Sub-actions for Anosov diffeomorphisms

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Abstract

We show a positive Livciz type theorem for $\mathcal{C}^2$ Anosov diffeomorphisms f on a compact boundaryless manifold M and Hölder observables A. Given $A : M \to \mathbb{R}$, α-Hölder, we show there exist $V : M \to \mathbb{R}$, β-Hölder, $\beta < \alpha$ and a probability measure μ, f-invariant such that

$$A \leq V \circ f - V + \int A\, d\mu.$$

We apply this inequality to prove the existence of an open set $\mathcal{G}_\beta$ of β-Hölder functions, β small, which admit a unique maximizing measure supported on a periodic orbit. Moreover the closure of $\mathcal{G}_\beta$, in the β-Hölder topology, contains all α-Hölder functions, α close to one.

Dedicated to Jacob Palis

1 Introduction

We consider a compact riemannian manifold M of dimension $d \geq 2$ without boundary and a $\mathcal{C}^2$ transitive Anosov diffeomorphism $f : M \to M$. The tangent bundle TM admits a continuous Tf-invariant splitting $TM = E^u \bigoplus E^s$ of expanding and contracting tangent vectors. We assume M is equiped with

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a riemannian metric and there exist constants $C(M)$, depending only on M and the metric and constants depending on f

$$0 < \Lambda_s < \lambda_s < 1 < \lambda_u < \Lambda_u$$

such that for all $n \in \mathbb{Z}$

$$\begin{cases} C(M)^{-1}\lambda_u^n \leq \|T_x f^n \cdot v\| \leq C(M)\Lambda_u^n & \text{for all } v \text{ in } E_x^u, \\ C(M)^{-1}\Lambda_s^n \leq \|T_x f^n \cdot v\| \leq C(M)\lambda_s^n & \text{for all } v \text{ in } E_x^s. \end{cases}$$

Livciz theorem [4] asserts that, if $A : M \to M$ is a given Hölder function and satisfies $\int A\, d\mu = 0$ for all f-invariant probability measure μ, then A is equal to a coboundary V (which is Hölder too), that is:

$$A = V \circ f - V.$$

What hapens if we only assume $\int A\, d\mu \geq 0$ for all f-invariant probability measure μ? We denote by $\mathcal{M}(f)$, the set of f-invariant probability measures. We prove the following:

Theorem 1 *Let $f : M \to M$ be a $\mathcal{C}^2$ transitive Anosov diffeomorphism on a compact manifold M without boundary. For any given α-Hölder function $A : M \to \mathbb{R}$, there exists a β-Hölder function $V : M \to \mathbb{R}$, that we call sub-action, such that:*

$$A \leq V \circ f - V + m(A, f),$$

where $m(A, f) = \sup\{ \int f\, d\mu \mid \mu \in \mathcal{M}(f) \}$, $\mathcal{M}(f)$ is the set of f-invariant probability measures and

$$\beta = \alpha \frac{\ln(1/\lambda_s)}{\ln(\Lambda_u/\lambda_s)}, \quad \text{Höld}_\beta(V) \leq \frac{C(M)}{\min(1 - \lambda_u^{-\alpha}, 1 - \lambda_s^{\alpha})^2} \text{Höld}_\alpha(A)$$

where $C(M)$ is some constant depending only on M and the metric.

By analogy with Hamiltonian mechanics and the way we define V from A, we may interpret A as a lagrangian and V as a sub-action. This result extends a similar one we obtained in [3] for expanding maps of the circle. Although the proof we give is specific for smooth systems, the same result holds for doubly infinite subshifts of finite type.

Corollary 2 *The hypothesis are the same as in theorem 1. The following statements are equivalent:*

(i) $A \geq V \circ f - V$ *for some bounded measurable function* V*,*

(ii) $\int A \, d\mu \geq 0$ *for all* f*-invariant probability measure* μ*,*

(iii) $\sum_{k=0}^{p-1} A \circ f^k(x) \geq 0$ *for all* $p \geq 1$ *and point* x *periodic of period* p*,*

