The quantum Boltzmann and BGK model near a global equilibrium

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Outline

The Boltzmann and BGK model

What is the Boltzmann equation?

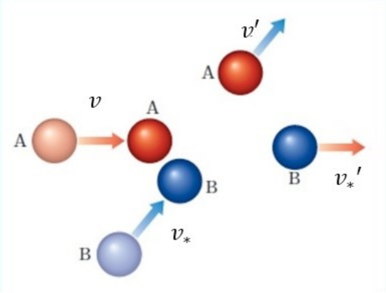


Figure: http://yjh-phys.tistory.com/1385

- Momentum conservation law: $\delta(v + v_* - v' - v'_*)$
- Energy conservation law: $\delta(|v|^2 + |v_*|^2 - |v'|^2 - |v'_*|^2)$

What is Boltzmann equation?

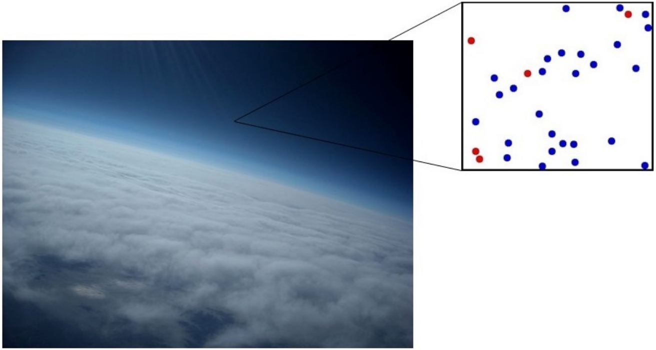


Figure: https://www.flickr.com/photos/23468143@N08/3332216506, https://en.wikipedia.org/wiki/Elasticcollision

What is Boltzmann equation?

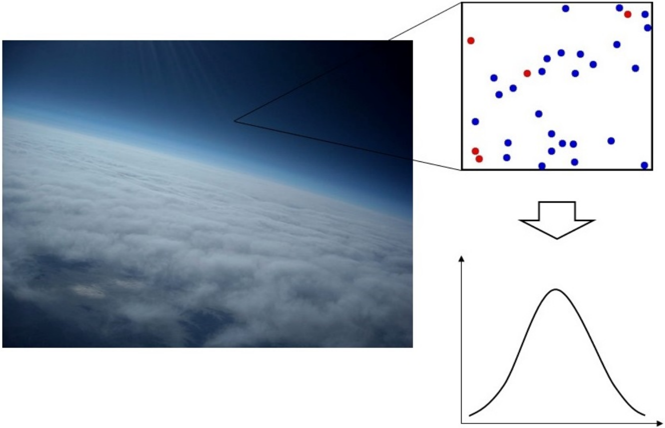


Figure: https://www.flickr.com/photos/23468143@N08/3332216506, https://en.wikipedia.org/wiki/Elasticcollision

Construction of the collision opeartor

- Collision operator is given by

$$Q(F,F) = \int_{\mathbb{R}^9} \delta(v + v_* - v' - v_*')\delta(|v|^2 + |v_*|^2 - |v'|^2 - |v_*'|^2) \times (F(v_*')F(v') - F(v_*)F(v))dv'dv_*'dv_*,$$

or

$$Q(F,F) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |v - v_*|)(F(v_*')F(v') - F(v_*)F(v))d\omega dv_*.$$

Boltzmann Equation (Ludwig Boltzmann (1872))

- Transport + collision → Boltzmann equation!

