{3.16}$$

It follows that

$$\Lambda_i(x) \leqq e^{-\lambda^* x} \varphi_i, \quad \forall i = 1, \ldots, n,$$

and functions u_i with $i = 1, \ldots, n$ satisfy the inequalities

$$d_{ui} u_i'' + cu_i' - \varphi_i e^{-\lambda^* x} u_i \leqq 0, \quad u_i(\pm X) = \underline{u_i}(\pm z).$$

Recalling that the function $\widehat{u}$ satisfies (3.6), we obtain from the maximum principle that

$$u = (u_1, \dots, u_n)^T \geqq \widehat{u}. \tag{3.17}$$

Since $u_i \geqq 0$ for each $i = 1, \dots, n$, we have that

$$u = (u_1, \dots, u_n)^T \geqq \underline{u} \;\; in(-z, z). \tag{3.18}$$

Finally, since the function $\widetilde{v}$ satisfies (3.7), $v_0(x) = v(0, x) \geqq \underline{v}(0, x)$ and $v(a, \pm z) \geqq \underline{v}(a, \pm z)$, once again the parabolic comparison principle implies that

$$v(a, x) \geqq \underline{v}(a, x) \quad \text{for any } (a, x) \in [0, \infty) \times [-z, z]. \tag{3.19}$$

Now we can easily conclude that the operator Φ maps E into E. Indeed, from (3.19) and $u_i \leqq 1$ for each $i = 1, \dots, n$, we have that

$$\Phi(v_0)(x) = (u_i \Lambda_i)_{i=1,\dots,n} \leqq (\Lambda_i)_{i=1,\dots,n} \leqq e^{-\lambda^*}\varphi.$$

Next, from inequality (3.19) and $u = (u_1, \dots, u_n)^T \geqq \underline{u}$ we obtain for each $i = 1, \dots, n$ that

$$\Phi(v_0)_i \geqq \underline{u}_i \sum_{j=1}^{n} \int_0^{\infty} k_{i,j}(a) \underline{v}_j(a, .)\, \mathrm{d}a \geqq \underline{v}_j(0, .).$$

This concludes the proof of $\Phi(E) \subset E$. Now from the Schauder theorem, we obtain the existence of a fixed point for the map Φ, that is, $v_0 \in E$ satisfying $v_0 = \Phi(v_0)$. Regularity of the solution follows from the bootstrapping argument.

It remains to be proven that the function u_i is increasing. From $\Lambda_i \geqq 0$ and $u_i \geqq 0$, we know that

$$d_{ui} u_i'' + c u_i' = u_i \Lambda_i(x) \geqq 0.$$

Therefore, the function u_i satisfies $(u_i'(x) e^{\frac{c}{d_{ui}} x})' \geqq 0$. Integrating this inequality from $-z$ to x yields

$$u_i'(x) e^{\frac{c}{d_{ui}} x} \geqq u_i'(-z) e^{-\frac{c}{d_{ui}} z}.$$

Recalling that $z > z_0$, we have $u_i(-z) = 0$. Since $u_i \geqq 0$, we obtain that $u_i'(-z) \geqq 0$ and $u_i'(x) \geqq 0$ for any $x \in [-z, z]$. The proof is completed. □

Finally, we consider a sequence $(z_q)_{q \geqq 0}$ of positive numbers such $z_q \to +\infty$ as $q \to +\infty$. We consider a solution $(\mathbf{u}_q, \mathbf{v}_q)$ of (3.10) on $I_q = (-z_q, z_q)$. We have the following estimates.

Proposition 3.5. *There exist $q_0 \geqq 0$ and $\widehat{M} > 0$ such that for each $q \geqq q_0$ we have*

$$\|\mathbf{v}_q\|_\infty \leqq \widehat{M}. \tag{3.20}$$

The proof of this crucial estimate is similar to the one in Ducrot and Magal [8]. For the sake of completeness we give the proof of this proposition. It is a consequence of the following lemma.

