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TWO-GROUP INFECTION AGE MODEL INCLUDING AN APPLICATION TO NOSOCOMIAL INFECTION*

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Abstract. In this article we analyze the global asymptotic behavior of a two-group SI (susceptible–infected) epidemic model with age of infection. We prove that the model exhibits the traditional threshold behavior where the disease-free equilibrium is globally asymptotically stable if the basic reproduction number is less than one, and the endemic equilibrium is globally asymptotically stable if the basic reproduction number is greater than one. We conclude the paper by presenting an application to nosocomial infections. Moreover some numerical simulations are presented for this application.

Key words. epidemic SI models, age of infection, global asymptotic stability

AMS subject classifications. 34K20, 92D30

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1. Introduction.

The modeling of epidemics began with Daniel Bernoulli [7] in 1760 to evaluate the effectiveness of variolation, which was being introduced to Europe as a means of conferring immunity to smallpox. Bernoulli's model included age-dependent mortality, with the goal of determining the change in life expectancy that would occur if smallpox were eliminated (see Dietz and Heesterbeek [18]). In 1916, Ross [49] studied a system of ordinary differential equations in order to build a theoretical framework for the mathematical analysis of epidemics of malaria; this was the origin of the modern susceptible–infected–recovered (SIR) compartmental model. In 1927, Kermack and McKendrick [32, 33, 34] extended Ross's ideas, proposing the cross quadratic term βSI. This linked the incidence to the sizes of the susceptible (S) and infective (I) populations and was based on a probabilistic analysis of the interactions between infective agents, or vectors, and hosts.

Epidemic models have been extended in several direction. We refer to Bailey [5], Brauer and Castillo-Chavez [8], Busenberg and Cooke [9], Capasso [10], Diekmann and Heesterbeek [16], Hethcote [29], Murray [47], Thieme [56], and the references cited therein for an overview of the topic.

The main body of our paper is devoted to the global analysis of a two-group version of the Kermack–McKendrick model, where both infectivity and recovery can depend on the duration of infection. As in the pioneer work of Kermack and McKendrick [32] (see Anderson [1] for a nice survey on Kermack–McKendrick models), we assume that each subgroup is divided into three classes: susceptible, infected, and recovered. For the infected population, we consider the age of infection, which is the time since individuals get infected. We consider a two-group model, where $S_j(t)$ and $R_j(t)$ give the number of susceptible and recovered individuals in group j at time t. The density of infected individuals in group j at time t that have been infected for duration a is given by $i_j(t,a)$.

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For $j = 1, 2$, the entering flux of new individuals into the jth group is λ_j, with all new individuals being susceptible, and the exit rate of individuals in group j is d_j. For infected individuals, $a \geq 0$ is the age of infection (i.e., the time since individuals were infected). For $j = 1, 2$, the recovery rate at infection age a is $m_j(a)$.

The primary motivation for the model being studied is nosocomial infection (i.e., infections taking place in a hospital setting), where the infection or contamination is passed between two groups: patients and health care workers (HCW), with no direct transmission within each group. For nosocomial infections, there are no recovered classes; however, we derive a separate model for nosocomial infections and show how that system can be transformed into a special case of the one studied in the main body of the paper. Our results demonstrate how different infection-age-dependent intervention strategies can be analyzed for effectiveness. This application is discussed in detail in section 7.

With the application to nosocomial infections in mind, we consider a *criss-cross* contamination process. Using mass action, new infections/contaminations in group j occur at incidence rate $S_j(t) \int_0^{+\infty} \beta_k(a) i_k(t,a) da$, where $j, k = 1, 2$ are distinct. The transfer flux is summarized in Figure 1.

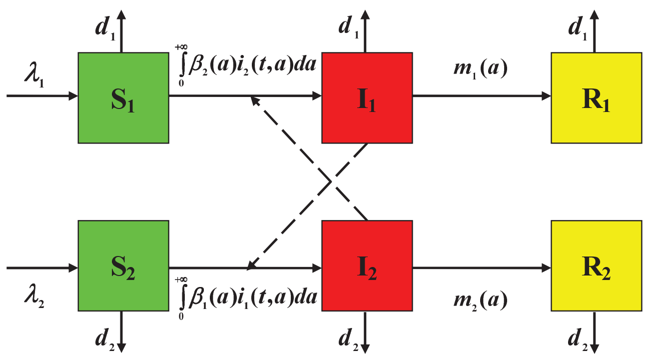

FIG. 1. *Diagram fluxes.*

We have the following two-group model with age of infection:

$$
\left\{
\begin{array}{l}
\dfrac{dS(t)}{dt} = \Lambda - DS(t) - \operatorname{diag}(S(t)) \int_0^{+\infty} B(a) i(t,a)\, da, \\
\dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\left[M(a) + D\right] i(t,a) \text{ for } a \geq 0, \\
i(t,0) = \operatorname{diag}(S(t)) \int_0^{+\infty} B(a) i(t,a)\, da, \\
\dfrac{dR(t)}{dt} = \int_0^{+\infty} M(a) i(t,a) da - DR(t), \\
S(0) = S_0 \in \mathbb{R}_+^2, \\
i(0,.) = i_0(.) \in L_+^1\left((0,+\infty), \mathbb{R}\right)^2, \\
R(0) = R_0 \in \mathbb{R}_+^2,
\end{array}
\right. \tag{1.1}
$$

where

$$
S(t) = \begin{pmatrix} S_1(t) \\ S_2(t) \end{pmatrix}, \quad i(t,a) = \begin{pmatrix} i_1(t,a) \\ i_2(t,a) \end{pmatrix}, \quad R(t) = \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix},
$$

$$
\Lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad M(a) = \begin{pmatrix} m_1(a) & 0 \\ 0 & m_2(a) \end{pmatrix},
$$

and the criss-cross transmission of the pathogen is described by

$$B(a) = \begin{pmatrix} 0 & \beta_2(a) \\ \beta_1(a) & 0 \end{pmatrix}.$$

We make the following assumption on the parameters.

ASSUMPTION 1.1. *We assume that*

(i) $\beta_1, \beta_2 \in L^\infty_+ ((0, +\infty), \mathbb{R}) \setminus \{0_{L^\infty}\}$;
(ii) $\lambda_1, \lambda_2 > 0$;
(iii) $d_1, d_2 > 0$;
(iv) $m_1, m_2 \in L^\infty_+ ((0, +\infty), \mathbb{R})$.

The function $\beta_j(a)$ can be regarded as the scaled probability of transmitting the contaminant to the other group for an individual that has been infected for a period of time $a > 0$. This function can have various shapes, and two prototype examples are presented in Figure 2.

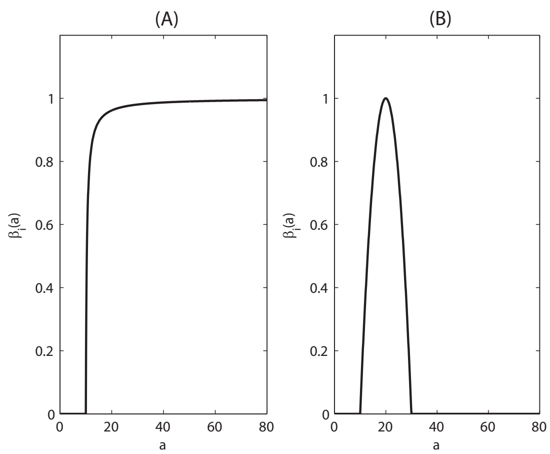

FIG. 2. *The function $\beta_i(a)$ describes the ability that infected individuals in group i have to transmit the contaminant to individuals in the other group.* (A) *corresponds to individuals that become infectious after a fixed period and remain infectious as long as they are present.* (B) *is similar but corresponds to scenarios in which individuals lose their infectiousness after a given duration.*

We now present important special cases that illustrate the utility of such systems.

Special case 1: Ordinary differential equations. Suppose

$$B(a) = B := \begin{bmatrix} 0 & \beta_2 \\ \beta_1 & 0 \end{bmatrix}$$

and

$$M(a) = M := \operatorname{diag}(m_1, m_2)$$

for all $a \geq 0$, with $\beta_1, \beta_2 > 0$ and $m_1, m_2 \geq 0$. Then we obtain the ordinary differential equation model

$$(1.2) \qquad \begin{cases} \dfrac{dS(t)}{dt} = \Lambda - DS(t) - \operatorname{diag}(S(t))\, BI(t), \\ \dfrac{dI(t)}{dt} = \operatorname{diag}(S(t))\, BI(t) - [M + D]\, I(t), \\ \dfrac{dR(t)}{dt} = MI(t) - DR(t), \\ S(0) = S_0 \in \mathbb{R}^2_+,\ I(0) = I_0 \in \mathbb{R}^2_+,\ R(0) = R_0 \in \mathbb{R}^2_+, \end{cases}$$

where

$$I(t) := \int_0^{+\infty} i(t,a)da.$$

The global asymptotic behavior of system (1.2) (and the n group case) has been studied in [25].

Special case 2: Delay. Suppose that for $j = 1, 2$ there exists $\tau_j > 0$ such that

$$\beta_j(a) = \beta_j 1_{[\tau_j,+\infty)}(a) \quad \text{for almost every } a \geq 0$$

and

$$M(a) = M = \operatorname{diag}(m_1, m_2) \quad \forall a \geq 0,$$

with $\beta_1, \beta_2 > 0$ and $m_1, m_2 \geq 0$. The total number of infectious individuals in group j is given by

$$I_j(t) = \int_{\tau_j}^{+\infty} i_j(t,a)\, da \quad \text{for } j = 1, 2.$$

For each $t \geq \max\{\tau_1, \tau_2\}$, we obtain the following system of delay differential equations:

$$\begin{cases} \dfrac{dS_1(t)}{dt} = \lambda_1 - d_1 S_1(t) - \beta_2 S_1(t) I_2(t), \\ \dfrac{dS_2(t)}{dt} = \lambda_2 - d_2 S_2(t) - \beta_1 S_2(t) I_1(t), \\ \dfrac{dI_1(t)}{dt} = \beta_2 e^{-(m_1+d_1)\tau_1} S_1(t-\tau_1) I_2(t-\tau_1) - (m_1+d_1)\, I_1(t), \\ \dfrac{dI_2(t)}{dt} = \beta_1 e^{-(m_2+d_2)\tau_2} S_2(t-\tau_2) I_1(t-\tau_2) - (m_2+d_2)\, I_2(t), \\ \dfrac{dR_1(t)}{dt} = m_1 I_1(t) - d_1 R_1(t), \\ \dfrac{dR_2(t)}{dt} = m_2 I_2(t) - d_2 R_2(t). \end{cases} \tag{1.3}$$

Further specializations of the delay case include the Ross–MacDonald model with delay, when the size of the kth group is constant, and an extended Ross–MacDonald model with delay. The above model is similar to the one found in Ruan, Xiao, and Beier [51].

One of the earliest multigroup epidemic models was proposed by Lajmanovich and Yorke [35] for the transmission of gonorrhea. The asymptotic stability of the endemic equilibrium for various multigroup epidemic models was analyzed in [6, 28, 30, 31, 38, 53, 56]; typically, however, the global behavior is not resolved for the full parameter space.

More recently, the global asymptotic behavior of certain multigroup models without age of infection has been completely resolved. In [25], Guo, Li and Shuai studied the behavior of a multigroup SIR model formulated in terms of ordinary differential equations. They perform a delicate analysis involving the weighted graph structure induced by the connections between the groups. Using arguments coming from the theory of positive linear operators (i.e., properties of irreducible linear C_0-semigroups), Thieme [57] extended this to allow for a continuum of groups, as would be the case for a nonmoving population spread over a spatial domain. Section 3 of this article is also devoted to the analysis of the linear problem.

In [36], Li and Shuai study coupled systems on networks, including a multigroup susceptible–infected (SI) distributed delay model. In [37], Li, Shuai, and Wang studied a model which is a multigroup analogue of the susceptible–exposed–infective–recovered (SEIR) age of infection model given by Rost and Wu in [50]; that model reduces to an infinite delay susceptible–exposed (SE) model. In each of these works, a Volterra-type Lyapunov functional is used to demonstrate that the endemic equilibrium is globally attracting (within an appropriate set) when it exists. These Lyapunov functionals are based on the ones used by McCluskey in [45] and [46] for single group models.

A single-group age of infection epidemic model was first considered by Kermack and McKendrick in [32]. The analysis of the local and the global asymptotic behavior of that and similar SI models has been studied in [14, 40, 58, 59]. In [42], the global behavior of a single-group age of infection model was fully resolved for all parameter values. It was shown that the model exhibits the traditional threshold behavior where the disease-free equilibrium is globally asymptotically stable if the basic reproduction number is less than one, and the endemic equilibrium is globally asymptotically stable if the basic reproduction number is greater than one. A key tool was a Volterra-type Lyapunov functional that included an integral over all infection ages, similar to the functionals used in [45, 46].

As for the single-class model, the Lyapunov functional used here is not defined on the entire state space. One may observe that similar difficulty also arises in [14, 42, 45, 57]. To circumvent this difficulty, we only use the Lyapunov functional on the global attractor for the semiflow restricted to the interior region. In general such a global attractor exists only in the sense of Magal and Zhao [44], and the uniform persistence property is required. Here we prove the uniform persistence by applying Theorem 4.2 in Hale and Waltman [27]. We also refer the reader to the books of Zhao [65] and Smith and Thieme [52] for more results on uniform persistence.

In sections 4 and 5 we analyze the extinction and uniform persistence based on comparison arguments for the i-equation with a linear system. Therefore, the first main difficulty for a system with age of infection is to understand the behavior (i.e., the spectral properties) of the linear system

$$\begin{cases} \dfrac{\partial i(t,a)}{\partial t}+\dfrac{\partial i(t,a)}{\partial a}=-\left[M(a)+D\right]i(t,a) \text{ for } a\geq 0,\\ i(t,0)=\operatorname{diag}\left(\overline{S}\right)\int_0^{+\infty}B(a)i\left(t,a\right)da,\\ i(0,.)=i_0\left(.\right)\in L^1_+\left(\left(0,+\infty\right),\mathbb{R}\right)^2, \end{cases}$$

where

$$\overline{S}=\left(\overline{S}_1,\overline{S}_2\right)\in\left(0,\infty\right)^2$$

is fixed.

