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CENTER-UNSTABLE MANIFOLDS FOR NONDENSELY DEFINED CAUCHY PROBLEMS AND APPLICATIONS TO STABILITY OF HOPF BIFURCATION

Dedicated to Professor Herbert I. Freedman on the occasion of his 70th birthday.

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ABSTRACT. Center-unstable manifolds are very useful in investigating nonlinear dynamics of nonlinear evolution equations. In this paper, we first present a center-unstable manifold theory for abstract semilinear Cauchy problems with nondense domain. We especially focus on the stability property of the center-unstable manifold. Then we study the stability of Hopf bifurcation, that is, stability of the bifurcating periodic orbits for the nondensely defined Cauchy problem. Our goal is to prove that the stability of a periodic orbit to the reduced system (i.e., restricted to the center-unstable manifold) implies the stability of the periodic orbit for the original system. As an application, we demonstrate that these results apply to differential equations with infinite delay.

1 Introduction The center manifold theory was first established by Pliss [**39**] and Kelley [**30**] and was developed and completed in Carr [**7**], Sijbrand [**40**], Vanderbauwhede [**46, 47**], etc. Due to its local invariance by the semiflow, the center manifold provides a considerable reduction of the dimension which leads to simple calculations and a better geometric insight on the dynamics. The center manifold theory has

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significant applications in studying other problems in dynamical systems, such as bifurcation, stability, perturbation, etc. See, for example, Bates and Jones [**6**], Chicone and Latushkin [**8**], Chow and Hale [**9**], Chow *et al.* [**11, 12**], Chow and Lu [**13**], Diekmann *et al.* [**19**], Faria *et al.* [**23**], Henry [**27**], Hirsch *et al.* [**28**], Krisztin [**31**], Vanderbauwhede and Iooss [**49**], Vanderbauwhede and van Gils [**48**], and the references cited therein. It has also been used to study various applied problems in biology, engineering, physics, etc. and we refer to Carr [**7**] and Hassard *et al.* [**26**].

Given a nonhyperbolic equilibrium, the center-unstable manifold is a locally invariant manifold by the semiflow and is tangent to the generalized eigenspace associated to the corresponding eigenvalues with nonnegative real parts (Kelley [**30**]). The local center-unstable manifold plays an important role in applications since it has some nice stability properties. Compared to center manifold, it is also easier to use in practice, since a point (locally around the equilibrium) belongs to the center-unstable manifold "only" if there exists a negative orbit (staying in some small neighborhood of the equilibrium) passing through the point at time $t = 0$, while for the center manifold a complete orbit is needed. Center-unstable manifolds in infinite dynamical systems have been studied by many researchers. For example, Armbruster *et al.* [**4**] investigated center-unstable manifolds in Kuramoto-Sivashinsky equation. Chow and Lu [**14**] discussed the existence and smoothness of global center-unstable manifolds for semilinear and fully nonlinear differential evolution equations. Dell'Antonio and D'Onofrio [**17**] studied center-unstable manifolds for the Navier-Stokes equation. Nakanishi and Schlag [**38**] established center-unstable and center-stable manifolds around soliton manifolds for the nonlinear Klein-Gordon equation. Turyn [**45**] obtained a center-unstable manifold theorem for parametrically excited surface waves. Stumpf [**41**] discussed center-unstable manifolds for differential equations with state-dependent delay.

The goal of this paper is to establish a center-unstable manifold theory for the abstract semilinear Cauchy problem with nondense domain:

$$\frac{du(t)}{dt} = Au(t) + F(u(t)),\ t \geq 0,\ u(0) = x \in \overline{D(A)}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space $(X, \|.\|)$, and $F : \overline{D(A)} \to X$ is a k-time ($k \geq 1$) continuous differentiable map locally around some equilibrium $\overline{u} \in D(A)$, that is, $A\overline{u} + F(\overline{u}) = 0$. Assume that

$$X_0 := \overline{D(A)} \neq X,$$

i.e., the domain of the operator A is not dense in the phase space X, so problem (1.1) is called a nondensely defined Cauchy problem.

Various examples of equations with nondense domain were given in Da Prato and Sinestrari [**16**] and Thieme [**42, 43**]. In fact, many biological and epidemiological models, such as age-structured models in population dynamics (Magal [**34**], Magal and Ruan [**35, 36, 37**]), population dynamic models described by reaction-diffusion equations with nonlinear boundary conditions (Chu *et al.* [**15**] and Ducrot *et al.* [**21**]), and models described by delay differential equations in L^p (Liu *et al.* [**32**] and Ducrot *et al.* [**20**]), can be written as nondensely defined semilinear Cauchy problems in form of (1.1).

(a) Age-structured models in population dynamics. Let $u(t,a)$ denote the population density of some species at time t and with age a. Consider the initial-boundary value problem (Magal and Ruan [**37**])

$$\begin{cases} \partial_t v(t,a) + \partial_a v(t,a) = -\mu v(t,a), \ \ t>0, \ a>0, \\ v(t,0) = f\left(\int_0^\infty \beta(a) v(t,a)da\right), \\ v(0,.) = v_0 \in L^1_+(0,\infty) \end{cases} \tag{1.2}$$

in the Banach space $X = \mathbb{R} \times L^1(0,\infty)$ and assume that the linear operator $A : D(A) \subset X \to X$ is defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}, \quad \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(A)$$

with $D(A) = \{0\} \times W^{1,1}(0,\infty)$. By setting $u(t) = \begin{pmatrix} 0 \\ v(t,.) \end{pmatrix}$, problem (1.2) can be reformulated as the Cauchy problem (1.1), where $x = \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \in \{0\} \times L^1(0,\infty)$ and the nonlinear map $F : \{0\} \times L^1(0,\infty) \to X$ is defined by

$$F\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} f\left(\int_0^\infty \beta(a)\varphi(a)da\right) \\ 0 \end{pmatrix}.$$

Notice that $\overline{D(A)} = \{0\} \times L^1(0,\infty) \neq X$, so (1.1) is an abstract Cauchy problem with the nondensely defined linear operator A.

(b) Size structured population models. Assume that the population density $u(t,s)$ depends on time t and its size s. Consider (Chu *et* *al.* [**15**])

$$(1.3)\qquad \begin{cases} \partial_t v(t,s) + \partial_s(g(s)v(t,s)) \\ \qquad\qquad\qquad = d\partial_s^2 v(t,s) - \mu v(t,s), \quad t>0,\ s>0, \\ -d\partial_s v(t,0) + g(0)v(t,0) = f\left(\int_0^\infty \beta(s)v(t,s)ds\right), \\ v(0,.) = v_0 \in L_+^1(0,\infty), \end{cases}$$

where the diffusion term represents a fluctuation in the population growth process. Consider the space $X := \mathbb{R} \times L^1(0,+\infty)$ endowed with the product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^1(0,+\infty)}.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varepsilon^2\varphi'(0) - \varphi(0) \\ \varepsilon^2\varphi'' - \varphi' - \mu\varphi \end{pmatrix}$$

with $D(A) = \{0\} \times W^{2,1}(0,+\infty)$. Then $X_0 := \overline{D(A)} = \{0\} \times L^1(0,+\infty)$ $\neq X$. Define $F : X_0 \to X$ by

$$F\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \alpha h(\int_0^{+\infty} \gamma(x)\varphi(x)dx) \\ 0 \end{pmatrix}.$$

By identifying $v(t)$ to $u(t) = \begin{pmatrix} 0 \\ v(t) \end{pmatrix}$, the partial differential equation (1.3) can be rewritten as the nondensely defined Cauchy problem (1.1).

(c) Delay equations with infinite delay. Consider the weighted space of continuous functions

$$C_\eta((-\infty,0],\mathbb{R}^n) := \left\{ \varphi \in C((-\infty,0],\mathbb{R}^n) : \sup_{\theta\le 0} e^{\eta\theta}\|\varphi(\theta)\| < +\infty \right\}$$

which is a Banach space endowed with the norm

$$\|\varphi\|_\eta := \sup_{\theta\le 0} e^{\eta\theta}\|\varphi(\theta)\|.$$

Let $x \in C\left(\left(-\infty, \tau\right], \mathbb{R}^n\right)$. Define $x_t \in C\left(\left(-\infty, 0\right], \mathbb{R}^n\right), \forall t \leq \tau$, by $x_t\left(\theta\right) = x(t+\theta), \forall \theta \leq 0$. Consider an equation with infinite delay (Hale and Kato [**25**], Hale and Verduyn Lunel [**24**], Auger and Ducrot [**5**]):

$$\begin{cases} \dfrac{dx(t)}{dt} = G(x_t), \\ x\left(\theta\right) = \varphi\left(\theta\right), \ \forall \theta \leq 0 \text{ with } \varphi \in C_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right), \end{cases} \tag{1.4}$$

where $G : C_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right) \to \mathbb{R}^n$ is k-time continuous differentiable (with $k \geq 1$), and $G(0_{C_\eta}) = 0_{\mathbb{R}^n}$. As in Liu *et al.* [**32**], by setting $u\left(t, \theta\right) = x_t\left(\theta\right)$, we can reformulate this problem as the following PDE with nonlinear and nonlocal boundary condition

$$\begin{cases} \partial_t v(t,\theta) - \partial_\theta v(t,\theta) = 0 \text{ for } \theta \leq 0 \text{ and } t \geq 0, \\ \partial_\theta v(t,0) = G(v(t,.)) \text{ for } t \geq 0, \\ v\left(0,.\right) = \varphi \in C_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right). \end{cases} \tag{1.5}$$

In order to incorporate the boundary condition into the state space, we consider the Banach space $X = \mathbb{R}^n \times C_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right)$ endowed with the product norm

$$\left\|(\alpha,\varphi)\right\| = \left\|\alpha\right\|_{\mathbb{R}^n} + \left\|\varphi\right\|_\eta.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) \\ \varphi' \end{pmatrix} \tag{1.6}$$

with $D(A) = \{0_{\mathbb{R}^n}\} \times C^1_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right)$, where

$$C^1_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right) := \left\{\varphi \in C^1\left(\left(-\infty, 0\right], \mathbb{R}^n\right) : \varphi, \varphi' \in C_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right)\right\}.$$

The closure of the domain is $X_0 := \overline{D(A)} = \{0_{\mathbb{R}^n}\} \times C_\eta\left(\left(-\infty, 0\right], \mathbb{R}^n\right) \neq X$. Therefore, A is nondensely defined. Consider the map $F : X_0 \to X$ defined by

$$F\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} G\left(\varphi\right) \\ 0_{C_\eta} \end{pmatrix}.$$

By identifying $v(t,.) = x_t$ to $u(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ v(t,.) \end{pmatrix}$, we can reformulate the infinite delay differential equation (1.4) as the Cauchy problem (1.1) with nondense domain.

For the nondensely defined Cauchy problem (1.1), since $X_0 := \overline{D(A)} \neq X$, the solutions of problem (1.1) become nonclassical whenever the range of F is not included in $\overline{D(A)}$. Based on a variation of constant formula obtained by using integrated semigroup theory, and by using Perron's method, a center manifold theory was obtained for nondensely defined semilinear Cauchy problems in Magal and Ruan [**37**]. The first part of this article is devoted to the existence and the smoothness of the center-unstable manifold for the nondensely defined semilinear Cauchy problem (1.1). We especially focus on the stability property of the center-unstable manifold. The second part of the paper deals with the stability of Hopf bifurcation, that is, stability of the bifurcating periodic orbits for the nondensely defined Cauchy problem (1.1). Our goal is to show that the stability of a periodic orbit to the reduced system (i.e. restricted to the center-unstable manifold) implies the stability of the periodic orbit for the original system. As an application, we demonstrate that these results apply to differential equations with infinite delay.

2 Existence and stability of center-unstable manifolds Recall that A is called a *Hille-Yosida operator* (densely defined or not) if there exists two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that

$$(\omega_A, +\infty) \subset \rho(A) \quad \text{(the resolvent set of } A)$$

and

$$\left\| (\lambda I - A)^{-n} \right\|_{\mathcal{L}(X)} \leq \frac{M_A}{(\lambda - \omega_A)^n}, \quad \forall \lambda > \omega_A,\ \forall n \geq 1.$$

In the Hille-Yosida case, solutions of the abstract Cauchy problem (1.1) was studied by Da Prato and Sinestrari [**16**] using an approach based on the sum of operators. Problem (1.1) has also been studied by using integrated semigroup theory, we refer to Arendt [**1, 2**], Arendt *et al.* [**3**], Kellermann and Hieber [**29**], and Thieme [**42, 43**] for related results. When A is not a Hille-Yosida operator, some recent progresses also permit us to use integrated semigroup theory to investigate the abstract Cauchy problem (1.1), we refer to Magal and Ruan [**35, 36, 37**] and Thieme [**44**] for more results on this topic.

Consider A_0, the part of A in X_0, which is the linear operator on X_0 defined by

$$A_0 x = Ax, \quad \forall x \in D(A_0)$$

and

$$D(A_0) := \{x \in D(A) : Ax \in X_0\}.$$

We first make the following assumption.

Assumption 2.1. Assume that $\rho(A)$ is nonempty and A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0.

