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A hyperbolic dispersion estimate, with applications to the linear Schrödinger equation

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Abstract. On a Hilbert space $\mathcal{H}$, consider the product $\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1$ of a large number of operators $\hat{P}_j$, with $\|\hat{P}_j\| = 1$. What kind of geometric considerations can serve to prove that the norm $\|\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1\|$ decays exponentially fast with n ? In the first part of this note, we will describe a situation in which $\mathcal{H} = L^2(\mathbb{R}^d)$, and the operators $\hat{P}_j$ are Fourier integral operators associated to a sequence of canonical transformations κ_j. We will give conditions, on the sequence of transformations κ_j and on the symbols of the operators $\hat{P}_j$, under which we can prove exponential decay. This technique was introduced to prove results related to the quantum unique ergodicity conjecture. In the second half of this paper, we will survey applications in scattering situations, to prove the existence of a gap below the real axis in the resolvent spectrum, and to get local smoothing estimates with loss, as well as Strichartz estimates.

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1. Introduction

On a Hilbert space $\mathcal{H}$, consider the product $\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1$ of a large number of operators $\hat{P}_j$, with $\|\hat{P}_j\| = 1$. Think, for instance, of the case where each operator $\hat{P}_j$ is an orthogonal projector, or a product of an orthogonal projector and a unitary operator. What kind of geometric considerations can be helpful to prove that the norm $\|\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1\|$ is strictly less than 1 ? or better, that it decays exponentially fast with n ? In Section 2, we will describe a situation in which $\mathcal{H} = L^2(\mathbb{R}^d)$, and the operators $\hat{P}_j$ are Fourier integral operators associated to a sequence of canonical transformations κ_j. We will give a "hyperbolicity" condition, on the sequence of transformations κ_j and on the symbols of the operators $\hat{P}_j$, under which we can prove exponential decay of the norm $\|\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1\|$.

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This technique was introduced in [1, 2], and was used in [1, 2, 3, 29, 4] to prove results related to the quantum unique ergodicity conjecture, for eigenfunctions of the laplacian on negatively curved manifolds : see Section 3. In the last section of this paper (Section 4), we will survey the work of Nonnenmacher-Zworski [27, 28], Christianson [8, 9, 10], Datchev [12], and Burq-Guillarmou-Hassell [6], who showed how to use the previous estimates in scattering situations, to prove the existence of a gap below the real axis in the resolvent spectrum, and to get local smoothing estimates with loss, as well as Strichartz estimates.

2. The hyperbolic dispersion estimate

In this section, $\mathbb{R}^d \times (\mathbb{R}^d)^*$ is endowed with the canonical symplectic form $\omega = \sum_{j=1}^d dx_j \wedge d\xi_j$, where dx_j denotes the projection of the j-th vector of the canonical basis in $\mathbb{R}^d$, and $d\xi_j$ is the projection of the j-th vector of the dual basis in $(\mathbb{R}^d)^*$. The space $\mathbb{R}^d$ will also be endowed with its usual scalar product, denoted $\langle .,. \rangle$, and we will use it to systematically identify $\mathbb{R}^d$ with $(\mathbb{R}^d)^*$.

We consider a sequence of smooth ($\mathcal{C}^\infty$) canonical transformations $\kappa_n : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$, preserving ω. We will only be interested in the restriction of κ_1 to a fixed relatively compact neighbourhood Ω of 0, and it is actually sufficient for us to assume that the product $\kappa_n \circ \kappa_{n-1} \circ \cdots \circ \kappa_1$ is well defined, for all n, on Ω. The Darboux-Lie theorem ensures that every lagrangian foliation can be mapped, by a symplectic change of coordinates, to the foliation of $\mathbb{R}^d \times \mathbb{R}^d$ by the "horizontal" leaves $\mathcal{L}_{\xi_0} = \{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \xi = \xi_0\}$. Thus, for our purposes, there is no loss of generality if we make the simplifying assumption that each symplectic transformation κ_n preserves this horizontal foliation. It means that κ_n is of the form $(x,\xi) \mapsto (x', \xi' = p_n(\xi))$ where $p_n : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a smooth function. In more elaborate words, κ_n has a generating function of the form $S_n(x, x', \theta) = \langle p_n(\theta), x' \rangle - \langle \theta, x \rangle + \alpha_n(\theta)$ (where $x, x', \theta \in \mathbb{R}^d$, and $\alpha_n : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a smooth function). We have the equivalence

$$\big[(x', \xi') = \kappa_n(x,\xi)\big] \Longleftrightarrow \big[\xi = -\partial_x S_n(x, x', \theta),\ \xi' = \partial_{x'} S_n(x, x', \theta),\ \partial_\theta S_n(x, x', \theta) = 0\big].$$

The product $\kappa_n \circ \ldots \circ \kappa_2 \circ \kappa_1$ also preserves the horizontal foliation, and it admits the generating function

$$\begin{aligned}\langle p_n \circ \ldots \circ p_1(\theta), x' \rangle - \langle \theta, x \rangle + \alpha_1(\theta) + \alpha_2(p_1(\theta)) + \ldots + \alpha_n(p_{n-1} \circ \ldots \circ p_1(\theta)) \\ = \langle p_n \circ \ldots \circ p_1(\theta), x' \rangle - \langle \theta, x \rangle + A_n(\theta),\end{aligned}$$

where the equality defines $A_n(\theta)$.

We will assume that the functions p_n are smooth diffeomorphisms, and that all the derivatives of p_n, of p_n^{-1} and of α_n are bounded uniformly in n. If p is a map $\mathbb{R}^d \longrightarrow \mathbb{R}^d$, we will denote ∇p the matrix $(\frac{\partial p_i}{\partial \theta_j})_{ij}$, which represents its differential in the canonical basis.

Assumptions (H) : We shall be interested in the following operators, acting on $L^2(\mathbb{R}^d)$:

$$\hat{P}_n f(x') = \frac{1}{(2\pi\hbar)^d} \int_{x\in\mathbb{R}^d,\theta\in\mathbb{R}^d} e^{\frac{iS_n(x,x',\theta)}{\hbar}} a^{(n)}(x,x',\theta,\hbar) f(x)dxd\theta,$$

where $\hbar > 0$ is a parameter destined to go to 0. We will assume that the functions $a^{(n)}(x,x',\theta,\hbar)$ have the following properties :

- For a given $\hbar > 0$, the function $(x,x',\theta) \mapsto a^{(n)}(x,x',\theta,\hbar)$ is of class $\mathcal{C}^\infty$;
- The function $a^{(1)}(x,x',\theta,\hbar)$ is supported in Ω with respect to the variable x;
- With respect to the variables (x',θ), the functions $a^{(n)}(x,x',\theta,\hbar)$ have a compact support $x' \in \Omega_1, \theta \in \Omega_2$, independent of n and $\hbar$;
- When $\hbar \longrightarrow 0$, each $a^{(n)}(x,x',\theta,\hbar)$ has an asymptotic expansion

  $$a^{(n)}(x,x',\theta,\hbar) \sim (\det \nabla p_n(\theta))^{1/2} \sum_{k=0}^{\infty} \hbar^k a_k^{(n)}(x,x',\theta),$$

  valid up to any order and in all the $\mathcal{C}^\ell$ norms. Besides, these asymptotic expansions are uniform with respect to n.
- If $(x',\theta') = \kappa_n(x,\theta)$, we have $|a_0^{(n)}(x,x',\theta)| \leq 1$. This condition ensures that $\|\hat{P}_n\|_{L^2 \longrightarrow L^2} \leq 1 + \mathcal{O}(\hbar)$.

The operators $\hat{P}_n$ are (semiclassical) *Fourier integral operators* associated with the transformations κ_n.

2.1. Propagation of a single plane wave.

The following theorem is essentially proved in [1]. We denote $e_{\xi_0,\hbar}$ the function $e_{\xi_0,\hbar}(x) = e^{\frac{i\langle \xi_0, x\rangle}{\hbar}}$.