One may observe that the analysis of the linear problem is more intricate than for a single-group model. A similar question was considered for chronological age by Feng and coauthors [21, 22, 23]. However, the current problem is not the same since we are considering infection age, and so it is necessary to provide a new analysis of the spectral properties of the above linear system. Also, in order to apply the Lyapunov techniques, we need more detailed analysis than just the irreducibility of the linear semigroup (see section 3), making the linear problem even more delicate to analyze.

For an n group system, our analysis can be extended to the case where $B(a)$ has the Leslie matrix form

$$B(a) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & \beta_n(a) \\ \beta_1(a) & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \beta_2(a) & 0 & & & 0 \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_{n-1}(a) & 0 \end{bmatrix}.$$

But since this article is motivated by an application to nosocomial infections (section 7), we restrict ourselves to the case where $n = 2$. Nevertheless, we basically provide the main ingredients for the linear problem described in section 3, as well as the Lyapunov function described in section 6, and our analysis can be extended to the case of an n group system having the above Leslie matrix form. This article can be regarded as a first step in analyzing multigroup systems with age of infection; further analysis is needed to understand the general case (i.e., with a full coupling in the matrix $B(a)$).

The plan of the paper is the following. In section 2, we reformulate the system as a Volterra equation and as a nondensely defined semilinear Cauchy problem (in order to apply integrated semigroup theory). We show that the system is dissipative, and we find the equilibria. In section 3, we study the irreducibility and some spectral properties of the linear problem. In section 4, we show that the disease dies out of the population if the basic reproduction number is less than one, and we show in section 5 that the disease is uniformly persistent if the basic reproduction number is greater than one. In section 6, we show that the endemic equilibrium is globally asymptotically stable when it exists. In section 7, we apply the results of the analysis to a model of nosocomial infection, where pathogens are transmitted back and forth between patients and HCWs.

2. Preliminary.

Dissipativity of the system. Let $I(t) = \int_0^\infty i(t,a)da$. By integrating the i-equation of system (1.1) with respect to the age a, we obtain

$$\frac{dI(t)}{dt} = \text{diag}(S(t)) \int_0^{+\infty} B\,(a)\, i(t,a)da - \int_0^{+\infty} M(a) i(t,a)da - DI(t).$$

Letting

$$N(t) = S(t) + I(t) + R(t),$$

we deduce that $N(t)$ satisfies the ordinary differential equation

$$\frac{dN(t)}{dt} = \Lambda - DN(t) \tag{2.1}$$

and therefore

$$\lim_{t\to+\infty} N(t) = D^{-1}\Lambda.$$

Furthermore, if $N(t) \leq D^{-1}\Lambda$ is satisfied for some $t = t_0 \in \mathbb{R}$, then it is satisfied for all $t \geq t_0$.

Since the R-equation can be decoupled, from here on we focus on the vector-valued SI model with age of infection:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - DS(t) - \mathrm{diag}(S(t)) \int_0^{+\infty} B(a)\, i(t,a) da, \\ (\partial_t + \partial_a)\, i(t,a) = -\left[D + M(a)\right] i(t,a), \\ i(t,0) = \mathrm{diag}(S(t)) \int_0^{+\infty} B(a)\, i(t,a) da, \\ S(0) = S_0 \in \mathbb{R}_+^2, \\ i(0,.) = i_0 \in L_+^1\left((0,+\infty), \mathbb{R}^2\right). \end{cases} \tag{2.2}$$

This system leaves the set

$$\left\{(S,i) \in \mathbb{R}_+^2 \times L_+^1\left((0,+\infty), \mathbb{R}^2\right) : S + \int_0^{+\infty} i(a) da \leq D^{-1}\Lambda\right\}$$

positively invariant.

Equation (2.2) can be reformulated as a Volterra equation and as an integrated semigroup. Each of these approaches is useful as there are many results pertaining to each formulation.

Volterra formulation. We write the problem as the following Volterra-type equation:

$$\frac{dS(t)}{dt} = \Lambda - DS(t) - \mathrm{diag}(S(t)) \int_0^{+\infty} B(a)\, i(t,a) da,$$

$$i(t,a) = \begin{cases} e^{-\int_{a-t}^{a} D+M(l)dl} i_0(a-t) \text{ if } a-t \geq 0, \\ e^{-\int_0^a D+M(l)dl} W(t-a) \text{ if } a-t < 0, \end{cases}$$

where the mapping $t \to W(t)$ for $t \geq 0$ is the unique continuous solution of the nonlinear Volterra integral equation

$$\begin{aligned} W(t) = \mathrm{diag}(S(t)) \Bigg[&\int_t^{+\infty} B(a)\, e^{-\int_{a-t}^{a} D+M(l)dl} i_0(a-t) da \\ &+ \int_0^t B(a) e^{-\int_0^a D+M(l)dl} W(t-a) da \Bigg]. \end{aligned} \tag{2.3}$$

This is derived using

$$W(t) = \mathrm{diag}(S(t)) \int_0^{+\infty} B(a)\, i(t,a) da = i(t,0)$$

for all $t \geq 0$. We note that solutions to (2.3) exist since $B \in L_+^\infty$, $i_0 \in L_+^1$, and $D+M$ is a diagonal matrix with diagonal entries bounded away from 0 so that $e^{-\int D+M(l)dl}$ is of exponential order.

Integrated semigroup formulation. We now use the approach introduced by Thieme in [54] to reformulate the problem as a semilinear Cauchy problem. In order to take care of the boundary condition, we extend the state space by considering

$$\mathcal{X} = \mathbb{R}^2 \times \mathbb{R}^2 \times L^1\left((0,+\infty), \mathbb{R}^2\right),$$

endowed with the usual product norm, and set

$$\mathcal{X}_0 := \mathbb{R}^2 \times \{0_{\mathbb{R}^2}\} \times L^1\left((0,+\infty), \mathbb{R}^2\right),$$
$$\mathcal{X}_+ := \mathbb{R}^2_+ \times \mathbb{R}^2_+ \times L^1_+\left((0,+\infty), \mathbb{R}^2\right),$$

and

$$\mathcal{X}_{0+} := \mathcal{X}_0 \cap \mathcal{X}_+.$$

We consider the linear operator $A : \mathrm{Dom}(A) \subset \mathcal{X} \to \mathcal{X}$ defined by

$$A\left(\begin{array}{c} S \\ \left(\begin{array}{c} 0_{\mathbb{R}^2} \\ i \end{array}\right) \end{array}\right) = \left(\begin{array}{c} -DS \\ \left(\begin{array}{c} -i(0) \\ -i' - [D+M]\, i \end{array}\right) \end{array}\right)$$

with

$$\mathrm{Dom}(A) = \mathbb{R}^2 \times \{0_{\mathbb{R}^2}\} \times W^{1,1}\left((0,+\infty), \mathbb{R}^2\right),$$

where $W^{1,1}$ is a Sobolev space, and we define $F : \mathcal{X}_0 \to \mathbb{R}^2$ by

$$F\left(\begin{array}{c} S \\ \left(\begin{array}{c} 0_{\mathbb{R}^2} \\ i \end{array}\right) \end{array}\right) = \left(\begin{array}{c} \Lambda - \mathrm{diag}(S(t)) \int_0^{+\infty} B(a)\, i(t,a)da \\ \left(\begin{array}{c} \mathrm{diag}(S(t)) \int_0^{+\infty} B(a)\, i(t,a)da \\ 0_{L^1} \end{array}\right) \end{array}\right).$$

Then by defining

$$v(t) = \left(\begin{array}{c} S(t) \\ \left(\begin{array}{c} 0_{\mathbb{R}^2} \\ i(t) \end{array}\right) \end{array}\right),$$

we can reformulate the partial differential equation problem (2.2) as the following abstract Cauchy problem:

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)) \text{ for } t \geq 0 \quad \text{and} \quad v(0) = x \in \mathcal{X}_{0+}. \tag{2.4}$$

By using the results in Thieme [54] and Magal [39] (see Magal and Ruan [41] for more results), we derive the existence and the uniqueness of the semiflow $\{U(t)\}_{t\geq 0}$ on $\mathcal{X}_{0+}$. By identifying $(S(t), 0_{\mathbb{R}^2}, i(t,.))$ with $(S(t), i(t,.))$ it can be proved that this semiflow coincides with the one generated by using the Volterra integral formulation. Moreover, by using (2.1), we deduce that the set

$$\widetilde{B} = \left\{\left(\begin{array}{c} S \\ \left(\begin{array}{c} 0_{\mathbb{R}^2} \\ i \end{array}\right) \end{array}\right) \in \mathcal{X}_{0+} : S + \int_0^{+\infty} i(a)da \leq D^{-1}\Lambda\right\}$$

is a positively invariant absorbing set under U, that is to say that

$$U(t)\widetilde{B} \subseteq \widetilde{B},$$

and for each $x = (S_0, 0_{\mathbb{R}^2}, i_0) \in \mathcal{X}_{0+}$,

$$d\left(U(t)x, \widetilde{B}\right) := \inf_{y\in\widetilde{B}} \|U(t)x - y\| \to 0 \text{ as } t \to +\infty.$$

It follows that $\{U(t)\}_{t\geq 0}$ is bounded dissipative on $\mathcal{X}_{0+}$ (see Hale [26]). Furthermore, the semiflow is asymptotically smooth (see Webb [61], Magal and Thieme [43], and Thieme and Vrabie [60]). As a consequence of the results on the existence of global attractors in Hale [26], we obtain the following theorem.

THEOREM 2.1. *Let Assumption* 1.1 *be satisfied. Problem* (2.4) *generates a unique continuous semiflow* $\{U(t)\}_{t\geq 0}$ *on* $\mathcal{X}_{0+}$ *that is asymptotically smooth and bounded dissipative. Therefore, U has a global attractor $\mathcal{A}$ in $\mathcal{X}_{0+}$ (which attracts the bounded sets of $\mathcal{X}_{0+}$).*

The attractor $\mathcal{A}$ consists of complete orbits of U, meaning that for any point $(S_0, 0_{\mathbb{R}^2}, i_0) \in \mathcal{A}$, there is a solution $\{(S(t), 0_{\mathbb{R}^2}, i(t, .)) : t \in \mathbb{R}\}$ of (2.4) which passes through $(S_0, 0_{\mathbb{R}^2}, i_0)$ at time $t = 0$.

Further estimate. Without loss of generality, we may restrict ourselves to the subdomain $\widetilde{B}$. Hence, we may assume that

$$S(t) + I(t) \leq D^{-1}\Lambda \quad \forall t \geq 0.$$

Then we obtain

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda - DS(t) - \operatorname{diag}(S(t)) \int_0^{+\infty} B\,(a)\, i(t, a)da \\ &\geq \Lambda - DS(t) - \|B\|_\infty \operatorname{diag}(S(t))PI(t),\end{aligned}$$

where

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \|B\|_\infty = \max\left\{\|\beta_1\|_\infty, \|\beta_2\|_\infty\right\}.$$

Since $I(t) \leq D^{-1}\Lambda$, we find

$$\begin{aligned}\frac{dS(t)}{dt} &\geq \Lambda - DS(t) - \|B\|_\infty \operatorname{diag}(S(t))PD^{-1}\Lambda \\ &= \Lambda - DS(t) - \|B\|_\infty \operatorname{diag}\left(PD^{-1}\Lambda\right) S(t).\end{aligned}$$

Letting

$$\widehat{D} := D + \|B\|_\infty \operatorname{diag}\left(PD^{-1}\Lambda\right),$$

we obtain

$$\frac{dS(t)}{dt} \geq \Lambda - \widehat{D}S(t), \tag{2.5}$$

and hence we obtain the following result.

LEMMA 2.2. *Let Assumption* 1.1 *be satisfied. The domain*

$$\widehat{B} = \left\{ \begin{pmatrix} S \\ \begin{pmatrix} 0_{\mathbb{R}^2} \\ i \end{pmatrix} \end{pmatrix} \in \mathcal{X}_{0+} : S \geq \widehat{D}^{-1}\Lambda \;\text{ and } S + \int_0^{+\infty} i(a)da \leq D^{-1}\Lambda \right\}$$

is positively invariant and is an absorbing set for U restricted to $\widetilde{B}$. Furthermore, $\mathcal{A} \subseteq \widehat{B}$.

Equilibrium solution. An equilibrium $\left(\overline{S}, \overline{\imath}\right) \in \mathbb{R}^2 \times W^{1,1}\left((0,+\infty), \mathbb{R}^2\right)$ with $\overline{S} \geq 0$ and $\overline{\imath} \geq 0$ must satisfy the following system:

$$\left\{\begin{array}{l} 0 = \Lambda - D\overline{S} - \operatorname{diag}(\overline{S}) \int_0^{+\infty} B(a)\, \overline{\imath}(a) da, \\ \overline{\imath}'(a) = -\left[D + M(a)\right] \overline{\imath}(a), \\ \overline{\imath}(0) = \operatorname{diag}(\overline{S}) \int_0^{+\infty} B(a)\, \overline{\imath}(a) da. \end{array}\right. \tag{2.6}$$

By using the second equation of system (2.6), we have

$$\overline{\imath}(a) = e^{-\int_0^a D + M(l) dl} \overline{\imath}(0).$$

Substituting into the third equation of system (2.6), we obtain

$$\overline{\imath}(0) = \operatorname{diag}(\overline{S}) \left(\int_0^{+\infty} B(a)\, e^{-\int_0^a D+M(l)dl} da\right) \overline{\imath}(0).$$

On the other hand, by combining the third and first equations of system (2.6) and solving for $\overline{S}$, we obtain

$$\overline{S} = D^{-1}\left(\Lambda - \overline{\imath}(0)\right).$$

Thus, we obtain the fixed point problem $\overline{\imath}(0) \in \mathbb{R}_+^2$,

$$\overline{\imath}(0) = \operatorname{diag}\left(D^{-1}\left(\Lambda - \overline{\imath}(0)\right)\right) G\, \overline{\imath}(0), \tag{2.7}$$

where

$$\begin{aligned} G &:= \int_0^{+\infty} B(a)\, e^{-\int_0^a D+M(l)dl} da \\ &= \begin{bmatrix} 0 & \int_0^{+\infty} \beta_2(a)\, e^{-\int_0^a d_2 + m_2(l) dl} da \\ \int_0^{+\infty} \beta_1(a)\, e^{-\int_0^a d_1 + m_1(l) dl} da & 0 \end{bmatrix}. \end{aligned}$$

In order to give the equilibria, we first state the basic reproduction number $\mathcal{R}_0$, which can be calculated using the next generation method of [17]. We have

$$\mathcal{R}_0 = \sqrt{\mathcal{R}_1 \mathcal{R}_2}, \tag{2.8}$$

where

$$\mathcal{R}_i := \frac{\lambda_i \int_0^{+\infty} \beta_i(a)\, e^{-\int_0^a d_i + m_i(l) dl} da}{d_i} \quad \text{for } i = 1, 2.$$

We obtain the following result.

