A-P scheme for the Vlasov-Poisson-Fokker-Planck equation

French-Korean IRL webinar

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The question at hand

- Consider the **Vlasov-Poisson-Fokker-Planck** model

$$\begin{cases} \epsilon\,\partial_t f^\epsilon + v\,\partial_x f^\epsilon - \underbrace{\partial_x \phi^\epsilon\,\partial_v f^\epsilon}_{\text{field interactions}} = \dfrac{1}{\epsilon}\,\underbrace{\partial_v\left[\,v\,f^\epsilon + \partial_v f^\epsilon\,\right]}_{\text{collisions}}, \\ -\,\partial_x^2 \phi^\epsilon = \rho^\epsilon - \rho_i\,, \quad \rho^\epsilon = \displaystyle\int_{\mathbb{R}} f^\epsilon\,dv\,. \end{cases}$$

- $f^\epsilon(t,x,v)$: density of particles at time t, position $x \in \mathbb{T}$, velocity $v \in \mathbb{R}$.

Possible dynamics:

$$\begin{tikzcd} f^\epsilon(t,x,v) \arrow[r, "t \to +\infty"] \arrow[dr, "\epsilon \to 0"'] & \rho_\infty(x)\,\mathcal{M}(v) \\ & \rho(t,x)\,\mathcal{M}(v) \arrow[u, "t \to +\infty"'] \end{tikzcd}$$

- **Gaussian** velocity distribution:

$$\mathcal{M}(v) = \frac{1}{\sqrt{2\,\pi}}\exp\left(-\frac{1}{2}|v|^2\right),$$

- $\rho(t,x)$ solves

$$\begin{cases} \partial_t\,\rho = \nabla_x \cdot \left[\rho\,\nabla_x\,\phi + \nabla_x\,\rho\right], \\ -\,\Delta_x\,\phi = \rho - \rho_i\,. \end{cases}$$

Linear setting

- Case of a given electric field $\partial_x \phi$ and $(x, v) \in \mathbb{T} \times \mathbb{R}$

$$\epsilon\, \partial_t f^\epsilon + v\, \partial_x f^\epsilon - \partial_x \phi\, \partial_v f^\epsilon = \frac{1}{\epsilon}\, \partial_v \left[v\, f^\epsilon + \partial_v f^\epsilon \right] .$$

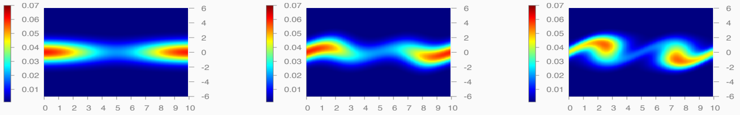
and $f^\epsilon(t, x, v) \to \rho(t, x)\, \mathcal{M}(v)$ as $\epsilon \to 0$ with

$$\partial_t \rho = \partial_x \left[\rho\, \partial_x \phi + \partial_x \rho \right] .$$

We design discrete approximations (f_n^h, ρ_n^h) of (f^ϵ, ρ) such that

Theorem (with F. Filbet, 22')

$$\left\| f_n^h - \rho_n^h \mathcal{M} \right\| \lesssim \epsilon\, (1 + \kappa \Delta t)^{-\frac{n}{2}} + \left(1 + \frac{\Delta t}{2\, \epsilon^2} \right)^{-\frac{n}{2}} . \tag{1}$$



From key estimate to functional space

Dissipation of the L^2-norm

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left|\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}-\sqrt{\rho_\infty}\right|^2\mathrm{d}x\mathcal{M}\mathrm{d}v=-\frac{2}{\epsilon^2}\int\left|\partial_v\left(\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\right)\right|^2\mathrm{d}x\mathcal{M}\mathrm{d}v$$

$\rightarrow$ Functional space :

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}\in L^2\left(\mathrm{d}x\,\mathcal{M}(v)\,\mathrm{d}v\right).$$