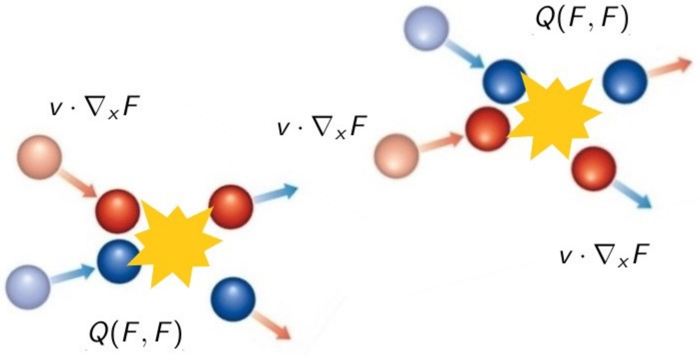
$$\partial_t F + \underbrace{v \cdot \nabla_x F}_{transport} = \underbrace{Q(F,F)}_{collision}.$$

$F(x,v,t)$: velocity distribution function in phase space $(x,v) \in (\Omega \times \mathbb{R}^3)$ and $t \in \mathbb{R}_+$.

The Boltzmann equation

- Transport + collision → Boltzmann equation!

$$\partial_t F + \underbrace{v \cdot \nabla_x F}_{transport} = \underbrace{Q(F,F)}_{collision}.$$



Conservation laws and H-theorem

- Conservation law

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3} F\left(1, v, |v|^2\right) dvdx = 0.$$

- H-Theorem

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3} F\ln F dvdx \leq 0.$$

- Local equilibrium

$$\mathcal{M}(F) = \frac{\rho}{\sqrt{2\pi T}^3} e^{-\frac{|v-U|^2}{2T}}.$$

The BGK model (Bhatnagar-Gross-Krook (1954))

Relaxation operator: $Q(F,F) \to \mathcal{M}(F) - F$

$$\partial_t F + v \cdot \nabla_x F = \mathcal{M}(F) - F,$$

where $\mathcal{M}(F)$ is the local Maxwellian:

$$\mathcal{M}(F) = \frac{\rho(x,t)}{\sqrt{2\pi T(x,t)}^3} \exp\left(-\frac{|v - U(x,t)|^2}{2T(x,t)}\right).$$

The macroscopic fields are defined by

$$\rho(x,t) = \int_{\mathbb{R}^3} F(x,v,t)dv,$$
$$\rho(x,t)U(x,t) = \int_{\mathbb{R}^3} F(x,v,t)vdv,$$
$$3\rho(x,t)T(x,t) = \int_{\mathbb{R}^3} F(x,v,t)|v - U(x,t)|^2 dv.$$

Global equilibrium

- Local equilibrium:

$$\mathcal{M}(F)(x,v,t) = \frac{\rho(x,t)}{\sqrt{2\pi T(x,t)}^3} \exp\left(-\frac{|v-U(x,t)|^2}{2T(x,t)}\right).$$

- Global equilibrium:

$$m(v) = \frac{1}{\sqrt{2\pi}^3} \exp\left(-\frac{|v|^2}{2}\right).$$

Effect of the BGK operator

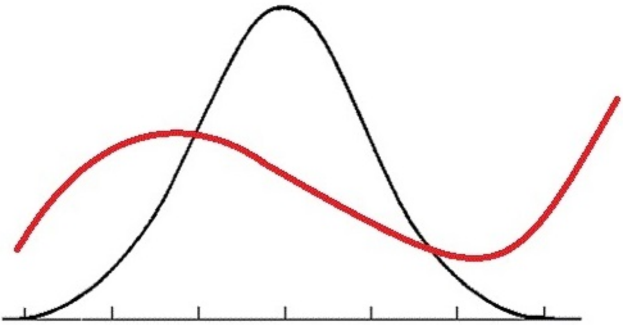


Figure: Operation of the BGK operator

Conservation laws and H-theorem

- Conservation law

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3} F\left(1, v, |v|^2\right) dvdx = 0.$$

- H-Theorem

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3} F\ln F dvdx = \frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3}(\mathcal{M} - F)\ln F dvdx \leq 0.$$

- Local equilibrium

$$\mathcal{M}(F) = \frac{\rho}{\sqrt{2\pi T}^3} e^{-\frac{|v-U|^2}{2T}}.$$

The quantum Boltzmann and BGK model

When do we consider the quantum effects?