Since $\rho(A) \neq \varnothing$, we have

$$\rho(A) = \rho(A_0).$$

Therefore, we can find two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that

$$\|T_{A_0}(t)\| \leq M_A e^{\omega_A t}, \quad \forall t \geq 0.$$

The integrated semigroup $\{S_A(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is the family of bounded linear operators on X solving the Cauchy problem

$$\frac{dS_A(t)y}{dt} = AS_A(t)y + y, \quad t \geq 0, \ S_A(0)y = 0.$$

It can be shown that

$$S_A(t)y = (\mu I - A_0) \int_0^t T_{A_0}(s)\, ds\, (\mu I - A)^{-1} y$$

for each $\mu \in \rho(A)$. One may observe that

$$S_A(t)X \subset X_0, \quad \forall t \geq 0.$$

Define

$$(S_A * f)(t) := \int_0^t S_A(t-s) f(s)\, ds$$

for each $f \in L^1((0,\tau), X)$, and

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t)$$

whenever $t \to (S_A * f)(t)$ is continuously differentiable.

We also make the following assumption.

Assumption 2.2. Assume that for each $\tau > 0$ and each function $f \in C([0,\tau], X)$, the map $t \to (S_A * f)(t)$ is continuously differentiable, and there exists a function $\delta : [0,+\infty) \to [0,+\infty)$ with

$$\lim_{t(>0)\to 0} \delta(t) = 0,$$

such that

$$\|(S_A \diamond f)(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \quad \forall t \in [0,\tau].$$

Remark 2.3. In applications, it is usually difficult to verify Assumption 2.2. The problem has been studied in Magal and Ruan [**35**] and Thieme [**44**] for the general hyperbolic case and by Ducrot *et al.* [**21**] for the parabolic case (i.e., for almost sectorial operators).

Similar to the densely defined case, a weak solution of the problem (1.1) will be a continuous function $u \in C([0,\tau],X)$ satisfying the following two properties

$$\int_0^t u(s)\,ds \in D(A), \quad \forall t \in [0,\tau],$$

$$u(t) = x + A\int_0^t u(s)\,ds + \int_0^t F(u(s))\,ds, \quad \forall t \in [0,\tau].$$

Moreover, it can be proved under the above assumptions that the notion of weak solutions is equivalent to the notion of mild solutions, that is,

$$u(t) = T_{A_0}(t)x + (S_A \diamond F(u(.)))(t), \quad \forall t \in [0,\tau].$$

The linearized equation around the equilibrium $\overline{u}$ is given by

$$\frac{dv(t)}{dt} = Av(t) + DF(\overline{u})v(t), \quad t \geq 0,\ v(0) = x \in X_0.$$

By using perturbation results, one may replace A by $A + DF(\overline{u})$ and the problem is unchanged. Therefore, we can assume that

$$\overline{u} = 0_X$$

and

$$F(0_X) = 0_X \quad \text{and} \quad DF(0_X) = 0_{\mathcal{L}(X)}.$$

The linearized equation becomes a Cauchy problem in X_0, namely,

$$\frac{dv(t)}{dt} = A_0 v(t), \quad t \geq 0,\ v(0) = x \in X_0.$$

Therefore, by making some appropriate assumption on the spectrum of A_0 one obtains a state space decomposition of X_0.

Assumption 2.4. Assume that X_0 has a state space decomposition

$$X_0 := X_{0s} \oplus X_{0cu},$$

where X_{0s} corresponds to the stable subspace and X_{0cu} corresponds to the center-unstable subspace, and satisfies

$$(\lambda I - A_0)^{-1} X_{0k} \subset X_{0k}, \quad \forall k = s, cu \text{ and } \forall \lambda \in \rho(A_0).$$

More precisely, we assume that

(i) X_{0cu} is a finite dimensional subspace of X_0 and

$$X_{0cu} = \underset{\lambda \in \sigma(A_0):\mathrm{Re}(\lambda) \geq 0}{\oplus} E_\lambda,$$

where E_λ is the generalized eigenspace of A_0 associated to $\lambda \in \sigma(A_0)$.

(ii) The growth rate of A_{0s}, the part of A_0 in X_{0s}, is strictly negative, that is,

$$\omega_0(A_{0s}) := \lim_{t \to +\infty} \frac{\ln\left(\|T_{A_0}(t)\|_{\mathcal{L}(X_{0s})}\right)}{t} < 0.$$

Remark 2.5. The above assumption is also equivalent to that

$$\sigma_{cu}(A_0) := \{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) \geq 0\}$$

is nonempty and the essential growth rate of $\{T_{A_0}(t)\}_{t \geq 0}$ is negative. That is,

$$\omega_{0,ess}(A_0) := \lim_{t \to +\infty} \frac{\ln(\|T_{A_0}(t)\|_{ess})}{t} < 0,$$

where

$$\|L\|_{ess} := \kappa(\{Lx : x \in B_{X_0}(0,1)\})$$

and

$$\kappa(B) := \inf \left\{ \begin{array}{c} \varepsilon > 0 : B \text{ can be covered by a finite number} \\ \text{of balls of radius } \leq \varepsilon \end{array} \right\}$$

is the Kuratovsky measure of noncompactness. We refer to Webb [**50**] and Engel and Nagel [**22**] for results on this topic.

Since the dimension of X_{0cu} is finite, by using Proposition 3.5 in Magal and Ruan [**37**], there exists a unique state space decomposition

$$X = X_s \oplus X_{cu},$$

where

$$X_{0cu} = X_{cu},$$

and

$$(\lambda I - A)^{-1} X_k \subset X_k, \quad \forall k = s, cu, \text{ and } \forall \lambda \in \rho(A).$$

One may observe that

$$X_{cu} \subset D(A_0).$$

Consider the linear operator $A_{cu} : X_{cu} \to X_{cu}$ defined by

$$A_{cu}x = Ax, \forall x \in X_{cu}.$$

Let $\Pi_{cu} \in L(X)$ be the bounded linear projector satisfying

$$\Pi_{cu}(X) = X_{cu} \quad \text{and} \quad (I - \Pi_{cu})(X) = X_s.$$

Set

$$\Pi_s := I - \Pi_{cu}.$$

We now study the existence and exponential stability of the center-unstable manifold for a nonlinear semiflow $\{U_F(t)\}_{t\geq 0}$ on X_0 generated by integrated solutions of (1.1). Since the existence completely parallels the case for center manifold given in Magal and Ruan [**37,** Chapter 3], we only give an outline of the theory for center-unstable manifold.

Recall that $u \in C(\mathbb{R}_-, X_0)$ is *a negative orbit* of $\{U_F(t)\}_{t\geq 0}$ if

$$u(t) = U_F(t-s)u(s), \quad \forall t, s \in \mathbb{R}_- \text{ with } t \geq s. \tag{2.1}$$

Note that equation (2.1) is also equivalent to

$$u(t) = u(s) + A\int_0^{t-s} u(s+r)\, dr + \int_0^{t-s} F(u(s+r))\, dr$$

for all $t, s \in (-\infty, 0]$ with $t \geq s$, or to

$$u(t) = T_{A_0}(t-s)u(s) + (S_A \diamond F(u(s+.)))(t-s) \tag{2.2}$$

for each $t, s \in (-\infty, 0]$ with $t \geq s$.

Let $(Y, \|.\|_Y)$ be a Banach space. Let $\eta \in \mathbb{R}$. Denote

$$BC^\eta(\mathbb{R}_-, Y) = \left\{ f \in C(\mathbb{R}_-, Y) : \sup_{t\leq 0} e^{-\eta|t|} \|f(t)\|_Y < +\infty \right\}.$$

It is well known that $BC^{\eta}(\mathbb{R}_-,Y)$ is a Banach space when it is endowed with the norm

$$\|f\|_{BC^{\eta}(\mathbb{R}_-,Y)} = \sup_{t\leq 0} e^{-\eta|t|}\,\|f(t)\|_Y\,.$$

Moreover, the family $\left\{\left(BC^{\eta}(\mathbb{R}_-,Y), \|.\|_{BC^{\eta}((-\infty,0],Y)}\right)\right\}_{\eta>0}$ forms a *scale of Banach spaces*, that is, if $0<\zeta<\eta$, then $BC^{\zeta}(\mathbb{R}_-,Y)\subset BC^{\eta}(\mathbb{R}_-,Y)$, and the embedding is continuous. More precisely, we have

$$\|f\|_{BC^{\eta}(\mathbb{R}_-,Y)} \leq \|f\|_{BC^{\zeta}(\mathbb{R}_-,Y)}\,, \quad \forall f\in BC^{\zeta}(\mathbb{R}_-,Y).$$

Let $(Z,\|.\|_Z)$ be a Banach space. Denote by $\mathrm{Lip}(Y,Z)$ (resp. $\mathrm{Lip}_B(Y,Z)$) the space of Lipschitz (resp. Lipschitz and bounded) maps from Y into Z, and set

$$\|F\|_{\mathrm{Lip}(Y,Z)} := \sup_{x,y\in Y: x\neq y} \frac{\|F(x)-F(y)\|_Z}{\|x-y\|_Y}.$$

From now on, we fix $\beta_-\in(0,-\omega_0(A_{0s}))\,.$

Definition 2.6. Let $\eta\in(0,\beta_-)$. The *η-center-unstable manifold* of (1.1), denoted by V_η^{cu}, is the set of all points $x\in X_0$ such that there exists $u\in BC^{\eta}(\mathbb{R}_-,X_0)\,,$ a negative orbit of $\{U_F(t)\}_{t\geq 0}\,,$ such that $u(0)=x$.

For each $\eta>0$, V_η^{cu} is invariant under the semiflow $\{U_F(t)\}_{t\geq 0}\,,$ that is,

$$U_F(t)\mathrm{V}_\eta^{\mathrm{cu}} = \mathrm{V}_\eta^{\mathrm{cu}},\ \ \forall t\geq 0.$$

Moreover, we say that $\{U_F(t)\}_{t\geq 0}$ is *reduced on* V_η^{cu} if there exists a map $\Psi_{cu}: X_{0cu}\to X_{0s}$ such that

$$\mathrm{V}_\eta^{\mathrm{cu}} = \mathrm{Graph}\left(\Psi_{cu}\right) = \left\{x_{cu}+\Psi_{cu}\left(x_{cu}\right) : x_{cu}\in X_{0cu}\right\}.$$

The following lemma was proved in Magal and Ruan [**37,** Lemma 4.6].

Lemma 2.7. *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Then*

(i) *For each* $\eta\in[0,\beta_-)\,,$ *each* $f\in BC^{\eta}(\mathbb{R}_-,X)\,,$ *and each* $t\in\mathbb{R}$,

$$K_s(f)(t) := \lim_{r\to-\infty} \Pi_{0s}\left(S_A\diamond f(r+.)\right)(t-r)\ \text{exists.}$$

(ii) *For each* $\eta \in [0, \beta_-)$, K_s *is a bounded linear operator from* $BC^\eta(\mathbb{R}_-, X)$ *into* $BC^\eta(\mathbb{R}_-, X_{0s})$. *More precisely, for each* $\nu \in (-\beta_-, 0)$, *there exists a constant* $\widehat{C}_{s,\nu} > 0$ *such that*

$$\|K_s\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-,X),BC^\eta(\mathbb{R}_-,X_{0s}))} \leq \widehat{C}_{s,\nu}, \quad \forall \eta \in [0, -\nu].$$

(iii) *For each* $\eta \in [0, \beta_-)$, *each* $f \in BC^\eta(\mathbb{R}_-, X)$, *and each* $t, s \in \mathbb{R}$ *with* $t \geq s$,

$$K_s(f)(t) - T_{A_{0s}}(t-s)K_s(f)(s) = \Pi_{0s}(S_A \diamond f(s + .))(t - s).$$

The following lemmas can be proved similarly as Lemmas 4.8 and 4.9 in Magal and Ruan [**37**].

Lemma 2.8. *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let* $\eta \in (0, \beta_-)$ *be fixed. For each* $x_{cu} \in X_{0cu}$, *each* $f \in BC^\eta(\mathbb{R}_-, X)$, *and each* $t \in (-\infty, 0]$, *denote*

$$K_1(x_{cu})(t) := e^{A_{cu}t}x_{cu}, \quad K_{cu}(f)(t) := \int_0^t e^{A_{cu}(t-s)}\Pi_{cu}f(s)\,ds,$$

where $\Pi_{cu} = \Pi_c + \Pi_u$. *Then* K_1 *is a bounded linear operator from* X_{0cu} *into* $BC^\eta(\mathbb{R}_-, X_{0cu})$ *and*

$$\|K_1\|_{\mathcal{L}(X_{0cu},BC^\eta(\mathbb{R}_-,X))} \leq \sup_{t\geq 0}\left\|e^{-(A_{cu}+\eta I)t}\right\| < +\infty,$$

$$\|K_{cu}\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-,X))} \leq \|\Pi_{cu}\|_{\mathcal{L}(X)}\int_0^{+\infty}\left\|e^{-(A_{cu}+\eta I)l}\right\|dl < +\infty.$$

Lemma 2.9. *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let* $\eta \in (0, \beta_-)$ *and* $u \in BC^\eta(\mathbb{R}_-, X_0)$ *be fixed. Then* u *is a complete orbit of* $\{U_F(t)\}_{t\geq 0}$ *if and only if for each* $t \in \mathbb{R}$,

$$u(t) = K_1(\Pi_{0cu}u(0))(t) + K_{cu}(F(u(.)))(t) + K_s(F(u(.)))(t), \tag{2.3}$$

where $\Pi_{0cu} = \Pi_{0c} + \Pi_{0u}$.

