2021-5-3

On the Fitting ideals of Selmer groups of modular forms

<u>Goal</u>: Reveal another relation of L-values ~~and~~ to Selmer gps.

<u>Setup</u>. p, an odd prime.

$f = \sum_{n \geq 1} a_n(f) \cdot q^n \in S_k(\Gamma_0(N))$ normalized newform.

$p \nmid N$ "good prime".

$F_\pi = \mathbb{Q}_p(\{a_n(f) : n\})$, $\mathcal{O}_\pi$, $\mathbb{F}_\pi$ res. field, π uniformizer.

Deligne, Eichler-Shimura

$\rightsquigarrow \rho_f : G_{\mathbb{Q}} \to GL_2(F_\pi) = \mathrm{Aut}(V_f)$

: cohomological convention. $(1-k, 0)$ HT wt.

Choose a Galois-stable lattice.

$T_f \subseteq V_f$, $W_f = V_f / T_f$.

$\bar{\rho}_f : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_\pi) = \mathrm{Aut}(T_f / \pi T_f)$

: <u>abs. irred.</u>

$K/\mathbb{Q}$ fin. abel. ext'n. $\quad \bar{\rho}_f$ is abs. irred.

Mazur–Tate.

$$\longrightarrow \underline{\theta_{K, k/2}(f)} \in \underline{\underline{\mathcal{O}_\pi}}[\mathrm{Gal}(K/\mathbb{Q})].$$

$$= \sum_{a \in} \cancel{c_a} \cdot \sigma_a.$$

~~organizing~~

$f \rightsquigarrow [\cancel{\mathbb{Q}}]^{\pm}_f$

$$f \overset{\text{E-S isom.}}{\rightsquigarrow} \underline{[\cdot]_f} : \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \longrightarrow \mathrm{Sym}^{k-2}(\mathbb{C}^2)$$

$$\widetilde{[\ ]}_f : \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to \mathrm{Sym}^{k-2}(\mathcal{O}_{\mathbb{Z}}^2)$$

$K = \mathbb{Q}(\xi_m)$

$$\theta_{\mathbb{Q}(\xi_m), k/2}(f) = \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} c_a \cdot \sigma_a.$$

where $c_a \longleftrightarrow \underbrace{\int_{i\infty}^{a/m} f(z) \cdot z^{k/2}\, dz.}_{\text{period.}}$

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organizing $L(f, \chi, k/2)$

for all $\chi : \mathrm{Gal}(K/\mathbb{Q}) \to \bar{\mathbb{Q}}^\times \begin{array}{l} \overset{i_p}{\nearrow} \bar{\mathbb{Q}}_p^\times \\ \underset{i_\infty}{\searrow} \mathbb{C}_{\mathbb{R}}^\times \end{array}$.

$\rightsquigarrow$ into an elt of a group ring.

$\underline{k/2}$? Our focus lies in the <u>central critical L-values</u>.

$s = 1, \ldots, k-1. \to s = k/2.$

- χ, a primitive character of $\mathrm{Gal}(K/\mathbb{Q})$

$$\chi(\theta_{K, k/2}(f)) = \tau(\bar{\chi}) \cdot \frac{L(f, \chi, k/2)}{\Omega_f^{\pm}}$$

$\chi(-1)$ and $k/2$.

- This is an analogue of Stickelberger elts for f.

<u>Quick review of Fitting ideals.</u>

R, Noetherian ring ; M, fin. pres. mod $/R$.

<u>Def.</u>

~~$R^s \to R^r$~~

$$R^s \xrightarrow{A} R^r \longrightarrow M \longrightarrow 0.$$

$A \leftarrow$ matrix.

$$\mathrm{Fitt}_R(M) = \begin{cases} \text{the ideal gen. by det. of all } (r \times r)\text{-minor matrices of } A. & (\text{if } s \geq r) \\ 0 & (\text{if } s < r) \end{cases}.$$

- $I \subseteq R$ an ideal.

$$\mathrm{Fitt}_R(R/I) = I.$$

$$\mathrm{Fitt}_R(R/I \oplus R/I) = I^2.$$

- $\mathrm{Fitt}_R(M) \subseteq \mathrm{Ann}_R(M)$.

<u>Conj.</u> (Mazur-Tate)

$$\theta_{K,k/2}(f) \in \mathrm{Fitt}_{\mathcal{O}_\pi[G]}(\mathrm{Sel}(K, \underline{\underline{W_f}}(k/2))^\vee).$$

where $G = \mathrm{Gal}(K/\mathbb{Q})$

<u>Conj.</u> $\Rightarrow$ an analogue of "the rank part of BSD"

$$\theta_{K,k/2}(f) \in I^r \setminus I^{r+1}.$$

where $I \subseteq \mathcal{O}_\pi[G]$ is the augmentation ideal.

$r = \mathrm{cork}_{\mathcal{O}_\pi} \mathrm{Sel}(\mathbb{Q}, W_f(k/2)).$

Selmer gps.

$\exists$ arithmetic info.

$\downarrow$

$\mathrm{Sel}(K, W_f(k/2))$: Bloch-Kato Selmer gps

$\cup$

$\mathrm{Sel}_0(K, W_f(k/2))$: "fine" Selmer gps.