(iv) $A \geq V \circ f - V$ *for some Hölder function* V*.*

The proof of that corollary is straitforward and uses (for (iii) $\Rightarrow$ (ii)) the fact that the convex hull of periodic measures is dense in the set of all f-invariant probability measures for topological dynamical systems satisfying the shadowing lemma (see Lemma 5). F. Labourie suggested to us the following corollary:

Corollary 3 *The hypothesis are the same as in theorem 1. If* A *satisfies* $\int A \, d\mu \geq 0$ *for all* $\mu \in \mathcal{M}(f)$ *and* $\sum_{k=0}^{p-1} A \circ f^k(x) > 0$ *for at least one periodic orbit* x *of period* p *then* $\int A \, d\lambda > 0$ *for all probability measure* λ *giving positive mass to any open set.*

Again the proof is straitforward: $R = A - V \circ f + V \geq 0$ for some continuous V and $\int R \, d\lambda = 0$ for such a measure λ implies $R = 0$ everywhere and in particular $\sum_{k=0}^{p-1} A \circ f(x) = 0$ for all periodic orbit x.

Any measure μ satisfying $\int A \, d\mu = m(A, f)$ is called a maximizing measure and since A is continuous, such a measure always exists. It is then natural to ask the following two questions : For which A, the set of maximizing measures is reduced to a single measure ? In the case there exists a unique maximizing measure, to what kind of compact set, the support of this measure looks like ?

The following theorem gives a partial answer for "generic" functions A.

Theorem 4 *Let* $f : M \to M$ *be a* $\mathcal{C}^2$ *transitive Anosov diffeomorphism and* $\beta < \ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)$*. Then there exists an open set* $\mathcal{G}_\beta$ *of* β*-Hölder functions (open in the* $\mathcal{C}^\beta$*-topology) such that:*

(i) any A *in* $\mathcal{G}_\beta$ *admits a unique maximizing measure* μ_A*;*

(ii) the support of μ_A *is equal to a periodic orbit and is locally constant with respect to* $A \in \mathcal{G}_\beta$*;*

(iii) any α-Hölder function with $\alpha > \beta \ln(\Lambda_u/\lambda_s)/\ln(1/\lambda_s)$ is contained in the closure of $\mathcal{G}_\beta$ (the closure is taken with respect to the $\mathcal{C}^\beta$-topology).

The proof of Theorem 4 is a simplification of what we gave in [3] in the one-dimensional setting. The existence of sub-actions is in both cases the main ingredient of the proof.

The plan of the proof of Theorem 1 is the following: Given a finite covering of M by open sets $\{U_1, \ldots, U_l\}$ with sufficiently small diameter, we construct a Markov covering (and not a Markov partition) $\{R_1, \ldots, R_l\}$ of rectangles: each R_i contains U_i and satisfies

$$x \in U_i \cap f^{-1}(U_j) \Rightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j),$$

where $W^s(x, R_i)$ denotes the local stable leaf through x restricted to R_i. We then associate to each R_i a local sub-action V_i, defined on R_i by:

$$V_i(x) = \sup\{S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \mid n \geq 0, \quad y \in W^s(x, R_i)\}$$

where $\Delta^s(y, x)$ is a kind of cocycle along the stable leaf $W^s(x)$:

$$\Delta^s(y, x) = \sum_{n \geq 0} (A \circ f^n(y) - A \circ f^n(x)).$$

This family $\{V_1, \ldots, V_l\}$ of local sub-actions satisfies the inequality:

$$x \in U_i \cap f^{-1}(U_j) \Rightarrow V_i(x) + A(x) - m \leq V_j \circ f(x)$$

and enable us to construct a global sub-action V:

$$V(x) = \sum_{i=1}^{l} \theta_i(x) V_i(x)$$

where $\{\theta_1, \ldots, \theta_l\}$ is a smooth partition of unity associated to the covering $\{U_1, \ldots, U_l\}$. The main difficulty is to prove that each V_i is Hölder on R_i.