Let $\eta \in (0, \beta_-)$ be fixed. Rewrite equation (2.3) as the following fixed point problem: To find $u \in BC^\eta(\mathbb{R}_-, X)$ such that

$$u = K_1(\Pi_{0cu}u(0)) + K_2\Phi_F(u), \tag{2.4}$$

where the nonlinear operator $\Phi_F \in \text{Lip}\left(BC^{\eta}\left(\mathbb{R}_{-}, X_0\right), BC^{\eta}\left(\mathbb{R}_{-}, X\right)\right)$ is defined by

$$\Phi_F(u)(t) = F(u(t)), \quad \forall t \in \mathbb{R},$$

and the linear operator $K_2 \in \mathcal{L}\left(BC^{\eta}\left(\mathbb{R}_{-}, X\right), BC^{\eta}\left(\mathbb{R}_{-}, X_0\right)\right)$ is defined by

$$K_2 = K_{cu} + K_s.$$

Moreover, we have the following estimates

$$\|K_1\|_{\mathcal{L}(X_{0cu}, BC^{\eta}(\mathbb{R}_{-}, X))} \leq \sup_{t \geq 0} \left\|e^{-(A_{cu}+\eta I)t}\right\|,$$

$$\|\Phi_F\|_{\text{Lip}} \leq \|F\|_{\text{Lip}},$$

and for each $\nu \in (-\beta_{-}, 0)$, we have

$$\|K_2\|_{\mathcal{L}(BC^{\eta}(\mathbb{R}, X))} \leq \gamma(\nu, \eta), \quad \forall \eta \in (0, -\nu],$$

where

$$\gamma(\nu, \eta) := \widehat{C}_{s,\nu} + \|\Pi_{cu}\|_{\mathcal{L}(X)} \int_0^{+\infty} \left\|e^{-(A_{cu}+\eta I)l}\right\| dl. \tag{2.5}$$

Furthermore, by Lemma 2.9, the η-center-unstable manifold is given by

$$\text{V}_{\eta}^{\text{cu}} = \big\{x \in X_0 : \exists u \in BC^{\eta}\left(\mathbb{R}_{-}, X_0\right) \text{ a solution of (2.4) and } u(0) = x\big\}. \tag{2.6}$$

We state the existence of center-unstable manifolds for the abstract semilinear Cauchy problem (1.1) with nondense domain which can be proved similarly as Theorem 4.10 in Magal and Ruan [**37**].

Theorem 2.10 (Global Center-Unstable Manifold)**.** *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let $\eta \in (0, \beta_{-})$ be fixed and $\delta_0 = \delta_0(\eta) > 0$ be such that*

$$\delta_0 \|K_2\|_{\mathcal{L}(BC^{\eta}(\mathbb{R}_{-}, X))} < 1.$$

Then for each $F \in \text{Lip}(X_0, X)$ with $\|F\|_{\text{Lip}(X_0,X)} \leq \delta_0$, there exists a Lipschitz continuous map $\Psi_{cu} : X_{0cu} \to X_{0s}$ such that

$$\text{V}_{\eta}^{\text{cu}} = \left\{x_{cu} + \Psi_{cu}(x_{cu}) : x_{cu} \in X_{0cu}\right\}.$$

Moreover, we have the following properties:

(i) $\sup_{x_{cu}\in X_{0cu}} \|\Psi_{cu}(x_{cu})\| \leq \|K_s\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-,X))} \sup_{x\in X_0} \|\Pi_s F(x)\|$.

(ii) *We have*

$$\|\Psi_{cu}\|_{\mathrm{Lip}(X_{0cu},X_{0s})} \leq \frac{\|K_s\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-,X))} \|F\|_{\mathrm{Lip}(X_0,X)} \|K_1\|_{\mathcal{L}(X_{0cu},BC^\eta(\mathbb{R}_-,X_0))}}{1-\|K_2\|_{\mathcal{L}(BC^\eta(\mathbb{R}_-,X))} \|F\|_{\mathrm{Lip}(X_0,X)}}. \tag{2.7}$$

We now state and prove the existence of local center-unstable manifolds.

Theorem 2.11 (Local Center-Unstable Manifold)**.** *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let $r>0$ and $F : B_{X_0}(0,r) \to X$ be a map. Assume that there exists an integer $k \geq 1$ such that F is k-time continuously differentiable in $B_{X_0}(0,r)$ with*

$$F(0) = 0 \quad \text{and} \quad DF(0) = 0.$$

Then there exists a neighborhood Ω of the origin in X_0 and a map $\Psi_{cu} \in C_b^k(X_{0cu}, X_{0s})$ with

$$\Psi_{cu}(0) = 0 \quad \text{and} \quad D\Psi_{cu}(0) = 0,$$

such that

$$M_{cu} = \{x_{cu} + \Psi_{cu}(x_{cu}) : x_{cu} \in X_{0cu}\}$$

is a locally invariant manifold by the semiflow generated by (1.1) *around* 0.

More precisely, the following properties hold:

(i) *If I is an interval of $\mathbb{R}$ and $x_{cu} : I \to X_{0cu}$ is a solution of*

$$\frac{dx_{cu}(t)}{dt} = A_{0cu}x_{cu}(t) + \Pi_{cu}F(x_{cu}(t) + \Psi_{cu}(x_{cu}(t))) \tag{2.8}$$

(reduced equation) *such that*

$$u(t) := x_{uc}(t) + \Psi_{cu}(x_{uc}(t)) \in \Omega, \quad \forall t \in I,$$

then for each $t, s \in I$ with $t \geq s$,

$$u(t) = u(s) + A\int_s^t u(l)dl + \int_s^t F(u(l))\,dl.$$

(ii) *If* $u : (-\infty, 0] \rightarrow X_0$ *is a map such that for each* $t, s \in (-\infty, 0]$ *with* $t \geq s$,

$$u(t) = u(s) + A \int_s^t u(l)\, dl + \int_s^t F\left(u(l)\right) dl$$

and

$$u(t) \in \Omega, \quad \forall t \in (-\infty, 0],$$

then

$$\Pi_s u(t) = \Psi_{cu}\left(\Pi_{cu} u(t)\right), \quad \forall t \in (-\infty, 0],$$

and $\Pi_{cu} u : (-\infty, 0] \rightarrow X_{0cu}$ *is a solution of* (2.8).

Proof. In order to prove the local center-unstable manifold theorem, we apply Theorem 2.10 to the Cauchy problem

$$\frac{du}{dt} = Au(t) + F_r\left(u(t)\right), \quad t \geq 0, \ u(0) = x \in X_0,$$

where $F_r : X_0 \to X$ is the following truncated function

$$F_r\left(x\right) = F(x)\chi_{cu}\left(r^{-1}\Pi_{0cu}(x)\right)\chi_s\left(r^{-1}\left\|\Pi_{0s}(x)\right\|\right), \quad \forall x \in X_0,$$

$\chi_{cu} : X_{0cu} \to [0, +\infty)$ is a C^∞ map with $\chi_{cu}\left(x\right) \leq 1$ and

$$\chi_{cu}\left(x\right) = \begin{cases} 1, & \text{if } \|x\| \leq 1, \\ 0, & \text{if } \|x\| \geq 2, \end{cases}$$

and $\chi_s : [0, +\infty) \to [0, +\infty)$ is a C^∞ map with $\chi_s\left(y\right) \leq 1, \ \forall y \geq 0$, and

$$\chi_s\left(y\right) = \begin{cases} 1, & \text{if } |y| \leq 1, \\ 0, & \text{if } |y| \geq 2. \end{cases}$$

The smoothness of Ψ_{cu} is obtained by applying the same arguments as in Magal and Ruan [**37**] to the above truncated system, and the result follows. □

The following theorem is the main result of this section. This result is proved for discrete time systems with bounded Lipschitz map F in Vanderbauwhede [**46**] and for ordinary differential equations in Vanderbauwhede [**47**] and Chow *et al.* [**10**].

Theorem 2.12 (Stability of the Center-Unstable Manifold). *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let* $\eta \in (0, \beta_-)$ *be fixed. Then there exists* $\delta_1(\eta) \in (0, \delta_0)$ *(where* $\delta_0 > 0$ *is the constant introduced in Theorem* 2.10*), such that for each* $F \in \mathrm{Lip}(X_0, X)$ *with* $\|F\|_{\mathrm{Lip}(X_0,X)} \leq \delta_1(\eta)$, *there exists a continuous map* $H_{cu} : X_0 \to \mathrm{V}_\eta^{\mathrm{cu}}$ *such that for each* $x \in X_0$,

$$\mathrm{V}_\eta^{\mathrm{cu}} \cap \widetilde{V}_\eta(x) = \{H_{cu}(x)\},$$

where

$$\widetilde{V}_\eta(x) = \Big\{ y \in X_0 : \sup_{t \geq 0} e^{\eta t} \|U_F(t)y - U_F(t)x\| < +\infty \Big\}.$$

More precisely, for each $x \in X_0$, *there is a constant* $M_\eta = M_\eta(x) > 0$ *such that*

$$\|U_F(t)H(x) - U_F(t)x\| \leq e^{-\eta t} M_\eta \|x - H(x)\|, \quad \forall t \geq 0.$$

Before proving the theorem we give some preliminary lemmas. Recall that

$$BC^{-\eta}(\mathbb{R}_+, X) = \Big\{ w \in C(\mathbb{R}_+, X) : \|w\|_\eta = \sup_{t \in \mathbb{R}_+} e^{\eta t} \|w(t)\| < +\infty \Big\}.$$

In order to determine $\widetilde{V}_\eta(x)$, we have to find all $w \in BC^{-\eta}(\mathbb{R}_+, X_0)$ such that $t \to U_F(t)x + w(t)$ is a solution of

$$u(t) = T_{A_0}(t)x + (S_A \diamond F(u(.)))(t), \quad \forall t \in [0, \tau]. \tag{2.9}$$

Lemma 2.13. *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let* $\eta \in (0, \beta_-)$ *be fixed and* $w \in BC^{-\eta}(\mathbb{R}_+, X_0)$. *Then the map* $t \to U_F(t)x + w(t)$ *is a solution of* (2.9) *if and only if for each* $t \geq 0$,

$$\begin{aligned} w(t) = {} & T_{A_{0s}}(t)\Pi_{0s} w(0) + (S_{A_s} \diamond \Pi_s [F(U(.)x + w(.)) - F(U(.)x)])(t) \\ & - \int_t^{+\infty} e^{A(t-s)} \Pi_{cu} [F(U(s)x + w(s)) - F(U(s)x)] \, ds. \end{aligned}$$

Proof. Let $w \in BC^{-\eta}(\mathbb{R}_+, X_0)$ be fixed. Assume first that $t \to U_F(t)x + w(t)$ is a solution of (2.9). Then we have for each $t, s \in [0, +\infty)$ with $t \geq s$ that

$$U_F(t)x + w(t) = T_{A_0}(t-s)\left(U(s)x + w(s)\right) \\ + \left(S_A \diamond F(U(s+.)x + w(s+.))\right)(t-s)$$

and

$$U_F(t)x = T_{A_0}(t-s)U(s)x + \left(S_A \diamond F(U(s+.)x)\right)(t-s).$$

Then

$$w(t) = T_{A_0}(t-s)w(s) + (S_A \diamond [F(U(s+.)x + w(s+.)) \\ - F(U(s+.)x)])(t-s). \tag{2.10}$$

By projecting the above equation on X_{cu}, we obtain for each $t, s \in [0, +\infty)$ with $t \geq s$ that

$$\Pi_{0cu}w(t) = e^{A_{cu}(t-s)}\Pi_{0cu}w(s) \\ + \int_s^t e^{A_{cu}(t-l)}\Pi_{cu}\left[F(U(l)x + w(l)) - F(U(l)x)\right] dl.$$

Then

$$\Pi_{0cu}w(s) = e^{-A_{cu}(t-s)}\Pi_{0cu}w(t) \\ - \int_s^t e^{A_{cu}(s-l)}\Pi_{cu}\left[F(U(l)x + w(l)) - F(U(l)x)\right] dl.$$

We have $\|e^{-A_{cu}(t-s)}\|_{\mathcal{L}(X_{0cu})} \leq \min\{e^{\frac{\eta}{2}|t-s|}M_{c,\frac{\eta}{2}},\ e^{-\eta_1(t-s)}M_u\}$, $\eta_1 > 0$, $\forall t \geq s$; here, η, $M_{c,\frac{\eta}{2}}$ and M_u are constants (see Magal and Ruan [**37**] for details). Since $w \in BC^{-\eta}(\mathbb{R}_+, X_0)$, we obtain for each $t, s \in [0, +\infty)$ with $t \geq s$ that

$$\left\|e^{-A_{cu}(t-s)}\Pi_{0cu}w(t)\right\| \\ \leq \min\left\{e^{\frac{\eta}{2}|t-s|}M_{c,\frac{\eta}{2}},\ e^{-\eta_1(t-s)}M_u\right\} \|\Pi_{cu}\|_{\mathcal{L}(X)} \|w\|_{\eta} e^{-\eta t}.$$

Then

$$\left\|e^{-A_{cu}(t-s)}\Pi_{0cu}w(t)\right\| \to 0 \quad \text{as} \quad t \to +\infty.$$