- Spectral decomp. in Hermite basis $(H_k)_{k\in\mathbb{N}}$ of $L^2\left(\mathcal{M}\,\mathrm{d}v\right)$

$$\frac{f^\epsilon}{\sqrt{\rho_\infty}\mathcal{M}}(t,x,v)\;=\;\sum_{k\in\mathbb{N}}D_k^\epsilon(t,x)\,H_k(v)\,.$$

- **No weight** with respect to $\mathrm{d}x$ so

$$D_k^\epsilon\in L^2\left(\mathrm{d}x\right).$$

Hermite decomposition

Vlasov-Fokker-Planck equation on $D^\epsilon = (D_k^\epsilon)_{k\in\mathbb{N}}$

$$\epsilon\,\partial_t D_k^\epsilon + \sqrt{k}\,\mathcal{A}\,D_{k-1}^\epsilon - \sqrt{k+1}\,\mathcal{A}^\star\,D_{k+1}^\epsilon = -\frac{k}{\epsilon}\,D_k^\epsilon\,, \quad \forall\, k\in\mathbb{N}\,,$$

with $\mathcal{A}u = \partial_x u + \frac{\partial_x\phi}{2}\,u$.

- Equilibrium is $D_{\infty,k} = \sqrt{\rho_\infty}\,\delta_{k=0}$.

Dissipation of the L^2-norm in Hermite basis

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^\epsilon - D_\infty\right\|_{L^2}^2 = -\frac{2}{\epsilon^2}\sum_{k\in\mathbb{N}^*} k\,\left\|D_k^\epsilon\right\|_{L^2}^2\,.$$

Discrete framework

Fully discrete scheme

$$\frac{D_k^{n+1} - D_k^n}{\Delta t} + \frac{1}{\epsilon}\left(\sqrt{k}\,\mathcal{A}_h\,D_{k-1}^{n+1} - \sqrt{k+1}\,\mathcal{A}_h^\star\,D_{k+1}^{n+1}\right) = -\frac{k}{\epsilon^2}\,D_k^{n+1},$$

for all $k \in \mathbb{N}$ where discrete operators $\mathcal{A}_h$ and $\mathcal{A}_h^\star$ verify

Properties	Preservation
$\langle \mathcal{A}_h u,\, v\rangle_{L^2} = \langle u,\, \mathcal{A}_h^\star v\rangle_{L^2}$	duality structure
$\mathcal{A}_h\,\sqrt{\rho}_\infty = 0$	equilibrium state
$\sum_j \Delta x_j\,(\mathcal{A}_h^\star u)_j\,\sqrt{\rho}_{\infty\,,\,j} = 0$	invariants
$\\|u\\|_{L^2} \leq C_d\,\\|\mathcal{A}_h\,u\\|_{L^2}$	macroscopic coercivity

for all $(u_j)_{j\in\mathcal{J}}$, $(v_j)_{j\in\mathcal{J}}$

Hypocoercivity

We go back to the L^2 estimate

$$\frac{\left\| D^{n+1} - D_\infty \right\|_{L^2}^2 - \left\| D^n - D_\infty \right\|_{L^2}^2}{\Delta t} \leq -\frac{2}{\epsilon^2} \left\| D_\perp^{n+1} \right\|_{L^2}^2 ,$$

with $D_\perp^{n+1} = \left(0\,,\, D_1^{n+1}\,,\, D_2^{n+1}\,,\, D_3^{n+1}, \ldots\right)$.

⚠ Lack of coercivity[1]

$$\boxed{\left\| D^{n+1} - D_\infty \right\|_{L^2}^2 \not\lesssim \left\| D_\perp^{n+1} \right\|_{L^2}^2}$$

[1] Villani (2009)

Illuminating example

Consider

$$\begin{cases} \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} x^\epsilon = + v^\epsilon \\ \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t} y^\epsilon = - x^\epsilon - \dfrac{1}{\epsilon} v^\epsilon \end{cases} .$$

- Relative entropy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|x^\epsilon(t)|^2 + |v^\epsilon(t)|^2\right) = -\frac{2}{\epsilon^2}|v^\epsilon(t)|^2 .$$