- de Broglie wavelength $>$ Characteristic size of the system

$$\lambda = \frac{h}{\sqrt{3mk_B T}} > d \quad \rightarrow \quad T < \frac{h^2}{3mk_B d^2}.$$

- When the temperature is extremly low.
- When the quantum gas is sufficiently rarefied.
- Dyanmics of electrons in conductor, semiconductor, graphene etc.

Quantum Boltzmann equation (Nordheim (1928))

- Quantum Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F).$$

The quantum collision operator is given by for boson,

$$Q(F,F) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |v - v_*|)\Big[F(v_*')F(v')(1 + F(v_*))(1 + F(v)) - F(v_*)F(v)(1 + F(v_*'))(1 + F(v'))\Big] dw dv_*,$$

for fermion.

$$Q(F,F) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |v - v_*|)\Big[F(v_*')F(v')(1 - F(v_*))(1 - F(v)) - F(v_*)F(v)(1 - F(v_*'))(1 - F(v'))\Big] dw dv_*.$$

Derivation: [Benedetto, Castella, Esposito, Pulvirenti 2005].

Conservation laws, H-theorem

- Conservation laws

$$\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}^3} F\left(1, v, |v|^2\right) dvdx = 0.$$

- H-Theorem

$$\frac{d}{dt}\int_{\Omega\times\mathbb{R}^3} F\ln F \mp (1\pm F)\ln(1\pm F)dvdx \leq 0.$$

- Local equilibrium

$$\mathcal{K}(F) = \frac{1}{e^{a|v-b|^2+c}\mp 1}.$$

(Upper sign is for boson, and lower sign is for fermion.)

Quantum BGK model

- Quantum BGK model:

$$\partial_t F + p \cdot \nabla_x F = \mathcal{K}(F) - F.$$

where $\mathcal{K}(F)$ is defined by

$$\mathcal{K}(F)(x,v,t) = \frac{1}{e^{a(x,t)|v-b(x,t)|^2+c(x,t)} \mp 1},$$

subject to

$$\int_{\mathbb{R}^3} \mathcal{K}(F)(x,v,t) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} N(x,t) \\ P(x,t) \\ E(x,t) \end{pmatrix}.$$

The equilibrium $\mathcal{K}$ denotes the Bose-Einstein distribution $\mathcal{B}$ for boson or the Fermi-Dirac distribution $\mathcal{F}$ for fermion.

Quantum equilibrium

Given (N, P, E), there exists equilibrium parameters a, c, k satisfying the following form of the equilibrium. [Escobedo, Mischler, Valle 2003]

- Fermi-Dirac distribution:

$$\mathcal{F}(F) = \begin{cases} \dfrac{1}{e^{a|v-\frac{P}{N}|^2+c}+1}, & \textit{if} \quad \dfrac{N}{\left(E-\frac{P^2}{N}\right)^{3/5}} < \dfrac{(4\pi)^{\frac{2}{5}}5^{\frac{3}{5}}}{3}, \\ \chi_{|v-\frac{P}{N}|\leq\left(\frac{3N}{4\pi}\right)^{\frac{1}{3}}}, & \textit{if} \quad \dfrac{N}{\left(E-\frac{P^2}{N}\right)^{3/5}} = \dfrac{(4\pi)^{\frac{2}{5}}5^{\frac{3}{5}}}{3}. \end{cases}$$

- Bose-Einstein distribution:

$$\mathcal{B}(F) = \begin{cases} \dfrac{1}{e^{a|v-\frac{P}{N}|^2+c}-1}, & \textit{if} \quad \dfrac{N}{\left(E-\frac{P^2}{N}\right)^{3/5}} \leq \beta_{\mathcal{B}}(0), \\ \dfrac{1}{e^{a|v-\frac{P}{N}|^2}-1} + k\delta_{p=\frac{P}{N}}, & \textit{if} \quad \dfrac{N}{\left(E-\frac{P^2}{N}\right)^{3/5}} > \beta_{\mathcal{B}}(0). \end{cases}$$

