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On a vector-host epidemic model with spatial structure

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Abstract

In this paper, we study a reaction–diffusion vector-host epidemic model. We define the basic reproduction number R_0 and show that R_0 is a threshold parameter: if $R_0 \leqslant 1$ the disease free equilibrium is globally stable; if $R_0 > 1$ the model has a unique globally stable positive equilibrium. Our proof combines arguments from monotone dynamical system theory, persistence theory, and the theory of asymptotically autonomous semiflows.

Keywords: reaction–diffusion, epidemic models, global stability, basic reproduction number
Mathematics Subject Classification numbers: 35B40, 35P05, 35Q92

1. Introduction

In recent years, many authors (e.g. [1, 5–8, 9–14, 16, 18, 20, 22, 23, 29, 34, 35, 37, 38, 41]) have proposed reaction–diffusion models to study the transmission of diseases in spatial settings. Among them, Fitzgibbon *et al* [11, 12] applied a reaction–diffusion system on non-coincident domains to describe the circulation of diseases between two locations; Lou and Zhao [22] proposed a reaction–diffusion model with delay and nonlocal terms to study the spatial spread of malaria; and Vaidya, Wang and Zou [34] studied the transmission of avian influenza in wild birds with a reaction–diffusion model with spatial heterogeneous coefficients.

New formulations of diffusive epidemic models have been used recently to study epidemics in spatial contexts. In [23] the spatial spread of influenza in Puerto Rico was analyzed using a diffusive SIR model based on geographical population data. In [14] the effectiveness of a diffusive vector-host epidemic model was demonstrated in understanding the recent Zika outbreak in Rio De Janeiro. In these works it was shown that the beginning location and magnitude of an epidemic can have significant impact on the spatial development and final size of the epidemic. The simulations in these works highlighted the limitations of incomplete spatial epidemic data in the applications of diffusive models to real world situations. Despite these limitations, spatial epidemic models offer the possibility of better understanding of the evolution of epidemic outbreaks in regions, and the possibility of mitigating their greater regional impact with intervention measures. Our objective in this manuscript is to provide an extended analysis of the reaction–diffusion spatial epidemic model proposed in [14]. A more complete understanding of the model in [14] can help to predict the possibility that current Zika epidemics will become regionally endemic.

Suppose that individuals are living in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $H_i(x,t), V_u(x,t)$ and $V_i(x,t)$ be the densities of infected hosts, uninfected vectors, and infected vectors at position x and time t, respectively. Then the model proposed in [14] to study the outbreak of Zika in Rio De Janerio is the following reaction–diffusion system

$$\begin{cases} \frac{\partial}{\partial t}H_i - \nabla\cdot\delta_1(x)\nabla H_i = -\lambda(x)H_i + \sigma_1(x)H_u(x)V_i, & x\in\Omega, t>0,\\ \frac{\partial}{\partial t}V_u - \nabla\cdot\delta_2(x)\nabla V_u = -\sigma_2(x)V_uH_i + \beta(x)(V_u+V_i) - \mu(x)(V_u+V_i)V_u, & x\in\Omega, t>0,\\ \frac{\partial}{\partial t}V_i - \nabla\cdot\delta_2(x)\nabla V_i = \sigma_2(x)V_uH_i - \mu(x)(V_u+V_i)V_i, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

with homogeneous Neumann boundary condition

$$\frac{\partial}{\partial n}H_i = \frac{\partial}{\partial n}V_u = \frac{\partial}{\partial n}V_i = 0, \quad x\in\partial\Omega, t>0, \tag{1.2}$$

and initial condition

$$(H_i(\cdot,0), V_u(\cdot,0), V_i(\cdot,0)) = (H_{i0}, V_{u0}, V_{i0}) \in C(\bar\Omega;\mathbb{R}^3_+), \tag{1.3}$$

where $\delta_1,\delta_2\in C^{1+\alpha}(\bar\Omega;\mathbb{R})$ are strictly positive, the functions $\lambda,\beta,\sigma_1,\sigma_2$ and μ are strictly positive and belong to $C^\alpha(\bar\Omega;\mathbb{R})$, and the function $H_u\in C^\alpha(\bar\Omega;\mathbb{R})$ is nonnegative and non-trivial. The flux of new infected humans is given by $\sigma_1(x)H_u(x)V_i(t,x)$ in which $H_u(x)$ is the density of susceptible population depending on the spatial location x. The main idea of this model is to assume that the susceptible human population is (almost) not affected by the epidemic during a relatively short period of time and therefore the flux of new infected is (almost) constant. Such a functional response mainly permits to take care of realistic density of population distributed in space. For Zika in Rio De Janerio the number of infected is fairly small in comparison with the number of the total population (less than 1% according to [3]). Therefore the density of susceptibles can be considered to be constant without being altered by the epidemic.

In section 2, we define the basic reproductive number R_0 as the spectral radius of $-CB^{-1}$, i.e. $R_0 = r(-CB^{-1})$, where $B: D(B)\subset C(\bar\Omega;\mathbb{R}^2)\to C(\bar\Omega;\mathbb{R}^2)$ and $C: C(\bar\Omega;\mathbb{R}^2)\to C(\bar\Omega;\mathbb{R}^2)$ are linear operators on $C(\bar\Omega;\mathbb{R}^2)$ with

$$B = \begin{pmatrix} \nabla\cdot\delta_1\nabla & 0\\ 0 & \nabla\cdot\delta_2\nabla \end{pmatrix} + \begin{pmatrix} -\lambda & \sigma_1 H_u\\ 0 & -\mu\hat V \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0\\ \sigma_2\hat V & 0 \end{pmatrix},$$

with the suitable domain $D(B)$ (see [25, 30]).

The equilibria of (1.1)–(1.3) are solutions of the following elliptic system:

$$\begin{cases} -\nabla \cdot \delta_1(x)\nabla H_i = -\lambda(x)H_i + \sigma_1(x)H_u(x)V_i, & x \in \Omega, \\ -\nabla \cdot \delta_2(x)\nabla V_u = -\sigma_2(x)V_uH_i + \beta(x)(V_u+V_i) - \mu(x)(V_u+V_i)V_u, & x \in \Omega, \\ -\nabla \cdot \delta_2(x)\nabla V_i = \sigma_2(x)V_uH_i - \mu(x)(V_u+V_i)V_i, & x \in \Omega, \\ \frac{\partial}{\partial n}H_i = \frac{\partial}{\partial n}V_u = \frac{\partial}{\partial n}V_i = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

The system always has one trivial equilibrium E_0 and a unique semi-trivial equilibrium $E_1 = (0, \hat{V}, 0)$. In section 2, we prove that E_1 is globally asymptotically stable if $R_0 < 1$ in theorem 2.4.

Our main result is in section 3, where we show that (1.1)–(1.3) has a unique globally asymptotically stable positive equilibrium $E_2 = (\hat{H}_i, \hat{V}_u, \hat{V}_i)$ if $R_0 > 1$ (see theorem 3.12). We remark that it is usually not an easy task to prove the global stability of the positive equilibrium for a three-equation parabolic system when there is no clear Lyapunov type functional. Our proof combines arguments from monotone dynamical system theory, persistence theory, and the theory of asymptotically autonomous semiflows.

We briefly summarize our idea of proof here. Adding up the second and third equations in (1.1) and letting $V := V_u + V_i$, V satisfies the diffusive logistic equation $\partial_t V - \nabla \cdot \delta_2 \nabla V = \beta V - \mu V^2$. Since this equation has a globally stable positive equilibrium $\hat{V}$, it is tempting to assume that the dynamics of (1.1)–(1.3) is determined by the limit system

$$\begin{cases} \frac{\partial}{\partial t}\tilde{H}_i - \nabla \cdot \delta_1 \nabla \tilde{H}_i = -\lambda \tilde{H}_i + \sigma_1 \tilde{H}_u \tilde{V}_i, & x \in \Omega, t > 0, \\ \frac{\partial}{\partial t}\tilde{V}_i - \nabla \cdot \delta_2 \nabla \tilde{V}_i = \sigma_2(\hat{V} - \tilde{V}_i)^+ \tilde{H}_i - \mu \hat{V} \tilde{V}_i, & x \in \Omega, t > 0. \end{cases} \tag{1.5}$$

However even for ordinary differential equation (ODE) systems, Thieme [33] gives many examples where the dynamics of the limit and original systems are quite different. A remedy to this is the theory of asymptotically autonomous semiflows (see [31, theorem 4.1]), which is generalized from the well-known theory by Markus on asymptotically autonomous ODE systems. Applying this theory, to prove the convergence of $(H_i(\cdot, t), V_i(\cdot, t))$, it suffices to show: (a) system (1.5) has a unique positive equilibrium $(\hat{H}_i, \hat{V}_i)$; (b) The equilibrium $(\hat{H}_i, \hat{V}_i)$ of (1.5) is globally stable in $W := \{(H_{i0}, V_{i0}) \in C(\bar{\Omega}; \mathbb{R}^2_+) : \ H_{i0} + V_{i0} \neq 0\}$; (c) The $\omega-$limit set of $(H_i(\cdot, t), V_i(\cdot, t))$ intersects W. The proof of (a) is given in section 3.1.1. The proof of (b) is provided in section 3.1.2, where we take advantage of the monotonicity of (1.5). To show (c), we use the uniform persistence theory in [15] to obtain $\liminf_{t\to\infty} \|H_i(\cdot, t)\|_\infty + \|V_i(\cdot, t)\|_\infty \geqslant \epsilon$ for some $\epsilon > 0$ (see lemma 3.11). Interested readers may read the appendix on the ODE system for the idea of the proof first.

In section 4, we prove the global stability of E_1 for the critical case $R_0 = 1$. Here the main difficulty is to prove the local stability of E_1 as the linearized system at E_1 has principal eigenvalue equaling zero. In section 5, we give some concluding remarks. In particular, we summarize our results on the basic reproduction number R_0, which will be presented in a forthcoming paper. We also remark that our idea is applicable to other models (e.g. [18, 19, 26, 28]).

2. Disease free equilibria

The objective of this section is to define the basic reproduction number and investigate the stability of the trivial and semi-trivial equilibria. The existence, uniqueness, and positivity of global classical solutions of (1.1)–(1.3) have been shown in [14]. Let $V = V_u + V_i$. Then $V(x, t)$ satisfies

$$\begin{cases} V_t - \nabla\cdot\delta_2(x)\nabla V = \beta(x)V - \mu(x)V^2, & x\in\Omega, t>0,\\ \frac{\partial}{\partial n}V = 0, & x\in\partial\Omega, t>0,\\ V(\cdot,0) = V_0\in C(\bar\Omega;\mathbb{R}_+). \end{cases} \tag{2.1}$$

The following result about (2.1) is well-known (see, e.g. [4]).

Lemma 2.1. *For any nonnegative nontrivial initial data $V_0\in C(\bar\Omega;\mathbb{R})$, (2.1) has a unique global classic solution $V(x,t)$. Moreover, $V(x,t)>0$ for all $(x,t)\in\bar\Omega\times(0,\infty)$ and*

$$\lim_{t\to\infty}\|V(\cdot,t)-\hat V\|_\infty = 0, \tag{2.2}$$

where $\hat V$ is the unique positive solution of the elliptic problem

$$\begin{cases} -\nabla\cdot\delta_2(x)\nabla V = \beta(x)V-\mu(x)V^2, & x\in\Omega,\\ \frac{\partial}{\partial n}V=0, & x\in\partial\Omega. \end{cases} \tag{2.3}$$

By lemma 2.1, $V_u(x,t)+V_i(x,t)\to\hat V(x)$ uniformly for $x\in\bar\Omega$ as $t\to\infty$ if $V_{u0}+V_{i0}\neq 0$.

As usual, we consider two types of equilibria for (1.1)–(1.2): disease free equilibrium (DFE) and endemic equilibrium (EE). A nonnegative solution $(\tilde H_i,\tilde V_u,\tilde V_i)$ of (1.4) is a DFE if $\tilde H_i=\tilde V_i=0$, and otherwise it is an EE. By lemma 2.1, we must have $\tilde V_u+\tilde V_i=\hat V$ or $\tilde V_u+\tilde V_i=0$. It is then not hard to show that (1.1) and (1.2) has two DFE: trivial equilibrium $E_0=(0,\ 0,\ 0)$ and semi-trivial equilibrium $E_1=(0,\hat V,0)$. We denote the EE by $E_2=(\hat H_i,\hat V_u,\hat V_i)$, which will be proven to be unique if exists.