2 Existence of sub-actions

We continue our description of the dynamics of transitive Anosov diffeomorphisms (for details information, see Bowen's monography [2]). All the results we are going to use depend on a small constant of expansiveness $\epsilon^* > 0$ depending on f and M in the following way:

$$\epsilon^* = C(M)^{-1} \min(\frac{\lambda_u - 1}{\|D^2 f\|_\infty}, \frac{1-\lambda_s}{\|D^2 f\|_\infty})$$

where $C(M) \geq 1$ is a constant depending only on M and the riemannian metric. At each point x, one can define its local stable manifold $W^s_\epsilon(x)$ for every $\epsilon < \epsilon^*$:

$$W^s_\epsilon(x) = \{ \, y \in M \mid d(f^n(x), f^n(y)) \leq \epsilon \quad \forall n \geq 0 \, \}$$

which are $\mathcal{C}^2$ embeded closed disks of dimension $d^s = \dim E^s_x$ and tangent to E^s_x. In the same manner, $W^u_\epsilon(x)$ is defined replacing f by f^{-1}. If two points x, y are close enough, $d(x,y) < \delta$, then $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ have a unique point in common, called $[x,y]$:

$$[x,y] = W^s_\epsilon(x) \cap W^u_\epsilon(y) = W^s_{\epsilon^*}(x) \cap W^u_{\epsilon^*}(y),$$

where $\epsilon = K^* \delta$ and K^* is again a large constant depending on M and f:

$$K^* = \frac{C(M)}{\min(1-\lambda_u^{-1}, 1-\lambda_s)}.$$

This estimate is in fact a particular case of Bowen's shadowing lemma:

Lemma 5 (Bowen) *If δ is small enough, $\delta < \epsilon^*/K^*$, if $(x_n)_{n\in\mathbb{Z}}$ is a bi-infinite δ-pseudo-orbit, that is, $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$, then there exists a unique true orbit $\{f^n(x)\}_{n\in\mathbb{Z}}$ which ϵ-shadow $(x_n)_{n\in\mathbb{Z}}$, that is $d(f^n(x), x_n) < \epsilon$ for all $n \in \mathbb{Z}$ with $\epsilon = K^*\delta$.*

This lemma is the main ingredient for constructing (dynamical) rectangles. A rectangle R is a closed set of diameter less than ϵ^*/K^* satisfying:

$$x, y \in R \Rightarrow [x,y] \in R.$$

We will not use the notion of proper rectangles but will use instead the notion of Markov covering.

Definition 6 Let $\mathcal{U} = \{U_1, \dots, U_l\}$ be a covering of M by open sets of diameter less than $\epsilon^*/(K^*)^2$. We call Markov covering associated to $\mathcal{U}$, a finite set $\mathcal{R} = \{R_1, \dots, R_l\}$ of rectangles of diameter less than ϵ^*/K^* satisfying:

$$\begin{aligned}
&U_i \subset R_i \\
&x \in U_i \cap f^{-1}(U_j) \Rightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j) \\
&y \in f(U_i) \cap U_j \quad \Rightarrow f^{-1}(W^u(y, R_j)) \subset W^u(f^{-1}(y), R_i)
\end{aligned}$$

where $W^s(x, R_i) = W^s_{\epsilon^*}(x) \cap R_i$ and $W^u(y, R_j) = W^u_{\epsilon^*}(y) \cap R_j$.

An easy consequence of the shadowing lemma shows there always exist such Markov coverings:

Proposition 7 *For every covering $\mathcal{U}$ of M by open sets such that the diameter of each U_i is less than $\epsilon^*/(K^*)^2$, there exists a Markov covering $\mathcal{R}$ by rectangles of diameter less than ϵ^*/K^*.*

Proof. Given $\mathcal{U} = \{U_1, \dots, U_l\}$ such a covering, we define the following compact space of $\epsilon^*/(K^*)^2$ pseudo-orbits:

$$\Sigma = \{\omega = (\dots, \omega_{-2}, \omega_{-1} \mid \omega_0, \omega_1, \dots) \quad \text{s.t.} \quad U_{\omega_n} \cap f^{-1}(U_{\omega_{n+1}}) \neq \emptyset\}.$$