Lemma 3.6. *Let $x_0 > 0$ be given. Suppose $p \in C^2([-x_0, x_0])$ and $\underline{p} \in C([-x_0, x_0])$ are differentiable at $x = x_0$, $w \in C^{1,2}([0, \infty) \times [-x_0, x_0])$, and $\underline{w} \in C([0, \infty) \times [-x_0, x_0])$ satisfy*

$$\begin{cases} \text{for any } a \in [0, \infty) \text{ the map } x \to \underline{w}(a, x) \text{ is differentiable at } x = x_0, \\ p \geqq \underline{p} \geqq 0, \quad w \geqq \underline{w} \geqq 0, \\ \partial_a w = d_2 \partial_x^2 w + c \partial_x w, \\ w(0, x) = d_1 p'' + c p', \quad w(a, -x_0) = 0, \\ w(a, x_0) = \underline{w}(a, x_0), \quad p(-x_0) = 0, \quad p(x_0) = \underline{p}(x_0). \end{cases} \tag{3.21}$$

Then we have for each $a \in [0, \infty)$ that

$$\int_{-x_0}^{x_0} w(a, x)\, \mathrm{d}x \leqq d_2 \int_0^a \partial_x \underline{w}(s, x_0)\, \mathrm{d}s + c \int_0^a \underline{w}(s, x_0)\, \mathrm{d}s + d_1 \underline{p}'(x_0) + c \underline{p}(x_0). \tag{3.22}$$

Moreover, if p is increasing then we have for any $(a, x) \in [0, \infty) \times (-x_0, x_0)$ that

$$\int_0^a w(s, x)\, \mathrm{d}s \leqq \frac{d_2}{c} \int_0^a \partial_x \underline{w}(s, x_0)\, \mathrm{d}s + \int_0^a \underline{w}(s, x_0)\, \mathrm{d}s + \frac{d_1}{c} \underline{p}'(x_0) + \underline{p}(x_0). \tag{3.23}$$

Proof. Let $a \in [0, \infty)$ be given. Then the map $W_a : [-x_0, x_0] \to \mathbb{R}$ defined by

$$W_a(x) = \int_0^a w(s, x)\, \mathrm{d}s,$$

satisfies the equation

$$w(a, x) - (d_1 p'' + c p')(x) = d_2 W_a''(x) + c W_a'(x). \tag{3.24}$$

Integrating this equality over $I = (-x_0, x_0)$ gives

$$\begin{aligned} &\int_I w(a, x)\, \mathrm{d}x - d_1 p'(x_0) + d_1 p'(-x_0) - c(p(x_0) - p(-x_0)) \\ &\quad = d_2 W_a'(x_0) - d_2 W_a'(-x_0) + c(W_a(x_0) - W_a(-x_0)). \end{aligned}$$

Therefore, using the boundary conditions for w and p, we obtain that

$$\begin{aligned} \int_I w(a, x)\, \mathrm{d}x + d_1 p'(-x_0) + d_2 W_a'(-x_0) &\leqq d_2 W_a'(x_0) \\ &\quad + c W_a(x_0) + d_1 p'(x_0) + c p(x_0). \end{aligned}$$

We also have

$$p(x_0) = \underline{p}(x_0) \text{ and } p \geqq \underline{p},$$

so that $p'(x_0) \leqq \underline{p}'(x_0)$. Similarly, we have

$$W_a'(x_0) \leqq \int_0^a \partial_x \underline{w}(s, x_0)\, \mathrm{d}s.$$

Therefore, we obtain

$$\begin{aligned} &\int_I w(a,x)\,\mathrm{d}x + d_1 p'(-x_0) + d_2 W_a'(-x_0) \\ &\quad \leqq d_2 \int_0^a \partial_x \underline{w}(s,x_0)\,\mathrm{d}s + c\int_0^a \underline{w}(s,x_0)\,\mathrm{d}s + d_1 \underline{p}'(x_0) + c\underline{p}(x_0). \end{aligned} \tag{3.25}$$

Finally, since $p(-x_0) = 0$ and $p \geqq 0$ we have $p'(-x_0) \geqq 0$. Similarly, we have $W_a'(-x_0) \geqq 0$. This concludes the proof of (3.22).