Lemma 2.3. *We have the following alternative:*

(i) *If $\mathcal{R}_0 \leq 1$, then the disease-free equilibrium*

$$\overline{E}_F = \left(\overline{S}_F, \overline{\imath}_F\right) = \left(D^{-1}\Lambda, 0_{L^1(0,+\infty)}\right)$$

is the unique equilibrium.

(ii) *If $\mathcal{R}_0 > 1$, then there are two equilibria: the disease-free equilibrium*

$$\overline{E}_F = \left(\overline{S}_F, \bar{\imath}_F\right) = \left(D^{-1}\Lambda, 0_{L^1(0,+\infty)}\right)$$

and the endemic equilibrium

$$E^* = (S^*, i^*) = \left(D^{-1}\left(\Lambda - i^*(0)\right), e^{-\int_0^a D+M(l)dl} i^*(0)\right),$$

where $i^*(0) \in (0,+\infty)^2$ is the unique positive solution of system (2.7).

Proof. To prove the result, it is sufficient to observe that system (2.7) is equivalent to finding $(i_1^*(0), i_2^*(0)) \in (0, \lambda_1) \times (0, \lambda_2)$ such that

$$\left(\int_0^{+\infty} \beta_2(a) e^{-\int_0^a d_2+m_2(l)dl} da\right)^{-1} \frac{d_1 i_1^*(0)}{\lambda_1 - i_1^*(0)} = i_2^*(0)$$

and

$$\left(\int_0^{+\infty} \beta_1(a) e^{-\int_0^a d_1+m_1(l)dl} da\right)^{-1} \frac{d_2 i_2^*(0)}{\lambda_2 - i_2^*(0)} = i_1^*(0).$$

On considering these curves in the $i_1^*(0)i_2^*(0)$-plane, and searching for intersections, the result follows. ∎

3. The linear problem. In this section, we investigate some properties of the i-equation whenever the mapping $t \to S(t)$ is constant. Therefore, we consider the linear age-structured system

$$\left\{\begin{array}{l} (\partial_t + \partial_a) i(t,a) = -\left[D + M(a)\right] i(t,a) \text{ for } a \geq 0, \\ i(t,0) = \bar{S}\int_0^{+\infty} B(a) i(t,a) da, \\ i(0,.) = i_0 \in L^1_+\left((0,+\infty), \mathbb{R}\right)^2, \end{array}\right. \tag{3.1}$$

where

$$\bar{S} = \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix} \quad \text{with} \quad \bar{S}_1, \bar{S}_2 > 0. \tag{3.2}$$

We need to study certain properties of system (3.1) in order to apply comparison arguments in section 5, which deals with uniform persistence. We consider the Banach space

$$\mathcal{Y} := \mathbb{R}^2 \times L^1\left((0,+\infty), \mathbb{R}^2\right),$$

endowed with the usual product norm, and

$$\mathcal{Y}_0 := \{0_{\mathbb{R}^2}\} \times L^1\left((0,+\infty), \mathbb{R}^2\right).$$

We also consider the usual positive cone

$$\mathcal{Y}_{0+} := \{0_{\mathbb{R}^2}\} \times L^1_+\left((0,+\infty), \mathbb{R}^2\right).$$

Let $\widehat{A} : \mathrm{Dom}(\widehat{A}) \subset \mathcal{Y} \to \mathcal{Y}$ be the linear operator defined by

$$\widehat{A}\begin{pmatrix} 0_{\mathbb{R}^2} \\ i \end{pmatrix} = \begin{pmatrix} -i(0) \\ -i' - [D+M] i \end{pmatrix}$$

with

$$\mathrm{Dom}(\widehat{A}) = \{0_{\mathbb{R}^2}\} \times W^{1,1}\left((0,+\infty), \mathbb{R}^2\right).$$

Then

$$\overline{\mathrm{Dom}(\widehat{A})} = \mathcal{Y}_0.$$

Let $L : \overline{\mathrm{Dom}(\widehat{A})} \to \mathcal{Y}$ be the linear operator defined by

$$L \begin{pmatrix} 0_{\mathbb{R}^2} \\ i \end{pmatrix} = \begin{pmatrix} \bar{S} \int_0^{+\infty} B(a)\, i(a) da \\ 0_{L^1} \end{pmatrix}.$$

Then by defining $u(t) = \binom{0_{\mathbb{R}^2}}{i(t,.)}$, we can rewrite system (3.1) as the following abstract Cauchy problem:

$$\frac{du(t)}{dt} = \left(\widehat{A} + L\right) u(t) \text{ for } t \geq 0 \quad \text{and} \quad u(0) = x \in \overline{\mathrm{Dom}(\widehat{A})}.$$

Note that [20] provides a discussion of how the notion of solutions to this Cauchy problem can be extended from $\mathrm{Dom}(\widehat{A})$ to its closure $\overline{\mathrm{Dom}(\widehat{A})}$.

Let $\left(\widehat{A} + L\right)_0$ be the part of $\widehat{A} + L$ in $\overline{\mathrm{Dom}(\widehat{A})}$. Recall that $\left(\widehat{A} + L\right)_0$ is the restriction of $\widehat{A} + L$ to those elements whose image is in $\mathcal{Y}_0$; that is,

$$\left(\widehat{A} + L\right)_0 \begin{pmatrix} 0_{\mathbb{R}^2} \\ i \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ -i' - [D + M]\, i \end{pmatrix}$$

and

$$\mathrm{Dom}\left(\left(\widehat{A} + L\right)_0\right) = \left\{ \begin{pmatrix} 0_{\mathbb{R}^2} \\ i \end{pmatrix} \in \mathrm{Dom}(\widehat{A}) : i(0) = \int_0^{+\infty} B(a)\, i(a) da \right\}.$$

Similar to the case of the nonlinear system (2.2), we now state a Volterra integral equation which is closely related to (3.1). This will allow us to connect $\left(\widehat{A} + L\right)_0$ to a semigroup on $\mathcal{Y}_0$ in the proposition that follows. Given $i_0 \in L^1\left((0,+\infty), \mathbb{R}^2\right)$, consider

$$\widetilde{W}(t) = F(t) + \bar{S} \int_0^t B(a) \exp\left(-\int_0^a [D + M(l)]\, dl\right) \widetilde{W}(t-a) da, \tag{3.3}$$

where

$$F(t) := \bar{S} \int_t^{+\infty} B(a) \exp\left(-\int_{a-t}^a [D + M(l)]\, dl\right) i_0(a-t) da \tag{3.4}$$

for $t \geq 0$. The following proposition is well known in the context of age-structured models (see Webb [61] for a detailed discussion of this topic, or see [39, section 6]).

Proposition 3.1. *Let Assumption* 1.1 *be satisfied. The linear operator* $\left(\widehat{A}+L\right)_0$ *is the infinitesimal generator of a strongly continuous semigroup* $\left\{T_{\left(\widehat{A}+L\right)_0}(t)\right\}_{t\geq 0}$ *of bounded linear operators on* $\mathcal{Y}_0$*. Moreover,*

$$T_{\left(\widehat{A}+L\right)_0}(t) \begin{pmatrix} 0_{\mathbb{R}^2} \\ i_0 \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \widehat{T}_{\left(\widehat{A}+L\right)_0}(t)\,(i_0) \end{pmatrix},$$

where

$$\widehat{T}_{\left(\widehat{A}+L\right)_0}(t)\left(i_0\right)(a)=\begin{cases} \exp\left(-\int_{a-t}^{a}\left[D+M(l)\right]dl\right) i_0(a-t)da \text{ if } a>t,\\ \exp\left(-\int_{0}^{a}\left[D+M(l)\right]dl\right)\widetilde{W}(t-a)da \text{ if } a\leq t,\end{cases}$$

where $\widetilde{W}(t)$ *is the unique solution of* (3.3).

For clarity, we state that the unique mild solution to (3.1) is

$$i(t,a)=\widehat{T}_{\left(\widehat{A}+L\right)_0}(t)\left(i_0\right)(a) \text{ for almost every } a\geq 0 \text{ and all } t\geq 0,$$

and the mapping $t\rightarrow i(t,\cdot)$ belongs to $C([0,+\infty),L^1(0,+\infty))$. Furthermore, the Volterra equation (3.3) is derived by using

$$\widetilde{W}(t)=\bar{S}\int_0^{+\infty}B\left(a\right)i(t,a)da=i(t,0)$$

for all $t\geq 0$. This is easily seen to be well-defined for each t since $\bar{S}$ is constant, $B(\cdot)$ is bounded, and $i(t,\cdot)\in L^1$.

3.1. Preliminary. In order to define the invariant sets for the uniform persistence analysis, we define

$$\overline{a}_k:=\sup\left\{a>0:\text{support}(\beta_k)\cap(a,\infty)\,\text{has positive measure}\right\} \tag{3.5}$$

for $k=1,2$, allowing that $\overline{a}_k$ may be infinite. Note that by Assumption 1.1, we have $\overline{a}_1,\overline{a}_2>0$. Let

$$\widehat{\mathcal{M}}_0=\left\{\begin{pmatrix} i_1\\ i_2\end{pmatrix}\in L^1_+\left((0,+\infty);\mathbb{R}\right)^2:\int_0^{\overline{a}_1}i_1(a)da>0 \text{ or } \int_0^{\overline{a}_2}i_2(a)da>0\right\}. \tag{3.6}$$

Then $\widehat{\mathcal{M}}_0$ consists of the distributions i that generate new infectives either now or at some time in the future. Set

$$\partial\widehat{\mathcal{M}}_0:=L^1_+\left((0,+\infty);\mathbb{R}\right)^2\setminus\widehat{\mathcal{M}}_0$$

and

$$d_{\min}:=\min\left\{d_1,d_2\right\}.$$

PROPOSITION 3.2. *Let Assumption* 1.1 *be satisfied. For each* $i_0\in\partial\widehat{\mathcal{M}}_0$*, the solution of* (3.1) *satisfies*

$$\int_0^{+\infty}B\left(a\right)i(t,a)da=0\quad\forall t\geq 0,$$

and therefore $i(t,a)$ *is also a solution of the system*

$$\begin{cases}\left(\partial_t+\partial_a\right)i(t,a)=-\left[D+M(a)\right]i(t,a) \text{ for } a\geq 0,\\ i(t,0)=0,\\ i(0,.)=i_0\in L^1_+\left((0,+\infty),\mathbb{R}\right)^2\end{cases}$$

and satisfies

$$\|i(t,.)\| \leq e^{-d_{\min} t} \|i_0\| \quad \forall t \geq 0.$$

Proof. Let $\widetilde{W}(t)$ be the solution of the Volterra integral equation (3.3). We begin by showing that F (given in (3.4)) is identically zero. Using i_{01} and i_{02} to denote the components of i_0, we observe that

$$\begin{aligned} F(t) &= \bar{S} \int_t^{+\infty} B(a)\, e^{-\int_{a-t}^{a} D+M(l)dl} i_0(a-t)da \\ &= \bar{S} \begin{pmatrix} \int_0^{+\infty} \beta_2(a+t)\, e^{-\int_a^{a+t} d_2+m_2(l)dl} i_{02}(a)da \\ \int_0^{+\infty} \beta_1(a+t)\, e^{-\int_a^{a+t} d_1+m_1(l)dl} i_{01}(a)da \end{pmatrix}. \end{aligned}$$

Since $i_0 \in \partial\widehat{\mathcal{M}}_0$, we deduce that F is identically zero. Thus, (3.3) becomes

$$\widetilde{W}(t) = \bar{S} \int_0^t B(a) e^{-\int_0^a D+M(l)dl} \widetilde{W}(t-a)da,$$

which has the unique solution

$$\widetilde{W}(t) = 0 \quad \forall t \geq 0.$$

It now follows from Proposition 3.1 that $i(t,a) = 0$ for $0 \leq a \leq t$. In particular, this holds for $a = 0$ and so i is a solution of the system given in the statement of this proposition. For $t < a$, we have

$$i(t,a) = \exp\left(-\int_{a-t}^{a} [D + M(l)]\, dl\right) i_0(a-t)da \leq e^{-d_{\min} t} i_0(a-t).$$

It follows that $\|i(t,.)\| \leq e^{-d_{\min} t} \|i_0\|$ for all $t \geq 0$. ∎

For $k = 1, 2$, let

$$a_k^* := \sup\left\{ a \geq 0 : \int_0^a \beta_k(s)ds = 0 \right\}.$$

In the following result, and throughout the paper, we use the notation $v \gg \delta$ to denote that each component of a vector $v \in \mathbb{R}^2$ is strictly greater than $\delta \in \mathbb{R}$.

LEMMA 3.3. *Let Assumption* 1.1 *be satisfied. Let* $\tau^* > \max\{a_1^*, a_2^*\}$. *Suppose there exists* $\delta > 0$

$$\int_0^{+\infty} B(a)\, i(t,a)da \gg \delta \quad \forall t \in [0, \tau^*]. \tag{3.7}$$

Then

$$\int_0^{+\infty} B(a)\, i(t,a)da \gg 0 \tag{3.8}$$

for all $t \geq 0$.