- Modified entropy: $\mathcal{H}(f^\epsilon) = |x^\epsilon|^2 + |v^\epsilon|^2 - \alpha\,\epsilon\, x^\epsilon\, v^\epsilon$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f^\epsilon) = -\frac{2}{\epsilon^2}|v^\epsilon(t)|^2 + \alpha\left(|v^\epsilon(t)|^2 - |x^\epsilon(t)|^2 + \frac{1}{\epsilon}x^\epsilon(t)v^\epsilon(t)\right).$$

We deduce $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f^\epsilon) = -\kappa\,\mathcal{H}(f^\epsilon)$ and $|x^\epsilon(t)|^2 + |v^\epsilon(t)|^2 \lesssim e^{-\kappa\, t}$.

Discrete hypocoercivity

Define a modified entropy functional

$$\mathcal{H}_0^n \,=\, \|D^n - D_\infty\|_{L^2}^2 \,+\, \alpha\,\epsilon\,\langle D_1^n, \mathcal{A}_h u_h^n\rangle\;.$$

where u_h^n solves the elliptic problem

$$\begin{cases} (\mathcal{A}_h^\star\,\mathcal{A}_h)\,u_h^n \;=\; D_0^n - D_{\infty,0}\,, \\[2mm] \displaystyle\sum_{j\in\mathcal{J}} \Delta x_j\, u_j\,\sqrt{\rho}_{\infty\,,\,j} \;=\; 0\,, \end{cases}$$

Macroscopic coercivity $\rightarrow$ we recover

$$\begin{cases} \|D^n - D_\infty\|_{L^2}^2 \lesssim \mathcal{H}_0^n \lesssim \|D^n - D_\infty\|_{L^2}^2\,, \\[2mm] \dfrac{\mathcal{H}_0^{n+1} - \mathcal{H}_0^n}{\Delta t} \lesssim -\dfrac{2}{\epsilon^2}(1-\alpha)\,\left\|D_\perp^{n+1}\right\|_{L^2}^2 \,-\, \alpha\left\|D_0^{n+1} - D_{\infty,0}\right\|_{L^2}^2\;. \end{cases}$$

Numerical experiments

Setting

We take $\Delta t = 10^{-3}$, 200 Hermite modes, 64 points in space and

$$\phi(x) \;=\; 0.1\,\cos\left(2\pi\, x\right) \;+\; 0.9\,\cos\left(4\pi\, x\right)\;.$$

First Test: $\epsilon \;=\; 1$ and

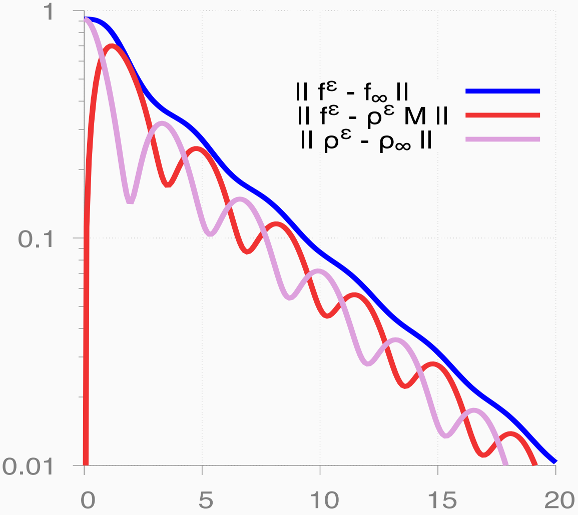
$$f_0(x,v) \;=\; \left(1 + 0.5\cos\left(2\pi\, x\right)\right)\,\exp\left(-|v|^2/2\right)\,/\sqrt{2\pi}\,,$$

Second Test: $\epsilon \;=\; 10^{-4}$ and

$$f_0(x,v) \;=\; \left(1 + 0.5\cos\left(2\pi\, x\right)\right)\,\exp\left(-|v-1|^2/2\right)\,/\sqrt{2\pi}\,.$$