Thus,

$$\Pi_{0cu}w(t) = -\int_t^{+\infty} e^{A_{cu}(t-l)}\Pi_{cu} \times [F(U(l)x + w(l)) - F(U(l)x)]\,dl, \quad \forall t \geq 0. \tag{2.11}$$

By projecting (2.10) on X_s we obtain for each $t \geq 0$ that

$$\Pi_{0s}w(t) = T_{A_{0s}}(t)\Pi_{0s}w(0) + (S_{A_s} \diamond \Pi_s [F(U(.)x + w(.)) - F(U(.)x)])(t). \tag{2.12}$$

So by summing up (2.11) and (2.12), we obtain (2.13). Conversely, assume that w satisfies (2.13). Then by projecting (2.13) on X_s we obtain for each $t \geq 0$ that

$$\Pi_s w(t) = \Pi_s T_{A_0}(t)w(0) + \Pi_s (S_A \diamond F(U(.)x + w(.)) - F(U(.)x))(t).$$

Then

$$\Pi_s(U_F(t)x + w(t)) = \Pi_s T_{A_0}(t)(w(0) + x) + \Pi_s (S_A \diamond F(U(.)x + w(.)))(t). \tag{2.13}$$

Furthermore, by projecting (2.13) on X_{0cu} we obtain for each $t \geq 0$ that

$$\Pi_{0cu}w(t) = -\int_t^{+\infty} e^{A(t-s)}\Pi_{cu} [F(U(s)x + w(s)) - F(U(s)x)]\,ds.$$

Thus,

$$\begin{aligned}
\Pi_{0cu}w(t) - e^{A_{cu}t}\Pi_{0cu}w(0) &= -\int_t^{+\infty} e^{A(t-s)}\Pi_{cu} [F(U(s)x + w(s)) - F(U(s)x)]\,ds \\
&\quad + \int_0^{+\infty} e^{A(t-s)}\Pi_{cu} [F(U(s)x + w(s)) - F(U(s)x)]\,ds \\
&= \int_0^t e^{A(t-s)}\Pi_{cu} [F(U(s)x + w(s)) - F(U(s)x)]\,ds.
\end{aligned}$$

Hence,

$$\text{(2.14)} \quad \Pi_{0cu}(U_F(t)x + w(t)) = e^{A_{cu}t}\Pi_{0cu}(x + w(0)) + \int_0^t e^{A(t-s)}\Pi_{cu}F(U(s)x + w(s))\,ds.$$

By summing up (2.13) and (2.14), we deduce that $t \to U_F(t)x + w(t)$ is a solution of (2.9). □

Rewrite (2.13) in the following abstract form

$$w = \widetilde{K}_1(w_s) + \widetilde{K}_2\widetilde{\Phi}(x, w),$$

where $\widetilde{K}_1 : X_{0s} \to BC^{-\eta}(\mathbb{R}_+, X_{0s})$, $\widetilde{K}_2 : BC^{-\eta}(\mathbb{R}_+, X) \to BC^{-\eta}(\mathbb{R}_+, X_0)$, and $\widetilde{\Phi} : X_0 \times BC^{-\eta}(\mathbb{R}_+, X_0) \to BC^{-\eta}(\mathbb{R}_+, X)$ are defined as follows

$$\widetilde{K}_1(x_s)(t) = T_{A_{0s}}(t)x_s, \quad t \in \mathbb{R}_+,$$

$$\widetilde{K}_2(f)(t) = (S_{A_s} \diamond \Pi_s f)(t) - \int_t^{+\infty} e^{A_{cu}(t-s)}\Pi_{cu}f(s)\,ds, \quad \forall t \in \mathbb{R}_+,$$

$$\widetilde{\Phi}(x, f)(t) = F(U_F(t)x + f(t)) - F(U_F(t)x), \quad \forall t \in \mathbb{R}_+.$$

One has

$$\text{(2.15)} \qquad \left\|\widetilde{\Phi}(x, f)(t)\right\| = \|F(U_F(t)x + f(t)) - F(U_F(t)x)\| \leq e^{-\eta t}\|F\|_{\mathrm{Lip}}\|f\|_\eta.$$

Lemma 2.14. *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let* $\eta \in (0, \beta_-)$ *be fixed. Then*

$$\widetilde{K}_1 \in \mathcal{L}(X_{0s}, BC^{-\eta}(\mathbb{R}_+, X_0))$$

and

$$\widetilde{K}_2 \in \mathcal{L}(BC^{-\eta}(\mathbb{R}_+, X), BC^{-\eta}(\mathbb{R}_+, X_0))$$

with

$$\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,X),BC^{-\eta}(\mathbb{R}_+,X_0))}$$

$$\leq \gamma(\eta) := \widehat{C}_{s,-\eta} + \|\Pi_{cu}\|_{\mathcal{L}(X)} \int_0^{+\infty} \left\| e^{-(A_{cu}+\eta I)l} \right\| dl,$$

where $\widehat{C}_{s,-\eta} > 0$ *is a constant, and*

$$\widetilde{\Phi}(x,0) = 0,$$

$$\widetilde{\Phi}(x,.) \in \operatorname{Lip}\left(BC^{-\eta}(\mathbb{R}_+, X_0), BC^{-\eta}(\mathbb{R}_+, X)\right), \quad \forall x \in X_0,$$

with

$$\left\|\widetilde{\Phi}(x,.)\right\|_{\operatorname{Lip}} \leq \|F\|_{\operatorname{Lip}}.$$

Proof. This proof is straightforward. □

Proof of Theorem 2.12. Let $\eta \in (0, \beta_-)$ and $x \in X_0$ be fixed. Let $\delta_0 > 0$ be the constant introduced in Theorem 2.10. Let $\delta_1^* \in (0, \delta_0)$ be such that

$$\delta_1^* \gamma(\eta) < 1. \tag{2.16}$$

Then for each $F \in \operatorname{Lip}(X_0, X)$ with $\|F\|_{\operatorname{Lip}} \leq \delta_1^*$, we obtain that for each $(x, w_s) \in X_0 \times X_{0s}$, there exists a unique solution $w = \widetilde{w}(x, w_s) \in BC^{-\eta}(\mathbb{R}_+, X_0)$ such that

$$w = \widetilde{K}_1(w_s) + \widetilde{K}_2 \widetilde{\Phi}(x, w)$$

and

$$w = (Id - \widetilde{K}_2 \widetilde{\Phi}(x,.))^{-1} \widetilde{K}_1(w_s).$$

We have

$$\|\widetilde{w}(x, w_s) - \widetilde{w}(x, \widetilde{w}_s)\|_\eta \leq l \|w_s - \widetilde{w}_s\|, \quad \forall x \in X_0, \ \forall w_s, \widetilde{w}_s \in X_{0s},$$

where l depends on η and $\|F\|_{\operatorname{Lip}}$ but stays bounded as $\|F\|_{\operatorname{Lip}} \to 0$. To see the continuous dependence of $\widetilde{w}(x, w_s)$ on $x \in X_0$, we remark that (2.16) and the continuity of $\gamma(\eta)$ imply that $\gamma(\zeta)\delta_1^* < 1$ for some $\zeta \in (\eta, \beta)$. Replacing η by ζ in the above argument, we conclude that $\widetilde{w}(x, w_s)$ belongs in fact to the space $BC^{-\zeta}(\mathbb{R}_+, X_0)$, which is continuously imbedded in $BC^{-\eta}(\mathbb{R}_+, X_0)$. More precisely, we have

$$\|\widetilde{w}(x, w_s)\|_\zeta \leq \left\|\widetilde{K}_1\right\|_{\mathcal{L}(X_s, BC^{-\zeta}(\mathbb{R}_+, X))} \|w_s\|$$

$$
\begin{aligned}
&\quad + \left\| \widetilde{K}_2 \right\|_{\mathcal{L}(BC^{-\zeta}(\mathbb{R}_+,X))} \left\| \widetilde{\Phi}(x+x_0,w) \right\|_\zeta \\
&\leq \left\| \widetilde{K}_1 \right\|_{\mathcal{L}(X_s,BC^{-\zeta}(\mathbb{R}_+,X))} \left\| w_s \right\| \\
&\quad + \left\| \widetilde{K}_2 \right\|_{\mathcal{L}(BC^{-\zeta}(\mathbb{R}_+,X))} \left\| F \right\|_{\mathrm{Lip}} \left\| \widetilde{w}(x,w_s) \right\|_\zeta .
\end{aligned}
$$

Therefore, we obtain an estimate independent of x,

$$
\left\| \widetilde{w}(x,w_s) \right\|_\zeta \leq \frac{\left\| \widetilde{K}_1 \right\|_{\mathcal{L}(X_s,BC^{-\zeta}(\mathbb{R}_+,X))} \left\| w_s \right\|}{1 - \left\| \widetilde{K}_2 \right\|_{\mathcal{L}(BC^{-\zeta}(\mathbb{R}_+,X))} \left\| F \right\|_{\mathrm{Lip}}} < +\infty.
$$

Moreover, we have

$$
\begin{aligned}
&\widetilde{w}(x+x_0,w_s) - \widetilde{w}(x_0,w_s) \\
&\quad = \widetilde{K}_1(w_s) + \widetilde{K}_2\widetilde{\Phi}(x+x_0,\widetilde{w}(x+x_0,w_s)) \\
&\qquad\qquad - \left[\widetilde{K}_1(w_s) + \widetilde{K}_2\widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s)) \right] \\
&\quad = \widetilde{K}_2\left[\widetilde{\Phi}(x+x_0,\widetilde{w}(x+x_0,w_s)) - \widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s)) \right] \\
&\quad = \widetilde{K}_2\left[\widetilde{\Phi}(x+x_0,\widetilde{w}(x+x_0,w_s)) - \widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s)) \right] \\
&\qquad\qquad + \widetilde{K}_2\left[\widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s)) - \widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s)) \right].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left\| \widetilde{w}(x+x_0,w_s) - \widetilde{w}(x_0,w_s) \right\|_\eta \\
&\quad \leq \left\| \widetilde{K}_2 \right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,X))} \left\| F \right\|_{\mathrm{Lip}} \left\| \widetilde{w}(x+x_0,w_s) - \widetilde{w}(x_0,w_s) \right\|_\eta \\
&\qquad + \left\| \widetilde{K}_2 \right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,X))} \left\| \widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s)) - \widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s)) \right\|_\eta .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\| \widetilde{w}(x+x_0,w_s) - \widetilde{w}(x_0,w_s) \right\|_\eta \\
&\qquad\qquad \leq \frac{\left\| \widetilde{K}_2 \right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,X))}}{1 - \left\| \widetilde{K}_2 \right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,X))} \left\| F \right\|_{\mathrm{Lip}}} \\
&\qquad\qquad\qquad \times \left\| \widetilde{\Phi}(x+x_0,\widetilde{w}(x_0,w_s)) - \widetilde{\Phi}(x_0,\widetilde{w}(x_0,w_s)) \right\|_\eta .
\end{aligned}
$$

For fixed $w \in BC^{-\zeta}(\mathbb{R}_+, X_0)$ we claim that the mapping $x \to \widetilde{\Phi}(x,w)$ is continuous from X_0 into $BC^{-\eta}(\mathbb{R}_+, X)$. In fact, by using (*2.16*), we have

$$\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta = \sup_{t\in\mathbb{R}_+} e^{\eta t}\left\|H(t)\right\|,$$

where

$$\begin{aligned} H(t) := {}& \left[F(U_F(t)\left(x+x_0\right)+w(t)) - F(U_F(t)x_0+w(t))\right] \\ & - \left[F(U_F(t)\left(x+x_0\right)) - F(U_F(t)x_0)\right]. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta \\ &\qquad = \max\left(\sup_{0\le t\le T} e^{\eta T}\left\|H(t)\right\|,\ 2e^{(\eta-\zeta)T}\left\|F\right\|_{\mathrm{Lip}}\left\|w\right\|_\zeta\right). \end{aligned}$$

By the continuity of $x \to U_F(t)\,(x)$ uniformly with respect to $t \in [0,T]$, we obtain

$$\limsup_{x\to 0}\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta \le 2e^{(\eta-\zeta)T}\left\|F\right\|_{\mathrm{Lip}}\left\|w\right\|_\zeta,\ T\ge 0.$$

So when T goes to $+\infty$, we obtain

$$\lim_{x\to 0}\left\|\widetilde{\Phi}(x+x_0,w)-\widetilde{\Phi}(x_0,w)\right\|_\eta = 0.$$

From this and the fact that $\widetilde{w}(x,w_s) \in BC^{-\zeta}(\mathbb{R}_+, X_0)$, it follows that $\widetilde{w} : X_0 \times X_s \to BC^{-\eta}\left(\mathbb{R}_+, X_0\right)$ is continuous.

