Some remarks on the continuity equation

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This text is the act of a talk given november 18 2008 at the seminar PDE of Ecole Polytechnique. The text is not completely faithfull to the oral exposition for I have taken this opportunity to present the proofs of some results that are not easy to find in the literature. On the other hand, I have been less precise on the material for which I found good references. Most of the novelties presented here come from a joined work with Luigi Ambrosio.

1 Introduction

We consider a Borel vector-field

$$V(t,x) :]0,T[\times \mathbb{R}^d \longrightarrow \mathbb{R}^d,$$

and the associated equations

$$\dot\gamma(t) = V(t,\gamma(t)) \tag{ODE}$$

and (with the notations $V_t(x) = V(t,x)$)

$$\partial_t \mu_t + \mathrm{d}iv(V_t\mu_t) = 0. \tag{PDE}$$

It is important to notice that $V(t,x)$ is a well-defined function, and not an equivalence class of functions. In order to avoid some technicalities we assume the bound

$$\|V\|_c := \int_0^T \|V_t\|_\infty dt < \infty. \tag{B}$$

Here $\|V_t\|_\infty$ is defined as the supremum of $\|V(t,x)\|$. A solution of (ODE) is an absolutely continuous curve $\gamma(t)$ such that $\dot\gamma(t) = V(t,\gamma(t))$ almost everywhere on $[0,T]$.

We consider solutions of (PDE) in the class $\mathcal{M}(\mathbb{R}^d)$ of bounded signed measures. It is necessary here to settle a couple of notations. We define the Banach space $C_0(\mathbb{R}^d)$ as the set of continuous functions which converge to zero at infinity. It is endowed with the uniform norm. The space $\mathcal{M}(\mathbb{R}^d)$ is the dual of $C_0(\mathbb{R}^d)$, we endow it with the weak-$*$ topology, that we will simply call the weak topology. We denote by $\mathcal{M}^+(\mathbb{R}^d)$ and $\mathcal{M}^{1+}(\mathbb{R}^d)$ the spaces of

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non-negative and probability Borel measures. Given a signed measure $\mu \in \mathcal{M}(\mathbb{R}^d)$, we denote by $|\mu| \in \mathcal{M}^+(\mathbb{R}^d)$ its total variation. The quantity $\|\mu\| := |\mu|(\mathbb{R}^d)$ defines a norm on $\mathcal{M}(\mathbb{R}^d)$, which coincides with the dual norm.

A solution of (PDE) is a weakly continuous curve $\mu_t \in C([0,T], \mathcal{M}(\mathbb{R}^d))$ such that, for each compactly supported smooth function u on $\mathbb{R}^d$, the function $t \longmapsto \int_{\mathbb{R}^d} u(x)d\mu_t(x)$ is absolutely continuous with derivative given by

$$\left(\int_{\mathbb{R}^d} u(x)d\mu_t(x)\right)' = \int_{\mathbb{R}^d} du_x(V(t,x))d\mu_t(x). \tag{D}$$

This relation then holds for each C^1 function u which is bounded and Lipschitz. That this definition is equivalent to the genuine definition in the sense of distributions (and in particular, that weak continuity is in fact a consequence of being a solution) is explained, for example, in [4], Chapter 8. Note that, for non-negative solutions, the norm $\|\mu_t\| = \int 1 d\mu_t$ is preserved. For signed solutions, however, this norm is not necessarily continuous and may not be bounded. We will restrict our attention to norm-bounded solutions (those for which the function $t \longmapsto \|\mu_t\|$ is bounded).

In the good cases, (ODE) can be solved by a flow:

Definition 1. *The Borel map*

$$X(t,s,x) : [0,T] \times [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

is called a flow of solutions of (ODE) if, for each fixed t and x, the curve $s \longmapsto X(t,s,x)$ is the only solution $\gamma(s)$ of (ODE) which satisfies the initial condition $\gamma(t) = x$. The maps $X_t^s : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ defined by $X_t^s(x) := X(t,s,x)$ then satisfy the Markov property

$$X_{t_1}^{t_2} \circ X_{t_0}^{t_1} = X_{t_0}^{t_2}.$$

It follows from Proposition 3 below that a flow of solutions of (ODE) exists if and only if, for each $S \in [0,T]$ and $x \in \mathbb{R}^d$, the Cauchy problem consisting of solving (ODE) with the initial data $\gamma(S) = x$ has one and only one solution. If $X(t,s,x)$ is the flow of solutions of (ODE), then it is easy to see that, for each given $S \in [0,T]$ and $\mu \in \mathcal{M}$, the expression

$$\mu_t := (X_S^t)_\sharp \mu$$

defines a solution of (PDE). We say that the flow X uniquely solves (PDE) if this is the only norm-bounded solution of (PDE) fulfilling the given initial value.