Next integrating (3.24) over $(-x_0, x)$, we obtain

$$\begin{aligned} &\int_{-x_0}^x w(a,y)\,\mathrm{d}y - d_1 p'(x) + d_1 p'(-x_0) - cp(x) \\ &\quad = d_2 W_a'(x) - d_2 W_a'(-x_0) + cW_a(x). \end{aligned}$$

Since p is positive and increasing, we obtain

$$\begin{aligned} d_2 W_a'(x) + cW_a(x) &= \int_{-x_0}^x w(a,y)\,\mathrm{d}y \\ &\quad - d_1 p'(x) + d_1 p'(-x_0) - cp(x) + d_2 W_a'(-x_0) \\ &\leqq \int_{-x_0}^{x_0} w(a,y)\,\mathrm{d}y + d_1 p'(-x_0) + d_2 W_a'(-x_0). \end{aligned} \tag{3.26}$$

From (3.25), we conclude that

$$\begin{aligned} d_2 W_a'(x) + cW_a(x) \leqq d_2 &\int_0^a \partial_x \underline{w}(s,x_0)\,\mathrm{d}s + c\int_0^a \underline{w}(s,x_0)\,\mathrm{d}s \\ &+ d_1 \underline{p}'(x_0) + c\underline{p}(x_0). \end{aligned}$$

We denote by $\widehat{C}(a)$ the right-hand side of the above inequality. Since $W_a(-x_0) = 0$, we get that

$$W_a(x) \leqq \frac{\widehat{C}(a)}{d_2} \int_{-x_0}^x e^{\frac{c}{d_2}(s-x)}\,\mathrm{d}s \leqq \frac{\widehat{C}(a)}{c}.$$

This completes the proof of the lemma. $\square$

Let us first obtain some estimates for the solution $(\mathbf{u}_q, \mathbf{v}_q)$. From Lemma 3.6, we obtain for each $i = 1, \ldots, n$ that

$$\begin{aligned} \int_0^\infty v_{i,q}(s,x)\,\mathrm{d}s \leqq \frac{d_{vi}}{c} &\int_0^\infty \partial_x \underline{v}_i(s,z_q)\,\mathrm{d}s + \int_0^\infty \underline{v}_i(s,z_q)\,\mathrm{d}s \\ &+ \frac{d_{ui}}{c} \underline{u}_i'(z_q) + \underline{u}_i(z_q). \end{aligned} \tag{3.27}$$

Thus, for each $i = 1, \ldots, n$,

$$v_{i,q}(0,x) \leqq \sum_{j=1}^{n} \|k_{i,j}\|_\infty \times \left(\frac{d_{vi}}{c} \int_0^\infty \partial_x \underline{v}_i(s, z_q)\, \mathrm{d}s + \int_0^\infty \underline{v}_i(s, z_q)\, \mathrm{d}s + \frac{d_{ui}}{c} \underline{u}_i'(z_q) + \underline{u}_i(z_q) \right).$$

Therefore, the parabolic comparison principle implies that there exists $\widehat{M} > 0$ such that for any q, we have $v_{i,q}(0,x) \leqq \widehat{M}$ and

$$v_{i,q}(a,x) \leqq \widehat{M}, \quad \forall (a,x) \in [0,\infty) \times (-z_q, z_q), \quad \forall i = 1, \ldots, n. \tag{3.28}$$

Now from Schauder estimates, we derive the following lemma.

Lemma 3.7. *Let $\alpha \in (0,1)$ be given. For each given $A > 0$ and $X > 0$ there exists some constant $M_{A,X} > 0$ such that for any q with $z_q > X$ we have*

$$\|\mathbf{v}_q\|_{C^{1+\alpha/2, 2+\alpha}([0,A]\times[-X,X])} + \|\mathbf{u}_q\|_{C^{2+\alpha}([-X,X])} \leqq M_{A,X}.$$

Remark 3.8. Here Schauder estimates can be used up to $a = 0$ by using a bootstrapping argument. Indeed the boundary condition at age $a = 0$ regularizes the initial data. For more details we refer to [7,8].

