Arithmetic Moduli of Elliptic Surfaces

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Webinar in Number Theory

Counting number fields

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One could consider a one-dimensional higher analogue which is,

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A: This is the influential Shafarevich's conjecture for algebraic curves first called to attention by Igor R. Shafarevich in his 1962 address at the International Congress in Stockholm.

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Question

How many exactly are there?

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Through the *global fields analogy*, we consider the geometric Shafarevich's conjecture where the number field $\mathbb{Q}$ is replaced by the global function field $\mathbb{F}_q(t)$.

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In joint works with **Dori Bejleri (Harvard)** and **Matthew Satriano (Waterloo)**, we show the following.

Counting elliptic curves with additive reductions

Let $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, we consider $\mathcal{Z}^{\Gamma}_{\mathbb{F}_q(t)}(\mathcal{B}) :=$

$|\{$Elliptic curves over the $\mathbb{P}^1_{\mathbb{F}_q}$ with $0 < ht(\Delta) \leq \mathcal{B}$ and one Γ reduction$\}|$

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Theorem (Dori–Matthew–June)

The counting $\mathcal{Z}^{\Gamma}_{\mathbb{F}_q(t)}(\mathcal{B})$, which counts the number of elliptic curves with one additive reduction of Γ type and the rest of the potential bad reductions are strictly multiplicative reductions, satisfies

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which are the sharp enumerations with the non-constant lower order terms $\mathcal{B}^{\frac{1}{2}}$ or $\mathcal{B}^{\frac{1}{3}}$.

Unstable/Semistable contrast of lower order terms

For strictly multiplicative reductions case, consider $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) :=$

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The counting $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B})$ by $ht(\Delta) = q^{12n} \leq \mathcal{B}$ satisfies

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Stability of moduli loci & Nature of lower order terms

Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Let us recall that $\overline{\mathcal{M}}_{1,1}$ is a smooth proper Deligne-Mumford stack of stable elliptic curves with a coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$. This $\mathbb{P}^1$ parametrizes the j–invariants of elliptic curves.

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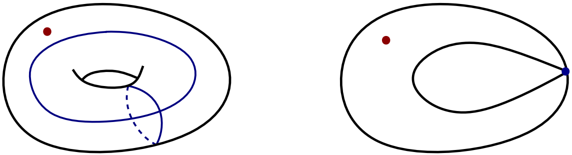

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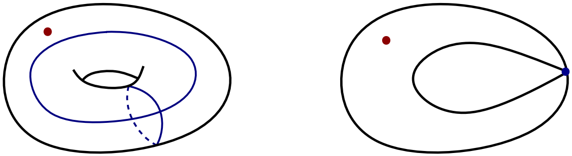

When the characteristic of the field K is not equal to 2 or 3, $(\overline{\mathcal{M}}_{1,1})_K \cong [(Spec\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] =: \mathcal{P}_K(4,6)$ through the short Weierstrass equation: $Y^2 = X^3 + a_4 X + a_6$

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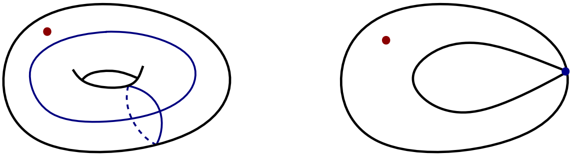

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Stabilizers are the orbifold points $[1:0]$ & $[0:1]$ with μ_4 & μ_6 respectively and the generic stacky points such as $[1:1]$ with μ_2

Moduli stack of stable elliptic fibrations

The fine moduli $\overline{\mathcal{M}}_{1,1}$ comes with universal family $p : \overline{\mathcal{E}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ of stable elliptic curves. Thus, a stable elliptic fibration $g : Y \to \mathbb{P}^1$ is induced from a morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ and vice versa.

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X is the non-singular semistable elliptic surface; Y is the stable elliptic fibration; $\nu : X \to Y$ is the minimal resolution.

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The moduli stack $\mathcal{L}_{1,12n}$ of stable elliptic fibrations over the $\mathbb{P}^1$ with $12n$ nodal singular fibers and section is the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ where $\varphi_f^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ for $n \in \mathbb{Z}_{\geq 1}$.

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There is a canonical equivalence of groupoids between $\mathcal{L}_{1,12n}(K)$ and the groupoid of semistable elliptic surfaces over K.

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Grothendieck class in $K_0(\mathrm{Stck}_K)$ with $\mathrm{char}(K) \neq 2, 3$,

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$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,12n}(\mathbb{F}_q)/\sim| \leq 2 \cdot \frac{(q^{11}-q^9)}{(q^{10}-1)} \cdot \left(\mathcal{B}^{\frac{5}{6}} - 1\right)$$

Theorem (Dori–Matthew–June)

Let $\mathrm{char}(K) \neq 2, 3$. *Given a minimal rational map* $\hat{\varphi}_f \colon C \dashrightarrow \overline{\mathcal{M}}_{1,1}$ *with* γ *vanishing constraints there exists a unique morphism* $\varphi_f \colon \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ *with* Γ *cyclotomic twistings and vice versa.*

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More generally, let $\mathcal{P}(\vec{\lambda})$ be the N-dimensional weighted projective stack with the weight vector $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$. Then over a base field K with $\mathrm{char}(K) \nmid \lambda_i$, $\mathrm{Rat}_n^{\gamma}(C, \mathcal{P}(\vec{\lambda})) \cong \mathrm{Hom}_n^{\Gamma}(\mathcal{C}, \mathcal{P}(\vec{\lambda}))$.