Here ω is a sequence of indices in $\{1, \dots, l\}$ and Σ is a subshift of finite type where $i \to j$ is a possible transition iff $U_i \cap f^{-1}(U_j)$ is not empty. Given such $\omega \in \Sigma$, we choose for all $n \in \mathbb{Z}$, $x_n \in U_{\omega_n}$ so that $f(x_n) \in U_{\omega_{n+1}}$. Then $(x_n)_{n\in\mathbb{Z}}$ is a $\epsilon^*/(K^*)^2$ pseudo-orbit which corresponds to a unique true orbit $(f^n(x))_{n\in\mathbb{Z}}$ satisfying:

$$d(f^n(x), U_{\omega_n}) < \epsilon^*/K^* \quad \forall n \in \mathbb{Z}.$$

Since ϵ^* is a constant of expansiveness, there can exists at most one point x satisfying the previous inequality for all n. We call that point $\pi(\omega)$ and notice that the map

$$\pi : \Sigma \to M$$

is surjective (for $\mathcal{U}$ is a covering), commutes with the left shift σ, $f \circ \pi = \pi \circ \sigma$, is continuous by expansiveness (in fact Hölder if Σ is equiped with the standard metric). Also notice that π may not be finite-to-one. We first construct a Markov cover on Σ as usual by the braket

$$[\omega, \omega'] = (\cdots, \omega'_{-2}, \omega'_{-1} \mid \omega_0, \omega_1, \cdots)$$

where $\omega = (\omega_n)_{n\in\mathbb{Z}}$, $\omega' = (\omega'_n)_{n\in\mathbb{Z}}$ and $\omega'_0 = \omega_0$. By uniqueness in the construction of $\pi(\omega)$, we get

$$\begin{aligned}
&\pi([\omega,\omega']) = [\pi(\omega),\pi(\omega')] \\
&\pi([i]) = R_i \quad \text{is a rectangle of } M \text{ containing } U_i \\
&\pi(W^s(\omega,[i])) = W^s(\pi(\omega),R_i) \quad \text{whenever } \omega_0 = i
\end{aligned}$$

where $[i]$, $i = 1,\cdots,l$, is the cylinder $\{\omega \in \Sigma \mid \omega_0 = i\}$ and $W^s(\omega,[i])$ is the symbolic stable set $\{\omega' \in \Sigma \mid \omega'_n = \omega_n \ \forall n \geq 0\}$. (For the proof of the last equality, we just notice : if $x = \pi(\omega)$, $y \in W^s(x,R_i)$ and $y = \pi(\omega')$ then $\pi([\omega,\omega']) = y$ and $[\omega,\omega'] \in W^s(\omega,[i])$.) To finish the proof we only show

$$x \in U_i \cap f^{-1}(U_j) \Rightarrow f(W^s(x,R_i)) \subset W^s(f(x),R_j).$$

Indeed, $x = \pi(\omega)$ for some $\omega = (\cdots,\omega_{-1} \mid i,j,\omega_2,\cdots)$ and

$$\sigma(W^s(\omega,[i]) \subset W^s(\sigma(\omega),[j]).$$

To conclude, we apply π on both sides. ■

Definition 8 Let $\mathcal{R} = \{R_1,\cdots,R_l\}$ be a Markov covering of M associated to some open covering $\mathcal{U} = \{U_1,\cdots,U_l\}$. We define a local sub-action by

$$V_i(x) = \sup\{\, S_n(A-m)\circ f^{-n}(y) + \Delta^s(y,x) \mid n \geq 0,\ y \in W^s(x,R_i)\,\}$$

where $S_nB = \sum_{k=0}^{n-1} B\circ f^k$, $\Delta^s(y,x) = \sum_{k\geq 0}(A\circ f^k(y) - A\circ f^k(x))$ and the supremum is taken over all $n \geq 0$ and points $y \in W^s(x,R_i)$.

Before showing V_i is a (finite!) Hölder function on each R_i, let's conclude the proof of Theorem 1:

Poof of Theorem 1. Let $\mathcal{U} = \{U_1,\cdots,U_l\}$ be an open covering of M, $\{R_1,\cdots,R_l\}$ a Markov covering associated to $\mathcal{U}$ and $\{\theta_1,\cdots,\theta_l\}$ a partition of unity adapted to $\mathcal{U}$. Let $\{V_1,\cdots,V_l\}$ constructed as above and

$$V = \sum_i \theta_i V_i.$$

Suppose we have proved that $x \in U_i \cap f^{-1}(U_j)$ implies

$$V_i(x) + (A-m)(x) \leq V_j \circ f(x).$$

Multiplying this inequality by $\theta_i(x)\theta_j \circ f(x)$ and summing over i and j, we get