Theorem 2.1. *Fix $\xi_0 \in \mathbb{R}^d$. In addition to the assumptions above, assume that*

$$\limsup_{n\longrightarrow+\infty} \frac{1}{n} \log \|\nabla(p_n \circ \ldots \circ p_2 \circ p_1)(\xi_0)\| \leq 0.$$

Fix $\mathcal{K} > 0$ arbitrary, and an integer $M \in \mathbb{N}$. Then we have, for $n = \mathcal{K}|\log \hbar|$,

$$\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}(x) = e^{i\frac{A_n(\xi_0)}{\hbar}} e_{\xi_n,\hbar}(x) (\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x,\xi_n)\right] + \mathcal{O}(\hbar^M).$$

The functions $b_k^{(n)}$, defined on $\mathbb{R}^d \times \mathbb{R}^d$, are smooth, and

$$b_0^{(n)}(x_n,\xi_n) = \prod_{j=1}^{n} a_0^{(j)}(x_j, x_{j+1}, \xi_j),$$

where we denote $\xi_n = p_n \circ \ldots \circ p_1(\xi_0)$, $x_n = x$ *and the other terms are defined by the relations* $(x_j, \xi_j) = \kappa_j \circ \ldots \circ \kappa_1(x_0, \xi_0)$.

The next terms $b_k^{(n)}$ *have the same support as* $b_0^{(n)}$. *We have* $|b_0^{(n)}(x_n, \xi_n)| \leq 1$, *and besides, we have bounds*

$$\|d^m b_k^{(n)}\| \leq C(k,m) n^{m+3k},$$

where $C(k,m)$ *does not depend on* n.

If n is fixed, and if we write $\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}(x)$ explicitly as an integral over $(\mathbb{R}^d)^{2n}$, this theorem is a straightforward application of the stationary phase method. If n is allowed to go to infinity as $\hbar \longrightarrow 0$, our result amounts to applying the method of stationary phase on a space whose dimension goes to ∞, and this is known to be very delicate. The theorem was first proved this way, in an unpublished version (available on request or on my webpage) of the paper [1]. A nicer proof is available in [1], and has also appeared under different forms in [2, 27]. In these papers, the proofs are written on a riemannian manifold, for $\hat{P}_n = e^{\frac{i\tau\hbar\Delta}{2}} \hat{\chi}_n$, where the operators $\hat{\chi}_n$ belong to a finite family of pseudodifferential operators, whose symbols are supported inside compact sets of small diameters, and where Δ is the laplacian and $\tau > 0$ is fixed. In local coordinates, the calculations done in [1, 2, 27] amount to the simpler statement presented here. In the unpublished version, the assumptions were much stronger; the transformations κ_j were assumed to be analytic, and the symbols $a^{(n)}$ were taken in a Gevrey class. The result was also much stronger, in that the conclusion held for $n = \hbar^{-\delta}$, for some $\delta > 0$.

In all the papers under review, the dynamical systems under study satisfy a uniform hyperbolicity condition, ensuring an exponential decay

$$\sup_{\xi \in \Omega_2} \|\nabla(p_n \circ \ldots \circ p_2 \circ p_1)(\xi)\| \leq C e^{-\lambda n}, \tag{1}$$

with fixed constants $C, \lambda > 0$. This is why, following [27], we call our result a *hyperbolic dispersion estimate*. Applications will be surveyed in Sections 3 and 4.

2.2. Estimating the norm of $\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1$.

We use the $\hbar$-Fourier transform

$$\mathcal{F}_\hbar u(\xi) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} u(x) e^{-\frac{i\langle \xi, x\rangle}{\hbar}} dx,$$

the inversion formula

$$u(x) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} \mathcal{F}_\hbar u(\xi) e^{\frac{i\langle \xi, x\rangle}{\hbar}} d\xi,$$

and the Plancherel formula $\|u\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}_\hbar u\|_{L^2(\mathbb{R}^d)}$. Using the Fourier inversion formula, Theorem 2.1 implies, in a straightforward manner, the following

Theorem 2.2. *In addition to the assumptions above, assume that*

$$\limsup_{n \longrightarrow +\infty} \frac{1}{n} \log \|\nabla(p_n \circ \ldots \circ p_2 \circ p_1)(\xi)\| \leq 0,$$

uniformly in $\xi \in \Omega_2$.

Fix $\mathcal{K} > 0$ *arbitrary. Then there exists* $\hbar_{\mathcal{K}} > 0$ *such that, for* $n = \mathcal{K}|\log \hbar|$, *and for* $\hbar < \hbar_{\mathcal{K}}$,

$$\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1\|_{L^2 \longrightarrow L^2} \leq \frac{|\Omega_2|^{1/2}}{(2\pi\hbar)^{d/2}} \sup_{\xi \in \Omega_2} |\det \nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2}(1 + \mathcal{O}(n^3\hbar)),$$

where $|\Omega_2|$ *denotes the volume of* Ω_2.

Of course, we always have the trivial bound $\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1\|_{L^2 \longrightarrow L^2} \leq 1 + \mathcal{O}(\hbar|\log \hbar|)$. Since we are working in the limit where $\hbar \longrightarrow 0$, our estimate can only have an interest if we have an upper bound of the form

$$\sup_{\xi \in \Omega_2} |\det \nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2} \leq C e^{-\lambda n}, \qquad \lambda > 0, \tag{2}$$

and if $\mathcal{K}$ is large enough. Note that (2) is weaker than the condition (1).

We now state a refinement of Theorem 2.2. We consider the same family $\hat{P}_i$, satisfying Assumptions (H). The multiplicative constants in our estimate have no importance, and in what follows we will omit them.

Theorem 2.3. *[4] In addition to the assumptions above, assume that*

$$\limsup_{n \longrightarrow +\infty} \frac{1}{n} \log \|\nabla(p_n \circ \ldots \circ p_2 \circ p_1)(\xi)\| \leq 0,$$

uniformly in $\xi \in \Omega_2$.

Let $r \leq d$, *and assume that the isotropic foliation by the leaves* $\{\xi_{r+1} = c_{r+1}, \ldots, \xi_d = c_r\}$ *is invariant by each canonical transformation* κ_n. *In other words, the map* p_n *is of the form*

$$p_n((\xi_1, \ldots, \xi_r), (\xi_{r+1}, \ldots, \xi_d)) = (m_n(\xi_1, \ldots, \xi_d), \tilde{p}_n(\xi_{r+1}, \ldots, \xi_d)),$$

where $m_n : \mathbb{R}^d \longrightarrow \mathbb{R}^r$ *and* $\tilde{p}_n : \mathbb{R}^{d-r} \longrightarrow \mathbb{R}^{d-r}$.

Fix $\mathcal{K} > 0$ *arbitrary. Then there exists* $\hbar_{\mathcal{K}} > 0$ *such that, for* $n = \mathcal{K}|\log \hbar|$, *and for* $\hbar < \hbar_{\mathcal{K}}$,

$$\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1\|_{L^2 \longrightarrow L^2} \leq \frac{1}{(2\pi\hbar)^{r/2}} \frac{\sup_{\xi \in \Omega_2} |(\det \nabla p_n \circ \ldots \circ p_1(\xi))|^{1/2}}{\inf_{\xi \in \Omega_2} |(\det \nabla \tilde{p}_n \circ \ldots \circ \tilde{p}_1(\xi))|^{1/2}} (1 + \mathcal{O}(n^3\hbar)).$$

Theorem 2.3 is an improvement of Theorem 2.2 in the case where we have

$$\frac{1}{(2\pi\hbar)^{d/2}} \sup_{\xi \in \Omega_2} |(\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2}| \gg 1$$

but

$$\frac{1}{(2\pi\hbar)^{r/2}} \frac{\sup_{\xi \in \Omega_2} |(\det \nabla p_n \circ \ldots \circ p_1(\xi))|^{1/2}}{\inf_{\xi \in \Omega_2} |(\det \nabla \tilde{p}_n \circ \ldots \circ \tilde{p}_1(\xi))|^{1/2}} \ll 1.$$

As a trivial example, when each κ_n is the identity, Theorem 2.2 gives a non-optimal bound, whereas we can take $r = 0$ in Theorem 2.3, and recover the (almost) optimal bound $\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1\|_{L^2 \longrightarrow L^2} \leq 1 + \mathcal{O}(\hbar|\log \hbar|^3)$. A less trivial example will appear in Section 3.

