Discrete velocity Boltzmann equations in the plane: stationary solutions.

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arXiv:2112.08640, to appear in Analysis and PDE

Aim of the talk

We prove existence of stationary mild solutions for normal discrete velocity Boltzmann equations

in the plane with no pair of colinear interacting velocities and given ingoing boundary values.

A discrete velocity model of a kinetic gas is a system of partial differential equations having the form,

$$\frac{\partial f_i}{\partial t}(t,z) + v_i \cdot \nabla_z f_i(t,z) = Q_i(f,f)(t,z),\ t>0,\ z\in\Omega,\ 1\leq i\leq p,$$

where $f_i(t,z)$, $1\leq i\leq p$, are phase space densities at time t, position z and velocity v_i.

The given discrete velocities are v_i, $1\leq i\leq p$.
For $f=(f_i)_{1\leq i\leq p}$, the collision operator $Q=(Q_i)_{1\leq i\leq p}$ with gain part Q^+, loss part Q^-, and collision frequency ν, is given by

$$Q_i(f,f) = \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm}(f_l f_m - f_i f_j) = Q_i^+(f,f) - f_i \nu_i(f),$$

$$\nu_i(f) = \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} f_j, \quad i=1,...,p.$$

The collision coefficients satisfy

$$\Gamma_{ij}^{lm} = \Gamma_{ji}^{lm} = \Gamma_{lm}^{ij} \geq 0,$$

$$\Gamma_{ij}^{lm} > 0 \Rightarrow v_i + v_j = v_l + v_m, \quad |v_i|^2 + |v_j|^2 = |v_l|^2 + |v_m|^2.$$

The discrete velocity model is called normal if any solution of the equations

$$\Psi(v_i) + \Psi(v_j) = \Psi(v_l) + \Psi(v_m),$$

where the indices $(i, j; l, m)$ take all possible values satisfying $\Gamma_{ij}^{lm} > 0$, is given by

$$\Psi(v) = a + b \cdot v + c|v|^2,$$

for some constants $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^d$. We consider

$$\begin{aligned} &\text{the generic case of normal coplanar velocity sets with} \\ &\text{no pair of colinear interacting velocities } (v_i, v_j), \end{aligned} \tag{1}$$

in a strictly convex bounded open subset $\Omega \subset \mathbb{R}^2$, with C^2 boundary $\partial\Omega$ and given boundary inflow.

Denote by $n(Z)$ the inward normal to $Z \in \partial\Omega$, and by

$\partial\Omega_i^+ = \{Z \in \partial\Omega\,;\, v_i \cdot n(Z) > 0\}$, (resp. $\partial\Omega_i^- = \{Z \in \partial\Omega\,;\, v_i \cdot n(Z) < 0\}$

the v_i-ingoing (resp. v_i-outgoing) part of the boundary. Let

$$s_i^+(z) = \inf\{s > 0\,;\, z - sv_i \in \partial\Omega_i^+\},$$
$$s_i^-(z) = \inf\{s > 0\,;\, z + sv_i \in \partial\Omega_i^-\}, \quad z \in \Omega.$$

Write

$$z_i^+(z) = z - s_i^+(z)v_i \quad (\text{resp. } z_i^-(z) = z + s_i^-(z)v_i) \qquad (2)$$

for the ingoing (resp. outgoing) point on $\partial\Omega$ of the characteristics through z in direction v_i.

The stationary boundary value problem

$$v_i \cdot \nabla f_i(z) = Q_i(f,f)(z), \quad z \in \Omega, \tag{3}$$

$$f_i(z) = f_{bi}(z), \quad z \in \partial\Omega_i^+, \qquad 1 \leq i \leq p, \tag{4}$$

is considered in L^1 in one of the following equivalent forms;
the exponential multiplier form,

$$f_i(z) = f_{bi}(z_i^+(z)) e^{-\int_0^{s_i^+(z)} \nu_i(f)(z_i^+(z)+sv_i)ds}$$
$$+ \int_0^{s_i^+(z)} Q_i^+(f,f)(z_i^+(z)+sv_i) e^{-\int_s^{s_i^+(z)} \nu_i(f)(z_i^+(z)+rv_i)dr} ds, \text{ a.a. } z \in \Omega,$$

