Persistence of a normally hyperbolic manifold for a system of non densely defined Cauchy problems

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Abstract

We consider a system of non densely defined Cauchy problems and we investigate the persistence of normally hyperbolic manifolds. The notion of exponential dichotomy is used to characterize the normal hyperbolicity and a generalized Lyapunov-Perron approach is used in order to prove our main result. The result presented in this article extend the previous results on the center manifold by allowing a nonlinear dynamic in the unperturbed central part of the system. We consider two examples to illustrate our results. The first example is a parabolic equation coupled with an ODE that can be considered as an interaction between an antimicrobial and bacteria while the second one is a Ross-Macdonald epidemic model with age of infection. In both examples we were able to reduce the infinite dimensional system to an ordinary differential equation.

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1 Introduction

The invariant manifolds theory plays an important role in understanding the asymptotic behavior of dynamical systems. It can be traced back to Poincaré [41], Hadamard [19], Lyapunov [28] and Perron [38, 39, 40]. Intuitively persistence of normally hyperbolic invariant manifolds (see Definition 3.9) can be understood as a generalization of the persistence of hyperbolic equilibrium point to invariant sets (for example a set of equilibrium points, heteroclinic orbit, homoclinic orbit) [4]. This generalization can be done if we know the dynamic in the invariant manifold as well as the dynamic on its normal directions. Roughly speaking in this paper the orbits lying in the manifold behave like in a center manifold while the orbits lying on the normal directions expend or contract

toward the manifold. To describe such a behavior the notion of exponential dichotomy will be used rather than the definition introduced in [4, 5, 13, 14, 15, 23]. Exponential dichotomy has been already used by several authors to describe normal hyperbolicity (see [6, 26, 35, 45, 46] and the references therein).
This theory has a long and rich history. It was popularized by Fenichel in the series of papers [13, 14, 15] and Hirsch Pugh and Shub [23] in the context of finite time dimensional settings. Since then many work have been done in finite and in infinite dimension so that it is very difficult to give an exhaustive list. We refer to [6, 15, 24, 26, 36, 45, 56] for different approaches and results on this subject for ordinary differential equations. By contrast of the finite dimensional setting, normally hyperbolic manifold have been hardly considered in the context of infinite dimensional dynamical systems. In the best of our knowledge Henry [21] obtained one of the earliest result on this topic for semilinear parabolic equations with a proof based on Lyapunov-Perron approach. A remarkable work in this direction can be attributed to Bates, Lu and Zeng [4, 5]. A closely related result to [4] can be found in [27] where the authors have investigated invariant manifolds for PDEs. In the present paper we will study the persistence of normally hyperbolic manifolds for a class of non densely defined problem. We consider the following system of equations

$$\begin{cases} \dot{u}(t) = F\left(u(t)\right) + K\left(u\left(t\right), v\left(t\right)\right), \ t > t_0 \ \text{ and } u(t_0) = x \in X, \\ \dot{v}(t) = \left[A + B(u(t))\right] v(t) + G\left(u(t), v(t)\right), \ t > 0 \ \text{ and } v(t_0) = y \in \overline{D(A)} \end{cases} \tag{1.1}$$

where the dot denote the derivative with respect to t. Here $A : D\left(A\right) \subset Y \to Y$ is an unbounded linear operator with possibly non dense domain while $B(.) : X \to \mathcal{L}(\overline{D\left(A\right)}, Y)$. The functions $K : X \times \overline{D\left(A\right)} \to X$ and $G : X \times \overline{D\left(A\right)} \to Y$ are non-linear bounded and Lipschitz continuous maps while $F : X \to X$ can be a bounded linear operator or a Lipschitz continuous non linear map. We note that the main complexity of such systems arise from the fact that in general we have $\overline{D(A)} \neq Y$.

In order to understand the notion of solution we will use the notion of integrated semigroups. Integrated semigroup was first introduced in the Hille-Yosida case by Ardent [1, 2, 3], Thieme [48] and other. See [3, 33] for a nice survey. This theory allows to deal with semilinear Cauchy problem (see [49, 29]). More recently this theory has been extended to the non Hille-Yosida case by Magal and Ruan [30, 31, 32] and Thieme [50]. We refer to Magal and Ruan [33] for a updated overview on the theory of abstract semilinear problem.

Next we briefly describe the main ideas used in order to obtain persistence of a normally hyperbolic manifold.
Step 1: System (1.1) might be considered as a perturbation of

$$\begin{cases} \dot{u}(t) = F\left(u(t)\right), \ t > t_0 \ \text{ and } u(t_0) = x \in X, \\ \dot{v}(t) = \left[A + B(u(t))\right] v(t), \ t > t_0 \ \text{ and } v(t_0) = y \in \overline{D(A)}. \end{cases} \tag{1.2}$$

If we set

$$\mathcal{M} := X \times \{0_Y\} = \{(x, \psi(x)) \in X \times \overline{D(A)} : x \in X\}$$

with

$$\psi(x) = 0, \ \forall x \in X$$

then $\mathcal{M}$ is an invariant manifold for (1.2) (see Definition 3.9). Thus the normal hyperbolicity of $\mathcal{M}$ is expressed in term of exponential growth condition for system (1.2). More precisely we assume that solution of the u-equation of (1.2) growth sub-exponentially (like in a center manifold) and the evolution family generated by the v-equation of (1.2) has an exponential dichotomy (see Definition 2.12).

Step 2: We consider

$$\begin{cases} \dot{u}(t) = F\left(u(t)\right) + K(u(t), v(t)), \ t > t_0 \ \text{ and } u(t_0) = x \in X, \\ \dot{v}(t) = \left[A + B(u(t))\right] v(t), \ t > t_0 \ \text{ and } v(t_0) = y \in \overline{D(A)}. \end{cases} \tag{1.3}$$

and show that if the Lipschitz norm of K is sufficiently small then the u-equation of (1.3) growth sub-exponentially and the evolution family generated by the v-equation of (1.3) has an exponential dichotomy.

Step 3: Finally we come back to system (1.1) to perform a fixed point problem that leads to the existence of the desired normally hyperbolic manifold

$$\hat{\mathcal{M}} = \{(x, \hat{\psi}(x)) \in X \times \overline{D(A)} : x \in X\}.$$

Note that it is not so restrictive to consider $X \times \{0_Y\}$ as an invariant manifold since in many situations a change of coordinates system allows to bring the study in this context. This is the case when the manifold is expressed as a graph of a function. We also refer to [21] where a coordinates change is proposed in order to bring the persistence of some general manifold to our context. In this paper we will show that under sufficient conditions on B, K, F and G, system (1.1) admits a unique normally hyperbolic invariant manifold.

Note that the assumptions on the operator A (See Assumptions 2.1 and 2.3) in system (1.1) allows to account several classes of differential equations. More precisely it incorporate retarded functional differential equations, parabolic differential equations with local or non local boundary conditions as well as hyperbolic differential equations with linear or non linear boundary conditions. Therefore the result presented in this paper can be applied to a wide class of differential equations (see [8, 30, 31] for more details).

To the best of our knowledge, the persistence of normally hyperbolic invariant manifold for system (1.1), with A non densely defined and non Hille-Yosida operator is not studied in the literature. When A is Hille-Yosida a resent result on the existence of unstable manifold was obtained by Jendoubi [25]. In [25] the results are based on the extrapolation method to define the notion of mild solution for the abstract Cauchy problem and a non-autonomous variation of constants formula obtain previously by Gühring and Räbiger [17]. Here since A is not Hille-Yosida such appraoch no longer applies. Instead, we extended Gühring and Räbiger's [17] non autonomous variation of constants in Magal and Seydi [34] (see Lemma 2.9).

This paper can be regarded as a first step for a singular perturbation theory in the context of non densely defined Cauchy problems. Also the results presented in this article can be regarded as an extension of the center manifold theorem presented in Magal and Ruan [32] whenever F is a bounded linear map on X and $B = 0$. To be more precise in [32] the authors proved the existence of a center manifold for the following system

$$\begin{cases} \dot{u}(t) = A_c u(t) + K\left(u\left(t\right), v\left(t\right)\right), \ t > t_0 \ \text{ and } u(t_0) = x \in X, \\ \dot{v}(t) = Av(t) + G\left(u(t), v(t)\right), \ t > t_0 \ \text{ and } v(t_0) = y \in \overline{D(A)} \end{cases} \tag{1.4}$$

where A_c is a bounded linear operator such that A_c is a bounded linear operator on X satisfying for each $\beta_0 > 0$,

$$\sup_{t \in \mathbb{R}} e^{-\beta_0 |t|} \| e^{A_c t} \|_{\mathcal{L}(X)} < \infty,$$

and the semigroup generated by A_0 (the part of A in $\overline{D(A)}$) has an exponential dichotomy. Compared to (1.4) the major difficulty arise from the v-equation of system (1.1) where the unperturbed part $[A+B(u(t))]v(t)$ may depends non linearly in $u(t)$. The foregoing discussion of our methodology in Step 1, 2, 3 will be used to generalize the Lyapunov-Perron method in order to deal with the persistence of the normally hyperbolic manifold.

The paper is organized as follow. In Section 2 we present notions that will be used in this paper. Namely the notion of mild solutions and exponential dichotomy. We also recall some results proved in [34]. In Section 3 we present our main result. In Section 4 we illustrate our results by considering two examples. The first example is a parabolic equation coupled with an ODE that can be considered as an interaction between an antimicrobial and bacteria while the second one is a Ross-Macdonald epidemic model with age of infection. In both examples we show that our results apply to these examples, and permit to reduce an infinite dimensional system to a scalar ordinary differential equation. The examples serve to illustrate the assumptions of our theoretical results and show how to verify such assumptions. In Section 5 we prove some crucial results related to the persistence of exponential dichotomy while Section 6 consists of the proof of the main result of this paper.

2 Preliminaries

In this section we will give the definitions that will be needed in the sequel. We will briefly discuss and recall some results proved in [34]. Since the domain of the linear operator A may be not dense in Y, we need to make some non classical assumption in order to deal with the non-homogeneous problem. Set $Y_0 := \overline{D(A)}$ and consider $A_0 : D(A_0) \subset Y_0 \to Y_0$ the part of A in Y_0 that is

$$A_0 y = Ay, \quad \forall y \in D(A_0),$$

and

$$D(A_0) = \{y \in D(A) : Ay \in Y_0\}.$$

Let $\bar{B} : \mathbb{R} \to \mathcal{L}(Y_0, Y)$ be given. Consider the following non homogeneous Cauchy problem

$$\dot{w}(t) = \left[A + \bar{B}(t)\right] w(t) + f(t), \quad t > t_0 \quad \text{and} \quad v(t_0) = y \in Y_0, \tag{2.1}$$

with $f \in L^1_{loc}(\mathbb{R}, Y)$. Let us also consider the linear non autonomous Cauchy problem

$$\dot{w}(t) = \left[A + \bar{B}(t)\right] w(t), \quad t > t_0 \quad \text{and} \quad w(t_0) = w_0 \in Y_0. \tag{2.2}$$

Assumption 2.1 *We assume that*

(i) *There exist two constants $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ (the resolvent set of A) and*

$$\left\| (\lambda I - A)^{-k} \right\|_{\mathcal{L}(Y_0)} \leq M_A \left(\lambda - \omega_A\right)^{-k}, \ \forall \lambda > \omega_A, \ k \geq 1.$$

(ii) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} y = 0, \ \forall y \in Y.$

It is important to note that Assumption 2.1 do not say that A is a Hille-Yosida linear operator since the operator norm in (i) is taken into $Y_0 \subseteq Y$ (where the inclusion can be strict) instead of Y. However from [32, Lemma 2.1] we have $\rho(A) = \rho(A_0)$. Note that

$$(\lambda I - A)^{-1} \mid_{Y_0} = (\lambda I - A_0)^{-1}$$

and Assumption 2.1-(ii) is equivalent $\overline{D(A)} = \overline{D(A_0)} = Y_0$.

Therefore we have the following lemma.

Lemma 2.2 *Assumption 2.1 is satisfied if and only if $\rho(A) \neq \emptyset$ and $(A_0, D(A_0))$ generates a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0} \subset \mathcal{L}(Y_0)$ with*

$$\|T_{A_0}(t)\|_{\mathcal{L}(Y_0)} \leq M_A e^{\omega_A t}, \quad \forall t \geq 0.$$

In order to obtain existence and uniqueness of solutions for (2.1) whenever f is a continuous map, we will require the following assumption.

Assumption 2.3 *Assume that for any $\tau > 0$ and $f \in C([0, \tau], Y)$ there exists $v_f \in C([0, \tau], Y_0)$ an integrated (mild) solution of*

$$\frac{dv(t)}{dt} = Av(t) + f(t), \ \forall t \in [0, \tau] \text{ and } v(0) = 0$$

that is to say that

$$\int_0^t v_f(r)dr \in D(A), \forall t \in [0, \tau]$$

and

$$v_f(t) = A \int_0^t v_f(r)dr + \int_0^t f(r)dr, \forall t \in [0, \tau].$$

Moreover we assume that there exists a non decreasing map $\delta : [0, +\infty) \to [0, +\infty)$ with $\delta(t) \to 0$ as $t \to 0^+$ such that for each $f \in C([0, \tau], Y)$ and each $\tau > 0$ we have

$$\|v_f(t)\| \leq \delta(t) \sup_{s \in [0,t]} \|f(s)\|, \forall t \in [0, \tau].$$

Remark 2.4 *In Assumption 2.3 the uniqueness of mild solutions is a consequence of a uniqueness result proved by Thieme [48]. Moreover it is a work in itself to check the existence of such mild solutions. It is important to observe that we require the mild solutions to exist only for the continuous function $t \to f(t)$. This together with the L^∞-estimation, this is the major difference with the standard Hille-Yosida case.*

In the general, necessary and sufficient condition has been obtain in Magal and Ruan [30] (by using a condition of the resolvent of A) and Thieme [50] (by using a condition of the integrated semigroup generated by A).

For abstract parabolic equation, the case of almost sectorial operator has been considered. One can use some sufficient estimation on the resolvent of A to derive Assumption 2.3. We refer to Ducrot, Magal and Prevost [8] and Ducrot and Magal [7].

We refer to the monograph of Magal and Ruan [33] for more detailed discussions and examples.

Remark 2.5 *Note that Assumptions 2.1 and 2.3 are satisfied whenever A is Hille-Yosida (see [33] for this result) and we have the following estimate*

$$\|v_f(t)\| \leq \int_0^t M_A e^{\omega_A s} ds \sup_{s\in[0,t]} \|f(s)\|, \ \forall t \geq 0.$$

Assumption 2.6 *Assume that $\{\bar{B}(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(Y_0, Y)$ is strongly continuous that is to say that for any $y \in Y_0$, the map $t \to \bar{B}(t)y$ is continuous. Assume in addition that $t \to \bar{B}(t)$ is locally bounded in norm of operator that is*

$$\sup_{t\in[-n,n]} \|\bar{B}(t)\|_{\mathcal{L}(Y_0,Y)} < +\infty, \ \forall n \in \mathbb{N}.$$

Define

$$\Delta := \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\}$$

Recall that $\{U_{\bar{B}}(t, t_0)\}_{(t,t_0)\in\Delta} \subset \mathcal{L}(Y_0)$ is an evolution family if and only if

$$U_{\bar{B}}(t, l)U_{\bar{B}}(l, t_0) = U_{\bar{B}}(t, t_0), \ \forall t, l, t_0 \in \mathbb{R} \text{ with } t \geq l \geq t_0,$$

and

$$U_{\bar{B}}(t, t)y = y, \ \forall t \in \mathbb{R} \ \text{ and } \ \forall y \in Y_0.$$

Definition 2.7 *Let $t_0 \in \mathbb{R}$ and $f \in C([t_0, +\infty), Y)$ be fixed. We say that $w \in C([t_0, +\infty), Y_0)$ is a mild solution of (2.1) if and only if*

$$\int_{t_0}^t w(r)dr \in D(A), \ \forall t \geq t_0$$

and

$$w(t) = y + A\int_{t_0}^t w(r)dr + \int_{t_0}^t \bar{B}(r)w(r)dr + \int_{t_0}^t f(r)dr, \ \forall t \geq t_0.$$

Definition 2.8 *Let $f \in C(\mathbb{R}, Y)$ be fixed. We say that $w \in C(\mathbb{R}, Y_0)$ is a mild solution of (2.1) if and only if*

$$\int_{t_0}^t w(r)dr \in D(A), \ \forall (t, t_0) \in \Delta$$

and

$$w(t) = w(t_0) + A\int_{t_0}^t w(r)dr + \int_{t_0}^t \bar{B}(r)w(r)dr + \int_{t_0}^t f(r)dr, \ \forall (t, t_0) \in \Delta.$$

The following result will play a crutial role in the analysis of the problem. This result has been proved in [34].

Lemma 2.9 (Non autonomous variation of constants formula) *Let Assumptions 2.1, 2.3 and 2.6 be satisfied. Then (2.2) generates a unique evolution family* $\{U_{\bar{B}}(t,t_0)\}_{(t,t_0)\in\Delta} \subset \mathcal{L}(Y_0)$ *with* $U_{\bar{B}}(\cdot,t_0)y \in C([t_0,+\infty),Y_0)$ *the fixed point of*

$$U_{\bar{B}}(t,t_0)y = T_{A_0}(t-t_0)y + \lim_{\lambda\to+\infty}\int_{t_0}^{t} T_{A_0}(t-s)\lambda R_\lambda(A)\bar{B}(s)U_{\bar{B}}(s,t_0)y ds,\ \forall t \geq t_0,$$

where $R_\lambda(A) := (\lambda I - A)^{-1}$ *is the resolvent of* A.

Moreover the following properties hold true :

(i) *For all* $f \in C(\mathbb{R},Y)$, $t_0 \in \mathbb{R}$ *and* $y \in Y_0$, *there exists a unique mild solution* $w \in C([t_0,+\infty),Y_0)$ *of (2.1). Moreover* w *is given by the non autonomous variation of constants formula*

$$w(t) = U_{\bar{B}}(t,t_0)y + \lim_{\lambda\to+\infty}\int_{t_0}^{t} U_{\bar{B}}(t,r)\lambda R_\lambda(A)f(r)dr.$$

(ii) *If* $\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(Y_0,Y)} < +\infty$ *then there exists some constants* $\hat{M} \geq 1$ *and* $\hat{\omega} \in \mathbb{R}$ *such that*

$$\|U_{\bar{B}}(t,t_0)\|_{\mathcal{L}(Y_0)} \leq \hat{M}e^{\hat{\omega}(t-t_0)}, \quad \forall (t,t_0) \in \Delta.$$

Remark 2.10 *The above result is non trivial since* $R_\lambda(A)$ *(the resolvent of* A*) and* $U_{\bar{B}}(t,s)$ *do not commute. One can also prove (see [32, Lemma 2.2]) that*

$$\lim_{\lambda\to+\infty}\lambda R_\lambda(A)y = y \Leftrightarrow y \in \overline{D(A)}.$$

Moreover the limit $\lim_{\lambda\to+\infty}\lambda R_\lambda(A)y$ *does not exist in general whenever* $y \in Y \setminus \overline{D(A)}$.

The following result is proved in [34, Proposition 5.5] and is a useful tool in studying mild solution for non Hille-Yosida operators in the non-autonomous case.

Proposition 2.11 *Let Assumptions 2.1, 2.3 and 2.6 be satisfied. Assume in addition that*

$$\bar{b} := \sup_{t\in\mathbb{R}}\|\bar{B}(t)\|_{\mathcal{L}(Y_0,Y)} < +\infty.$$

Then there exists a non decreasing map $\delta^* := \delta^*(A,\omega_A,M_A,\bar{b},\delta) : [0,+\infty) \to [0,+\infty)$ *with* $\delta^*(t) \to 0$ *as* $t \to 0^+$ *such that for each* $f \in C(\mathbb{R},Y)$ *the map*

$$w(t,t_0) = \lim_{\lambda\to+\infty}\int_{t_0}^{t} U_{\bar{B}}(t,s)\lambda R_\lambda(A)f(s)ds,\ (t,t_0) \in \Delta$$

satisfies

$$\|w(t,t_0)\| \leq \delta^*(t-t_0)\sup_{s\in[t_0,t]}\|f(s)\|,\ \forall (t,t_0) \in \Delta.$$

In order to state the next result we first recall the notion of exponential dichotomy.