Tate's correspondence

Theorem (Dori–Matthew–June)

Let $\mathrm{char}(K) \neq 2, 3$ *then we have the following correspondence between* γ *vanishing constraint and* Γ *cyclotomic twisting.*

Reduction	j	$\gamma = (\nu(a_4),\ \nu(a_6))$	$\Gamma = \chi(1) \Rightarrow \mu_r < \mu_j$
II	*0*	$(\geq 1, 1)$	$1 \mapsto 1 \Rightarrow \mu_6 < \mu_6$
IV	*0*	$(\geq 2, 2)$	$1 \mapsto 2 \Rightarrow \mu_3 < \mu_6$
I_0^*	*0*	$(\geq 3, 3)$	$1 \mapsto 3 \Rightarrow \mu_2 < \mu_6$
IV^*	*0*	$(\geq 3, 4)$	$1 \mapsto 4 \Rightarrow \mu_3 < \mu_6$
II^*	*0*	$(\geq 4, 5)$	$1 \mapsto 5 \Rightarrow \mu_6 < \mu_6$
III	*1728*	$(1, \geq 2)$	$1 \mapsto 1 \Rightarrow \mu_4 < \mu_4$
I_0^*	*1728*	$(2, \geq 4)$	$1 \mapsto 2 \Rightarrow \mu_2 < \mu_4$
III^*	*1728*	$(3, \geq 5)$	$1 \mapsto 3 \Rightarrow \mu_4 < \mu_4$
I_0^* $\mathrm{I}_{k>0}^*$	$\neq 0, 1728$ ∞	$(2, 3)$	$1 \mapsto 1 \Rightarrow \mu_2 < \mu_2$

Motives of minimal elliptic surfaces moduli stacks

Theorem (Dori–Matthew–June)

Let $\mathrm{char}(K) \neq 2, 3$. The motives for the moduli stack of minimal elliptic surfaces over the parameterized $\mathbb{P}^1$ with section and discriminant degree $12n$ having one additive reduction of type Γ

Reduction	j	$\{\mathrm{Rat}_n^\gamma(\mathbb{P}^1, \mathcal{P}(4,6))\}/\{\mathrm{PGL}_2\}$
II	*0*	$\mathbb{L}^{10n-3} - \mathbb{L}^{4n-2}$
IV	*0*	$\mathbb{L}^{10n-5} - \mathbb{L}^{4n-3}$
I_0^*	*0*	$\mathbb{L}^{10n-7} - \mathbb{L}^{4n-4}$
IV^*	*0*	$\mathbb{L}^{10n-8} - \mathbb{L}^{4n-4}$
II^*	*0*	$\mathbb{L}^{10n-10} - \mathbb{L}^{4n-5}$
III	*1728*	$\mathbb{L}^{10n-4} - \mathbb{L}^{6n-3}$
I_0^*	*1728*	$\mathbb{L}^{10n-7} - \mathbb{L}^{6n-5}$
III^*	*1728*	$\mathbb{L}^{10n-9} - \mathbb{L}^{6n-6}$
I_0^* $\mathrm{I}_{k>0}^*$	$\neq 0, 1728$ ∞	$(\mathbb{L} - 1) \cdot (\mathbb{L}^{10n-7} - \mathbb{L}^{6n-5} - \mathbb{L}^{4n-4})$

Heuristic through global fields analogy

Switching to the number field realm with $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, we have the following conjecture.

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Conjecture (Dori–Matthew–June)

The function $\mathcal{Z}_{\mathbb{Q}}^{\Gamma}(\mathcal{B})$, which counts the number of elliptic curves over $\mathbb{Z}$ with $0 < ht(\Delta) \leq \mathcal{B}$, has the following asymptotic behavior:

$$a \cdot \mathcal{B}^{\frac{5}{6}} + b \cdot \mathcal{B}^{\frac{1}{2}} + c \cdot \mathcal{B}^{\frac{1}{3}} + o(\mathcal{B}^{\frac{1}{3}})$$

with the main leading term $\mathcal{O}\left(\mathcal{B}^{\frac{5}{6}}\right)$, the secondary term $\mathcal{O}\left(\mathcal{B}^{\frac{1}{2}}\right)$ and the tertiary term $\mathcal{O}\left(\mathcal{B}^{\frac{1}{3}}\right)$.

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The lower order term of the order $\mathcal{O}\left(\mathcal{B}^{(7-\frac{5}{27}+\epsilon)/12}\right)$ was suggested by the work of S. Baier for stable elliptic curves. However, his proof relies on the assumption of the Riemann Hypothesis for Dirichlet L-functions.

The end

Thank you for listening!