It is not hard to show that E_0 is always unstable. Linearizing (1.1) around E_1, we arrive at the following eigenvalue problem:

$$\begin{cases} \kappa\varphi &= \nabla\cdot\delta_1\nabla\varphi-\lambda\varphi+\sigma_1 H_u\psi, & x\in\Omega,\\ \kappa\phi &= \nabla\cdot\delta_2\nabla\phi-\sigma_2\hat V\varphi+\beta(\phi+\psi)-2\mu\hat V\phi-\mu\hat V\psi, & x\in\Omega,\\ \kappa\psi &= \nabla\cdot\delta_2\nabla\psi+\sigma_2\hat V\varphi-\mu\hat V\psi, & x\in\Omega,\\ \frac{\partial}{\partial n}\varphi &= \frac{\partial}{\partial n}\phi=\frac{\partial}{\partial n}\psi=0, & x\in\partial\Omega. \end{cases} \tag{2.4}$$

Since the second equation of (2.4) is decoupled from the system, we consider the problem

$$\begin{cases} \kappa\varphi &= \nabla\cdot\delta_1\nabla\varphi-\lambda\varphi+\sigma_1 H_u\psi, & x\in\Omega,\\ \kappa\psi &= \nabla\cdot\delta_2\nabla\psi+\sigma_2\hat V\varphi-\mu\hat V\psi, & x\in\Omega,\\ \frac{\partial}{\partial n}\varphi &= \frac{\partial}{\partial n}\psi=0, & x\in\partial\Omega. \end{cases} \tag{2.5}$$

Problem (2.5) is cooperative, so it has a principal eigenvalue κ_0 associated with a positive eigenvector (φ_0,ψ_0) (e.g. see [17]).

For $\delta\in C^1(\bar\Omega;\mathbb{R})$ being strictly positive on $\bar\Omega$ and $f\in C(\bar\Omega;\mathbb{R})$, let $\kappa_1(\delta,f)$ be the principal eigenvalue of

$$\begin{cases} \kappa\phi = \nabla\cdot\delta(x)\nabla\phi+f\phi, & x\in\Omega,\\ \frac{\partial}{\partial n}\phi=0, & x\in\partial\Omega. \end{cases} \tag{2.6}$$

It is well known that $\kappa_1(\delta,f)$ is the only eigenvalue associated with a positive eigenvector, and it is monotone in the sense that if $f_1\geqslant(\neq)f_2$ then $\kappa_1(\delta,f_1)>\kappa_2(\delta,f_2)$.

Lemma 2.2. *E_1 is locally asymptotically stable if $\kappa_0 < 0$ and unstable if $\kappa_0 > 0$.*

Proof. Since $\hat{V}$ is a positive solution of (2.3), we have $\kappa_1(\delta_2, \beta - \mu\hat{V}) = 0$. Therefore, $\kappa_1(\delta_2, \beta - 2\mu\hat{V}) < 0$.

Suppose $\kappa_0 < 0$. Let κ be an eigenvalue of (2.4). Then κ is an eigenvalue of either (2.5) or the following eigenvalue problem:

$$\begin{cases} \kappa\phi & = \nabla \cdot \delta_2 \nabla\phi + \beta\phi - 2\mu\hat{V}\phi, \qquad x \in \Omega, \\ \frac{\partial}{\partial n}\phi & = 0, \qquad\qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{cases}$$

Since $\kappa_0 < 0$ and $\kappa_1(\delta_2, \beta - 2\mu\hat{V}) < 0$, the real part of κ is less than zero. Since κ is arbitrary, E_1 is linearly stable. By the principle of linearized stability, E_1 is locally asymptotically stable.

Suppose $\kappa_0 > 0$. Let (φ_0, ψ_0) be a positive eigenvector associated with κ_0. By $\kappa_1(\delta_2, \beta - 2\mu\hat{V}) < 0$ and the Fredholm alternative, the following problem has a unique solution ϕ_0:

$$\begin{cases} \kappa_0\phi & = \nabla \cdot \delta_2 \nabla\phi - \sigma_2\hat{V}\varphi_0 + \beta(\phi + \psi_0) - 2\mu\hat{V}\phi - \mu\hat{V}\psi_0, \qquad x \in \Omega, \\ \frac{\partial}{\partial n}\phi & = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{cases}$$

Hence (2.4) has an eigenvector $(\varphi_0, \phi_0, \psi_0)$ corresponding to eigenvalue $\kappa_0 > 0$. So E_1 is linearly unstable. By the principle of linearized instability, E_1 is unstable. ■

We adopt the approach of [32, 36] to define the basic reproduction number of (1.1). Let $B : C(\bar{\Omega}; \mathbb{R}^2) \to C(\bar{\Omega}; \mathbb{R}^2)$ be the operator such that

$$D(B) := \left\{ (\varphi, \psi) \in \bigcap_{p \geqslant 1} W^{2,p}(\Omega; \mathbb{R}^2) : \frac{\partial}{\partial n}\varphi = \frac{\partial}{\partial n}\psi = 0 \text{ on } \partial\Omega \text{ and } B(\varphi, \psi) \in C(\bar{\Omega}; \mathbb{R}^2) \right\}$$

and

$$B(\varphi, \psi) = \begin{pmatrix} \nabla \cdot \delta_1 \nabla\varphi \\ \nabla \cdot \delta_2 \nabla\psi \end{pmatrix} + \begin{pmatrix} -\lambda & \sigma_1 H_u \\ 0 & -\mu\hat{V} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \quad (\varphi, \psi) \in D(B).$$

Define

$$C = \begin{pmatrix} 0 & 0 \\ \sigma_2\hat{V} & 0 \end{pmatrix}.$$

Let $A = B + C$. Then A and B are resolvent positive (see [32] for the definition), and A is a positive perturbation of B. It is easy to check that the spectral bound of B is negative, i.e. $s(B) < 0$. By [32, theorem 3.5], $\kappa_0 = s(A)$ has the same sign with $r(-CB^{-1}) - 1$, where $r(-CB^{-1})$ is the spectral radius of $-CB^{-1}$. Then we define the *basic reproduction number* R_0 by

$$R_0 = r(-CB^{-1}).$$

We immediately have the following result:

Lemma 2.3. *$R_0 - 1$ and κ_0 have the same sign. Moreover, E_1 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

We then consider the global dynamics of the model when $R_0 < 1$.

Theorem 2.4. *If $R_0 < 1$, then E_1 is globally asmyptototically stable, i.e. E_1 is locally stable and, for any initial data $(H_{i0}, V_{u0}, V_{i0}) \in C(\bar{\Omega}; \mathbb{R}^3_+)$ with $V_{u0} + V_{i0} \neq 0$, we have*

$$\lim_{t\to\infty} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - E_1\|_\infty = 0. \tag{2.7}$$

Proof. By lemma 2.3, E_1 is locally asymptotically stable and $\kappa_0 < 0$. Then we can choose $\epsilon > 0$ small such that the following eigenvalue problem

$$\begin{cases} \kappa\varphi &= \nabla\cdot\delta_1\nabla\varphi - \lambda\varphi + \sigma_1 H_u\psi, & x\in\Omega, \\ \kappa\psi &= \nabla\cdot\delta_2\nabla\psi + \sigma_2(\hat{V}+\epsilon)\varphi - \mu(\hat{V}-\epsilon)\psi, & x\in\Omega, \\ \frac{\partial}{\partial n}\varphi &= \frac{\partial}{\partial n}\psi = 0, & x\in\partial\Omega, \end{cases}$$

has a principal eigenvalue $\kappa_\epsilon < 0$ with a corresponding positive eigenvector $(\varphi_\epsilon, \psi_\epsilon)$. By $V_{u0} + V_{i0} \neq 0$ and lemma 2.1, we know that $V_u(x,t) + V_i(x,t) \to \hat{V}(x)$ uniformly on $\bar{\Omega}$ as $t \to \infty$. Hence there exists $t_0 > 0$ such that $\hat{V}(x) - \epsilon < V_u(x,t) + V_i(x,t) < \hat{V}(x) + \epsilon$ for $x \in \bar{\Omega}$ and $t > t_0$. It then follows that

$$\begin{cases} \frac{\partial}{\partial t}H_i - \nabla\cdot\delta_1\nabla H_i &= -\lambda H_i + \sigma_1 H_u(x)V_i, & x\in\Omega, t>t_0, \\ \frac{\partial}{\partial t}V_i - \nabla\cdot\delta_2\nabla V_i &\leqslant \sigma_2(\hat{V}+\epsilon)H_i - \mu(\hat{V}-\epsilon)V_i, & x\in\Omega, t>t_0. \end{cases}$$

So (H_i, V_i) is a lower solution of the following problem

$$\begin{cases} \frac{\partial}{\partial t}\hat{H}_i - \nabla\cdot\delta_1\nabla\hat{H}_i = -\lambda\hat{H}_i + \sigma_1 H_u\hat{V}_i, & x\in\Omega, t>t_0, \\ \frac{\partial}{\partial t}\hat{V}_i - \nabla\cdot\delta_2\nabla\hat{V}_i = \sigma_2(\hat{V}+\epsilon)\hat{H}_i - \mu(\hat{V}-\epsilon)\hat{V}_i, & x\in\Omega, t>t_0, \\ \frac{\partial}{\partial n}\hat{H}_i = \frac{\partial}{\partial n}\hat{V}_i = 0, & x\in\partial\Omega, t>t_0, \\ \hat{H}_i(x,t_0) = M\varphi_\epsilon(x), \quad \hat{V}_i(x,t_0) = M\psi_\epsilon(x), & x\in\Omega, \end{cases} \tag{2.8}$$

where M is large such that $H_i(x,t_0) \leqslant \hat{H}_i(x,t_0)$ and $V_i(x,t_0) \leqslant \hat{V}_i(x,t_0)$. By the comparison principle for cooperative systems (e.g. [27]), $H_i(x,t) \leqslant \hat{H}_i(x,t)$ and $V_i(x,t) \leqslant \hat{V}_i(x,t)$ for all $x \in \bar{\Omega}$ and $t \geqslant t_0$. It is easy to check that the unique solution of the linear problem (2.8) is $(\hat{H}_i(x,t), \hat{V}_i(x,t)) = (M\varphi_\epsilon(x)e^{\kappa_\epsilon(t-t_0)}, M\psi_\epsilon(x)e^{\kappa_\epsilon(t-t_0)})$. Since $\kappa_\epsilon < 0$, we have $\hat{H}_i(x,t) \to 0$ and $\hat{V}_i(x,t) \to 0$ uniformly for $x \in \bar{\Omega}$ as $t \to \infty$. Hence $H_i(x,t) \to 0$ and $V_i(x,t) \to 0$ uniformly for $x \in \bar{\Omega}$ as $t \to \infty$. By $V_u(\cdot,t) + V_i(\cdot,t) \to \hat{V}$ in $C(\bar{\Omega};\mathbb{R})$, we have $V_u(x,t) \to \hat{V}(x)$ uniformly for $x \in \bar{\Omega}$ as $t \to \infty$. ■

3. Global dynamics when $R_0 > 1$

The objective in this section is to prove the convergence of solutions of (1.1)–(1.3) to the unique positive equilibrium when $R_0 > 1$.

3.1. The limit problem

By lemma 2.1, we have $V_u(\cdot,t)+V_i(\cdot,t)\to\hat{V}$ in $C(\bar{\Omega};\mathbb{R})$ as $t\to\infty$ if $V_{u0}+V_{i0}\neq 0$. This suggests us to study the following limit problem of (1.1)–(1.3):

$$\begin{cases} \frac{\partial}{\partial t}H_i-\nabla\cdot\delta_1\nabla H_i=-\lambda H_i+\sigma_1 H_u V_i, & x\in\Omega, t>0,\\ \frac{\partial}{\partial t}V_i-\nabla\cdot\delta_2\nabla V_i=\sigma_2(\hat{V}-V_i)^+H_i-\mu\hat{V}V_i, & x\in\Omega, t>0,\\ \frac{\partial}{\partial n}H_i=\frac{\partial}{\partial n}V_i=0, & x\in\partial\Omega, t>0,\\ H_i(x,0)=H_{i0}(x),\ V_i(x,0)=V_{i0}(x), & x\in\Omega. \end{cases} \tag{3.1}$$

The equilibria of (3.1) are nonnegative solutions of the problem:

$$\begin{cases} -\nabla\cdot\delta_1\nabla H_i=-\lambda H_i+\sigma_1 H_u V_i, & x\in\Omega,\\ -\nabla\cdot\delta_2\nabla V_i=\sigma_2(\hat{V}-V_i)^+H_i-\mu\hat{V}V_i, & x\in\Omega,\\ \frac{\partial}{\partial n}H_i=\frac{\partial}{\partial n}V_i=0, & x\in\partial\Omega. \end{cases} \tag{3.2}$$

Clearly $(0,0)$ is an equilibrium. In this section, we prove that if a positive equilibrium of (3.1) exists, it is globally stable in $\{(H_{i0},V_{i0})\in C(\bar{\Omega};\mathbb{R}^2_+): H_{i0}+V_{i0}\neq 0\}$.