Definition 2.12 *We say that an evolution family* $\{U(t,t_0)\}_{(t,t_0)\in\Delta} \subset \mathcal{L}(Y_0)$ *has an* ***exponential dichotomy*** *on* $\mathbb{R}$ *with constant* $\kappa \geq 1$ *and exponent* $\beta > 0$ *if and only if the following properties are satisfied*

(i) *There exist two strongly continuous families of projections* $\{\Pi^+(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(Y_0)$ *and* $\{\Pi^-(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(Y_0)$ *such that*

$$\Pi^+(t) + \Pi^-(t) = I_{\mathcal{L}(Y_0)}, \quad \forall t \in \mathbb{R}.$$

Then we define for all $(t,t_0) \in \Delta$

$$U^+(t,t_0) := U(t,t_0)\Pi^+(t_0) \quad \text{and} \quad U^-(t,t_0) := U(t,t_0)\Pi^-(t_0).$$

(ii) *For all* $(t,t_0) \in \Delta$ *we have* $\Pi^+(t)U(t,t_0) = U(t,t_0)\Pi^+(t_0)$.

(iii) *For all* $(t,t_0) \in \Delta$ *the restricted linear operator* $U(t,t_0)\Pi^-(t_0)$ *is invertible from* $\Pi^-(t_0)(Y_0)$ *into* $\Pi^-(t)(Y_0)$ *with inverse denoted by* $\bar{U}^-(t_0,t)$ *and we set*

$$U^-(t_0,t) := \bar{U}^-(t_0,t)\Pi^-(t).$$

(iv) *For all* $(t,t_0) \in \Delta$

$$\|U^+(t,t_0)\|_{\mathcal{L}(Y_0)} \leq \kappa e^{-\beta(t-t_0)} \quad \text{and} \quad \|U^-(t_0,t)\|_{\mathcal{L}(Y_0)} \leq \kappa e^{-\beta(t-t_0)}.$$

If $\Pi^-(t) = 0_{\mathcal{L}(Y_0)}$ *for all* $t \in \mathbb{R}$ *we say that the evolution family is* ***exponentially stable***.

The following results are proved in [34, Theorem 1.10].

Theorem 2.13 *Let Assumptions 2.1, 2.3 and 2.6 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|\bar{B}(t)\|_{\mathcal{L}(Y_0,Y)} < +\infty.$$

Then the following assertions are equivalent

(i) *The evolution family* $\{U_{\bar{B}}(t,s)\}_{(t,s)\in\Delta}$ *has an exponential dichotomy.*

(ii) *For each* $f \in BC(\mathbb{R},Y)$, *there exists a unique integrated solution* $u \in BC(\mathbb{R},Y_0)$ *of* (2.1).

From now on define for $Z = Y_0, Y, X$

$$BC^\gamma(\mathbb{R},Z) := \left\{ f \in C(\mathbb{R},Z) : \|f\|_\gamma := \sup_{t\in\mathbb{R}} e^{-\gamma|t|}\|f(t)\| < +\infty \right\}, \forall \gamma \geq 0.$$

Then we have the following result which was proved in [34, Theorem 1.11].

Theorem 2.14 *Let Assumptions 2.1, 2.3 and 2.6 be satisfied. Assume in addition that*

$$\sup_{t\in\mathbb{R}} \|\bar{B}(t)\|_{\mathcal{L}(Y_0,Y)} < +\infty.$$

If $U_{\bar{B}}$ has an exponential dichotomy with exponent $\beta > 0$, then for each $\gamma \in [0,\beta)$ and each $f \in BC^\gamma(\mathbb{R}, Y)$ there exists a unique integrated solution $u \in BC^\gamma(\mathbb{R}, Y_0)$ of (2.1) which is given by

$$u_f(t) = \lim_{\lambda\to+\infty} \left[\int_{-\infty}^{t} U_{\bar{B}}^{+}(t,s)\lambda R_\lambda(A) f(s) ds - \int_{t}^{+\infty} U_{\bar{B}}^{-}(t,s)\lambda R_\lambda(A) f(s) ds\right], \ \forall t \in \mathbb{R}. \tag{2.3}$$

Moreover the following properties hold true

(i) *The limit (2.3) exists uniformly on compact subset of $\mathbb{R}$.*

(ii) *If f is bounded and uniformly continuous with relatively compact range then the limit (2.3) is uniform on $\mathbb{R}$.*

(iii) *For each $\nu \in (-\beta, 0)$ there exists $C(\nu,\kappa,\beta) > 0$ such that*

$$\|u_f\|_\gamma \le C(\nu,\kappa,\beta)\|f\|_\gamma, \ \forall \gamma \in [0,-\nu].$$

Remark 2.15 *It is important to point out in Section 6 we will use the fact that the constant $C(\nu,\kappa,\beta)$ does not depend explicitly of $\bar{B}$. But $C(\nu,\kappa,\beta)$ depends on*

$$\bar{b} := \sup_{t\in\mathbb{R}} \|\bar{B}(t)\|_{\mathcal{L}(Y_0,Y)}.$$

3 Main result

Throughout this section the following series of assumptions will be required.

Assumption 3.1 *Assume that $F \in C^1(X)$ and there exists a constant $L_F \ge 0$ such that for all $x, \bar{x} \in X$*

$$\begin{cases} \|F(x) - F(\bar{x})\| \le L_F\|x - \bar{x}\|, \\ \|DF(x) - DF(\bar{x})\|_{\mathcal{L}(X)} \le L_F\|x - \bar{x}\|. \end{cases}$$

Assumption 3.2 *Assume that there exist $\beta_0 \ge 0$ and $\kappa_0 \ge 1$ such that if $u_1, u_2 \in C^1(\mathbb{R}, X)$ satisfy*

$$\dot{u}(t) = F(u(t)), \ \forall t \in \mathbb{R}$$

then

$$\|u_1(t) - u_2(t)\| \le \kappa_0 e^{\beta_0|t-l|}\|u_1(l) - u_2(l)\|, \quad \forall t, l \in \mathbb{R}. \tag{3.1}$$

Remark 3.3 *Assumption 3.2 means that the solutions $t \to u(t)$ growth sub-exponentially. A simple case in which Assumption 3.2 is always satisfied is*

$$F(x) = x_0 + A_c x, \ \forall x \in X$$

with $x_0 \in X$ and A_c is a bounded linear operator on X such that

$$\sup_{t\in\mathbb{R}} e^{-\beta_0|t|}\|e^{A_c t}\|_{\mathcal{L}(X)} < \infty.$$

This last condition is satisfied whenever the dimension of X is finite and $\sigma(A_c) \subset i\mathbb{R}$.

Assumption 3.4 *Assume that there exist $\beta > \beta_0 \geq 0$ and $\kappa \geq 1$ such that if $u \in C^1(\mathbb{R}, X)$ satisfies*

$$\dot{u}(t) = F(u(t)), \ \forall t \in \mathbb{R}$$

then the evolution family generated by

$$\dot{w}(t) = [A + B(u(t))]\, w(t), \ t > t_0 \text{ and } w(t_0) = w_0 \in Y_0,$$

has an exponential dichotomy with constant κ and exponent β.

Remark 3.5 *When the map $u \to B(u)$ is constant, then the exponential dichotomy can be expressed in term of spectral properties. To be more precise it has been proved in [31] that the evolution family generated by*

$$\dot{w}(t) = [A + B]\, w(t), \ t > t_0 \text{ and } w(t_0) = w_0 \in Y_0,$$

has an exponential dichotomy if the following conditions are satisfied :

(i) *A satisfies Assumption 2.1.*

(ii) *$\sigma((A + B)_0)) \cap i\mathbb{R} = \emptyset$.*

(iii) *$\{\lambda \in \sigma((A+B)_0)) : \mathcal{R}e(\lambda) > 0\}$ is non empty and the essential growth rate of $\left\{T_{(A+B)_0}(t)\right\}_{t \geq 0}$ is negative. That is to say that*

$$\omega_{0,ess}((A + B)_0) := \lim_{t \to +\infty} \frac{\ln(\|T_{(A+B)_0}(t)\|_{ess})}{t} < 0 \tag{3.2}$$

where $\|L\|_{ess}$ of a bounded linear operator $L \in \mathcal{L}(Y_0)$ is defined by

$$\|L\|_{ess} = Mes\left(\{Ly : y \in Y_0, \ \|y\| \leq 1\}\right)$$

and

$$Mes(\Omega) := \inf\left\{\varepsilon > 0 : \Omega \text{ can be covered by a finite number of balls of radius} \leq \varepsilon\right\}.$$

Here $Mes(\cdot)$ is the measure of non compactness of Kuratovsky. We refer to Webb [55], Engel and Nagel [12] and Magal and Ruan [33] for more results about the spectral theory for strongly continuous semigroups of bounded linear operators.

We will also need some regularity conditions on the perturbations K, G and the map B.

Assumption 3.6 *Assume that there exist $\zeta > 0$, $\eta > 0$, $\sigma > 0$ and $L_G > 0$ such that if we set*

$$B_{Y_0}(0, \zeta) := \{y \in Y : \|y\| \leq \zeta\}$$

then

(i) *The map $(x, y) \to K(x, y)$ is continuously differentiable on an open neighborhood of $X \times B_{Y_0}(0, \zeta)$ and for all $(x, y), (\bar{x}, \bar{y}) \in X \times B_{Y_0}(0, \zeta)$*

$$\|K(x, y)\| \leq \eta \quad \text{and} \quad \|K(x, y) - K(\bar{x}, \bar{y})\| \leq \eta\|x - \bar{x}\| + \eta\|y - \bar{y}\|.$$

(ii) *The map* $(x, y) \to G(x, y)$ *is continuous on* $X \times B_{Y_0}(0, \zeta)$ *and for all* $(x, y), (\bar{x}, \bar{y}) \in X \times B_{Y_0}(0, \zeta)$

$$\|G(x, y)\| \leq \hat{\sigma} \quad \text{and} \quad \|G(x, y) - G(\bar{x}, \bar{y})\| \leq L_G\|x - \bar{x}\| + \sigma\|y - \bar{y}\|.$$

Assumption 3.7 *The map* $x \to B(x)$ *is continuous on* X *and there exists a constant* $L_B \geq 0$ *such that for all* $x, \bar{x} \in X$

$$\|B(x)\|_{\mathcal{L}(Y_0, Y)} \leq L_B \quad \text{and} \quad \|B(x) - B(\bar{x})\|_{\mathcal{L}(Y_0, Y)} \leq L_B\|x - \bar{x}\|.$$

Before giving our main result we define what we call mild solution for system (1.1) and normally hyperbolic manifold.

Definition 3.8 *We say that* $(u, v) \in C^1(\mathbb{R}, X) \times C(\mathbb{R}, Y_0)$ *is a mild solution of (1.1) on* $\mathbb{R}$ *if and only if*

$$\begin{cases} \dot{u}(t) = F(u(t)) + K(u(t), v(t)), \quad t \in \mathbb{R} \\ v(t) = v(t_0) + A\displaystyle\int_{t_0}^{t} v(l)dl + \int_{t_0}^{t} B(u(l))v(l)dl + \int_{t_0}^{t} G(u(l), v(l))dl, \ \forall (t, t_0) \in \Delta. \end{cases}$$

Definition 3.9 *Let* $\hat{\psi} : X \to Y_0$ *be a map. Let* $\hat{\mathcal{M}} = \{(x, \hat{\psi}(x)) \in X \times Y_0 : x \in X\}$ *be a given manifold. We say that* $\hat{\mathcal{M}}$ *is a normally hyperbolic invariant manifold for* (1.1) *with constants* $\hat{\kappa} \geq 1$, $\hat{\kappa}_0 \geq 1$ *and exponents* $\hat{\beta} > 0$, $\hat{\beta}_0 \in [0, \hat{\beta})$ *if the following properties are satisfied*

(i) *For each* $(x, \hat{\psi}(x)) \in \hat{\mathcal{M}}$ *there exists a unique mild solution* $(u, v) \in C^1(\mathbb{R}, X) \times BC(\mathbb{R}, Y_0)$ *of (1.1) such that*

$$(u(0), v(0)) = (x, \hat{\psi}(x)) \quad \text{and } v(t) = \hat{\psi}(u(t)), \forall t \in \mathbb{R}.$$

(ii) *If* $(u, v) \in C^1(\mathbb{R}, X) \times BC(\mathbb{R}, Y_0)$ *is a mild solution of (1.1) in* $\hat{\mathcal{M}}$ *then the evolution family generated by*

$$\dot{w}(t) = [A + B(u(t))]w(t), \ t > t_0 \text{ and } w(t_0) = w_0 \in Y_0$$

has an exponential dichotomy with constant $\hat{\kappa} \geq 1$ *and exponent* $\hat{\beta} > 0$.

(iii) *If* $(u, v), (\bar{u}, \bar{v}) \in C^1(\mathbb{R}, X) \times BC(\mathbb{R}, Y_0)$ *are mild solutions of (1.1) in* $\hat{\mathcal{M}}$ *then*

$$\|u(t) - \bar{u}(t)\| \leq \hat{\kappa}_0 e^{\hat{\beta}_0|t-l|}\|u(l) - \bar{u}(l)\|, \quad \forall t, l \in \mathbb{R}.$$

Remark 3.10 *Observe that according to this definition and Assumptions 3.2, 3.4*

$$\mathcal{M} := X \times \{0_Y\} = \{(x, \psi(x)) \in X \times Y_0 : x \in X\}$$

with

$$\psi(x) = 0, \ \forall x \in X$$

is a normally hyperbolic invariant manifold for (1.2) with constants κ, κ_0 *and exponents* $\beta > 0$, $\beta_0 \in [0, \beta)$.

Remark 3.11 *Since the map G is only assumed to be Lipschitz continuous, we cannot use the derivative of G to define the usual normal hyperbolic invariant manifold in the property ii) of Definition 3.9. When G is C^1 and $\|G\|_{Lip}$ is small enough the above property will implies that (see [34])*

$$\dot{w}(t) = [A + B(u(t))]w(t) + \partial_y G(u(t), v(t))w(t), \ t > t_0 \text{ and } w(t_0) = w_0 \in Y_0$$

has an exponential dichotomy with constant $\hat{\kappa} \geq 1$ and exponent $\hat{\beta} > 0$. Due to the fact that G is not assumed to be differentiable (since G is only Lipischitz continuous) the property ii) of normal hyperbolicity in the above definition is not the usual one.

The main result of this article is the following.

Theorem 3.12 *Let Assumptions 2.1, 2.3 and Assumptions 3.1, 3.2, 3.4, 3.6 and 3.7 be satisfied. Let $\beta > \beta_0 \geq 0$, $\kappa \geq 1$ and $\kappa_0 \geq 1$ the constants defined in Assumptions 3.1, 3.2, 3.4, 3.6 and 3.7. Let two constants $\hat{\beta}, \hat{\beta}_0 \in (\beta_0, \beta)$ with $\hat{\beta}_0 < \hat{\beta}$. Then there exist $\hat{\kappa} \geq \kappa$, $\sigma_0 > 0$, $\eta_0 > 0$ and $\hat{C}_0 > 0$ such that if*

$$0 \leq \sigma \leq \sigma_0, \quad 0 \leq \eta < \eta_0 \text{ and } \quad 0 \leq \hat{\sigma}\hat{C}_0 \leq \zeta$$

then there exists a Lipschitz continuous map $\hat{\psi} : X \to Y_0$ with

$$\sup_{x \in X} \|\hat{\psi}(x)\| \leq \hat{\sigma}\hat{C}_0$$

and the following properties are satisfied

(i) *$\hat{\mathcal{M}} = \{(x, \hat{\psi}(x)) \in X \times Y_0 : x \in X\}$ is a normally hyperbolic invariant manifold for (1.1) with constants $\hat{\kappa}$, κ_0 and exponents $\hat{\beta}$, $\hat{\beta}_0$. Moreover if we consider $u(t)$ the solution of the ordinary differential equation*

$$\dot{u}(t) = F(u(t)) + K(u(t), \hat{\psi}(u(t))), \forall t \in \mathbb{R}, \text{ with } u(0) = x \in X,$$

then $\Big(u(t), \hat{\psi}(u(t))\Big)$ is a mild solution of system (1.1).

(ii) *If $(u, v) \in C^1(\mathbb{R}, X) \times BC(\mathbb{R}, Y_0)$ is a mild solution of (1.1) on $\mathbb{R}$ and*

$$\|v(t)\| \leq \zeta, \forall t \in \mathbb{R},$$

then $(u(t), v(t)) \in \hat{\mathcal{M}}$, $\forall t \in \mathbb{R}$ which means that

$$v(t) = \hat{\psi}(u(t)), \ \forall t \in \mathbb{R}.$$

4 Applications

4.1 A parabolic equation coupled with an ODE

In order to illustrate the application of our results we first consider the following system of parabolic equation with non local boundary conditions coupled with an ordinary differential equation

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon r u(t)\left[1-\theta^{-1}u(t)\right] - \varepsilon k u(t)\displaystyle\int_0^1 c(t,x)dx \\ \dfrac{\partial c(t,x)}{\partial t} = \dfrac{\partial^2 c(t,x)}{\partial x^2} - \mu c(t,x) - \gamma u(t) c(t,x) \\ \dfrac{\partial c(t,0)}{\partial x} = 0, \quad \dfrac{\partial c(t,1)}{\partial x} = 1 - \varepsilon f\left(\displaystyle\int_0^1 c(t,x)dx\right) \\ u(0) = u_0 \in [0,\theta], \ c(0,\cdot) = c_0 \in L^p_+([0,1],\mathbb{R}) \end{cases} \tag{4.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is given by

$$f(x) = \frac{k_0 x^2}{1+x^2}, \ \forall x \in \mathbb{R}, \tag{4.2}$$

with $k_0 \in (0,1]$. All the parameters of the system are assume to be positive that is $\gamma > 0, \mu > 0, \theta > 0$ and $k > 0$.

The model (4.1) is a toy model which is inspried by [47]. Here $u(t)$ is a number of bacteria and $c(t,x)$ is the density of antimicrobial. In order to simplify the analysis and the presentation we use some non local mass action interaction between the bacteria population and the antimicrobial. The non linear and non-local boundary condition corresponds to a negative feedback to control the quantity of antimicrobial injected through the boundary at $x = 1$. A more realistic model with spatially distributed population of bacteria and with local mass action law will be considered in future work.

The unperturbed system (obtained by fixing $\varepsilon = 0$ in (4.1)) is the following

$$\begin{cases} \dfrac{du(t)}{dt} = 0 \\ \dfrac{\partial c(t,x)}{\partial t} = \dfrac{\partial^2 c(t,x)}{\partial x^2} - \mu c(t,x) - \gamma u(t) c(t,x) \\ \dfrac{\partial c(t,0)}{\partial x} = 0, \quad \dfrac{\partial c(t,1)}{\partial x} = 1. \end{cases} \tag{4.3}$$

The unperturbed system (4.3) do not have an invariant manifold of the form $\mathbb{R}\times\{0_{L^p}\}$ as in the abstract settings. Therefore we will first find an invariant manifold for system (4.3) as a graph of a function and make a change of variables. To obtain this manifold we solve the steady states problem

$$\begin{cases} u(t) \equiv u_0 \\ \dfrac{\partial^2 c(t,x)}{\partial x^2} = (\mu + \gamma u_0) c(t,x) \\ \dfrac{\partial c(t,0)}{\partial x} = 0, \quad \dfrac{\partial c(t,1)}{\partial x} = 1. \end{cases} \tag{4.4}$$

Equilibria: Consider the map $\vartheta : \left(-\mu\gamma^{-1}, +\infty\right) \to L^p([0,1],\mathbb{R})$ by

$$\vartheta(u)(x) = \frac{1}{\sqrt{\mu+\gamma u}} \frac{\cosh\left(x\sqrt{\mu+\gamma u}\right)}{\sinh\left(\sqrt{\mu+\gamma u}\right)}, \forall x \in [0,1], \ \forall u > -\mu\gamma^{-1}. \tag{4.5}$$

Then the equilibria of (4.4) (with $u \in [0, \theta]$) are given by

$$E := \{(u, \vartheta(u)) : u \in [0, \theta]\}.$$

We will prove the following result.