Therefore, up to a subsequence, we can suppose that there exists two maps

$$\mathbf{v} \in C^{1,2}([0,\infty) \times \mathbb{R}, \mathbb{R}^n), \quad \mathbf{u} \in C^2(\mathbb{R}, \mathbb{R}^n),$$

such that $(\mathbf{v}_q, \mathbf{u}_q) \to (\mathbf{v}, \mathbf{u})$ in the topology of $C^{1,2}_{loc}([0,\infty)\times\mathbb{R}, \mathbb{R}^n) \times C^2_{loc}(\mathbb{R}, \mathbb{R}^n)$. We can show that $(\mathbf{u}, \mathbf{v})$ satisfies system (1.3)–(1.4). It remains to check the limit behavior (1.4).

For that purpose let us notice that, for any q, we have

$$\underline{u} \leqq \mathbf{u}_q \leqq \overline{u}, \quad \underline{v} \leqq \mathbf{v}_q \leqq \overline{v}.$$

Then we deduce that these inequalities remain true when $q \to +\infty$, that is

$$\underline{u} \leqq \mathbf{u} \leqq \overline{u}, \quad \underline{v} \leqq \mathbf{v} \leqq \overline{v}. \tag{3.29}$$

This allows us to conclude that

$$\lim_{x\to\infty} u_i(x) = 1, \quad \lim_{x\to\infty} v_i(a,x) = 0 \text{ uniformly for } a \in [0,\infty).$$

Moreover, from (3.27), we have for any $(a,x) \in [0,\infty) \times \mathbb{R}$ that

$$\int_0^a v_i(s,x)\, \mathrm{d}s \leqq 1, \quad \forall i = 1, \ldots, n. \tag{3.30}$$

In addition, from Lemma 3.6, the sequence $(\mathbf{v}_q)$ satisfies

$$\int_{-z_q}^{z_q} v_{i,q}(a,x)\, \mathrm{d}x \leqq d_{vi} \int_0^a \partial_x \underline{v}_i(s, z_q)\, \mathrm{d}s + c \int_0^a \underline{v}_i(s, z_q)\, \mathrm{d}s + d_{ui} \underline{u}_i'(z_q) + c \underline{u}_i(z_q). \tag{3.31}$$

Letting $q \to +\infty$ we obtain that for any $a \in [0, \infty)$

$$\int_{\mathbb{R}} v_i(a, x)\,\mathrm{d}x \leqq c, \quad \forall i = 1, \ldots, n. \tag{3.32}$$

Therefore, we have

$$\mathbf{v}(a, x) \to 0 \ \text{ when } x \to -\infty \text{ in } C_{loc}([0, \infty)).$$

Finally, since the function $\mathbf{u}_q$ is increasing, it follows that $\mathbf{u}$ is nondecreasing and has a limit $\mathbf{u}^\infty$ at $x = -\infty$.

To complete the proof of Theorem 2.1 it remains to prove the estimate (2.6). For this purpose let us notice that from the maximum principle, we have

$$u_i(x) \geqq u_i^\infty \ \ \forall i = 1, \ldots, n, \quad \forall x \in \mathbb{R}.$$

Therefore, for any $x \in \mathbb{R}$, we obtain that

$$v_i(0, x) \geqq u_i^\infty \sum_{j=1}^{n} \int_0^\infty k_{i,j}(a) v_j(a, x)\,\mathrm{d}a.$$

By noticing that $\int_{\mathbb{R}} v_i(a, x)\,\mathrm{d}x = \int_{\mathbb{R}} v_i(0, x)\,\mathrm{d}x$ for any $a \in [0, \infty)$ and any $i = 1, \ldots, n$, we obtain that

$$\int_{\mathbb{R}} v_i(0, x)\,\mathrm{d}x \geqq u_i^\infty \sum_{j=1}^{n} \int_0^\infty k_{i,j}(a)\,\mathrm{d}a \int_{\mathbb{R}} v_j(0, x)\,\mathrm{d}x.$$

Set $I = \left(\int_{\mathbb{R}} v_1(0, x)\,\mathrm{d}x, \ldots, \int_{\mathbb{R}} v_n(0, x)\,\mathrm{d}x\right)^T \geqq 0$. We obtain that

$$I \geqq \begin{pmatrix} u_1^\infty & 0 & \cdots & 0 \\ 0 & \ddots & & \\ 0 & \cdots & 0 & u_n^\infty \end{pmatrix} MI,$$

which implies the estimate (2.6) and completes the proof of Theorem 2.1.