$$V(x) + (A - m)(x) \le V \circ f(x) \quad (\forall x \in M).$$

We now prove the local sub-cohomological equation: if $x \in U_i \cap f^{-1}(U_j)$ and $y \in W^s(x, R_i)$, then $f(y) \in W^s(f(x), R_j)$ and

$$\begin{aligned} &S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) + (A - m)(x) \\ &\qquad = S_{n+1}(A - m) \circ f^{-(n+1)} \circ f(y) + \Delta^s(f(y), f(x)) \le V_j \circ f(x). \end{aligned}$$

Taking the supremum over all $n \ge 0$ and all $y \in W^s(x, R_i)$, we get indeed

$$V_i(x) + (A - m)(x) \le V_j \circ f(x).$$

That finishes the proof of theorem 1. ■

We now come to our main technical lemma. We notice that, even in the case where A is Lipschitz, we only obtain a Hölder sub-action.

Lemma 9 *If A is α-Hölder on M, R is a rectangle and V is defined as in Definition 8, then V is β-Hölder on R with exponent*

$$\beta = \alpha \frac{|\ln \lambda_s|}{\Lambda_u + |\ln \lambda_s|} < \alpha.$$

Proof. We divide the proof into four steps:
Step one. If $d(x, x') < \epsilon^*$ and x, x' are on the same stable leaf, then

$$\Delta^s(x, x') \le \sum_{n \ge 0} |A \circ f^n(x) - A \circ f^n(x')| \le C(M) \frac{\mathrm{Höld}_\alpha(A)}{1 - \lambda_s^\alpha} d(x, x')^\alpha,$$

for some constant $C(M)$ depending only on M and the metric.

Indeed, it follows from the contraction $d(f^k(x), f^k(x')) \le C(M)\lambda_s^k d(x, x')$ for $k \ge 0$ and the fact that A is α-Hölder.

Step two. For every $n \ge 1$, $x, x' \in M$ such that $d(f^k(x), f^k(x')) < \epsilon^*/K^*$ for all $0 \le k \le n$, then

$$\sum_{k=0}^{n-1} |A \circ f^k(x) - A \circ f^k(x')| \le K(M, f) \max(d(x, x')^\alpha, d(f^n(x), f^n(x'))^\alpha),$$

$$\text{where} \quad K(M, f) = C(M) \frac{\mathrm{Höld}_\alpha(A)}{\min(1 - \lambda_u^{-\alpha}, 1 - \lambda_s^\alpha)^2}.$$

Indeed, one can build $w = [x, x']$; then on the one hand, $d(x, w) \leq \epsilon^*$ and x, w are on the same stable leaf; on the other hand, $d(f^n(w), f^n(x')) \leq \epsilon^*$ and $f^n(w)$ and $f^n(x')$ are on the same unstable leaf. We conclude by applying step one and the estimates:

$$d(x, w) \leq K^* d(x, x'), \quad d(f^n(w), f^n(x')) \leq K^* d(f^n(x), f^n(x')).$$

Step three. We show that $V(x)$ is finite for every $x \in R$. It is precisely here that the choice of the normalizing constant $m(A, f)$ is important.

Indeed, since a transitive Anosov diffeomorphism is mixing, there exists an integer $\tau^* \geq 1$ such that, for every finite orbit $\{f^{-n}(y), \cdots, f^{-1}(y), y\}$, n arbitrary, $f^{\tau^*}(B(y, \epsilon^*/K^*))$ contains $f^{-n}(y)$. Thanks to the shadowing lemma, there exists a periodic orbit z, of period $n + \tau^*$, satisfying

$$d(f^{-k}(z), f^{-k}(y)) \leq \epsilon^* \qquad (\forall k = 0, 1, \cdots, n).$$

Using step two, $\sum_{k=1}^{n} (A \circ f^{-k}(y) - A \circ f^{-k}(z))$ is uniformly bounded in n by some constant $C(M, f)$ and using $\sum_{k=1}^{n+\tau^*} A \circ f^{-k}(z) \leq (n + \tau^*) m(A, f)$, we get

$$\begin{aligned} \sum_{k=1}^{n} A \circ f^{-k}(y) &\leq C(M, f) + \sum_{k=1}^{n+\tau^*} A \circ f^{-k}(z) + \tau^* \|A\|_\infty \\ &\leq C(M, f) + 2\tau^* \|A\|_\infty. \end{aligned}$$