3. An application to the quantum unique ergodicity conjecture

3.1. Statement of the conjecture. Let X be a d-dimensional compact riemannian manifold, let Δ denote the Laplace-Beltrami operator on X, and let V be a smooth function on X. In the most general framework, the question of "quantum ergodicity" asks about the behaviour of the solutions of the stationary Schrödinger equation

$$\left(-\hbar^2\frac{\Delta}{2}+V\right)\psi_\hbar = E_\hbar\psi_\hbar, \tag{3}$$

in the limit $\hbar \longrightarrow 0$ and assuming the eigenvalue $E_\hbar$ converges to a fixed value E. We will always assume that the eigenfunction $\psi_\hbar$ is normalized in $L^2(X, \mathrm{Vol})$. Quantum ergodicity asks about the weak limits of the family of probability measures $|\psi_\hbar(x)|^2 d\mathrm{Vol(x)}$. Actually, people are interested in a family of distributions $\mu_\hbar$ on the cotangent bundle T^*X, that contain more information, defined as follows :

$$\forall a \in \mathcal{C}_c^\infty(T^*X), \langle \mu_\hbar, a\rangle = \langle \psi_\hbar, \mathrm{Op}_\hbar(a)\psi_\hbar\rangle_{L^2(X)}, \tag{4}$$

where $\mathrm{Op}_\hbar(a)$ is a semiclassical pseudodifferential operator with principal symbol a (if $a = a(x,\xi)$, then $\mathrm{Op}_\hbar(a) = a(x, -i\hbar\partial_x)$, and this can be defined properly using the Weyl calculus in local coordinates). The distribution $\mu_\hbar$ appears under various names in the literature, depending on the specific context : Wigner transform, semiclassical/microlocal defect measure, microlocal lift of $\psi_\hbar$... Although the definition of $\mu_\hbar$ depends on the choice of local coordinates, the collection of weak limits of $\mu_\hbar$, as $\hbar \longrightarrow 0$, is well defined, independently on any choices. Besides, the definition (4) can be extended to the case whn a is a function on T^*X depending only on the base point x, and in that case $\mathrm{Op}_\hbar(a)$ is the multiplication operator by a. We see that the projection of $\mu_\hbar$ on X is the probability measure $|\psi_\hbar(x)|^2 d\mathrm{Vol(x)}$ that we were originally interested in. The distribution $\mu_\hbar$ contains more information, it tells us something about the local directions of oscillations of $\psi_\hbar$.

The following is a form of the theorem of propagation of singularities, due to Hörmander. Define the function $H(x,\xi) = \frac{\|\xi\|_x^2}{2} + V(x)$, on T^*X – where $\|.\|_x^2$ is the norm on T_x^*X dual to the riemannian metric. Denote (Φ_H^t) the hamiltonian flow defined by H, acting on T^*X. In local coordinates, the flow (Φ_H^t) is defined by the following first order differential equation :

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial \xi} \\ \\ \dot{\xi} = -\frac{\partial H}{\partial x}\ . \end{cases} \tag{5}$$

We will denote by Y_H, or simply Y, the vector field on T^*X associated with this flow.

Theorem 3.1. *(i) Given any sequence $\hbar_n \longrightarrow 0$, one can extract from the sequence $(\mu_{\hbar_n})$ a converging subsequence in $\mathcal{D}'(T^*X)$.*

We will call limits of such subsequences "semiclassical measures" associated with the family $(\psi_\hbar)$.

(ii) Let μ be a semiclassical measure. Then μ is a probability measure, carried by the level set $\{H = E\}$.

(iii) In addition, μ is invariant by the hamiltonian flow (Φ_H^t) : we have $(\Phi_H^t)_*\mu = \mu$, for all t.

This theorem, in general, does not suffice to characterize a unique limit μ, as there are generally many invariant measures under (Φ_H^t). A hamiltonian flow on T^*X always preserves the Liouville measure, defined in local coordinates by $dxd\xi$: this measure, or more precisely its disintegration on $\{H = E\}$, is a candidate to be a semiclassical measure. If the flow (Φ_H^t) has periodic orbits on the energy level $\{H = E\}$, each of them carries an invariant measure, which is also a candidate to be a semiclassical measure. Characterizing the set of semiclassical measures is, in such generality, an open question. The two most studied cases are *completely integrable* hamiltonian flows on the one hand, *"chaotic"* flows on the other hand. In what follows we will focus on the "chaotic" case, and will give a more precise definition of this term.

Until the end of this section, we turn to a special case which has been most studied, and is a source of numerous open questions : the case when $V = 0$. In this case, (Φ_H^t) is the geodesic flow; we shall simply denote it by (Φ^t). We consider the case of a non-singular energy level, in other words $E \neq 0$, and since in this case the function H is homogeneous with respect to ξ, we may decide without loss of generality to take $E = \frac{1}{2}$. Then the level set $\{H = E\}$ is the unit cotangent bundle S^*X. Letting $\lambda = \frac{E_\hbar}{\hbar^2}$, equation (3) amounts to studying the eigenfunctions of the laplacian,

$$-\Delta\phi_\lambda = \lambda\phi_\lambda,$$

in the limit $\lambda \longrightarrow +\infty$. We recall that, on a compact manifold, the eigenvalues λ form a discrete set. We denote $\mu_\lambda \in \mathcal{D}'(T^*X)$ the distribution defined previously, by $\langle\mu_\lambda, a\rangle = \langle\phi_\lambda, \mathrm{Op}_{\lambda^{-1/2}}(a)\phi_\lambda\rangle$. We can rephrase our question by asking : among the invariant probability measures of the geodesic flow, which ones can be obtained as limits of the family (μ_λ) ? does the answer depend on the geometry ? The following theorem is referred to as "the Shnirelman theorem", or "the quantum ergodicity theorem". It was later extended to more general hamiltonian flows [20], and to the case of manifolds with a boundary (when X has a boundary, one has to impose boundary conditions to the eigenfunctions) [18].

Theorem 3.2. *[33, 38, 11] Let X be a compact riemannian manifold. Let (ϕ_n) be an orthonormal basis of $L^2(X)$ formed by eigenfunctions of the laplacian ($-\Delta\phi_n = \lambda_n\phi_n$, with $\lambda_n \longrightarrow +\infty$). Denote $\mu_n = \mu_{\lambda_n}$.*

Assume that the geodesic flow, acting on the unit cotangent bundle S^*X, is ergodic with respect to the Liouville measure. Then, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density 1, such that

$$\mu_n \stackrel{n\in\mathcal{S}}{\longrightarrow} \text{Liouville},$$

the convergence taking place in $\mathcal{D}'(T^*X)$.

The set $\mathcal{S}$ being of density 1 means that $\frac{\sharp \mathcal{S}\cap[0,N]}{N} \underset{N\to+\infty}{\longrightarrow} 1$.

It is a difficult question to know whether the whole sequence (μ_n) converges, or if there can be exceptional subsequences converging to a measure other than Liouville. Of course, the answer depends on the geometry. A particularly frustrating example is the case where X is a euclidean domain in $\mathbb{R}^2$ in the shape of a stadium, called the Bunimovich stadium. In this example, it is quite clear in numerical simulations that, although Shnirelman's theorem holds, there are also exceptional subsequences concentrating on the periodic trajectories that bounce back and forth between the two parallel sides of the stadium. The first breakthrough in that direction was made in 2008 by Hassell [19], who showed, for "almost all stadia", that there are exceptional subsequences of eigenfunctions.

If X is a compact riemannian manifold with negative sectional curvatures, Rudnick and Sarnak conjectured that, for any orthonormal basis of eigenfunctions (ϕ_n), the whole sequence (μ_n) should converge to the Liouville measure : this is referred to as the quantum unique ergodicity conjecture [31]. A special case of this conjecture, called arithmetic quantum unique ergodicity, was proved by Lindenstrauss [25, 5], with the final touch by Soundararajan in the case of the modular surface [36]. They deal with the case of certain hyperbolic surfaces, called arithmetic congruence surfaces; and the eigenfunctions (ϕ_n) are assumed to be common eigenfunctions of Δ and of the Hecke operators ([36] shows that there is no escape of mass to infinity, in the case of noncompact finite volume arithmetic surfaces, such as the modular surface). The methods therein are a very powerful mixture of number theory and ergodic theory. They give, unfortunately, no clue as to the general conjecture.