When the vector-field $V(t,x)$ is smooth, there exists a smooth flow $X(t,s,x)$ which uniquely solves (ODE) and (PDE). One of our goals in the present text is to present more general classes of vector-fields for which both (PDE) and (ODE) are uniquely solved by a flow. Before this, we settle some measurability issues in Section 2, and then describe, following Ambrosio, Gigli and Savaré, some important relations between (ODE) and (PDE) which hold in full generality. They imply in particular that a flow solving (ODE) always solves (PDE) uniquely in the class of non-negative measures. The situation is more intricate in the class of signed measures. In order to understand this fact, it is important to realize that the positive part of a signed solution is not necessarily a solution in general. For example, assume that there exists two solutions $x(t)$ and $y(t)$ of (ODE) and a time $S \in]0,T[$ such that $x(S) = y(S)$. Then we can define the signed solution μ_t by $\mu_t = 0$ for $t \leqslant S$ and $\mu_t = \delta_{x(t)} - \delta_{y(t)}$ for $t \geqslant S$. It is not hard to see that this is a solution of (PDE), but that the positive part is not. This example also illustrates the non-continuity of the norm $t \longmapsto \|\mu_t\|$ for signed solutions.

In order to study the existence of flows, we then focus our attention to vector-fields which are continuous in the space variable. We denote by $\mathcal{C}([0,T] \times \mathbb{R}^d)$ (or simply $\mathcal{C}$) the set of Borel vector-fields $V(t,x)$ such that, for each t, the map $V_t : x \longmapsto V(t,x)$ is continuous, and such that, in addition, the estimate (B) holds. The quantity $\|V\|_c$ defined by $\|V\|_c := \int_0^T \|V_t\|_\infty dt$ is a norm on $\mathcal{C}$, and $\mathcal{C}$ endowed with this norm is a Banach space. If $V \in \mathcal{C}$, it is well-known that, for each S and x, there exists a solution γ to (ODE) satisfying $\gamma(S) = x$. The existence of a flow of solutions of (ODE) is then equivalent to the uniqueness for each Cauchy data. In Section 4, we will prove:

Theorem 1. *The set of vector-fields $V \in \mathcal{C}$ for which both (ODE) and (PDE) are uniquely solved by a flow is generic in $\mathcal{C}$ in the sense of Baire.*

This result is rather easy, but we do not know any reference, and will therefore provide a complete proof. A different but similar genericity result is proved in [14]. Our proof is very different from the one in [14], but a similar one could possibly be used.

Next we try to derive the existence of a flow from regularity estimates. We recall that a modulus of continuity is a continuous non-decreasing function $\rho : [0,1) \longrightarrow [0,\infty)$, such that $\rho(0) = 0$. A modulus of continuity ρ is said to be Osgood if

$$\int_0^1 \frac{1}{\rho(s)} ds = +\infty.$$

We will always extend the moduli of continuity to $[1,\infty)$ by $\rho = \infty$. Typical examples of Osgood moduli of continuity are $\rho(s) = s$ and $\rho(s) = s(1 - \ln(s))$. Note that the moduli $\rho(s) = s^\alpha$, $\alpha \in (0,1)$, are not Osgood.

It is known that (ODE) is solved by a unique flow (which is a flow of homeomorphisms) provided there exists an Osgood modulus of continuity ρ and $C(t) \in L^1(0,T)$ such that

$$|V(t,x) - V(t,y)| \leqslant C(t)\rho(|x-y|) \tag{O}$$

for all x, $y \in \mathbb{R}^d$, and all $t \in]0,T[$. We do not know if, in general, the existence of a flow of homeomorphisms solving (ODE) implies that this flow also uniquely solves (PDE) (although we know that it uniquely solves (PDE) in the class of non-negative measures, see Section 3). If $V \in \mathcal{C}$ satisfies (O), then this is true:

Theorem 2. *If $V \in \mathcal{C}$ satisfies (O), then there exists a flow of homeomorphisms uniquely solving (ODE) and (PDE).*

This result was proved in [5]. It had been proved earlier in [6] in the case where $\rho(s) = s(1 - \ln(s))$ and where V is incompressible. The method was strikingly different. The result can be considered standard in the case $\rho(s) = s$. We give some indications of proof in Section 5.

2 Some measurability issues

Proposition 2. *If V is a Borel vector-field, then the set of solutions of (ODE) is Borel in $C([0,T], \mathbb{R}^d)$.*

Proof. The curve $\gamma \in C([0,T], \mathbb{R}^d)$ is a solution if and only if

$$\gamma(t) - \gamma(0) = \int_0^t V(s, \gamma(s)) ds$$

for each $t \in \mathbb{Q} \cap [0,T]$. So in order to prove the Proposition, it is enough to prove that, for a given $t \in [0,T]$, the map $\gamma \longmapsto \int_0^t V(s,\gamma(s))ds$ is Borel. We claim that, for each non-negative Borel function $f(s,x) : [0,t] \times \mathbb{R}^d \longrightarrow [0,\infty)$, the map

$$\gamma \longmapsto \int_0^t f(s, \gamma(s)) ds$$

is Borel. The claim clearly implies the desired result.