Equilibrium parameter a and c

We have

$$\frac{N}{\left(E-\frac{P^2}{N}\right)^{\frac{3}{5}}}=\frac{\int_{\mathbb{R}^3}\frac{1}{e^{|v|^2+c}\mp 1}dv}{\left(\int_{\mathbb{R}^3}\frac{|v|^2}{e^{|v|^2+c}\mp 1}dv\right)^{\frac{3}{5}}}\equiv\beta(c).$$

If β is monotone function, c is well defined. After then, we can recover a:

$$a(x,t)=\left(\int_{\mathbb{R}^3}\frac{1}{e^{|v|^2+c(x,t)}\mp 1}dv\right)^{\frac{2}{3}}N(x,t)^{-\frac{2}{3}}.$$

References

- Existence: [Dolbeault 1994], [Lu 2000-2017], [Nouri 2008]
- Finite time blow up: [Escobedo, Velazquez 2014, 2015], [Lu 2014]
- Interact with condensation: [Allemand 2009], [Arkeryd Nouri, 2013, 2015], [Alonso, Gamba, Tran 2018], [Soffer Tran 2018]
- Near equilibrium: [Bae, Yun 2020], [Bae, Jang, Yun 2021], [Ouyang, Wu 2022]

The quantum BGK model near a global equilibrium

Pros and cons of the Boltzmann and BGK model

- Boltzmann model: Relatively high numeric cost, bilinear
- BGK model: Relatively low numeric cost, highly nonlinear

Linearization of the Boltzmann equation

- Linearization of the Boltzmann equation.

$$\partial_t F + v \cdot \nabla_x F = Q(F, F).$$

We substitute $F = m + \sqrt{m} f$ on the equation to have

$$\partial_t f + v \cdot \nabla_x f = Lf + \Gamma(f, f).$$

Linearization of the BGK model

- Linearization of the BGK model.

$$\partial_t F + v \cdot \nabla_x F = \mathcal{M}(F) - F,$$

But the local equilibrium is highly nonlinear.

$$\mathcal{M}(F) = \frac{\rho}{\sqrt{2\pi T}^3} \exp\left(-\frac{|v-U|^2}{2T}\right).$$

The macroscopic fields (ρ, U, T):

$$\rho = \int_{\mathbb{R}^3} F dv, \qquad U = \frac{\rho U}{\rho} = \frac{\int_{\mathbb{R}^3} F v dv}{\int_{\mathbb{R}^3} F dv},$$

$$T = \left(\frac{3\rho T + \rho|U|^2}{3\rho} - \frac{(\rho U)^2}{3\rho^2}\right) = \left(\frac{\int_{\mathbb{R}^3} F|v|^2 dv}{3\int_{\mathbb{R}^3} F dv} - \frac{(\int_{\mathbb{R}^3} F v dv)^2}{3(\int_{\mathbb{R}^3} F dv)^2}\right).$$

Linearization of the BGK model

- [2010 Yun, S.-B.] We substitute $F = m + \sqrt{m} f$ on the BGK model:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{M}(F) - F,$$

then we have

$$\partial_t f + v \cdot \nabla_x f = Pf - f + \Gamma(f, f),$$

where Pf is projection onto L^2_v space with the following orthonormal basis:

$$\left\{ \sqrt{m}, \quad v\sqrt{m}, \quad \frac{|v|^2 - 3}{\sqrt{6}} \sqrt{m} \right\}.$$

So that

$$\langle Pf - f, f \rangle_{L^2_v} = -\|(I - P)f\|^2_{L^2_v}.$$

Linearization of the BGK model

- The idea of the linearization:

$$\mathcal{M}(F) = m + \textit{linear term} + \textit{nonlinear term}.$$

Use Taylor expansion!

$$\mathcal{M}(1) = \mathcal{M}(0) + \mathcal{M}'(0) + \int_0^1 \mathcal{M}''(\theta)(1-\theta)d\theta.$$

We want to make a $\mathcal{M}(\theta)$ which satisfies

$$\mathcal{M}(1) = \mathcal{M}(F), \quad \textit{and} \quad \mathcal{M}(0) = m,$$

and that $\mathcal{M}'(0)$ is linear with respect to f.