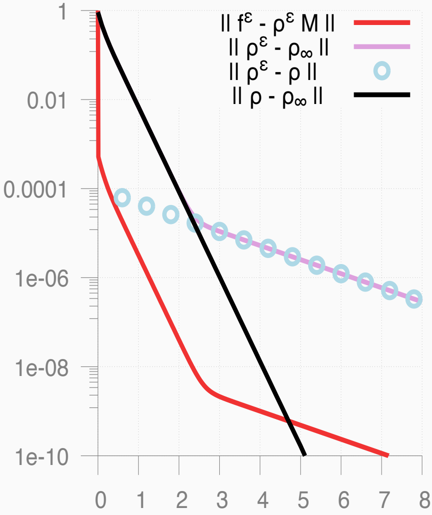
First Test, $\epsilon = 1$

Time evolution (log-scale): $\|f^\epsilon - f_\infty\|_{L^2(f_\infty^{-1})}$ (blue), $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink)



Second Test: $\epsilon = 10^{-4}$

Time evolution (log scale) of $\|f^\epsilon - \rho^\epsilon \mathcal{M}\|_{L^2(f_\infty^{-1})}$ (red), $\|\rho^\epsilon - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (pink), $\|\rho^\epsilon - \rho\|_{L^2(\rho_\infty^{-1})}$ (blue points) and $\|\rho - \rho_\infty\|_{L^2(\rho_\infty^{-1})}$ (black)



We have proven that

$$\|D_\perp^n\| \le \|D_\perp^0\| \left(1 + \frac{\Delta t}{2\,\epsilon^2}\right)^{-\frac{n}{2}} + \epsilon C \left\|D^0 - D_\infty\right\| (1 + \kappa\,\Delta t)^{-\frac{n}{2}},$$

and

$$\left\|D_0^n - \overline{D}_0^n\right\| \le C\epsilon \left\|D^0 - D_\infty\right\| (1 + \kappa\Delta t)^{-\frac{n}{2}},$$

and

$$\left\|\overline{D}^n - D_\infty\right\| \le \left\|\overline{D}^0 - D_\infty\right\| (1 + \tilde{\kappa}\Delta t)^{-\frac{n}{2}},$$

Linearized equation

Linear setting

- Consider the linearized equation

$$\begin{cases} \epsilon\,\partial_t f^\epsilon + v\partial_x f^\epsilon - \partial_x\phi_\infty\partial_v f^\epsilon - \partial_x\phi^\epsilon\partial_v f_\infty = \partial_v\left[\, v\, f^\epsilon + \partial_v f^\epsilon\,\right], \\ \\ -\partial_x^2\phi^\epsilon = \rho^\epsilon - \rho_\infty\,, \quad \rho^\epsilon = \displaystyle\int_{\mathbb{R}^d} f^\epsilon\, dv\,, \end{cases}$$

with $f_\infty(x,v) = \rho_\infty(x)\,\mathcal{M}(v)$ and

$$\begin{cases} \rho_\infty = e^{-\phi_\infty}\,, \\ \\ -\partial_x^2\,\phi_\infty = \rho_\infty - \rho_i\,. \end{cases}$$

We design discrete approximations (f_n^h, ρ_∞^h) of $(f^\epsilon, \rho_\infty)$ such that

Theorem (ongoing)

$$\left\| f_n^h - \rho_\infty^h\mathcal{M} \right\| \lesssim \left(1 + \kappa\frac{\Delta t}{\epsilon}\right)^{-\frac{n}{2}}. \tag{2}$$

Numerical experiments

Plasma echoes

Consider the fully non linear equation

$$\partial_t f + v\partial_x f - \partial_x \phi_\infty \partial_v f - \partial_x \phi \, \partial_v f_\infty - \partial_x \phi \partial_v (f - f_\infty) = \frac{1}{\tau_0} \partial_v \left[vf + \partial_v f \right] ,$$

$$-\partial_x^2 \phi \; = \; \rho - \rho_\infty \, , \quad \rho \; = \; \int_{\mathbb{R}^d} f \, dv \, .$$