We prove the claim by a monotone class argument. Let $\mathcal{F}$ be the set of functions which satisfy the desired conclusion. We observe that $\mathcal{F}$ is stable under addition, multiplication by a non-negative real number, and monotone convergence. Moreover, $\mathcal{F}$ obviously contains bounded continuous functions. By standard monotone class arguments, we conclude that $\mathcal{F}$ contains all non-negative Borel functions (see *e. g.* [8], Lemma 39). □

Proposition 3. *Let B be a Borel set in $[0,T] \times \mathbb{R}^d$ such that, for each $(S,x) \in B$, there exists one and only one solution $\gamma(t)$ of (ODE) which satisfies $\gamma(S) = x$. Then the flow map $X : [0,T] \times [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is well-defined and Borel on the set*

$$\tilde{B} := \{(S,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d \quad | \quad (S,x) \in B\}.$$

Proof. Let $\mathcal{A} \subset C([0,T], \mathbb{R}^d)$ be the (Borel) set of solutions of (ODE). The map

$$ev : [0,T] \times \mathcal{A} \longrightarrow [0,T] \times \mathbb{R}^d$$

given by $ev(t,\gamma) = (t, \gamma(t))$ is continuous hence Borel. As a consequence, the set $\mathcal{A}_R := ev^{-1}(R)$ is Borel. Our hypothesis is precisely that the map ev is a bijection between $\mathcal{A}_R$ and R. By (non trivial) general results of measure theory (see [13] or [16], Theorem 3.9), the inverse map ev^{-1} is then Borel. Now the conclusion follows from the formula

$$X(S,t,x) = \pi \circ ev(t, \pi \circ ev^{-1}(S,x)),$$

where we have denoted by the same letter π two different projections consisting in forgetting the time. □

3 The uniqueness question for probability measures

We describe several links between (ODE) and (PDE) which hold when V is only Borel. The content of this section is due to Ambrosio, Gigli and Savaré (see [4], chapter 8), but is also closely related to earlier works of Smirnov ([17]) as will be made more apparent in section 5.

Let us first recall that every solution of (ODE) can be seen as a non-negative solution of (PDE). Indeed, if $\gamma(t)$ solves (ODE), then the measures $\mu_t := \delta_{\gamma(t)}$ solves (PDE). We call the solutions of (PDE) which can be obtained this way elementary. The following statement, established by Ambrosio, Gigli and Savaré in the line of anterior works of Smirnov, roughly states that the set of solutions of (PDE) in the class of probability measures coincides with the closed convex envelop of the set of elementary solutions.

Theorem 3. *Let $\mu_t \in C([0,T], \mathcal{M}^{1+})$ be a solution of (PDE). Then there exists a Borel probability measure ν on $C([0,T], \mathbb{R}^d)$ with the following properties:*

1. *ν is concentrated on the Borel set of solutions of (ODE),*
2. *$(ev_t)_\sharp \nu = \mu_t$ for each $t \in [0,T]$, where $ev_t : C([0,T], \mathbb{R}^d) \longrightarrow \mathbb{R}^d$ is the evaluation map $\gamma \longmapsto \gamma(t)$.*

This theorem can be equivalently stated as follows:

Theorem 4. *Let $\mu_t \in C([0,T], \mathcal{M}^{1+})$ be a solution of (PDE). Then there exists a stochastic process $Z(t,\omega)$ such that :*

1. *almost each sample path $t \longmapsto Z(t,\omega)$ is a solution of (ODE)*

2. *The law of the random variable $\omega \longmapsto Z(t,\omega)$ is μ_t.*

These results, called the superposition principle, sum up what is known in general concerning the relations between (ODE) and(PDE). Excellent proofs can be found in [4, 2, 3], see also [8] for another approach, and [15, 9] for related material. The following corollary is obvious:

Corollary 4. *Let $\mu_t \in C([0,T],\mathcal{M}^{1+})$ be a solution of (PDE). Then, for each $S \in [0,T]$ and for μ_S-almost every point $x \in \mathbb{R}^d$, there exists a solution $\gamma(t)$ of (ODE) such that $\gamma(S) = x$.*

Corollary 5. *If V_t is volume-preserving for each t, then, for each $S \in [0,T]$ and for almost every point $x \in \mathbb{R}^d$, there exists a solution $\gamma(t)$ of (ODE) such that $\gamma(S) = x$.*

Proof. This corollary does not immediately follow from Corollary 4 because the Lebesgue measure is not bounded. We denote by λ the Lebesgue measure, and consider a positive and bounded function $v : \mathbb{R}^d \longrightarrow]0,1]$ such that $\int v d\lambda = 1$, so that $\mu = f\lambda$ is a probability measure. The corollary holds if there exists a solution $\mu_t \in C([0,T],\mathcal{M}^+)$ such that $\mu_S = \mu$. This is the content of the following Lemma: □