Linearization of the BGK model

We make a convex combination with respect to these quantities.

$$\rho_\theta = \theta\rho + (1-\theta), \quad \rho_\theta U_\theta = \theta\rho U,$$

and

$$G_\theta = 3\rho_\theta T_\theta + \rho_\theta |U_\theta|^2 = \theta(3\rho T + \rho|U|^2).$$

Then, we have the following linear term:

$$\mathcal{M}'(0) = \left(\frac{d(\rho_\theta, \rho_\theta U_\theta, G_\theta)}{d\theta}\right)^T \left(\frac{\partial(\rho_\theta, \rho_\theta U_\theta, G_\theta)}{\partial(\rho_\theta, U_\theta, T_\theta)}\right)^{-1} \times \left(\nabla_{(\rho_\theta, U_\theta, T_\theta)}\mathcal{M}(\theta)\right)\Big|_{\theta=0}.$$

Coercivity estimate

The linearized Boltzmann type equation:

$$\partial_t f + v \cdot \nabla_x f = Lf + \Gamma(f, f).$$

The linear term of the Boltzmann type equation satisfy the coercivity:

$$\langle Lf, f\rangle_{L^2_{x,v}} \leq -C\langle (I-P)f, f\rangle_{L^2_{x,v}}.$$

We take $\int f(\cdot)dxdv$ on each sides, then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_{x,v}} &\leq -C\langle (I-P)f, f\rangle_{L^2_{x,v}} + \langle \Gamma(f,f), f\rangle_{L^2_{x,v}} \\ &\leq -C\|(I-P)f\|^2_{L^2_p} \\ &\underbrace{\leq -C\|f\|^2_{L^2_{x,v}}}_{our\ hope}\end{aligned}$$

It can give exponential decay of the L^2 energy norm.

Global existence of the quantum BGK model

Theorem (Bae and Yun: SIMA)

If an initial perturbation is sufficiently small, $\sum_{|\alpha|\leq n} \|\partial^\alpha f_0\|^2_{L^2_{x,v}} \leq \delta$, *for* $n \geq 3$, *(*$\partial^\alpha = \partial_t^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$*) and* $0 \leq F_0(x,v)$, *and* $N_0 E_0^{-3/5} < C$, *then there exists a unique global solution* F *satisfying*

- *The solution is bounded, and not saturated or not condensate*

$$0 \leq F(x,v,t) \qquad\qquad N\left(E - \frac{|P|^2}{N}\right)^{-3/5} < C,$$

- *The perturbation decays exponentially.*

$$\sum_{|\alpha|\leq n} \|\partial^\alpha f(t)\|^2_{L^2_{x,v}} \leq Ce^{-\epsilon t}.$$

Quantum case

The quantum BGK model:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{K}(F) - F. \tag{1}$$

We can linearize the quantum equilibrium

$$\mathcal{K}(F) = m + \sqrt{m \pm m^2} Pf + \sqrt{m \pm m^2} \Gamma(f).$$

Then substituting $F = m + \sqrt{m \pm m^2} f$ on (1) yields

$$\partial_t f + v \cdot \nabla_x f = Pf - f + \Gamma(f).$$

Now the global equilibrium is

$$m(v) = \frac{1}{e^{a_0 |v|^2 + c_0} \mp 1}.$$

Linearization of the quantum equilibrium

We define

$$(N_\theta, P_\theta, E_\theta) = \theta(N, P, E) + (1-\theta)(N_0, P_0, E_0).$$