- Initial data: resolution on $[-30, 0]$ with $\tau_0 \; = \; 10^6$ and

$$f(-30, x, v) \; = \; f_\infty(x, v) + 0.01 \, cos(x)$$

- Plasma echoes : resolution on $[0, 120]$ with variable τ_0 and

$$\tilde{f}(0, x, v) \; = \; f(0, x, v) + 0.01 \, cos(2 \, x)$$

Construction of the initial data

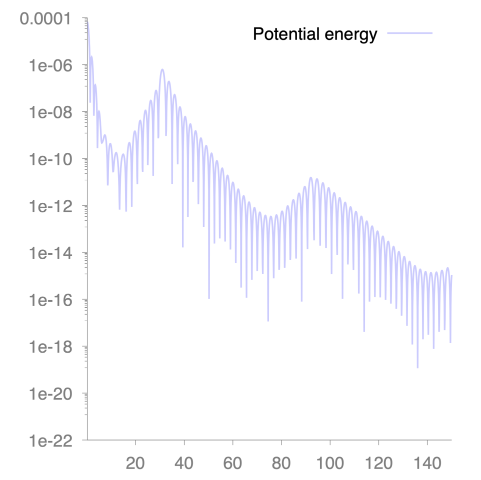
resolution on $[-30, 0]$ with $\tau_0 = 10^6$ (weakly collisional setting)
ANIMATIONS HAVE BEEN REMOVED

Phase space representation of the solution

resolution on $[0, 120]$ with $\tau_0 = 10^6$ (weakly collisional setting)
ANIMATIONS HAVE BEEN REMOVED

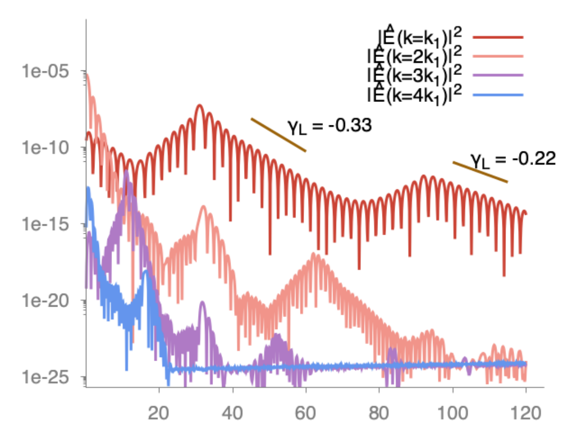
Potential energy

Time evolution of $\mathcal{E}_p(t) = \frac{1}{2} \left\| \partial_x \left(\phi_\infty + \phi \right) \right\|_{L^2}^2$ with $\tau_0 = 10^6$



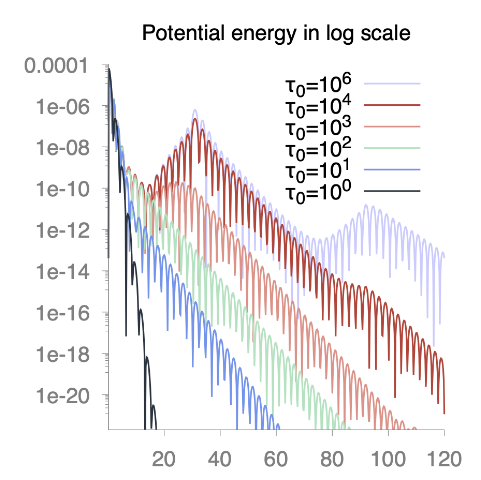
First modes of the electric field

Time evolution of the Fourier modes of the field with $\tau_0 = 10^6$



Suppression of plasma echoes

Time evolution of $\mathcal{E}_p(t) = \frac{1}{2} \left\| \partial_x \left(\phi_\infty + \phi \right) \right\|_{L^2}^2$ with variable τ_0



Perspectives

• quantitative numerical results for the non-linear model in a perturbative setting;

• extending the method/analysis to similar models (second order Kuramato models, Newtonian interactions...);

• quantitative long-time behavior of the non-linear model in non-perturbative setting;

• including collision operators closer to physics (ex: Landau[2])

[2] S. Chaturvedi, J. Luk, T. Nguyen