the mild form,

$$f_i(z) = f_{bi}(z_i^+(z)) + \int_0^{s_i^+(z)} Q_i(f,f)(z_i^+(z)+sv_i)ds, \text{ a.a. } z \in \Omega,\ i \leq p,$$

the renormalized form,

$$v_i \cdot \nabla \ln(1+f_i)(z) = \frac{Q_i(f,f)}{1+f_i}(z),\ z \in \Omega, \qquad f_i(z) = f_{bi}(z),\ z \in \partial\Omega_i^+,$$

in the sense of distributions.

Let $L^1_+(\Omega)$ be the set of non-negative integrable functions on Ω.

Define the entropy (resp. entropy dissipation) of $f = (f_i)_{1\leq i\leq p}$ by

$$\sum_{i=1}^{p} \int_\Omega f_i \ln f_i(z)dz, \quad \Big(\text{resp.} \quad \sum_{i,j,l,m=1}^{p} \Gamma_{ij}^{lm} \int_\Omega (f_l f_m - f_i f_j) \ln \frac{f_l f_m}{f_i f_j}(z)dz\Big).$$

Theorem

Consider a non-negative ingoing boundary value f_b with mass and entropy inflows bounded,

$$\int_{\partial\Omega_i^+} v_i \cdot n(z)\, f_{bi}(1+\ln f_{bi})(z)d\sigma(z) < +\infty, \quad 1 \leq i \leq p.$$

For the boundary value problem

$$\begin{aligned} &v_i \cdot \nabla f_i(z) = Q_i(f,f)(z), \quad z \in \Omega, \\ &f_i(z) = f_{bi}(z), \quad z \in \partial\Omega_i^+, \qquad 1 \leq i \leq p, \end{aligned}$$

there exists a stationary mild solution in $\left(L^1_+(\Omega)\right)^p$ with finite mass and entropy-dissipation.

FORMER RESULTS

In 1997 Bobylev, Palczewski and Schneider proved the consistency with the Boltzmann equation of some discrete velocity models.

Most mathematical results for stationary discrete velocity models of the Boltzmann equation have been obtained in one space dimension.

A discussion of normal discrete velocity models, i.e. conserving nothing but mass, momentum and energy, can be found in (Bobylev-Vinerean-Windfall 2010).

Half-space problems (Bernhoff 2021) and weak shock waves (Bernhoff 2007) for discrete velocity models have also been studied.

The Broadwell model, not included in the present results, is a four-velocity model, with $v_1 + v_2 = v_3 + v_4 = 0$ and v_1, v_3 orthogonal.

Special classes of solutions to the Broadwell model in the plane are given in (Bobylev-Toscani 1996), (Bobylev 1996) and (Ilyin 2014).

The existence of continuous solutions to the two-dimensional stationary Broadwell model with continuous boundary data in a rectangle is proven in (Cercignani-Illner-Shinbrot 1988).

We solve that problem in an L^1-setting in (Arkeryd-Nouri 2020).

For every normal model, there is a priori control of entropy dissipation, mass and entropy flows through the boundary.

The main difficulties are to prove that for a sequence of approximations, weak L^1 compactness holds and that the limit of the collision operator equals the collision operator of the limit.

The argument used in the stationary paper (Arkeryd-Nouri 1994) in the continuous velocity case for obtaining control of entropy, hence weak L^1 compactness of a sequence of approximations from the control of entropy dissipation, does not work in a discrete velocity case because the number of velocities is finite.

In kinetic equations with continuous velocities, an important use is made of averaging lemmas like (Golse, Saint-Raymond 2002)

Lemma

Let $(f_n)_{n\in\mathbb{N}}$ be weakly compact in L^1 and such that $(v\cdot\nabla_x f_n)$ is bounded in $L^1_{loc}(\mathbb{R}^d\times\mathbb{R}^d)$.
For any $\psi\in L^\infty_{compact}(\mathbb{R}^d)$, the sequence $\big(\int\psi(v)f_n(\cdot,v)dv\big)$ is compact in $L^1(\mathbb{R}^d)$.