Proof. Since $\tau^* > \max\{a_1^*, a_2^*\}$, the matrix

$$\bar{S} \int_0^{\tau^*} B(a) e^{-\int_0^a D+M(l)dl} da$$

is irreducible, and there exists $\varepsilon \in (0, \tau^*)$ such that

$$Q := \bar{S} \int_{\varepsilon}^{\tau^*} B(a) e^{-\int_0^a D+M(l)dl} da$$

is irreducible. For $t \geq \tau^*$, we have

$$\begin{aligned}
\widetilde{W}(t) &= F(t) + \bar{S} \int_0^t B(a) e^{-\int_0^a D+M(l)dl} \widetilde{W}(t-a) da \\
&\geq \bar{S} \int_{\varepsilon}^{\tau^*} B(a) e^{-\int_0^a D+M(l)dl} \widetilde{W}(t-a) da \\
&\geq \left(\inf_{\sigma \in [t-\tau^*, t-\varepsilon]} \min \left\{ \widetilde{W}_1(\sigma), \widetilde{W}_2(\sigma) \right\} \right) \bar{S} \int_{\varepsilon}^{\tau^*} B(a) e^{-\int_0^a D+M(l)dl} \begin{pmatrix} 1 \\ 1 \end{pmatrix} da \\
&= \left(\inf_{\sigma \in [t-\tau^*, t-\varepsilon]} \min \left\{ \widetilde{W}_1(\sigma), \widetilde{W}_2(\sigma) \right\} \right) Q \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

Considering $t \in [\tau^*, \tau^* + \varepsilon]$, we have

$$\widetilde{W}(t) \geq \left(\inf_{\sigma \in [0, \tau^*]} \min \left\{ \widetilde{W}_1(\sigma), \widetilde{W}_2(\sigma) \right\} \right) Q \begin{pmatrix} 1 \\ 1 \end{pmatrix} \geq \delta Q \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Recalling that $\int_0^{+\infty} B(a)\, i(t,a) da = \bar{S}^{-1} i(t,0) = \bar{S}^{-1} \widetilde{W}(t)$, it follows that (3.8) is satisfied for all $t \in [\tau^*, \tau^* + \varepsilon]$. Using induction arguments, it follows that the result holds on $[\tau^* + n\varepsilon, \tau^* + (n+1)\varepsilon]$. ∎

3.2. Irreducibility property. Recall that the essential growth rate is a key notion in applying spectral theory to a linear strongly continuous semigroup of bounded linear operators (see Webb [61, 62] and Engel and Nagel [20]).

LEMMA 3.4. *Let Assumption* 1.1 *be satisfied. Then the essential growth rate of* $\left\{ T_{(\widehat{A}+L)_0}(t) \right\}_{t \geq 0}$ *satisfies*

$$\omega_{0,ess}\left(\left(\widehat{A} + L \right)_0 \right) \leq -d_{\min}.$$

Proof. Since L is compact, bounded, and linear, it follows from [19, Theorem 1.2] that

$$\omega_{0,ess}\left(\left(\widehat{A} + L \right)_0 \right) \leq \omega_{0,ess}\left(\widehat{A}_0 \right) \leq \omega_0\left(\widehat{A}_0 \right) := \lim_{t \to +\infty} \frac{\ln\left(\left\| T_{\widehat{A}_0}(t) \right\| \right)}{t}, \tag{3.9}$$

where $\widehat{A}_0$ is the part of $\widehat{A}$ in the closure of its domain $\overline{\text{Dom}(\widehat{A})}$. Recall that $\widehat{A}_0$ is the infinitesimal generator of a strongly continuous semigroup $\left\{ T_{\widehat{A}_0}(t) \right\}_{t \geq 0}$ of bounded linear operators which is defined by

$$T_{\widehat{A}_0}(t) \begin{pmatrix} 0_{\mathbb{R}^2} \\ i_0 \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \widehat{i}(t,.) \end{pmatrix},$$

where

$$\widehat{i}(t,a) = \begin{cases} e^{-\int_{a-t}^a D+M(l)dl} i_0(a-t) & \text{if } a \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\left\|\widehat{i}(t,.)\right\| \leq e^{-d_{\min}t}\left\|i_0(.)\right\|.$$

Combining this with (3.9), we deduce that

$$\omega_{0,ess}\left(\left(\widehat{A}+L\right)_0\right) \leq \lim_{t\to+\infty}\frac{\ln\left(\left\|T_{\widehat{A}_0}(t)\right\|\right)}{t} \leq \lim_{t\to+\infty}\frac{\ln\left(e^{-d_{\min}t}\right)}{t} = -d_{\min}. \qquad \square$$

Let

$$\Omega := \left\{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) > -d_{\min}\right\}.$$

By directly calculating the resolvent of $\widehat{A}$, we obtain the following lemma.

LEMMA 3.5. *Let Assumption* 1.1 *be satisfied. The resolvent set $\rho(\widehat{A})$ of $\widehat{A}$ contains Ω, and for each $\lambda \in \Omega$ the resolvent of $\widehat{A}$ is given by the formula*

$$\left(\lambda I - \widehat{A}\right)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^2}\\ \varphi\end{pmatrix},$$

where

$$\varphi(a) = e^{-\int_0^a \lambda I + D + M(l)dl}\alpha + \int_0^a e^{-\int_s^a \lambda I + D + M(l)dl}\psi(s)ds.$$

Consider the characteristic function $\Delta(\lambda)$ defined by

$$\Delta(\lambda) := I - \bar{S}\int_0^{+\infty} B(a)\, e^{-\int_0^a \lambda I + D + M(l)dl}da.$$

LEMMA 3.6. *Let Assumption* 1.1 *be satisfied. The spectrum of $\widehat{A}+L$ in Ω satisfies*

$$\sigma\left(\widehat{A}+L\right) = \left\{\lambda \in \Omega : \det\left(\Delta(\lambda)\right) = 0\right\}.$$

Moreover, for each $\lambda \in \Omega$ with $\det(\Delta(\lambda)) \neq 0$, the resolvent of $\widehat{A}+L$ is given by the formula

$$\left(\lambda I - \left(\widehat{A}+L\right)\right)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^2}\\ \varphi\end{pmatrix},$$

where

$$\varphi(a) = e^{-\int_0^a \lambda I + D + M(l)dl}\Delta(\lambda)^{-1}\left[\alpha + \bar{S}\int_0^{+\infty} B(r)\int_0^r e^{-\int_s^r \lambda I + D + M(l)dl}\psi(s)dsdr\right]$$
$$+ \int_0^a e^{-\int_s^a \lambda I + D + M(l)dl}\psi(s)ds.$$

Proof. Let $\lambda \in \Omega$. Then $\lambda I - \widehat{A}$ is invertible and

$$\lambda I - \widehat{A} - L = \left(I - L\left(\lambda I - \widehat{A}\right)^{-1}\right)\left(\lambda I - \widehat{A}\right).$$

Thus, $\lambda I - \widehat{A} - L$ is invertible if and only if $I - L\big(\lambda I - \widehat{A}\big)^{-1}$ is invertible, with

$$\left(\lambda I - \widehat{A} - L\right)^{-1} = \left(\lambda I - \widehat{A}\right)^{-1}\left(I - L\left(\lambda I - \widehat{A}\right)^{-1}\right)^{-1}. \tag{3.10}$$

Suppose

$$\left(I - L\left(\lambda I - \widehat{A}\right)^{-1}\right)\begin{pmatrix}\alpha_1\\ \psi_1\end{pmatrix} = \begin{pmatrix}\alpha_2\\ \psi_2\end{pmatrix}.$$

Then $\psi_2 = \psi_1$ and

$$\begin{aligned}\alpha_2 &= \left(I - \bar{S}\int_0^{+\infty} B(a)\, e^{-\int_0^a \lambda I + D + M(l)dl} da\right)\alpha_1\\ &\quad - \bar{S}\int_0^{+\infty} B(a)\int_0^a e^{-\int_s^a \lambda I + D + M(l)dl}\psi_1(s)dsda\\ &= \Delta(\lambda)\alpha_1 - \bar{S}\int_0^{+\infty} B(r)\int_0^r e^{-\int_s^r \lambda I + D + M(l)dl}\psi_2(s)dsdr.\end{aligned}$$

Solving for α_1 and ψ_1, we deduce that $I - L\big(\lambda I - \widehat{A}\big)^{-1}$ is invertible if and only if $\Delta(\lambda)$ is invertible. Moreover, if $\Delta(\lambda)$ is invertible, we have

$$\begin{aligned}&\left(I - L\left(\lambda I - \widehat{A}\right)^{-1}\right)^{-1}\begin{pmatrix}\alpha_2\\ \psi_2\end{pmatrix} = \begin{pmatrix}\alpha_1\\ \psi_1\end{pmatrix}\\ &\qquad = \begin{pmatrix}\Delta(\lambda)^{-1}\left[\alpha_2 + \bar{S}\int_0^{+\infty} B(r)\int_0^r e^{-\int_s^r \lambda I + D + M(l)dl}\psi_2(s)dsdr\right]\\ \psi_2\end{pmatrix}.\end{aligned}$$

The result follows. $\square$

For each $\lambda \in \Omega$ with $\det(\Delta(\lambda)) \neq 0$,

$$\Delta(\lambda)^{-1} = \frac{1}{\det(\Delta(\lambda))}\Gamma(\lambda), \tag{3.11}$$

with

$$\Gamma(\lambda) := \begin{bmatrix} 1 & \bar{S}_2\int_0^{+\infty}\beta_1(a)e^{-\int_0^a \lambda + d_1 + m_1(l)dl}da\\ \bar{S}_1\int_0^{+\infty}\beta_2(a)e^{-\int_0^a \lambda + d_2 + m_2(l)dl}da & 1\end{bmatrix}$$

and

$$\det(\Delta(\lambda)) = 1 - \bar{S}_1\bar{S}_2\int_0^{+\infty}\beta_1(a)e^{-\int_0^a \lambda + d_1 + m_1(l)dl}da\int_0^{+\infty}\beta_2(a)e^{-\int_0^a \lambda + d_2 + m_2(l)dl}da.$$

We observe that as λ approaches infinity, the matrices Δ and Γ approach the identity. However, for any finite λ, the entries of Γ are strictly positive. Define

$$R_0(\bar{S}) := \bar{S}_1\bar{S}_2\int_0^{+\infty}\beta_1(a)e^{-\int_0^a d_1 + m_1(l)dl}da\int_0^{+\infty}\beta_2(a)e^{-\int_0^a d_2 + m_2(l)dl}da. \tag{3.12}$$

Lemma 3.7. *Let Assumption* 1.1 *be satisfied. If $R_0(\bar{S}) > 1$, then there exists a unique $\lambda_0 > 0$ such that*

$$\det(\Delta(\lambda_0)) = 0.$$

Proof. Suppose $R_0(\bar{S}) > 1$. Then $\det(\Delta(0)) < 0$. On the other hand, as λ tends to infinity, $\det(\Delta(\lambda))$ tends to 1. Thus, there exists $\lambda_0 > 0$ such that the determinant is zero. Furthermore, calculation shows that $\frac{d}{d\lambda}\det(\Delta(\lambda))$ is positive for $\lambda > 0$. Thus, λ_0 is unique. □

Let

$$C(\lambda_0) := \left.\frac{d\det(\Delta(\lambda))}{d\lambda}\right|_{\lambda=\lambda_0} > 0. \tag{3.13}$$

Then for λ sufficiently close to λ_0 we have the approximation

$$\det(\Delta(\lambda)) \approx \lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)\, C(\lambda_0).$$

Rearranging, and combining with (3.11), we obtain

$$\begin{aligned}\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\,\Delta(\lambda)^{-1} &= \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\,\tfrac{1}{\det(\Delta(\lambda))}\Gamma(\lambda)\\ &= \tfrac{1}{C(\lambda_0)^{-1}}\Gamma(\lambda_0).\end{aligned} \tag{3.14}$$

This is used in the following lemma, along with the resolvent formula given in Lemma 3.6, to calculate a projector associated with λ_0. We refer to Yosida [64, Theorem 3, p. 299] for more results about this topic.

LEMMA 3.8. *Let Assumption* 1.1 *be satisfied. If* $R_0(\bar{S}) > 1$*, then* λ_0 *is simple eigenvalue of* $\left(\widehat{A}+L\right)$ *and the corresponding projector on the generalized eigenspace of* $\left(\widehat{A}+L\right)$ *is given by*

$$\Pi_{\lambda_0}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^2}\\ \varphi\end{pmatrix},$$

where

$$\varphi(a) = e^{-\int_0^a \lambda_0 I + D + M(l)dl} C(\lambda_0)^{-1}\Gamma(\lambda_0)\left[\alpha + \bar{S}\int_0^{+\infty} B(r)\int_0^r e^{-\int_s^r \lambda_0 I + D + M(l)dl}\psi(s)dsdr\right].$$

Furthermore, if $\alpha \geq 0$ *and* $\psi \geq 0$*, and at least one is nonzero, then* $\varphi(a) \gg 0$ *for all* $a > 0$.