Theorem 4.1 *Let $\zeta > 0$ and $2\delta \in \left(0, \mu\gamma^{-1}\right)$ be fixed. Then we can find two constants $\varepsilon^* > 0$ and $\hat{C}_0 > 0$ such that if $\varepsilon \in (0, \varepsilon^*)$ then there exists a Lipschitz continuous map $\hat{\vartheta}_\varepsilon : \mathbb{R} \to L^p((0,1), \mathbb{R})$ such that*

$$\sup_{u \in \mathbb{R}} \|\hat{\vartheta}_\varepsilon(u)\|_{L^p} \leq \varepsilon \hat{C}_0$$

and the following properties are satisfied:

(i) *For each $u_0 \in [0, \theta]$,*

$$\int_0^1 [\vartheta(u_0)(x) + \hat{\vartheta}_\varepsilon(u_0)(x)]dx > 0.$$

(ii) *The subset*

$$E_\varepsilon = \left\{ \Big(u, \vartheta(u) + \hat{\vartheta}_\varepsilon(u)\Big) : u \in [0, \theta] \right\}.$$

is locally positively invariant by the semiflow generated by (4.1). That is to say that if we choose $(u_0, c_0) \in [0, \theta] \times L^p_+([0,1], \mathbb{R})$ with $c_0 = \vartheta(u_0) + \hat{\vartheta}_\varepsilon(u_0)$. Let $I \subset \mathbb{R}_+$ be the maximal interval such that $t \to (u(t), c(t, \cdot))$ the mild solution of (4.1) satisfies

$$\|c(t, \cdot) - \vartheta(u(t))\|_{L^p} \leq \zeta, \forall t \in I.$$

Then

$$(u(t), c(t, \cdot)) \in E_\varepsilon, \forall t \in I.$$

(iii) *For each $u_0 \in [0, \theta]$, there exists a unique solution $u \in C^1([0, \infty), \mathbb{R})$ of the scalar ordinary differential equation*

$$\frac{du(t)}{dt} = \varepsilon \left[ru(t)(1 - \theta^{-1}u(t)) - ku(t) \left(\int_0^1 [\vartheta(u(t))(x) + \hat{\vartheta}_\varepsilon(u(t))(x)]dx \right) \right]$$

satisfying

$$u(t) \in [0, \theta], \forall t \geq 0.$$

Then $t \to \Big(u(t), \vartheta(u(t)) + \hat{\vartheta}_\varepsilon(u(t))\Big)$ is a mild solution of (4.1).

Abstract reformulation : To incorporate the boundary condition into the state variable, we consider

$$Y := \mathbb{R}^2 \times L^p([0,1], \mathbb{R})$$

which is a Banach space endowed with the usual product norm and we set

$$Y_0 := \{0_{\mathbb{R}^2}\} \times L^p([0,1], \mathbb{R}).$$

Let $A : D(A) \subset Y \to Y$ be the linear operator defined by

$$A \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} := \begin{pmatrix} \varphi'(0) \\ -\varphi'(1) \\ \varphi'' \end{pmatrix}$$

with

$$D(A) := \{0_{\mathbb{R}^2}\} \times W^{2,p}([0,1], \mathbb{R}).$$

Define $R_1 : \mathbb{R} \times Y_0 \to \mathbb{R}$ by

$$R_1 \left(u, \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} \right) = ru \left(1 - \theta^{-1} u\right) - ku \int_0^1 \varphi(x)dx \tag{4.6}$$

$R_2 : \mathbb{R} \times Y_0 \to Y$ by

$$R_2 \left(u, \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \\ 0_{L^p} \end{pmatrix} \tag{4.7}$$

$R_3 : \mathbb{R} \times Y_0 \to Y$ by

$$R_3 \left(u, \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} \right) = \begin{pmatrix} 0 \\ f\left(\int_0^1 \varphi(x)dx \right) \\ 0_{L^p} \end{pmatrix} \tag{4.8}$$

and $\hat{B} : \mathbb{R} \to \mathcal{L}(Y_0, Y)$ by

$$\hat{B}(u) \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -[\mu + \gamma u]\varphi \end{pmatrix}. \tag{4.9}$$

Thus by setting

$$w(t) := \begin{pmatrix} 0_{\mathbb{R}^2} \\ c(t, \cdot) \end{pmatrix}, \ t > 0 \text{ and } w(0) := \begin{pmatrix} 0_{\mathbb{R}^2} \\ c_0 \end{pmatrix} = w_0,$$

system (4.1) rewrites as

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon R_1(u(t), w(t)), \ t > 0 \\ \dfrac{dw(t)}{dt} = [A + \hat{B}(u(t))]w(t) + R_2(u(t), w(t)) + \varepsilon R_3(u(t), w(t)), \ t > 0 \\ u(0) = u_0 \in [0, \theta] \ \text{ and } \ w(0) = w_0 \in Y_0 \end{cases} \tag{4.10}$$

and we observe that

$$\begin{cases} \dfrac{du(t)}{dt} = 0, \ t > 0 \\ \dfrac{dw(t)}{dt} = [A + \hat{B}(u(t))]w(t) + R_2(u(t), w(t)), \ t > 0 \\ u(0) = u_0 \in [0, \theta] \ \text{ and } \ w(0) = w_0 \in Y_0 \end{cases} \tag{4.11}$$

corresponds exactly to the abstract formulation of (4.3). This implies that

$$[A + \hat{B}(u)] \begin{pmatrix} 0_{\mathbb{R}^2} \\ \vartheta(u) \end{pmatrix} + R_2 \left(u, \begin{pmatrix} 0_{\mathbb{R}^2} \\ \vartheta(u) \end{pmatrix} \right) = 0, \ \forall u \in (\mu\gamma^{-1}, +\infty). \tag{4.12}$$

Properties of the linear operator A **:** Observe that by construction A_0 the part of A in Y_0 coincides with the usual formulation for the parabolic equation with homogeneous boundary conditions. More precisely we have $A_0 : D(A_0) \subset Y_0 \to Y_0$ is the linear operator on Y_0 defined by

$$A_0 \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi'' \end{pmatrix}$$

with

$$D(A_0) = \{0_{\mathbb{R}^2}\} \times \left\{\varphi \in W^{2,p}\left((0,1), \mathbb{R}\right) : \varphi'(0) = \varphi'(1) = 0\right\}.$$

The next Lemmas 4.2, 4.3 and 4.4 can be found in [34].

Lemma 4.2 *The linear operator A_0 is the infinitesimal generator of $\{T_{A_0}(t)\}_{t \geq 0}$ an analytic semigroup of bounded linear operator on Y_0. Moreover $T_{A_0}(t)$ is compact for each $t > 0$ and $(0, +\infty) \subset \rho(A_0)$. The spectrum of A_0 is given by*

$$\sigma(A_0) = \left\{-(\pi k)^2 : k \in \mathbb{N}\right\}$$

and each eigenvalue $\lambda_k := -(\pi k)^2$ is associated to the eigenfunction

$$e_k(x) := sin(\pi k x), \ k \geq 1, \ e_0(x) = 1.$$

Moreover each eigenvalue λ_k is simple and the projector on the generalized eigenspace associated to this eigenvalue is given by

$$\Pi_{k,0} \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} := \begin{pmatrix} 0_{\mathbb{R}^2} \\ \frac{\int_0^1 e_k(r)\varphi(r)dr}{\int_0^1 e_k(r)^2 dr} e_k \end{pmatrix}.$$

Set

$$\Omega_\omega = \{\lambda \in \mathbb{C} : \mathcal{R}e\,(\lambda) > \omega\}, \ \forall \omega \in \mathbb{R},$$

and define for $\lambda \in \mathbb{C}$,

$$\Delta\,(\lambda) := \mu^2 (e^{\mu} - e^{-\mu}),$$

where

$$\mu := \sqrt{\lambda}.$$

Next we give the explicit formula of the resolvent of A.

Lemma 4.3 *For each $\omega \geq 0$, we have*

$$\Omega_\omega \subset \{\lambda \in \mathbb{C} : \Delta\,(\lambda) \neq 0\} \subset \rho\,(A)\,,$$

and the resolvent of A is given for each $\lambda := \mu^2 \in \Omega_\omega$ by

$$\begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = (\lambda I - A)^{-1} \begin{pmatrix} y_0 \\ y_1 \\ \varphi_0 \end{pmatrix} \Leftrightarrow$$

$$\begin{aligned} \varphi(x) = &\frac{\Delta_1(x)}{\Delta\,(\lambda)} \frac{1}{\mu} y_0 + \frac{\Delta_2(x)}{\Delta\,(\lambda)} \frac{1}{\mu} y_1 + \frac{\Delta_1(x)}{\Delta\,(\lambda)} \frac{1}{2\mu} \int_0^1 e^{-\mu s} \varphi_0(s) ds \\ &+ \frac{\Delta_2(x)}{\Delta\,(\lambda)} \frac{1}{2\mu} \int_0^1 e^{-\mu(1-s)} \varphi_0(s) ds + \frac{1}{2\mu} \int_0^1 e^{-\mu|x-s|} \varphi_0(s) ds \end{aligned}$$

where

$$\Delta_1(x) = \mu^2 \left[e^{\mu(1-x)} + e^{-\mu(1-x)}\right] \ \textit{and} \ \ \Delta_2(x) = \mu^2 \left[e^{-\mu x} + e^{\mu x}\right].$$

Lemma 4.4 *The linear operator A satisfies the following estimate*

$$0<\liminf_{\lambda\to+\infty}\lambda^{\frac{1}{p^*}}\left\|(\lambda I-A)^{-1}\right\|_{\mathcal{L}(Y)}\le\limsup_{\lambda\to+\infty}\lambda^{\frac{1}{p^*}}\left\|(\lambda I-A)^{-1}\right\|_{\mathcal{L}(Y)}<+\infty,\tag{4.13}$$

with $p^*=\dfrac{2p}{1+p}$.

From (4.13) we see that A can not be a Hille-Yosida linear operator when $p>1$ since

$$\lim_{\lambda\to+\infty}\lambda\left\|(\lambda I-A)^{-1}\right\|_{\mathcal{L}(Y)}=+\infty,\quad \text{if } p>1,\tag{4.14}$$

which is one of the main difference with respect to the Hille-Yosida case. In fact if A is Hille-Yosida the above limit (4.14) is finite. However we still have

$$\lim_{\lambda\to+\infty}(\lambda I-A)^{-1}y=0,\ \forall y\in Y.\tag{4.15}$$

By using Lemma 4.3 and the estimate (4.13) we deduce that Assumption 3.4 in Ducrot, Magal and Prevost [8] is satisfied. Therefore by applying Theorem 3.11 in [8] we obtain the following lemma.

Lemma 4.5 *The linear operator A satisfies Assumption 2.1 and Assumption 2.3.*

Remark 4.6 *Since $\rho(A)\neq\emptyset$ one has $\sigma(A_0)=\sigma(A)$ (see [32]).*

A positively invariant set : Here we will show that the semiflow generated by (4.1) leaves positively invariant the set

$$\Omega=\left\{(u_0,c_0):u_0\in[0,\theta],\ c_0\in L^p_+([0,1],\mathbb{R})\right\}.$$

Note that this equivalent to prove that the semiflow generated by (4.10) leaves positively invariant $[0,\theta]\times Y_{0+}$ where we have set

$$Y_+:=\mathbb{R}^2_+\times L^p_+([0,1],\mathbb{R})\ \text{ and } Y_{0+}:=Y_0\cap Y_+=\{0_{\mathbb{R}^2}\}\times L^p_+([0,1],\mathbb{R}).$$

In fact using Lemma 4.3 one can see that A is resolvent positive. More precisely we have

$$(0,+\infty)\subset\rho(A)\ \text{ and }\ (\lambda I-A)^{-1}Y_+\subset Y_{0+},\ \forall\lambda>0.$$

Furthermore for any constant $L>0$ there exists $\lambda:=\lambda(L)>0$ such that for any $u\in\mathbb{R}_+$ and $y\in Y_{0+}$ with $|u|+\|y\|<L$ we have

$$\lambda y+R_2(u,y)+\varepsilon R_3(u,y)\in Y_+\ \text{ and } \lambda u+R_1(u,y)\in\mathbb{R}_+,\ \forall\varepsilon\in(0,1).$$

Therefore using the results in [31] it follow that for each $u_0\in\mathbb{R}_+$ and $v_0\in Y_{0+}$ there exists a maximally defined mild solution of (4.10) with values in $\mathbb{R}_+\times Y_{0+}$. Once the positivity of the solutions is obtained one can use standard blowup arguments to prove that the solutions are globally defined in $[0,+\infty)$.

Moreover by using the explicit form of the u-equation of (4.10)

$$\frac{du(t)}{dt} = \varepsilon r u(t)(1-\theta^{-1}u(t)) - ku(t)\int_0^1 c(t,x)dx.$$

we deduce that $[0,\theta]\times Y_{0+}$ is positively invariant with respect to (4.10).

Transformed system : To apply our results we will use an alternative system and extend the perturbations in order to obtain global Lipschitz properties. To do so we first define the map $H:\mathbb{R}\to Y_0$ by

$$H(u) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \vartheta(u) \end{pmatrix}, \ \forall u \in (\mu\gamma^{-1}, +\infty)$$

and recalling (4.12) we have by construction

$$[A+B(u)]H(u) + R_2(u, H(u)) = 0, \ \forall u \in (\mu\gamma^{-1}, +\infty). \tag{4.16}$$

Let $u_0 \in \mathbb{R}_+$ and $w_0 \in Y_{0+}$ be given. Then $u(t) \geq 0$ for all $t \geq 0$ and $H(u(t))$ is well defined for all $t \geq 0$. Hence we can make the following change of variables

$$v(t) = w(t) - H(u(t)) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ c(t,x) - \vartheta(u(t))(x) \end{pmatrix}, \forall t \geq 0.$$

By using (4.16) together with the fact that the map $(u,w)\to R_2(u,w)$ is the constant function defined in (4.7) one can see that $t\to(u(t),v(t))$ will satisfies

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon R_1(u(t), v(t) + H(u(t))), \ t > 0 \\ \dfrac{dv(t)}{dt} = [A + \hat{B}(u(t))]v(t) + \varepsilon R_3(u(t), v(t) + H(u(t))) \\ \qquad\qquad -\varepsilon \partial_u H(u(t)) R_1(u(t), v(t) + H(u(t))), \ t > 0 \\ u(0) = u_0 \in \mathbb{R}_+ \ \text{ and } \ v(0) = w_0 - H(u_0) \in Y_0. \end{cases} \tag{4.17}$$

The above abstract equation corresponds to the following partial differential equation

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon r u(t)\left[1-\theta^{-1}u(t)\right] - \varepsilon k u(t)\displaystyle\int_0^1 c(t,x) + \vartheta(u(t))(x)dx \\ \dfrac{\partial c(t,x)}{\partial t} = \dfrac{\partial^2 c(t,x)}{\partial x^2} - \mu c(t,x) - \gamma u(t)c(t,x) \\ \qquad -\varepsilon\partial_u\vartheta(u(t))\left(ru(t)\left[1-\theta^{-1}u(t)\right] - \varepsilon k u(t)\int_0^1 c(t,x)+\vartheta(u(t))(x)dx\right) \\ \dfrac{\partial c(t,0)}{\partial x} = 0, \quad \dfrac{\partial c(t,1)}{\partial x} = -\varepsilon f\left(\displaystyle\int_0^1 c(t,x) + \vartheta(u(t))(x)dx\right) \\ u(0) = u_0 \in [0,\theta], \ c(0,\cdot) = c_0 - \vartheta(u_0) \in L^p([0,1],\mathbb{R}). \end{cases} \tag{4.18}$$

By using the fact that $[0,\theta]\times L^p_+(0,1)$ is positively invariant by the semiflow generated by (4.10), we deduce that system (4.17) leaves positively invariant the subset

$$\widetilde{M} = \left\{(u,v)\in\mathbb{R}\times Y_0 : u\in[0,\theta] \text{ and } v + H(u) \geq 0\right\}.$$

Truncated system: Let $2\delta \in (0, \mu\gamma^{-1})$ be given and fixed. In order to extend the system and overcome the singularity of H at $\mu\gamma^{-1}$ we introduce the smooth cut-off function $\xi \in C^\infty(\mathbb{R}, [0,1])$ defined by

$$\xi(u) = \begin{cases} 1 & \text{if} \quad -\delta \le u \le \theta + \delta \\ 0 & \text{if} \quad u < -2\delta \text{ or } u > \theta + 2\delta. \end{cases}$$

Therefore by setting for all $(u,v) \in \mathbb{R} \times Y_0$

$$\begin{cases} K_\varepsilon(u,v) := \varepsilon\xi(u) R_1(u, v + H(u)) \\ B(u)v := \hat{B}(u\xi(u))v \\ G_\varepsilon(u,v) := \varepsilon\xi(u)\left[R_3(u, v + H(u)) - \partial_u H(u) R_1(u, v + H(u)\right] \end{cases}$$

we consider the following extended system

$$\begin{cases} \dfrac{du(t)}{dt} = K_\varepsilon(u(t), v(t)) \\ \dfrac{dv(t)}{dt} = [A + B(u(t)))]v(t) + G_\varepsilon(u(t), v(t)) \\ u(0) = u_0 \in \mathbb{R} \text{ and } v(0) = v_0 \in Y_0. \end{cases} \tag{4.19}$$

The corresponding unperturbed system to (4.19) is

$$\begin{cases} \dfrac{du(t)}{dt} = 0, \ t > t_0 \\ \dfrac{dv(t)}{dt} = [A + B(u(t))]v(t), \ t > t_0 \\ u(0) = u_0 \in \mathbb{R} \text{ and } v(0) = v_0 \in Y_0. \end{cases} \tag{4.20}$$

Since we replaced $\hat{B}(u)v$ by $B(u)v = \hat{B}(u\xi(u))v$ we deduce that

$$\mathcal{M} := \mathbb{R} \times \{0_Y\}$$

is invariant for (4.20).

Lemma 4.7 *Let $\zeta > 0$ be given and define*

$$B_{Y_0}(0,\zeta) := \{y \in Y_0 : \|y\| \le \zeta\}.$$

Then the following properties hold true

(i) *The map $(u,y) \to K_\varepsilon(u,y)$ is continuously differentiable on any open neighborhood of $\mathbb{R} \times B_{Y_0}(0,\zeta)$ and there exists a constant $L_K > 0$ such that for all $(u,y), (\bar{u}, \bar{y}) \in \mathbb{R} \times B_{Y_0}(0,\zeta)$*

$$\|K_\varepsilon(u,y)\| \le \varepsilon L_K \quad \text{and} \quad \|K_\varepsilon(u,y) - K_\varepsilon(\bar{u},\bar{y})\| \le \varepsilon L_K |u - \bar{u}| + \varepsilon L_K \|y - \bar{y}\|.$$

(ii) *The map $(u,y) \to G_\varepsilon(u,y)$ is continuous on $\mathbb{R} \times B_{Y_0}(0,\zeta)$ and there exists a constant $L_G > 0$ such that for all $(u,y), (\bar{u},\bar{y}) \in \mathbb{R} \times B_{Y_0}(0,\zeta)$*

$$\|G_\varepsilon(u,y)\| \le \varepsilon L_G \quad \text{and} \quad \|G_\varepsilon(u,y) - G_\varepsilon(\bar{u},\bar{y})\| \le \varepsilon L_G |u - \bar{u}| + \varepsilon L_G \|y - \bar{y}\|.$$

(iii) *The map* $u \to B(u)$ *is continuous from* $\mathbb{R}$ *into* $\mathcal{L}(Y_0, Y)$ *and there exists a constant* $L_B > 0$ *such that for all* $u, \bar{u} \in \mathbb{R}$

$$\|B(u)\|_{\mathcal{L}(Y_0,Y)} \leq L_B \quad \text{and} \quad \|B(u) - B(\bar{u})\|_{\mathcal{L}(Y_0,Y)} \leq L_B|u - \bar{u}|.$$

Remark 4.8 *Note that* (4.19) *is in the general form of* (1.1) *with* $F = 0$ *so that Assumption 3.2 is satisfied for* (4.19) *with*

$$\beta_0 = 0 \quad \text{and} \quad \kappa_0 = 1.$$

The following lemma shows that Assumption 3.4 is satisfied for (4.19).

Lemma 4.9 *Let* $\beta \in (0, \mu - 2\delta\gamma)$ *be given. If*

$$\frac{du(t)}{dt} = 0, \ \forall t \in \mathbb{R}, \ u(t_0) = u_0$$

then the evolution family generated by

$$\frac{dv(t)}{dt} = [A + B(u(t))]v(t), \ t > t_0, \ v(t_0) = v_0$$

is exponentially stable with constant $\kappa \geq 1$ *and exponent* $\beta > 0$*.*

Proof. Since u is constant in time we have

$$\frac{dv(t)}{dt} = [A + B(u_0)]v(t), \ t > t_0, \ v(t_0) = v_0. \tag{4.21}$$

By using Lemma 4.3 and the perturbation result in [31] it follows that $(A + B(u_0))_0$ the part of $A + B(u_0)$ in Y_0 generates a strongly continuous semigroup in Y_0. Let us denote it by $\{T(t)\}_{t\geq 0}$. Then its corresponding evolution family is given by

$$U(t, s) = T(t - s), \ \forall t \geq s.$$

Then recalling that

$$B(u_0)v = [\mu + u_0\xi(u_0)\gamma], \ \forall v \in Y_0$$

and $2\delta \in (0, \gamma^{-1}\mu)$ we obtain

$$\sigma(A + B(u_0)) = -(\mu + u_0\xi(u_0)\gamma) + \sigma(A) = \left\{-\mu - u_0\xi(u_0)\gamma - (k\pi)^2 : k \in \mathbb{N}\right\}$$

and we deduce that

$$\begin{array}{rcl} \omega_0(A) & = & \sup\{Re(\lambda) : \lambda \in \sigma(A + B(u_0))\} \\ & \leq & -\mu + 2\delta\gamma\xi(u_0) \\ & \leq & -\mu + 2\delta\gamma < 0. \end{array}$$

Therefore if $\beta \in (0, \mu - 2\delta\gamma)$ then the evolution family generated by (4.20) is exponentially stable with some constant $\kappa \geq 1$ and exponent $\beta > 0 = \beta_0$. ∎

We can now apply our results to system (4.19).