This lemma does not hold in a discrete velocity frame.

We instead prove L^1 compactness of integrated collision frequency

$$\int_{-s_i^+(z)}^{s_i^-(z)}\nu_i(z+sv_i)ds$$

with the Kolmogorov-Riesz theorem.

In (Arkeryd-Nouri 2020), weak L^1 compactness of a sequence of approximations was obtained with assumption (1) together with the assumption that all velocities v_i point out into the same half-plane.

In this talk, only assumption (1) is kept, the second assumption is removed and a new proof of weak L^1 compactness of approximations is provided.

Assumption (1) is also crucial for proving L^1 compactness of the integrated collision frequencies, that is important for the convergence procedure.

Primary approximations.

Let $\alpha \in]0,1[$ and μ_α a smooth mollifier in $\mathbb{R}^2$ with support in the ball centered at the origin of radius α.

Lemma

For any $\alpha > 0$ *and* $k \in \mathbb{N}^*$, *there is a solution* $F^{\alpha,k} \in (L^1_+(\Omega))^p$ *to*

$$\alpha F_i^{\alpha,k} + v_i \cdot \nabla F_i^{\alpha,k} = \sum_{j,l,m=1}^{p} \Gamma_{ij}^{lm} \Big(\frac{F_l^{\alpha,k}}{1 + \frac{F_l^{\alpha,k}}{k}} \frac{F_m^{\alpha,k} * \mu_\alpha}{1 + \frac{F_m^{\alpha,k} * \mu_\alpha}{k}} - \frac{F_i^{\alpha,k}}{1 + \frac{F_i^{\alpha,k}}{k}} \frac{F_j^{\alpha,k} * \mu_\alpha}{1 + \frac{F_j^{\alpha,k} * \mu_\alpha}{k}} \Big),$$

$$F_i^{\alpha,k}(z) = \min\{ f_{bi}^k(z), k\}, \quad z \in \partial\Omega_i^+, 1 \le i \le p.$$

Let $k \in \mathbb{N}^*$ be given.
Each component of $F^{\alpha,k}$ is bounded by a multiple of k^2.
Therefore $(F^{\alpha,k})_{\alpha \in]0,1[}$ is weakly compact in $(L^1(\Omega))^p$.

For a subsequence in α, the convergence is strong in $(L^1(\Omega))^p$ as stated in the following lemma.

Lemma

There is a sequence $(\beta(q))_{q \in \mathbb{N}}$ tending to zero when $q \to +\infty$ and a function $F^k \in L^1$, such that $(F^{\beta(q),k})_{q \in \mathbb{N}}$ strongly converges in $(L^1(\Omega))^p$ to F^k when $q \to +\infty$.

Lemma

F^k is a non-negative solution to

$$v_i \cdot \nabla F_i^k = Q_i^{+k} - F_i^k \nu_i^k \,, \tag{5}$$

$$F_i^k(z) = f_{bi}^k(z), \quad z \in \partial\Omega_i^+, \quad 1 \le i \le p. \tag{6}$$

Solutions $(F^k)_{k\in\mathbb{N}^*}$ to (5)-(6) have mass and entropy dissipation bounded from above uniformly with respect to k. Moreover,

$$\sum_{i=1}^{p} \int_{\partial\Omega_i^-, F_i^k \le k} \mid v_i \cdot n(Z) \mid F_i^k \ln F_i^k(Z) d\sigma(Z)$$
$$+ \ln \frac{k}{2} \int_{\partial\Omega_i^-, F_i^k > k} \mid v_i \cdot n(Z) \mid F_i^k d\sigma(Z) \le c_b. \tag{7}$$

L^1 compactness properties of the approximations. Main lines.

In Lemma 6, weak L^1 compactness of $(F^k)_{k\in\mathbb{N}^*}$ is proven.