3.1.1. Uniqueness of positive equilibrium. In the following lemmas, we prove that the positive equilibrium of (3.1) is unique if it exists. We are essentially using the fact that (3.2) is cooperative and sublinear, and similar ideas can be found in [2, 42].

Lemma 3.1. *If $(\hat{H}_i,\hat{V}_i)$ is a nontrivial nonnegative equilibrium of (3.1), then $\hat{H}_i(x),\hat{V}_i(x)>0$ for all $x\in\bar{\Omega}$ and $\hat{V}_i(x_0)<\hat{V}(x_0)$ for some $x_0\in\bar{\Omega}$.*

Proof. Since $(\hat{H}_i,\hat{V}_i)$ is nontrivial, $\hat{H}_i\not\equiv 0$ or $\hat{V}_i\not\equiv 0$. Since $(\lambda-\nabla\cdot\delta_1\nabla)\hat{H}_i=\sigma_1 H_u\hat{V}_i$, we must have $\hat{H}_i\not\equiv 0$ and $\hat{V}_i\not\equiv 0$. By the maximum principle, we have $\hat{H}_i(x),\hat{V}_i(x)>0$ for all $x\in\bar{\Omega}$. Assume to the contrary that $\hat{V}_i(x)\geqslant\hat{V}(x)$ for all $x\in\bar{\Omega}$, then

$$-\nabla\cdot\delta_2\nabla\hat{V}_i=\sigma_2(\hat{V}-\hat{V}_i)^+\hat{H}_i-\mu\hat{V}\hat{V}_i=-\mu\hat{V}\hat{V}_i.$$

This implies $\hat{V}_i=0$, which is a contradiction. ■

By the previous lemma, any nontrivial nonnegative equilibrium must be positive. For any $C_1,C_2>0$, define

$$S=\{V_i\in C(\bar{\Omega};\mathbb{R}_+):\ \|V_i\|_\infty\leqslant C_1 \text{ and } V_i(x_0)<\hat{V}(x_0) \text{ for some } x_0\in\bar{\Omega}\},$$

and $f:S\subset C(\bar{\Omega})\to C(\bar{\Omega})$ by

$$f(V_i)=(C_2-\nabla\cdot\delta_2\nabla)^{-1}\left[\sigma_2(\hat{V}-V_i)^+(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1 H_u V_i+(C_2-\mu\hat{V})V_i\right],\quad V_i\in S.$$

Lemma 3.2. *If $(\hat{H}_i,\hat{V}_i)$ is a positive equilibrium, then there exists $C_1^*>0$ such that $\hat{V}_i$ is a nontrivial fixed point of f for all $C_1>C_1^*$ and $C_2>0$.*

Proof. By the first equation of (3.2), $\hat{H}_i=(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1 H_u\hat{V}_i$. Substituting it into the second equation, we obtain

$$-\nabla\cdot\delta_2\nabla\hat{V}_i=\sigma_2(\hat{V}-V_i)^+(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1H_u\hat{V}_i-\mu\hat{V}\hat{V}_i.$$

By lemma 3.1, V_i is a nontrivial fixed point of f if C_1 is large. ■

Lemma 3.3. *For any $C_1>0$, there exists $C_2^*>0$ such that f is monotone for all $C_2>C_2^*$ in the sense that $f(V_i)\leqslant f(\tilde{V}_i)$ for all $V_i,\tilde{V}_i\in S$ with $V_i\leqslant\tilde{V}_i$.*

Proof. It suffices to prove that $f(V_i)\leqslant f(V_i+h)$ for any $V_i\in S$ and $0\leqslant h\leqslant\hat{V}-V_i$. Define

$$\tilde{f}(V_i)=\sigma_2(\hat{V}-V_i)^+(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1H_uV_i+(C_2-\mu\hat{V})V_i.$$

Then, we have

$$\begin{aligned}\tilde{f}(V_i+h)-\tilde{f}(V_i)&=\sigma_2((\hat{V}-V_i-h)^+-(\hat{V}-V_i)^+)(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1H_uV_i\\&\quad+\sigma_2(\hat{V}-V_i-h)^+(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1H_uh+(C_2-\mu\hat{V})h\\&\geqslant h[-\sigma_2(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1H_uV_i+C_2-\mu\hat{V}],\end{aligned}$$

where we have used

$$|(\hat{V}-V_i-h)^+-(\hat{V}-V_i)^+|\leqslant h.$$

By the elliptic estimate, the following set is bounded:

$$\{(\lambda-\nabla\cdot\delta_1\nabla)^{-1}\sigma_1H_uV_i,\ \ V_i\in S\}.$$

Hence, $\tilde{f}(V_i+h)-\tilde{f}(V_i)\geqslant 0$ if C_2 is large. Therefore, $f(V_i+h)-f(V_i)\geqslant 0$, and f is monotone if C_2 is large. ■

For any $f_1,f_2\in C(\bar{\Omega};\mathbb{R})$, we say $f_1\ll f_2$ if $f_1(x)<f_2(x)$ for all $x\in\bar{\Omega}$.

Lemma 3.4. *For any $k\in(0,1)$ and $V_i\in S$ with $V_i\gg 0$, $kf(V_i)\ll f(kV_i)$.*

Proof. By the definition of S, there exists $x_0\in\bar{\Omega}$ such that $\hat{V}(x_0)>V_i(x_0)$. So $(\hat{V}(x_0)-V_i(x_0))^+<(\hat{V}(x_0)-kV_i(x_0))^+$ and $(\hat{V}(x)-V_i(x))^+\leqslant(\hat{V}(x)-kV_i(x))^+$ for all $x\in\bar{\Omega}$. It then follows that $k\tilde{f}(V_i)(x_0)<\tilde{f}(kV_i)(x_0)$ and $k\tilde{f}(V_i)\leqslant\tilde{f}(kV_i)$. The assertion now just follows from the fact that $(C_2-\nabla\cdot\delta_2\nabla)^{-1}$ is strongly positive (i.e. if $g\in C(\bar{\Omega};\mathbb{R})$ such that $g\geqslant 0$ and $g(x_0)>0$ for some $x_0\in\bar{\Omega}$, then $(C_2-\nabla\cdot\delta_2\nabla)^{-1}g\gg 0$). ■

Lemma 3.5. *The positive equilibrium of (3.1), if exists, is unique.*

Proof. Suppose to the contrary that (H_i^1,V_i^1) and (H_i^2,V_i^2) are two distinct positive equilibria. Then $V_i^1\neq V_i^2$ by the first equation of (3.2). Without loss of generality, we may assume $V_i^1\nleqslant V_i^2$. Define

$$k=\max\{\tilde{k}\geqslant 0:\ \ \tilde{k}V_i^1\leqslant V_i^2\}.$$

Then $k\in(0,1)$. By the definition of k, $kV_i^1\leqslant V_i^2$ and $kV_i^1(x_0)=V_i^2(x_0)$ for some $x_0\in\bar{\Omega}$. We can choose C_1 and C_2 such that V_i^1 and V_i^2 are fixed points of f, i.e. $f(V_i^1)=V_i^1$ and $f(V_i^2)=V_i^2$. By the previous lemmas, we have

$$kV_i^1 = kf(V_i^1) \ll f(kV_i^1) \leqslant f(V_i^2) = V_i^2.$$

Thus $kV_i^1 \ll V_i^2$, which contradicts $kV_i^1(x_0) = V_i^2(x_0)$. ■

Remark 3.6. It is possible to improve lemma 3.1 by proving $\hat{V}_i(x) < \hat{V}(x)$ for all $x \in \bar{\Omega}$, which means that a positive equilibrium of (3.1) is always a positive equilibrium of (1.1). To see this, let $\hat{V}_u = \hat{V} - \hat{V}_i$. Since $\hat{V}$ satisfies $-\nabla \cdot \delta_2 \nabla \hat{V} = \beta \hat{V} - \mu \hat{V}^2$ and $\hat{V}_i$ satisfies $-\nabla \cdot \delta_2 \nabla \hat{V}_i = \sigma_2 (\hat{V} - \hat{V}_i)^+ H_i - \mu \hat{V} \hat{V}_i$, we have

$$\begin{cases} -\nabla \cdot \delta_2 \nabla \hat{V}_u = \beta \hat{V} - \sigma_2 H_i \hat{V}_u^+ - \mu \hat{V} \hat{V}_u, & x \in \Omega, \\ \frac{\partial \hat{V}_u}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Let $x_0 \in \bar{\Omega}$ such that $\hat{V}_u(x_0) = \min_{x \in \bar{\Omega}} \hat{V}_u(x)$. Assume to the contrary that $\hat{V}_u(x_0) \leqslant 0$. By a comparison principle due to Lou and Ni [21], we have $\beta(x_0)\hat{V}(x_0) - \sigma_2(x_0)H_i(x_0)$ $\hat{V}_u^+(x_0) - \mu(x_0)\hat{V}(x_0)\hat{V}_u(x_0) \leqslant 0$, which implies $\hat{V}_u(x_0) \geqslant \beta(x_0)/\mu(x_0) > 0$. This contradicts the assumption $\hat{V}_u(x_0) \leqslant 0$. Therefore, $\hat{V}_i(x) < \hat{V}(x)$ for all $x \in \bar{\Omega}$.

3.1.2. Global stability of positive equilibrium. Let $F_1(H_i, V_i) = -\lambda H_i + \sigma_1 H_u V_i$ and $F_2(H_i, V_i) = \sigma_2(\hat{V} - V_i)^+ H_i - \mu \hat{V} V_i$. Since $\partial F_1/\partial V_i \geqslant 0$ and $\partial F_2/\partial H_i \geqslant 0$, system (3.1) is cooperative. Let $\tilde{\Phi}(t) : C(\bar{\Omega}; \mathbb{R}^2) \to C(\bar{\Omega}; \mathbb{R}^2)$ be the semiflow induced by the solution of (3.1), i.e. $\tilde{\Phi}(t)(H_{i0}, V_{i0}) = (H_i(\cdot, t), V_i(\cdot, t))$ for all $t \geqslant 0$. Then $\tilde{\Phi}(t)$ is monotone (e.g. see [27]).

Lemma 3.7. *For any nonnegative nontrivial initial data (H_{i0}, V_{i0}), the solution of (3.1) satisfies that $H_i(x,t) > 0$ and $V_i(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$.*

Proof. By the comparison principle for cooperative systems, $H_i(x,t) \geqslant 0$ and $V_i(x,t) \geqslant 0$ for all $x \in \bar{\Omega}$ and $t \geqslant 0$. Suppose $V_{i0} \neq 0$. Noticing

$$\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2 \nabla V_i \geqslant -\mu \hat{V} V_i \tag{3.3}$$

and by the comparison principle, we have $V_i(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. Then,

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geqslant -\lambda H_i,$$

where the inequality is strict for some $x \in \bar{\Omega}$ as H_u is nontrivial. So by the comparison principle, $H_i(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$.

Suppose $V_{i0} = 0$. Since (H_{i0}, V_{i0}) is nontrivial, we have $H_{i0} \neq 0$. By

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geqslant -\lambda H_i,$$

and the comparison principle, we have $H_i(x, t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. By the continuity of $V_i(x,t)$ and $V_i(x,0) = 0$, $(\hat{V} - V_i(x,t))^+ > 0$ for all $(x,t) \in \bar{\Omega} \times (0, t_0]$ for some $t_0 > 0$. Then by

$$\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2 \nabla V_i > -\mu \hat{V} V_i, \quad x \in \bar{\Omega}, t \in (0, t_0]$$

and the comparison principle, we have $V_i(x,t) > 0$ for all $(x,t) \in \bar{\Omega} \times (0, t_0]$. Finally by (3.3), we have $V_i(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. ■

Lemma 3.8. *For any nonnegative initial data (H_{i0}, V_{i0}), there exists $M > 0$ such that the solution of (3.1) satisfies*

$$0 \leqslant H_i(x,t), V_i(x,t) \leqslant M, \quad \text{for all } x \in \bar{\Omega}, t > 0.$$

Proof. Let $M_1 = \max\{\|\hat{V}\|_\infty, \|V_{i0}\|_\infty\}$. By the second equation of (3.1) and the comparison principle, we have $V_i(x,t) \leqslant M_1$ for all $x \in \bar{\Omega}$ and $t > 0$. Then by the first equation of (3.1), we have

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1(x) \nabla H_i \leqslant -\lambda(x) H_i + \sigma_1(x) H_u(x) M_1, \quad x \in \Omega, t > 0.$$

So H_i is a lower solution of the problem:

$$\begin{cases} \frac{\partial}{\partial t} w - \nabla \cdot \delta_1(x) \nabla w = -\lambda(x) w + \sigma_1(x) H_u(x) M_1, & x \in \Omega, t > 0, \\ \frac{\partial}{\partial n} w = 0, & x \in \partial\Omega, t > 0, \\ w(x,0) = H_{i0}(x), & x \in \Omega. \end{cases}$$

Let $M_2 = \max\{\|\sigma_1\|_\infty \|H_u\|_\infty M_1/\lambda_m, \ \|H_{i0}\|_\infty\}$, where $\lambda_m = \min\{\lambda(x) : x \in \bar{\Omega}\}$. Then we have $0 \leqslant w(x,t) \leqslant M_2$ for all $(x,t) \in \bar{\Omega} \times (0, \infty)$. Hence by the comparison principle, we have $0 \leqslant H_i(x,t) \leqslant w(x,t) < M_2$. Therefore, the claim holds for $M = \max\{M_1, M_2\}$. ■

Lemma 3.9. *If the positive equilibrium $(\hat{H}_i, \hat{V}_i)$ of (3.1) exists, it is globally asymptotically stable, i.e. it is locally stable and, for any nonnegative nontrivial initial data (H_{i0}, V_{i0}),*

$$\lim_{t\to\infty} H_i(\cdot, t) = \hat{H}_i \ \text{ and } \ \lim_{t\to\infty} V_i(\cdot, t) = \hat{V}_i \quad \text{in } C(\bar{\Omega}; \mathbb{R}).$$

Proof. By lemma 3.7, we have $H_i(x,t) > 0$ and $V_i(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. So without loss of generality, we may assume $H_{i0}(x) > 0$ and $V_{i0}(x) > 0$ for all $x \in \bar{\Omega}$.