Proof. Since $C(\lambda_0) > 0$ and the nullspace of $\Delta(\lambda_0)$ has dimension one, it follows that λ_0 is a simple eigenvalue. Furthermore, the projector Π_{λ_0} onto the generalized eigenspace associated with λ_0 is given by

$$\Pi_{\lambda_0} = \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\left(\lambda I - \left(\widehat{A}+L\right)\right)^{-1}.$$

Using (3.14) and the resolvent formula given in Lemma 3.6, we obtain the given formula for φ. Finally, the entries of $\Gamma(\lambda_0)$ are all strictly positive and each of the other matrices in the formula for φ are nonnegative and have full rank, so $\varphi(a) \gg 0$ for all $a > 0$ if $(\alpha,\psi)^T$ is nonnegative and nonzero. □

In order to prove that λ_0 is a dominant eigenvalue, we use a result for irreducible semigroups. Whenever

$$\min\{\overline{a}_1, \overline{a}_2\} < +\infty$$

the semigroup generated by problem (3.1) is not irreducible. The irreducibility property is obtained by considering the following restricted problem:

$$\left\{\begin{array}{l}(\partial_t+\partial_a)\, i_1(t,a)=-\left[d_1+m_1(a)\right] i_1(t,a) \text{ for } a\in(0,\overline{a}_1)\,,\\ (\partial_t+\partial_a)\, i_2(t,a)=-\left[d_2+m_2(a)\right] i_2(t,a) \text{ for } a\in(0,\overline{a}_2)\,,\\ i_1(t,0)=\bar{S}_1\int_0^{\overline{a}_2}\beta_2\,(a)\, i_2(t,a)da,\\ i_2(t,0)=\bar{S}_2\int_0^{\overline{a}_1}\beta_1\,(a)\, i_1(t,a)da,\\ i_1(0,.)=i_0^1\in L_+^1\left((0,\overline{a}_1)\,,\mathbb{R}\right),\\ i_2(0,.)=i_0^2\in L_+^1\left((0,\overline{a}_2)\,,\mathbb{R}\right).\end{array}\right. \tag{3.15}$$

From the above resolvent formula, combined with results on irreducible positive semigroups in Banach lattices (see Webb [62]) applied to system (3.15), we obtain the following theorem. We also refer the reader to Arino [3, 4], Nagel [48], and Thieme [55] for more results about this topic.

THEOREM 3.9. *Let Assumption* 1.1 *be satisfied. Suppose* $R_0(\bar{S})>1$. *Then* $\lambda_0>0$ *is a simple dominant eigenvalue of* $\left(\widehat{A}+L\right)$, *or equivalently*

$$T_{\left(\widehat{A}+L\right)_0}(t)\Pi_{\lambda_0}=\Pi_{\lambda_0}T_{\left(\widehat{A}+L\right)_0}(t)=e^{\lambda_0 t}\Pi_{\lambda_0}\quad \forall t\geq 0,$$

and there exist constants $\varepsilon>0$ *and* $\eta>0$ *such that*

$$\left\|T_{\left(\widehat{A}+L\right)_0}(t)\left(I-\Pi_{\lambda_0}\right)\right\|\leq \eta e^{(\lambda_0-\varepsilon)t}\left\|(I-\Pi_{\lambda_0})\right\|\quad \forall t\geq 0.$$

As a consequence of the above theorem, we obtain the following result.

COROLLARY 3.10. *Let Assumption* 1.1 *be satisfied. Let* $i(t,a)$ *be a solution of* (3.1) *with* $i_0\in\widehat{\mathcal{M}}_0$. *If* $R_0(\bar{S})>1$, *then there exists* $t^*=t^*(i_0)>0$ *such that*

$$\int_0^{+\infty}B\,(a)\, i(t,a)da\gg 0\quad \forall t\geq t^*.$$

Proof. Let

$$\begin{pmatrix}0_{\mathbb{R}^2}\\ \varphi\end{pmatrix}:=\Pi_{\lambda_0}\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}.$$

Since $i_0\in\widehat{\mathcal{M}}_0$, it follows from Lemma 3.8 that $\varphi(a)\gg 0$ for all $a>0$. Thus,

$$\int_0^{+\infty}B\,(a)\,\varphi(a)da\gg 0.$$

Also, using Theorem 3.9, we find that

$$\begin{aligned}\begin{pmatrix}0_{\mathbb{R}^2}\\ i(t,.)\end{pmatrix}&=T_{\left(\widehat{A}+L\right)_0}(t)\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}\\ &=T_{\left(\widehat{A}+L\right)_0}(t)\Pi_{\lambda_0}\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}+T_{\left(\widehat{A}+L\right)_0}(t)\left(I-\Pi_{\lambda_0}\right)\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}\\ &=e^{\lambda_0 t}\Pi_{\lambda_0}\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}+T_{\left(\widehat{A}+L\right)_0}(t)\left(I-\Pi_{\lambda_0}\right)\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}\\ &=e^{\lambda_0 t}\begin{pmatrix}0_{\mathbb{R}^2}\\ \varphi\end{pmatrix}+T_{\left(\widehat{A}+L\right)_0}(t)\left(I-\Pi_{\lambda_0}\right)\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}.\end{aligned}$$

Noting, from Theorem 3.9, that

$$\lim_{t\to+\infty}\left\|e^{-\lambda_0 t}T_{\left(\widehat{A}+L\right)_0}(t)\left(I-\Pi_{\lambda_0}\right)\right\|=0,$$

we obtain

$$\lim_{t\to+\infty}e^{-\lambda_0 t}\int_0^{+\infty}B(a)\,i(t,a)da=\int_0^{+\infty}B(a)\,\varphi(a)da\gg 0.$$

The result follows. $\square$

3.3. Extra property. We start this subsection by showing that the conclusion of Corollary 3.10 holds in the case $R_0(\bar{S})\leq 1$.

PROPOSITION 3.11. *Let Assumption* 1.1 *be satisfied. Let* $i(t,a)$ *be a solution of* (3.1) *with* $i_0\in\widehat{\mathcal{M}}_0$. *Then there exists* $t^*=t^*(i_0)>0$ *such that*

$$\int_0^{+\infty}B(a)\,i(t,a)da\gg 0\quad\forall t\geq t^*.$$

Proof. Let $i(t,a)$ be a solution of (3.1). Let

$$\widehat{i}(t,a)=e^{\alpha(t-a)}i(t,a)$$

for all $t,a\geq 0$, where $\alpha\geq 0$ is yet to be determined. Then

$$(\partial_t+\partial_a)\,\widehat{i}(t,a)=e^{\alpha(t-a)}\,(\partial_t+\partial_a)\,i(t,a)=-\left[D+M(a)\right]\widehat{i}(t,a)$$

and

$$\widehat{i}(t,0)=e^{\alpha t}i(t,0)=e^{\alpha t}\bar{S}\int_0^{+\infty}B(a)\,i(t,a)da=\bar{S}\int_0^{+\infty}B(a)\,e^{\alpha a}\widehat{i}(t,a)da.$$

Therefore, $\widehat{i}(t,a)$ is a solution of the following system:

$$\text{(3.16)}\qquad\begin{cases}(\partial_t+\partial_a)\,\widehat{i}(t,a)=-\left[D+M(a)\right]\widehat{i}(t,a)\text{ for }a\geq 0,\\ \widehat{i}(t,0)=\bar{S}\int_0^{+\infty}B(a)\,e^{\alpha a}\widehat{i}(t,a)da,\\ \widehat{i}(0,.)=e^{-\alpha a}i_0\in L^1_+\left((0,+\infty),\mathbb{R}\right)^2.\end{cases}$$

It is possible that $B(a)\,e^{\alpha a}\notin L^\infty\left((0,+\infty),\mathbb{M}_2(\mathbb{R})\right)$. However, we can always find $\alpha\geq 0$ and $\overline{a}^*\geq\max\{\overline{a}_1,\overline{a}_2\}$ such that

$$\begin{aligned}\widehat{R}_0(\bar{S}):=\bar{S}_1\bar{S}_2\int_0^{\overline{a}^*}\beta_1(a)e^{\alpha a}e^{-\int_0^a d_1+m_1(l)dl}da\int_0^{\overline{a}^*}\beta_2(a)e^{\alpha a}e^{-\int_0^a d_2+m_2(l)dl}da\\ >1.\end{aligned}$$

Then

$$\widehat{i}(t,a)\geq\widetilde{i}(t,a),$$

where $\widetilde{i}(t,a)$ is the solution of

$$\text{(3.17)}\qquad\begin{cases}(\partial_t+\partial_a)\,\widetilde{i}(t,a)=-\left[D+M(a)\right]\widetilde{i}(t,a)\text{ for }a\geq 0,\\ \widetilde{i}(t,0)=\bar{S}\int_0^{\overline{a}^*}B(a)\,e^{\alpha a}\widetilde{i}(t,a)da,\\ \widetilde{i}(0,.)=e^{-\alpha a}i_0\in L^1_+\left((0,+\infty),\mathbb{R}\right)^2.\end{cases}$$

Then, the result follows by applying Corollary 3.10 to system (3.17). ∎

As a consequence of Proposition 3.11 we obtain the following invariance property.

COROLLARY 3.12. *Let Assumption* 1.1 *be satisfied. Then*

$$T_{(A+L)_0}(t)\left(\{0_{\mathbb{R}^2}\}\times\widehat{\mathcal{M}}_0\right)\subseteq\{0_{\mathbb{R}^2}\}\times\widehat{\mathcal{M}}_0 \quad \forall t\geq 0.$$

Proof. Let $i_0\in\widehat{\mathcal{M}}_0$. Assume that there exists $t_1>0$ such that

$$T_{(A+L)_0}(t_1)\begin{pmatrix}0_{\mathbb{R}^2}\\ i_0\end{pmatrix}\in\{0_{\mathbb{R}^2}\}\times\partial\widehat{\mathcal{M}}_0.$$

Then by Proposition 3.2, we have

$$\int_0^{+\infty}B(a)\,i(t,a)da=0 \quad \forall t\geq t_1,$$

and we obtain a contraction with Proposition 3.11. Thus, no such t_1 exists. ∎

4. Extinction. Recalling the definition of $\widehat{\mathcal{M}}_0$ given in (3.6), we define

$$\mathcal{M}:=\mathcal{X}_{0+}=[0,+\infty)\times\{0_{R^2}\}\times L^1_+\left((0,+\infty),\mathbb{R}\right)^2,$$

$$\mathcal{M}_0:=[0,+\infty)\times\{0_{R^2}\}\times\widehat{\mathcal{M}}_0,$$

and

$$\partial\mathcal{M}_0:=\mathcal{M}\setminus\mathcal{M}_0.$$

LEMMA 4.1. *Let Assumption* 1.1 *be satisfied. There exists* $S^+\in(0,+\infty)^2$, *such that*

$$S(t)\leq S^+ \quad \forall t\geq 0, \tag{4.1}$$

and for each $t_0>0$, *there exists* $S^-=S^-(t_0)\in(0,+\infty)^2$ *such that*

$$S(t)\geq S^-\gg 0 \quad \forall t\geq t_0. \tag{4.2}$$

Proof. As in (2.1), we have

$$\frac{dN(t)}{dt}=\Lambda-DN(t). \tag{4.3}$$

Since D is diagonal, a consequence of (4.3) is that

$$S(t)\leq N(t)\leq N^+:=\begin{pmatrix}\max\left\{\frac{\lambda_1}{d_1},N_1(0)\right\}\\ \max\left\{\frac{\lambda_2}{d_2},N_2(0)\right\}\end{pmatrix}$$

for all $t\geq 0$. Letting $S^+=N^+$, the first part of the lemma is proven.

Additionally,

$$\int_0^{+\infty}i(t,a)da\leq N(t)\leq N^+,$$

and so

$$\int_0^{+\infty} B(a)\, i(t,a)da \leq \int_0^{+\infty} \|B\|_{L^\infty} i(t,a)da \leq \|B\|_{L^\infty} N^+$$

for all $t \geq 0$. Therefore,

$$\begin{aligned}\frac{dS(t)}{dt} &= \Lambda - DS(t) - \operatorname{diag}(S(t)) \textstyle\int_0^{+\infty} B(a)\, i(t,a)da \\ &= \Lambda - DS(t) - \operatorname{diag}\left(\textstyle\int_0^{+\infty} B(a)\, i(t,a)da\right) S(t) \\ &\geq \Lambda - \overline{D}S(t),\end{aligned}$$

where $\overline{D} = D + \|B\|_{L^\infty} \operatorname{diag}(N^+)$. Since $\overline{D}$ is diagonal, it follows that for a given $t_0 > 0$,

$$S(t) \geq S^- := \begin{pmatrix} \min\left\{\frac{\lambda_1}{d_1+\|B\|_{L^\infty} N_1^+}, S_1(t_0)\right\} \\ \min\left\{\frac{\lambda_2}{d_2+\|B\|_{L^\infty} N_2^+}, S_2(t_0)\right\} \end{pmatrix}$$

for all $t \geq t_0$. ∎

LEMMA 4.2. *Let Assumption* 1.1 *be satisfied. Then* $\partial\mathcal{M}_0$ *is positively invariant under* $\{U(t)\}_{t\geq 0}$ *(the semiflow generated by* (2.4)*). Moreover,* $\overline{E}_F = \left(\overline{S}_F, \overline{\imath}_F\right) = \left(D^{-1}\Lambda, 0_{\mathbb{R}^2}, 0_{L^1(0,+\infty)}\right)$ *is globally exponentially stable for* U *restricted to* $\partial\mathcal{M}_0$.

Proof. Let $(S_0, 0_{\mathbb{R}^2}, i_0) \in \partial\mathcal{M}_0$. Then $i_0 \in \partial\widehat{\mathcal{M}}_0$ and we have

$$\begin{cases} (\partial_t + \partial_a)\, i(t,a) = -\left[D + M(a)\right] i(t,a), \\ i(t,0) = \operatorname{diag}(S(t)) \int_0^{+\infty} B(a)\, i(t,a)da, \\ i(0,.) = i_0. \end{cases}$$

By using (4.1) we have

$$i(t,a) \leq \widehat{i}(t,a),$$

where

$$\begin{cases} (\partial_t + \partial_a)\, \widehat{i}(t,a) = -\left[D + M(a)\right] \widehat{i}(t,a), \\ \widehat{i}(t,0) = \operatorname{diag}(S^+) \int_0^{+\infty} B(a)\, i(t,a)da, \\ \widehat{i}(0,.) = i_0, \end{cases}$$

and by applying Proposition 3.2 to the above linear system, we deduce that

$$0 \leq \int_0^{+\infty} B(a)\, i(t,a)da \leq \int_0^{+\infty} B(a)\, \widehat{i}(t,a)da = 0,$$

and the result follows. ∎

THEOREM 4.3. *Let Assumption* 1.1 *be satisfied. Assume that*

$$\mathcal{R}_0 \leq 1.$$

The disease-free equilibrium is global asymptotically stable.