Finally, we have the following nonexistence result.

Proposition 3.9. *If* $\rho(M) < 1$ *and* $(\mathbf{u}, \mathbf{v})$ *is a solution of* (1.3)–(1.4) *with* $\mathbf{v}(0, .) \in L^1(\mathbb{R}, \mathbb{R}^n)$, *then*

$$\mathbf{u} \equiv e, \quad \mathbf{v} \equiv 0.$$

Proof. We argue by contradiction. Assume that $\mathbf{v}$ is not identically zero. Since the PDE for $\mathbf{v}$ preserves the total mass, we have

$$\int_{\mathbb{R}} \mathbf{v}(a, x)\,\mathrm{d}x = \int_{\mathbb{R}} \mathbf{v}(0, x)\,\mathrm{d}x, \quad \forall a > 0.$$

Next since $u_i \in [0, 1]$ for any $i = 1, \dots, n$, we obtain by integrating the boundary condition over $\mathbb{R}$ that

$$\int_{\mathbb{R}} v_i(0, x)\,\mathrm{d}x \leqq \sum_{j=1}^{n} \int_0^{\infty} k_{i,j}(a)\,\mathrm{d}a \int_{\mathbb{R}} v_j(0, x)\,\mathrm{d}x.$$

Set $I = \int_{\mathbb{R}} \mathbf{v}(0, x)\,\mathrm{d}x \in \mathbb{R}^n$. We obtain that

$$I \geqq 0, \quad I \neq 0, \quad I \leqq MI.$$

This is in contradiction with $\rho(M) < 1$. $\square$

4. Numerical simulations

In this section we carry out some numerical simulations of the problem (1.1) with $u_i(t, x) = \theta_i(t, x), \quad v_i(t, a, x) = \psi_i(t, a, x)e^{\int_0^a \mu_i(s)\,\mathrm{d}s}$. We only consider the case $n = 2$ which applies to FIV propagation and some sexual transmission diseases [9].

Consider an age-structured epidemic model with two population groups:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_{ui}\dfrac{\partial^2 u_i}{\partial x^2} - u_i(t, x)\displaystyle\sum_{j=1}^{2}\int_0^{a_\dagger} k_{i,j}(a)v_j(t, a, x)\,\mathrm{d}a, \quad x \in (0, L), \quad t > 0 \\ \dfrac{\partial v_i}{\partial t} + \dfrac{\partial v_i}{\partial a} = d_{vi}\dfrac{\partial^2 v_i}{\partial x^2}, \quad x \in (0, L), \quad a > 0, \quad t > 0 \quad \text{for} \quad i = 1, 2 \\ v_i(t, 0, x) = u_i(t, x)\displaystyle\sum_{j=1}^{2}\int_0^{a_\dagger} k_{i,j}(a)v_j(t, a, x)\,\mathrm{d}a, \quad t > 0, \quad x \in (0, L). \end{cases} \tag{4.1}$$

This system is supplemented with the boundary conditions

$$\begin{aligned} &\frac{\partial u_i(t, x)}{\partial x} = 0, \quad t > 0, \ x \in \{0, L\}, \\ &\frac{\partial v_i(t, a, x)}{\partial x} = 0, \quad t > 0, \quad a > 0, \ x \in \{0, L\} \end{aligned} \tag{4.2}$$

and the initial conditions

$$u_i(0, x) = u_i^0, \quad v_i(0, a, x) = v_i^0(a, x), \quad a > 0, \quad x \in (0, L).$$

This problem is numerically solved by using semi-implicit finite differences for the parabolic equations while the hyperbolic equations are solved using the characteristics.