We make quantum equilibrium with respect to N_θ, P_θ and E_θ.

$$\mathcal{K}(\theta) = \frac{1}{e^{a_\theta|v-\frac{P_\theta}{N_\theta}|^2+c_\theta} \mp 1}.$$

By Taylor expansion around $\theta = 0$, we have

$$\mathcal{K}(1) = \mathcal{K}(0) + \mathcal{K}'(0) + \int_0^1 \mathcal{K}''(\theta)(1-\theta)d\theta.$$

Linearization of the quantum equilibrium

The chain rule gives

$$\mathcal{K}'(0) = \left(\frac{\partial N_\theta}{\partial \theta}\frac{\partial \mathcal{K}(\theta)}{\partial N_\theta} + \frac{\partial P_\theta}{\partial \theta}\frac{\partial \mathcal{K}(\theta)}{\partial P_\theta} + \frac{\partial E_\theta}{\partial \theta}\frac{\partial \mathcal{K}(\theta)}{\partial E_\theta} \right)\bigg|_{\theta=0}$$
$$= (N - N_0)\frac{\partial \mathcal{K}(\theta)}{\partial N_\theta}\bigg|_{\theta=0} + P\frac{\partial \mathcal{K}(\theta)}{\partial P_\theta}\bigg|_{\theta=0} + (E - E_0)\frac{\partial \mathcal{K}(\theta)}{\partial E_\theta}\bigg|_{\theta=0}.$$

- The equilibrium parameters a and c are related to the macroscopic fields:

$$a_\theta = a(N_\theta, P_\theta, E_\theta), \qquad c_\theta = c(N_\theta, P_\theta, E_\theta).$$

Sketch of the linearization process

Finally, we can have

$$\mathcal{K}'(0) = \sqrt{m \pm m^2}Pf,$$

where $Pf \equiv \sum_{i=1}^{5} \langle f, e_i \rangle_{L^2_v} e_i$, and e_i is orthonormal basis generated by

$$\left\{ \sqrt{m \pm m^2}, v\sqrt{m \pm m^2}, |v|^2\sqrt{m \pm m^2} \right\},$$

which gives coercivity property:

$$\langle Pf - f, f \rangle_{L^2_{x,v}} = -\|(I-P)f\|^2_{L^2_{x,v}}.$$

Linearization

- Perturbed equation:

$$\partial_t f + v \cdot \nabla_x f = Pf - f + \Gamma(f, f),$$
$$f(x, v, 0) = f_0(x, v).$$

The relativistic quantum Boltzmann equation near a global equilibrium

Special relativity

1. The laws of physics are invariant in all inertial frames of reference.
2. The speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.

Special relativity

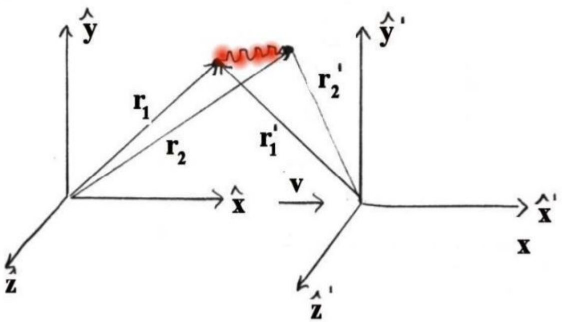


Figure: https://web.mst.edu/ hale/courses/Physics357457/Notes /Lecture.3.Relativity.Lorentz.Invariance/Lecture3.pdf