Lemma 7 splits Ω into a set of i-characteristics with arbitrary small measure and its complement, where both the approximations and their integrated collision frequencies are bounded.

In Lemma 8, the strong L^1 compactness of integrated collision frequency is proven.

Lemma

The sequence $(F^k)_{k\in\mathbb{N}^*}$ *solution to* (5)-(6) *is weakly compact in* L^1.

Proof. First $(F^k)_{k\in\mathbb{N}^*}$ is uniformly bounded in $(L^1(\Omega))^p$.

Given (7) and the following bound on F^k,

$$F_i^k(z) \leq F_i^k(z + s_i^-(z)v_i) \exp\Big(\Gamma \sum_{j\in J_i} \int_{-s_i^+(z)}^{s_i^-(z)} F_j^k(z + rv_i)dr\Big),\ z \in \Omega, \tag{8}$$

the weak L^1 compactness of $(F^k)_{k\in\mathbb{N}^*}$ will follow from the uniform boundedness in $L^\infty(\partial\Omega_i^+)$ of

$$\Big(\int_0^{s_i^-(Z)} F_j^k(Z + rv_i)dr\Big)_{j\in J_i, k\in\mathbb{N}}, \tag{9}$$

where J_i denotes $\{j \in \{1,\cdots,p\}; (v_i, v_j) \text{ are interacting velocities}\}$.

By (1), there exists $\eta > 0$ such that for all interacting velocities (v_i, v_j),

$$|\sin(\widehat{v_i, v_j})| > \eta.$$

Let $i \in \{1, \cdots, p\}$ and $Z \in \partial\Omega_i^+$. Multiply the equation satisfied by F_j^k by $\frac{v_i^\perp \cdot v_j}{|v_i|}$ and integrate it on one of the half domains defined by the segment $[Z, Z + s_i^-(Z)v_i]$. Summing over $j \in \{1, \cdots, p\}$ implies that

$$\sum_{j=1}^{p} \sin^2(\widehat{v_i, v_j}) \int_0^{s_i^-(Z)} F_j^k(Z + sv_i)ds \leq c_b, \quad Z \in \partial\Omega_i^+.$$

This leads to the control of $\int_{-s_i^+(z)}^{s_i^-(z)} F_j(z + rv_i)dr$ in L^∞.

Recall the exponential multiplier form for the approximations $(F^k)_{k\in\mathbb{N}^*}$,

$$F_i^k(z) = f_{bi}^k(z_i^+(z))e^{-\int_{-s_i^+(z)}^0 \nu_i^k(z+sv_i)ds} + \int_{-s_i^+(z)}^0 Q_i^{+k}(z+sv_i)e^{-\int_s^0 \nu_i^k(F^k)(z+rv_i)dr}ds, \quad \text{a.a. } z\in\Omega,\ i\leq p.$$

An i-characteristics is a segment of points $[Z - s_i^+(Z)v_i, Z]$, where $Z\in\partial\Omega_i^-$.

Denote by $\Gamma = \max_{i,j,l,m} \Gamma_{ij}^{lm}$.

Lemma

For $i \in \{1, ..., p\}$, $k \in \mathbb{N}^*$ and $\epsilon > 0$, there is a subset $\Omega_i^{k,\epsilon}$ of i-characteristics of Ω with measure smaller than $c_b\epsilon$, such that for any $z \in \Omega \setminus \Omega_i^{k,\epsilon}$,

$$F_i^k(z) \leq \frac{1}{\epsilon^2} \exp\left(\frac{p\Gamma}{\epsilon^2}\right), \quad \int_{-s_i^+(z)}^{s_i^-(z)} \nu_i^k(z + sv_i)ds \leq \frac{p\Gamma}{\epsilon^2}. \qquad (10)$$

Given $i \in \{1, ..., p\}$ and $\epsilon > 0$, let $\chi_i^{k,\epsilon}$ denote the characteristic function of the complement of $\Omega_i^{k,\epsilon}$.