3.2. Entropy of semiclassical measures on hyperbolic manifolds. The papers [1, 2, 3] deal with the question of quantum unique ergodicity by studying the *Kolmogorov-Sinai entropy* of semiclassical measures. This entropy, denoted h_{KS} in this paper, is a functional going from the set $\mathcal{M}^1_\Phi(S^*X)$ of Φ^t-invariant probability measures on S^*X, to $\mathbb{R}_+$. The shortest definition of the entropy results from a theorem due to Brin and Katok [7]. For any time $T>0$, introduce a distance on S^*X,

$$d_T(\rho,\rho') = \max_{t\in[-T/2,T/2]} d(\Phi^t\rho, \Phi^t\rho'),$$

where d is the distance built from the Riemannian metric. For $\epsilon>0$, denote by $B_T(\rho,\epsilon)$ the ball of centre ρ and radius ϵ for the distance d_T. When ϵ is fixed and T goes to infinity, it looks like a thinner and thinner tubular neighbourhood of the geodesic segment $[g^{-\epsilon}\rho, g^{+\epsilon}\rho]$ (this tubular neighbourhood is of radius $e^{-T/2}$ if the curvature of X is constant and equal to -1).

Let μ be a Φ^t–invariant probability measure on S^*X. Then, for μ-almost every

ρ, the limit

$$\lim_{\epsilon\longrightarrow 0}\liminf_{T\longrightarrow+\infty} -\frac{1}{T}\log\mu\big(B_T(\rho,\epsilon)\big)$$
$$= \lim_{\epsilon\longrightarrow 0}\limsup_{T\longrightarrow+\infty} -\frac{1}{T}\log\mu\big(B_T(\rho,\epsilon)\big) \stackrel{\text{def}}{=} h_{KS}(\mu,\rho)$$

exists and it is called the local entropy of the measure μ at the point ρ (it is independent of ρ if μ is ergodic). The Kolmogorov-Sinai entropy is the average of the local entropies: $h_{KS}(\mu) = \int h_{KS}(\mu,\rho)d\mu(\rho)$.

We recall the following (non obvious) facts :

- if $\mu \in \mathcal{M}^1_\Phi(S^*X)$ is carried by a periodic trajectory of Φ^t, then $h_{KS}(\mu) = 0$.
- for all $\mu \in \mathcal{M}^1_\Phi(S^*X)$, we have $0 \leq h_{KS}(\mu) \leq \int_{S^*X}\sum_{j=1}^{d-1}\lambda_j^+(\rho)d\mu(\rho)$, where the numbers $\lambda_j^+(\rho)$ are the nonnegative Lyapunov exponents of $\rho \in S^*X$ for the geodesic flow (the Ruelle-Pesin inequality). Note that S^*X has dimension $2d-1$. Because the flow is symplectic, there can be at most $d-1$ positive Lyapunov exponents and $d-1$ negative ones.
- If X has negative sectional curvatures, there is equality in the Ruelle-Pesin inequality if and only if μ is the Liouville measure [24].
- the functional h_{KS} is affine.

If the sectional curvature of X is constant equal to -1, the Ruelle-Pesin inequality takes the simpler form : $h_{KS}(\mu) \leq d-1$, with equality if and only if μ is the Liouville measure.

The assumption on the curvature implies that the action of (Φ^t) on S^*X is (uniformly) hyperbolic. This means that, for any $\rho \in S^*X$, the tangent space to S^*X at ρ splits into flow direction, unstable and stable subspaces : there exist $C, \lambda > 0$, and at each $\rho \in S^*X$ a splitting $T_\rho(S^*X) = \mathbb{R}Y(\rho) \oplus E^+_\rho \oplus E^-_\rho$, $\dim E^\pm_\rho = d-1$, such that

(i) For all $\rho \in S^*X$, $d\Phi^t_\rho E^\pm_\rho = E^\pm_{\Phi^t(\rho)}$ for all $t \in \mathbb{R}$;

(ii) For all $\rho \in S^*X$, for all $v \in E^\mp_\rho$, $\|d\Phi^t_\rho.v\| \leq Ce^{-\lambda|t|}\|v\|$, for $\pm t > 0$.

Uniform hyperbolicity is a very strong, and very well understood, form of "chaos".

Let us define the unstable jacobian by

$$\exp\Lambda^+_t(\rho) = \det(d\Phi^t\rceil_{E^+_\rho});$$

for t large enough, we have $\Lambda^+_t(\rho) > 0$ for all ρ.

The following form of Theorem 2.2 is used in [1, 2]. Fix $\delta > 0$, and consider a finite family $\chi_1, \ldots, \chi_K$ of smooth compactly supported functions on T^*X, such that $\sum_{j=1}^K \chi_j \equiv 1$ on $H^{-1}[\frac12 - \delta, \frac12 + \delta]$. For all j, assume the function χ_j is supported on a set W_j of diameter $\leq \varepsilon$ (that will be chosen small enough). Also assume that each χ_j vanishes outside $H^{-1}[\frac12 - 2\delta, \frac12 + 2\delta]$. Consider the associated pseudodifferential operators, defined by the Weyl calculus in local coordinates : $\hat{\chi}_j = \chi_j(x, -i\hbar\partial_x)$. Define $\hat{P}_j = e^{\frac{i\tau\hbar\Delta}{2}}\hat{\chi}_j$, for some fixed time step $\tau > 0$. The following theorem amounts to Theorem 2.2 if one works in adapted coordinates in each set W_j :

Theorem 3.3. *In the definition of $\hat{\chi}_j$, we can fix ε, δ small enough, so that the following holds.*

Fix $\mathcal{K} > 0$ arbitrary. Then there exists $\hbar_{\mathcal{K}} > 0$ such that, for $n = \mathcal{K}|\log \hbar|$, for any sequence $(\alpha_1, \ldots, \alpha_n) \in \{1, \ldots, K\}^n$, and for all $\hbar < \hbar_{\mathcal{K}}$,

$$\|\hat{P}_{\alpha_n} \circ \ldots \circ \hat{P}_{\alpha_2} \circ \hat{P}_{\alpha_1}\|_{L^2 \longrightarrow L^2} \leq \frac{1}{(2\pi\hbar)^{d/2}} \prod_{j=1}^{n} e^{\frac{S_\tau(W_{\alpha_j})}{2}},$$

where $S_\tau(W_j) = -\inf_{\rho \in W_j} \Lambda_\tau^+(\rho)$.

If τ is chosen large enough, then the hyperbolicity condition implies that $S_\tau(W_j) < 0$, and that $\prod_{j=1}^n e^{\frac{S_\tau(W_{\alpha_j})}{2}}$ decays exponentially with n. If the sectional curvature of X is constant, equal to -1, the estimate takes a simpler form :

Theorem 3.4. *Assume that the sectional curvature of X is constant, equal to -1. In the definition of $\hat{\chi}_j$, we can fix ε, δ small enough, so that the following holds.*

Fix $\mathcal{K} > 0$ arbitrary. Then there exists $\hbar_{\mathcal{K}} > 0$ such that, for $n = \mathcal{K}|\log \hbar|$, for any sequence $(\alpha_1, \ldots, \alpha_n) \in \{1, \ldots, K\}^n$, and for all $\hbar < \hbar_{\mathcal{K}}$,

$$\|\hat{P}_{\alpha_n} \circ \ldots \circ \hat{P}_{\alpha_2} \circ \hat{P}_{\alpha_1}\|_{L^2 \longrightarrow L^2} \leq \frac{1}{(2\pi\hbar)^{d/2}} e^{-\left(\frac{d-1}{2}\right)n}(1 + \mathcal{O}(\delta))^n.$$

In [1, 2, 3], we showed how these estimates imply the following lower bound on the entropy of semiclassical measures.

Theorem 3.5. *Let X be a compact d-dimensional riemannian manifold, with negative sectional curvatures. Let (ϕ_λ) be a family of normalized eigenfunctions of the laplacian, $\Delta\phi_\lambda = -\lambda\phi_\lambda$, with $\lambda \longrightarrow +\infty$, and let μ be an associated semiclassical measure. Then :*

[1] We have $h_{KS}(\mu) > 0$.

[2] If the sectional curvature of X is constant, equal to -1, we have $h_{KS}(\mu) \geq \frac{d-1}{2}$.