Special relativity

1. Time is relative (Time dilation)

$$\triangle t' = \frac{\triangle t}{\sqrt{1 - \frac{|v|^2}{c^2}}}.$$

2. Length is relative (Length contraction)

$$L_{rel} = L_0 \sqrt{1 - \frac{|v|^2}{c^2}}.$$

3. Mass is relative

$$m_{rel} = \frac{m_0}{\sqrt{1 - \frac{|v|^2}{c^2}}}.$$

4. Conserved energy

$$E = \sqrt{|p|^2 c^2 + (m_0 c^2)^2}.$$

(c: speed of light, m_0: rest mass)

Relativistic framework

We introduce the energy-momentum 4-vector

$$v^\mu = (v^0, v^1, v^2, v^3) = \left(\frac{E}{c}, v^1, v^2, v^3\right),$$

That is,

$$v^0 = \sqrt{1+|v|^2}.$$

We define a inner product of energy-momentum 4-vector as

$$v^\mu v_{*\mu} = -v^0 v_*^0 + \sum_{i=1}^{3} v^i v_*^i.$$

Then we have $v^\mu v_\mu = -1$ and inner product of energy-momentum 4-vectors is Lorentz invariant $v^\mu v_{*\mu} = \Lambda v^\mu \Lambda v_{*\mu}$.

Relativistic framework

- Momentum conservation law

$$v + v_* = v' + v'_*.$$

- Energy conservation law

$$v^0 + v_*^0 = v'^0 + v_*'^0,$$

where $v^0 = \sqrt{1 + |v|^2}$.

Relativistic framework

The relativistic post-collisional momenta v', v'_* satisfying the conservation laws:

$$v' = \frac{v+v_*}{2} + \frac{g}{2}\left(w - \left(\frac{v^0+v^0_*}{\sqrt{s}} - 1\right)(v+v_*)\frac{(v+v_*)\cdot w}{|v+v_*|^2}\right),$$
$$v'_* = \frac{v+v_*}{2} - \frac{g}{2}\left(w - \left(\frac{v^0+v^0_*}{\sqrt{s}} - 1\right)(v+v_*)\frac{(v+v_*)\cdot w}{|v+v_*|^2}\right),$$

where $w \in \mathbb{S}^2$.

Relativistic quantum Boltzmann equation

- Relativistic quantum Boltzmann equation:

$$\partial_t F + \frac{v}{v^0} \cdot \nabla_x F = Q(F, F, F, F),$$

where

$$\begin{aligned} Q(F_1, F_2, F_3, F_4) = \int_{\mathbb{R}^3} \frac{dv}{v^0} \int_{\mathbb{R}^3} \frac{dv_*}{v_*^0} \int_{\mathbb{R}^3} \frac{dv'}{v'^0} \int_{\mathbb{R}^3} \frac{dv'_*}{v_*'^0} s\omega(g, \theta) \\ \times \delta^{(4)}(v^\mu + v_*^\mu - v'^\mu - v_*'^\mu) \\ \times \big[F_1(v')F_2(v'_*)(1 \pm F_3(v))(1 \pm F_4(v_*)) \\ - (1 \pm F_1(v'))(1 \pm F_2(v'_*))F_3(v)F_4(v_*)\big]. \end{aligned}$$

- The relativistic quantum collision operator satisfies

$$\int_{\mathbb{R}^3} dv \; Q(F,F,F,F) \begin{pmatrix} 1 \\ v^\mu \end{pmatrix} = 0,$$

- Conservation laws:

$$\frac{d}{dt} \int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dv \; F \begin{pmatrix} 1 \\ v^\mu \end{pmatrix} = 0.$$

- *H*-theorem:

$$\frac{d}{dt} \int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dv \; F \ln F \mp (1 \pm F) \ln(1 \pm F) \leq 0.$$

Linearization of the relativistic quantum Boltzmann equation

- Global equilibrium:

$$m(v) = \frac{1}{e^{av^0+c} \mp 1}.$$

- Substituting $F = m + \sqrt{m \pm m^2} f$, we can have the linearized RQBE:

$$\partial_t f + \hat{p} \cdot \nabla_x f + Lf = \Gamma(f),$$
$$f(x, v, 0) = f_0(x, v),$$

where $Lf = \nu(v)f + K_1 f - K_2 f$. $\Gamma(f)$ denotes the nonlinear terms.