Proof. It is sufficient to prove that the global attractor $\mathcal{A}$ is equal to the singleton

$$\left\{ \left(\begin{array}{c} \overline{S}_F \\ \left(\begin{array}{c} 0_{\mathbb{R}^2} \\ 0_{L^1(0,+\infty)} \end{array} \right) \end{array} \right) \right\},$$

where

$$\overline{S}_F = D^{-1}\Lambda.$$

Consider a total solution

$$Y(t) = \left(\begin{array}{c} S(t) \\ \left(\begin{array}{c} 0_{\mathbb{R}^2} \\ i(t,.) \end{array} \right) \end{array} \right) \in \mathcal{A}.$$

By Lemma 2.2, we have

$$S \leq \overline{S}_F.$$

Define the function $\mathcal{L} : L^1_+\left((0,+\infty);\mathbb{R}^2\right) \to \mathbb{R}^2_+$ by

$$\mathcal{L}[\Psi] = \int_0^\infty B(a) \int_0^a e^{-\int_s^a D+M(l)dl} \Psi(s) ds\, da. \tag{4.4}$$

Then

$$\begin{aligned} &\frac{d}{dt}\mathcal{L}\left[i(t,.)\right] \\ &= \int_0^\infty B(a) \int_0^a e^{-\int_s^a D+M(l)dl} \frac{\partial}{\partial t} i(t,s) ds\, da \\ &= \int_0^\infty B(a) \int_0^a e^{-\int_s^a D+M(l)dl} \left[-\frac{\partial}{\partial s} i(t,s) - (D+M(s))\, i(t,s) \right] ds\, da. \end{aligned}$$

Performing the integration with respect to s, we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}\left[i(t,.)\right] &= -\int_0^\infty B(a) \left[e^{-\int_s^a D+M(l)dl} i(t,s) \right]_{s=0}^a da \\ &= \int_0^\infty B(a) e^{-\int_0^a D+M(l)dl} da\, i(t,0) - \int_0^\infty B(a) i(t,a) da. \end{aligned}$$

Using the boundary condition to replace $i(t,0)$ with $\mathrm{diag}(S(t)) \int_0^{+\infty} B(a)i(t,a)da$, and defining

$$A_S = \int_0^\infty B(a) e^{-\int_0^a D+M(l)dl} da\, \mathrm{diag}\left(\overline{S}_F\right),$$

we have

$$\frac{d}{dt}\mathcal{L}\left[i(t,.)\right] = \left[A_S \mathrm{diag}\left(\overline{S}_F\right)^{-1} \mathrm{diag}(S(t)) - I \right] \int_0^\infty B(a) i(t,a) da.$$

One may observe that Assumption 1.1 implies that A_S is a 2×2 irreducible nonnegative matrix. Moreover, $\mathcal{R}_0 \leq 1$ implies $r\left(A_S\right) \leq 1$ (where $r\left(A_S\right)$ is the spectral radius of the matrix A_S). It follows that there exists $w = \binom{w_1}{w_2} \gg 0$ such that

$$w^T \overline{A}_{\bar{S}} = r\left(A_S\right) w^T.$$

Thus,

$$\frac{d}{dt}w^T\mathcal{L}\left[i(t,.)\right]=\left[r\left(A_S\right)w^T\operatorname{diag}\left(\overline{S}_F\right)^{-1}\operatorname{diag}\left(S(t)\right)-w^T\right]\int_0^\infty B(a)i(t,a)da. \tag{4.5}$$

Using the fact that $S(t)\leq\overline{S}_F$ and $r\left(A_S\right)\leq 1$, we deduce that

$$\frac{d}{dt}w^T\mathcal{L}\left[i(t,.)\right]\leq 0\quad\forall t\in\mathbb{R}. \tag{4.6}$$

Since (4.5) is coupled with

$$\frac{dS(t)}{dt}=\Lambda-DS(t)-\operatorname{diag}(S(t))\int_0^{+\infty}B\left(a\right)i(t,a)da,$$

equation (4.6) achieves equality only if $\int_0^\infty B(a)\widetilde{i}(t,a)da$ is zero.

Consider a point Z_0 in the alpha limit set of $Y(t)$. Let

$$Z(t)=\begin{pmatrix}\widetilde{S}(t)\\ \begin{pmatrix}0_{\mathbb{R}^2}\\ \widetilde{i}(t,.)\end{pmatrix}\end{pmatrix}$$

be the solution that passes through Z_0 at time 0. Then $w^T\mathcal{L}\left[\widetilde{i}(t,.)\right]$ is constant along $Z(.)$. It follows that $\int_0^\infty B(a)\widetilde{i}(t,a)da$ is identically zero. Thus, $Z(t)\subseteq\partial\mathcal{M}_0$, and by Lemma 4.2 the disease-free equilibrium is globally asymptotically stable in $\partial\mathcal{M}_0$, and we deduce that the alpha limit set of $Y(t)$ consists solely of the disease-free equilibrium. Finally since $t\to w^T\mathcal{L}\left[i(t,.)\right]$ is decreasing and is zero at the alpha limit set, we deduce that it is identically zero, and the result follows. ∎

5. Uniform persistence.

LEMMA 5.1. *Let Assumption* 1.1 *be satisfied. Then $\mathcal{M}_0$ is positively invariant under $\{U(t)\}_{t\geq 0}$ (the semiflow generated by* (2.4)*). Moreover, for each $(S_0,0_{\mathbb{R}^2},i_0)\in\mathcal{M}_0$, there exists $t^*>0$, such that*

$$\int_0^{+\infty}B\left(a\right)i(t,a)da\gg 0\quad\forall t\geq t^*,$$

or equivalently

$$\int_0^{+\infty}\beta_k\left(a\right)i_k(t,a)da>0\quad\forall t\geq t^*\ \textit{for}\ k=1,2.$$

Proof. The first part of this proof allows for the case that one or more components of $S(0)$ may be zero. Since

$$i(t,a)\geq\widehat{i}(t,a),$$

where

$$\begin{cases}\left(\partial_t+\partial_a\right)\widehat{i}(t,a)=-\left[D+M(a)\right]\widehat{i}(t,a),\\ \widehat{i}(t,0)=0,\\ \widehat{i}\left(0,.\right)=i_0,\end{cases}$$

we deduce that there exists sufficiently small $t_0 > 0$ such that

$$i(t_0,.) \in \widehat{\mathcal{M}}_0.$$

Now, by (4.2), we have

$$S(t) \geq S^- \gg 0 \quad \forall t \geq t_0.$$

Let

$$\widetilde{i}(t,.) = i(t+t_0,.).$$

Then

$$\begin{cases} (\partial_t + \partial_a)\widetilde{i}(t,a) = -\left[D + M(a)\right]\widetilde{i}(t,a), \\ \widetilde{i}(t,0) \geq \operatorname{diag}(S^-)\int_0^{+\infty} B(a)\widetilde{i}(t,a)da, \\ \widetilde{i}(0,.) = i(t_0,.) \in \widehat{\mathcal{M}}_0. \end{cases} \tag{5.1}$$

Let $i^+(t,.)$ be the solution to the system that is obtained by replacing the inequality in (5.1) with an equality. Then

$$\widetilde{i}(t,.) \geq i^+(t,.),$$

and the result follows by applying Proposition 3.11 and Corollary 3.12 to $i^+(t,.)$ and the system that it solves. ∎

By combining Theorem 4.2 in Hale and Waltman [27] and Theorem 3.7 in Magal and Zhao [44] (see also Theorem 2.11 in Magal [40] for both discrete and continuous time versions of this results), we are able to prove the following theorem.

Theorem 5.2 (uniform persistence). *Let Assumption 1.1 be satisfied. Suppose $\mathcal{R}_0 > 1$. Then the semiflow U is uniformly persistent in $\mathcal{M}_0$, (with respect to the decomposition $(\partial\mathcal{M}_0, \mathcal{M}_0)$). That is, there exists $\varepsilon > 0$, such that for each $(S_0, 0_{\mathbb{R}^2}, i_0) \in \mathcal{M}_0$,*

$$\liminf_{t\to+\infty} \|i(t,.)\| \geq \varepsilon.$$

Furthermore, the semiflow U has a compact global attractor $\mathcal{A}_0$ in $\mathcal{M}_0$.

Proof. Recall that the disease-free equilibrium is given by $\overline{E}_F = \left(\overline{S}_F, \overline{\imath}_F\right) = \left(D^{-1}\Lambda, 0_{L^1(0,+\infty)}\right)$ and is globally exponentially stable in $\partial\mathcal{M}_0$. Denote the components of $\overline{S}_F$ as follows:

$$\overline{S}_F = \begin{pmatrix} \overline{S}_{F1} \\ \overline{S}_{F2} \end{pmatrix}.$$

Since $\mathcal{R}_0 > 1$, there exists $\varepsilon > 0$ such that

$$\left(\overline{S}_{F1} - \varepsilon\right)\left(\overline{S}_{F2} - \varepsilon\right)\int_0^{+\infty} \beta_1(a)e^{-\int_0^a d_1 + m_1(l)dl}da \int_0^{+\infty} \beta_2(a)e^{-\int_0^a d_2 + m_2(l)dl}da > 1. \tag{5.2}$$

Assume that there exists $(S_0, 0_{\mathbb{R}^2}, i_0) \in \mathcal{M}_0$ such that

$$\left\|(S(t), 0_{\mathbb{R}^2}, i(t,.)) - \left(\overline{S}_F, 0_{\mathbb{R}^2}, 0_{L^1(0,+\infty)}\right)\right\| \leq \varepsilon \quad \forall t \geq 0. \tag{5.3}$$

Then we have

$$S(t) \geq \begin{pmatrix} \overline{S}_{F1} - \varepsilon \\ \overline{S}_{F2} - \varepsilon \end{pmatrix} =: S_\varepsilon^- \quad \forall t \geq 0,$$

and hence

$$\begin{cases} (\partial_t + \partial_a)\, i(t,a) = -\left[D + M(a)\right] i(t,a), \\ i(t,0) \geq \operatorname{diag}(S_\varepsilon^-) \int_0^{+\infty} B\,(a)\, i(t,a)da, \\ i\,(0,.) = i_0 \in \widehat{\mathcal{M}}_0. \end{cases}$$

Therefore

$$i(t,a) \geq \widehat{i}(t,a),$$

where

$$\begin{cases} (\partial_t + \partial_a)\, \widehat{i}(t,a) = -\left[D + M(a)\right] \widehat{i}(t,a), \\ \widehat{i}(t,0) = \operatorname{diag}(S_\varepsilon^-) \int_0^{+\infty} B\,(a)\, \widehat{i}(t,a)da, \\ \widehat{i}\,(0,.) = i_0 \in \widehat{\mathcal{M}}_0. \end{cases} \tag{5.4}$$

Recalling (5.2), and applying Theorem 3.9 to (5.4), we deduce that

$$\Pi_{\lambda_0} \begin{pmatrix} 0_{\mathbb{R}^2} \\ \widehat{i}(t,.) \end{pmatrix} = e^{\lambda_0 t} \Pi_{\lambda_0} \begin{pmatrix} 0_{\mathbb{R}^2} \\ i_0 \end{pmatrix}$$

for some $\lambda_0 > 0$, where Π_{λ_0} is given in the statement of Lemma 3.8. It follows that

$$\lim_{t \to +\infty} \left\| \Pi_{\lambda_0} \begin{pmatrix} 0_{\mathbb{R}^2} \\ \widehat{i}(t,.) \end{pmatrix} \right\| = +\infty. \tag{5.5}$$

Note that the norm $|\cdot|$ defined by

$$\left| \begin{pmatrix} 0_{\mathbb{R}^2} \\ \widehat{i} \end{pmatrix} \right| = \max \left\{ \left\| \Pi_{\lambda_0} \begin{pmatrix} 0_{\mathbb{R}^2} \\ \widehat{i} \end{pmatrix} \right\|, \left\| (I - \Pi_{\lambda_0}) \begin{pmatrix} 0_{\mathbb{R}^2} \\ \widehat{i} \end{pmatrix} \right\| \right\}$$

is equivalent to $\|\cdot\|$. Thus, recalling that $i(t,a) \geq \widehat{i}(t,a)$, we see that (5.5) contradicts (5.3). Therefore, the stable manifold of $\overline{E}_F$ does not intersect $\mathcal{M}_0$.

We already know (see Theorem 2.1) that the semiflow U is asymptotically smooth, point dissipative, and that the forward trajectory of a bounded set is bounded. Furthermore, the $\overline{E}_F$ is globally asymptotically stable in $\partial\mathcal{M}_0$. Thus, Theorem 4.2 of Hale and Waltman [27] implies that U is uniformly persistent. $\square$

The last key ingredient in order to apply our Lyapunov method is the following proposition.

PROPOSITION 5.3. *Let Assumption* 1.1 *be satisfied. There exists* $\delta > 0$ *such that for each* $(S, 0_{\mathbb{R}^2}, i) \in \mathcal{A}_0$,

$$S \geq \delta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \int_0^{+\infty} B\,(a)\, i(a)da \geq \delta \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Proof. Let $(S(t), 0_{\mathbb{R}^2}, i(t,.))$ be the solution to (2.4) with initial condition $(S_0, 0_{\mathbb{R}^2}, i_0) \in \mathcal{A}_0$. Since this solution is in the attractor, Lemma 2.2 implies

$$S(t) \geq \widehat{D}^{-1}\Lambda \gg 0$$

for all $t \geq 0$. Thus, for sufficiently small δ, we have $S(t) \geq \delta\binom{1}{1}$ for all $t \geq 0$. Furthermore,

$$i\,(t,.) \geq \widehat{i}\,(t,.)\,,$$

where

$$\begin{cases} (\partial_t + \partial_a)\,\widehat{i}(t,a) = -\left[D + M(a)\right]\widehat{i}(t,a), \\ \widehat{i}(t,0) \geq \mathrm{diag}(D^{-1}\Lambda)\int_0^{+\infty} B\,(a)\,\widehat{i}(t,a)da, \\ \widehat{i}\,(0,.) = i_0 \in \widehat{\mathcal{M}}_0. \end{cases}$$

Thus, by Proposition 3.11, there exists $t^* = t^*\,(i_0) > 0$ such that

$$\int_0^{+\infty} B\,(a)\,i(t,a)da \gg 0 \quad \forall t \geq t^*.$$

Let $\tau^* > \max\{a_1^*, a_2^*\}$. Recall that the mapping $(t, (S, 0_{\mathbb{R}^2}, i)) \rightarrow U(t)\,(S, 0_{\mathbb{R}^2}, i)$ is continuous. Also, the mapping $i \rightarrow \int_0^{+\infty} B(a)i(a)$ is continuous (in the L^1 norm).