Remark. *In the case of arithmetic congruence surfaces; and assuming the eigenfunctions (ϕ_λ) are common eigenfunctions of Δ and of the Hecke operators, Bourgain and Lindenstrauss [5] proved the following bound on the measures μ_λ : for any ρ, and all $\epsilon > 0$ small enough,*

$$\mu_\lambda\big(B_T(\rho, \epsilon)\big) \leq Ce^{-T/9}, \tag{6}$$

where the constant C does not depend on ρ or λ. This immediately yields that any semiclassical measure associated with these eigenmodes satisfies $\mu(B_T(\rho, \epsilon)) \leq Ce^{-T/9}$, which implies that any ergodic component of μ has entropy $\geq \frac{1}{9}$. The *measure classification result of [25] then implies that* μ *has to be the Liouville measure.*

In [2], Theorem 3.4 is used to prove an estimate that can, in a non rigourous but intuitive manner, be formulated as follows :

$$\mu_\lambda\big(B_T(\rho,\epsilon)\big) \leq C\,\lambda^{\frac{d-1}{4}}\,e^{-\frac{(d-1)T}{2}}. \tag{7}$$

This bound only becomes non-trivial for times $T \gg \log\lambda$*. For this reason, we cannot directly deduce bounds on the weights* $\mu(B_T(\rho,\epsilon))$*; the link between* (7) *and the entropic bounds of Theorem 3.5 is less direct and uses some specific features of quantum mechanics.*

By the properties of entropy, our Theorem 3.5 implies :

Corollary 3.6. *Under the same assumptions,*

[1] If X *has (variable) negative sectional curvature, and if* γ *is a periodic trajectory of* Φ^t*, then* $\mu(\gamma) < 1$.

[2] If the sectional curvature of X *is constant, equal to* -1*, and if* γ *is a periodic trajectory of* Φ^t*, then* $\mu(\gamma) \leq \frac{1}{2}$.

Corollary 3.7. *[1] If the sectional curvature of* X *is constant, equal to* -1*, then the Hausdorff dimension of the support of* μ *is* $\geq d$.

If X has (variable) negative sectional curvature, we conjectured the following explicit bound for any semiclassical measure μ :

$$h_{KS}(\mu) \geq \frac{1}{2}\int_{S^*X}\sum_{j=1}^{d-1}\lambda_j^+(x,\xi)d\mu(x,\xi). \tag{8}$$

However, in variable curvature, we were not able to push our method that far. This inequality has been proved in the case $d = 2$ by G. Rivière, who was able to extend the proof to nonpositively curved surfaces [29, 30]. In this case, the inequality implies that μ cannot be entirely concentrated on an exponentially unstable closed geodesic.

Proving the quantum unique ergodicity conjecture would be equivalent to getting rid of the $\frac{1}{2}$ factor in (8). This is still far from reach, and would require some new insight into the problem, as there exists an example of a discrete time quantum dynamical system (namely, the "quantum cat-map" [16]) for which equality is reached in (8). This example, however, comes from a symplectic map that is not hamiltonian; see [16] for details.

At the moment, it is not known how to prove (8) when E^+ or E^- have dimension $d-1 > 1$; or for general non-uniformly hyperbolic systems. The Bunimovich stadium would be a particularly interesting example : the inequality would imply that μ cannot be entirely concentrated on an exponentially unstable periodic trajectory. It would be also be interesting to prove (8) for systems that have some zero Lyapunov exponents. This is one of the motivations for the following paragraph.

3.3. Generalization to higher rank symmetric spaces of non-positive curvature.

Let G be a connected semisimple Lie group with finite center, let K be a maximal compact subgroup, and G/K the corresponding symmetric space. Let Γ be a cocompact lattice in G, and $X = \Gamma\backslash G/K$.

Example. *Taking $G = \mathrm{SO}_o(1, d; \mathbb{R})$, $K = \mathrm{SO}(d; \mathbb{R})$, one finds that G/K is the d-dimensional real hyperbolic space, which was already treated in the previous paragraph. In this section, one should keep in mind the case $G = \mathrm{SL}(n; \mathbb{R})$, $K = \mathrm{SO}(n; \mathbb{R})$. For $n = 2$, G/K is again the 2-dimensional real hyperbolic space, but from now on we will mostly be interested in $n \geq 3$.*

We will denote by $\mathfrak{g}$ the Lie algebra of G; it is endowed with the Killing bilinear form, which allows to endow G/K with a riemannian metric. We keep using similar calligraphy for Lie subalgebras of $\mathfrak{g}$.

The spectral problem. We look at the algebra $\mathcal{D}$ of G-invariant differential operators on G/K. As a consequence of the structure of semisimple Lie algebras, it is known that $\mathcal{D}$ is commutative, finitely generated. The number of generators r coincides with the *real rank* of G/K, the dimension of a maximal *flat totally geodesic submanifold*; or with the dimension of $\mathfrak{a}$, a maximal *abelian subalgebra* of $\mathfrak{g}$ contained in $\mathfrak{k}^{\perp}$.

Note that $\mathcal{D}$ always contains the laplacian. If $r = 1$, $\mathcal{D}$ is generated by the laplacian, but we will mostly be interested in the case $r \geq 2$.

Example. *For $G = \mathrm{SL}(n, \mathbb{R}), K = \mathrm{SO}(n, \mathbb{R})$, the subalgebra $\mathfrak{a}$ is the set of diagonal matrices with vanishing trace. We will denote by A the connected subgroup of G generated by $\mathfrak{a}$, it consists of diagonal matrices with determinant 1 and nonnegative entries. The rank is the dimension of $\mathfrak{a}$, $r = n - 1$. We denote the Weyl group by W, in this example it is the group of permutation matrices. It acts on $\mathfrak{a}$ (and on its dual$\mathfrak{a}^*$).*

We look at the common eigenfunctions of $\mathcal{D}$ on $X = \Gamma\backslash G/K$. The "eigenvalue" is now an r-dimensional vector. In fact, an eigenfunction of $\mathcal{D}$ generates a spherical irreducible representation of G, and these are naturally parametrized by $\nu \in \mathfrak{a}^*/W$. In what follows, the "eigenvalue" will be parametrized by the spectral parameter $\nu \in \mathfrak{a}^*/W$.

The semiclassical limit (as proposed by Silberman-Venkatesh [34]). It consists in the limit

$$\|\nu\| \longrightarrow +\infty, \qquad \frac{\nu}{\|\nu\|} \longrightarrow \nu_\infty. \tag{9}$$

To keep semiclassical notations, one can define $\hbar = \|\nu\|^{-1}$.

We are again interested in the question of quantum ergodicity, which consists in studying a sequence of L^2-normalized eigenfunctions ϕ_ν, of spectral parameters ν, in the asymptotic regime described above. We want to understand the behaviour of the measures $|\phi_\nu(x)|^2 d\mathrm{Vol}(x)$.

The "classical" dynamical system. Consider the algebra $\mathcal{H}$ of G-invariant smooth hamiltonians (i.e. functions) on the cotangent bundle $T^*(G/K)$, that are polynomial in the fibers of the projection $T^*(G/K) \longrightarrow G/K$. Again by the structure of semisimple Lie algebras, $\mathcal{H}$ is commutative under the Poisson bracket, generated by r functions. The algebra $\mathcal{H}$ always contains the quadratic form associated with the Killing metric. Common energy levels of $\mathcal{H}$ are naturally parametrized by $\nu \in \mathfrak{a}^*/W$. We will denote by $\mathcal{E}_\nu$ the energy layer corresponding to the value ν.

We will restrict our attention to non-singular energy levels, in the sense that the generators of $\mathcal{H}$ must have everywhere independent differentials. This is equivalent to ν not being fixed by any element of W : in this case we will say that ν is regular.

The microlocal lift. The measures $|\phi_\nu(x)|^2 d\mathrm{Vol}(x)$ are defined on X. Just as in (4), we study the distributions $\mu_\nu(a) = \langle \phi_\nu, \mathrm{Op}_\hbar(a)\phi_\nu\rangle$ (with $\hbar = \|\nu\|^{-1}$), $\mu_\nu \in \mathcal{D}'(T^*X)$, which project on X to the measure $|\phi_\nu(x)|^2 d\mathrm{Vol}(x)$.

If $a = H \in \mathcal{H}$, then $\mathrm{Op}_\hbar(H)$ is in $\mathcal{D}$, and the isomorphism $H(-i\hbar\bullet) \longleftrightarrow \mathrm{Op}_\hbar(H)$ is the Harish-Chandra isomorphism between $\mathcal{H}$ and $\mathcal{D}$.

The analogue of Theorem 3.1 reads :

Theorem 3.8. *(i) Given any sequence (ν_n) satisfying (9), one can extract from the sequence (μ_{ν_n}) a converging subsequence in $\mathcal{D}'(T^*X)$.*

We will call limits of such subsequences "semiclassical measures" associated with the family (ϕ_{ν_n}), or also "semiclassical measures in the direction ν_∞".