Lemma 6. *Let $V(t,x)$ be a Borel vector-field such that $\|V\|_c < \infty$ and $div(V_t) = 0$ for each t. Let $S \in [0,T]$ be a fixed time and let $v : \mathbb{R}^d \longrightarrow [0,1]$ be an integrable function (normalized to $\int v d\lambda = 1$). Then there exists a non-negative solution $\mu_t \in C([0,T],\mathcal{B})$ of (PDE) with $\mu_S = v\lambda$ and such that, for each time $t \in [0,T]$, we have $\mu_t = v_t\lambda$ for some integrable function $v_t : \mathbb{R}^d \longrightarrow [0,1]$.*

Proof. We mollify V by

$$W^n(t,x) := n^d \int V(t,y) g(n(x-y))dy, \tag{M}$$

where g is a compactly supported smooth kernel. We have $W^n \longrightarrow W$ in L^1_{loc}, and W^n satisfy the estimate (O) with a Lipschitz modulus. As a consequence, for each n, there exists a flow of homeomorphisms solving (ODE), and therefore there exists a Borel function $v^n(t,x) : [0,T] \times \mathbb{R}^d \longrightarrow [0,1]$ such that $\mu^n_t := v^n_t\lambda$ is a solution of (PDE) (with the vector-field W^n) and such that $\mu^n_S = v\lambda$. Observe also that $\|W^n\|_c \leqslant \|V^n\|_c$, from which follows, using (E) in the appendix, that the sequence μ^n_t is equicontinuous. As a consequence, we can assume that μ^n_t converges uniformly to a limit $\mu_t \in \mathcal{C}([0,T],\mathcal{B})$. Note that $\mu_S = v\lambda$. For each fixed t, the measure μ^n_t has a density v^n_t with values in $[0,1]$ hence the limit μ_t has a density v_t with values in $[0,1]$, and $v^n_t \longrightarrow v_t$ weakly-$*$ in $L^\infty(\mathbb{R}^d)$. We have to prove that μ_t solves (PDE).

We have

$$\int_0^T f'(t)\int u(x)d\mu^n_t(x) + f(t)\int du_x(W^n(t,x))d\mu^n_t(x) \quad dt = 0 \tag{1}$$

for each compactly supported smooth function u on $\mathbb{R}^d$ and each smooth compactly supported function f on $]0,T[$. For each fixed t the functions $du_x(W^n_t(x))$ strongly converge to $du_x(V_t(x))$ in L^1. Since in addition v^n_t converges to v_t weakly-$*$ in $L^\infty(\mathbb{R}^d)$, we have:

$$\int du_x(W^n(t,x))v^n_t(x)d\lambda(x) \longrightarrow \int du_x(V(t,x))v_t(x)d\lambda(x).$$

By the dominated convergence theorem (using that $\|W^n\|_c$ is bounded), we can pass to the limit in (1) and get

$$\int_0^T f'(t)\int u(x)d\mu_t(x) + f(t)\int du_x(W(t,x))d\mu_t(x) \quad dt = 0$$

which says exactly that the measures μ_t solve (PDE). □

On the side of uniqueness, we have:

Corollary 7. *Let $S \in [0,T]$ be fixed, and let $B \in \mathbb{R}^d$ be a Borel set such that, for each $x \in B$, there exists at most one solution $\gamma(t)$ of (ODE) satisfying $\gamma(S) = B$. Then, if μ is a probability measure concentrated on B, there exists at most one solution $\mu_t \in C([0,T], \mathcal{M}^+)$ of (PDE) satisfying $\mu_S = \mu$.*

Note that, in general, there may exist other solutions in $C([0,T], \mathcal{M})$.
Proof. Let μ_t and $\tilde{\mu}_t$ be two solutions in $C([0,T], \mathcal{M}^+)$ satisfying $\tilde{\mu}_S = \mu = \mu_S$. Let ν and $\tilde{\nu}$ be the decompositions given by Theorem 3. We claim that $\nu = \tilde{\nu}$, and therefore that $\mu_t = \tilde{\mu}_t$ for each t. In order to prove the claim, we consider the Borel subset $Q \subset C([0,T], \mathbb{R}^d)$ formed by solutions γ of (ODE) which satisfy $\gamma(S) \in B$. That this set is Borel follows from Proposition 2. Our hypothesis is that the restriction to Q of the evaluation map ev_S is one-to-one. As a consequence (by Theorem 3.9 in [16]) the image $B' = ev_S(Q)$ is Borel and the inverse map $ev_S^{-1} : B' \longrightarrow Q$ is Borel. By Theorem 3, we have $\nu(Q) = 1$, so the measure ν can be identified with its restriction to Q. Since $(ev_S)_\sharp \nu = \mu$, we have $\mu(B') = \nu(Q) = 1$, so that the measure μ coincides with its restriction to B'. As a consequence, we obtain $\nu = (ev_S^{-1})_\sharp \mu$. Similarly, we have $\nu' = (ev_S^{-1})_\sharp \mu$. □

Corollary 8. *If (ODE) is uniquely solved by the flow X, then, for each $S \in [0,T]$ and each probability measure μ, the curve $\mu_t = (X_S^t)_\sharp \mu$ is the only solution of (PDE) in $C([0,T], \mathcal{M}^+)$ which satisfies $\mu_S = \mu$.*

There may exist other norm-bounded solutions in $C([0,T], \mathcal{M})$. It is not easy to give good extensions of the theory presented in this section for the case of signed measures. I shall present some recent works in that direction in Section 5.