Lemma

The sequences $\left(\int_{-s_i^+(z)}^0 \nu_i^k(z+sv_i)ds \right)_{k\in\mathbb{N}^*}$, $1 \leq i \leq p$, *are strongly compact in* $L^1(\Omega)$.

Proof. Take $\Gamma_{ij}^{lm} > 0$. By (1), v_i and v_j span $\mathbb{R}^2$. Denote by (a,b) the corresponding coordinate system, (a^-, a^+) defined by

$$a^- = \min\{a \in \mathbb{R}; (a,b) \in \Omega \text{ for some } b\},$$
$$a^+ = \max\{a \in \mathbb{R}; (a,b) \in \Omega \text{ for some } b\},$$

and by D the Jacobian of the change of variables $z \longrightarrow (a,b)$. The uniform bound for the mass of $(F^k)_{k\in\mathbb{N}^*}$ implies that

$$\left(\int_\Omega \int_{-s_i^+(z)}^0 \nu_i^k(z+sv_i)dsdz \right)_{k\in\mathbb{N}^*}$$

is bounded in L^1 uniformly with respect to k.

Indeed, for some $(b^-(a), b^+(a))$, $a \in [a^-, a^+]$,

$$\begin{aligned}
&\int_\Omega \int_{-s_i^+(z)}^0 F_j^k(z + sv_i)dsdz \\
&= D \int_{a^-}^{a^+} \int_{b^-(a)}^{b^+(a)} \int_{-s_i^+(bv_j)}^{a} F_j^k(bv_j + sv_i)ds\,db\,da \\
&\leq D \int_{a^-}^{a^+} \int_{b^-(a)}^{b^+(a)} \int_{-s_i^+(bv_j)}^{s_i^-(bv_j)} F_j^k(bv_j + sv_i)ds\,db\,da \\
&\leq c \int_\Omega F_j^k(z)dz, \qquad\qquad j \in J_i.
\end{aligned}$$

By the Kolmogorov-Riesz theorem, the compactness of

$$\Big(\int_{-s_i^+(z)}^{0} \nu_i^k(z+sv_i)ds \Big)_{k\in\mathbb{N}^*}$$

will follow from its translational equicontinuity in $L^1(\Omega)$.

Equicontinuity is proven in the direction v_i and in the direction v_j with the mild form for F_j^k.

Here the assumption (1) becomes crucial.

Equicontinuity in the v_i-direction.

It follows from $s_i^+(z+hv_i) = s_i^+(z) + h$ that

$$\int_\Omega \mid \int_{-s_i^+(z+hv_i)}^0 F_j^k(z+hv_i+sv_i)ds - \int_{-s_i^+(z)}^0 F_j^k(z+sv_i)ds \mid dz$$
$$= \int_\Omega \int_{s\in I(0,h)} F_j^k(z+sv_i)ds\, dz$$
$$\leq c \mid h \mid .$$

Equicontinuity in the v_j-direction.

By the weak L^1 compactness of $(F_j^k)_{k\in\mathbb{N}^*}$, it is sufficient to prove the translational equi-continuity in the v_j-direction of

$$\Big(\int_{s_i^+(z)}^0 \chi_j^{k,\epsilon} F_j^k(z+sv_i)ds\Big)_{k\in\mathbb{N}^*}.$$

Expressing $F_j^k(z+hv_j+sv_i)$ (resp. $F_j^k(z+sv_i)$) as integral along its v_j-characteristics, it holds that

$$
\begin{aligned}
&\mid \int_{-s_i^+(z+hv_j)}^0 \chi_j^{k,\epsilon} F_j^k(z+hv_j+sv_i)ds - \int_{-s_i^+(z)}^0 \chi_j^{k,\epsilon} F_j^k(z+sv_i)ds \mid \\
&\leq \mid A_{ij}^k(z,h) \mid + \mid B_{ij}^k(z,h) \mid,
\end{aligned}
$$

where $A_{ij}^k(z,h)$ is a difference of boundary terms, and

$$
\begin{aligned}
B_{ij}^k(z,h) = &\int_{-s_i^+(z+hv_j)}^0 \int_{-s_j^+(z+hv_j+sv_i)}^0 \chi_j^{k,\epsilon} Q_j^k(z+hv_j+sv_i+rv_j)drds \\
&- \int_{-s_i^+(z)}^0 \int_{-s_j^+(z+sv_i)}^0 \chi_j^{k,\epsilon} Q_j^k(z+sv_i+rv_j)drds.
\end{aligned}
$$