Suppose that $(\hat{H}_i, \hat{V}_i)$ is a positive equilibrium of (3.1), which is unique by lemma 3.5. Let $(\underline{H}_i, \underline{V}_i) = (\epsilon \hat{H}_i, \epsilon \hat{V}_i)$ for some $\epsilon > 0$. We may choose ϵ small such that the following is satisfied:

$$\begin{cases} -\nabla \cdot \delta_1(x) \nabla \underline{H}_i \leqslant -\lambda(x) \underline{H}_i + \sigma_1(x) H_u(x) \underline{V}_i, & x \in \Omega, \\ -\nabla \cdot \delta_2(x) \nabla \underline{V}_i \leqslant \sigma_2(x) (\hat{V} - \underline{V}_i)^+ \underline{H}_i - \mu(x) \hat{V} \underline{V}_i, & x \in \Omega, \\ \frac{\partial}{\partial n} \underline{H}_i = \frac{\partial}{\partial n} \underline{V}_i = 0, & x \in \partial\Omega, \\ \underline{H}_i(x) \leqslant H_{i0}(x), \ \underline{V}_i(x) \leqslant V_{i0}(x), & x \in \Omega. \end{cases} \tag{3.4}$$

Hence by [27, corollary 7.3.6], $\tilde{\Phi}(t)(\underline{H}_i, \underline{V}_i)$ is monotone increasing in t and converges to a positive equilibrium of (3.1). Since $(\hat{H}_i, \hat{V}_i)$ is the unique positive equilibrium of (3.1), we must have $\tilde{\Phi}(t)(\underline{H}_i, \underline{V}_i) \to (\hat{H}_i, \hat{V}_i)$ in $C(\bar{\Omega}; \mathbb{R}^2)$ as $t \to \infty$.

Similarly, we may define $(\overline{H}_i, \overline{V}_i) = (k\hat{H}_i, k\hat{V}_i)$ with k large such that (3.4) is satisfied with inverse inequalities, and then $\tilde{\Phi}(t)(\overline{H}_i, \overline{V}_i) \to (\hat{H}_i, \hat{V}_i)$ in $C(\bar{\Omega}; \mathbb{R}^2)$ as $t \to \infty$. Since $(\underline{H}_i, \underline{V}_i) \leqslant (H_{i0}, V_{i0}) \leqslant (\overline{H}_i, \overline{V}_i)$ and $\tilde{\Phi}(t)$ is monotone, we have $\tilde{\Phi}(t)(\underline{H}_i, \underline{V}_i) \leqslant \tilde{\Phi}(t)(H_{i0}, V_{i0}) \leqslant \tilde{\Phi}(t)(\overline{H}_i, \overline{V}_i)$ for all $t \geqslant 0$. Therefore, $\tilde{\Phi}(t)(H_{i0}, V_{i0}) \to (\hat{H}_i, \hat{V}_i)$ in $C(\bar{\Omega}; \mathbb{R}^2)$ as $t \to \infty$.

For any $\epsilon' > 0$ and initial data (H_{i0}, V_{i0}) satisfying $(1-\epsilon')(\hat{H}_i, \hat{V}_i) \leqslant (H_{i0}, V_{i0}) \leqslant (1+\epsilon')(\hat{H}_i, \hat{V}_i)$, similar to the previous arguments, we can show $(1-\epsilon')(\hat{H}_i, \hat{V}_i) \leqslant (H_i(\cdot, t), V_i(\cdot, t)) \leqslant (1+\epsilon')(\hat{H}_i, \hat{V}_i)$ for all $t \geqslant 0$. Therefore, $(\hat{H}_i, \hat{V}_i)$ is locally stable. This proves the lemma. ■

3.2. Global stability of E_2

In this section, we prove the convergence of solutions of (1.1)–(1.3) to the unique positive equilibrium E_2 when $R_0 > 1$. We begin by proving the ultimate boundedness of the solutions.

Lemma 3.10. *There exists $M > 0$, independent of initial data, such that any solution (H_i, V_u, V_i) of (1.1)–(1.3) satisfies that*

$$0 \leqslant H_i(x,t), V_u(x,t), V_i(x,t) \leqslant M, \quad x \in \bar{\Omega}, t \geqslant t_0,$$

where t_0 is dependent on initial data.

Proof. By lemma 2.1, we have $V_u(x,t) + V_i(x,t) \to \hat{V}(x)$ uniformly on $\bar{\Omega}$ as $t \to \infty$ if $V_{u0} + V_{i0} \neq 0$. Hence there exists $t_1 > 0$ depending on initial data such that $V_u(x,t) + V_i(x,t) \leqslant \|\hat{V}\|_\infty + 1$ for $t > t_1$ and $x \in \bar{\Omega}$. By the first equation of (1.1) and the comparison principle, we have $H_i \leqslant \hat{H}_i$ on $\bar{\Omega} \times [t_1, \infty)$, where $\hat{H}_i$ is the solution of the problem

$$\begin{cases} \frac{\partial}{\partial t}\hat{H}_i - \nabla \cdot \delta_1(x)\nabla \hat{H}_i = -\lambda(x)\hat{H}_i + \sigma_1(x)H_u(x)(\|\hat{V}\|_\infty + 1), & x \in \Omega, t > t_1, \\ \frac{\partial}{\partial n}\hat{H}_i = 0, & x \in \partial\Omega, t > t_1, \\ \hat{H}_i(x, t_1) = H_i(x, t_1), & x \in \Omega. \end{cases}$$

We know that $\hat{H}_i(x,t) \to \hat{H}_i^*(x)$ uniformly on $\bar{\Omega}$ as $t \to \infty$, where $\hat{H}_i^*$ is the unique solution of the problem

$$\begin{cases} -\nabla \cdot \delta_1(x)\nabla \hat{H}_i = -\lambda(x)\hat{H}_i + \sigma_1(x)H_u(x)(\|\hat{V}\|_\infty + 1), & x \in \Omega, \\ \frac{\partial}{\partial n}\hat{H}_i = 0, & x \in \partial\Omega. \end{cases}$$

Therefore there exists $t_0 > t_1$ such that $H_i(x,t) \leqslant \hat{H}_i(x,t) < \|\hat{H}_i^*\|_\infty + 1$ for all $x \in \bar{\Omega}$ and $t \geqslant t_0$. Therefore, the claim holds with $M = \max\{\|\hat{V}\|_\infty + 1, \|\hat{H}_i^*\|_\infty + 1\}$. ■

Let (X, d) be a complete metric space and $\Phi(t) : X \to X$ be a continuous semiflow. The distance from a point $z \in X$ to a subset A of X is defined as $d(z, A) := \inf_{x \in A} d(z, x)$. Suppose that $X = \bar{X}_0$, where X_0 is an open subset of X. Then $X = X_0 \cup \partial X_0$ with the boundary $\partial X_0 = X - X_0$ being closed in X. The semiflow $\Phi(t)$ is said to be uniformly persistent with respect to $(X_0, \partial X_0)$ if there is an $\epsilon > 0$ such that $\liminf_{t\to\infty} d(T(t)x, \partial X_0) \geqslant \epsilon$ for all $x \in X_0$.

In the following of this section, let $X = C(\bar{\Omega}; \mathbb{R}^3_+)$ with the metric induced by the supremum norm $\|\cdot\|_\infty$. Define

$$\partial X_0 := \{(H_i, V_u, V_i) \in X : H_i + V_i = 0 \text{ or } V_u + V_i = 0\}$$

and

$$X_0 := \{(H_i, V_u, V_i) \in X : H_i + V_i > 0 \text{ and } V_u + V_i > 0\}.$$

Then $X = X_0 \cup \partial X_0$, X_0 is relatively open with $\bar{X}_0 = X$, and ∂X_0 is relatively closed in X. Let $w(x,t) = (H_i(x,t), V_u(x,t), V_i(x,t))$ be the solution of (1.1)–(1.3) with initial data $w_0 = (H_{i0}, V_{u0}, V_{i0}) \in X$. Let $\Phi(t) : X \to X$ be the semiflow induced by the solution of (1.1)–(1.3), i.e. $\Phi(t)w_0 = w(\cdot, t)$ for $t \geqslant 0$. Then $\Phi(t)$ is point dissipative by lemma 3.10 (see, e.g. [15] for the definition). Moreover, $\Phi(t)$ is compact for any $t > 0$, since (1.1)–(1.3) is a standard reaction–diffusion system.

We prove the following persistence result when $R_0 > 1$, which is necessary for proving the convergence of solutions to the positive equilibrium.

Lemma 3.11. *If $R_0 > 1$, then (1.1)–(1.3) is uniformly persistent in the sense that there exists $\epsilon > 0$ such that, for any initial data $(H_{i0}, V_{u0}, V_{i0}) \in X_0$,*

$$\liminf_{t\to\infty} \inf_{w\in\partial X_0} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - w\|_\infty \geqslant \epsilon. \tag{3.5}$$

Moreover, (1.1)–(1.3) has at least one EE.

Proof. We prove this result in several steps.

Step 1. X_0 is invariant under $\Phi(t)$.

Let $w_0 = (H_{i0}, V_{u0}, V_{i0}) \in X_0$. Then $H_{i0} + V_{i0} > 0$ and $V_{u0} + V_{i0} > 0$. Suppose $V_{i0} = 0$. Then $H_{i0} \neq 0$ and $V_{u0} \neq 0$. By the first equation of (1.1), we have

$$\frac{\partial}{\partial t} H_i - \nabla \cdot \delta_1 \nabla H_i \geqslant -\lambda H_i.$$

Then by $H_{i0} \neq 0$ and the maximum principle, we have $H_i(x, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. By the second equation of (1.1), we have

$$\frac{\partial}{\partial t} V_u - \nabla \cdot \delta_2 \nabla V_u \geqslant V_u(-\sigma_2 H_i + \beta - \mu(V_u + V_i)).$$

Then by $V_{u0} \neq 0$ and the maximum principle, we have $V_u(x,t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. Noticing the third equation of (1.1), we have

$$\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2 \nabla V_i > -\mu(V_u + V_i)V_i, \quad x \in \Omega, t > 0.$$

Then by the maximum principle, we have $V_i(x,t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$.

Suppose $V_{i0} \neq 0$. Noticing

$$\frac{\partial}{\partial t} V_i - \nabla \cdot \delta_2 \nabla V_i \geqslant -\mu(V_u + V_i)V_i$$

and by the maximum principle, we have $V_i(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. By the first equation of (1.1), we have

$$\frac{\partial}{\partial t}H_i - \nabla \cdot \delta_1 \nabla H_i \geqslant -\lambda H_i, \quad x \in \Omega, t > 0,$$

where the inequality is strict for some $x \in \bar{\Omega}$ as H_u is nontrivial. So by the comparison principle, $H_i(x, t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. By the second equation of (1.1), we have

$$\frac{\partial}{\partial t}V_u - \nabla \cdot \delta_2 \nabla V_u > V_u(-\sigma_2 H_i + \beta - \mu(V_u + V_i)), \quad x \in \Omega, t > 0,$$

which implies $V_u(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$. Therefore, we have $\Phi(t)w_0 \in X_0$ for all $t > 0$. Hence X_0 is invariant under $\Phi(t)$.