4 Generic existence of a flow

In this section, we prove Theorem 1. In order to prove that both (ODE) and (PDE) are solved by a flow for a given vector-field V, it is enough to prove that any norm-bounded solution μ_t of (PDE) which satisfies $\mu_0 = 0$ or $\mu_T = 0$ must vanish identically. Indeed, assume that there exists two solutions μ_t and μ'_t which are not equal, and some $S \in [0,T]$ such that $\mu_S = \mu'_S$. Let us assume for instance that there exists $t \geqslant S$ such that $\mu_t \neq \mu'_t$. Then we can define a new solution $\tilde{\mu}_t$ by $\tilde{\mu}_t = 0$ for $t \leqslant S$ and $\tilde{\mu}_t = \mu_t - \mu'_t$ for $t \geqslant S$. The solution $\tilde{\mu}_t$ vanishes at $t = 0$ and it is not identically zero.

So what we have to prove now is that, for a generic vector-field V, there is no non-trivial solution of (PDE) satisfying $\mu_0 = 0$ (the analogous statement for μ_T is similar).

Let us define the set-valued mapping $\mathcal{S}$ which, to each vector-field $V \in \mathcal{C}$, associates the subset $\mathcal{S}(V) \subset C([0,T], \mathcal{B})$ formed by those solutions μ_t of (PDE) which vanish at time $t = 0$. As explained in Appendix A, we embed $C([0,T], \mathcal{B})$ into a compact metric space $\mathcal{Y}$, and consider $\mathcal{S}$ as a set-valued map between $\mathcal{C}$ and $\mathcal{Y}$. We refer to Appendix B for the terminology on set-valued maps.

Lemma 9. *The set-valued map $\mathcal{S}$ has closed graph (or equivalently it is upper semi-continuous)*

Proof. Let $V^n \in \mathcal{C}$ be a sequence of vector-fields converging to V in $\mathcal{C}$, and let $\mu_t^n \in C([0,T], \mathcal{B})$ be a sequence of solutions of (PDE) with vector-fields V^n converging to a limit $\mu_t \in \mathcal{Y}$ (see Appendix A for the definition of $\mathcal{Y}$). We have

$$\big| \|V_t^n\|_\infty - \|V_t\|_\infty \big| \leqslant \|V_t^n - V_t\|_\infty$$

and therefore the functions $t \longmapsto \|V_t^n\|_\infty$ converge to $\|V_t\|_\infty$ in $L^1([0,T])$. As a consequence, the functions $\|V_t^n\|_\infty$ are equi-integrable, and we conclude from inequality (E) in Appendix A that the curves μ_t^n are equi-continuous. As a consequence, the limit μ_t belongs to $C([0,T],\mathcal{B})$. We have to prove that $\mu_t \in \mathcal{S}(V)$. Recall that $\mu_t^n \in \mathcal{S}(V^n)$ if and only if the equality

$$\int_0^T \int f'(t)u(x)d\mu_t^n(x)dt + \int_0^T \int f(t)du_x(V^n(t,x))d\mu_t^n(x)dt = 0 \tag{2}$$

holds for each compactly supported smooth function u on $\mathbb{R}^d$ and each smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $f(T)=0$. Since μ_t^n converge to μ_t in $\mathcal{Y}$, we have

$$\int_0^T \int f'(t)u(x)d\mu_t^n(x)dt \longrightarrow \int_0^T \int f'(t)u(x)d\mu_t^n(x)dt$$

and

$$\int_0^T \int f(t)du_x(V(t,x))d\mu_t^n(x)dt \longrightarrow \int_0^T \int f(t)du_x(V(t,x))d\mu_t(x)dt.$$

On the other hand, we have

$$\left|\int_0^T \int f(t)du_x(V(t,x))d\mu_t^n(x)dt - \int_0^T \int f(t)du_x(V^n(t,x))d\mu_t^n(x)dt\right| \leqslant C\|V_n - V\|_c \longrightarrow 0$$

hence, at the limit in (2), we obtain

$$\int_0^T \int f'(t)u(x)d\mu_t(x)dt + \int_0^T \int f(t)du_x(V(t,x))d\mu_t(x)dt = 0$$

for each compactly supported smooth function u on $\mathbb{R}^d$ and each smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $f(T)=0$. This implies that $\mu_t \in \mathcal{S}(V)$. □