In order to provide numerical simulations for this model, we use a finite maximal infection age $a_\dagger < +\infty$. Moreover, we use a latency period model for functions $k_{i,j}$, that is

$$k_{i,j}(a) = \frac{\beta_{i,j}}{\mu_i}\left(\exp(-\mu_i \tau_{i,j}) - \exp(-\mu_i a_\dagger)\right) 1_{[\tau_{i,j}, a_\dagger]}(a).$$

Table 1. Parameter values for the numerical simulations

β	$i=1$	$i=2$	τ	$i=1$	$i=2$
$j=1$	5	0.5	$j=1$	0.2	0.5
$j=2$	0.5	0.5	$j=2$	0.5	0.2

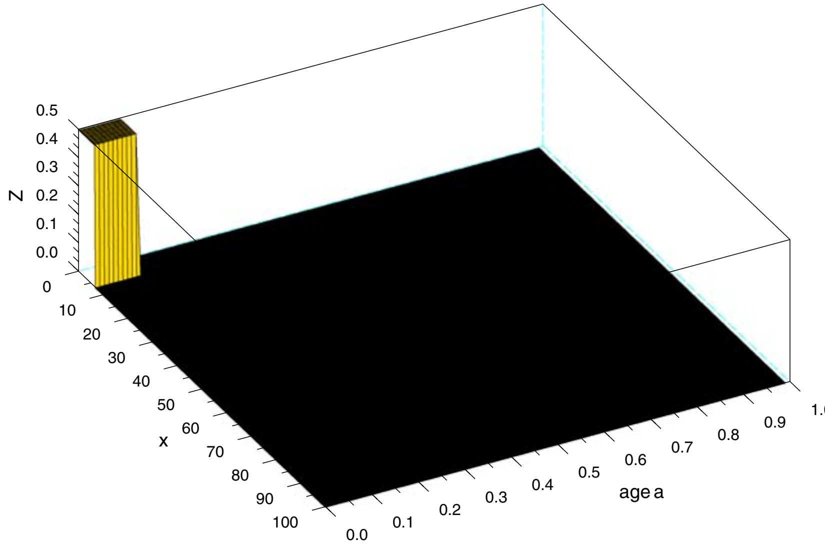

Fig. 1. Initial distribution of infected individuals v_1

Here $\tau_{i,j}$ corresponds to the latency period of infection from the jth population to the ith group. Parameters are chosen such that the matrix M reads

$$a_\dagger = 1, \quad \mu_1 = \mu_2 = 1, \quad L = 100, \tag{4.3}$$

while the other parameters are given in Table 1.

Using the parameter set given in (4.3) and Table 1, we have the following incidence matrix

$$M = \begin{pmatrix} R_{11}^0 & R_{12}^0 \\ R_{21}^0 & R_{22}^0 \end{pmatrix} = \begin{pmatrix} 2.254 & 0.119 \\ 0.119 & 0.225 \end{pmatrix}.$$

We obtain $R_0 = \rho(M) \approx 2.26 > 1$ and by the results of the previous sections we obtain the existence of a wave solution. To do the numerical simulations, we suppose that the susceptible populations are initially homogeneous in space and constant to one in the domain $(0, L)$. Then we introduce a small perturbation of v_1 shown in Fig. 1 while the density v_2 is initially taken as the zero function. The form of the solution of the evolution problem (4.1)–(4.2) is shown in Figs. 2 and 3. We observe the propagation of a wave solution. The density of infected individuals exhibits a pulse-like structure, while the density of susceptible individuals is a front solution.

Note that in the absence of the crisscross infection, the disease is not able to propagate within the second population. Indeed its basic reproduction number is

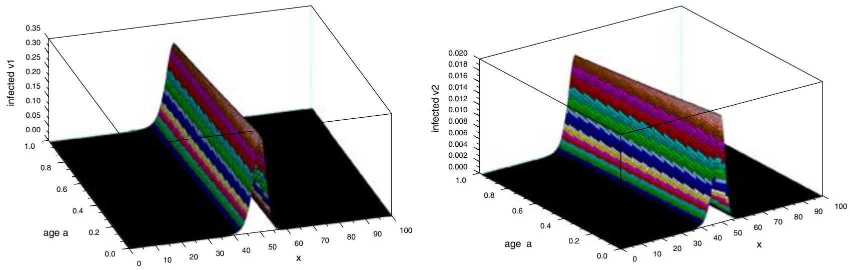

Fig. 2. Snapshot of the evolution of infected individuals v_1 (*left*) and v_2 (*right*)