Step 2. ∂X_0 is invariant under $\Phi(t)$. For any $w_0 \in \partial X_0$, the ω-limit set $\omega(w_0)$ is either $\{E_0\}$ or $\{E_1\}$.

Suppose $w_0 = (H_{i0}, V_{u0}, V_{i0}) \in \partial X_0$. Then, $H_{i0} + V_{i0} = 0$ or $V_{u0} + V_{i0} = 0$. If $H_{i0} + V_{i0} = 0$ and $V_{u0} \neq 0$, then we have $H_i(\cdot,t) = V_i(\cdot,t) = 0$ for all $t \geqslant 0$ by the first and third equations of (1.1). Then the second equation of (1.1) is

$$\frac{\partial}{\partial t}V_u - \nabla \cdot \delta_2 \nabla V_u = V_u(\beta - \mu V_u).$$

Hence by lemma 2.1, we have $V_u(x,t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$, and $V_u(\cdot,t) \to \hat{V}$ uniformly on $\bar{\Omega}$ as $t \to \infty$. So $\Phi(t)w_0 \in \partial X_0$ with $\omega(w_0) = \{E_1\}$.
If $V_{u0} + V_{i0} = 0$, then by the second and third equations of (1.1), we have $V_u(\cdot,t) = V_i(\cdot,t) = 0$ for all $t \geqslant 0$. Then the first equation of (1.1) is

$$\frac{\partial}{\partial t}H_i - \nabla \cdot \delta_1 \nabla H_i = -\lambda H_i,$$

which implies that $H_i(x,t) \to 0$ uniformly on $\bar{\Omega}$ as $t \to 0$. Therefore, we have $\Phi(t)w_0 \in \partial X_0$ with $\omega(w_0) = \{E_0\}$.
By Step 2, the semiflow $\Phi_\partial(t) := \Phi(t)|_{\partial X_0}$, the restriction of $\Phi(t)$ on ∂X_0, admits a compact global attractor A_∂. Moreover, it is clear that

$$\tilde{A}_\partial := \cup_{w_0 \in A_\partial} \omega(w_0) = \{E_0, E_1\}.$$

Step 3. $\tilde{A}_\partial$ has an acyclic covering $M = \{E_0\} \cup \{E_1\}$.

It suffices to show that $\{E_1\} \not\to \{E_0\}$, i.e. $W^u(E_1) \cap W^s(E_0) = \varnothing$. Suppose to the contrary that there exists $w_0 = (H_{i0}, V_{u0}, V_{i0}) \in W^u(E_1) \cap W^s(E_0)$. Let $(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t))$ be a complete orbit through w_0. By $w_0 \in W^s(E_0)$ and lemma 2.1, we have $V_{u0} = V_{i0} = 0$, and hence $V_u(\cdot,t) = V_i(\cdot,t) = 0$ for all $t \in (-\infty, \infty)$. Therefore $V_u(\cdot,t) \not\to \hat{V}$ as $t \to -\infty$, contradicting $w_0 \in W^u(E_1)$. Therefore $M = \{E_0\} \cup \{E_1\}$ is an acyclic covering of $\tilde{A}_\partial$.

Step 4. $W^s(E_0) \cap X_0 = \varnothing$ and $W^s(E_1) \cap X_0 = \varnothing$.

We will actually show:

$$W^s(E_0) = \{(H_{i0}, V_{u0}, V_{i0}) \in \partial X_0 : \ V_{u0} = V_{i0} = 0\} \tag{3.6}$$

and

$$W^s(E_1) = \{(H_{i0}, V_{u0}, V_{i0}) \in \partial X_0 : \ H_{i0} = V_{i0} = 0 \ \text{ and } \ V_{u0} \not\equiv 0\}.$$

By the proof of step 2, it suffices to show that there exists $\epsilon > 0$ such that, for any initial data $(H_{i0}, V_{u0}, V_{i0}) \in X_0$, we have

$$\limsup_{t\to\infty} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - E_0\|_\infty \geqslant \epsilon \tag{3.7}$$

and

$$\limsup_{t\to\infty} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - E_1\|_\infty \geqslant \epsilon. \tag{3.8}$$

We first prove (3.8). By lemma 2.3 and $R_0 > 1$, we have $\kappa_0 > 0$. Hence there exists $\epsilon_0 > 0$ such that the following problem has a principal eigenvalue $\kappa_{\epsilon_0} > 0$ corresponding to a positive eigenvector $(\phi_{\epsilon_0}, \psi_{\epsilon_0})$

$$\begin{cases} \kappa\varphi = \nabla\cdot\delta_1\nabla\varphi - \lambda\varphi + \sigma_1 H_u\psi, & x\in\Omega,\\ \kappa\psi = \nabla\cdot\delta_2\nabla\psi + \sigma_2(\hat V - \epsilon_0)\varphi - \mu(\hat V + 2\epsilon_0)\psi, & x\in\Omega,\\ \frac{\partial}{\partial n}\varphi = \frac{\partial}{\partial n}\psi = 0, & x\in\partial\Omega. \end{cases}$$

Assume to the contrary that (3.8) does not hold. Then there exists some $w_0 = (H_{i0}, V_{u0}, V_{i0}) \in X_0$ such that the corresponding solution satisfies

$$\limsup_{t\to\infty} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - E_1\|_\infty < \epsilon_0.$$

Hence there exists $t_0 > 0$ such that $\hat V - \epsilon_0 < V_u(\cdot,t) < \hat V + \epsilon_0$ and $V_i(\cdot,t) < \epsilon_0$ for all $t \geqslant t_0$. It then follows from the second and third equations of (1.1) that

$$\begin{cases} \frac{\partial}{\partial t}H_i - \nabla\cdot\delta_1\nabla H_i = -\lambda H_i + \sigma_1 H_u V_i, & x\in\Omega, t\geqslant t_0,\\ \frac{\partial}{\partial t}V_i - \nabla\cdot\delta_2\nabla V_i \geqslant \sigma_2(\hat V - \epsilon_0)H_i - \mu(\hat V + 2\epsilon_0)V_i, & x\in\Omega, t\geqslant t_0. \end{cases}$$

In Step 1, we have shown that $H_i(x,t), V_i(x,t) > 0$ for all $x \in \bar\Omega$ and $t > 0$. Thus we can choose $m > 0$ small such that $H_i(\cdot,t_0) \geqslant m\phi_{\epsilon 0}$ and $V_i(\cdot,t_0) \geqslant m\psi_{\epsilon 0}$. Hence (H_i, V_i) is an upper solution of the problem

$$\begin{cases} \frac{\partial}{\partial t}\bar H_i - \nabla\cdot\delta_1\nabla\bar H_i = -\lambda\bar H_i + \sigma_1 H_u\bar V_i, & x\in\Omega, t\geqslant t_0,\\ \frac{\partial}{\partial t}\bar V_i - \nabla\cdot\delta_2\nabla\bar V_i = \sigma_2(\hat V - \epsilon_0)\bar H_i - \mu(\hat V + 2\epsilon_0)\bar V_i, & x\in\Omega, t\geqslant t_0,\\ \frac{\partial}{\partial n}\bar H_i = \frac{\partial}{\partial n}\bar V_i = 0, & x\in\partial\Omega, t\geqslant t_0,\\ \bar H_i(\cdot,t_0) = m\phi_{\epsilon 0},\quad \bar V_i(\cdot,t_0) = m\psi_{\epsilon 0}. \end{cases}$$

We observe that the solution of this problem is $(\bar H_i, \bar V_i) = m e^{\kappa_{\epsilon 0}(t-t_0)}(\phi_{\epsilon_0}, \psi_{\epsilon_0})$. By the comparison principle of cooperative systems, we have $H_i(\cdot,t) \geqslant \bar H_i(\cdot,t)$ and $V_i(\cdot,t) \geqslant \bar V_i(\cdot,t)$ for $t \geqslant t_0$. Since $\kappa_{\epsilon 0} > 0$, we have $H_i(\cdot,t) \to \infty$ and $V_i(\cdot,t) \to \infty$ as $t \to \infty$, which contradicts the boundedness of the solution. This proves (3.8).

We then prove (3.7). Suppose to the contrary that (3.7) does not hold. Then for given small $\epsilon_1 > 0$, there exists initial data $(H_{i0}, V_{u0}, V_{i0}) \in X_0$ such that

$$\limsup_{t\to\infty} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - E_0\|_\infty < \epsilon_1.$$

Hence there exists $t_1 > 0$ such that $V_u(\cdot,t) < \epsilon_1$ and $V_i(\cdot,t) < \epsilon_1$ for all $t \geqslant t_1$. However by lemma 2.1, we know that $V_u(\cdot,t) + V_i(\cdot,t) \to \hat V$ uniformly on $\bar\Omega$ as $t \to \infty$, which is a contradiction as ϵ_1 is small.

Finally by steps 1–4 and [15, theorem 4.1], there exists $\epsilon > 0$ such that (3.5) holds. Moreover by [42, theorem 1.3.7], (1.1)–(1.3) has an EE. ∎

Combing lemmas 3.9 and 3.11, we can prove the main result in this section.

Theorem 3.12. *If $R_0 > 1$, then for any initial data $(H_{i0}, V_{u0}, V_{i0}) \in X_0$, the solution (H_i, V_u, V_i) of (1.1)–(1.3) satisfies that*

$$\lim_{t\to\infty} (H_i(x,t), V_u(x,t), V_i(x,t)) = (\hat{H}_i, \hat{V}_u, \hat{V}_i) \quad \text{uniformly on } \bar{\Omega},$$

where $E_2 = (\hat{H}_i, \hat{V}_u, \hat{V}_i)$ is the unique EE of (1.1).

Proof. By lemma 3.11, there exists an EE, $E_2 := (\hat{H}_i, \hat{V}_u, \hat{V}_i)$, of (1.1)–(1.3) when $R_0 > 1$. By lemma 2.1, $\hat{V}_u + \hat{V}_i = \hat{V}$. So $(\hat{H}_i, \hat{V}_i)$ is a positive solution of (3.2), which is unique by lemma 3.5. Hence, E_2 is the unique EE of (1.1)–(1.3).

Let $(H_{i0}, V_{u0}, V_{i0}) \in X_0$. Then $V_{u0} + V_{i0} \neq 0$ and $H_{i0} + V_{i0} \neq 0$. By lemma 2.1, we have $V_u(\cdot,t) + V_i(\cdot,t) \to \hat{V}$ in $C(\bar{\Omega};\mathbb{R})$ as $t \to \infty$. By lemma 3.11, there exists $\epsilon > 0$ such that

$$\liminf_{t\to\infty} \|H_i(\cdot,t)\|_\infty + \|V_i(\cdot,t)\|_\infty \geqslant \epsilon. \tag{3.9}$$

We focus on the first and third equations of (1.1) and rewrite them as:

$$\begin{cases} \frac{\partial}{\partial t}H_i - \nabla\cdot\delta_1\nabla H_i = -\lambda H_i + \sigma_1 H_u V_i, & x \in \Omega, t > 0, \\ \frac{\partial}{\partial t}V_i - \nabla\cdot\delta_2\nabla V_i = \sigma_2(\hat{V} - V_i)^+ H_i - \mu\hat{V}V_i + F(x,t), & x \in \Omega, t > 0, \\ \frac{\partial}{\partial n}H_i = \frac{\partial}{\partial n}V_i = 0, & x \in \partial\Omega, t > 0, \\ H_i(x,0) = H_{i0}(x),\ V_i(x,0) = V_{i0}(x), & x \in \Omega, \end{cases} \tag{3.10}$$

where

$$F(\cdot,t) = \sigma_2(V_u(\cdot,t) - (\hat{V} - V_i(\cdot,t))^+)H_i - \mu(V_u(\cdot,t) + V_i(\cdot,t) - \hat{V})V_i(\cdot,t).$$

Since

$$|V_u(\cdot,t) - (\hat{V} - V_i(\cdot,t))^+| \leqslant |V_u(\cdot,t) + V_i(\cdot,t) - \hat{V}|,$$

we have $F(\cdot,t) \to 0$ in $C(\bar{\Omega};\mathbb{R})$ as $t \to \infty$. Then by [24, proposition 1.1], (3.10) is asymptotically autonomous with limit system (3.1). By (3.9), the ω−limit set of (3.10) is contained in $W := \{(H_i, V_i) \in C(\bar{\Omega};\mathbb{R}^2_+) :\ H_i + V_i \neq 0\}$. By lemma 3.9, W is the stable set (or basin of attraction) of the equilibrium $(\hat{H}_i, \hat{V}_i)$ of (3.1). Hence by the theory of asymptotically autonomous semiflows (originally due to Markus. See [31, theorem 4.1] for the generalization to asymptotically autonomous semiflows), we have $(H_i(\cdot,t), V_i(\cdot,t)) \to (\hat{H}_i, \hat{V}_i)$ in $C(\bar{\Omega};\mathbb{R}^2)$ as $t \to \infty$. Moreover, by $V_u(\cdot,t) + V_i(\cdot,t) \to \hat{V}$ and $\hat{V}_i + \hat{V}_u = \hat{V}$, we have $V_u(\cdot,t) \to \hat{V}_u$ in $C(\bar{\Omega};\mathbb{R})$ as $t \to \infty$. This completes the proof. ∎

4. Global stability when $R_0 = 1$

In this section, we prove the global stability of E_1 for the critical case $R_0 = 1$. The following result is well known. Since we can not locate a reference and for the convenience of readers, we attach a proof.