Define a map $\Gamma : X_0 \times X_{0s} \to X_{0cu}$ by

$$\Gamma\left(x,w_s\right) = \Pi_{cu}(Id-\widetilde{K}_2\widetilde{\Phi}(x,.))^{-1}\widetilde{K}_1(w_s)(0), \quad \forall x\in X_0,\ w_s\in X_{0s}.$$

Notice that $\Gamma : X_0 \times X_{0s} \to X_{0cu}$ is continuous and Γ is Lipschitz continuous with respect to w_s with

$$\left\|\Gamma(x,.)\right\|_{\mathrm{Lip}} \le \left\|\Pi_{0cu}\right\|_{\mathcal{L}(X)}\frac{\left\|\widetilde{K}_1\right\|_{\mathcal{L}(X_s,BC^{-\eta}(\mathbb{R}_+,X))}}{1-\left\|\widetilde{K}_2\right\|_{\mathcal{L}(BC^{-\eta}(\mathbb{R}_+,X))}\left\|F\right\|_{\mathrm{Lip}}}.$$

We have by construction that

$$y \in \widetilde{V}_\eta(x) \Longleftrightarrow y = x+w \quad \text{with} \quad \Pi_{0cu}w = \Gamma\left(x,\Pi_s w\right).$$

Then

$$\begin{aligned}\widetilde{V}_\eta(x) &= \{x_s + w_s + x_{cu} + \Gamma(x, w_s) : w_s \in X_{0s}\}\\ &= \{z + x_{cu} + \Gamma(x, z - x_s) : z \in X_{0s}\}.\end{aligned}$$

Consider the map $\Theta : X_0 \times X_{0s} \to X_{0cu}$ defined by

$$\Theta(x, z) = x_c + \Gamma(x, z - x_s), \quad \forall x \in X_0,\ z \in X_{0s},$$

we have $\widetilde{V}_\eta(x) = \{z + \Theta(x, z) : z \in X_{0s}\}$. Since $\Gamma : X_0 \times X_{0s} \to X_{0cu}$ is continuous and $\Gamma(x, w_s)$ is Lipschitz continuous with respect to w_s, so is Θ, and $\|\Theta\|_{\mathrm{Lip}} \leq \|\Gamma\|_{\mathrm{Lip}}$. Finally, we look for $y \in X_0$, such that

$$\Pi_{0s}y = \Psi_{cu}(\Pi_{0cu}y) \quad \text{and} \quad \Pi_{0cu}y =: \Theta(x, \Pi_{0s}y).$$

But by (2.7), we deduce that $\|\Psi_{cu}\|_{\mathrm{Lip}} \to 0$ as $\|F\|_{\mathrm{Lip}} \to 0$. So (2.7) and (2.16) imply that there exists $\delta_1 \in (0, \delta_1^*)$ such that for each $F \in \mathrm{Lip}(X_0, X)$ with $\|F\|_{\mathrm{Lip}} \leq \delta_1$,

$$\|\Theta(x, .)\|_{\mathrm{Lip}} \|\Psi_{cu}\|_{\mathrm{Lip}} < 1.$$

Thus, there exists for each $x \in X_0$ a unique $\widetilde{y}_{cu}(x) \in X_{0cu}$ such that

$$\Theta(x, \Psi_{cu}(\widetilde{y}_{cu}(x))) = \widetilde{y}_{cu}(x)$$

and the map $\widetilde{y}_c : X_0 \to X_{0cu}$ is continuous. By setting $H_{cu}(x) = \widetilde{y}_{cu}(x) + \Psi_{cu}(\widetilde{y}_{cu}(x))$ the result follows. □

Theorem 2.15 (Local Uniform Convergence). *Let Assumptions* 2.1, 2.2 *and* 2.4 *be satisfied. Let* $\eta \in (0, \beta_-)$ *be fixed. Then there exists* $\delta_1(\eta) \in (0, \delta_0)$ *(where* $\delta_0 > 0$ *is the constant introduced in Theorem* 2.10*), such that for each* $F \in \mathrm{Lip}(X_0, X)$ *with* $\|F\|_{\mathrm{Lip}(X_0,X)} \leq \delta_1(\eta)$*, the following holds: for each* $\widetilde{x} \in V_\eta^{cu}$ *and for each* $\varepsilon > 0$*, there exists some* $\delta > 0$ *such that*

$$\|U_F(t)x - U_F(t)H_{cu}(x)\| \leq \varepsilon e^{-\eta t}, \quad \forall t \geq 0, \tag{2.17}$$

for all $x \in X_0$ *with* $\|x - \widetilde{x}\| < \delta$.

Proof. Let $\widetilde{x} \in V_\eta^{cu}$ be fixed. The proof of Theorem 2.12 implies that

$$\begin{aligned}&\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_{0s}x)(t)\\ &\qquad = U_F(t)(H_{cu}(x)) - U_F(t)(x), \quad \forall t \geq 0, \quad \forall x \in X_0,\end{aligned}$$

where

$$H_{cu}(x) = \widetilde{y}_c(x) + \Psi_{cu}(\widetilde{y}_c(x)) \in V_\eta^{cu}.$$

It is clear that $H_{cu}(\widetilde{x}) = \widetilde{x}$ if $\widetilde{x} \in V_\eta^{cu}$ and hence,

$$\begin{aligned}&\widetilde{w}(\widetilde{x}, \Psi_{cu}(\widetilde{y}_c(\widetilde{x})) - \Pi_{0s}\widetilde{x})\\ &\qquad = U_F(t)(H_{cu}(\widetilde{x})) - U_F(t)(\widetilde{x}) = 0, \quad \forall t \geq 0, \ \forall \widetilde{x} \in V_\eta^{cu}.\end{aligned}$$

Let $\widetilde{x} \in V_\eta^{cu}$ and $\varepsilon > 0$. By the continuity of $\widetilde{w} : X_0 \times X_{0s} \to BC^{-\eta}(\mathbb{R}_+, X_0)$ and $\widetilde{y}_c : X_0 \to X_{0cu}$, we can find some $\delta > 0$ such that

$$||\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_{0s}x) - \widetilde{w}(\widetilde{x}, \Psi_{cu}(\widetilde{y}_c(\widetilde{x})) - \Pi_{0s}\widetilde{x})||_\eta \leq \varepsilon$$

whenever $x \in X_0$ and $\|x - \widetilde{x}\| < \delta$. Therefore,

$$\begin{aligned}&\sup_{t\in\mathbb{R}_+} e^{\eta t}\|U_F(t)(H_{cu}(x)) - U_F(t)(x)\|\\ &\qquad = \sup_{t\in\mathbb{R}_+} e^{\eta t}\|\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_{0s}x)(t)\|\\ &\qquad = \sup_{t\in\mathbb{R}_+} e^{\eta t}\|\widetilde{w}(x, \Psi_{cu}(\widetilde{y}_c(x)) - \Pi_{0s}x)(t)\\ &\qquad\qquad - \widetilde{w}(\widetilde{x}, \Psi_{cu}(\widetilde{y}_c(\widetilde{x})) - \Pi_{0s}\widetilde{x})(t)\|\\ &\qquad \leq \varepsilon \text{ if } x \in X_0 \text{ and } \|x - \widetilde{x}\| < \delta.\end{aligned}$$

The proof is complete. □

Remark 2.16. Our presentations focused on center-unstable manifolds. However, similar results can be established for center-stable manifolds. In fact, we will use a center-stable result to discuss the stability of Hopf bifurcation next section.

3 Applications to stability of Hopf bifurcation The existence of Hopf bifurcation and the stability of the bifurcating periodic orbits was studied in Hassard *et al.* [**26**] for ordinary differential equations and has been extended to various types of equations. We also refer to Desch and Schappacher [**18**] for an interesting stability result for nonlinear semigroups of continuous differential operators.

Recently, we [**33**] presented a Hopf bifurcation theorem for the non-densely defined abstract Cauchy problem

$$\begin{aligned}\frac{du(t)}{dt} &= Au(t) + F(\mu, u(t)) \\ &= f(\mu, u(t)), \quad \forall t \geq 0,\ u(0) = x \in \overline{D(A)},\end{aligned} \tag{3.1}$$

where $F : \mathbb{R} \times \overline{D(A)} \to X$ is C^k map with $k \geq 2$, and $\mu \in \mathbb{R}$ is the bifurcation parameter. The goal of this section is to use the center-stable theory to study the stability of Hopf bifurcation, i.e., stability of the bifurcating periodic orbits for abstract semilinear Cauchy problem (1.1) with nondense domain

In order to introduce the Hopf bifurcation theorem for parametrized differential equation (3.1), we need the following assumption.

Assumption 3.1. Let $\varepsilon > 0$ and $F \in C^k\left((-\varepsilon, \varepsilon) \times B_{X_0}(0, \varepsilon); X\right)$ for some $k \geq 4$. Assume that the following conditions are satisfied:

(a) $F(\mu, 0) = 0, \forall \mu \in (-\varepsilon, \varepsilon)$, and $\partial_x F(0, 0) = 0$.

(b) **(Transversality condition)** For each $\mu \in (-\varepsilon, \varepsilon)$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_x F(\mu, 0))_0$, denoted by $\lambda(\mu)$ and $\overline{\lambda(\mu)}$, such that

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu),$$

the map $\mu \to \lambda(\mu)$ is continuously differentiable,

$$\omega(0) > 0, \quad \alpha(0) = 0, \quad \frac{d\alpha(0)}{d\mu} \neq 0,$$

and

$$\sigma(A_0) \cap i\mathbb{R} = \left\{\lambda(0), \overline{\lambda(0)}\right\}.$$

(c) The essential growth rate of $\{T_{A_0}(t)\}_{t \geq 0}$ is strictly negative, that is,

$$\omega_{0,ess}(A_0) < 0.$$

The main result in [**33**] is the following Hopf bifurcation theorem.

Theorem 3.2 (Hopf Bifurcation)**.** *Let Assumptions* 2.1, 2.2 *and* 3.1 *be satisfied. Then there exist* $\varepsilon^* > 0$, *three* C^{k-1} *maps,* $\varepsilon \to \mu(\varepsilon)$ *from* $(0,\varepsilon^*)$ *into* $\mathbb{R}$, $\varepsilon \to x_\varepsilon$ *from* $(0,\varepsilon^*)$ *into* $\overline{D(A)}$, *and* $\varepsilon \to \gamma(\varepsilon)$ *from* $(0,\varepsilon^*)$ *into* $\mathbb{R}$, *such that for each* $\varepsilon \in (0,\varepsilon^*)$ *there exists a* $\gamma(\varepsilon)$*-periodic function* $u_\varepsilon \in C^k(\mathbb{R}, X_0)$, *which is an integrated solution of* (3.1) *with the parameter value equals* $\mu(\varepsilon)$ *and the initial value equals* x_ε. *So for each* $t \geq 0$, u_ε *satisfies*

$$u_\varepsilon(t) = x_\varepsilon + A\int_0^t u_\varepsilon(l)\,dl + \int_0^t F(\mu(\varepsilon), u_\varepsilon(l))\,dl.$$

Moreover, we have the following properties

(i) *There exist a neighborhood* N *of* 0 *in* X_0 *and an open interval* I *in* $\mathbb{R}$ *containing* 0, *such that for* $\widehat{\mu} \in I$ *and any periodic solution* $\widehat{u}(t)$ *in* N *with minimal period* $\widehat{\gamma}$ *close to* $\frac{2\pi}{\omega(0)}$ *of* (3.1) *for the parameter value* $\widehat{\mu}$, *there exists* $\varepsilon \in (0,\varepsilon^*)$ *such that* $\widehat{u}(t) = u_\varepsilon(t+\theta)$ *(for some* $\theta \in [0, \gamma(\varepsilon))$*),* $\mu(\varepsilon) = \widehat{\mu}$, *and* $\gamma(\varepsilon) = \widehat{\gamma}$.

(ii) *The map* $\varepsilon \to \mu(\varepsilon)$ *is a* C^{k-1} *function and we have the Taylor expansion*

$$\mu(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \mu_{2n}\varepsilon^{2n} + O(\varepsilon^{k-1}), \quad \forall \varepsilon \in (0,\varepsilon^*),$$

where $[\frac{k-2}{2}]$ *is the integer part of* $\frac{k-2}{2}$.

(iii) *The period* $\gamma(\varepsilon)$ *of* $t \to u_\varepsilon(t)$ *is a* C^{k-1} *function and*

$$\gamma(\varepsilon) = \frac{2\pi}{\omega(0)}\left[1 + \sum_{n=1}^{[\frac{k-2}{2}]} \gamma_{2n}\varepsilon^{2n}\right] + O(\varepsilon^{k-1}), \quad \forall \varepsilon \in (0,\varepsilon^*),$$

where $\omega(0)$ *is the imaginary part of* $\lambda(0)$ *defined in Assumption* 3.1.