Step four. We finally prove that V is Hölder on R. Let $n \geq 0$, $x, x' \in R$, $y \in W^s(x, R)$ and define $y' = [x', y]$ belonging to R since R is a rectangle and to the same local unstable manifold as y. Then for some N we are going to choose soon: let $B = A - m(A, f)$,

$$\begin{aligned} S_n B \circ f^{-n}(y) + \Delta^s(y, x) \leq\ & S_n B \circ f^{-n}(y') + \Delta^s(y', x') \\ & + \sum_{k=-n}^{N-1} |A \circ f^k(y) - A \circ f^k(y')| \qquad (= \Sigma_1) \\ & + \sum_{k=0}^{N-1} |A \circ f^k(x) - A \circ f^k(x')| \qquad (= \Sigma_2) \\ & + |\Delta^s(f^N(y), f^N(x))| \qquad (= \Sigma_3) \\ & + |\Delta^s(f^N(y'), f^N(x'))| \qquad (= \Sigma_4) \end{aligned}$$

We now bound from above each Σ_i with respect to $d(x,x')$:

$$\begin{aligned}
\Sigma_1 &\le C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_u^{-\alpha}}d(f^N(y),f^N(y'))^\alpha,\\
\Sigma_2 &\le C(M)\frac{\text{Höld}_\alpha(A)}{\min(1-\lambda_u^{-\alpha},1-\lambda_s^\alpha)^2}\max(d(x,x')^\alpha,d(f^N(x),f^N(x'))^\alpha),\\
\Sigma_3 &\le C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_s^\alpha}d(f^N(y),f^N(x)),\\
\Sigma_4 &\le C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_s^\alpha}d(f^N(y'),f^N(x'))^\alpha.
\end{aligned}$$

We now choose $N=N(x,x')$ by $\lambda_s^t\epsilon^*=\Lambda_u^t d(x,x')$, $N=[t]+1$ and then choose $\tilde\epsilon\ge\epsilon^*$ so that $\lambda_s^N\tilde\epsilon=\Lambda_u^N d(x,x')$. Then

$$\begin{gathered}
d(f^N(x),f^N(x'))\le C(M)\Lambda_u^N d(x,x')\le C(M)\lambda_s^N\tilde\epsilon,\\
d(f^N(y),f^N(x)) \text{ or } (f^N(y'),f^N(x'))\le C(M)\lambda_s^N\epsilon^*\le C(M)\lambda_s^N\tilde\epsilon.
\end{gathered}$$

In particuliar, we get first $d(f^N(y),f^N(y'))\le 3C(M)\lambda_s^N\tilde\epsilon$ and next:

$$\begin{gathered}
\Sigma_1+\cdots+\Sigma_4\le 6C(M)\frac{\text{Höld}_\alpha(A)}{\min(1-\lambda_u^{-\alpha},1-\lambda_s^\alpha)^2}(\lambda_s^N\tilde\epsilon)^\alpha=K(M,f)(\lambda_s^N\tilde\epsilon)^\alpha,\\
S_nB\circ f^{-n}(y)+\Delta^s(y,x)\le S_nB\circ f^{-n}(y')+\Delta^s(y',x')+K(M,f)(\lambda_s^N\tilde\epsilon)^\alpha,\\
V(x)\le V(x')+K(M,f)(\lambda_s^N\tilde\epsilon)^\alpha.
\end{gathered}$$

But

$$\lambda_s^N\tilde\epsilon=d(x,x')^{\ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)}.$$

■

Remark 10 We have not used explicitely the fact that the stable foliation W^s is Hölder but our proof (step four) is close to showing W^s is Hölder of exponent $\gamma=\ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