Lemma 4.1. *The positive equilibrium $\hat{V}$ of (2.1) is exponentially asymptotically stable.*

Proof. It is easy to see that $\hat{V}$ is locally asymptotically stable. To see this, linearizing (2.1) around $\hat{V}$, we obtain

$$\begin{cases} \kappa\phi = \nabla\cdot\delta_2\nabla\phi + \beta\phi - 2\mu\hat{V}\phi, & x \in \Omega, \\ \frac{\partial}{\partial n}\phi = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Since $\hat{V}$ satisfies (2.3), we have $\kappa_1(\delta_2, \beta - \mu\hat{V}) = 0$. Hence $a := \kappa_1(\delta_2, \beta - 2\mu\hat{V}) < 0$, i.e. the principal eigenvalue of (4.1) is negative. Therefore, $\hat{V}$ is linearly stable. By the principle of linearized stability, it is locally asymptotically stable.

Let $\epsilon > 0$ be given. Since $\hat{V}$ is locally asymptotically stable, there exists $\delta > 0$ such that $\|V(\cdot, t) - \hat{V}\|_\infty < \epsilon$ for all $V_0 \in C(\bar{\Omega}; \mathbb{R}_+)$ with $\|V_0 - \hat{V}\|_\infty < \delta$. Let $w(\cdot, t) = V(\cdot, t) - \hat{V}$. Then w satisfies

$$\begin{cases} w_t = \nabla\cdot\delta_2\nabla w + (\beta - 2\mu\hat{V})w - 2\mu w^2, & x \in \Omega, t > 0, \\ \frac{\partial}{\partial n}w = 0, & x \in \partial\Omega, t > 0, \\ w(x, 0) = V_0 - \hat{V}, & x \in \Omega. \end{cases} \tag{4.2}$$

Let $S(t)$ be the semigroup generated by $\nabla\cdot\delta_2\nabla + (\beta - 2\mu\hat{V})$ (associated with Neumann boundary condition) in $C(\bar{\Omega}; \mathbb{R})$. Then there exists $M_1 > 0$ such that $\|S(t)\| \leqslant M_1 \mathrm{e}^{-at}$ for all $t \geqslant 0$. Then by (4.2), we have

$$w(\cdot, t) = S(t)w(\cdot, 0) - \int_0^t S(t-s)\mu w(\cdot, s)^2 \mathrm{d}s.$$

It then follows that

$$\begin{aligned} \|w(\cdot, t)\|_\infty &\leqslant \|S(t)w(\cdot, 0)\|_\infty + \int_0^t \|S(t-s)\mu w(\cdot, s)^2\|_\infty \mathrm{d}s \\ &\leqslant M_1 \mathrm{e}^{-at}\|w(\cdot, 0)\|_\infty + \epsilon M_1 \|\mu\|_\infty \int_0^t \mathrm{e}^{-a(t-s)}\|w(\cdot, t)\|_\infty \mathrm{d}s. \end{aligned}$$

By the Gronwall's inequality, if $\epsilon \leqslant a/2\|\mu\|_\infty M_1$, we have

$$\|w(\cdot, t)\|_\infty \leqslant M_1\|V_0 - \hat{V}\|_\infty \mathrm{e}^{(M_1\|\mu\|_\infty \epsilon - a)t} \leqslant M_1\|V_0 - \hat{V}\|_\infty \mathrm{e}^{-at/2}.$$

Therefore, $\hat{V}$ is exponentially asymptotically stable. ∎

We then prove the local stability of E_1 when $R_0 = 1$.

Lemma 4.2. *If $R_0 = 1$, then E_1 is locally stable.*

Proof. Let $\epsilon > 0$ be given. Denote $V = V_u + V_i$. By lemma 4.1, there exist $\delta, M_1, b > 0$ such that, if $\|V_{u0} + V_{i0} - \hat{V}\|_\infty < 2\delta$, then

$$\|V - \hat{V}\|_\infty \leqslant M_1 \|V_{u0} + V_{i0} - \hat{V}\|_\infty \mathrm{e}^{-bt}. \tag{4.3}$$

Suppose that (H_{i0}, V_{u0}, V_{i0}) satisfies $\|H_{i0}\|_\infty \leqslant \delta$, $\|V_{u0} - \hat{V}\|_\infty \leqslant \delta$ and $\|V_{i0}\|_\infty \leqslant \delta$ such that (4.3) holds.

Since κ_0 has the same sign with $R_0 - 1$, we have $\kappa_0 = 0$. Let $T(t)$ be the positive semi-group generated by $A = B + C$ in $C(\bar{\Omega}; \mathbb{R}^2)$. Then there exists $M_2 > 0$ such that $\|T(t)\| \leqslant M_2$ for all $t \geqslant 0$ [39, propositon 4.15]. By (1.1)–(1.3), we have

$$\begin{aligned}\begin{pmatrix} H_i(\cdot,t) \\ V_i(\cdot,t) \end{pmatrix} &= T(t)\begin{pmatrix} H_{i0} \\ V_{i0} \end{pmatrix} + \int_0^t T(t-s)\begin{pmatrix} 0 \\ \sigma_2(V_u(\cdot,s) - \hat{V})H_i(\cdot,s) - \mu(V(\cdot,s) - \hat{V})V_i(\cdot,s) \end{pmatrix} \mathrm{d}s \\ &\leqslant T(t)\begin{pmatrix} H_{i0} \\ V_{i0} \end{pmatrix} + \int_0^t T(t-s)\begin{pmatrix} 0 \\ \sigma_2(V(\cdot,s) - \hat{V})H_i(\cdot,s) - \mu(V(\cdot,s) - \hat{V})V_i(\cdot,s) \end{pmatrix} \mathrm{d}s.\end{aligned}$$

Let $u(t) = \max\{\|H_i(\cdot,t)\|_\infty, \|V_i(\cdot,t)\|_\infty\}$. By (4.3), we have

$$\begin{aligned} u(t) &\leqslant M_2 u(0) + 2M_2 \max\{\|\sigma_2\|_\infty, \|\mu\|_\infty\} \int_0^t \|V(\cdot,s) - \hat{V}\|_\infty u(s)\mathrm{d}s \\ &\leqslant M_2\delta + \delta C \int_0^t \mathrm{e}^{-bs} u(s)\mathrm{d}s \end{aligned}$$

where $C = 4M_1M_2 \max\{\|\sigma_2\|_\infty, \|\mu\|_\infty\}$. Then by Gronwall's inequality,

$$u(t) = \max\{\|H_i(\cdot,t)\|_\infty, \|V_i(\cdot,t)\|_\infty\} \leqslant M_2 \mathrm{e}^{C\delta/b}\delta. \tag{4.4}$$

Moreover, by (4.3), we have

$$\|V_u(\cdot,t) - \hat{V}\|_\infty \leqslant \|V_u(\cdot,t) + V_i(\cdot,t) - \hat{V}\|_\infty + \|V_i(\cdot,t)\|_\infty \leqslant 2M_1\delta + M_2\mathrm{e}^{C\delta/b}\delta. \tag{4.5}$$

Combining (4.4) and (4.5), we can find $\delta = \delta(\epsilon) > 0$ such that

$$\|H_i(\cdot,t)\|_\infty \leqslant \epsilon, \|V_u(\cdot,t) - \hat{V}\|_\infty \leqslant \epsilon, \text{ and } \|V_i(\cdot,t)\|_\infty \leqslant \epsilon.$$

Since $\epsilon > 0$ is arbitrary, E_1 is locally stable. ■

We then prove the global attractivity of E_1 when $R_0 = 1$.

Theorem 4.3. *If $R_0 = 1$, then E_1 is globally stable in the sense that it is locally stable and, for any nonnegative initial data (H_{i0}, V_{u0}, V_{i0}) with $V_{u0} + V_{i0} \neq 0$,*

$$\lim_{t\to\infty} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - E_1\|_\infty = 0.$$

Proof. Let

$$\mathbb{M} = \{(H_{i0}, V_{u0}, V_{i0}) \in C(\bar{\Omega}; \mathbb{R}^3_+) : \ V_{u0} + V_{i0} = \hat{V}\}.$$

It suffices to show: (a) E_1 is a locally stable equilibrium of (1.1)–(1.3); (b) the stable set (or basin of attraction) of E_1 contains $\mathbb{M}$; (c) the $\omega-$limit set of (H_{i0}, V_{u0}, V_{i0}) with $V_{u0} + V_{i0} \neq 0$ is contained in $\mathbb{M}$.

By lemma 4.2, E_1 is locally stable, which gives (a). If $V_{u0} + V_{i0} \neq 0$, we have $V_u(\cdot, t) + V_i(\cdot, t) \to \hat{V}$ in $C(\bar{\Omega}; \mathbb{R})$ as $t \to \infty$, which implies (c).

To prove (b), suppose $(H_{i0}, V_{u0}, V_{i0}) \in \mathbb{M}$. Then the solution of (1.1)–(1.3) satisfies $V_u(x, t) + V_i(x, t) = \hat{V}(x)$ for all $x \in \bar{\Omega}$ and $t \geqslant 0$. Hence $(H_i(x, t), V_i(x, t))$ is the solution of the limit problem (3.1).

Since $R_0 = 1$, we have $\kappa_0 = 0$. Let (φ_0, ϕ_0) be a positive eigenvector associated with κ_0 of the eigenvalue problem (2.5). Motivated by [7, 40], for any $w_0 := (H_{i0}, V_{i0})$, we define

$$c(t; w_0) := \inf\{\tilde{c} \in \mathbb{R} : H_i(\cdot, t) \leqslant \tilde{c}\varphi_0 \text{ and } V_i(\cdot, t) \leqslant \tilde{c}\phi_0\}.$$

Then $c(t;w_0) > 0$ for all $t > 0$. We now claim that $c(t;w_0) > 0$ is strictly decreasing. To see that, fix $t_0 > 0$, and we define $\bar{H}_i(x, t) = c(t_0; w_0)\varphi_0(x)$ and $\bar{V}_i(x, t) = c(t_0; w_0)\phi_0(x)$ for all $t \geqslant t_0$ and $x \in \bar{\Omega}$. Then $(\bar{H}_i(x, t), \bar{V}_i(x, t))$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}\bar{H}_i - \nabla \cdot \delta_1 \nabla \bar{H}_i = -\lambda \bar{H}_i + \sigma_1 H_u \bar{V}_i, & x \in \Omega, t \geqslant t_0, \\ \frac{\partial}{\partial t}\bar{V}_i - \nabla \cdot \delta_2 \nabla \bar{V}_i > \sigma_2(\hat{V} - \bar{V}_i)^+ \bar{H}_i - \mu \hat{V} \bar{V}_i, & x \in \Omega, t \geqslant t_0, \\ \frac{\partial}{\partial n}\bar{H}_i = \frac{\partial}{\partial n}\bar{V}_i = 0, & x \in \partial\Omega, t \geqslant t_0, \\ \bar{H}_i(\cdot, t_0) \geqslant H_i(\cdot, t_0), \quad \bar{V}_i(\cdot, t_0) \geqslant V_i(\cdot, t_0). \end{cases} \tag{4.6}$$

By the comparison principle for cooperative systems, we have $(\bar{H}_i(x, t), \bar{V}_i(x, t)) \geqslant (H_i(x, t), V_i(x, t))$ for all $x \in \bar{\Omega}$ and $t \geqslant t_0$. By the second equation of (4.6), we have

$$\frac{\partial}{\partial t}\bar{V}_i - \nabla \cdot \delta_2 \nabla \bar{V}_i > \sigma_2(\hat{V} - \bar{V}_i)^+ H_i - \mu \hat{V} \bar{V}_i.$$

By the comparison principle, $\bar{V}_i(x, t) > V_i(x, t)$ for all $x \in \bar{\Omega}$ and $t > t_0$. Then by the first equation of (4.6),

$$\frac{\partial}{\partial t}\bar{H}_i - \nabla \cdot \delta_1 \nabla \bar{H}_i \geqslant -\lambda \bar{H}_i + \sigma_1 H_u V_i,$$

where the inequality is strict for some $x \in \bar{\Omega}$ as H_u is nontrivial. By the comparison principle, we have $\bar{H}_i(x, t) > H_i(x, t)$ for all $x \in \bar{\Omega}$ and $t > t_0$. Therefore, $c(t_0; w_0)\varphi_0(x) > H_i(x, t)$ and $c(t_0; w_0)\phi_0(x) > V_i(x, t)$ for all $x \in \bar{\Omega}$ and $t > t_0$. By the definition of $c(t;w_0)$, $c(t_0; w_0) > c(t; w_0)$ for all $t > t_0$. Since $t_0 \geqslant 0$ is arbitrary, $c(t;w_0)$ is strictly decreasing for $t \geqslant 0$.