Energy norm

Definition

We define a energy norm as

$$\mathcal{E}(f(t)) = \frac{1}{2}\sum_{|\alpha|\leq n}\|\partial^\alpha f(t)\|^2_{L^2_{x,v}} + \int_0^t \sum_{|\alpha|\leq n}\|\partial^\alpha f(s)\|^2_{x,\nu}ds.$$

Theorem (Bae, Jang and Yun, ARMA)

Let $n \geq 3$. Suppose that the initial data F_0 satisfies

$$\begin{cases} 0 \leq F_0(x,v) \leq 1 & \text{for fermions,} \\ 0 \leq F_0(x,v) & \text{for bosons.} \end{cases}$$

Then there exist $\delta > 0$ and $C > 0$ such that if $\mathcal{E}(f_0) \leq \delta$ then there exists a unique global-in-time solution such that

1. *The distribution function $F(x,v,t)$ has the following bounds:*

$$\begin{cases} 0 \leq F(x,v,t) \leq 1 & \text{for fermions,} \\ 0 \leq F(x,v,t) & \text{for bosons.} \end{cases}$$

2. *The energy norm satisfies*

$$\sup_{t\in\mathbb{R}^+} \mathcal{E}(f(t)) \leq C\mathcal{E}(f_0), \qquad \sum_{|\alpha|\leq n} \|\partial^\alpha f(t)\|^2_{L^2_{x,v}} \leq Ce^{-\epsilon t}.$$

Estimate of the nonlinear term

• There appear nonlinear terms involving all possible combination of pre- and post-collisional momenta at the same time:

$$\left|\Gamma(f,h)\right| \leq C \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} dw \; v_{ø}\sigma(g,\theta)J(v_*^0)J(v'^0/2)|f(v)||h(v')|.$$

• The determinant of Jacobian $|\partial v'/\partial v|$ and $|\partial v'_*/\partial v|$ coming from the change of variables $v \mapsto v'$ or v'_* has no uniform lower-bound.

Estimate of the nonlinear term

We lift the dw integral to $dv'dv'_*$ integral imposing a 4-dimensional Dirac-delta function:

$$\left|\langle\Gamma(f,h),\eta\rangle_{L^2_v}\right| \leq C\int_{\mathbb{R}^3}\frac{dv}{v^0}\int_{\mathbb{R}^3}\frac{dv_*}{v^0_*}\int_{\mathbb{R}^3}\frac{dv'}{v'^0}\int_{\mathbb{R}^3}\frac{dv'_*}{v'^0_*}s\omega(g,\theta)$$
$$\times\,\delta^{(4)}(v^\mu+v^\mu_*-v'^\mu-v'^\mu_*)J(v^0_*)J(v'^0/2)|f(v)||h(v')||\eta(v)|.$$

We also lift the dv_* and dv'_* integrals to the relativistic energy-momentum 4-vector integral dv^μ_* and dv'^μ_*:

$$B=\int_{\mathbb{R}^4}dv^\mu_*\int_{\mathbb{R}^4}dv'^\mu_*s\omega(g,\theta)\delta^{(4)}(v^\mu+v^\mu_*-v'^\mu-v'^\mu_*)$$
$$\times\,J(v^0_*)u(v^0_*)u(v'^0_*)\delta(v^\mu_*v_{*\mu}+1)\delta(v'^\mu_*v'_{*\mu}+1).$$

Estimate of the nonlinear term

Finally, we can have

$$\left|\langle \Gamma(f,h),\eta\rangle_{L^2_v}\right| \leq C\left(\|f\|_{L^2_v}\|h\|_\nu + \|f\|_\nu\|h\|_{L^2_v}\right)\|\eta\|_\nu.$$

Thank you for attention!