Proof. We show that if $\epsilon<\epsilon^*/K^*$, $d(x,x')\le\epsilon$, $y\in W^s_\epsilon(x)$, $y'\in W^s_\epsilon(x')$ and $y\in W^u_{\epsilon^*}(y')$ then

$$d(y,y')\le 3C(M)^2d(x,x')^\gamma$$

where $\gamma=\ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

Indeed we choose $t > 0$ real such that $\lambda_s^t \epsilon = \Lambda_u^t d(x, x')$, $N = [t] + 1$, and $\tilde{\epsilon}$ close to ϵ so that $\lambda_s^N \tilde{\epsilon} = \Lambda_u^N d(x, x')$ where $\tilde{\epsilon}/\epsilon$ varies between 1 and Λ_u/λ_s. Then

$$d(f^N(x), f^N(y)) \text{ or } d(f^N(x'), f^N(y')) \text{ or } d(f^N(x), f^N(x')) \leq C(M)\lambda_s^N \tilde{\epsilon},$$
$$d(f^N(y), f^N(y')) \leq 3C(M)\lambda_s^N \tilde{\epsilon},$$
$$d(y, y') \leq 3C(M)^2(\lambda_s/\lambda_u)^N \tilde{\epsilon} = 3C(M)^2 d(x, x')^\gamma.$$

■

3 Maximizing periodic measures

The proof of Theorem 4 requires two ingredients: the first one is the notion of sub-actions we have already studied, the second is the notion of strongly non-wandering points we are going to explain.

Definition 11 Given $A \in \mathcal{C}^\beta(M)$ and $m = m(A, f)$, a point $x \in M$ is said to be strongly non-wandering with respect to A, if for any $\epsilon > 0$, there exist $n \geq 1$ and $y \in M$ such that

$$y \in B(x, \epsilon), \quad f^n(y) \in B(x, \epsilon) \quad \text{and} \quad |\sum_{k=0}^{n-1} (A - m) \circ f^k(y)| < \epsilon$$

where $B(x, \epsilon)$ denotes the ball centered at x and radius ϵ. We call $\Omega(A, f)$ the set of strongly non-wandering points.

The first non-trivial but easy observation is that $\Omega(A, f)$ is non-empty; more precisely:

Lemma 12 *The set $\Omega(A, f)$ is compact forward and backward f-invariant and contains the support of any maximizing measure.*

Proof. If μ is maximizing, by Atkinson's theorem [1], for almost μ-point x, the Birkhoff's sums $\sum_{k=0}^{n-1}(A - m) \circ f^k$ are recurrent (in the sense of random walk theory) to $\int (A - m)\, d\mu = 0$: that is, for any Borel set B of positive μ-measure and for any $\epsilon > 0$, the set

$$\left\{ x \in B \mid \exists\, n \geq 1 \quad f^n(x) \in B \quad \text{and} \quad |\sum_{k=0}^{n-1} (A - m) \circ f^k(x)| < \epsilon \right\}$$

has positive μ-measure. Since by definition of the support of a measure, any ball $B(x,\epsilon)$ has positive μ-measure, we have proved that $\mathrm{supp}(\mu)$ is included in $\Omega(A,f)$. ■

The second observation is that any Hölder function A is cohomologuous to $m(A,f)$ on $\Omega(A,f)$, more precisely:

Lemma 13 *Let A be a $\mathcal{C}^0$-function and assume A admits a $\mathcal{C}^0$ sub-action V, then*

$$\Omega(A,f) \subseteq \Sigma_V(A,f) = \big\{x \in M \mid A - m = V\circ f - V\big\}$$

and any f-invariant measure μ whose support in contained in $\Omega(A,f)$ is maximizing.

The set $\Sigma_V(A,f)$ will play an important role later and it is convenient to to give it a name:

Definition 14 Let A be a $\mathcal{C}^0$-function and V be a sub-action of A.

(i) We call the set $\Sigma_V(A,f) = \big\{x \in M \mid A - m = V\circ f - V\big\}$, the V-action-set of A.

(ii) Two points x, y of the V-action-set are said to be V-connected and we shall write $x \stackrel{V}{\to} y$, if for every $\epsilon > 0$, there exist $n \geq 1$ and $z \in M$ (not necessarily in $\Sigma_V(A,f)$) such that

$$x \in B(z,\epsilon), \quad y \in B(f^n(z),\epsilon), \quad |S_N(A-m)(z) - (V(y)-V(x))| < \epsilon.$$

Notice that, if V is β-Hölder for some $\beta > 0$, using the shadowing lemma, one can prove that $x \stackrel{V}{\to} y$ and $y \stackrel{V}{\to} z$ imply $x \stackrel{V}{\to} z$.