Thus, there exists $\varepsilon = \varepsilon\big(S_0, 0_{\mathbb{R}^2}, i_0\big) > 0$ such that if $\big(\widehat{S}_0, 0_{\mathbb{R}^2}, \widehat{i}_0\big) \in \mathcal{A}_0$ with

$$\left\|(\widehat{S}_0, 0_{\mathbb{R}^2}, \widehat{i}_0) - (S_0, 0_{\mathbb{R}^2}, i_0)\right\| \leq \varepsilon,$$

and the solution to (2.4) with initial condition $\big(\widehat{S}_0, 0_{\mathbb{R}^2}, \widehat{i}_0\big)$ is denoted by $\big(\widehat{S}(t), 0_{\mathbb{R}^2}, \widehat{i}(t,.)\big)$, then

$$\int_0^{+\infty} B\,(a)\,\widehat{i}(t,a)da \gg 0 \quad \forall t \in \left[t^*, t^* + \tau^*\right].$$

Now by using Lemma 3.3 we deduce that

$$\int_0^{+\infty} B\,(a)\,\widehat{i}(t,a)da \gg 0 \quad \forall t \geq t^*.$$

Therefore, using the compactness of $\mathcal{A}_0$ and a finite cover, we deduce that there exists $\widehat{t} > 0$, such that for each $(S_0, 0_{\mathbb{R}^2}, i_0) \in \mathcal{A}_0$,

$$\int_0^{+\infty} B\,(a)\,i(t,a)da \gg 0 \quad \forall t \geq \widehat{t},$$

where $(S(t), 0, i(t,.))$ is the solution to (2.4) with initial condition $(S_0, 0, i_0)$. Now by using the fact that $\mathcal{A}_0$ is invariant under U, it follows that for each $(S_0, 0_{\mathbb{R}^2}, i_0) \in \mathcal{A}_0$,

$$\int_0^{+\infty} B\,(a)\,i_0(a)da \gg 0.$$

Now using the continuity of $i_0 \rightarrow \int_0^{+\infty} B(a)i_0(a)$ again, and the compactness of $\mathcal{A}_0$, the result follows. $\square$

6. Global asymptotic stability of the endemic equilibrium. In this section, we use a Lyapunov functional to show that, for the case $\mathcal{R}_0 > 1$, all solutions of (1.1) for which the disease is initially present tend to the endemic equilibrium E^*.

Throughout this section, we use the variables x_k, y_k, and z_k for $k = 1, 2$, defined by

$$x_k(t) := \frac{S_k(t)}{S_k^*}, \quad y_k(t,a) := \frac{i_k(t,a)}{i_k^*(a)}, \quad z_k(t) := \frac{i_k(t,0)}{i_k^*(0)}.$$

We begin by proving two identities that can be used to simplify the Lyapunov calculation that appears later in this section. In these identities, and throughout the section, we use the notation that $j, k \in \{1, 2\}$ are distinct. In these propositions, we note that the function $H(t)$ may be given as a function of x_k or z_k, but not y_k since y_k depends on a.

Proposition 6.1. *If $\mathcal{R}_0 > 1$, then*

$$\int_0^\infty \beta_k(a) i_k^*(a) \left(y_k x_j - z_j\right) H \, da = 0$$

for any $H = H(t)$.

Proof. We first note that

$$\begin{aligned}
\int_0^\infty \beta_k(a) i_k^*(a) y_k x_j da &= \int_0^\infty \beta_k(a) i_k^*(a) \frac{i_k(t,a)}{i_k^*(a)} \frac{S_j(t)}{S_j^*} da \\
&= \frac{1}{S_j^*} \int_0^\infty \beta_k(a) i_k(t,a) S_j(t) da \\
&= \frac{i_j^*(0)}{S_j^*} \frac{i_j(t,0)}{i_j^*(0)} \\
&= \int_0^\infty \beta_k(a) i_k^*(a) z_j da.
\end{aligned}$$

Subtracting the right-hand side from the left-hand side gives the result for the special case where $H = 1$. Then, since H does not depend on a, multiplying both sides by a general $H(t)$ gives the desired result. ∎

In the previous proposition, and in the next one, the condition that $\mathcal{R}_0$ be greater than one appears in order that the endemic equilibrium be defined.

The next proposition will be used in the Lyapunov calculation to move terms that do not depend on a from one integral to another.

Proposition 6.2. *If $\mathcal{R}_0 > 1$, then*

$$\int_0^\infty \beta_2(a) i_2^*(a) H da = \frac{i_1^*(0) S_2^*}{i_2^*(0) S_1^*} \int_0^\infty \beta_1(a) i_1^*(a) H da$$

for any $H = H(t)$.

Proof. We first note that without H, we have

$$\begin{aligned}
\int_0^\infty \beta_2(a) i_2^*(a) da &= \frac{i_1^*(0)}{S_1^*} \\
&= \frac{i_1^*(0) S_2^*}{i_2^*(0) S_1^*} \frac{i_2^*(0)}{S_2^*} \\
&= \frac{i_1^*(0) S_2^*}{i_2^*(0) S_1^*} \int_0^\infty \beta_1(a) i_1^*(a) da.
\end{aligned}$$

Then, since H does not depend on a, multiplying both sides by H gives the desired result. ∎

The main theorem of this section is the following.

THEOREM 6.3. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium E^* is globally asymptotically stable.*

Proof. Let $v(t) = (S(t), i(t,.))$ be a complete solution to (1.1) that lies in the attractor $\mathcal{A}_0 \subset \mathcal{M}_0$. From Proposition 5.3, we know there exist $\delta_1, \delta_2 > 0$ such that

$$\delta_1 < \frac{S_k(t)}{S_k^*} < \delta_2 \quad \text{and} \quad \delta_1 < \frac{i_k(t,a)}{i_k^*(a)} < \delta_2 \tag{6.1}$$

for $k = 1, 2$, $t \in \mathbb{R}$, and $a \geq 0$.

Let $g(u) = u - 1 - \ln(u)$. Then $g : (0, \infty) \to \mathbb{R}_+$ is decreasing for $x \in (0,1)$, is increasing for $x > 1$, and has global minimum $g(1) = 0$. It follows from (6.1) that $g\big(\frac{S_k(t)}{S_k^*}\big)$ and $g\big(\frac{i_k(t,a)}{i_k^*(a)}\big)$ are bounded.

Let

$$V_{S_k}(t) = g\left(\frac{S_k(t)}{S_k^*}\right), \qquad V_{i_k}(t) = \int_0^\infty \alpha_k(a) g\left(\frac{i_k(t,a)}{i_k^*(a)}\right) da,$$

where

$$\alpha_k(a) = \int_a^\infty \beta_k(\tau) i_k^*(\tau)\, d\tau. \tag{6.2}$$

Since β_k is bounded and $0 < i_k^*(\tau) \leq i_k^*(0)\mathrm{e}^{-D\tau}$, it follows that $\alpha_k(a)$ is bounded above by a decaying exponential. Then, since $g\big(\frac{i_k(t,.)}{i_k^*(.)}\big)$ is bounded, it can be shown that V_{i_k} is finite for each t, $k = 1, 2$.

We consider the Lyapunov functional

$$V(t) = V_{S_1}(t) + K\left(V_{i_1}(t) + V_{S_2}(t)\right) + V_{i_2}(t), \tag{6.3}$$

where $K = \frac{i_1^*(0)}{i_2^*(0)} \frac{S_2^*}{S_1^*}$. When necessary, we interpret V as a function of the state variables S and i. We note that, when restricted to $\mathcal{A}_0$, the function V is bounded. Furthermore, V obtains its minimum value of 0 only at E^*.

For clarity of presentation, we first find the derivatives of V_{S_k} and V_{i_k}, and then combine them to get the derivative of V. In the following calculation, we use the substitution

$$\lambda_k = d_k S_k^* + \int_0^\infty \beta_j(a) i_j^*(a) S_k^*\, da,$$

which comes from expanding $\frac{dS_k}{dt} = 0$ at the endemic equilibrium. Using this to replace λ_k, we obtain

$$\begin{aligned}
\frac{dV_{S_k}}{dt} &= \left(1 - \frac{S_k^*}{S_k}\right) \frac{1}{S_k^*} \frac{dS_k}{dt} \\
&= \left(1 - \frac{S_k^*}{S_k}\right) \frac{1}{S_k^*} \left[d_k\left(S_k^* - S_k\right) + \int_0^\infty \beta_j(a)\left(i_j^*(a) S_k^* - i_j(t,a) S_k\right) da\right] \\
&= -d_k \frac{\left(S_k - S_k^*\right)^2}{S_k S_k^*} + \left(1 - \frac{1}{x_k}\right) \int_0^\infty \beta_j(a) i_j^*(a) \left(1 - y_j x_k\right) da \\
&= -d_k \frac{\left(S_k - S_k^*\right)^2}{S_k S_k^*} + \int_0^\infty \beta_j(a) i_j^*(a) \left[1 - \frac{1}{x_k} - y_j x_k + y_j\right].
\end{aligned} \tag{6.4}$$

Next, we calculate the derivative of V_{i_k}.

$$\begin{aligned}\frac{dV_{i_k}}{dt} &= \frac{d}{dt}\int_0^\infty \alpha_k(a) g\left(\frac{i_k(t,a)}{i_k^*(a)}\right) da\\ &= \frac{d}{dt}\int_0^\infty \alpha_k(a) g\left(\frac{i_k(t-a,0)}{i_k^*(0)}\right) da.\end{aligned}$$

Using $\sigma = t - a$ to replace a, we obtain

$$\begin{aligned}\frac{dV_{i_k}}{dt} &= \frac{d}{dt}\int_{-\infty}^t \alpha_k(t-\sigma) g\left(\frac{i_k(\sigma,0)}{i_k^*(0)}\right) d\sigma\\ &= \alpha_k(0) g\left(\frac{i_k(t,0)}{i_k^*(0)}\right) + \int_{-\infty}^t \alpha_k'(t-\sigma) g\left(\frac{i_k(\sigma,0)}{i_k^*(0)}\right) d\sigma\\ &= \alpha_k(0) g\left(\frac{i_k(t,0)}{i_k^*(0)}\right) + \int_0^\infty \alpha_k'(a) g\left(\frac{i_k(t,a)}{i_k^*(a)}\right) da.\end{aligned}$$

Now, using (6.2) to replace $\alpha_k(0)$ and $\alpha_k'(a)$, and combining the integrals, we find

$$\begin{aligned}\frac{dV_{i_k}}{dt} &= \int_0^\infty \beta_k(a) i_k^*(a)\left(g\left(\frac{i_k(t,0)}{i_k^*(0)}\right) - g\left(\frac{i_k(t,a)}{i_k^*(a)}\right)\right) da\\ &= \int_0^\infty \beta_k(a) i_k^*(a)\left(g\left(z_k\right) - g\left(y_k\right)\right) da \qquad (6.5)\\ &= \int_0^\infty \beta_k(a) i_k^*(a)\left(z_k - y_k + \ln y_k - \ln z_k\right) da.\end{aligned}$$

Paying careful attention to the subscripts, we now use (6.4) and (6.5) to calculate the derivative of V, defined in (6.3), obtaining

$$\begin{aligned}\frac{dV}{dt} &= -d_1\frac{(S_1 - S_1^*)^2}{S_1 S_1^*} - K d_2 \frac{(S_2 - S_2^*)^2}{S_2 S_2^*}\\ &\quad + \int_0^\infty \beta_2(a) i_2^*(a)\left[1 - \frac{1}{x_1} - y_2 x_1 + z_2 + \ln y_2 - \ln z_2\right] da\\ &\quad + K\int_0^\infty \beta_1(a) i_1^*(a)\left[1 - \frac{1}{x_2} - y_1 x_2 + z_1 + \ln y_1 - \ln z_1\right] da.\end{aligned}$$

We now use Proposition 6.1 twice, with $H = 1 - \frac{1}{z_1}$ for $k = 2$ and with $H = 1 - \frac{1}{z_2}$ for $k = 1$, to add terms to the two integrals without changing the value of the expression. This gives

$$\begin{aligned}\frac{dV}{dt} &= -d_1\frac{(S_1 - S_1^*)^2}{S_1 S_1^*} - K d_2 \frac{(S_2 - S_2^*)^2}{S_2 S_2^*}\\ &\quad + \int_0^\infty \beta_2(a) i_2^*(a)\left[1 - \frac{1}{x_1} - z_1 + 1 - \frac{y_2 x_1}{z_1} + z_2 + \ln y_2 - \ln z_2\right] da\\ &\quad + K\int_0^\infty \beta_1(a) i_1^*(a)\left[1 - \frac{1}{x_2} - z_2 + 1 - \frac{y_1 x_2}{z_2} + z_1 + \ln y_1 - \ln z_1\right] da.\end{aligned}$$

Next, we interpret Proposition 6.2 as allowing an expression that does not depend on a to be subtracted from one integral and added to the other. We subtract $H =$

$z_2 - z_1 - \ln z_2 + \ln z_1$ from the first integral and add it to the second, obtaining

$$\begin{aligned}\frac{dV}{dt} &= -d_1\frac{(S_1-S_1^*)^2}{S_1S_1^*} - Kd_2\frac{(S_2-S_2^*)^2}{S_2S_2^*}\\&\quad+\int_0^\infty \beta_2(a)i_2^*(a)\left[1-\frac{1}{x_1}+1-\frac{y_2x_1}{z_1}+\ln y_2-\ln z_1\right]da\\&\quad+K\int_0^\infty \beta_1(a)i_1^*(a)\left[1-\frac{1}{x_2}+1-\frac{y_1x_2}{z_2}+\ln y_1-\ln z_2\right]da.\end{aligned}$$