From Kuratowski Theorem (Theorem 7 in Appendix B), we conclude that the set of points of continuity of $\mathcal{S}$ is generic. In order to end the proof, it is enough to see that $\mathcal{S}(V)=\{0\}$ when V is a point of continuity of $\mathcal{S}$. Let $W^n(t,x)$ be the sequence of mollified approximations of V defined in (M). We have $W^n \longrightarrow V$ in $\mathcal{C}$. On the other hand, we have $\mathcal{S}(W^n)=\{0\}$. Since V is a point of continuity of $\mathcal{S}$, for each solution $\mu_t \in \mathcal{S}(V)$, there exists a sequence $\mu_t^n \in \mathcal{S}(W^n)$ such that $\mu_t^n \longrightarrow \mu_t$ in $\mathcal{Y}$. This implies that $\mathcal{S}(V)=\{0\}$. □

5 The uniqueness question for signed measures

Let us first recall the following well-known result:

Theorem 5. *Consider a vector-field $V \in \mathcal{C}$ which satisfies (O). Then there exists a unique flow of homeomorphisms X_s^t solving (ODE).*

Our main issue here is to prove Theorem 2, that is to prove that the vector-field which solves (ODE) also uniquely solves (PDE). As far as non-negative solutions of (PDE) are concerned, this is implied by Section 3. So it is natural to start with a new superposition principle adapted to signed solutions.

Our method to de so is to consider the extended vector-field $\tilde{V}(t,x) := (1, V(t,x))$ on $[0,T]\times \mathbb{R}^d$. Now if the measure μ_t solve (PDE), then setting $\tilde{\mu} := dt \otimes \mu_t$ (extended by zero outside of $]0,T[\times\mathbb{R}^d)$, we have

$$div(\tilde{V}\tilde{\mu}) = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0$$

on $\mathbb{R}^{d+1}$. Here the important object is the product $\tilde{V}\tilde{\mu}$ which is a vector-valued measure and even a normal one-current (a vector-valued measure whose divergence is also a vector-valued measure). From general results of Smirnov (see [17]) on the decomposition of normal one-currents, we infer (see [5]) the following superposition principle for the signed solutions of (PDE):

Theorem 6. *Let $V(t,x)$ be a Borel vector-field satisfying (B), let $\mu_t \in C([0,T],\mathcal{M})$ be a solution of (PDE), and let $C(t)$ be a given positive integrable function on $[0,T]$. Then there exists a Borel probability measure ν on $C([0,1],\mathbb{R}^{d+1})$ such that :*

1. $\delta_T \otimes \mu_T - \delta_0 \otimes \mu_0 = (ev_1)_\sharp \nu - (ev_0)_\sharp \nu$
2. *ν-almost each curve $\gamma(s) = (t(s), x(s))$ is one to one and Lipschitz, it satisfies the estimate $\int_0^1 C(t(s))|\dot{t}(s)|ds < \infty$, it takes values in $[0,T]\times\mathbb{R}^d$, and solves the equation*
$$\dot{x}(s) = \dot{t}(s)V(t(s), x(s)) \tag{R}$$
for almost every s.

The equality in 1 is global, it is not true in general that $\delta_T \otimes \mu_T = (ev_1)_\sharp \nu$ or $\delta_0 \otimes \mu_0 = (ev_0)_\sharp \nu$.

This superposition principe is far less appealing than Theorem 3. This kind of complication seems unavoidable. The price paid from the existence of a non-constant sign of the solution is the fact that the time component $t(s)$ of the curves appearing in the decomposition is not necessarily monotone. In order to understand the role of the equation (R), it is worth noticing that, if $x(t)$ is a solution of (ODE) and if $t(s) : [0,1] \longrightarrow [0,T]$ is any Lipschitz function, then the curve $\gamma(s) = (t(s), x(t(s)))$ satisfies (R).

Conversely, if we could prove that (R), seen as the ODE
$$\dot{x}(s) = P(s, x(s)) \tag{R'}$$
with $P(s,x) : [0,1]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ given by
$$P(s,x) := \dot{t}(s)V(t(s), x)$$
satisfies uniqueness, then we would conclude that
$$x(s) = X(t(s_0), t(s), x(s_0))$$
for each s_0 and s in $[0,1]$, where X is the flow solving (ODE) with the vector-field $V(t,x)$. In general, given $t(s)$, we do not see any reason why uniqueness should hold for (R') even if it holds for (ODE). However, we have:

Lemma 10. *Under the hypotheses of Theorem 6, if in addition the vector-field V satisfies (O) with the same function $C(t)$ as in Theorem 6, then ν-almost each curve $\gamma(s) = (t(s), x(s))$ satisfies*
$$x(s) = X(t(0), t(s), x(0)).$$