$\uparrow$ the p-strict local condition.

appears in the formulation of Kato's IMC.

Thm (K.)

- $2 \le k \le p-1$.
- $\mathrm{Im}(\rho_f)$ contains a conjugate of $SL_2(\mathbb{Z}_p)$

Tate conj. $\Rightarrow$

- $X^2 - a_p(f)\cdot X + p^{k-1}$ has distinct roots.
- $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is irred. (non-ordinary)
- $\mathrm{Sel}_0(\mathbb{Q}(\zeta_{p^\infty}), W_f(k/2))^\vee$ has no nontrivial finite Iwasawa submodule.

Then

$$\theta_{\mathbb{Q}(\zeta_{p^n}), k/2}(f) \in (\nu_{j,n}(\theta_{\mathbb{Q}(\zeta_{p^j}), k/2}(f)) : 0 \le j \le n)$$

$$\subseteq \mathrm{Fitt}_{\mathcal{O}_\pi[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})]}(\mathrm{Sel}(\mathbb{Q}(\zeta_{p^n}), W_f(k/2))^\vee)$$

for all $n \ge 1$.

$0 \le j \le n.$

where $\nu_{j,n} : \mathcal{O}_\lambda[\underbrace{\mathrm{Gal}(\mathbb{Q}(\zeta_{p^j})/\mathbb{Q})}_{}] \to \mathcal{O}_\lambda[\underbrace{\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})}_{}]$

$\tau \longmapsto \sum_{\sigma \mapsto \tau} \sigma$

projection.

<u>Rem</u>

- When $f \longleftrightarrow E$ and $a_p(E) = 0$, a similar result is obtained by K-Kurihara via a completely different method. ($\pm$-Iwasawa theory)

- $\exists$ a more general version. (but with error terms) without $\left(\begin{array}{l} 2 \le k \le p-1. \\ \text{the condition on } \mathrm{Sel}_0. \end{array} \right.$

If $p \le k$, we lose integrality.

"$\theta \in \mathrm{Fitt}(\mathrm{Sel}^\vee)$" $\times$.

"$p^{\square} \cdot \theta \in \mathrm{Fitt}\, \mathrm{Sel}^\vee)$" $\circ$.

If the condition on Sel_0 breaks, then

$\left(\mathrm{Sel}_0(\mathbb{Q}(\zeta_{p^\infty}), W_f(k/2))^\vee\right)_{\underline{mft}}$

max. fin. torsion.

· $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is irred. $\Rightarrow$ non-ord. case.

ord. case is easier.
"control thm + α":

· Emerton – Pollack – Weston announced
a similar result around 2010.
(based on p-adic LLC)

Tools.

· Euler system construction of Mazur–Tate elts.
$\theta_{\mathbb{Q}(\zeta_{p^n}), k/2}(f)$.
(from Kato's Euler systems)

· Finite layer version of Perrin-Riou's alg. θ-elts
+ Fitting ideal techniques

· Kato's divisibility of the IMC.

· Kato's E.S. $\to$ M-T elts.

We can construct "finite layer Coleman map"

$$\exists\, \mathrm{Col}_{\mathbb{Q}(\zeta_{p^n}),\, k/2} : H^1(\mathbb{Q}(\zeta_{p^n})\otimes\mathbb{Q}_p,\ T_f(k/2))$$
$$\to \mathcal{O}_{\mathcal{K}}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})]$$

s.t. $\mathrm{Col}_{\mathbb{Q}(\zeta_{p^n}),\, k/2}(\mathrm{loc}_p(z_{\mathbb{Q}(\zeta_{p^n}),\, k/2}))$

$$= \theta_{\mathbb{Q}(\zeta_{p^n}),\, k/2}(f).$$

(Kurihara, Otsuki, Ota, Kataoka.) elliptic curves

: based ~~on~~ heavily on integral p-adic Hodge thy.

(Wach modules / Fontaine-Laffaille modules)

<u>The real issue</u>: Compare 3 <u>different integralities.</u>

1. integrality of $\theta_{\mathbb{Q}(\zeta_{p^n}),\, k/2}(f)$

   $\longleftrightarrow$. periods

2. integrality of $\mathrm{Col}_{\mathbb{Q}(\zeta_{p^n}),\, k/2}$.

3. integrality of Kato's E.S. : $z_{\mathbb{Q}(\zeta_{p^n}),\, k/2}$.

· Finite layer alg. p-adic L-fun. à la Perrin-Riou.

Thm (Kato) ρ_f has large img. ($\mathrm{Im}(\rho_f) \supseteq$ a conjugate of $SL_2(\mathbb{Z}_p)$)

$$H^1_{Iw}(\mathbb{Q}, T_f(k/2)) \text{ is free of rk } 1 \text{ over } \Lambda = \mathcal{O}_{\mathbb{Z}}[[Gal(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]].$$

$$\in \qquad \in$$

$$z_{\mathbb{Q}(\zeta_{p^\infty}), k/2} \smile w.$$

Keep track the difference.

$\rightsquigarrow$

Can compare M-T elts.

with P-R's alg. Θ elts.