Let $\tilde{\Phi}(t)$ be the semiflow induced by the solution of the limit problem (3.1). Let $\omega := \omega(w_0)$ be the omega limit set of w_0. We claim that $\omega = \{(0, 0)\}$. Assume to the contrary that there exists a nontrivial $w_1 \in \omega$. Then there exists $\{t_k\}$ with $t_k \to \infty$ such that $\tilde{\Phi}(t_k)w_0 \to w_1$. Let $c_* = \lim_{t \to \infty} c(t; w_0)$. We have $c(t; w_1) = c_*$ for all $t \geqslant 0$. Actually this follows from the fact that $\tilde{\Phi}(t)w_1 = \tilde{\Phi}(t)\lim_{t_k \to \infty} \tilde{\Phi}(t_k)w_0 = \lim_{t_k \to \infty} \tilde{\Phi}(t + t_k)w_0$. However since w_1 is nontrivial, we can repeat the previous arguments to show that $c(t;w_1)$ is strictly decreasing. This is a contraction. Therefore $\omega = \{(0, 0)\}$, and $(H_i(\cdot, t), V_i(\cdot, t)) \to (0, 0)$ in $C(\bar{\Omega}; \mathbb{R}^2)$ as $t \to \infty$. Since $V_u(\cdot, t) + V_i(\cdot, t) = \hat{V}$, we have $V_u(\cdot, t) \to \hat{V}$ in $C(\bar{\Omega}; \mathbb{R})$ as $t \to \infty$. This completes the proof. ■

5. Concluding remarks

In this paper, we define a basic reproduction number R_0 for the model (1.1)–(1.3), and show that it serves as the threshold value for the global dynamics of the model: if $R_0 \leqslant 1$, then disease free equilibrium E_1 is globally asymptotically stable; if $R_0 > 1$, the model has a unique endemic equilibrium E_2, which is globally attractive.

As shown in theorem A.4, the global dynamics of the corresponding ODE model of (1.1)–(1.3) is determined by the magnitude of $\sigma_1\sigma_2 H_u/\lambda\mu$. This motivates us to define the local basic reproduction number for model (1.1)–(1.3):

$$R(x) := R_1(x)R_2(x) = \frac{\sigma_1(x)H_u(x)}{\lambda(x)}\frac{\sigma_2(x)}{\mu(x)}.$$

Since R_0 is difficult to visualize, it is natural to ask: are there any connections between R_0 and R? As the global dynamics of both models are determined by the magnitude of the basic reproduction number, this is equivalent to ask: how the diffusion rates change the dynamics of the model (1.1)–(1.3), and what is the relation between the reaction–diffusion model (1.1)–(1.3) and the corresponding reaction system (the model without diffusion)? We will explore these questions in a forthcoming paper. Our main ingredient is the formula:

$$R_0 = r(L_1R_1L_2R_2)$$

with $L_1 := (\lambda - \nabla \cdot \delta_1 \nabla)^{-1}\lambda$ and $L_2 := (\mu\hat{V} - \nabla \cdot \delta_2\nabla)^{-1}\mu\hat{V}$. This formula establishes an interesting connection between R_0 and R as we can prove

$$r(L_1L_2) = r(L_1) = r(L_2) = 1.$$

Consequences of this formula are:

1. If $R_i(x)$, $i = 1, 2$, is constant, then $R_0 = R$;
2. $R_0 > 1$ if $R_i(x) > 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$ and $R_0 < 1$ if $R_i(x) < 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$.

Furthermore, when the diffusion coefficients δ_1 and δ_2 are constant, we prove

- $\lim_{(\delta_1,\delta_2)\to(\infty,\infty)} R_0 = \frac{\int_\Omega \lambda R_1 \mathrm{d}x}{\int_\Omega \lambda \mathrm{d}x}\frac{\int_\Omega \mu R_2 \mathrm{d}x}{\int_\Omega \mu \mathrm{d}x}$;
- $\lim_{\delta_1\to 0}\lim_{\delta_2\to 0} R_0 = \lim_{\delta_2\to 0}\lim_{\delta_1\to 0} R_0 = \max\{R(x) : x \in \bar{\Omega}\}$.

Finally, we remark that our approach is applicable to several other reaction–diffusion models (e.g. [18, 19, 26, 28]). For example, the reaction–diffusion within-host model of viral dynamics studied in [26, 28] is

$$\begin{cases} \frac{\partial}{\partial t}T - \nabla \cdot \delta_1(x)\nabla T = \lambda(x) - \mu T - k_1TV(-k_2TI), & x \in \Omega, t > 0, \\ \frac{\partial}{\partial t}I - \nabla \cdot \delta_2(x)\nabla I = k_1TV(+k_2TI) - \mu_i I & x \in \Omega, t > 0, \\ \frac{\partial}{\partial t}V - \nabla \cdot \delta_3(x)\nabla V = N(x)I - \mu_v V, & x \in \Omega, t > 0, \end{cases} \tag{5.1}$$

where T, I and V denote the densities of healthy cells, infected cells and virions, respectively. If $\delta_1 = \delta_2$ and $\mu = \mu_i$, then $E := T + I$ satisfies

$$\frac{\partial}{\partial t}E - \nabla \cdot \delta_1(x)\nabla E = \lambda(x) - \mu E.$$

This equation has a unique positive equilibrium $\hat{E}$ and $E(\cdot, t) \to \hat{E}$ in $C(\bar{\Omega})$ as $t \to \infty$. Therefore (5.1) also has a limit system which is monotone:

$$\begin{cases} \frac{\partial}{\partial t}I - \nabla \cdot \delta_2(x)\nabla I = k_1(\hat{E} - I)^+ V(+k_2(\hat{E} - I)^+ I) - \mu_i I & x \in \Omega, t > 0, \\ \frac{\partial}{\partial t}V - \nabla \cdot \delta_3(x)\nabla V = N(x)I - \mu_v V, & x \in \Omega, t > 0. \end{cases}$$

For the models in [18, 19], our method is applicable when there are no chemotaxis. The analysis of the basic reproduction number of all these models can also be done similarly.

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Appendix

Let $H_i(t), V_u(t)$ and $V_i(t)$ be the densities of infected hosts, uninfected vectors, and infected vectors at time t respectively. Then the model is

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}H_i(t) = -\lambda H_i(t) + \sigma_1 H_u V_i(t), & t > 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}V_u(t) = -\sigma_2 V_u(t)H_i(t) + \beta(V_u(t) + V_i(t)) - \mu(V_u(t) + V_i(t))V_u(t), & t > 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}V_i(t) = \sigma_2 V_u(t)H_i(t) - \mu(V_u(t) + V_i(t))V_i(t), & t > 0 \end{cases} \tag{A.1}$$

with initial value

$$(H_i(0), V_u(0), V_i(0)) \in M := \mathbb{R}^3_+.$$

The basic reproduction number R_0 is defined as

$$R_0 := \frac{\sigma_1\sigma_2 H_u}{\lambda\mu}.$$

The equilibria of (A.1) are $ss_0 = (0{,}0{,}0)$, $ss_1 = (0, \beta/\mu, 0)$, and

$$\begin{aligned} ss_2 &= \left(\frac{\beta(H_u\sigma_1\sigma_2 - \lambda\mu)}{\lambda\mu\sigma_2}, \frac{\beta\lambda}{H_u\sigma_1\sigma_2}, \frac{\beta(H_u\sigma_1\sigma_2 - \lambda\mu)}{H_u\mu\sigma_1\sigma_2}\right) \\ &= \left(\frac{\beta(R_0 - 1)}{\sigma_2}, \frac{\beta}{R_0\mu}, \frac{\lambda\beta(R_0 - 1)}{H_u\sigma_1\sigma_2}\right) \\ &:= (\hat{H}_i, \hat{V}_u, \hat{V}_i), \end{aligned}$$

which exists if and only if $R_0 > 1$.

If we add the last two equations of (A.1) then $N(t) := V_u(t) + V_i(t)$ satisfies the logistic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \beta N(t) - \mu N^2(t). \tag{A.2}$$

We decompose the domain $M := \mathbb{R}^3_+$ into the partition

$$M = \partial M_0 \cup M_0,$$

where

$$\partial M_0 := \{(H_i, V_u, V_i) \in M : H_i + V_i = 0 \text{ or } V_u + V_i = 0\}$$

and

$$M_0 := \{(H_i, V_u, V_i) \in M : H_i + V_i > 0 \text{ and } V_u + V_i > 0\} = M \setminus \partial M_0.$$

Biologically, we can interpret ∂M_0 as the states without vectors or infected individuals. The subregions ∂M_0 and M_0 are both positively invariant with respect to the semiflow generated by (A.1). We can also decompose M with respect to the subdomain

$$\partial M_1 := \{(H_i, V_u, V_i) \in M : V_u + V_i = 0\}$$

and

$$M_1 := \{(H_i, V_u, V_i) \in M : V_u + V_i > 0\}.$$

Since $N(t) := V_u(t) + V_i(t)$ always satisfies the logistic equation (A.2), the subregions ∂M_1 and M_1 are both positively invariant with respect to the semiflow generated by (A.1).

Lemma A.1. *Both ∂M_1 and M_1 are positively invariant by the semiflow generated (A.1). Moreover,*

1. *if $(H_i(0), V_u(0), V_i(0)) \in \partial M_1$, then*
$$\lim_{t\to\infty} (H_i(t), V_u(t), V_i(t)) = (0, 0, 0);$$
2. *if $(H_i(0), V_u(0), V_i(0)) \in M_1$, then*
$$\lim_{t\to\infty} V_u(t) + V_i(t) = \frac{\beta}{\mu}.$$

If $(H_i(0), V_u(0), V_i(0)) \in M_1$ the long time behavior of (A.1) is characterized by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}H_i(t) = -\lambda H_i + \sigma_1 H_u V_i, & t > 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}V_i(t) = \sigma_2(\beta/\mu - V_i)^+ H_i - \beta V_i, & t > 0, \\ H_i(0) = H_{i0} \geqslant 0, \ V_i(0) = V_{i0} \geqslant 0. \end{cases} \tag{A.3}$$

Lemma A.2. *Suppose $R_0 > 1$. Then (A.3) has a unique positive equilibrium $(\hat{H}_i, \hat{V}_i)$. Moreover, $(\hat{H}_i, \hat{V}_i)$ is locally asymptotically stable, and if $H_{i0} + V_{i0} \neq 0$, then the solution (H_i, V_i) of (A.3) satisfies*

$$\lim_{t\to\infty} (H_i(t), V_i(t)) = (\hat{H}_i, \hat{V}_i).$$

Proof. The uniqueness of the positive equilibrium $(\hat{H}_i, \hat{V}_i)$ can be checked directly when $R_0 > 1$. Let $D = \mathbb{R}^2_+$. Then D is invariant for (A.3). It is not hard to show that the solution of (A.3) is bounded.

Let $F_1(H_i, V_i) = -\lambda H_i + \sigma_1 H_u V_i$ and $F_2(H_i, V_i) = \sigma_2(\beta/\mu - V_i)^+ H_i - \beta V_i$. Then $\partial F_1/\partial V_i \geqslant 0$ and $\partial F_2/\partial H_i \geqslant 0$ on D. So (A.3) is cooperative. Let $\tilde{\Phi}(t) : D \to D$ be the semiflow generated by the solution of (A.3). Then $\tilde{\Phi}(t)$ is monotone.

If $H_{i0} + V_{i0} \neq 0$, then $H_i(t) > 0$ and $V_i(t) > 0$ for all $t > 0$. So without loss of generality, we may assume $H_{i0} > 0$ and $V_{i0} > 0$. We can choose δ small such that $F_1(\delta\hat{H}_i, \delta\hat{V}_i) \geqslant 0$, $F_2(\delta\hat{H}_i, \delta\hat{V}_i) \geqslant 0$, $H_{i0} \geqslant \delta\hat{H}_i$, and $V_{i0} \geqslant \delta\hat{V}_i$. By [27, proposition 3.2.1], $\tilde{\Phi}(t)(\delta\hat{H}_i, \delta\hat{V}_i)$ is nondecreasing for $t \geqslant 0$ and converges to a positive equilibrium as $t \to \infty$. Since $(\hat{H}_i, \hat{V}_i)$ is the unique positive equilibrium, we must have $\tilde{\Phi}(t)(\delta\hat{H}_i, \delta\hat{V}_i) \to (\hat{H}_i, \hat{V}_i)$ as $t \to \infty$.