In order to apply the reduction technics, we first incorporate the parameter into the state variable by considering the following system

$$\begin{cases} \dfrac{d\mu(t)}{dt} = 0, \\ \dfrac{du(t)}{dt} = Au(t) + F(\mu(t), u(t)), \\ (\mu(0), u(0)) = (\mu_0, u_0) \in (-\varepsilon, \varepsilon) \times \overline{D(A)}. \end{cases} \tag{3.2}$$

Note that F is only defined in a neighborhood of $(0,0) \in \mathbb{R}\times X$. In order to rewrite (3.2) as an abstract Cauchy problem, consider the Banach space $\mathcal{X} := \mathbb{R}\times X$ endowed with the usual product norm

$$\left\| \begin{pmatrix} \mu \\ x \end{pmatrix} \right\| = |\mu| + \|x\| ,$$

and the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{A}\begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ Ax + \partial_\mu F(0,0)\,\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \partial_\mu F(0,0) & A \end{pmatrix}\begin{pmatrix} \mu \\ x \end{pmatrix}$$

with

$$D(\mathcal{A}) = \mathbb{R}\times D(A).$$

Then

$$\mathcal{X}_0 := \overline{D(\mathcal{A})} = \mathbb{R}\times\overline{D(A)}.$$

Observe that by Assumption 3.1-(a) we have $\partial_x F(0,0) = 0$, and the linear operator $\mathcal{A}$ is the generator of the linearized equation of system (3.2) at $(0,0)$. Consider the function $\mathcal{F} : (-\varepsilon,\varepsilon)\times B_{X_0}(0,\varepsilon) \to \mathcal{X}$ defined by

$$\mathcal{F}\begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ F(\mu,x) - \partial_\mu F(0,0)\,\mu \end{pmatrix}.$$

Using the variable $v(t) = \begin{pmatrix} \mu(t) \\ u(t) \end{pmatrix}$, we can rewrite system (3.2) as the following abstract Cauchy problem

$$\frac{dv(t)}{dt} = \mathcal{A}v(t) + \mathcal{F}(v(t)), \quad t \geq 0,\ v(0) = v_0 \in \overline{D(\mathcal{A})}. \tag{3.3}$$

We first observe that $\mathcal{F}$ is defined on $B_{\mathcal{X}}(0,\varepsilon)$ and is k-time continuously differentiable with $k \geq 4$. Moreover, by using Assumption 3.1-(a), we have

$$\mathcal{F}(0) = 0 \quad \text{and} \quad D\mathcal{F}(0) = 0.$$

We now study the spectral properties of the linear operator $\mathcal{A}$. From Assumption 3.1-(b) and (c), we know that

$$\sigma(A_0) \cap i\mathbb{R} = \left\{\lambda(0), \overline{\lambda(0)}\right\} \quad \text{and} \quad \omega_{0,\mathrm{ess}}(A_0) < 0.$$

The following results are obtained in Liu *et al.* [**33**].

Lemma 3.3. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Then*

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_0) = \sigma(A_0) \cup \{0\} = \sigma(A) \cup \{0\},$$

and for each $\lambda \in \rho(\mathcal{A})$,

$$(\lambda I - \mathcal{A})^{-1} \begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} \lambda^{-1}\mu \\ (\lambda I - A)^{-1}\left[x + \partial_\mu F(0,0)\lambda^{-1}\mu\right] \end{pmatrix}.$$

Lemma 3.4. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. The linear operator* $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{R}\times X \to \mathbb{R}\times X$ *satisfies Assumptions* 2.1 *and* 2.2. *Moreover, we have*

$$T_{\mathcal{A}_0}(t)\begin{pmatrix} \mu \\ x \end{pmatrix} := \begin{pmatrix} \mu \\ T_{A_0}(t)x + S_A(t)\partial_\mu F(0,0)\mu \end{pmatrix}$$

and

$$S_{\mathcal{A}}(t)\begin{pmatrix} \mu \\ x \end{pmatrix} := \begin{pmatrix} t\mu \\ S_A(t)x + \int_0^t S_A(l)\partial_\mu F(0,0)\mu\, dl \end{pmatrix}.$$

Furthermore,

$$\omega_{0,\mathrm{ess}}(\mathcal{A}_0) = \omega_{0,\mathrm{ess}}(A_0).$$

Lemma 3.5. *Let Assumptions* 2.1, 2.2 *and* 3.1 *be satisfied. We have the following:*

(i) *The projector on the generalized eigenspace of* $\mathcal{A}$ *associated to*

$$\lambda_0 \in \{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}(\lambda) > 0\},$$

a pole of order k_0 *of the resolvent of* $\mathcal{A}$, *is given by*

$$B^{\mathcal{A}}_{-1,\lambda_0}\begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ B^{A}_{-1,\lambda_0}x + \sum\limits_{j=-k_0}^{-1} \frac{(-1)^{-1-j}}{\lambda_0^{-j}} B^{A}_{j,\lambda_0}\partial_\mu F(0,0)\mu \end{pmatrix}.$$

(ii) $\lambda(0)$ *and* $\overline{\lambda(0)}$ *are nonnull simple eigenvalues of* $\mathcal{A}$ *and the projectors on the generalized eigenspace of* $\mathcal{A}$ *associated to* $\lambda(0)$ *and* $\overline{\lambda(0)}$ *are given by*

$$B^{\mathcal{A}}_{-1,\gamma}\begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ B^{A}_{-1,\gamma}\left[x + \gamma^{-1}\partial_\mu F(0,0)\mu\right] \end{pmatrix}$$

for $\gamma = \lambda(0)$ *or* $\gamma = \overline{\lambda(0)}$.

The projector on the generalized eigenspace of $\mathcal{A}$ associated to 0 is given in the following lemma.

Lemma 3.6. *0 is a simple eigenvalue of $\mathcal{A}$ and the projector on the generalized eigenspace of $\mathcal{A}$ associated to 0 is given by*

$$B_{-1,0}^{\mathcal{A}} \begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} \mu \\ (-A)^{-1} \partial_\mu F(0,0)\,\mu \end{pmatrix}.$$

From the above results, we obtain a state space decomposition with respect to the spectral properties of the linear operator $\mathcal{A}$. More precisely, the projector on the unstable linear manifold is given by

$$\Pi_u^{\mathcal{A}} = \sum_{\lambda \in \sigma(A) : \mathrm{Re}(\lambda) > 0} B_{-1,\lambda}^{\mathcal{A}}$$

and the projector on the linear center manifold is defined by

$$\Pi_c^{\mathcal{A}} = B_{-1,0}^{\mathcal{A}} + B_{-1,\lambda(0)}^{\mathcal{A}} + B_{-1,\overline{\lambda(0)}}^{\mathcal{A}}.$$

Moreover, we have

$$\begin{aligned} \Pi_c^{\mathcal{A}} \begin{pmatrix} \mu \\ x \end{pmatrix} &= \begin{pmatrix} \mu \\ (-A)^{-1} \partial_\mu F(0,0)\,\mu \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 \\ B_{-1,\lambda(0)}^{A} \left[x + \lambda(0)^{-1} \partial_\mu F(0,0)\,\mu \right] \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 \\ B_{-1,\overline{\lambda(0)}}^{A} \left[x + \overline{\lambda(0)}^{-1} \partial_\mu F(0,0)\,\mu \right] \end{pmatrix} \\ &= \begin{pmatrix} \mu \\ \Pi_c^{A} x + \mu h \end{pmatrix}. \end{aligned}$$

Hence,

$$\Pi_c^{\mathcal{A}} \begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} \mu \\ \Pi_c^{A} x + \mu h \end{pmatrix},$$

where

$$\Pi_c^{A} = B_{-1,\lambda(0)}^{A} + B_{-1,\overline{\lambda(0)}}^{A}$$

and

$$h = (-A)^{-1}\,\partial_\mu F(0,0) + \lambda(0)^{-1}\, B^A_{-1,\lambda(0)}\partial_\mu F(0,0) \\ + \overline{\lambda(0)}^{-1} B^A_{-1,\overline{\lambda(0)}}\partial_\mu F(0,0).$$

Since $\Pi_c^{\mathcal{A}}$ is a projector, we must have $\Pi_c^A h = 0$. Therefore,

$$h \in R(I - \Pi_c^A).$$

Set

$$\Pi_s^{\mathcal{A}} := I - \left(\Pi_c^{\mathcal{A}} + \Pi_u^{\mathcal{A}}\right), \quad \Pi_h^{\mathcal{A}} = \Pi_s^{\mathcal{A}} + \Pi_u^{\mathcal{A}},$$
$$\mathcal{X}_{0c} = \Pi_c^{\mathcal{A}}\left(\mathbb{R}\times\overline{D(A)}\right) \quad \text{and} \quad \mathcal{X}_{0h} = \left(I - \Pi_c^{\mathcal{A}}\right)\left(\mathbb{R}\times\overline{D(A)}\right).$$

For each $r > 0$ we set

$$N_r := \left\{v \in \mathcal{X}_0 : \left\|\Pi_c^{\mathcal{A}}v\right\| \le r,\ \left\|\Pi_h^{\mathcal{A}}v\right\| \le r\right\}.$$

Since $\mathcal{F}$ is k-time continuously differentiable on $B_{\mathbb{R}\times X_0}(0,\varepsilon)$ with $k \ge 4$, we can find some $r_0 > 0$ such that $N_{2r_0} \subset B_{\mathbb{R}\times X_0}(0,\varepsilon)$. Set

$$\varrho(r) := \sup_{v\in N_r} \|D\mathcal{F}(v)\|, \quad \forall r \in [0, 2r_0].$$

Then we have

$$\varrho(r) \to 0 \text{ as } r \to 0$$

and

$$\|\mathcal{F}(v)\| \le 2r\varrho(r) \quad \text{for } v \in N_r.$$

Let $\chi_c : \mathcal{X}_{0c} \to [0,+\infty)$ be a C_b^k map with $\chi_c(v_c) \le 1$ and

$$\chi_c(v_c) = \begin{cases} 1, & \text{if } \|v_c\| \le 1, \\ 0, & \text{if } \|v_c\| \ge 2, \end{cases}$$

and $\chi_h : [0,+\infty) \to [0,+\infty)$ be a C_b^∞ map with $\chi_h(y) \le 1,\ \forall y \ge 0$, and

$$\chi_h(y) = \begin{cases} 1, & \text{if } |y| \le 1, \\ 0, & \text{if } |y| \ge 2. \end{cases}$$

Set

$$\chi(v) := \chi_c\left(\Pi_c^{\mathcal{A}}(v)\right)\chi_h\left(\left\|\Pi_h^{\mathcal{A}}(v)\right\|\right), \quad \forall v \in \mathcal{X}_0.$$

We see that $\chi \in C_b^0(\mathcal{X}_0;\mathbb{R}) \cap \mathrm{Lip}(\mathcal{X}_0;\mathbb{R})$. Set

$$\mathcal{F}_r(v) = \chi(r^{-1}v)\mathcal{F}(v), \quad \forall x \in \mathcal{X}_0,\ \forall r > 0.$$

Then $\mathcal{F}_r \in C_b^0(\mathcal{X}_0;\mathcal{X}) \cap \mathrm{Lip}(\mathcal{X}_0;\mathcal{X})$. Since $\mathcal{F}_r(v) = \mathcal{F}(v)\chi_c\left(r^{-1}\Pi_c^{\mathcal{A}}(v)\right)$ for $x \in V_r = \left\{x \in \mathcal{X}_0 : \left\|\Pi_h^{\mathcal{A}}(x)\right\| \le r\right\}$, we conclude that $\mathcal{F}_r|_{V_r} \in C_b^k(V_r;\mathcal{X})$ for each $r \in (0, r_0]$. One can also verify that

$$|\mathcal{F}_r|_0 \le 4r\varrho(2r)\,|\chi|_0 \quad \text{and} \quad |\mathcal{F}_r|_{\mathrm{Lip}} \le \varrho(2r)\big(\,|\chi|_0 + 4\,|\chi|_{\mathrm{Lip}}\,\big).$$

Now fix $r_* \in (0, r_0]$ sufficiently small such that $\mathcal{F}_{r_*}$ satisfies the condition $|\mathcal{F}_{r_*}|_{\mathrm{Lip}} < \delta_1$ in Theorem 2.12 and that

$$\frac{dv(t)}{dt} = \mathcal{A}v(t) + \mathcal{F}_{r_*}(v(t)), \quad t \ge 0,\ v(0) = v_0 \in \overline{D(\mathcal{A})}, \tag{3.4}$$

has the unique global center manifold

$$M_\psi = \{x_c + \psi(x_c) : x_c \in \mathcal{X}_{0c}\}, \quad \psi \in C_b^k(\mathcal{X}_{0c}, \mathcal{X}_{0h}) \tag{3.5}$$

with the properties described in Theorem 2.10. Let

$$\Omega_{r^*} := \left\{ \begin{pmatrix} \mu \\ x \end{pmatrix} \in \mathcal{X}_0 : \left\|\Pi_c^{\mathcal{A}} \begin{pmatrix} \mu \\ x \end{pmatrix}\right\| \le r_*,\ \ \left\|\Pi_h^{\mathcal{A}} \begin{pmatrix} \mu \\ x \end{pmatrix}\right\| \le r_* \right\}.$$

Then

$$\mathcal{F}_{r_*}(x) = \mathcal{F}(x) \text{ for each } x \in \Omega_{r^*},$$

and (3.5) is the local center manifold of (3.3) with the properties described in Theorem 2.11.