Lemma 4.10 *Let $\zeta > 0$ be given and fixed such that Lemma 4.9 holds true. Let $\beta \in (0, \mu - 2\delta\gamma)$ be given. Let $\hat{\beta}, \hat{\beta}_0 \in (0, \beta)$ be two constants with $\hat{\beta}_0 < \hat{\beta}$. Then there exist $\hat{\kappa} \geq \kappa$, $\varepsilon_0 > 0$ and $\hat{C}_0 > 0$ such that if*

$$0 \leq \varepsilon \leq \varepsilon_0$$

then there exists a Lipschitz continuous map $\hat{\psi}_\varepsilon : \mathbb{R} \to Y_0$ with

$$\sup_{u \in \mathbb{R}} \|\hat{\psi}_\varepsilon(u)\| \leq \hat{C}_0 \varepsilon.$$

Moreover $\hat{\mathcal{M}}_\varepsilon = \{(u, \hat{\psi}_\varepsilon(u)) \in \mathbb{R} \times Y_0 : u \in \mathbb{R}\}$ is a normally hyperbolic invariant manifold for (4.19) with constants $\hat{\kappa}$, κ_0 and exponents $\hat{\beta}$, $\hat{\beta}_0$. Hence for each $(u_0, \hat{\psi}_\varepsilon(u_0)) \in \hat{\mathcal{M}}_\varepsilon$ there exists a unique mild solution $(u, v) \in C^1(\mathbb{R}, \mathbb{R}) \times BC(\mathbb{R}, Y_0)$ of (4.19) such that

$$(u(0), v(0)) = (u_0, \hat{\psi}(u_0)) \quad \text{and } v(t) = \hat{\psi}(u(t)), \forall t \in \mathbb{R}.$$

Note that since $\hat{\psi}_\varepsilon : \mathbb{R} \to Y_0$ there exists $\hat{\vartheta}_\varepsilon : \mathbb{R} \to L^p([0,1], \mathbb{R})$ such that

$$\hat{\psi}_\varepsilon(u) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \hat{\vartheta}_\varepsilon(u) \end{pmatrix}, \ \forall u \in \mathbb{R}$$

and we have

$$\sup_{u \in \mathbb{R}} \|\hat{\vartheta}_\varepsilon(u)\|_{L^p} = \sup_{u \in \mathbb{R}} \|\hat{\psi}_\varepsilon(u)\| \leq \hat{C}_0 \varepsilon. \tag{4.22}$$

In order to get back to system (4.1) we will need the following lemma.

Lemma 4.11 *Let*

$$0 < \varepsilon < \varepsilon^* := \min \left\{ \varepsilon_0, \frac{1}{\hat{C}_0 \sqrt{\mu + \beta\theta}} \frac{1}{\sinh\left(\sqrt{\mu + \beta\theta}\right)} \right\}. \tag{4.23}$$

Then we have

$$\int_0^1 [\vartheta(u)(x) + \hat{\vartheta}_\varepsilon(u)(x)] dx > 0, \ \forall u \in [0, \theta].$$

Proof. Note that

$$\cosh(x\sqrt{\mu + \beta u}) \geq 1, \ \forall x \in [0,1], \ \forall u \geq 0$$

providing that

$$\vartheta(u)(x) \geq \frac{1}{\sqrt{\mu + \beta u}} \frac{1}{\sinh\left(\sqrt{\mu + \beta u}\right)}, \ \forall x \in [0,1], \ \forall u \geq 0.$$

Since the map $u \to u \sinh(u)$ and $u \to \sqrt{\mu + \beta u}$ are strictly increasing in $[0, +\infty)$ we obtain

$$\vartheta(u)(x) \geq \frac{1}{\sqrt{\mu + \beta\theta}} \frac{1}{\sinh\left(\sqrt{\mu + \beta\theta}\right)}, \ \forall x \in [0,1], \ \forall u \in [0, \theta]$$

and we set

$$\delta_0 := \frac{1}{\sqrt{\mu + \beta\theta}} \frac{1}{\sinh\left(\sqrt{\mu + \beta\theta}\right)}.$$

By the Holder inequality in $[0,1]$ we have

$$\left|\int_0^1 \hat{\vartheta}_\varepsilon(u)(x)dx\right| \leq \|\hat{\vartheta}_\varepsilon(u)\|_{L^1} \leq \|\hat{\vartheta}_\varepsilon(u)\|_{L^p} \leq \hat{C}_0\varepsilon < \delta_0$$

and we obtain

$$\int_0^1 \vartheta(u)(x)dx + \int_0^1 \hat{\vartheta}_\varepsilon(u)(x)dx \geq \delta_0 + \int_0^1 \hat{\vartheta}_\varepsilon(u)(x)dx > 0.$$

■

We are ready to prove Theorem 4.1.

Proof of Theorem 4.1.

From Lemma 4.10 if we consider $u(t)$ the solution of the ordinary differential equation

$$\frac{du(t)}{dt} = K_\varepsilon(u(t), \hat{\psi}_\varepsilon(u(t))), \; \forall t \in \mathbb{R}, \text{ with } u(0) = u_0 \in \mathbb{R},$$

then $t \in \mathbb{R} \to \left(u(t), \hat{\psi}_\varepsilon(u(t))\right)$ is a mild solution of system (4.19). Moreover we explicitly have the following form for the u-equation

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon\xi(u(t))\left[ru(t)(1-\theta^{-1}u(t)) - ku(t)\left(\displaystyle\int_0^1 [\vartheta(u(t))(x) + \hat{\vartheta}_\varepsilon(u(t))(x)]dx\right)\right] \\ u(0) = u_0 \in \mathbb{R}. \end{cases}$$

By classical fixed point arguments we know that if $u_0 \geq 0$ then $u(t) \geq 0$ for all $t \geq 0$. Therefore if $u_0 \in [0, \theta]$ then by Lemma 4.11 we have

$$u(t) \in [0, \theta], \; \forall t \geq 0.$$

The result now follows by applying Theorem 3.12 to the truncated system (4.19). ■

4.2 An epidemic model with age of infection

In order to describe malaria, Ross [43] (1911) first and later Macdonald [37] (1957) introduced several class of epidemic including human and mosquito infection. We refer to Ruan, Xiao and Beier [44] for a nice survey on this topic. Here we consider a Ross-McDonald's model with age of infection for the mosquitoes

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon\left[\gamma(1-u(t))\displaystyle\int_0^{+\infty} \Pi(a)i(t,a)da - \alpha u(t)\right] \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\mu i(t,a) \\ i(t,0) = \beta u(t)\left(1 - \displaystyle\int_0^{+\infty} i(t,a)da\right), \end{cases} \tag{4.24}$$

with initial conditions

$$u(0) = u_0 \in [0,1] \text{ and } i(0,\cdot) = i_0 \in L^1_+((0,+\infty),\mathbb{R}).$$

Here $u(t)$ is the fraction of infected human at time t, while $i(t,a)$ is the density of infected mosquitoes at time t with respect to the age of infection a. The age of infection is the time since the mosquitoes become infected, and the term density of population means that

$$\int_{a_1}^{a_2} i(t,a)da$$

is the fraction of infected mosquito with age of infection between a_1 and a_2. The fraction of infected mosquitoes is

$$I(t) = \int_0^{+\infty} i(t,a)da \in [0,1].$$

In this model the function $a \to \Pi(a)$ is the probability for a mosquito to be infectious at the age of infection a. We assume that

$$\Pi \in L^\infty_+((0,+\infty),\mathbb{R}).$$

From the i-equation we deduce that $I(t)$ satisfies

$$I'(t) = \beta u(t)\left(1 - I(t)\right) - \mu I(t)$$

therefore the system (4.24) generated a continuous semiflow on

$$\Omega := \left\{(u,i) \in [0,1] \times L^1_+((0,+\infty),\mathbb{R}) : \int_0^{+\infty} i(t,a)da \leq 1\right\}.$$

When $\varepsilon = 0$ we formally obtain the following unperturbed system

$$\begin{cases} \dfrac{du(t)}{dt} = 0 \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\mu i(t,a) \\ i(t,0) = \beta u(t)\left(1 - \displaystyle\int_0^{+\infty} i(t,a)da\right), \end{cases} \tag{4.25}$$

$$u(0) = u_0 \in [0,1] \text{ and } i(0,\cdot) = i_0 \in L^1_+((0,+\infty),\mathbb{R}).$$

For each $u(t) = u_0$ fixed, the equilibria for the i-equation is given by

$$\bar{i}(a) = e^{-\mu a}\bar{i}_0$$

and $\bar{i}_0$ is a real number satisfying

$$\bar{i}_0 = \beta u_0 \left(1 - \int_0^{+\infty} \bar{i}_0 e^{-\mu a} da\right)$$

thus

$$\left(1 + \frac{\beta u_0}{\mu}\right)\bar{i}_0 = \beta u_0$$

therefore for each $u_0 \in \left(-\mu\beta^{-1}, +\infty\right)$ there exists a unique equilibrium for the i-equation which is given by

$$\bar{i}(a) = \frac{\mu\beta u_0}{\mu + \beta u_0} e^{-\mu a}.$$

Let $\vartheta : \left(-\mu\beta^{-1}, +\infty\right) \to L^1((0,+\infty),\mathbb{R})$ be defined for all $u \in \left(-\mu\beta^{-1}, +\infty\right)$ by

$$\vartheta(u)(a) = \frac{\mu\beta u}{\mu + \beta u} e^{-\mu a}, \text{ for almost every} a \geq 0.$$

By construction the subset

$$M = \{(u, \vartheta(u)) : u \in [0,1]\}$$

is invariant under the semiflow generated by (4.25). In order to apply our results we will first rewrite system (4.24) as an abstract Cauchy problem. Consider the Banach space

$$Y = \mathbb{R} \times L^1((0,+\infty),\mathbb{R})$$

endowed with the usual product norm and set

$$Y_0 = \{0\} \times L^1((0,+\infty),\mathbb{R}).$$

Let $A : D(A) \subset Y \to Y$ be the linear operator with

$$D(A) = \{0\} \times W^{1,1}((0,+\infty),\mathbb{R})$$

and

$$A \begin{pmatrix} 0 \\ i \end{pmatrix} = \begin{pmatrix} -i(0) \\ -i' - \mu i \end{pmatrix}.$$

Let $R_1 : \mathbb{R} \times Y_0 \to \mathbb{R}$ and $R_2 : \mathbb{R} \times Y_0 \to Y$ be defined by

$$R_1 \left(u, \begin{pmatrix} 0 \\ i \end{pmatrix} \right) = \gamma(1-u) \int_0^{+\infty} \Pi(a) i(a) da - \alpha u$$

and

$$R_2 \left(u, \begin{pmatrix} 0 \\ i \end{pmatrix} \right) = \begin{pmatrix} \beta u \left(1 - \int_0^{+\infty} i(a) da \right) \\ 0 \end{pmatrix}.$$

By identifying $w(t)$ and $\begin{pmatrix} 0 \\ i(t,\cdot) \end{pmatrix}$ system (4.24) rewrites as the following abstract Cauchy system

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon R_1(u(t), w(t)) \\ \dfrac{dw(t)}{dt} = Aw(t) + R_2(u(t), w(t)) \\ u(0) = u_0 \in [0,1] \text{ and } w(0) = w_0. \end{cases} \tag{4.26}$$

Consider the map $H : (-\mu\beta^{-1}, +\infty) \to Y_0$ defined by

$$H(u) = \begin{pmatrix} 0_{\mathbb{R}} \\ \vartheta(u) \end{pmatrix}, \ \forall u > -\mu\beta^{-1}.$$

Then we have by construction

$$AH(u) + R_2(u, H(u)) = 0, \ \forall u > -\mu\beta^{-1}.$$

Note that for each $u \in \left(-\mu\beta^{-1}, +\infty\right)$ we have

$$\partial_2 R_2(u, H(u)) \begin{pmatrix} 0 \\ i \end{pmatrix} = \begin{pmatrix} -\beta u \int_0^{+\infty} i(a)da \\ 0 \end{pmatrix}, \ \forall \begin{pmatrix} 0 \\ i \end{pmatrix} \in Y_0.$$

Next let us use the following change of variable

$$v(t) = w(t) - H(u(t)), t \in \mathbb{R}$$

so that $t \to (u(t), v(t))$ satisfies the following system as long as $u(t) \in \left(-\mu\beta^{-1}, +\infty\right)$

$$\begin{cases} \dfrac{du(t)}{dt} = \varepsilon R_1(u(t), v(t) + H(u(t))) \\ \dfrac{dv(t)}{dt} = Av(t) + R_2(u(t), v(t) + H(u(t))) - R_2\left(u(t), H(u(t))\right) \\ \qquad\quad -\varepsilon H'(u(t)) R_1(u(t), v(t) + H(u(t))) \\ u(0) = u_0 \text{ and } v(0) = w_0 - H(u_0) \in Y_0. \end{cases} \tag{4.27}$$

with

$$H'(u) = \begin{pmatrix} 0_{\mathbb{R}} \\ \vartheta'(u) \end{pmatrix}, \ \forall u > -\mu\beta^{-1}$$

and

$$\vartheta'(u)(a) = \frac{\mu^2\beta}{(\mu + \beta u)^2} e^{-\mu a}, \text{ for almost every} a \geq 0.$$

In order to avoid the singularity at $u = -\mu\beta^{-1}$ we need to truncate the system. Let $\delta \in (0, \mu\beta^{-1})$. Let $\xi \in C^\infty(\mathbb{R}, [0,1])$ be a cut-off function such that

$$\xi(u) = \begin{cases} 1 & \text{if } \ 0 \leq u \leq 1 \\ 0 & \text{if } \ u < -\delta \text{ or } u > 1 + \delta. \end{cases}$$

Set for all $(u, v) \in \mathbb{R} \times Y_0$

$$\begin{cases} K_\varepsilon(u, v) := \varepsilon\xi(u) R_1(u, v + H(u)) \\ B(u)v := \xi(u)\partial_2 R_2(u, H(u))v \\ G_\varepsilon(u, v) := \xi(u) \left[R_2(u, v + H(u)) - R_2(u, H(u)) - \partial_2 R_2(u, H(u))v - \varepsilon \dfrac{dH(u)}{du} R_1(u, v + H(u)) \right]. \end{cases}$$

Observe that or all $(u, v) \in \mathbb{R} \times Y_0$ we have

$$R_2(u, v + H(u)) - R_2(u, H(u)) - \partial_2 R_2(u, H(u))v = 0$$

so that

$$G_\varepsilon(u, v) := -\varepsilon\xi(u) \frac{dH(u)}{du} R_1(u, v + H(u)), \ \forall (u, v) \in \mathbb{R} \times Y_0.$$

Consider the following system

$$\begin{cases} \dfrac{du(t)}{dt} = K_\varepsilon(u(t), v(t)) \\ \dfrac{dv(t)}{dt} = [A + B(u(t))]v(t) + G_\varepsilon(u(t), v(t)) \\ u(0) = u_0 \in \mathbb{R} \text{ and } v(0) = v_0 \in Y_0. \end{cases} \tag{4.28}$$

System (4.28) is in the framework of system (1.1) with $F = 0$ and Assumption 3.2 is satisfied with $\beta_0 = 0$ and $\kappa_0 = 1$. Then one can verifies that there exist L_K, L_G and L_B such that for all $\zeta \in (0, 1)$, for all $(u_1, v_1), (u_2, v_2) \in \mathbb{R} \times B_{Y_0}(0, \zeta)$ we have

$$\begin{cases} |K_\varepsilon(u_1, v_1) - K_\varepsilon(u_2, v_2)| \leq L_K \varepsilon(|u_1 - u_2| + \|v_1 - v_2\|) \\ |K_\varepsilon(u_1, v_1)| \leq L_K \varepsilon \\ \|B(u_1) - B(u_2)\|_{\mathcal{L}(Y_0)} \leq L_B |u_1 - u_2| \\ \|B(u_1)\|_{\mathcal{L}(Y_0)} \leq L_B \end{cases}$$

and

$$\begin{cases} \|G_\varepsilon(u_1, v_1) - G_\varepsilon(u_2, v_2)\| \leq L_G \varepsilon[|u_1 - u_2| + \|v_1 - v_2\| \\ \|G_\varepsilon(u_1, v_1)\| \leq L_G \varepsilon. \end{cases}$$

By setting

$$\hat{\sigma} = \sigma = L_G \varepsilon \text{ and } \eta = L_K \varepsilon.$$

it is clear that Assumption 3.6 holds true. In the sequel $\zeta \in (0, 1)$ is fixed. Assumption 3.1 is also trivially satisfied. We now show that Assumption 3.4 is satisfied. Consider the following unperturbed system

$$\begin{cases} \dfrac{du(t)}{dt} = 0 \\ \dfrac{dv(t)}{dt} = [A + B(u(t))]v(t) \\ u(0) = u_0 \text{ and } v(0) = v_0 \in Y_0. \end{cases}$$

This last system is equivalent to look at the following partial derivative equation

$$\begin{cases} \dfrac{du(t)}{dt} = 0 \\ \dfrac{\partial i(t, a)}{\partial t} + \dfrac{\partial i(t, a)}{\partial a} = -\mu i(t, a) \\ i(t, 0) = -\xi(u(t))\beta u(t) \displaystyle\int_0^{+\infty} i(t, a)da, \\ u(0) = u_0 \text{ and } i(0, .) = i_0 \in L^1(0, +\infty). \end{cases}$$

Let $\{U_B(t, s)\}_{t \geq s}$ be the evolution family generated by

$$\begin{cases} \dfrac{dv(t)}{dt} = [A + B(u_0)]v(t) \\ v(0) = v_0 \in Y_0. \end{cases}$$

More precisely for any $\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in Y_0$ we have

$$U_B(t, s) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ i(t - s, \cdot) \end{pmatrix} \tag{4.29}$$

where $t \to i(t, \cdot) \in L^1((0, +\infty), \mathbb{R})$ is the unique mild solution of

$$\begin{cases} \dfrac{\partial i(t, a)}{\partial t} + \dfrac{\partial i(t, a)}{\partial a} = -\mu i(t, a) \\ i(t, 0) = -\beta u_0 \xi(u_0) \displaystyle\int_0^{+\infty} i(t, a)da, \\ i(0, \cdot) = \varphi \in L^1((0, +\infty, \mathbb{R}). \end{cases} \tag{4.30}$$

Next we give an estimate for the evolution family. Set $I(t) := \int_0^{+\infty} i(t,a)da$ then we have

$$I'(t) = -(\beta u_0 \xi(u_0))I(t) - \mu I(t)$$

therefore

$$I(t) = e^{-(\mu+\beta u_0 \xi(u_0))t} \int_0^{+\infty} i_0(a)da, \forall t \geq 0. \tag{4.31}$$

Moreover we have the following estimate

$$\|i(t,.)\|_{L^1} \leq e^{-\mu t}\|i_0\|_{L^1} + \int_0^t e^{-\mu(t-s)}|\beta u_0 \xi(u_0)||I(s)|ds, \forall t \geq 0,$$

hence by using (4.31)

$$\|i(t,.)\|_{L^1} \leq e^{-\mu t}\|i_0\|_{L^1}\left[1 + \int_0^t e^{-(\beta u_0 \xi(u_0))s}|\beta u_0 \xi(u_0)|ds\right], \forall t \geq 0,$$

and by distinguishing the case u_0 positive and the case u_0 negative, we obtain that

$$\|i(t,.)\|_{L^1} \leq e^{-\mu t}\|i_0\|_{L^1}\left[1 + \int_0^t e^{-\min(0,(\beta u_0 \xi(u_0)))s}|\beta u_0 \xi(u_0)|ds\right], \forall t \geq 0,$$

and $-\beta u_0 \xi(u_0) \leq \beta\delta$ whenever $u_0 \leq 0$, therefore we can find a constant $\kappa^* \geq 1$ such that

$$\|i(t,.)\|_{L^1} \leq \kappa^* e^{-\mu t}\|i_0\|_{L^1}\left[1 + \int_0^t \beta\delta e^{\beta\delta s}ds\right], \forall t \geq 0,$$

and by setting $\beta^* := \mu - \beta\delta > 0$ we obtain

$$\|i(t,.)\|_{L^1} \leq \kappa^* e^{-\beta^* t}\|i_0\|_{L^1}, \forall t \geq 0.$$

It follows that the evolution family $\{U_B(t,s)\}_{t\geq s}$ is exponentially stable with constant $\kappa^* \geq 1$ and exponent $\beta^* > 0$.