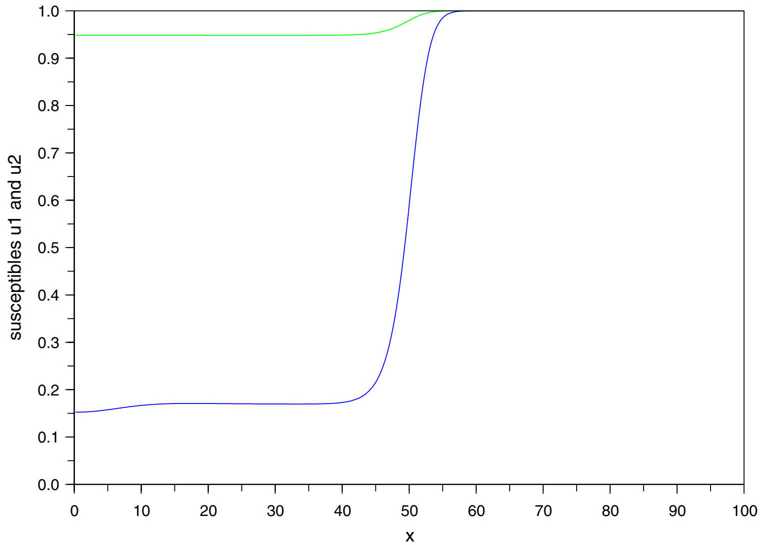

Fig. 3. Snapshot of the evolution of susceptible individuals u_1 (*bottom*) and u_2 (*upper curve*)

$R^0_{22} = 0.676 < 1$. Therefore, crisscross infection can sustain the infection, whereas the intra-specific epidemiological threshold is less than unity.

We now give some remarks about the minimal wave speed. From a theoretical point of view, we expect that the linear approximation provides a good approximation of the wave speed. We introduce the map L defined on the domain $\Omega \subset \mathbb{R}^2$:

$$\Omega = \left\{ (c, \lambda) \in \mathbb{R}^2 : \quad c \geqq 0, \quad 0 \leqq \lambda \leqq \frac{c}{\max\{d_{vj} : \ j = 1, \dots, n\}} \right\},$$
$$L(\lambda, c) = \rho \left(\int_0^\infty k_{i,j}(a) e^{(d_{vj}\lambda^2 - c\lambda)a} \, \mathrm{d}a \right)_{(i,j)=1,\dots,n}, \quad (\lambda, c) \in \Omega.$$

The classical linear approximation implies that the minimal wave speed is given by the resolution of the system:

$$L(\lambda^*, c^*) = 1, \quad \frac{\partial L}{\partial \lambda}(\lambda^*, c^*) = 0 \ \text{ with } \ (\lambda^*, c^*) \in \Omega.$$

Table 2. Comparison of the numerical (c) and theoretical (c_{th}) wave speeds

	$d_{v1} = 0.5$	$d_{v1} = 0.8$	$d_{v1} = 1$	$d_{v1} = 1.5$
$d_{v2} = 1$	$c = 1.72$	$c = 2.47$	$c = 2.22$	$c = 3.04$
	$c_{th} = 1.78$	$c_{th} = 2.53$	$c_{th} = 2.26$	$c_{th} = 3.10$

We now compare this heuristic expression of the minimal wave speed together with that of the stable numerical waves. For that purpose we use the parameter set (4.3) together with Table 1 and vary the diffusion coefficient d_{v1}, while $d_{v2} = 1$. The results are shown in Table 2.

Here we observe a good accuracy between the numerical wave speed and that given by the linear approximation. There is numerical evidence of the linear approximation conjecture, but it remains as an open theoretical problem.

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UMR CNRS 5466 MAB and INRIA Futurs Anubis,
Université of Bordeaux 2,
146 rue Léo Saignat,
33076 Bordeaux, France

and

Faculte des Sciences et Techniques,
Université du Havre,
B. P. 540, 76058 Le Havre, France

and

Department of Mathematics,
University of Miami,
Coral Gables,
FL 33124-4250, USA.
e-mail: ruan@math.miami.edu

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