Proof. We have $|P(s,y) - P(s,x)| \leqslant D(s)|x-y|$, with $D(s) = |\dot{t}(s)|C(t(s))$. For ν-almost every curve $\gamma(s) = (t(s), x(s))$, we know that the function $D(s) = |\dot{t}(s)|C(t(s))$ is integrable (this is one of the conclusions of Theorem 6). In other words, the vector-field P satisfies (O) and therefore, by Theorem 5, we have uniqueness for (R') with the given function $t(s)$, and we conclude that $x(s) = X(t(0), t(s), x(0))$. □

Corollary 11. *Under the hypotheses of Theorem 6, if in addition the vector-field V satisfies (O), then $\mu_T = (X_0^T)_\sharp \mu_0$.*

PROOF. Let $\mathcal{L} \subset C([0,1], \mathbb{R}^{d+1})$ be a Borel set of full ν-measure formed by Lipschitz curves which satisfy all the properties of 2 in Theorem 6. If $\gamma(s) = (t(s), x(s))$ is a curve in $\mathcal{L}$, then γ is one-to-one and, by Lemma 10, it is of the form $\gamma(s) = (t(s), x(t(s))$. We conclude that $t(s)$ is one to one and thus monotone. By 1 in Theorem 6 we can assume in addition that $t(0) \in \{0, T\}$ and $t(1) \in \{0, T\}$.

Denoting by $\mathcal{L}^+$ the Borel subset of $\mathcal{L}$ formed by curves $\gamma = (t, x)$ such that t is increasing on $[0,1]$ and satisfies $t(0) = 0$ and $t(1) = S$, and by $\mathcal{L}^-$ the Borel subset of $\mathcal{L}$ formed by curves $\gamma = (t, x)$ such that t is decreasing on $[0,1]$ and satisfies $t(0) = T$ and $t(1) = 0$, we conclude that $\mathcal{L}^+ \cup \mathcal{L}^- = \mathcal{L}$. We denote by $\nu^\pm$ the restrictions of ν to $\mathcal{L}^\pm$. The measures $\nu^\pm$ are mutually singular, non-negative, and $\nu = \nu^+ + \nu^-$. Let

$$B_i : \mathcal{L}^+ \cup \mathcal{L}^- \longrightarrow \mathbb{R}^d$$

be the Borel map defined by $B_i(\gamma) = x(0)$ if $\gamma \in \mathcal{L}^+$ and $B_i(\gamma) = x(1)$ if $\gamma \in \mathcal{L}^-$. Similarly, we define

$$B_f : \mathcal{L}^+ \cup \mathcal{L}^- \longrightarrow \mathbb{R}^d$$

by $B_i(\gamma) = x(0)$ if $\gamma \in \mathcal{L}^-$ and $B_i(\gamma) = x(1)$ if $\gamma \in \mathcal{L}^+$. Note that

$$B_f = X_0^T \circ B_i$$

on $\mathcal{L}$. We have the identities $(ev_1)_\sharp \nu^+ = \delta_T \otimes (B_f)_\sharp \nu^+, (ev_0)_\sharp \nu^+ = \delta_0 \otimes (B_i)_\sharp \nu^+, (ev_1)_\sharp \nu^- = \delta_0 \otimes (B_i)_\sharp \nu^-$, and $(ev_0)_\sharp \nu^- = \delta_T \otimes (B_f)_\sharp \nu^-$. It follows that

$$\delta_T \otimes \mu_T - \delta_0 \otimes \mu_0 = (ev_1)_\sharp(\nu^+ + \nu^-) - (ev_0)_\sharp(\nu^+ + \nu^-) = \delta_T \otimes (B_f)_\sharp(\nu^+ - \nu^-) + \delta_0 \otimes (B_i)_\sharp(\nu^+ - \nu^-).$$

We conclude that $\mu_0 = (B_i)_\sharp(\nu^- - \nu^+)$ and $\mu_S = (B_f)_\sharp(\nu^- - \nu^+)$. As a consequence, we have

$$\mu_T = (X_0^T)_\sharp \mu_0.$$

□

PROOF OF THEOREM 2 : We want to prove that $\mu_t = (X_s^t)_\sharp \mu_s$ for each s and t in $[0, T]$. Since $X_t^s = (X_s^t)^{-1}$, it is enough to prove the statement when $s < t$. In order to do so, it is enough to apply Corollary 11 on the time interval $[s, t]$ instead of $[0, T]$. □

A Topology on the spaces of measures

Let us first define the separable Banach space $C_0(\mathbb{R}^d)$ formed by the continuous functions which converge to zero at infinity, with the uniform norm. The space $\mathcal{M}(\mathbb{R}^d)$ of bounded Borel measures can be identified with the topological dual of $C_0(\mathbb{R}^d)$, and the dual norm is $\|\mu\| = |\mu(\mathbb{R}^d)|$. We endow this dual space with the weak-$*$ topology, it is known that the unit ball $\mathcal{B} := \{\mu \in \mathcal{M} | \|\mu\| \leqslant 1\}$ is compact and metrizable. It is useful to work with a specific distance. In order to define this distance, we consider a sequence u_n of compactly supported smooth functions on $\mathbb{R}^d$ which generates a dense vector subspace in $C_0(\mathbb{R}^d)$. We assume in addition that $\|u_n\|_{C^1} = 1$. A distance on $\mathcal{B}$ can be defined by the formula

$$d(\mu, \eta) = \sum_{n \in \mathbb{N}} \frac{\left| \int u_n d\mu - \int u_n d\eta \right|}{2^n}.$$