Then as a consequence of Theorem 3.12 we deduce that there exists $\varepsilon_0 > 0$ small enough such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a globally Lipschitz continuous map $\hat{\vartheta}_\varepsilon : \mathbb{R} \to L^1((0,+\infty),\mathbb{R})$ such that the manifold

$$\hat{\mathcal{M}}_\varepsilon := \left\{\left(u, \begin{pmatrix} 0 \\ \hat{\vartheta}_\varepsilon(u) \end{pmatrix}\right) \in \mathbb{R} \times Y_0 : u \in \mathbb{R}\right\}$$

is invariant by the semiflow generated by system (4.28). Moreover from Theorem 3.12 we also have

$$\|\hat{\vartheta}_\varepsilon(u)\|_{L^1} \leq L_G \varepsilon \hat{C}_0 \leq \zeta, \ \forall u \in \mathbb{R}$$

with $\hat{C}_0 > 0$ the positive constant of Theorem 3.12. Next the semiflow generated by (4.24) restricted to Ω is asymptotically smooth by using the results of Thieme and Vrabie [51]. Therefore by using the result of Hale [20] system (4.24) restricted to Ω has a connected global attractor $\mathcal{A}_\varepsilon \subset \Omega$. Note that for $\varepsilon = 0$ the global attractor is

$$\mathcal{A}_0 = M = \{(u, \vartheta(u)) : u \in [0,1]\} \subset \Omega.$$

Denote by L_ϑ the Lipschitz norm of ϑ in $[0,1]$. By using the lower semi-continuity of the attractor we know that there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for each $\varepsilon \in (0, \varepsilon_1)$ the global attractor $\mathcal{A}_\varepsilon \subset \Omega$ is contained in the $\zeta(1+L_\vartheta)^{-1}-$ neighborhood of $\mathcal{A}_0 = M$.
Let $t \in \mathbb{R} \to (u(t), i(t,\cdot)$ be a complete orbit of (4.24) in the attractor $\mathcal{A}_\varepsilon$ with $\varepsilon \in (0, \varepsilon_1)$. Then we have

$$u(t) \in [0,1],\ \forall t \in \mathbb{R},\ i(t,\cdot) \in L^1_+((0,+\infty), \mathbb{R}),\ \forall t \in \mathbb{R}$$

and

$$\|i(t,\cdot) - \vartheta(u(t))\|_{L^1} \leq \zeta,\ \forall t \in \mathbb{R}.$$

Therefore the map

$$t \in \mathbb{R} \to (u(t), v(t)) \in [0,1] \times Y_0$$

with

$$v(t) = \begin{pmatrix} 0 \\ i(t,\cdot) - \vartheta(u(t)) \end{pmatrix},\ \forall t \in \mathbb{R}$$

is a complete orbit of (4.27) and then a complete orbit of (4.28) because system (4.27) and system (4.28) coincide as long as $u(t) \in [0,1]$.
Now by Theorem 3.12, all bounded complete orbit (u,v) of system (4.28) satisfying $\|v\|_\infty \leq \zeta$ are contained in $\hat{\mathcal{M}}_\varepsilon$. Therefore we must have

$$v(t) = \begin{pmatrix} 0 \\ i(t,\cdot) - \vartheta(u(t)) \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\vartheta}_\varepsilon(u(t)) \end{pmatrix},\ \forall t \in \mathbb{R} \Rightarrow i(t,\cdot) = \vartheta(u(t)) + \hat{\vartheta}_\varepsilon(u(t)),\ \forall t \in \mathbb{R}.$$

Our result reads as follows.

Theorem 4.12 *There exists a Lipschitz continuous map $u \to \hat{\vartheta}_\varepsilon(u)(.)$ from $\mathbb{R}$ into $L^1((0,+\infty), \mathbb{R})$ such that for each $\varepsilon > 0$ small enough, system (4.24) can be reduced to a single human equation on the global attractor $\mathcal{A}_\varepsilon \subset \Omega$ that is*

$$u'(t) = \varepsilon \left[\gamma(1-u(t)) \int_0^{+\infty} \Pi(a) \left(\hat{\vartheta}_\varepsilon(u(t))(a) + \vartheta(u(t))(a) \right) da - \alpha u(t) \right]$$

and $i(t,a) = \hat{\vartheta}_\varepsilon(u(t))(a) + \vartheta(u(t))(a)$ is a mild solution of the i-equation of system (4.24). Moreover we have

$$\|\hat{\vartheta}_\varepsilon(u)\|_{L^1} \leq L_G \varepsilon \hat{C}_0,\ \forall u \in \mathbb{R}.$$

Therefore there are two cases whenever $\varepsilon > 0$ is small enough

(i) *If there is no positive interior equilibrium solution in Ω, the global attractor is reduced to the trivial equilibrium $\{0\}$;*

(ii) *If there is one positive interior equilibrium solution in Ω, the global attractor $\mathcal{A}_\varepsilon$ contains both equilibria and an heteroclinic orbit joining both equilibria.*

5 Growth estimates under small perturbation

In this section we consider the unperturbed equation

$$\dot{u}(t) = F(u(t)) \ t > t_0 \text{ and } u(t_0) = x \in X, \tag{5.1}$$

and the perturbed one

$$\frac{d\hat{u}(t)}{dt} = F(\hat{u}(t)) + \hat{F}(\hat{u}(t), t), \ t > t_0 \text{ and } \hat{u}(t_0) = x \in X. \tag{5.2}$$

We prove that when $\hat{F}$ is sufficiently small then the solutions of (5.2) inherit the properties of the solutions of (5.1).

As consequence of the estimations obtained in this section, we will prove that the linear evolution family generated by

$$\dot{w}(t) = [A + B(\hat{u}(t))]w(t), \ t > t_0, \ w(t_0) = w_0 \in Y_0 \tag{5.3}$$

has an exponential dichotomy whenever $\hat{F}$ is small enough.

5.1 Nonlinear variation of constants formula

For convenience in the following, we rewrite the system (5.1) as a non autonomous system. Note that Assumption 3.1 guaranties for each $x \in X$ and $t_0 \in \mathbb{R}$ the existence and uniqueness of a solution $u \in C^1(\mathbb{R}, X)$ of (5.1) with $u(t_0) = x$. For each $x \in X$ and $t_0 \in \mathbb{R}$ let

$$t \in \mathbb{R} \to \Psi(x, t, t_0)$$

be the unique solution of (5.1) with $\Psi(x, t_0, t_0) = x$. Let us now collect some properties of $\{\Psi(\cdot, t, t_0) : X \to X\}_{(t,t_0)\in\mathbb{R}^2}$. Indeed by using the uniqueness of the solutions of (5.1) it follows that $\{\Psi(\cdot, t, t_0)\}_{(t,t_0)\in\mathbb{R}^2}$ has the evolutionary properties that is

$$\begin{cases} \Psi(\cdot, t, l) \circ \Psi(\cdot, l, t_0) = \Psi(\cdot, t, t_0), \ \forall (t, l), (l, t_0) \in \mathbb{R}^2 \\ \Psi(x, t, t) = x, \ \forall t \in \mathbb{R} \text{ and } x \in X. \end{cases} \tag{5.4}$$

Observe that under Assumption 3.1 one has for any $(t, t_0) \in \mathbb{R}^2$ the map

$$x \in X \to \Psi(x, t, t_0)$$

is differentiable on X. Moreover (5.4) implies that for any $(t, t_0) \in \mathbb{R}^2$ the map $x \in X \to \Psi(x, t, t_0)$ is a diffeomorphism on X with inverse function $x \in X \to \Psi(x, t_0, t)$. In particular for all $x \in X$ and $(t, t_0) \in \mathbb{R}^2$ the linear map

$$\partial_x \Psi(x, t, t_0) \in \mathcal{L}(X)$$

is invertible on X with inverse

$$\partial_x \Psi(x, t, t_0)^{-1} = \partial_x \Psi(\hat{x}, t_0, t) \in \mathcal{L}(X) \quad \text{with} \quad \hat{x} := \Psi(x, t, t_0)$$

and $\{\partial_x \Psi(x,t,t_0)\}_{(t,t_0)\in\Delta} \subset \mathcal{L}(X)$ is the linear evolution family generated by

$$\frac{dz(t)}{dt} = DF(\Psi(x,t,t_0))z(t),\ t > t_0 \text{ and } z(t_0) = z_0 \in X. \tag{5.5}$$

Furthermore one also has for each $t_0 \in \mathbb{R}$

$$\partial_l \Psi(x,t,l) = -\partial_x \Psi(x,t,l)F(x),\ \forall t \geq l \geq t_0 \text{ and } x \in X. \tag{5.6}$$

Lemma 5.1 (Nonlinear variation of constants formula) *Let Assumption 3.1 be satisfied. If $\hat{u} \in C^1(\mathbb{R}, X)$ satisfies*

$$\frac{d\hat{u}(t)}{dt} = F(\hat{u}(t)) + \hat{F}(\hat{u}(t),t),\ t \in \mathbb{R}$$

then

$$\hat{u}(t) = \Psi(\hat{u}(t_0),t,l) + \int_{t_0}^{t} \partial_x \Psi(\hat{u}(r),t,r)\hat{F}(\hat{u}(r),r)dr,\ \forall t \geq t_0.$$

Proof. Note that for each $t \geq l \geq t_0$ we have

$$\frac{d\Psi(\hat{u}(l),t,l)}{dl} = \partial_x \Psi(\hat{u}(l),t,l)\frac{d\hat{u}(l)}{dl} + \partial_l \Psi(\hat{u}(l),t,l)$$

and since $\hat{u}(l) \in X$ we obtain from (5.6)

$$\partial_l \Psi(\hat{u}(l),t,l) = -\partial_x \Psi(\hat{u}(l),t,l)F(\hat{u}(l))$$

so that by the linearity of $\partial_x \Psi(\hat{u}(l),t,l)$

$$\begin{aligned} \frac{d\Psi(\hat{u}(l),t,l)}{dl} &= \partial_x \Psi(\hat{u}(l),t,l)\left[\frac{d\hat{u}(l)}{dl} - F(\hat{u}(l))\right] \\ &= \partial_x \Psi(\hat{u}(l),t,l)\hat{F}(\hat{u}(l),l). \end{aligned}$$

Then integrating $\frac{d\Psi(\hat{u}(l),t,l)}{dl}$ between t_0 and t yields

$$\Psi(\hat{u}(t),t,t) - \Psi(\hat{u}(t_0),t,t_0) = \int_{t_0}^{t} \partial_x \Psi(\hat{u}(r),t,r)\hat{F}(\hat{u}(r),r)dr$$

and the result follows by using the fact that $\Psi(\hat{u}(t),t,t) = \hat{u}(t)$. ∎

5.2 Growth estimates for the perturbed equation

We start by giving estimates for $\{\partial_x \Psi(x,t,t_0)\}_{(t,t_0)\in\mathbb{R}^2} \subset \mathcal{L}(X)$ on the operator norm.

Lemma 5.2 *Let Assumptions 3.1 and 3.2 be satisfied. Then the following estimates hold*

$$\|\partial_x \Psi(x,t,t_0)\|_{\mathcal{L}(X)} \leq \kappa_0\ e^{\beta_0|t-t_0|},\ \forall (t,t_0) \in \mathbb{R}^2 \text{ and } x \in X, \tag{5.7}$$

and

$$\|\partial_x \Psi(x,t,t_0)^{-1}\|_{\mathcal{L}(X)} \leq \kappa_0\ e^{\beta_0|t-t_0|},\ \forall (t,t_0) \in \mathbb{R}^2 \text{ and } x \in X. \tag{5.8}$$

Proof. Let $(t,t_0) \in \mathbb{R}^2$ and $x \in X$ be given. Then using (3.1) one has for each $s>0$, $\hat{x} \in X$ and $z \in X$

$$\begin{array}{rcl} \frac{1}{s}\|\Psi(\hat{x}+sz,t,t_0)-\Psi(\hat{x},t,t_0)\| & \leq & \frac{1}{s}\kappa_0 e^{\beta_0|t-t_0|}\|\Psi(\hat{x}+sz,t_0,t_0)-\Psi(\hat{x},t_0,t_0)\| \\ & \leq & \kappa_0 e^{\beta_0|t-t_0|}\|z\| \end{array}$$

so that letting $s \to 0^+$ provides

$$\|\partial_x \Psi(\hat{x},t,t_0)z\| \leq \kappa_0 \; e^{\beta_0|t-t_0|}\|z\|, \; \forall (t,t_0) \in \mathbb{R}^2.$$

Hence (5.7) holds by setting $\hat{x}=x$ while (5.8) is obtained from (5.7) by setting $\hat{x}=\Psi(x,t,t_0)$ and using the fact that

$$\partial_x \Psi(x,t,t_0)^{-1} = \partial_x \Psi(\hat{x},t_0,t).$$

■

In what follows our goal is to prove a lemma analogous to Lemma 5.2 but for system (5.2). We will need the following assumption on $\hat{F}$

Assumption 5.3 *Assume that for each $t \in \mathbb{R}$, the map $x \to \hat{F}(x,t)$ belongs to $C^1(X)$ and there exists $\hat{\eta}>0$ such that*

$$\|\hat{F}(x_1,t)-\hat{F}(x_2,t)\| \leq \hat{\eta}\|x_1-x_2\|, \; \forall x_1,x_2 \in X \text{ and } t \in \mathbb{R}.$$

Assume in addition that for any $z \in X$, the map

$$(t,x) \in \mathbb{R}\times X \to \partial_x \hat{F}(x,t)z \in X$$

is continuous on $\mathbb{R}\times X$ and for each $t \in \mathbb{R}$ the map

$$x \in X \to \partial_x \hat{F}(x,t) \in \mathcal{L}(X)$$

is continuous on X.

Assumptions 3.1 and 5.3 ensure that for each $x \in X$ and $t_0 \in \mathbb{R}$ there exists a unique solution $\hat{u} \in C(\mathbb{R},X)$ of (5.2) with $u(t_0)=x$. For any $x \in X$ and $t_0 \in \mathbb{R}$ let

$$t \in \mathbb{R} \to \hat{\Psi}(x,t,t_0)$$

be the unique solution of (5.2) satisfying $\hat{\Psi}(x,t_0,t_0)=x$. Then we have

$$\begin{cases} \hat{\Psi}(\cdot,t,l)\circ\hat{\Psi}(\cdot,l,t_0)=\hat{\Psi}(\cdot,t,t_0), \; \forall (t,l),(l,t_0) \in \mathbb{R}^2 \\ \hat{\Psi}(x,t,t)=x, \; \forall t \in \mathbb{R} \text{ and } x \in X. \end{cases} \tag{5.9}$$

Furthermore Assumptions 3.1 and 5.3 together with (5.9) implies that for any $(t,t_0) \in \mathbb{R}^2$ the map

$$x \in X \to \hat{\Psi}(x,t,t_0)$$

is a diffeomorphism on X with inverse function $x \in X \to \hat{\Psi}(x,t_0,t)$. Hence for each $x \in X$ and $(t,t_0) \in \mathbb{R}^2$, the linear map

$$\partial_x \hat{\Psi}(x,t,t_0) : X \to X$$

is invertible on X with inverse

$$\partial_x \hat{\Psi}(x,t,t_0)^{-1} = \partial_x \hat{\Psi}(\hat{x},t_0,t) \quad \text{with} \quad \hat{x} := \hat{\Psi}(x,t,t_0)$$

and $\{\partial_x \hat{\Psi}(x,t,t_0)\}_{(t,t_0)\in\Delta} \subset \mathcal{L}(X)$ is the linear evolution family generated by

$$\frac{d\hat{z}(t)}{dt} = DF(\hat{\Psi}(x,t,t_0))\hat{z}(t) + \partial_x \hat{F}(\hat{\Psi}(x,t,t_0),t)\hat{z}(t),\ t > t_0 \text{ and } \hat{z}(t_0) = z_0 \in X. \tag{5.10}$$

By using comparison principle for ordinary differential equation we have the following result.

Lemma 5.4 *Let $\alpha : [a,b] \to [0,+\infty)$ with $a < b$ be a continuous function on $[a,b]$. Let $\gamma \in \mathbb{R}$ and $c_1 \geq 1$ and $c_2 \geq 0$ be given. Then the inequality*

$$\alpha(t) \leq c_1 \alpha(a) e^{\gamma(t-a)} + c_2 \int_a^t e^{\gamma(t-r)} \alpha(r) dr, \forall t \in [a,b], \tag{5.11}$$

implies that

$$\alpha(t) \leq c_1 \alpha(a) e^{(\gamma+c_2)(t-a)}, \forall t \in [a,b], \tag{5.12}$$

and the inequality

$$\alpha(t) \leq c_1 \alpha(b) e^{\gamma(b-t)} + c_2 \int_t^b e^{\gamma(r-t)} \alpha(r) dr, \forall t \in [a,b], \tag{5.13}$$

implies that

$$\alpha(t) \leq c_1 \alpha(b) e^{(\gamma+c_2)(b-t)}, \forall t \in [a,b]. \tag{5.14}$$

Lemma 5.5 *Let Assumptions 3.1, 3.2 and 5.3 be satisfied. Assume in addition that*

$$\sup_{(x,t)\in X\times\mathbb{R}} \|\hat{F}(x,t)\| \leq \hat{\eta}. \tag{5.15}$$

Then for each $x \in X$, each $m > 1$ and $(t,t_0) \in \mathbb{R}^2$ we have

$$\begin{cases} \|\partial_x \hat{\Psi}(x,t,t_0)\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\left(\frac{\ln(\kappa_0)}{m} + \beta_0 + \hat{\eta}\tilde{\kappa}_0 (m+1) e^{\beta_0 m}\right)|t-t_0|} \\ \|\partial_x \hat{\Psi}(x,t,t_0)^{-1}\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\left(\frac{\ln(\kappa_0)}{m} + \beta_0 + \hat{\eta}\tilde{\kappa}_0 (m+1) e^{\beta_0 m}\right)|t-t_0|} \end{cases} \tag{5.16}$$

where

$$\tilde{\kappa}_0 := \max(\kappa_0^2 L_F, \kappa_0).$$

Proof. First of all observe that condition

$$\|\hat{F}(x_1,t) - \hat{F}(x_2,t)\| \leq \hat{\eta}\|x_1 - x_2\|,\ \forall x_1, x_2 \in X \text{ and } t \in \mathbb{R}$$

implies that

$$\sup_{(x,t)\in X\times\mathbb{R}} \|\partial_x \hat{F}(x,t)\|_{\mathcal{L}(X)} \leq \hat{\eta}. \tag{5.17}$$

Let $x \in X$ and $m > 0$ be given and fixed. Let $t_0 \in \mathbb{R}$ and $z_0 \in X$. Define

$$\hat{z}(t) := \partial_x \hat{\Psi}(x,t,t_0) z_0,\ \forall t \geq t_0.$$

Then $t \in [t_0, +\infty) \to \hat{z}(t)$ satisfies (5.10) which can be rewritten as

$$\frac{d\hat{z}(t)}{dt} = DF(\Psi(x,t,t_0))\hat{z}(t) + \left[DF(\hat{\Psi}(x,t,t_0)) - DF(\Psi(x,t,t_0))\right]\hat{z}(t) + \partial_x \hat{F}(\hat{\Psi}(x,t,t_0),t)\hat{z}(t),$$

for $t > t_0$ and

$$\hat{z}(t_0) = z_0.$$

Recalling that $\{\partial_x \Psi(x,t,t_0)\}_{(t,t_0)\in\Delta}$ is the linear evolution family generated by (5.5) we can use a variation of constants formula to obtain on one hand

$$\begin{aligned}\hat{z}(t) = \partial_x\Psi(x,t,l)\hat{z}(l) + \int_l^t \partial_x\Psi(x,t,r)\left[DF(\hat{\Psi}(x,r,l)) - DF(\Psi(x,r,l))\right]\hat{z}(r)dr \\ + \int_l^t \partial_x\Psi(x,t,r)\partial_x\hat{F}(\hat{\Psi}(x,r,l),r)\hat{z}(r)dr, \ \forall t \geq l \geq t_0.\end{aligned} \tag{5.18}$$