(ii) Let μ be a semiclassical measure in the direction ν_∞. Then μ is a probability measure, carried by the level set $\mathcal{E}_{\nu_\infty}$.

(iii) In addition, for all $H \in \mathcal{H}$, μ is invariant by the hamiltonian flow (Φ_H^t) : we have $(\Phi_H^t)_*\mu = \mu$, for all t.

One can extend the quantum unique ergodicity conjecture to this new situation : is it true that the only semiclassical measure in the direction ν_∞ is the Liouville measure on the energy level $\mathcal{E}_{\nu_\infty}$?

Analogously to (8), I would expect the following inequality to hold, for any semiclassical measure μ and all $H \in \mathcal{H}$:

$$h_{KS}(\mu, \Phi_H^t) \geq \frac{1}{2}\sum_j \lambda_j^+(\Phi_H^t).$$

Here $h_{KS}(\mu, \Phi_H^t)$ is the entropy of μ for the flow generated by H, and the $\lambda_j^+(\Phi_H^t)$ are the nonnegative Lyapunov exponents for that flow (since we are on a homogeneous space, each $\lambda_j^+(\Phi_H^t)$ is a constant function). However, the method of [2], so far, can only be pushed to prove the bound :

$$h_{KS}(\mu, \Phi_H^t) \geq \sum_j \left(\lambda_j^+(\Phi_H^t) - \frac{\lambda_{\max}(\Phi_H^t)}{2}\right), \qquad (10)$$

where the sum is over all j, and $\lambda_{\max}(\Phi_H^t)$ is the largest of the Lyapunov exponents $\lambda_j^+(\Phi_H^t)$. The right-hand side is, in general, negative, and the lower bound is trivial.

In [4], we are able to prove an explicit, non-trivial lower bound. To do so, we need to get rid of the low Lyapunov exponents in (10). This is where we use the refined norm estimate Theorem 2.3. From now on, we assume that ν_∞ regular.

Theorem 3.9. *[4] Let μ be a semiclassical measure associated to the limit* (9)*. Assume that ν_∞ regular.*

For any $H \in \mathcal{H}$,

$$h_{KS}(\mu, \Phi_H^t) \geq \sum_{j, \lambda_j^+(\Phi_H^t) \geq \frac{\lambda_{\max}(\Phi_H^t)}{2}} \left(\lambda_j^+(\Phi_H^t) - \frac{\lambda_{\max}(\Phi_H^t)}{2} \right).$$

We note that, unless H is a constant function, the entropy lower bound given by Theorem 3.9 is always positive.

One reason to study this problem is that, when the rank r is ≥ 2, the commuting flows (Φ_H^t) $(H \in \mathcal{H})$ are expected to have few joint invariant measures. As a consequence, quantum unique ergodicity should be easier to prove.

To explain what is known about the joint invariant measures of the family (Φ_H^t), we translate everything from the language of hamiltonian flows to the language of group actions. For simplicity we stick to the case $G = \mathrm{SL}(n, \mathbb{R}), K = \mathrm{SO}(n, \mathbb{R})$. If $\mathcal{E}_{\nu_\infty} \subset T^*X$ is a regular energy level of $\mathcal{H}$, it is known that there is a G-equivariant identification between $\mathcal{E}_{\nu_\infty}$ and $\Gamma\backslash G/M$, where M is the group of diagonal matrices of determinant 1 and entries ± 1. Under this identification, the action of the flows $(\Phi_H^t)_{H \in \mathcal{H}}$ on $\mathcal{E}_{\nu_\infty}$ is transported to the right action of the group A on $\Gamma\backslash G/M$. More precisely, if $H \in \mathcal{H}$ is seen as a polynomial function on $\mathfrak{g}^*$, the hamiltonian flow (Φ_H^t) is transported to the 1-parameter subgroup e^{tZ} of A, with $Z = dH(\nu_\infty) \in \mathfrak{a}$ (see [21] for a detailed proof of this fact). In particular, a semiclassical measure μ can be seen as a probability measure on $\Gamma\backslash G/M$, invariant under the right-action of A (in [34], Silberman-Venkatesh constructed a microlocal lift of ϕ_ν, that is directly defined on $\Gamma\backslash G/M$ instead of T^*X, and their construction has the advantage of being equivariant). The Liouville measure on $\mathcal{E}_{\nu_\infty}$ corresponds to the Haar measure on $\Gamma\backslash G/M$.

Margulis' conjecture (see [23]) : Let G be a semisimple Lie group with finite center, $\Gamma < G$ a lattice, $A < G$ a maximal split torus. Let μ be an A-invariant and ergodic Borel probability measure on $\Gamma\backslash G$. Then there exists a subgroup L of G, containing A, closed and connected, and a closed orbit $xL \subset \Gamma\backslash G$, such that μ is supported on xL. Also, except possibly when L has a factor of rank 1, μ is **algebraic**, that is the L-invariant measure on xL.

Here is what is known about this conjecture. Let us denote μ_{Haar} the Haar measure on $\Gamma\backslash G$.

- [14], Theorem 4.1 : Let G be an $\mathbb{R}$-split simple group. There exists $0 < c < 1$ such that, if Γ is a lattice of G, and if μ is an A-invariant and ergodic probability measure on $\Gamma\backslash G$ satisfying $h_{KS}(\mu) \geq c\, h_{KS}(\mu_{Haar})$ for every 1-parameter subgroup of A, then μ is the Haar measure on $\Gamma\backslash G$.

In the case $G = \mathrm{SL}(n, \mathbb{R})$: if μ has positive entropy for each 1-parameter subgroup of A, then μ is the Haar measure on $\Gamma\backslash G$.

- [15] If $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, or if Γ is a lattice of "inner type"; if μ is ergodic and has positive entropy for some 1-parameter subgroup of A, then μ is algebraic.
- [26, 37] In the latter case, L must be of a certain form : it must be conjugate, via a permutation matrix, to the connected component of identity in $\mathrm{GL}(t, \mathbb{R})^s \cap \mathrm{SL}(n, \mathbb{R})$; where $n = ts$ and $\mathrm{GL}(t, \mathbb{R})^s$ denotes the block-diagonal embedding of s copies of $\mathrm{GL}(t, \mathbb{R})$ into $\mathrm{GL}(n, \mathbb{R})$.

Here is a reformulation of Theorem 3.9 :

Theorem 3.10. *Let μ be a semiclassical measure associated to the limit* (9)*. Assume that ν_∞ regular.*

Then for any 1*-parameter flow e^{tZ} in A (with $Z \in \mathfrak{a}$),*

$$h_{KS}(\mu, e^{tZ}) \geq \sum_{j, \lambda_j^+(e^{tZ}) \geq \frac{\lambda_{\max}(e^{tZ})}{2}} \left(\lambda_j^+(e^{tZ}) - \frac{\lambda_{\max}(e^{tZ})}{2} \right).$$

Unless $Z = 0$, this lower bound is positive. In the case $G = \mathrm{SL}(n, \mathbb{R})$, if we knew that μ was ergodic, we could deduce from the result of [14] that μ is the Haar measure, and quantum unique ergodicity would be proved. Unfortunately, nothing tells us that μ is ergodic. However, our entropic lower bound is explicit, and we can use the more precise measure classification results listed above, to prove the following :

Theorem 3.11. *[4] Let $G = \mathrm{SL}(3, \mathbb{R})$, and Γ be any cocompact lattice in G. Let μ be a semiclassical measure associated to the limit* (9)*. Assume that ν_∞ regular.*

Then μ has a Haar component, of weight $\geq \frac{1}{4}$. In other words, there exists an A-invariant probability measure ν on $\Gamma\backslash G/M$, such that

$$\mu = \frac{1}{4}\mu_{Haar} + \frac{3}{4}\nu,$$

where μ_{Haar} denotes the Haar measure on $\Gamma\backslash G/M$.

Theorem 3.12. *[4] Let $G = \mathrm{SL}(n, \mathbb{R})$, with $n \geq 3$, and let Γ be a lattice associated to a division algebra over $\mathbb{Q}$. Let μ be a semiclassical measure associated to the limit* (9)*. Assume that ν_∞ regular.*

Then μ has a Haar component, of weight $\geq \frac{n-1}{n-d}\left(\frac{1}{2} - \frac{d-1}{n-1}\right)$, where d is the largest proper divisor of n.