It is well-known that the topology associated to this distance is the weak-$*$ topology. $(\mathcal{B}, d)$ is a compact metric space. Now let μ_t satisfy (D) for each of the functions u_n. Then, given two times $s \leqslant t$, we have

$$\int u_n d\mu_t - \int u_n d\mu_s = \int_s^t \int_{\mathbb{R}^d} d(u_n)_x (V(\sigma, x)) d\mu_\sigma d\sigma$$

and therefore

$$d(\mu_s, \mu_t) \leqslant \int_s^t \|V_\sigma\|_\infty d\sigma. \tag{E}$$

We deduce that, given a Borel vector-field V satisfying (B), the set of all solutions of (PDE) in $C([0,T],\mathcal{B})$ is equi-continuous.

In order to put a topology on the space of measure-valued solutions of (PDE), a small digression is needed. We define $L^1([0,T],C_0)$ as the set of Borel maps $u : [0,T] \longrightarrow C_0(\mathbb{R}^d)$ such that $\int_0^T \|u(t)\|_\infty dt < \infty$. This is a separable banach space, which coincides with the space of Borel measurable functions $v(t,x) : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ such that, for each t, the map v_t belongs to $C_0(\mathbb{R}^d)$ and such that, in addition, $\int_0^T \|v_t\|_\infty dt$ is finite. All this is classical, see for example [7]. We denote by $\mathcal{Y}$ the unit ball in the dual of $L^1([0,T],C_0)$ endowed with the weak-$*$ topology. This dual can be naturally identified with $L^\infty([0,T],\mathcal{M})$, and so it can be seen as a space of Young measures, hence the notation. $\mathcal{Y}$ is a compact metrizable space. There is a natural embedding of $C([0,T],\mathcal{B})$ into $\mathcal{Y}$. If μ_t^n is an equi-continuous sequence in $C([0,T],\mathcal{B})$ which converges in $\mathcal{Y}$ to the limit $\mu_t \in \mathcal{Y}$, then $\mu_t \in C([0,T],\mathcal{B})$ and the convergence is uniform in $C([0,T],\mathcal{B})$.

B Set valued maps

The classical reference for this material is the book of Kuratowski, [13]. Let X be a complete metric space, and K be a compact metric space. A set-valued map S associates to each point $x \in X$ a subset $S(x)$ of K. The set-valued map S is called upper semi-continuous if its graph

$$\{(x,y) \in X \times K \quad | \quad y \in S(x)\}$$

is closed. We consider from now on an upper semi-continuous set function S. Given $U \subset K$, we define $S^{-1}(U)$ as the set of points $x \in B$ such that $S(x) \subset U$. It is easy to see that $S^{-1}(U)$ is open for each open set U (recall that S is upper semi-continuous). Since every closed set is a G_δ (a countable intersection of open sets), we get:

Lemma 12. *If S is upper semi-continuous, then $S^{-1}(F)$ is a G_δ for each closed set $F \subset K$.*

We say that x is a point of continuity of S if, for each $y \in S(x)$ and each sequence $x_n \longrightarrow x$ in X, there exists a sequence $y_n \longrightarrow y$ such that $y_n \in S(x_n)$.

Theorem 7. *If S is an upper semi-continuous set function, then the set of points of continuity of S is a dense G_δ.*

Proof. Let U_k be a countable base of open sets, and let F_k be the complement of U_k . We claim that the set of points of continuity is

$$\bigcap_{k \in N} \left[\left(\overline{S^{-1}(F_k)}\right)^c \cup S^{-1}(F_k) \right].$$

Each of the sets

$$\left(\overline{S^1(F_k)}\right)^c \cup S^1(F_k)$$

is a G_δ because $S^{-1}(F_k)$ is a G_δ and $\left(\overline{S^{-1}(F_k)}\right)^c$ is open. In addition, it is clearly dense. By the Baire property, we conclude that the set of continuity points is a dense G_δ. We now have to check the claim. The point x is not a point of continuity if and only if there exists an open set U_k such that $U_k \cap S(x)$ is not empty and a sequence $x_n \longrightarrow x$ such that $S(x_n) \cap U_k$ is empty.

This amounts to say that $x \notin S^{-1}(F_k)$ an $x \in \overline{S^{-1}(F_k)}$. As a consequence, the complement of the set of continuity points is

$$\bigcup_k \big(\overline{S^{-1}(F_k)} - S^{-1}(F_k).\big)$$

□

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