On the other hand since for any $r \in [l,t] \subset [t_0,t]$

$$\partial_x\Psi(x,t,l) = \partial_x\Psi(x,t,r)\partial_x\Psi(x,r,l) \Rightarrow \partial_x\Psi(x,t,l)^{-1} = \partial_x\Psi(x,r,l)^{-1}\partial_x\Psi(x,t,r)^{-1}$$

by applying $\partial_x\Psi(x,t,l)^{-1}$ to the left hand side of (5.18) we obtain

$$\begin{aligned}\hat{z}(l) = \partial_x\Psi(x,t,l)^{-1}\hat{z}(t) - \int_l^t \partial_x\Psi(x,r,l)^{-1}\left[DF(\hat{\Psi}(x,r,l)) - DF(\Psi(x,r,l))\right]\hat{z}(r)dr \\ - \int_l^t \partial_x\Psi(x,r,l)^{-1}\partial_x\hat{F}(\hat{\Psi}(x,r,t_0),r)\hat{z}(r)dr, \ \forall t \geq l \geq t_0.\end{aligned} \tag{5.19}$$

We now divide the proof into three steps.
Step 1 : In this step we will prove that

$$\begin{cases} \|\partial_x\hat{\Psi}(x,t,t_0)\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\left(\beta_0 + \hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}, & \forall t \in [t_0, t_0+m] \\ \|\partial_x\hat{\Psi}(x,t,t_0)^{-1}\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\left(\beta_0 + \hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}, & \forall t \in [t_0, t_0+m]. \end{cases} \tag{5.20}$$

Using (5.18) we infer from Lemma 5.2 combined together with Assumption 3.1 and (5.17) that

$$\begin{aligned}\|\hat{z}(t)\| \leq \ & \kappa_0 e^{\beta_0(t-t_0)}\|\hat{z}(t_0)\| + \int_{t_0}^t \kappa_0 e^{\beta_0(t-r)} L_F \|\hat{\Psi}(x,r,t_0) - \Psi(x,r,t_0)\|\|\hat{z}(r)\|dr \\ & + \int_{t_0}^t \kappa_0 e^{\beta_0(t-r)}\hat{\eta}\|\hat{z}(r)\|dr, \ \forall t \in [t_0, t_0+m].\end{aligned} \tag{5.21}$$

Similarly we obtain from (5.19) that for each $t \in [t_0, t_0+m]$

$$\begin{aligned}\|\hat{z}(l)\| \leq \ & \kappa_0 e^{\beta_0(t-l)}\|\hat{z}(t)\| + \int_l^t \kappa_0 e^{\beta_0(r-l)} L_F \|\hat{\Psi}(x,r,l) - \Psi(x,r,l)\|\|\hat{z}(r)\|dr \\ & + \int_l^t \kappa_0 e^{\beta_0(r-l)}\hat{\eta}\|\hat{z}(r)\|dr, \ \forall l \in [t_0, t].\end{aligned} \tag{5.22}$$

Next by using Lemma 5.1 and (5.15) one obtains for each $r \in [t_0, t_0+m]$

$$\begin{aligned}
\|\hat{\Psi}(x,r,t_0)-\Psi(x,r,t_0)\| &\leq \int_{t_0}^{r} \kappa_0 e^{\beta_0(r-l)}\|\hat{F}(\hat{\Psi}(x,l,t_0),l)\|dl \\
&\leq \int_{t_0}^{r} \kappa_0 e^{\beta_0(r-l)}\hat{\eta}dl \\
&\leq \kappa_0(r-t_0)e^{\beta_0(r-t_0)}\hat{\eta} \\
&\leq \kappa_0 m e^{\beta_0 m}\hat{\eta}
\end{aligned} \tag{5.23}$$

and by plugging (5.23) into (5.21) we obtain for each $t \in [t_0, t_0+m]$

$$\begin{aligned}
\|\hat{z}(t)\| &\leq \kappa_0 e^{\beta_0(t-t_0)}\|\hat{z}(t_0)\| + \int_{t_0}^{t} \hat{\eta}\left[L_F\kappa_0^2 m e^{\beta_0 m}+\kappa_0\right] e^{\beta_0(t-r)}\|\hat{z}(r)\|dr \\
&\leq \kappa_0 e^{\beta_0(t-t_0)}\|\hat{z}(t_0)\| + \hat{\eta}\left[\kappa_0^2 L_F m+\kappa_0\right] e^{\beta_0 m}\int_{t_0}^{t} e^{\beta_0(t-r)}\|\hat{z}(r)\|dr
\end{aligned} \tag{5.24}$$

Recalling that

$$\tilde{\kappa}_0 = \max(\kappa_0^2 L_F, \kappa_0),$$

we obtain from (5.24)

$$\|\hat{z}(t)\| \leq \kappa_0 e^{\beta_0(t-t_0)}\|\hat{z}(t_0)\| + \hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\int_{t_0}^{t} e^{\beta_0(t-r)}\|\hat{z}(r)\|dr,\ \forall t \in [t_0, t_0+m]$$

and by similar arguments we also obtain

$$\|\hat{z}(l)\| \leq \kappa_0 e^{\beta_0(t-l)}\|\hat{z}(t)\| + \hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\int_{l}^{t} e^{\beta_0(r-l)}\|\hat{z}(r)\|dr,\ \forall l \in [t_0, t].$$

Therefore by Lemma 5.4

$$\begin{cases}
\|\hat{z}(t)\| \leq \kappa_0 e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|\hat{z}(t_0)\|,\ \forall t \in [t_0, t_0+m] \\
\|\hat{z}(t_0)\| \leq \kappa_0 e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|\hat{z}(t)\|,\ \forall t \in [t_0, t_0+m]
\end{cases}$$

that is

$$\begin{cases}
\|\partial_x\hat{\Psi}(x,t,t_0)z_0\| \leq \kappa_0 e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|z_0\|,\ \forall t \in [t_0, t_0+m] \\
\|z_0\| \leq \kappa_0 e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|\partial_x\hat{\Psi}(x,t,t_0)z_0\|,\ \forall t \in [t_0, t_0+m]
\end{cases}$$

and (5.20) follows.

Step 2 : In this step we prove the lemma whenever $(t, t_0) \in \Delta$. Observe when $0 \leq t-t_0 \leq m$ then the estimates (5.16) follows from Step 1. Now assume that $t-t_0 > m$. Then there exists a positive integer n and a non negative real number r with $0 \leq r < m$ such that

$$t - t_0 = nm + r.$$

For more convenience in the notations set

$$t_k := t_0 + (k-1)m + r,\ 1 \leq k \leq n$$

so that

$$\begin{cases} t_{n+1} = t \\ t_{k+1} - t_k = m, \quad \text{if } 1 \le k \le n \\ t_1 - t_0 = r < m. \end{cases}$$

Since $\{\partial_x \hat{\Psi}(x,t,t_0)\}_{(t,t_0)\in\Delta}$ is a linear evolution family we have

$$\partial_x \hat{\Psi}(x,t,t_0)z_0 = \partial_x \hat{\Psi}(x,t_{n+1},t_0)z_0 = \partial_x \hat{\Psi}(x,t_{n+1},t_n)\partial_x \hat{\Psi}(x,t_n,t_{n-1})\cdots\partial_x \hat{\Psi}(x,t_1,t_0)z_0$$

and by Step 1

$$\begin{aligned} \|\partial_x \hat{\Psi}(x,t,t_0)z_0\| &\le \kappa_0 e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t_{n+1}-t_n)}\cdots\kappa_0 e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t_1-t_0)}\|z_0\| \\ &\le \kappa_0\kappa_0^n e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t_{n+1}-t_0)}\|z_0\| \\ &\le \kappa_0\kappa_0^n e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|z_0\|. \end{aligned}$$

Then observing that

$$n = \frac{t-t_0-r}{m} \le \frac{t-t_0}{m}$$

we obtain

$$\|\partial_x \hat{\Psi}(x,t,t_0)z_0\| \le \kappa_0\kappa_0^{\frac{t-t_0}{m}} e^{\left(\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|z_0\| = \kappa_0 e^{\left(\frac{\ln(\kappa_0)}{m}+\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|z_0\|.$$

Since

$$\partial_x \hat{\Psi}(x,t,t_0)^{-1}z_0 = \partial_x \hat{\Psi}(x,t_1,t_0)^{-1}\cdots\partial_x \hat{\Psi}(x,t_n,t_{n-1})^{-1}\partial_x \hat{\Psi}(x,t_{n+1},t_n)^{-1}z_0$$

by using similar arguments one also obtains

$$\|\partial_x \hat{\Psi}(x,t,t_0)^{-1}z_0\| \le \kappa_0 e^{\left(\frac{\ln(\kappa_0)}{m}+\beta_0+\hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)(t-t_0)}\|z_0\|.$$

Step 3 : We can now complete the proof of our lemma by using Step 2. Indeed if $(t,t_0)\in\Delta$ and $x \in X$ then by setting

$$\hat{x} = \hat{\Psi}(x,t_0,t) \in X$$

we obtain

$$\partial_x \hat{\Psi}(x,t_0,t) = \partial_x \hat{\Psi}(\hat{x},t,t_0)^{-1} \Longleftrightarrow \partial_x \hat{\Psi}(x,t_0,t)^{-1} = \partial_x \hat{\Psi}(\hat{x},t,t_0)$$

and the result follows from Step 2. ∎

As a consequence of Lemma 5.5 we obtain the following analogous of Assumption 3.2 for the perturbed equation (5.2).

Lemma 5.6 *Let Assumptions 3.1, 3.2 and 5.3 be satisfied. For each $\hat{\beta}_0 > \beta_0$ there exists $\hat{\eta}_0 > 0$ such that if*

$$\sup_{(x,t)\in X\times\mathbb{R}} \|\hat{F}(x,t)\| \le \hat{\eta}_0$$

then any two solutions $\hat{u}_1, \hat{u}_2 \in C^1(\mathbb{R},X)$ of (5.2) satisfies

$$\|\hat{u}_1(t)-\hat{u}_2(t)\| \le \kappa_0 e^{\hat{\beta}_0|t-t_0|}\|\hat{u}_1(t_0)-\hat{u}_2(t_0)\|, \ \forall (t,t_0)\in\mathbb{R}^2.$$

Proof. Let $\hat{u}_1, \hat{u}_2 \in C^1(\mathbb{R}, X)$ be two solutions of (5.2). Then we have

$$\hat{u}_i(t) = \hat{\Psi}(\hat{u}_i(t_0), t, t_0),\ \forall (t, t_0) \in \mathbb{R}^2,\ i = 1, 2.$$

Hence

$$\begin{aligned} \hat{u}_1(t) - \hat{u}_2(t) &= \hat{\Psi}(\hat{u}_1(t_0), t, t_0) - \hat{\Psi}(\hat{u}_2(t_0), t, t_0) \\ &= \int_0^1 \partial_x \hat{\Psi}(r\hat{u}_1(t_0) + (1-r)\hat{u}_2(t_0), t, t_0)dr\ (\hat{u}_1(t_0) - \hat{u}_2(t_0)) \end{aligned}$$

and from Lemma 5.5 if

$$\sup_{(x,t) \in X \times \mathbb{R}} \|\hat{F}(x,t)\| \leq \hat{\eta}$$

then for each $m > 1$

$$\|\hat{\Psi}(r\hat{u}_1(t_0) + (1-r)\hat{u}_2(t_0), t, t_0)\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\left(\frac{\ln(\kappa_0)}{m} + \beta_0 + \hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)|t - t_0|},\ \forall r \in [0, 1]$$

and we obtain

$$\|\hat{u}_1(t) - \hat{u}_2(t)\| \leq \kappa_0 e^{\left(\frac{\ln(\kappa_0)}{m} + \beta_0 + \hat{\eta}\tilde{\kappa}_0(m+1)e^{\beta_0 m}\right)|t - t_0|} \|\hat{u}_1(t_0) - \hat{u}_2(t_0)\|,\ \forall (t, t_0) \in \mathbb{R}^2.$$

The result follows by taking firstly $m > 1$ large enough and secondly $\hat{\eta} > 0$ sufficiently small. ■

5.3 Persistence of exponential dichotomy

In this subsection we will show that if $\hat{u} \in C^1(\mathbb{R}, X)$ is a solution of (5.2) for $\hat{F}$ sufficiently small then the evolution family generated by (5.3) has an exponential dichotomy.

Proposition 5.7 *Let Assumptions 2.1, 2.3, 3.1, 3.2, 3.4, 3.7 and 5.3 be satisfied. Let $\hat{\beta} \in (\beta_0, \beta)$ be given. There exists $\eta_1 > 0$ such that if*

$$\sup_{(x,t) \in X \times \mathbb{R}} \|\hat{F}(x,t)\| \leq \eta_1$$

and $\hat{u} \in C^1(\mathbb{R}, X)$ is solution of (5.2) then the evolution family generated by (5.3) has an exponential dichotomy with some constant $\hat{\kappa} \geq 1$ and exponent $\hat{\beta}$.

This proposition is crucial in proving our main result. Before giving its proof we first introduce some notions and prove a technical lemma.

Definition 5.8 *Let $\hat{u} \in C^1(\mathbb{R}, X)$ be a solution of (5.2) and $(t_k)_{k \in \mathbb{Z}} \subset \mathbb{R}$ be a non-decreasing sequence. We say that the sequence of functions $(u_k)_{k \in \mathbb{Z}} \subset C^1(\mathbb{R}, X)$ is a t_k-approximation of $\hat{u}$ if u_k satisfies*

$$\dot{u}_k(t) = F(u_k(t)), \quad \forall t \in \mathbb{R} \quad \text{and} \quad u_k(t_k) = \hat{u}(t_k),$$

for each $k \in \mathbb{Z}$.

The following lemma gives a relationship between $\hat{u}$ and its approximation.

Lemma 5.9 *Let Assumptions 3.1 and 5.3 be satisfied. Assume in addition that*

$$\sup_{(x,t)\in X\times\mathbb{R}} \|\hat{F}(x,t)\| \leq \hat{\eta}.$$

Let $\hat{u} \in C^1(\mathbb{R}, X)$ be a solution of (5.2) and $(t_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a non-decreasing sequence. If $(u_k)_{k\in\mathbb{Z}} \subset C^1(\mathbb{R}, X)$ is a t_k-approximation of $\hat{u}$ then

$$\|u(t) - u_k(t)\| \leq \hat{\eta}(t-t_k)e^{L_F(t-t_k)}, \quad \forall t \geq t_k. \tag{5.25}$$

Proof. We have

$$\begin{cases} \hat{u}(t) = \hat{u}(t_k) + \displaystyle\int_{t_k}^t F(\hat{u}(r))dr + \int_{t_k}^t \hat{F}(\hat{u}(r), r)dr, \ t \geq t_k \\ u_k(t) = u_k(t_k) + \displaystyle\int_{t_k}^t F(u_k(t))dr, \ t \geq t_k \end{cases}$$

and since $\hat{u}(t_k) = u_k(t_k)$ we obtain

$$\|\hat{u}(t) - u_k(t)\| \leq L_F \int_{t_k}^t \|\hat{u}(r) - u_k(r)\|dr + \hat{\eta}(t-t_k)$$

and (5.25) follows by Gronwall's inequality. ∎

In the proof of Proposition 6.3 we will use the following notion of exponential dichotomy for discrete time systems.

Definition 5.10 *Let $\boldsymbol{C} = \{C_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(Y_0)$ be a family of bounded linear operators on Y_0. Define*

$$U_{\boldsymbol{C}}(n,m) := \begin{cases} C_{n-1}\dots C_m, & \text{if } n > m \\ I_{\mathcal{L}(Y_0)}, & \text{if } n = m. \end{cases}$$

We say that $\boldsymbol{C} = \{C_n\}_{n\in\mathbb{Z}}$ is exponentially dichotomic on $\mathbb{Z}$ with constant $\kappa \geq 1$ and exponent $\beta > 0$ if and only if the following properties are satisfied

(i) *There exist two families of projections $\{\Pi_n^+\}_{n\in\mathbb{Z}} \subset \mathcal{L}(Y_0)$ and $\{\Pi_n^-\}_{n\in\mathbb{Z}} \subset \mathcal{L}(Y_0)$ on Y_0 such that*

$$\Pi_n^+ + \Pi_n^- = I_{\mathcal{L}(Y_0)}, \quad \forall n \in \mathbb{Z}.$$

Then we define for all $n \geq m$

$$U_{\boldsymbol{C}}^+(n,m) := U_{\boldsymbol{C}}(n,m)\Pi_n^+ \quad \text{and} \quad U_{\boldsymbol{C}}^-(n,m) := U(n,m)\Pi_m^-.$$

(ii) *For all $n \geq m$ we have $\Pi_n^+ U_{\boldsymbol{C}}(n,m) = U_{\boldsymbol{C}}(n,m)\Pi_m^+$.*

(iii) *For all $n \geq m$ the restricted linear operator $U_{\boldsymbol{C}}(n,m)\Pi_m^-$ is invertible from $\Pi_m^-(Y_0)$ into $\Pi_n^-(Y_0)$ with inverse denoted by $\bar{U}_{\boldsymbol{C}}^-(m,n)$ and we set*

$$U_{\boldsymbol{C}}^-(m,n) := \bar{U}_{\boldsymbol{C}}^-(m,n)\Pi_n^-.$$

(iv) *For all* $n \geq m$

$$\|U^+_{\boldsymbol{C}}(n,m)\|_{\mathcal{L}(Y_0)} \leq \kappa e^{-\beta(n-m)} \quad \text{and} \quad \|U^-_{\boldsymbol{C}}(m,n)\|_{\mathcal{L}(Y_0)} \leq e^{-\beta(n-m)}.$$

Now we are in position to prove the main result of this section.

Proof of Proposition 5.7 . Let $\hat{u} \in C^1(\mathbb{R}, X)$ be a solution of (5.2). Denote by $\{U_{\hat{u}}(t,l)\}_{t\geq l} \subset \mathcal{L}(Y_0)$ the evolution family generated by

$$\dot{w}(t) = [A + B(\hat{u}(t))]w(t),\ t > l,\ w(l) = w_0 \in Y_0.$$

The existence of such evolution family is ensured by Assumptions 2.1, 2.3 and 3.7 together with Lemma 2.9. Let $t_0 \in \mathbb{R}$ be given. Define the non-decreasing sequence

$$t_k := t_0 + k, \quad k \in \mathbb{Z}. \tag{5.26}$$

Let $(u_k)_{k\in\mathbb{Z}} \subset C^1(\mathbb{R}, X)$ be a t_k-approximation of u with t_k defined in (5.26). For each $k \in \mathbb{Z}$ denote by $\{U_{u_k}(t,l)\}_{t\geq l} \subset \mathcal{L}(Y_0)$, the evolution family generated by

$$\dot{w}(t) = [A + B(u_k(t))]\, w(t), \quad t > l \ \text{ and } w(l) = w_0 \in Y_0.$$

Then due to Assumption 3.4 the evolution family $\{U_{u_k}(t,l)\}_{t\geq l} \subset \mathcal{L}(Y_0)$, $k \in \mathbb{Z}$, has an exponential dichotomy with constants κ and exponent β. Moreover from Assumption 3.7 and Lemma 2.9 we also have for each $k \in \mathbb{Z}$ and $t \geq l$

$$\|U_{u_k}(t,l)\|_{\mathcal{L}(Y_0)} \leq M_1 e^{\omega_1(t-l)} \quad \text{and} \quad \|U_{\hat{u}}(t,l)\|_{\mathcal{L}(Y_0)} \leq M_1 e^{\omega_1(t-l)}, \tag{5.27}$$

for some $M_1 \geq 1$ and $\omega_1 \in \mathbb{R}$ independent of $\hat{u}$ and u_k.
Next define the following sequences of bounded linear operators

$$C_n := U_{\hat{u}}(t_{n+1}, t_n) \in \mathcal{L}(Y_0), \quad \forall n \in \mathbb{Z},$$

and

$$C^k_n := U_{u_k}(t_{n+1}, t_n), \quad \forall n \in \mathbb{Z}.$$

Then for each $k \in \mathbb{Z}$, it is easy to see that $\{C^k_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(Y_0)$ is exponentially dichotomic on $\mathbb{Z}$ with constant κ and exponent β.
Next let $y \in Y_0$ and $n \in \mathbb{Z}$ be given. Then $t \in [t_n, +\infty) \to U_{\hat{u}}(t, t_n)y$ is the mild solution of

$$\dot{w}(t) = [A + B(\hat{u}(t))]w(t),\ t > t_n \text{ and } w(t_n) = y \in Y_0$$

which can be rewritten for $k \in \mathbb{Z}$ as

$$\dot{w}(t) = [A + B(u_k(t))]w(t) + [B(\hat{u}(t)) - B(u_k(t))]w(t),\ t > t_n \text{ and } w(t_n) = y.$$

Then using lemma 2.9 with $f(t) = [B(\hat{u}(t)) - B(u_k(t))]w(t)$ we obtain

$$w(t) = U_{u_k}(t, t_n)y + \lim_{\lambda\to+\infty} \int_{t_n}^{t} U_{u_k}(t,r)\lambda R_\lambda(A)\,[B(\hat{u}(r)) - B(u_k(r))]\, w(r)dr,\ t \geq t_n$$

or equivalently

$$U_{\hat{u}}(t,t_n)y = U_{u_k}(t,t_n)y + \lim_{\lambda\to+\infty}\int_{t_n}^{t} U_{u_k}(t,r)\lambda R_\lambda(A)\left[B(\hat{u}(r)) - B(u_k(r))\right]U_{\hat{u}}(r,t_n)ydr,\ t\geq t_n.$$