For some $\omega_h(z) \subset \Omega$ of measure or order $\mid h \mid$ uniformly with respect to $z \in \Omega$,

$$B_{ij}^k(z,h) = \int_{\omega_h(z)} \chi_j^{k,\epsilon} Q_j^k(Z) dZ.$$

The sequence $(\chi_j^{k,\epsilon} Q_j^k)_{k\in\mathbb{N}^*}$ is weakly compact in L^1. Indeed,

$$\begin{aligned} \chi_j^{k,\epsilon} Q_j^k &\leq \frac{1}{\ln \Lambda} \tilde{D}_k + \Gamma\Lambda \Big(\sum_{i\in J_j} F_i^k \Big) (\chi_j^{k,\epsilon} F_j^k) \\ &\leq \frac{1}{\ln \Lambda} \tilde{D}_k + \frac{\Gamma\Lambda}{\epsilon^2} \exp\big(\frac{p\Gamma}{\epsilon^2}\big) \Big(\sum_{i\in J_j} F_i^k \Big), \quad \Lambda > 1, \end{aligned}$$

with $(\tilde{D}_k)_{k\in\mathbb{N}^*}$ uniformly bounded in L^1 and $(F_i^k)_{k\in\mathbb{N}^*}$ weakly compact in L^1. Hence,

$$\lim_{h\to 0} \int_\Omega \mid B_{ij}^k(z,h) \mid dz = 0, \quad \text{uniformly with respect to } k.$$

Let f be the weak L^1 limit of a subsequence of the solutions $(F^k)_{k\in\mathbb{N}^*}$.

For proving that f is a mild solution of the problem, it is sufficient to prove that for any $\eta > 0$ and $i \in \{1, \cdots, p\}$, there is a set X_i^η of i-characteristics with complementary set of measure smaller than $c\eta$, such that

$$\int_\Omega \varphi\chi_i^\eta f_i(z)dz = \int_\Omega \varphi\chi_i^\eta f_{bi}(z_i^+(z))dz + \int_\Omega \int_{-s_i^+(z)}^0 \big(\varphi\chi_i^\eta Q_i(f,f) + \chi_i^\eta f_i\, v_i \cdot \nabla\varphi\big)(z+sv_i)ds\, dz,$$

where χ_i^η denotes the characteristic function of X_i^η.

Define the set X_i^η as follows. For every $\epsilon > 0$, pass to the limit when $k \to +\infty$ in

$$\chi_i^{k,\epsilon} F_i^k(z) \leq \chi_i^{k,\epsilon} F_i^k(z_i^-(z)) \exp\Big(\int_{-s_i^+(z)}^{s_i^-(z)} \nu_i^k(z + sv_i) ds \Big), \quad \text{a.a. } z \in \Omega,$$

It implies that

$$F_i^\epsilon(z) \leq F_i^\epsilon(z_i^-(z)) \exp\Big(\int_{-s_i^+(z)}^{s_i^-(z)} \nu_i(f)(z + sv_i) ds \Big), \quad \text{a.a. } z \in \Omega, \quad \epsilon \in]0$$

where F_i^ϵ is the limit of a subsequence of $(\chi_i^{k,\epsilon} F_i^k)_{k\in\mathbb{N}^*}$ and $\nu_i(f) = \sum_{j,l,m=1}^p \Gamma_{ij}^{lm} f_j$. By the monotonicity in ϵ of $(F^\epsilon)_{\epsilon\in]0,1[}$ (resp. $\big(F^\epsilon(z_i^-(z))\big)_{\epsilon\in]0,1[}$) and the uniform boundedness of their masses, it holds that

$$f_i(z) \leq f_i(z_i^-(z)) \exp\Big(\int_{-s_i^+(z)}^{s_i^-(z)} \nu_i(f)(z + sv_i) ds \Big), \quad \text{a.a. } z \in \Omega.$$