We now add and subtract $\ln x_1$ to the first integral and $\ln x_2$ to the second. Then, by manipulating logarithms, we find

$$\begin{aligned}\frac{dV}{dt} &= -d_1\frac{(S_1-S_1^*)^2}{S_1S_1^*} - Kd_2\frac{(S_2-S_2^*)^2}{S_2S_2^*}\\&\quad+\int_0^\infty \beta_2(a)i_2^*(a)\left[1-\frac{1}{x_1}+\ln\frac{1}{x_1}+1-\frac{y_2x_1}{z_1}+\ln\frac{y_2x_1}{z_1}\right]da\\&\quad+K\int_0^\infty \beta_1(a)i_1^*(a)\left[1-\frac{1}{x_2}+\ln\frac{1}{x_2}+1-\frac{y_1x_2}{z_2}+\ln\frac{y_1x_2}{z_2}\right]da\\&= -d_1\frac{(S_1-S_1^*)^2}{S_1S_1^*} - Kd_2\frac{(S_2-S_2^*)^2}{S_2S_2^*}\\&\quad-\int_0^\infty \beta_2(a)i_2^*(a)\left[g\left(\frac{1}{x_1}\right)+g\left(\frac{y_2x_1}{z_1}\right)\right]da\\&\quad-K\int_0^\infty \beta_1(a)i_1^*(a)\left[g\left(\frac{1}{x_2}\right)+g\left(\frac{y_1x_2}{z_2}\right)\right]da \le 0,\end{aligned}$$

with equality if and only if

$$S_k = S_k^* \quad\text{and}\quad y_j = z_k \tag{6.6}$$

for $j,k = 1,2$ with $j \neq k$. We have now shown that the function V is nonincreasing along any complete solution $v(t)$ in the attractor $\mathcal{A}_0$. Consider a point $P = (S_P, 0_{\mathbb{R}^2}, i_P)$ in the alpha limit set of $v(t)$. We deduce that V is constant along any complete orbit $v^P(t) = \left(S^P(t), 0_{\mathbb{R}^2}, i^p(t)\right)$ passing through $P \in \mathcal{A}_0$. By (6.6), applied to v^P, we have

$$S^P = S^*$$

and

$$\frac{i_k^P(t,0)}{i_k^*(0)} = z_k(t) = y_j(t,a) = \frac{i_j^P(t,a)}{i_j^*(a)} = \frac{i_j^P(t-a,0)}{i_j^*(0)}.$$

Thus,

$$i_k^P(t,0) = i_j^P(t-a,0)\frac{i_k^*(0)}{i_j^*(0)}$$

for all $t \in \mathbb{R}$ and all $a \geq 0$. Since the left-hand side does not depend on a, it follows that the right-hand side must take on the same value for all $a \geq 0$. It then follows that both sides of the equation remain fixed as a and t are changed. Thus $i_k^P(t,0)$ is constant for $k = 1, 2$. Combined with $S^P = S^*$, we see that the complete solution v^P is constant. Hence

$$v^P(t) = E^* \quad \forall t \in \mathbb{R}.$$

Therefore the alpha limit set of $v(t)$ is simply $\{E^*\}$. Similarly, the omega limit set of $v(t)$ is also $\{E^*\}$. Since $t \to V(v(t))$ is nonincreasing function, we deduce that

$$V(v(t)) = V\left(E^*\right) \quad \forall t \in \mathbb{R}.$$

That is, V is constant along the complete solution v. We now apply to v the argument that was just used for v^P, concluding that

$$v(t) = E^* \quad \forall t \in \mathbb{R}.$$

We have now shown that the arbitrary complete solution v in the attractor A_0 must be the endemic equilibrium solution. Thus, we have shown that

$$\mathcal{A}_0 = \{E^*\}. \qquad \square$$

7. Application to nosocomial infection. In this section, we consider an epidemic of bacteria arising from the interaction between patients and health care workers (HCWs) in a hospital. This study is motivated by some earlier models introduced in D'Agata et al. [13] describing nosocomial infections. Such problems have been considered previously in several articles [12, 14, 13, 15, 24, 63]. The results of [13] suggest that including both antibiotic resistant and nonresistant strains of pathogen does not play a major role in the population level infection dynamics. Here we only focus on the transmission of the resistant strain. The age of infection is introduced in such a context to account for antibiotic treatment in the model.

Let $H_U(t)$ be the number of uncolonized HCWs, $H_C(t)$ the number of colonized HCWs, and $S(t)$ the number of susceptible patients at time t. Let $i(t, a)$ be the density of infected patients who have been infected for duration a at time t. Susceptible patients may become newly infected through interaction with a colonized HCW. Typically, the colonization of HCWs is of a superficial form, such as dirty hands that carry the pathogen. Thus, for colonized HCWs, we assume that the transmissibility does not depend on the age of infection. The rate ν_V at which contacts between staff and patients occur is taken to be constant. When a susceptible patient has contact with an HCW, the probability that it is with a contaminated HCW is equal to the fraction $\frac{H_C}{N_H}$ of HCWs that are colonized, where N_H is the total number of HCWs. Finally, given a contact between a susceptible patient and a contaminated HCW, the probability that the patient becomes infected is $P_I \in (0, 1]$. Thus, the rate of incidence of new infections in the patient population is $\frac{\nu_V P_I}{N_H} S H_C$. All newly infected patients enter the infected population with infection age 0.

For infected patients, we assume that the recovery rate ν_R is independent of the infection age. Upon recovery, patients either become susceptible or they leave the hospital and are replaced by new susceptibles. Here, we have assumed that the size of the hospital patient population remains constant. We now have the following equations to describe the patient population:

$$\begin{aligned}
&\frac{dS(t)}{dt} = \nu_R \int_0^{+\infty} i(t,a)da - \frac{\nu_V P_I}{N_H} S(t) H_C(t),\\
&\frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -\nu_R i(t,a),\\
&i(t,0) = \frac{\nu_V P_I}{N_H} S(t) H_C(t),\\
&S(0) = S_0 \geq 0,\\
&i(0,.) = i_0 \in L^1_+ (0,+\infty).
\end{aligned}$$

Next, we determine equations for the HCWs, beginning with the incidence. As in the patient equations, contacts occur at rate ν_V. Let $P_C \in (0,1]$ be the maximum probability that a contact between an infected patient and an uncontaminated HCW leads to a new contamination. The relative infectivity of patients of infection age a is $\gamma(a)$ and the density of contacts with patients of infection age a is $\frac{i(t,a)}{N_P}$, where N_P is the total number of patients. Thus, the incidence of new contaminations in the HCW population is $\frac{\nu_V P_C}{N_P} H_U \int_0^{+\infty} \gamma(a) i(t,a)da$. The decontamination rate for HCWs is ν_H. We now have the following equations to describe the HCW population:

$$\begin{aligned}
&\frac{dH_U(t)}{dt} = \nu_H H_C(t) - \frac{\nu_V P_C}{N_P} H_U(t) \int_0^{\infty} \gamma(a) i(t,a)da,\\
&\frac{dH_C(t)}{dt} = \frac{\nu_V P_C}{N_P} H_U(t) \int_0^{\infty} \gamma(a) i(t,a)da - \nu_H H_C(t),\\
&H_U(0) = H_{U0} \geq 0,\\
&H_C(0) = H_{C0} \geq 0.
\end{aligned}$$

It arises from these equations that N_P and N_H are fixed.

We now transform these equations into a form that is a special case of (1.1). To do so, in the equations for S and H_U, we make the substitutions

$$\int_0^{+\infty} i(t,a)da = N_P - S(t) \qquad \text{and} \qquad H_C(t) = N_H - H_U(t)$$

to obtain

$$\text{(7.1)} \qquad \begin{matrix}\text{Patient}\\ \text{equation}\end{matrix} \begin{cases}
\dfrac{dS(t)}{dt} = \nu_R N_P - \nu_R S(t) - \dfrac{\nu_V P_I}{N_H} S(t) H_C(t),\\
\dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\nu_R i(t,a),\\
i(t,0) = \dfrac{\nu_V P_I}{N_H} S(t) H_C(t),\\
S(0) = S_0 \geq 0,\\
i(0,.) = i_0 \in L^1_+ (0,+\infty)
\end{cases}$$

and
(7.2)

$$\text{HCW equation}\quad \begin{cases} \dfrac{dH_U(t)}{dt} = \nu_H N_H - \nu_H H_U(t) - \dfrac{\nu_V P_C}{N_P} H_U(t) \displaystyle\int_0^\infty \gamma(a) i(t,a) da, \\ \dfrac{dH_C(t)}{dt} = \dfrac{\nu_V P_C}{N_P} H_U(t) \displaystyle\int_0^\infty \gamma(a) i(t,a) da - \nu_H H_C(t), \\ H_U(0) = H_{U0} \geq 0, \\ H_C(0) = H_{C0} \geq 0. \end{cases}$$

The meaning of the parameters, as well as the values used in simulations, are listed in Table 1. In spite of the small sizes of typical patient and HCW populations, this model presents an invaluable opportunity to rigorously study different infection-age-dependent intervention strategies by observing the impact that different choices of $\gamma(\cdot)$ have on the behavior of the system.

Equations (7.1) and (7.2) combine to give a special case of (1.1), where group 1 is the patients and group 2 is the HCWs. Because infected HCWs are assumed to have a constant decontamination rate and a constant level of infectiousness, the infection age of this group plays no role. More formally, taking $H_C = \int_0^\infty i_2(t,a)da$, we see that all of the results obtained for (1.1) also apply to the nosocomial infection system being studied in this section.

Using (2.8) to calculate the basic reproduction number for this special case, we find that

$$\mathcal{R}_0 = \sqrt{\frac{\nu_V^2}{\nu_H} P_I P_C \int_0^{+\infty} \gamma(a) e^{-\nu_R a} da}.$$

Antibiotic treatments can be incorporated into the model through the function $\gamma(a)$. Remember that we consider only infection with the resistant strain. Therefore, $\gamma(a)$ can be interpreted as the relative likelihood that a patient infected with the resistant strain transmits the resistant pathogen.

Table 1

The parameter values are taken from [13] *and are used in the numerical simulations. Values marked with * were estimated for Beth Israel Deaconess Medical Center, Boston. Values marked with ** were estimated for Cook County Hospital, Chicago.*

Symbol	Interpretation	Value	Units
N_P	total number of patients	400*	-
N_H	total number of HCWs	100*	-
$\frac{1}{\nu_H}$	average time during which an HCW stays colonized	1*	hours
$\frac{1}{\nu_V}$	average duration of visit to a patient by an HCW plus time to the next visit	1.5*	hours
$\frac{1}{\nu_R}$	average time spent in the hospital for an infected patient	28*	days
P_I	probability for a patient to be infected by an HCW per visit	0.06**	-
P_C	probability for an HCW to be colonized by a patient per visit	0.4**	-
$\gamma(a)$	relative infectivity of patients of infection age a		-

One reasonable possibility is to assume that

$$\gamma(a) = \begin{cases} 1, \ a \in [\tau_1, \tau_2)\,, \\ 0 \text{ otherwise,} \end{cases}$$

where $0 < \tau_1 < \tau_2$ are two positive numbers. This corresponds to the biological assumptions that patients do not become infectious until a duration τ_1 has passed since becoming infected, and that the infection will be detected and effectively treated after a further time $\tau_2 - \tau_1$ has elapsed. Then we find that

$$\mathcal{R}_0 = \sqrt{\frac{\nu_V^2}{\nu_H \nu_R} P_I P_C \left(e^{-\nu_R \tau_1} - e^{-\nu_R \tau_2}\right)}.$$

The above formula suggests that the parameters τ_1 and τ_2 play a crucial role for the persistence (or the invasion) of resistant pathogens. Clearly, these parameters are related to antibiotic treatment (see [13]). At the level of a single patient, antibiotic treatment provides an in-host environment that selects in favor of the resistant strain. As a consequence, due to antibiotic treatment, patients may becomes more likely to transmit resistant pathogens. But the effects of treatments for a single patient is a fairly complex system. Some mechanisms involved in such problems have been described in [11, 2] (see also the references therein).

Figure 3 illustrates the main results of the paper in the context of nosocomial infections. In the first case, we find $\mathcal{R}_0 \approx 0.81 < 1$ and the disease dies out, as predicted by Theorem 4.3. In the second case, we find $\mathcal{R}_0 \approx 1.13 > 1$ and the system tends to the endemic equilibrium, as predicted by Theorem 6.3.

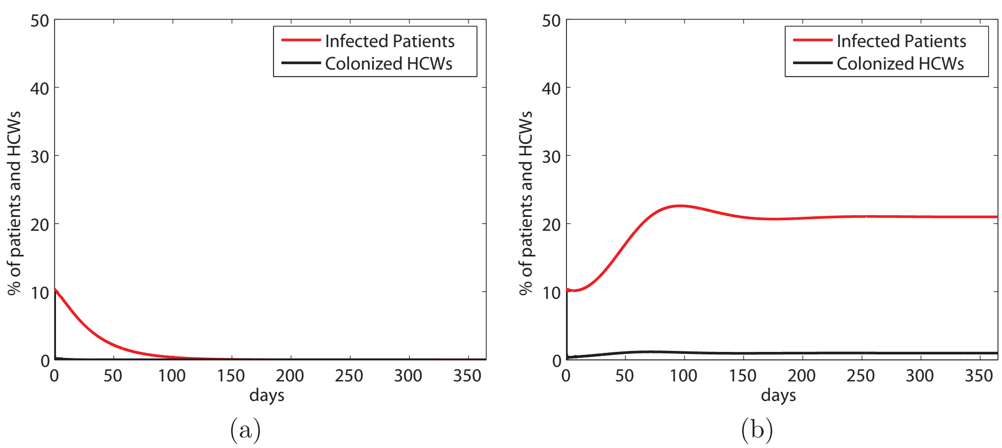

FIG. 3. *In these figures we use the values of the parameters in Table* 1*, and subfigure* (a) *corresponds to* $\tau_1 = 3$ *(days) and* $\tau_2 = 6$ *giving* $\mathcal{R}_0 \approx 0.81$*, while subfigure* (b) *corresponds to* $\tau_1 = 2$ *and* $\tau_2 = 8$*, giving* $\mathcal{R}_0 \approx 1.13$.

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