Then by Proposition 2.11 we obtain

$$\|U_{\hat{u}}(t,t_n)y - U_{u_k}(t,t_n)y\| \leq \delta^*(t-t_n)\sup_{r\in[t_n,t]}\|\left[B(\hat{u}(r)) - B(u_k(r))\right]U_{\hat{u}}(r,t_n)y\|,\ \forall t\geq t_n$$

where $\delta^*:[0,+\infty)\to[0,+\infty)$ is a non decreasing function depending only on A, ω_A, M_A, L_B and δ. Therefore by using Assumption 3.7, Lemma 5.9 and (5.27) we obtain for each $t\geq t_n$

$$\begin{aligned}\|U_{\hat{u}}(t,t_n)y - U_{u_k}(t,t_n)y\| &\leq L_B\delta^*(t-t_n)M_1e^{\omega_1(t-t_n)}\|y\|\sup_{r\in[t_n,t]}\|\hat{u}(r)-u_k(r)\|\\ &\leq L_B\delta^*(t-t_n)M_1e^{\omega_1(t-t_k)}\hat{\eta}(t-t_n)e^{L_F(t-t_n)}\|y\|.\end{aligned}$$

providing that

$$\|U_{\hat{u}}(t,t_n) - U_{u_k}(t,t_n)\|_{\mathcal{L}(Y_0)} \leq L_B\delta^*(t-t_n)M_1e^{\omega_1(t-t_n)}\hat{\eta}(t-t_n)e^{L_F(t-t_n)}.$$

It follows that for each $k,n\in\mathbb{Z}$

$$\begin{aligned}\|C_n - C_n^k\|_{\mathcal{L}(Y_0)} &= \|U_{\hat{u}}(t_{n+1},t_n) - U_{u_k}(t_{n+1},t_n)\|_{\mathcal{L}(Y_0)}\\ &\leq L_B\delta^*(t_{n+1}-t_n)M_1e^{\omega_1(t_{n+1}-t_n)}\hat{\eta}(t_{n+1}-t_n)e^{L_F(t_{n+1}-t_n)}\end{aligned}$$

so that for each $p\in\mathbb{N}$ we obtain

$$\sup_{n\in[k,k+p]}\|C_n - C_n^k\|_{\mathcal{L}(Y_0)} \leq L_B\delta^*(p)M_1e^{\omega_1 p}\hat{\eta}pe^{L_F p}.$$

Furthermore since we have

$$\sup_{n\in\mathbb{Z}}\|C_n\|_{\mathcal{L}(Y_0)} \leq M_1e^{\omega_1} \quad\text{and}\quad \sup_{n\in\mathbb{Z}}\|C_n^k\|_{\mathcal{L}(Y_0)} \leq M_1e^{\omega_1},$$

we infer from [9, Proposition 6.4] that for $\hat{\eta}$ sufficiently small $\{C_n\}_{n\in\mathbb{Z}}\subset\mathcal{L}(Y_0)$ is exponentially dichotomic on $\mathbb{Z}$ with some constant $\widetilde{\kappa}\geq\kappa$ (independent of t_0) and exponent $0<\hat{\beta}<\beta$. Now using the fact that $\widetilde{\kappa}$ and $\hat{\beta}$ do not depend on t_0 we are in the situation of [22, Theorem 1.3] so we conclude that $\{U_{\hat{u}}(t,l)\}_{t\geq l}\subset\mathcal{L}(Y_0)$ has an exponential dichotomy with constant $\hat{\kappa}\geq\kappa$ (possibly larger than $\widetilde{\kappa}$) and exponent $\hat{\beta}$. ■

6 Proof of the main result

Let $\hat{\beta}\in(\beta_0,\beta)$ and $\hat{\beta}_0\in(\beta_0,\hat{\beta})$ be given. Let $\hat{\eta}_0>0$ and $\hat{\eta}_1>0$ be the positive constants such that Lemma 5.5 and Proposition 5.7 hold true. In what follows we always assume that the positive constant $\eta>0$ in Assumption 3.6 satisfies

$$0<\eta<\min(\hat{\eta}_0,\hat{\eta}_1), \tag{6.1}$$

where $\hat{\eta}_0$ (respectively $\hat{\eta}_1$) is defined in Lemma 5.6 (respectively in Proposition 5.7).

6.1 Estimates of the u-equation

Let $x \in X$, $t_0 \in \mathbb{R}$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ be given where $\zeta > 0$ is the positive constant in Assumption 3.6. Consider

$$\dot{u}(t) = F(u(t)) + K(u(t), v(t)), \quad \forall t > t_0, \quad u(t_0) = x \in X. \tag{6.2}$$

By Assumption 3.1 and Assumption 3.6-(i) for each $x \in X$, $t_0 \in \mathbb{R}$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ there exists a unique solution $u \in C^1(\mathbb{R}, X)$ of (6.2) such that $u(t_0) = x$. Therefore for any $x \in X$, $t_0 \in \mathbb{R}$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ we define

$$u(t) := \Lambda_0(x, v)(t, t_0), \ \forall t \in \mathbb{R} \Longleftrightarrow u \in C^1(\mathbb{R}, X) \text{ satisfies (6.2) with } u(t_0) = x. \tag{6.3}$$

Hence for any $x \in X$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ the family of maps $\{\Lambda_0(x, v)(t, l)\}_{(t,l)\in\mathbb{R}^2}$ satisfies

$$\Lambda_0(\cdot, v)(t, l) \circ \Lambda_0(\cdot, v)(l, t_0) = \Lambda_0(\cdot, v)(t, t_0), \ \forall (t, l), (l, t_0) \in \mathbb{R}^2$$

and

$$\Lambda_0(x, v)(t, t) = x, \forall t \in \mathbb{R} \text{ and } x \in X.$$

Furthermore for each $(t, t_0) \in \mathbb{R}^2$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ the map

$$x \to \Lambda_0(x, v)(t, t_0)$$

is a diffeomorphism in X with inverse function

$$x \to \Lambda_0(x, v)(t_0, t)$$

and $\{\partial_x \Lambda_0(x, v)(t, t_0)\}_{(t,t_0)\in\Delta}$ is the linear evolution family generated by

$$\dot{z}(t) = [DF(\Lambda_0(x, v)(t, t_0)) + \partial_x K(\Lambda_0(x, v)(t, t_0), v(t))]z(t), \ \forall t > t_0 \text{ and } z(t_0) = z_0 \in X.$$

The next lemma is a direct consequence of Lemma 5.6 applied with $\hat{F}(u(t), t) := K(u(t), v(t))$.

Lemma 6.1 *Let Assumptions 3.1, 3.2 and 3.6-(i) be satisfied and $\eta > 0$ satisfying (6.1). Let $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ be given. Then for each $x_1, x_2 \in X$ and $(t, t_0) \in \mathbb{R}^2$ we have*

$$\|\Lambda_0(x_1, v)(t, t_0) - \Lambda_0(x_2, v)(t, t_0)\| \leq \kappa_0 e^{\hat{\beta}_0 |t - t_0|} \|x_1 - x_2\|.$$

The following lemma is obtained by using the same arguments as in the proof of Lemma 5.2.

Lemma 6.2 *Let Assumptions 3.1, 3.2 and 3.6-(i) be satisfied and $\eta > 0$ satisfying (6.1). Then for each $x \in X$, $(t, t_0) \in \mathbb{R}^2$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ we have*

$$\|\partial_x \Lambda_0(x, v)(t, t_0)\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\hat{\beta}_0 |t - t_0|} \quad \text{and} \quad \|\partial_x \Lambda_0(x, v)(t, t_0)^{-1}\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\hat{\beta}_0 |t - t_0|}.$$

The proof of the next proposition is taken from [21]. This is a key ingredient for our fixed point arguments.

Proposition 6.3 *Let Assumptions 3.1, 3.2 and 3.6-(i) be satisfied and $\eta > 0$ satisfying (6.1). Then for each $x, \bar{x} \in X$, $(t, t_0) \in \Delta$ and $v, \bar{v} \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ and $\|\bar{v}\|_\infty \leq \zeta$ we have*

$$\begin{cases} \|\Lambda_0(x, v)(t, t_0) - \Lambda_0(\bar{x}, \bar{v})(t, t_0)\| \leq \kappa_0 e^{\hat{\beta}_0 (t-t_0)} \|x - \bar{x}\| + \eta \int_{t_0}^{t} \kappa_0 e^{\hat{\beta}_0 (t-r)} \|v(r) - \bar{v}(r)\| dr \\ \|\Lambda_0(x, v)(t_0, t) - \Lambda_0(\bar{x}, \bar{v})(t_0, t)\| \leq \kappa_0 e^{\hat{\beta}_0 (t-t_0)} \|x - \bar{x}\| + \eta \int_{t_0}^{t} \kappa_0 e^{\hat{\beta}_0 (r-t_0)} \|v(r) - \bar{v}(r)\| dr. \end{cases} \tag{6.4}$$

Proof. Define

$$v_s := sv + (1-s)\bar{v}, \ \forall s \in [0,1]$$

Let $x_0, x_1 \in X$ be given. Define

$$x_s := s x_1 + (1-s) x_0 \in X, \ \forall s \in [0,1].$$

Since $\|v_s\| \leq \zeta$ and $x_s \in X$ for all $s \in [0,1]$ we can define for each $(t, t_0) \in \mathbb{R}$ the map

$$s \to \Lambda_0(x_s, v_s)(t, t_0),$$

so that by Lemma 6.2 we have for each $s \in [0,1]$ and $(t, t_0) \in \mathbb{R}^2$

$$\|\partial_x \Lambda(x_s, v_s)(t, t_0)\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\hat{\beta}_0 |t-t_0|} \text{ and } \|\partial_x \Lambda(x_s, v_s)(t, t_0)^{-1}\|_{\mathcal{L}(X)} \leq \kappa_0 e^{\hat{\beta}_0 |t-t_0|}. \tag{6.5}$$

Note that condition

$$\|K(x,y) - K(\bar{x}, \bar{y})\| \leq \eta(\|x - \bar{x}\| + \|y - \bar{y}\|), \ \forall (x,y), (\bar{x}, \bar{y}) \in X \times B_{Y_0}(0, \zeta)$$

with

$$B_{Y_0}(0, \zeta) = \{y \in Y_0 : \|y\| \leq \zeta\}$$

implies that

$$\sup_{(x,y) \in X \times B_{Y_0}(0,\zeta)} \|\partial_x K(x,y)\|_{\mathcal{L}(X)} \leq \eta \quad \text{and} \quad \sup_{(x,y) \in X \times B_{Y_0}(0,\zeta)} \|\partial_y K(x,y)\|_{\mathcal{L}(Y_0, X)} \leq \eta. \tag{6.6}$$

Recalling that the map $F : X \to X$ and $K : X \times Y_0 \to X$ are respectively differentiable on X and $X \times B_{Y_0}(0, \zeta)$ combined with the fact that $s \to v_s$ is differentiable with respect to s we infer from Gronwall [18] that

$$s \to \Lambda_0(x_s, v_s)(t, t_0)$$

is differentiable with respect to s. Moreover (see [18]) if we set

$$\begin{cases} z_s(t) := \partial_s \Lambda_0(x_s, v_s)(t, t_0), \ \forall t \geq t_0 \\ u_s(t) := \Lambda_0(x_s, v_s)(t, t_0), \quad \forall t \geq t_0 \end{cases}$$

then for any $s \in (0,1)$, the map $t \in [t_0, +\infty) \to z_s(t)$ is the solution of

$$\begin{cases} \dfrac{dz_s(t)}{dt} = [DF(u_s(t)) + \partial_x K(u_s(t), v_s(t))] z_s(t) + \partial_y K(u_s(t), v_s(t)) \partial_s v_s(t), \ t > t_0 \\ z_s(t_0) = \partial_s u_s(t_0) \end{cases}$$

that is

$$\begin{cases} \dfrac{dz_s(t)}{dt} = [DF(u_s(t)) + \partial_x K(u_s(t), v_s(t))]z_s(t) + \partial_y K(u_s(t), v_s(t))(v(t) - \bar{v}(t)), \ t > t_0 \\ z_s(t_0) = x_1 - x_0. \end{cases}$$

Since $\{\partial_x \Lambda_0(x, v_s)(t, t_0)\}_{(t,t_0)\in\Delta}$ is the evolution family generated by

$$\frac{dz(t)}{dt} = [DF(u_s(t)) + \partial_x K(u_s(t), v_s(t))]z(t), \ t > t_0 \text{ and } z(t_0) = z_0 \in X,$$

is an invertible family of linear operators we obtain by the variation of constants formula

$$z_s(t) = \partial_x \Lambda_0(x_s, v_s)(t, t_0)(x_1 - x_0) + \int_{t_0}^{t} \partial_x \Lambda_0(x_s, v_s)(t, r)\partial_y K(u_s(r), v_s(r))(v(r) - \bar{v}(r))dr, \ \forall t \in \mathbb{R}. \tag{6.7}$$

Integrating equation (6.7) with respect to s between 0 and 1 gives

$$\begin{aligned} \Lambda_0(x_1, v)(t, t_0) - \Lambda_0(x_0, \bar{v})(t, t_0) &= \int_0^1 \partial_x \Lambda_0(x_s, v_s)(t, t_0)(x_1 - x_0)ds \\ &+ \int_0^1 \left[\int_{t_0}^{t} \partial_x \Lambda_0(x_s, v_s)(t, r)\partial_y K(u_s(r), v_s(r))(v(r) - \bar{v}(r))dr \right] ds, \ \forall t \in \mathbb{R} \end{aligned}$$

so that by using (6.5) and (6.6) we obtain for $x_1 = x$ and $x_0 = \bar{x}$

$$\|\Lambda_0(x, v)(t, t_0) - \Lambda_0(\bar{x}, \bar{v})(t, t_0)\| \le \kappa_0 e^{\hat{\beta}_0|t-t_0|}\|x - \bar{x}\| + \eta \left| \int_{t_0}^{t} \kappa_0 e^{\hat{\beta}_0|t-t_0|}\|v(r) - \bar{v}(r)\|dr \right|, \ \forall t \in \mathbb{R}.$$

The result follows. ∎

An immediate consequence of Proposition 6.3 is the following.

Corollary 6.4 *Let Assumptions 3.1, 3.2 and 3.6-(i) be satisfied and $\eta > 0$ satisfying (6.1). Then the following properties hold*

(i) *For each $x, \bar{x} \in X$ and $v, \bar{v} \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \le \zeta$ and $\|\bar{v}\|_\infty \le \zeta$ we have*

$$\sup_{(t,t_0)\in\mathbb{R}^2} \left[e^{-\hat{\beta}_0|t-t_0|}\|\Lambda_0(x, v)(t, t_0) - \Lambda_0(\bar{x}, \bar{v})(t, t_0)\| \right] \le \kappa_0\|x - \bar{x}\| + \frac{\eta\kappa_0}{\hat{\beta}_0}\|v - \bar{v}\|_\infty.$$

(ii) *If $\hat{\beta}_0 < \gamma_0$ then for each $x, \bar{x} \in X$ and $v, \bar{v} \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \le \zeta$ and $\|\bar{v}\|_\infty \le \zeta$ we have*

$$\sup_{(t,t_0)\in\mathbb{R}^2} e^{-\gamma_0|t-t_0|}\|v(t) - \bar{v}(t)\| \le \|v - \bar{v}\|_\infty < +\infty$$

and

$$\sup_{(t,t_0)\in\mathbb{R}^2} e^{-\gamma_0|t-t_0|}\|\Lambda_0(x, v)(t, t_0) - \Lambda_0(\bar{x}, \bar{v})(t, t_0)\| \le \kappa_0\|x - \bar{x}\| + \frac{\eta\kappa_0}{\gamma_0 - \hat{\beta}_0} \sup_{(t,t_0)\in\mathbb{R}^2} e^{-\gamma_0|t-t_0|}\|v(t) - \bar{v}(t)\|.$$

The following lemma is a direct an application of Proposition 5.7 with $\hat{F}(u(t), t) := K(u(t), v(t))$.

Lemma 6.5 *Let Assumptions 2.1, 2.3, 3.1, 3.2, 3.4, 3.6-(i) and 3.7 be satisfied and and* $\eta > 0$ *satisfying* (6.1)*. Let* $x \in X$*,* $t_0 \in \mathbb{R}$ *and* $v \in BC(\mathbb{R}, Y_0)$ *with* $\|v\|_\infty \leq \zeta$ *be given. If*

$$u(t) := \Lambda_0(x, v)(t, t_0), \ \forall t \in \mathbb{R},$$

then the evolution family generated by

$$\dot{w}(t) = [A + B(u(t))]w(t), \ t > t_0 \ \text{ and } \ w(t_0) = w_0 \in Y_0$$

has an exponential dichotomy with constant $\hat{\kappa} \geq \kappa$ *and exponent* $\hat{\beta} \in (\beta_0, \beta)$*.*

6.2 Fixed point formulation

Let $x \in X$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ be given. Define

$$\mathcal{K}_c(x, v)(t) := \Lambda_0(x, v)(t, 0), \ \forall t \in \mathbb{R}. \tag{6.8}$$

We introduce the following intermediate subset of $C(\mathbb{R}, X)$ to simplify the notations.

Notation 6.6 *We write*

$$u \in ED(\hat{\kappa}, \hat{\beta})$$

if and only if the evolution family $\{U_u(t, t_0)\}_{(t,t_0)\in\Delta}$ *generated by*

$$\dot{w}(t) = [A + B(u(t))]w(t), \ t > t_0 \ \text{ and } \ w(t_0) = w_0 \in Y_0$$

has an exponential dichotomy with constant $\hat{\kappa} \geq \kappa$ *and exponent* $\hat{\beta} \in (\beta_0, \beta)$*.*

Note that $ED(\hat{\kappa}, \hat{\beta})$ is not empty as long as the conditions of Lemma 6.5 are fulfilled. In particular we have the following lemma.

Lemma 6.7 *Let Assumptions 2.1, 2.3, 3.1, 3.2, 3.4, 3.6-(i) and 3.7 be satisfied and* $\eta > 0$ *satisfying* (6.1)*. Then*

$$\mathcal{K}_c(x, v) \in ED(\hat{\kappa}, \hat{\beta})$$

for all $x \in X$ *and* $v \in BC(\mathbb{R}, Y_0)$ *with* $\|v\|_\infty \leq \zeta$*.*

Remember that the space $BC^\gamma(\mathbb{R}, Z)$ is defined for $Z = Y_0, Y, X$ by

$$BC^\gamma(\mathbb{R}, Z) := \left\{ f \in C(\mathbb{R}, Z) : \|f\|_\gamma := \sup_{t\in\mathbb{R}} e^{-\gamma|t|}\|f(t)\| < +\infty \right\}, \ \forall \gamma \geq 0.$$

Therefore

$$BC^0(\mathbb{R}, Z) = BC(\mathbb{R}, Z) \text{ and } BC(\mathbb{R}, Z) \subset BC^\gamma(\mathbb{R}, Z), \ \forall \gamma \geq 0.$$

In particular we have

$$\|f\|_\gamma \leq \|f\|_\infty, \ \forall \gamma \geq 0 \text{ and } f \in BC(\mathbb{R}, Z).$$

The following result is obtained from Theorem 2.13 and Theorem 2.14.