We cannot prove quantum unique ergodicity, that says that the only semiclassical measure is the Haar measure. But we have a partial result, saying that any semiclassical measure has a Haar component. For $n = 3$ the result holds for any lattice Γ in $\mathrm{SL}(n,\mathbb{R})$, whereas for $n \geq 4$ we need to assume that Γ is associated to a division algebra over $\mathbb{Q}$ to apply the results of [15, 26, 37].

As a comparison, for n prime, Γ coming from a division algebra over $\mathbb{Q}$, and assuming that the ϕ_ν were also eigenfunctions of the Hecke operators, Silberman-Venkatesh [34, 35] generalized the inequality (6), and improved it by estimating the measures of tubular neighbourhoods of orbits of subgroups. If μ is a semiclassical measure associated to a regular direction ν_∞, their result implies that every ergodic component of μ has positive entropy, with respect to all 1-parameter subgroups of A. This generalizes the result of [5], and implies that μ is the Haar measure.

4. Resonances, local smoothing and Strichartz estimates

Nonnenmacher and Zworski used a variant of Theorem 2.2 in order to prove spectral estimates in scattering theory [27]. For simplicity, we just state their results in a special case. On $\mathbb{R}^d$, consider a Schrödinger operator of the form

$$P(\hbar) = -\hbar^2 \frac{\Delta}{2} + V(x), \quad V \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R}),$$

where Δ is the euclidean laplacian. The resonances of $P(\hbar)$ are defined as poles of the meromorphic continuation of the resolvent

$$R(z,\hbar) \stackrel{\text{def}}{=} (P(\hbar) - z)^{-1} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d), \quad \Im m(z) > 0,$$

through the continuous spectrum $[0, +\infty)$. More precisely,

$$R(z,\hbar) : L^2_{comp}(\mathbb{R}^d) \longrightarrow L^2_{loc}(\mathbb{R}^d), \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

is a meromorphic family of operators (here L^2_{comp} and L^2_{loc} denote functions which are compactly supported and in L^2, and functions which are locally in L^2). The poles are called resonances, and their set is denoted by $\mathrm{Res}(P(\hbar))$. They are counted according to their multiplicities.

The classical hamiltonian flow is given by Newton's equations :

$$\Phi^t(x,\xi) \stackrel{\text{def}}{=} (x(t), \xi(t)),$$

$$\dot{x}(t) = \xi(t), \dot{\xi}(t) = -dV(x(t)), x(0) = x, \xi(0) = \xi.$$

We will denote $Y = Y_H = \frac{d\Phi^t}{dt}_{t=0}$ the corresponding vector field. This flow preserves the classical hamiltonian

$$H(x,\xi) \stackrel{\text{def}}{=} \frac{\|\xi\|^2}{2} + V(x), \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

and it leaves invariant the level sets $\mathcal{E}_E = H^{-1}(E)$. The incoming and outgoing sets at energy E are defined as

$$\Gamma_E^\pm = \{\rho \in \mathcal{E}_E, \Phi^t(\rho) \not\longrightarrow \infty, t \longrightarrow \mp\infty\}.$$

The trapped set at energy E is $K_E = \Gamma_E^+ \cap \Gamma_E^-$. It is a compact invariant set for Φ^t. We will always assume that K_E is non empty.

The fundamental assumption in [27] is that K_E contains no fixed points of the flow, and that the dynamics of (Φ^t) on K_E is (uniformly) hyperbolic. This means that, for any $\rho \in K_E$, the tangent space to $\mathcal{E}_E$ at ρ splits into flow direction, unstable and stable subspaces : there exist $C, \lambda > 0$, and at each $\rho \in K_E$ a splitting $T_\rho \mathcal{E}_E = \mathbb{R}Y(\rho) \oplus E_\rho^+ \oplus E_\rho^-$, $\quad \dim E_\rho^\pm = d-1$, such that

(i) For all $\rho \in K_E$, $d\Phi_\rho^t E_\rho^\pm = E_{\Phi^t(\rho)}^\pm$ for all $t \in \mathbb{R}$;

(ii) For all $\rho \in K_E$, for all $v \in E_\rho^\mp$, $\|d\Phi_\rho^t.v\| \leq Ce^{-\lambda|t|}\|v\|$, for $\pm t > 0$.

Hyperbolicity implies structural stability, and in particular $K_{E'}$ is also a non empty hyperbolic set, for E' close enough to E.

Let us introduce the unstable jacobian, defined by

$$\exp \Lambda_t^+(\rho) = \det(d\Phi^t\rceil_{E_\rho^+});$$

for t large enough, we have $\Lambda_t^+(\rho) > 0$ for all ρ.

By assumption, there exists $R > 0$ such that V is supported inside the ball $B(0,R)$. Fix $\delta > 0$. The technique of complex scaling, used in [27] (but which we don't explain in detail here), allows to construct a deformation $P_\theta(\hbar)$ of $P(\hbar)$ with the following properties : (i) $P_\theta(\hbar)$ is a non self-adjoint deformation of $P(\hbar)$, such that the propagator $e^{-it\frac{P_\theta(\hbar)}{\hbar}}$ damps very rapidly the functions supported away from $B(0,3R)$; (ii) $P_\theta(\hbar)$ coincides with $P(\hbar)$ inside $B(0,2R)$; (iii) the resonances of $P(\hbar)$ close to the real axis are the eigenvalues of $P_\theta(\hbar)$, with the same multiplicities.

The following form of Theorem 2.2 is used in [27]. With the same $\delta > 0$ as previously, consider a finite family $\chi_1, \ldots, \chi_K$ of smooth compactly supported functions on $\mathbb{R}^d \times \mathbb{R}^d$, such that $\sum_{j=1}^K \chi_j \equiv 1$ on $H^{-1}[E-\delta, E+\delta] \cap T^*B(0,R)$. For all j, assume the function χ_j is supported on a set W_j of diameter $\leq \varepsilon$ (that will be chosen small enough). Also assume that each χ_j vanishes outside $H^{-1}[E-2\delta, E+2\delta] \cap T^*B(0,2R)$. Consider the associated pseudodifferential operators, defined by the Weyl calculus : $\hat{\chi}_j = \chi_j(x, -i\hbar\partial_x)$. Define $\hat{P}_j = e^{-i\tau\frac{P(\hbar)}{\hbar}}\hat{\chi}_j$, for some fixed time step $\tau > 0$ (in this definition, it is indifferent to take $P(\hbar)$ or $P_\theta(\hbar)$, since they coincide inside $B(0,2R)$). The following theorem is a variant of Theorem 2.2.

Theorem 4.1. *In the definition of $\hat{\chi}_j$, we can fix ε, δ small enough, so that the following holds.*

Fix $\mathcal{K} > 0$ arbitrary. Then there exists $\hbar_\mathcal{K} > 0$ such that, for $n = \mathcal{K}|\log \hbar|$, for any sequence $(\alpha_1, \ldots, \alpha_n) \in \{1, \ldots, K\}^n$, and for all $\hbar < \hbar_\mathcal{K}$,

$$\|\hat{P}_{\alpha_n} \circ \ldots \circ \hat{P}_{\alpha_2} \circ \hat{P}_{\alpha_1}\|_{L^2 \longrightarrow L^2} \leq \frac{1}{(2\pi\hbar)^{d/2}} \prod_{j=1}^n e^{\frac{S_\tau(W_{\alpha_j})}{2}},$$

where $S_\tau(W_j) = -\inf_{|E'-E|\leq 2\delta, \rho \in W_j \cap K_{E'}} \Lambda_\tau^+(\rho)$.

If τ is chosen large enough, then the hyperbolicity condition implies that $S_\tau(W_j) < 0$, and that $\prod_{j=1}^n e^{\frac{S_\tau(W_{\alpha_j})}{2}}$ decays exponentially with n.

To study the spectral theory of $P_\theta(\hbar)$, we can write

$$e^{-in\tau\frac{P_\theta(\hbar)}{\hbar}}$$
$$= \sum_{(\alpha_1,\ldots,\alpha_n)} \hat{P}_{\alpha_n}\circ\ldots\circ\hat{P}_{\alpha_2}\circ\hat{P}_{\alpha_1} + \left(e^{-in\tau\frac{P_\theta(\hbar)}{\hbar}} - \sum_{(\alpha_1,\ldots,\alpha_n)} \hat{P}_{\alpha_n}\circ\ldots\circ\hat{P}_{\alpha_2}\circ\hat{P}_{\alpha_1}\right),$$

where the sum runs over all $(\alpha_1,\ldots,\alpha_n)\in\{1,\ldots,K\}^n$.