Note that (3.4) can be written as the following form

$$\begin{cases} \dfrac{d\mu(t)}{dt} = 0, \\ \dfrac{du(t)}{dt} = Au(t) + G(\mu(t), u(t)), \\ (\mu(0), u(0)) = (\mu_0, u_0) \in \mathbb{R} \times \mathbb{R}^n. \end{cases} \tag{3.6}$$

By applying formally $\Pi_c^{\mathcal{A}}$ to both sides of (3.3), we obtain the reduced system in $\mathcal{X}_{0c} = \Pi_c^{\mathcal{A}}(\mathbb{R} \times X)$:

$$(3.7)\quad \frac{d}{dt}\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix} = \mathcal{A}_{0c}\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix} + \Pi_c^{\mathcal{A}}\mathcal{F}\left(\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix} + \psi\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix}\right),$$

where

$$\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix} = \Pi_c^{\mathcal{A}}\begin{pmatrix}\mu(t)\\ u(t)\end{pmatrix}.$$

We denote the reduced system (3.7) as

$$(3.8)\qquad \begin{cases}\mu' = 0,\\ u_c'(t) = A_{0c}u_c(t) + G_c(\mu, u_c(t)) =: f(\mu, u_c).\end{cases}$$

Similarly, by applying $\Pi_c^{\mathcal{A}}$ to both sides of (3.4), we obtain the (globally) reduced system of (3.4) as

$$\frac{d}{dt}\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix} = \mathcal{A}_{0c}\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix} + \Pi_c^{\mathcal{A}}\mathcal{F}_{r_*}\left(\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix} + \psi\begin{pmatrix}\mu(t)\\ u_c(t)\end{pmatrix}\right).$$

As before, this system can be rewritten as

$$(3.9)\qquad \begin{cases}\mu' = 0,\\ u_c'(t) = A_{0c}u_c(t) + G_{c,r^*}(\mu, u_c(t)) =: f_{r^*}(\mu, u_c).\end{cases}$$

By the local invariance property of the center manifold, a family of bifurcating periodic solutions for (3.3) necessarily belongs to the center manifold and therefore corresponds to bifurcating periodic solutions for (3.8).

In the remaining part of this paper, we investigate how to relate the stability of the bifurcating periodic solutions for (3.3) to the stability of those solutions for its reduced system. Before presenting the main result, we first introduce the following definitions.

Definition 3.7. Let $\{V(t)\}_{t\geq 0}$ be a continuous semiflow on the Banach space $(E, \|.\|_E)$. Assume that $\{V(t)\}_{t\geq 0}$ admits a nontrivial ω-periodic orbit $\{p(t)\}_{t\in\mathbb{R}}$ (with $\omega > 0$), that is,

$$p(t) = V(t-s)p(s), \quad \forall t \geq s,$$

$$p(t+\omega) = p(t), \quad \forall t \in \mathbb{R},$$

and

$$p(t) \neq p(0), \quad \forall t \in (0, \omega).$$

Consider the orbit

$$\gamma := \bigcup_{t \in \mathbb{R}} \{p(t)\}.$$

Recall that the *Haussdorff semi-distance* is defined by

$$\delta(x, \gamma) := \inf_{y \in \gamma} \|x - y\|_{E}.$$

If there exists $\varsigma > 0$ such that for each $\varepsilon > 0$, there exists $\eta > 0$, such that for each $x \in N(\gamma, \eta) := \{x \in E : \delta(x, \gamma) < \eta\}$, there exists $\widehat{t} \in [0, \omega]$, such that

$$\left\| V(t)x - p(t + \widehat{t}) \right\| \leq \varepsilon e^{-\varsigma t}, \quad \forall t \geq 0,$$

then $p(t)$ is said to be *exponentially asymptotically stable*.

The main result of this section is the following theorem. Here we consider the center-stable case and investigate the relationship between the asymptotic stability of the periodic orbit of (3.3) close enough to the origin and the asymptotic stability of the corresponding periodic orbit for its reduced system. In the context of ordinary differential equations, this result is similar to the Pliss's reduction principle theorem (see Chow *et al.* [**10,** Theorem 4.13, p. 45]).

Theorem 3.8. *Let Assumptions* 2.1, 2.2 *and* 3.1 *be satisfied. Assume in addition that*

$$\sigma(A_0) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\}.$$

Let $\mu \in \mathbb{R}$. Assume that there exists an ω-periodic orbit $t \to p^{\mu}(t)$ of the abstract Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(\mu, u(t)), \quad t \geq 0, \ u(0) = x \in \overline{D(A)}, \tag{3.10}$$

and define

$$\begin{pmatrix} \mu \\ p_c^{\mu}(t) \end{pmatrix} := \Pi_c^{\mathcal{A}} \begin{pmatrix} \mu \\ p(t) \end{pmatrix}, \quad \forall t \in \mathbb{R}.$$

Then $t \to p_c^{\mu}(t)$ is an ω-periodic orbit of the reduced system

$$u_c'(t) = f(\mu, u_c(t)). \tag{3.11}$$

Moreover, if μ, $t \to p^{\mu}(t)$ *and* $t \to p_c^{\mu}(t)$ *are close enough to* 0, *then* $t \to p^{\mu}(t)$ *is exponentially asymptotically stable for* (3.10) *if and only if* $t \to p_c^{\mu}(t)$ *is exponentially asymptotically stable for* (3.11).

Proof. **Proof of the first implication** ($\Rightarrow$). Suppose that $p^{\mu}(t)$ is exponentially asymptotically stable for system (3.10) (which is close enough to 0). Then

$$\begin{pmatrix} \mu \\ p_c^{\mu}(t) \end{pmatrix} = \Pi_c^{\mathcal{A}} \begin{pmatrix} \mu \\ p^{\mu}(t) \end{pmatrix}$$

is a solution of the reduced system and $t \to p_c^{\mu}(t)$ is a periodic orbit of (3.11). Moreover, assume that

$$\begin{pmatrix} \mu \\ p_c^{\mu}(t) + v_c(t) \end{pmatrix} \text{ for } t \geq 0$$

is a positive orbit of the reduce system (which stays in some small neighborhood of 0 for any positive time), then

$$\begin{aligned} \begin{pmatrix} \mu \\ p^{\mu}(t) + v(t) \end{pmatrix} = & \begin{pmatrix} \mu \\ p_c^{\mu}(t) + v_c(t) \end{pmatrix} \\ & + \psi\left(\begin{pmatrix} \mu \\ p_c^{\mu}(t) + v_c(t) \end{pmatrix}\right) \text{ for } t \geq 0 \end{aligned}$$

is a solution of the system (3.3). Now since

$$\begin{aligned} & \begin{pmatrix} \mu \\ p_c^{\mu}(t) + v_c(t) \end{pmatrix} - \begin{pmatrix} \mu \\ p_c^{\mu}(t) \end{pmatrix} \\ & \quad = \begin{pmatrix} \mu \\ p^{\mu}(t) + v(t) \end{pmatrix} - \begin{pmatrix} \mu \\ p^{\mu}(t) \end{pmatrix} \\ & \qquad - \left[\psi\left(\begin{pmatrix} \mu \\ p_c^{\mu}(t) + v_c(t) \end{pmatrix}\right) - \psi\left(\begin{pmatrix} \mu \\ p_c^{\mu}(t) \end{pmatrix}\right)\right] \end{aligned}$$

and ψ is Lipschitz continuous, we have

$$\begin{aligned} & \left\| \begin{pmatrix} \mu \\ p_c^{\mu}(t) + v_c(t) \end{pmatrix} - \begin{pmatrix} \mu \\ p_c^{\mu}(t) \end{pmatrix} \right\| \\ & \quad \leq \left\| \begin{pmatrix} \mu \\ p^{\mu}(t) + v(t) \end{pmatrix} - \begin{pmatrix} \mu \\ p^{\mu}(t) \end{pmatrix} \right\| \end{aligned}$$

$$+\|\psi\|_{\mathrm{Lip}} \left\| \begin{pmatrix} \mu \\ p_c^\mu(t) + v_c(t) \end{pmatrix} - \begin{pmatrix} \mu \\ p_c^\mu(t) \end{pmatrix} \right\|.$$

By using the fact that the Lipschitz norm

$$\|\mathcal{F}_{r_*}\|_{\mathrm{Lip}} \to 0 \text{ as } r^* \to 0,$$

we deduce by Theorem 2.5 that

$$\|\psi\|_{\mathrm{Lip}} \to 0 \text{ as } r^* \to 0.$$

Therefore, we can fix $r^* > 0$ such that

$$\|\psi\|_{\mathrm{Lip}} < 1.$$

This implies that

$$\begin{aligned} &\left\| \begin{pmatrix} \mu \\ p_c^\mu(t) + v_c(t) \end{pmatrix} - \begin{pmatrix} \mu \\ p_c^\mu(t) \end{pmatrix} \right\| \\ &\qquad \le \left(1 - \|\psi\|_{\mathrm{Lip}}\right)^{-1} \left\| \begin{pmatrix} \mu \\ p^\mu(t) + v(t) \end{pmatrix} - \begin{pmatrix} \mu \\ p^\mu(t) \end{pmatrix} \right\|. \end{aligned}$$

Then it becomes clear that the exponentially asymptotic stability of the periodic orbit for the original system implies the exponentially asymptotic stability of the periodic orbit for the reduced system.

Proof of the converse implication ($\Leftarrow$). We first observe that since

$$\sigma(A_0) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \le 0\},$$

and by Assumption 3.1

$$\partial_x F(0,0) = 0,$$

we have

$$\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \le 0\}.$$

Therefore,

$$\Pi_u^{\mathcal{A}} = 0$$

and the center and the center-unstable manifold coincide, that is,

$$V_\eta^c = V_\eta^{cu}.$$

It follows that we can apply Theorems 2.12 and 2.15 to system (3.4) by using V_η^c instead of V_η^{cu}.

Suppose $p_c^\mu(t)$ is an exponentially asymptotically stable ω-periodic solution (which is close enough to 0) for system (3.11). Then

$$\begin{pmatrix} \mu \\ p^\mu(t) \end{pmatrix} = \begin{pmatrix} \mu \\ p_c^\mu(t) \end{pmatrix} + \psi\left(\begin{pmatrix} \mu \\ p_c^\mu(t) \end{pmatrix}\right)$$

is an ω-periodic orbit of system (3.10). Note that when μ is small enough and $p^\mu(t)$ is close enough to the origin, $p^\mu(t)$ is also the solution of the second equation in (3.6); that is,

$$\frac{du(t)}{dt} = Au(t) + G(\mu, u(t)). \tag{3.12}$$

In fact, we only need to prove that $p^\mu(t)$ is exponentially asymptotically stable as a solution of (3.12). Then restricting to Ω, we obtain that $p^\mu(t)$ is exponentially asymptotically stable as a solution of (3.10).

In the following we will prove that $p^\mu(t)$ is exponentially asymptotically stable as a solution of (3.12) by two steps. Firstly, we will prove that $p^\mu(t)$ is exponentially asymptotically stable on the space restricted to the center manifold V_η^c. Then we prove that $p^\mu(t)$ is exponentially asymptotically stable on the whole space X_0.

Step 1 (Exponential stability on the center manifold). In this part we consider the semiflow restricted to the center manifold and investigate the stability of the periodic orbit $\gamma := \bigcup_{t\in\mathbb{R}} \{p^\mu(t)\}$ on the center-unstable manifold V_η^{cu} (see Figure 1).

Let $u_0 \in X_0$ with $\widehat{v} = \begin{pmatrix} \mu \\ u_0 \end{pmatrix} \in V_\eta^c$. Since $\widehat{v} = \begin{pmatrix} \mu \\ u_0 \end{pmatrix} \in V_\eta^c$, $U_{\mathcal{F}_{r_*}}(t)\widehat{v} = \Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(t)\widehat{v} + \psi\left(\Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(t)\widehat{v}\right)$. Notice that $U_{\mathcal{F}_{r_*}}(t)\widehat{v} = \begin{pmatrix} \mu \\ u^\mu(t,u_0) \end{pmatrix}$, where $u^\mu(t,u_0)$ is the solution of (3.12) satisfying $u(0) = u_0$. We have $\Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(t)\widehat{v} = \Pi_c^{\mathcal{A}} \begin{pmatrix} \mu \\ u^\mu(t,u_0) \end{pmatrix} = \begin{pmatrix} \mu \\ \Pi_c^A u^\mu(t,u_0) + \mu h \end{pmatrix}$. Hence,

$$\left|\Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(0)\widehat{v} - \Pi_c^{\mathcal{A}} \begin{pmatrix} \mu \\ p^\mu(t) \end{pmatrix}\right| \le \left\|\Pi_c^{\mathcal{A}}\right\| \left|\widehat{v} - \begin{pmatrix} \mu \\ p^\mu(t) \end{pmatrix}\right|.$$

Let $\gamma_c := \bigcup_{t\in\mathbb{R}} \{p_c^\mu(t)\}$. By using the stability property of $p_c^\mu(t)$, we obtain that there exists $\varsigma > 0$ such that for each $\varepsilon > 0$, there exists $\eta > 0$, such

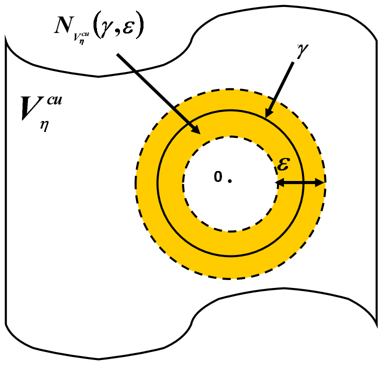

FIGURE 1: Schematic representation of a neighborhood of the periodic orbit γ on the center-unstable manifold V_η^{cu}.

that for each $\Pi_c^{\mathcal{A}} u_0 + \mu h \in N(\gamma_c, \eta)$, there exists $\widehat{t} \in [0, \omega]$, such that

$$\left| \Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(t)\widehat{v} - \begin{pmatrix} \mu \\ p_c^\mu(t+\widehat{t}) \end{pmatrix} \right| < \frac{\varepsilon e^{-\varsigma t}}{2(1+\|\psi\|_{\mathrm{Lip}})} \quad \text{for each } t \geq 0.$$