Lemma

f is a subsolution of the problem, i.e.

$$\int_\Omega \varphi \chi_i^\eta f_i(z)dz \leq \int_\Omega \varphi f_{bi}(z_i^+(z))dz + \int_\Omega \int_{-s_i^+(z)}^0 \chi_i^\eta f_i\, v_i \cdot \nabla\varphi(z+sv_i)ds\, dz + \int_\Omega \int_{-s_i^+(z)}^0 \varphi Q_i(f,f)(z+sv_i)ds\, dz, \quad 1 \leq i \leq p, \quad \varphi \in ($$

Let us pass to the limit when $k \to +\infty$ in any term of the loss term of the approximate equation, denoted by $\Gamma_{ij}^{lm} L^k$, where

$$L^k := \int_\Omega \chi_i^\eta \chi_i^{k,\epsilon}(z) \int_{-s_i^+(z)}^0 \varphi \frac{F_i^k}{1+\frac{F_i^k}{k}} \frac{F_j^k}{1+\frac{F_j^k}{k}}(z+sv_i)ds\,dz, \quad j \in J_i.$$

By integration by parts, L_k equals

$$\int_\Omega \int_{-s_i^+(z)}^0 \chi_i^\eta \chi_i^{k,\epsilon}\left(\varphi(Q_i^{+k} - F_i^k \nu_i^k) + (v_i \cdot \nabla\varphi)F_i^k\right)(z+sv_i)$$

$$\left(\int_s^0 \chi_i^{k,\epsilon} \frac{F_j^k}{(1+\frac{F_i^k}{k})(1+\frac{F_j^k}{k})}(z+rv_i)dr\right)ds\,dz$$

$$+\int_\Omega \chi_i^\eta \chi_i^{k,\epsilon} \varphi \frac{f_{bi}^k}{1+\frac{f_{bi}^k}{k}}(z_i^+(z)) \int_{-s_i^+(z)}^0 \frac{F_j^k}{1+\frac{F_j^k}{k}}(z+sv_i)ds\,dz.$$

In order to end the prove of the lemma, we prove that each

$$\Gamma_{ij}^{lm}\int_{\Omega}\int_{-s_i^+(z)}^{0}\varphi\chi_i^{\eta}\chi_i^{k,\epsilon}\frac{F_l^k}{1+\frac{F_l^k}{k}}\frac{F_m^k}{1+\frac{F_m^k}{k}}(z+sv_i)ds\,dz,\quad j\in J_i,$$

term from the gain term, converges when $k\to+\infty$ to a limit smaller than

$$\Gamma_{ij}^{lm}\int_{\Omega}\int_{-s_i^+(z)}^{0}\varphi\chi_i^{\eta}F_l^{\epsilon'}f_m(z+sv_i)ds\,dz+\alpha(\epsilon'),\ \epsilon'\in]0,1[,$$

with

$$\lim_{\epsilon'\to 0}\alpha(\epsilon')=0.$$

To end the proof of the passage to the limit in the approximate equations, it is proven that

$$\int_\Omega \varphi\chi_i^\eta f_i(z)dz \geq \int_\Omega \varphi\chi_i^\eta f_{bi}(z_i^+(z))dz$$
$$+ \int_\Omega \int_{-s_i^+(z)}^0 \chi_i^\eta f_i\, v_i \cdot \nabla\varphi(z+sv_i)ds\,dz$$
$$+ \int_\Omega \int_{-s_i^+(z)}^0 \varphi\,\chi_i^\eta Q_i(f,f)(z+sv_i)ds\,dz, \quad 1 \leq i \leq p, \quad \varphi$$

THANK YOU FOR YOUR ATTENTION!