Similarly, we may choose $k > 0$ such that $F_1(k\hat{H}_i, k\hat{V}_i) \leqslant 0$, $F_2(k\hat{H}_i, k\hat{V}_i) \leqslant 0$, $H_{i0} \leqslant k\hat{H}_i$, and $V_{i0} \leqslant k\hat{V}_i$. Then $\tilde{\Phi}(t)(\delta\hat{H}_i, \delta\hat{V}_i)$ is non-increasing for $t \geqslant 0$ and $\tilde{\Phi}(t)(k\hat{H}_i, k\hat{V}_i) \to (\hat{H}_i, \hat{V}_i)$ as $t \to \infty$. By the monotonicity of $\tilde{\Phi}(t)$, we have $\tilde{\Phi}(t)(\delta\hat{H}_i, \delta\hat{V}_i) \leqslant \tilde{\Phi}(t)(H_{i0}, V_{i0}) \leqslant \tilde{\Phi}(t)(k\hat{H}_i, k\hat{V}_i)$ for $t \geqslant 0$. It then follows that $\tilde{\Phi}(t)(H_{i0}, V_{i0}) \to (\hat{H}_i, \hat{V}_i)$ as $t \to \infty$.

For any $\epsilon' > 0$ and initial data (H_{i0}, V_{i0}) satisfying $(1-\epsilon')(\hat{H}_i, \hat{V}_i) \leqslant (H_{i0}, V_{i0}) \leqslant (1+\epsilon')(\hat{H}_i, \hat{V}_i)$, similar to the previous arguments, we can show $(1-\epsilon')(\hat{H}_i, \hat{V}_i) \leqslant (H_i(t), V_i(t)) \leqslant (1+\epsilon')(\hat{H}_i, \hat{V}_i)$ for all $t \geqslant 0$. Therefore, $(\hat{H}_i, \hat{V}_i)$ is locally stable. This proves the lemma. ■

We now present a uniform persistence result.

Lemma A.3. *If $R_0 > 1$, then the semiflow generated by (A.1) is uniformly persistent with respect to $(M_0, \partial M_0)$ in the sense that there exists $\epsilon > 0$ such that, for any $(H_i(0), V_u(0), V_i(0)) \in M_0$, we have*

$$\liminf_{t\to\infty} \inf_{w\in\partial M_0} |(H_i(t), V_u(t), V_i(t)) - w| \geqslant \epsilon. \tag{A.4}$$

Proof. We apply [15, theorem 4.1] to prove this result. Let $\Phi(t) : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ be the semiflow generated by (A.1), i.e. $\Phi(t)w_0 = (H_i(t), V_u(t), V_i(t))$ for $t \geqslant 0$, where $(H_i(t), V_u(t), V_i(t))$ is the solution of (A.1) with initial condition $w_0 = (H_i(0), V_u(0), V_i(0)) \in \mathbb{R}^3_+$.

The semiflow $\Phi(t)$ is point dissipative in the sense that there exists $M > 0$ such that $\limsup_{t\to\infty} \|\Phi(t)w_0\| \leqslant M$ for any $w_0 \in \mathbb{R}^3_+$. Actually, lemma A.1 implies that $\limsup_{t\to\infty} V_u(t) \leqslant \beta/\mu$ and $\limsup_{t\to\infty} V_i(t) \leqslant \beta/\mu$. By the first equation of (A.1), we have $\limsup_{t\to\infty} H_i(t) \leqslant \sigma_1\beta H_u/\mu\lambda$.

We note that M_0 and ∂M_0 are both invariant with respect to $\Phi(t)$. Moreover, the semiflow $\Phi_\partial(t) := \Phi(t)|_{\partial M_0}$, i.e. the restriction of $\Phi(t)$ on ∂M_0, admits a compact global attractor A_∂. If $w_0 = (H_i(0), V_u(0), V_i(0)) \in \partial M_0$, then the $\omega-$limit set of w_0 is $\omega(w_0) = \{ss_0\}$ if $w_0 \in \partial M_1$ and $\omega(w_0) = \{ss_1\}$ if $w_0 \in \partial M_0 \setminus \partial M_1$. Hence we have

$$\tilde{A}_\partial := \cup_{w_0\in A_\partial}\omega(w_0) = \{ss_0\} \cup \{ss_1\}.$$

This covering is acyclic since $\{ss_1\} \not\to \{ss_0\}$, i.e. $W^u(ss_1) \cap W^s(ss_0) = \varnothing$. To see this, suppose $w_0 = (H_i(0), V_u(0), V_i(0)) \in W^u(ss_1) \cap W^s(ss_0)$. Since $w_0 \in W^s(ss_0)$, we have $w_0 \in \partial M_1$. Let $(H_i(t), V_u(t), V_i(t))$ be the complete orbit through w_0, then $V_u(t) = V_i(t) = 0$ for $t \in \mathbb{R}$. So $w_0 \notin W^u(ss_1)$, which is a contradiction.

We then show that $W^s(ss_0) \cap M_0 = \varnothing$ and $W^s(ss_1) \cap M_0 = \varnothing$. By lemma A.1, $W^s(ss_0) = \partial M_1 \subseteq \partial M_0$, and hence $W^s(ss_0) \cap M_0 = \varnothing$. To see $W^s(ss_1) \cap M_0 = \varnothing$, it suffices to prove that there exists $\epsilon > 0$ such that, for any $w_0 = (H_i(0), V_u(0), V_i(0)) \in M_0$, the following inequality holds

$$\limsup_{t\to\infty} |\Phi(t)w_0 - ss_1| \geqslant \epsilon. \tag{A.5}$$

Assume to the contrary that (A.5) does not hold. Let $\epsilon_0 > 0$ be given. Then there exists $w_0 \in M_0$ such that

$$\limsup_{t\to\infty} |\Phi(t)w_0 - ss_1| < \epsilon_0.$$

So there exists $t_0 > 0$ such that $\beta/\mu - \epsilon_0 \leqslant V_u(t) \leqslant \beta/\mu + \epsilon_0$ and $V_i(t) \leqslant \epsilon_0$ for $t \geqslant t_0$.

By (A.1), we have

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}H_i(t) = -\lambda H_i + \sigma_1 H_u V_i, & t > t_0, \\ \frac{\mathrm{d}}{\mathrm{d}t}V_i(t) \geqslant \sigma_2(\frac{\beta}{\mu} - \epsilon_0)H_i - \mu(\frac{\beta}{\mu} + 2\epsilon_0)V_i, & t > t_0. \end{cases} \tag{A.6}$$

The matrix associated with the right hand side of (A.6) is

$$A_{\epsilon_0} := \begin{bmatrix} -\lambda & \sigma_1 H_u \\ \sigma_2(\frac{\beta}{\mu} - \epsilon_0) & -\mu(\frac{\beta}{\mu} + 2\epsilon_0) \end{bmatrix},$$

whose eigenvalues λ_1 and λ_2 satisfy that $\lambda_1 + \lambda_2 = -\lambda - \mu(\beta/\mu + 2\epsilon_0) < 0$ and $\lambda_1\lambda_2 = \lambda\mu(\beta/\mu + 2\epsilon_0) - \sigma_1\sigma_2 H_u(\beta/\mu - \epsilon_0)$. Since $R_0 > 1$, we can choose ϵ_0 small such that $\lambda_1\lambda_2 < 0$. Hence either $\lambda_1 > 0 > \lambda_2$ or $\lambda_2 > 0 > \lambda_1$. Without loss of generality, suppose $\lambda_1 > 0 > \lambda_2$. Then by the Perron–Frobenius theorem, there is an eigenvector (ϕ, ψ) associated with λ_1 such that $\phi > 0$ and $\psi > 0$.

Let $(\tilde{H}_i(t), \tilde{V}_i(t))$ be the solution of the following problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}_i(t) = -\lambda \tilde{H}_i(t) + \sigma_1 H_u \tilde{V}_i(t), & t > t_0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\tilde{V}_i(t) = \sigma_2(\frac{\beta}{\mu} - \epsilon_0)\tilde{H}_i(t) - \mu(\frac{\beta}{\mu} + 2\epsilon_0)\tilde{V}_i(t), & t > t_0, \\ \tilde{H}_i(t_0) = \delta\phi, \tilde{V}_i(t_0) = \delta\psi, \end{cases} \tag{A.7}$$

where δ is small such that $H_i(t_0) \geqslant \tilde{H}_i(t_0)$ and $V_i(t_0) \geqslant \tilde{V}_i(t_0)$. By (A.6) and the comparison principle for cooperative systems, we have $(H_i(t), V_i(t)) \geqslant (\tilde{H}_i(t), \tilde{V}_i(t))$ for $t \geqslant t_0$. We can check that the solution of (A.7) is $(\tilde{H}_i(t), \tilde{V}_i(t)) = (\delta\phi \mathrm{e}^{\lambda_1(t-t_0)}, \delta\psi \mathrm{e}^{\lambda_1(t-t_0)})$. It then follows from $\lambda_1 > 0$ that $\lim_{t\to\infty} H_i(t) = \infty$ and $\lim_{t\to\infty} V_i(t) = \infty$, which contradicts the boundedness of the solution.

Our conclusion now just follows from [15, theorem 4.1]. ■

We now present the result about the global dynamics of (A.1).

Theorem A.4. *The following statements hold.*

1. *ss_0 is unstable; If $(H_i(0), V_u(0), V_i(0)) \in \partial M_1$, then*

$$\lim_{t\to\infty} (H_i(t), V_u(t), V_i(t)) = ss_0.$$

2. *Suppose $R_0 < 1$. Then ss_1 is globally asymptotically stable, i.e. ss_1 is locally asymptotically stable and if $(H_i(0), V_u(0), V_i(0)) \in M_1$, then*

$$\lim_{t\to\infty} (H_i(t), V_u(t), V_i(t)) = ss_1.$$

3. *Suppose $R_0 > 1$. Then ss_1 is unstable, and if $(H_i(0), V_u(0), V_i(0)) \in \partial M_0 \setminus \partial M_1$, then*

$$\lim_{t\to\infty} (H_i(t), V_u(t), V_i(t)) = ss_1.$$

Moreover, ss_2 is globally asymptotically stable in the sense that ss_2 is locally asymptotically stable and for any $(H_i(0), V_u(0), V_i(0)) \in M_0$,

$$\lim_{t\to\infty} (H_i(t), V_u(t), V_i(t)) = ss_2.$$

Proof. We only prove the second convergence result in part 3 (see [14] and lemma A.1 for the other parts). Since the solution of (A.3) is bounded, the omega limit set of the solution of (A.1) exists.

Suppose $(H_i(0), V_u(0), V_i(0)) \in M_0$. Then the solution $(H_i(t), V_u(t), V_i(t))$ of (A.1) satisfies that $H_i(t), V_u(t), V_i(t) > 0$ for all $t > 0$. Since $V_u(0) + V_i(0) \neq 0$, we have $V_u(t) + V_i(t) \to \beta/\mu$ as $t \to \infty$. So,

$$f(t) := \sigma_2[V_u(t) - (\beta/\mu - V_i(t))^+]H_i(t) + (\beta - \mu(V_u(t) + V_i(t)))V_i(t) \to 0 \text{ as } t \to \infty,$$

and the limit system of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}H_i(t) = -\lambda H_i + \sigma_1 H_u V_i, & t > 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}V_i(t) = \sigma_2 V_u H_i - \mu(V_u + V_i)V_i = \sigma_2(\beta/\mu - V_i)^+ H_i - \beta V_i + f(t), & t > 0, \end{cases} \tag{A.8}$$

is (A.3). By lemma A.3 and $V_u(t) + V_i(t) \to \beta/\mu$, there exists $\epsilon > 0$ such that

$$\liminf_{t\to\infty} |H_i(t)| + |V_i(t)| \geqslant \epsilon.$$

Hence the omega limit set of (A.8) is contained in $W := \{(H_{i0}, V_{i0}) \in R^2_+ : \ H_{i0} + V_{i0} \neq 0\}$. By lemma A.2, W is the stable set of the stable equilibrium $(\hat{H}_i, \hat{V}_i)$ of (A.3). By the theory of asymptotic autonomous systems, we have $H_i(t) \to \hat{H}_i$ and $V_i(t) \to \hat{V}_i$ as $t \to \infty$. Moreover since $V_u(t) + V_i(t) \to \beta/\mu = \hat{V}_u + \hat{V}_i$, we have $V_u(t) \to \hat{V}_u$ as $t \to \infty$. ∎

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