Thus, for each $u_0 \in N_{V_\eta^c}\left(\gamma, \frac{\eta}{\|\Pi_c^{\mathcal{A}}\|}\right)$ with $\widehat{v} = \begin{pmatrix} \mu \\ u_0 \end{pmatrix} \in V_\eta^c$, there exists $\widehat{t} \in [0, \omega]$, such that

$$\begin{aligned}
\left| u^\mu(t, u_0) - p^\mu(t+\widehat{t}) \right| &= \left| U_{\mathcal{F}_{r_*}}(t)\widehat{v} - \begin{pmatrix} \mu \\ p^\mu(t+\widehat{t}) \end{pmatrix} \right| \\
&\leq \left| \begin{array}{c} \Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(t)\widehat{v} + \psi\left(\Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(t)\widehat{v}\right) \\ -\left(\begin{pmatrix} \mu \\ p_c^\mu(t+\widehat{t}) \end{pmatrix} + \psi\begin{pmatrix} \mu \\ p_c^\mu(t+\widehat{t}) \end{pmatrix}\right) \end{array} \right| \\
&\leq (1+\|\psi\|_{\mathrm{Lip}}) \left| \Pi_c^{\mathcal{A}} U_{\mathcal{F}_{r_*}}(t)\widehat{v} - \begin{pmatrix} \mu \\ p_c^\mu(t+\widehat{t}) \end{pmatrix} \right| \\
&< \frac{\varepsilon e^{-\varsigma t}}{2} \quad \text{for each } t \geq 0.
\end{aligned}$$

Step 2 (Exponential stability on the whole space). Now we consider the general case. Since $\mathcal{F}_{r_*}$ satisfies the conditions in Theorem 2.12, there exists a Lipschitz continuous mapping $H_{cu} : \mathcal{X}_0 \to V_\eta^c$. Note

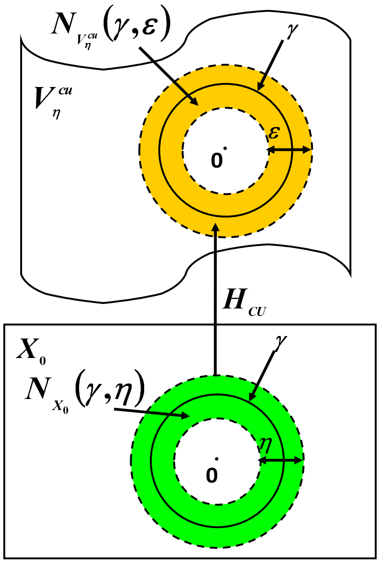

FIGURE 2: Schematic representation of the fact that $H_{cu} : X_0 \to V_\eta^{cu}$ maps a neighborhood of the periodic orbit γ in X_0 into a neighborhood of γ on the center-unstable manifold V_η^{cu}.

that

$$H_{cu}\begin{pmatrix} \mu \\ p^\mu(t) \end{pmatrix} = \begin{pmatrix} \mu \\ p^\mu(t) \end{pmatrix}, \quad t \in [0,\omega].$$

We deduce that there exists $\delta_1 > 0$ such that

$$H_{cu}\begin{pmatrix} \mu \\ N_{X_0}(\gamma,\delta_1) \end{pmatrix} \sqsubset \begin{pmatrix} \mu \\ N_{V_\eta^c}\left(\gamma, \frac{\eta}{\|\Pi_c^{\mathcal{A}}\|}\right) \end{pmatrix}.$$

Let $\overline{u}_0 \in X_0$ and $\overline{u}_0 \in N_{X_0}(\gamma,\delta_1)$ with $\overline{v} = \begin{pmatrix} \mu \\ \overline{u}_0 \end{pmatrix} \in \mathcal{X}_0$. Then $H_{cu}(\overline{v}) = \widehat{v} = \begin{pmatrix} \mu \\ u_0 \end{pmatrix} \in V_\eta^c$ and $u_0 \in N_{V_\eta^c}\left(\gamma, \frac{\eta}{\|\Pi_c^{\mathcal{A}}\|}\right)$. Furthermore, for any $\begin{pmatrix} \mu \\ p^\mu(\tau) \end{pmatrix}$, $\tau \in [0,\omega]$, we can find a $\delta_\tau > 0$ such that

$$\|U_F(t)v - U_F(t)H_{cu}(v)\| \le \frac{\varepsilon}{2} e^{-\eta t}, \quad \forall t \ge 0,$$

for all $v = \begin{pmatrix} \mu \\ u \end{pmatrix} \in \mathcal{X}_0$ with $\left\| v - \begin{pmatrix} \mu \\ p^\mu(\tau) \end{pmatrix} \right\| = \|u - p^\mu(\tau)\| < \delta_\tau$. By the compactness of γ there exists a finite subsystem $\{N\left(p^\mu(\tau_i), \delta_{\tau_i}\right);$ $i = 1, 2, \cdots, s\}$ which covers γ. Then there exists $0 < \delta < \delta_1$ such that $N_{X_0}(\gamma, \delta) \subset \bigcup_{i=1}^{s} N\left(p^\mu(\tau_i), \delta_{\tau_i}\right)$. Now we obtain that for any $\overline{u}_0 \in X_0$ and $\overline{u}_0 \in N_{X_0}(\gamma, \delta)$ with $\overline{v} = \begin{pmatrix} \mu \\ \overline{u}_0 \end{pmatrix} \in \mathcal{X}_0$, there exists $\widehat{t} \in [0, \omega]$, such that

$$
\begin{aligned}
&\left|u^\mu(t, \overline{u}_0) - p^\mu(t + \widehat{t})\right| \\
&\quad = \left| U_{\mathcal{F}_{r_*}}(t)\overline{v} - \begin{pmatrix} \mu \\ p^\mu(t + \widehat{t}) \end{pmatrix} \right| \\
&\quad \le \left|U_{\mathcal{F}_{r_*}}(t)\overline{v} - U_{\mathcal{F}_{r_*}}(t)\widehat{v}\right| + \left| U_{\mathcal{F}_{r_*}}(t)\widehat{v} - \begin{pmatrix} \mu \\ p^\mu(t + \widehat{t}) \end{pmatrix} \right| \\
&\quad = \left|U_{\mathcal{F}_{r_*}}(t)\overline{v} - U_{\mathcal{F}_{r_*}}(t)\widehat{v}\right| + \left|u^\mu(t, u_0) - p^\mu(t + \widehat{t})\right| \\
&\quad \le \frac{\varepsilon}{2} e^{-\eta t} + \frac{\varepsilon}{2} e^{-\varsigma t} \le \varepsilon e^{-\kappa t} \quad \text{for } t \ge 0,
\end{aligned}
$$

where $\kappa = \min\{\eta, \varsigma\}$ and $U_{\mathcal{F}_{r_*}}(t)\overline{v} = \begin{pmatrix} \mu \\ u^\mu(t, \overline{u}_0) \end{pmatrix}$ and $u^\mu(t, \overline{u}_0)$ is the solution of (3.12) satisfying $u(0) = \overline{u}_0$. This gives the stability property of $p^\mu(t)$ as a solution of (3.12). □

4 Application to equations with infinite delay In Section 1, we showed that the infinite delay equation (1.4) can be written as an abstract semilinear Cauchy problem (1.1) with nondense domain. Consider the linear operator A defined in (1.6) for the delay differential equation (1.4). The resolvent of A satisfies the following properties

$$(0, +\infty) \subset \rho(A)$$

and for each $\lambda > 0$,

$$(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix} \iff$$

$$\psi(\theta) = e^{\lambda\theta} \frac{[\varphi(0) + \alpha]}{\lambda} + \int_\theta^0 e^{\lambda(\theta - s)} \varphi(s)\, ds.$$

Moreover, we have for each $\lambda > 0$ that

$$\begin{aligned}
&\left\| (\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| \\
&\quad \leq \frac{\|\alpha\|}{\lambda} + \sup_{\theta \leq 0} e^{(\eta+\lambda)\theta} \left[\frac{\|\varphi(0)\|}{\lambda} + \int_\theta^0 e^{-\lambda s} \|\varphi(s)\| \, ds \right] \\
&\quad \leq \frac{\|\alpha\|}{\lambda} + \sup_{\theta \leq 0} e^{(\eta+\lambda)\theta} \left[\frac{\|\varphi(0)\|}{\lambda} + \int_\theta^0 e^{-(\lambda+\eta)s} ds \, \|\varphi\|_\eta \right] \\
&\quad = \frac{\|\alpha\|}{\lambda} + \sup_{\theta \leq 0} e^{(\eta+\lambda)\theta} \left[\frac{1}{\lambda} + \frac{e^{-(\lambda+\eta)\theta} - 1}{(\lambda+\eta)} \right] \|\varphi\|_\eta \\
&\quad \leq \frac{\|\alpha\|}{\lambda} + \sup_{\theta \leq 0} e^{(\eta+\lambda)\theta} \left[\frac{1}{\lambda} + \frac{e^{-(\lambda+\eta)\theta} - 1}{\lambda} \right] \|\varphi\|_\eta .
\end{aligned}$$

Hence,

$$\left\| (\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| \leq \frac{1}{\lambda} \left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\|.$$

Lemma 4.1. *The linear operator $A : D(A) \subset X \to X$ is a Hille-Yosida operator.*

Since A is Hille-Yosida, the abstract Cauchy problem (1.1) admits at most one weak solution, and by using the same arguments as in Liu *et al.* [**32**], we deduce that if $x : (-\infty, \tau] \to \mathbb{R}^n$ is a solution of (1.4), then

$$t \to \begin{pmatrix} 0_{\mathbb{R}^n} \\ x_t \end{pmatrix}$$

satisfies

$$\begin{aligned}
\begin{pmatrix} 0_{\mathbb{R}^n} \\ x_t \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + A \int_0^t \begin{pmatrix} 0_{\mathbb{R}^n} \\ x_s \end{pmatrix} ds& \\
+ \int_0^t \begin{pmatrix} G(x_s) \\ 0_{C_\eta} \end{pmatrix} ds \quad &\text{for all } t \geq 0.
\end{aligned}$$

Therefore, the problems (1.4) and (1.1) coincide.

The linear operator A_0 is defined by

$$A_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}$$

with
$$D(A_0) = \{0_{\mathbb{R}^n}\} \times C^1_{\eta,0}\left((-\infty,0],\mathbb{R}^n\right),$$
where
$$C^1_{\eta,0}\left((-\infty,0],\mathbb{R}^n\right) := \left\{\varphi \in C^1_{\eta}\left((-\infty,0],\mathbb{R}^n\right) : \varphi'(0) = 0\right\}.$$

Lemma 4.2. *The linear operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0. Moreover,*
$$T_{A_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{A_0}(t)\varphi \end{pmatrix},$$
where
$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = \begin{cases} \varphi(0), & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta), & \text{if } t+\theta \leq 0. \end{cases}$$

The only difference compared with the finite delay differential equations is the following property.

Lemma 4.3. *The essential growth rate of $\{T_{A_0}(t)\}_{t\geq 0}$ satisfies*
$$\omega_{0,\text{ess}}(A_0) \leq -\eta.$$

Proof. The linear operator $B : C_\eta\left((-\infty,0],\mathbb{R}^n\right) \to C_\eta\left((-\infty,0],\mathbb{R}^n\right)$ defined by
$$B(\varphi) = \varphi(0)$$
is compact. Therefore,
$$\left\|\widehat{T}_{A_0}(t)\right\|_{\text{ess}} = \left\|\widehat{T}_{A_0}(t) - B\right\|_{\text{ess}} \leq \left\|\widehat{T}_{A_0}(t) - B\right\|_{\mathcal{L}(C_\eta)}.$$
Now we have
$$\left\|\widehat{T}_{A_0}(t)\varphi - B\varphi\right\|_{\mathcal{L}(C_\eta)} = \sup_{\theta \leq -t} e^{\eta\theta}\left\|\varphi(t+\theta) - \varphi(0)\right\|.$$
Set $\sigma := t+\theta$, we obtain
$$\left\|\widehat{T}_{A_0}(t)\varphi - B\varphi\right\|_{\mathcal{L}(C_\eta)} = \sup_{\sigma \leq 0} e^{\eta(\sigma - t)}\left\|\varphi(\sigma) - \varphi(0)\right\|.$$
Hence,
$$\left\|\widehat{T}_{A_0}(t) - B\right\|_{\mathcal{L}(C_\eta)} \leq 2e^{-\eta t}$$
and the result follows. □

Therefore, by using similar arguments as in Liu *et al.* [**32**], we can provide a spectral theory for this class of problems. In particular, we can apply the center manifold theory in Magal and Ruan [**37**] and the center-unstable theory presented in section 2 around the equilibrium

$$0_X = \begin{pmatrix} 0_{\mathbb{R}^n} \\ 0_{C_\eta} \end{pmatrix}$$

of system (1.1). One can also obtain a Hopf bifurcation theorem as in Liu *et al.* [**33**] for the infinite delay equation (1.4), an example was presented in Auger and Ducrot [**5**], and the stability of Hopf bifurcation in such equations can be obtained using the results in Section 3.

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