The term $\left(e^{-in\tau\frac{P_\theta(\hbar)}{\hbar}} - \sum_{(\alpha_1,\ldots,\alpha_n)} \hat{P}_{\alpha_n}\circ\ldots\circ\hat{P}_{\alpha_2}\circ\hat{P}_{\alpha_1}\right)$ only takes into account classical trajectories that, at some time, exit $H^{-1}[E-\delta,E+\delta]\cap T^*B(0,R)$. The trajectories that start inside $H^{-1}[E-\delta,E+\delta]\cap T^*B(0,R)$, but later exit that set, are very rapidly damped by $e^{-in\tau\frac{P_\theta(\hbar)}{\hbar}}$; an important part of [27] is devoted to showing that this term is not relevant when one wants to study the resonance spectrum near $\{\Re e(z)=E\}$. Concerning the other term, we know that each operator $\hat{P}_{\alpha_n}\circ\ldots\circ\hat{P}_{\alpha_2}\circ\hat{P}_{\alpha_1}$ has a norm that decays exponentially fast with n, but on the other hand there is an exponential number of terms in the sum $\sum_{(\alpha_1,\ldots,\alpha_n)}$. To measure the competition between the exponential number of terms, and the exponential decay of each term, it is natural to introduce the following quantity

$$\mathcal{P}_E(s) = \lim_{\delta\longrightarrow 0}\lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow+\infty}\frac{1}{n\tau}\log Z_{n\tau}(s,(W_j)),$$

where

$$Z_{n\tau}(s,(W_j)) = \inf_B\left\{\sum_{(\alpha_1,\ldots,\alpha_n)\in B}\prod_{j=1}^n e^{\frac{S_\tau(W_{\alpha_j})}{2}}\right\},$$

and the inf is taken over all $B\subset\{1,\ldots,K\}^n$, such that $K_{E'}\subset\cup_{(\alpha_1,\ldots,\alpha_n)\in B}W_{\alpha_1}\cap\Phi^{-\tau}W_{\alpha_2}\cap\Phi^{-(n-1)\tau}W_{\alpha_n}$ for $|E'-E|\leq\delta$.

The function $s\mapsto\mathcal{P}_E(s)$ is called the topological pressure associated with the unstable jacobian. It is strictly decreasing with s.

Corollary 4.2. *Fix $\eta>0$ arbitrary. Then we can find $\tau>0$ large enough, ε,δ small enough, and a partition of unity (χ_j) satisfying all the conditions above, such that the following holds.*

For $\mathcal{K}>0$ arbitrary, there exists $\hbar_{\mathcal{K}}>0$ such that, for $n=\mathcal{K}|\log\hbar|$, and for all $\hbar<\hbar_{\mathcal{K}}$,

$$\|\sum_{(\alpha_1,\ldots,\alpha_n)}\hat{P}_{\alpha_n}\circ\ldots\circ\hat{P}_{\alpha_2}\circ\hat{P}_{\alpha_1}\|_{L^2\longrightarrow L^2}\leq\frac{1}{(2\pi\hbar)^{d/2}}e^{n\tau\mathcal{P}_E(\frac12)}(1+\eta)^{n\tau}.$$

We see that this upper bound is non trivial only if $\mathcal{P}_E(\frac12)<0$, which means in some sense that the trapped set K_E is rather small. In dimension $d=2$, this condition is equivalent to saying that the Hausdorff dimension of the trapped set is <2.

One of the main results in [27] is to deduce from Corollary 4.2 the existence of a spectral gap in the resonance spectrum :

Theorem 4.3. *[27] Assume that $\mathcal{P}_E(\frac{1}{2}) < 0$.*

Then there exists $\delta > 0$ such that, for any γ satisfying

$$0 < \gamma < \min_{|E'-E|\leq\delta} \left(-\mathcal{P}_{E'} \left(\frac{1}{2} \right) \right),$$

there exists $\hbar_{\delta,\gamma} > 0$ such that

$$0 < \hbar < \hbar_{\delta,\gamma} \Longrightarrow \mathrm{Res}(P(\hbar)) \cap ([E-\delta, E+\delta] - i[0, \hbar\gamma]) = \emptyset.$$

This means that if the trapped set is small enough, the resonances stay away from the real axis. This question has been present in the physics literature at least since the seminal paper by Gaspard and Rice [17]. We note that the analogous result for scattering by a disjoint union of convex obstacles was proved in 1988 by Ikawa [22]. One can say that Ikawa's paper contained, in a hidden form and in a specific geometric situation, the idea expressed by Theorem 2.1.

One important consequence of Corollary 4.2 is the following estimate on the resolvent. It is proved in [27], using the relation between the resolvent and the propagator.

Theorem 4.4. *[27] Assume that $\mathcal{P}_E(\frac{1}{2}) < 0$. Then, for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, there exists $C > 0$ such that*

$$\|\chi(P(\hbar) - E)^{-1}\chi\|_{L^2 \longrightarrow L^2} \leq \frac{C \log|\hbar|}{\hbar},$$

for $\hbar$ small enough.

These theorems hold for more general operators : see [27] for a more general set of assumptions. An interesting situation is when there is no potential ($V = 0$) and one studies the resonance spectrum of the laplacian for a riemannian metric that is euclidean ouside a compact set[1]. Since the hamiltonian is homogeneous, one can without loss of generality consider the case $E = \frac{1}{2}$, that is, our hamiltonian flow is the unit geodesic flow. In this situation, the resolvent estimate above was extended by Datchev to the case of asymptotically conic manifolds, also called scattering manifolds [12] It is shown in [9, 12] how such resolvent estimates imply a local smoothing estimate :

Theorem 4.5. *[27, 9, 12] Let (X, g) be a riemannian manifold that is euclidean outside a compact set; or asymptotically conic (see [12] for the definition). Let Δ denote the associated Laplace-Beltrami operator.*

Assume that the trapped set K of the unit speed geodesic flow is compact, hyperbolic, and that the pressure of the unstable jacobian on K satisfies $\mathcal{P}(\frac{1}{2}) < 0$.

Then, for any $\eta > 0$, for any $T > 0$ and any $\chi \in \mathcal{C}_c^\infty(M)$, there exists $C > 0$ such that

$$\int_0^T \|\chi e^{it\Delta} u\|^2_{H^{1/2-\eta}} dt \leq C\|u\|^2_{L^2}. \tag{11}$$

[1] In the case of convex-cocompact hyperbolic manifolds, the existence of a gap in the resonance spectrum, if the limit set has small dimension, seems to have been known before.

The local smoothing effect usually refers to the inequality

$$\int_0^T \|\chi e^{it\Delta}u\|^2_{H^{1/2}} dt \leq C\|u\|^2_{L^2},$$

which is known to hold when the trapped set for the geodesic flow is empty. Doi [13] showed, in a variety of geometric situations, that the absence of trapped geodesics is also a necessary condition for (11) to hold with $\eta = 0$. According to Theorem 4.5, if the trapped set is hyperbolic and small enough, (11) holds for all $\eta > 0$, which is called "local smoothing with loss".

Burq-Guillarmou-Hassell [6] showed how the combination of Theorem 4.5 and the norm estimate of Corollary 4.2 yields a Strichartz estimate *without* loss :

$$\|e^{it\Delta}u\|_{L^p((0,1),L^q(M))} \leq C\|u\|_{L^2(M)},$$

for $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad p > 2, q \geq 2, \quad (p,q) \neq (2,\infty)$. This estimate holds for riemannian manifolds that are asymptotically conic, assuming that the trapped set of the unit geodesic flow is compact, hyperbolic and satisfies $\mathcal{P}(\frac{1}{2}) < 0$.

Finally, Christianson [10] and Nonnenmacher-Zworski [28] show how to extend the resolvent estimate of Theorem 4.4 to the analytic extension of the cut-off resolvent in a small strip below the real axis. As an application, Christianson [10] proves exponential decay of the local energy, under the action of the wave group, on a riemannian manifold that it euclidean outside a compact set, assuming once again that the trapped set of the unit geodesic flow is hyperbolic and satisfies $\mathcal{P}(\frac{1}{2}) < 0$.

We refer the reader to the work of Emmanuel Schenck [32], who used similar ideas to study the spectrum and the energy decay for the damped wave equation.

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