Lemma 6.8 *Let Assumptions 2.1, 2.3, 3.1, 3.2, 3.4, 3.6-(i) and 3.7 be satisfied and $\eta > 0$ satisfying (6.1). Let $u \in ED(\hat{\kappa}, \hat{\beta})$ (see Notation 6.6) be given. Then for any $f \in BC(\mathbb{R}, Y)$ there exists a unique mild solution $w \in BC(\mathbb{R}, Y_0)$ of*

$$\dot{w}(t) = [A + B(u(t))]w(t) + f(t), \; t \in \mathbb{R}$$

given by

$$w(t) = \lim_{\lambda \to +\infty} \left[\int_{-\infty}^{t} U_u^{+}(t,s)\lambda R_\lambda(A) f(s) ds - \int_{t}^{+\infty} U_u^{-}(t,s)\lambda R_\lambda(A) f(s) ds \right], \; \forall t \in \mathbb{R}.$$

Moreover if $\gamma_0 > 0$ with

$$-\gamma_0 \in (-\hat{\beta}, -\hat{\beta}_0]$$

then there exists $C(\gamma_0, \hat{\kappa}, \hat{\beta}) > 0$ (independent of u) such that

$$\|w\|_\gamma \leq C(\gamma_0, \hat{\kappa}, \hat{\beta}) \|f\|_\gamma, \; \forall \gamma \in [0, \hat{\beta}_0].$$

The foregoing lemma will allows us to define our second operator for our fixed point problem. Before proceeding let us note that for any $u \in C(\mathbb{R}, X)$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ the map

$$t \to G(u(t), v(t))$$

belongs to $BC(\mathbb{R}, Y)$ and by Assumption 3.6 it satisfies

$$\|G(u(\cdot), v(\cdot))\|_\infty \leq \hat{\sigma}. \tag{6.9}$$

Hence one can define for each $u \in ED(\hat{\kappa}, \hat{\beta})$ and $v \in BC(\mathbb{R}, Y_0)$ with $\|v\|_\infty \leq \zeta$ the map

$$\mathcal{K}_h(u,v)(t) := \lim_{\lambda \to +\infty} \left[\int_{-\infty}^{t} U_u^{+}(t,s)\lambda R_\lambda(A) G(u(s), v(s)) ds - \int_{t}^{+\infty} U_u^{-}(t,s)\lambda R_\lambda(A) G(u(s), v(s)) ds \right], \tag{6.10}$$

for all $t \in \mathbb{R}$.
Thus due to Lemma 6.8 and (6.9) the map $\mathcal{K}_h$ is well defined and we have the following properties.

Lemma 6.9 *Let Assumptions 2.1, 2.3, 3.1, 3.2, 3.4, 3.6-(i) and 3.7 be satisfied and $\eta > 0$ satisfying (6.1). Let $\gamma_0 > 0$ with*

$$-\gamma_0 \in (-\hat{\beta}, -\hat{\beta}_0]$$

be given and fixed. Let $u_1, u_2 \in ED(\hat{\kappa}, \hat{\beta})$ and $v_1, v_2 \in BC(\mathbb{R}, Y_0)$ with $\|v_1\|_\infty \leq \zeta$ and $\|v_2\|_\infty \leq \zeta$. Then the following properties hold

(i) *For each $\gamma \in [0, \gamma_0]$ and $i, j = 1, 2$ we have $\mathcal{K}_h(u_i, v_j) \in BC^\gamma(\mathbb{R}, Y_0)$ and*

$$\|\mathcal{K}_h(u_i, v_j)\|_\gamma \leq C(\gamma_0, \hat{\kappa}, \hat{\beta})\hat{\sigma}.$$

(ii) *For each $\gamma \in [0, \gamma_0]$*

$$\|\mathcal{K}_h(u_1, v_1) - \mathcal{K}_h(u_1, v_2)\|_\gamma \leq C(\gamma_0, \hat{\kappa}, \hat{\beta})\sigma \|v_1 - v_2\|_\gamma.$$

(iii) *For each* $\gamma \in [0, \gamma_0]$

$$G(u_1(\cdot), v_2(\cdot)) - G(u_2(\cdot), v_2(\cdot)) \in BC^\gamma(\mathbb{R}, Y) \quad \textit{and} \quad [B(u_1(\cdot)) - B(u_2(\cdot))]\mathcal{K}_h(u_2, v_2) \in BC^\gamma(\mathbb{R}, Y)$$

and

$$\begin{aligned} \|\mathcal{K}_h(u_1, v_2) - \mathcal{K}_h(u_2, v_2)\|_\gamma \quad &\le \quad C(\gamma_0, \hat{\kappa}, \hat{\beta})\|G(u_1(\cdot), v_2(\cdot)) - G(u_2(\cdot), v_2(\cdot))\|_\gamma \\ &\quad + C(\gamma_0, \hat{\kappa}, \hat{\beta})\|[B(u_1(\cdot)) - B(u_2(\cdot))]\mathcal{K}_h(u_2, v_2)\|_\gamma. \end{aligned} \tag{6.11}$$

If in addition $u_1, u_2 \in BC^\gamma(\mathbb{R}, X)$ *then*

$$\|\mathcal{K}_h(u_1, v_2) - \mathcal{K}_h(u_2, v_2)\|_\gamma \le C(\gamma_0, \hat{\kappa}, \hat{\beta}) \left[L_G + C(\gamma_0, \hat{\kappa}, \hat{\beta}) L_B \hat{\sigma}\right] \|u_1 - u_2\|_\gamma. \tag{6.12}$$

Proof. Let us first note that Assumption 3.6 implies that

$$G(u_i(\cdot), v_j(\cdot)) \in BC(\mathbb{R}, Y), \quad \text{for } i, j = 1, 2$$

so that

$$G(u_i(\cdot), v_j(\cdot)) \in BC^\gamma(\mathbb{R}, Y), \ \forall \gamma \in [0, \gamma_0] \text{ and } i, j = 1, 2.$$

Therefore

$$\|G(u_i(\cdot), v_j(\cdot))\|_\gamma \le \|G(u_i(\cdot), v_j(\cdot))\|_\infty \le \hat{\sigma}, \ \forall \gamma \in [0, \gamma_0] \text{ and } i, j = 1, 2 \tag{6.13}$$

Next we infer from Lemma 6.8 that for $i, j = 1, 2$

$$\|\mathcal{K}_h(u_i, v_j)\|_\gamma \le C(\gamma_0, \hat{\kappa}, \hat{\beta})\|G(u_i(\cdot), v_j(\cdot))\|_\gamma, \ \forall \gamma \in [0, \gamma_0]$$

and (i) follows from (6.13). To prove (ii) we observe that by Lemma 6.8 and the definition of $\mathcal{K}_h$ we have

$$\|\mathcal{K}_h(u_1, v_1) - \mathcal{K}_h(u_1, v_2)\|_\gamma \le C(\gamma_0, \hat{\kappa}, \hat{\beta})\|G(u_1(\cdot), v_1(\cdot)) - G(u_1(\cdot), v_2(\cdot))\|_\gamma, \ \forall \gamma \in [0, \gamma_0].$$

Since G is Lipschitz continuous with respect to its second variable with Lispchitz constant σ it is easy to see that

$$\|G(u_1(\cdot), v_1(\cdot)) - G(u_1(\cdot), v_2(\cdot))\|_\gamma \le \sigma \|v_1 - v_2\|_\gamma, \ \forall \gamma \in [0, \gamma_0],$$

and (ii) follows.
It remains to prove (iii). To do so note that by (i) and Assumption 3.7 we have

$$[B(u_1(\cdot)) - B(u_2(\cdot))]\mathcal{K}_h(u_2, v_2) \in BC(\mathbb{R}, Y)$$

and it is also clear that

$$G(u_1(\cdot), v_1(\cdot)) - G(u_2(\cdot), v_2(\cdot)) \in BC(\mathbb{R}, Y).$$

Next note that by Lemma 6.8, $\mathcal{K}_h(u_i, v_2) \in BC(\mathbb{R}, Y)$, $i = 1, 2$ is the unique mild solution of

$$\dot{w}(t) = [A + B(u_i(t)]w(t) + f_i(t), \ t \in \mathbb{R}$$

with

$$f_i(t) = G(u_i(t), v_2(t)),\ \forall t \in \mathbb{R}.$$

This is equivalent to say that

$$\mathcal{K}_h(u_i, v_2)(t) = \mathcal{K}_h(u_i, v_2)(t_0) + A\int_{t_0}^{t} \mathcal{K}_h(u_i, v_2)(r)dr + \int_{t_0}^{t} B(u_i(r))\mathcal{K}_h(u_i, v_2)(r)dr + \int_{t_0}^{t} f_i(r)dr$$

for all $(t, t_0) \in \Delta$ and $i = 1, 2$. Thus by setting

$$w(t) := \mathcal{K}_h(u_1, v_2)(t) - \mathcal{K}_h(u_2, v_2)(t),\ \forall t \in \mathbb{R}$$

we obtain for each $(t, t_0) \in \Delta$

$$w(t) = w(t_0) + A\int_{t_0}^{t} w(r)dr + \int_{t_0}^{t} B(u_1(r))w(r)dr + \int_{t_0}^{t} f(r)dr$$

with

$$f(\cdot) := G(u_1(t), v_1(t)) - G(u_2(t), v_1(t)) + [B(u_1(\cdot)) - B(u_2(\cdot))]\mathcal{K}_h(u_2, v_2) \in BC(\mathbb{R}, Y)$$

that is to say that w is the unique globally defined mild solution of

$$\dot{w}(t) = [A + B(u_1(t))]w(t) + f(t),\ t \in \mathbb{R}.$$

Hence by Lemma 6.8 we obtain

$$\|w\|_\gamma \leq C(\gamma_0, \hat{\kappa}, \hat{\beta})\|f\|_\gamma,\ \forall \gamma \in [0, \gamma_0],$$

and (6.11) follows. The remaining estimate (6.12) is easily obtain from Assumption 3.6 and Assumption 3.7. ■

6.3 Proof of Theorem 3.12

All the materials are now completed in order to give the proof of Theorem 3.12. Let $\gamma_0 > 0$ with

$$-\gamma_0 \in (-\hat{\beta}, -\hat{\beta}_0)$$

be given and fixed. Define

$$\Omega(\zeta, \gamma_0) := \{v \in BC(\mathbb{R}, Y_0) : \|v\|_{\gamma_0} \leq \zeta \text{ and } \|v\|_\infty \leq \zeta\}.$$

It is easy to prove that $\Omega(\zeta, \gamma_0)$ is a closed subset of $BC^{\gamma_0}(\mathbb{R}, Y_0)$.
From now on assume that

$$0 < \eta < \min(\hat{\eta}_0, \hat{\eta}_1),$$

where $\hat{\eta}_0$ (respectively $\hat{\eta}_1$) is defined in Lemma 5.6 (respectively in Proposition 5.7).
Then Lemma 6.5 and Lemma 6.8 ensure that for each $(x, v) \in X \times \Omega(\zeta, \gamma_0)$ we have

$$\mathcal{K}_c(x, v) \in ED(\hat{\kappa}, \hat{\beta})$$

so the map

$$\mathcal{K}(x,v) := \mathcal{K}_h(\cdot,v)\circ\mathcal{K}_c(x,v)$$

is well defined.
In order to prove that the map $\mathcal{K}$ maps $\Omega(\zeta,\gamma_0)$ into itself is a contraction with respect to its second variable we now use the properties of the maps $\mathcal{K}_c$ and $\mathcal{K}_h$. Recall that $\mathcal{K}_c(x,v)(t) = \Lambda_0(x,v)(t,0)$ we have from Corollary 6.4 and Lemma 6.7

$$\mathcal{K}_c(x,v) \in ED(\hat{\kappa},\hat{\beta}) \cap BC^{\gamma_0}(\mathbb{R},X),\ \forall (x,v) \in X\times\Omega(\zeta,\gamma_0)$$

and

$$\|\mathcal{K}_c(x_1,v_1)-\mathcal{K}_c(x_2,v_2)\|_{\gamma_0} \le \kappa_0\|x_1-x_2\| + \frac{\eta\kappa_0}{\gamma_0-\hat{\beta}_0}\|v_1-v_2\|_{\gamma_0},\ \forall (x_1,v_1),(x_2,v_2)\in X\times\Omega(\zeta,\gamma_0). \tag{6.14}$$

Next by using condition (i) in Lemma 6.9 it is clear that if

$$C(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma} \le \zeta$$

then

$$\|\mathcal{K}(x,v)\|_{\gamma} \le C(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma} \le \zeta,\ \forall (x,v)\in X\times\Omega(\zeta,\gamma_0),\ \forall \gamma\in[0,\gamma_0]$$

that is to say that $\mathcal{K}(\cdot,\cdot)$ maps $X\times\Omega(\zeta,\gamma_0)$ into $\Omega(\zeta,\gamma_0)$ as long as $C(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma}\le\zeta$. We now prove that $\mathcal{K}(\cdot,\cdot)$ is Lipschitz continuous in $X\times\Omega(\zeta,\gamma_0)$. To do this let $(x_1,v_1),(x_2,v_2)\in X\times\Omega(\zeta,\gamma_0)$ be given. Set

$$u_1 := \mathcal{K}_c(x_1,v_1)\in ED(\hat{\kappa},\hat{\beta})\cap BC^{\gamma_0}(\mathbb{R},X)\ \text{ and }\ u_2 := \mathcal{K}_c(x_2,v_2)\in ED(\hat{\kappa},\hat{\beta})\cap BC^{\gamma_0}(\mathbb{R},X).$$

Then

$$\begin{aligned}\mathcal{K}(x_1,v_1)-\mathcal{K}(x_2,v_2) &= \mathcal{K}_h(u_1,v_1)-\mathcal{K}_h(u_2,v_2)\\ &= \mathcal{K}_h(u_1,v_1)-\mathcal{K}_h(u_1,v_2)+\mathcal{K}_h(u_1,v_2)-\mathcal{K}_h(u_2,v_2)\end{aligned}$$

so that conditions (ii) and (iii) of Lemma 6.9

$$\|\mathcal{K}(x_1,v_1)-\mathcal{K}(x_2,v_2)\|_{\gamma_0} \le C(\gamma_0,\hat{\kappa},\hat{\beta})\sigma\|v_1-v_2\|_{\gamma_0} + C(\gamma_0,\hat{\kappa},\hat{\beta})[L_G+L_BC(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma}]\|u_1-u_2\|_{\gamma_0}.$$

Hence we infer from (6.14) that

$$\|u_1-u_2\|_{\gamma_0} = \|\mathcal{K}_c(x_1,v_1)-\mathcal{K}_c(x_2,v_2)\|_{\gamma_0} \le \kappa_0\|x_1-x_2\| + \frac{\eta\kappa_0}{\gamma_0-\hat{\beta}_0}\|v_1-v_2\|_{\gamma_0}$$

providing that

$$\begin{aligned}\|\mathcal{K}(x_1,v_1)-\mathcal{K}(x_2,v_2)\|_{\gamma_0} \le\ & C(\gamma_0,\hat{\kappa},\hat{\beta})[L_G+L_BC(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma}]\kappa_0\|x_1-x_2\|\\ &+C(\gamma_0,\hat{\kappa},\hat{\beta})\left[\sigma+\frac{\eta\kappa_0}{\gamma_0-\hat{\beta}_0}[L_G+L_BC(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma}]\right]\|v_1-v_2\|_{\gamma_0}.\end{aligned}$$

Therefore we let $\sigma>0$ and $\eta>0$ small enough such that

$$0 < C(\gamma_0,\hat{\kappa},\hat{\beta})\left[\sigma+\frac{\eta\kappa_0}{\gamma_0-\hat{\beta}_0}[L_G+L_BC(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma}]\right] < \frac{1}{2}.$$

Then for $\sigma > 0$, $\eta > 0$ and $\hat{\sigma} > 0$ small enough we have the following properties for $\mathcal{K}_h$

$$\begin{cases} \mathcal{K}_h(x,v) \in \Omega(\zeta,\gamma_0), \ \forall (x,v) \in X \times \Omega(\zeta,\gamma_0) \\ \|\mathcal{K}(x,v_1) - \mathcal{K}(x,v_2)\|_{\gamma_0} \leq \dfrac{1}{2}\|v_1 - v_2\|_{\gamma_0}, \ \forall x \in X \text{ and } v_1, v_2 \in \Omega(\zeta,\gamma_0) \end{cases} \tag{6.15}$$

and for each $(x_1,v_1),(x_2,v_2) \in X \times \Omega(\zeta,\gamma_0)$

$$\|\mathcal{K}(x_1,v_1) - \mathcal{K}(x_2,v_2)\|_{\gamma_0} \leq C(\gamma_0,\hat{\kappa},\hat{\beta})[L_G + L_B C(\gamma_0,\hat{\kappa},\hat{\beta})\hat{\sigma}]\kappa_0\|x_1 - x_2\| + \frac{1}{2}\|v_1 - v_2\|_{\gamma_0}. \tag{6.16}$$

We divide the remaining part into four steps.
Step 1 (Existence) : Observe that on one hand $(u,v) \in C^1(\mathbb{R},X) \times \Omega(\zeta,\gamma_0)$ is a mild solution of (1.1) with $u(0) = x$ if and only if

$$\begin{cases} u = \mathcal{K}_c(x,v) \\ v = \mathcal{K}(x,v) = \mathcal{K}_h(\cdot,v) \circ \mathcal{K}_c(x,v) \end{cases}$$

On the other hand by using (6.15) we infer from the contraction mapping fixed point theorem that for each $(x,v) \in X \times \Omega(\zeta,\gamma_0)$ there exists a unique $v \in \Omega(\zeta,\gamma_0)$ such that

$$v = \mathcal{K}(x,v).$$

Hence we define for each $x \in X$ the map

$$\hat{\psi}(x) := \mathcal{K}(x,v)(0) \tag{6.17}$$

with $v \in \Omega(\zeta,\gamma_0)$ the unique function satisfying

$$v = \mathcal{K}(x,v)$$

and we set

$$\hat{\mathcal{M}} := \left\{(x,\hat{\psi}(x)) : x \in X\right\}. \tag{6.18}$$

Step 2 (Invariance) : It is clear from the definition of $\hat{\mathcal{M}}$ that if $(x,\hat{\psi}(x)) \in \hat{\mathcal{M}}$ then there exists a unique mild solution $(u,v) \in C^1(\mathbb{R},X) \times \Omega(\zeta,\gamma_0)$ of (1.1). We now prove that any given solution $(u,v) \in C^1(\mathbb{R},X) \times \Omega(\zeta,\gamma_0)$ of (1.1) satisfies

$$v(t) = \hat{\psi}(u(t)), \ \forall t \in \mathbb{R}.$$

Let $t_0 \in \mathbb{R}$ be given. Define

$$\hat{u} := u(\cdot + t_0) \ \text{ and } \hat{v} := v(\cdot + t_0).$$

Then $(\hat{u},\hat{v}) \in C^1(\mathbb{R},X) \times \Omega(\zeta,\gamma_0)$ is a mild solution of (1.1) with $\hat{u}(0) = u(t_0)$ which is equivalent to say that

$$\begin{cases} \hat{u} = \mathcal{K}_c(u(t_0),\hat{v}) \\ \hat{v} = \mathcal{K}(u(t_0),\hat{v}). \end{cases}$$

Therefore it follows that

$$\hat{v}(0) = v(t_0) = \mathcal{K}(u(t_0),\hat{v})(0) = \hat{\psi}(u(t_0)).$$

Step 3 (Normal Hyperbolicity) : By using Lemma 6.1 and Lemma 6.7 it is straightforward that $\hat{\mathcal{M}}$ is normally hyperbolic with constants $\hat{\kappa} \geq 1$, $\kappa_0 \geq 1$, exponents $\hat{\beta} \in (\beta_0, \beta)$ and $\hat{\beta}_0 \in (\beta_0, \hat{\beta})$.
Step 4 (Lipschitz continuity) : Let $x_1, x_2 \in X$ be given. Denote by $v_1 \in \Omega(\gamma_0, \zeta)$ and $v_2 \in \Omega(\gamma_0, \zeta)$ the unique maps such that

$$v_1 = \mathcal{K}(x_1, v_1) \text{ and } v_2 = \mathcal{K}(x_2, v_2).$$

Then (6.16) gives

$$\|v_1 - v_2\|_{\gamma_0} \leq C(\gamma_0, \hat{\kappa}, \hat{\beta})[L_G + L_B C(\gamma_0, \hat{\kappa}, \hat{\beta})\hat{\sigma}]\kappa_0 \|x_1 - x_2\| + \frac{1}{2}\|v_1 - v_2\|_{\gamma_0}$$

and we obtain

$$\|v_1 - v_2\|_{\gamma_0} \leq 2C(\gamma_0, \hat{\kappa}, \hat{\beta})[L_G + L_B C(\gamma_0, \hat{\kappa}, \hat{\beta})\hat{\sigma}]\kappa_0 \|x_1 - x_2\|$$

and since we have

$$\|\psi(x_1) - \psi(x_2)\| = \|v_1(0) - v_2(0)\| \leq \|v_1 - v_2\|_{\gamma_0} \leq 2C(\gamma_0, \hat{\kappa}, \hat{\beta})[L_G + L_B C(\gamma_0, \hat{\kappa}, \hat{\beta})\hat{\sigma}]\kappa_0 \|x_1 - x_2\|$$

it follows that $\hat{\psi}$ is Lipschitz continuous on X. Finally we prove that $\hat{\psi}$ is uniformly bounded on X. This is achieved by observing that for each $x \in X$ if we denote by $v \in \Omega(\gamma_0, \zeta)$ the unique solution of the fixed point problem

$$v = \mathcal{K}_h(x, v)$$

in $\Omega(\gamma_0, \zeta)$ then

$$\|\hat{\psi}(x)\| = \|\mathcal{K}_h(x, v)(0)\| \leq C(\gamma_0, \hat{\kappa}, \hat{\beta})\hat{\sigma}.$$

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