Proof of Lemma 13. Define $R = V\circ f - V - A + m$ and choose $x \in \Omega(A,f)$. Then $\sum_{k=0}^{n_i-1}(A-m)\circ f^k(y_i)$ converges to 0 for a sequence of points y_i and a sequence of integers n_i such that y_i converges to x, n_i converges to $+\infty$ and $f^{n_i}(y_i)$ converges to x. Since R is non-negative,

$$0 \leq R(y_i) \leq \sum_{k=0}^{n_i-1} R\circ f^k(y_i) = V\circ f^{n_i}(y_i) - V(y_i) - \sum_{k=0}^{n_i-1}(A-m)\circ f^k(y_i)$$

converges to 0 and by continuity of R : $R(x) = 0$. ■

Definition 15 For any $\beta > 0$, define

$$\mathcal{G}_\beta = \{A \in \mathcal{C}^\beta(M) \mid \Omega(A, f) \text{ is a periodic orbit } \}.$$

Our next goal is to show that $\mathcal{G}_\beta$ is open in $\mathcal{C}^\beta$. We could have choosen a bigger set : the set of A in $\mathcal{C}^\beta(M)$ such that $\Omega(A, f)$ is minimal and is dynamically isolated (i.e. there exists U, open, containing $\Omega(A, f)$ as the only f-invariant compact set inside U) and the proof below would again be the same.

Lemma 16 *For any $\beta > 0$, $\mathcal{G}_\beta$ is open in $\mathcal{C}^\beta$ and $\Omega(A, f)$ is locally constant as a function of A in $\mathcal{G}_\beta$.*

Proof. Let $A \in \mathcal{G}_\beta$. We want to show that $\Omega(A, f) = \Omega(B, f)$ whenever B is sufficiently close to A in the $\mathcal{C}^\beta$ topology. By contradiction : let U be an isolating open set of the periodic orbit $\Omega(A, f) = \mathrm{orb}(p)$ and $\{A_n\}$ be a sequence of β-Hölder observables converging to A in the $\mathcal{C}^\beta$ topology such that $\Omega(A, f)$ is not included in U for each n.

Each A_n admits (Theorem 1) a γ-Hölder subaction V_n with γ-Hölder norm uniformly bounded and $\gamma = \beta \ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)$. By Ascoli, $\{V_n\}$ admits a subsequence converging in the $\mathcal{C}^0$ topology to some γ-Hölder function V. Since the set of non-empty compact sets is compact with respect to the Hausdorff topology, we may assume that $\{\Omega(A_n, f)\}$ has a sub-sequence converging to some compact invariant set K. Each A_n satisfies :

$$\begin{aligned} A_n - m(A_n, f) &\leq V_n \circ f - V_n \qquad (\forall\, x \in M), \\ A_n - m(A_n, f) &= V_n \circ f - V_n \qquad (\forall\, x \in \Omega(A_n, f)). \end{aligned}$$

By continuity of $m(A, f)$ with respect to A (for the $\mathcal{C}^0$ topology),

$$\begin{aligned} A - m(A, f) &\leq V \circ f - V \qquad (\forall\, x \in M) \\ A - m(A, f) &= V \circ f - V \qquad (\forall\, x \in K). \end{aligned}$$

We have assumed that each $\Omega(A_n, f) \setminus U$ is not empty, then $K \setminus U$ is not empty too. Let $x_0 \in K \setminus U$, the ω-limit set $\omega(x_0)$ and the α-limit set $\alpha(x_0)$ of x_0 are compact invariant sets included in $\Omega(A, f)$, necessarily :

$$\omega(x_0) = \alpha(x_0) = \mathrm{orb}(p) \subset \overline{\mathrm{orb}(x_0)} \subset \Sigma_V(A, f).$$

Since p is V-connected to x_0 and x_0 is V-connected to p, x_0 is V-connected to itself which is equivalent to $x_0 \in \Omega(A, f)$. We just have obtained a contradiction. ■

Proof of Theorem 4. Let β given and A, α-Hölder with:

$$\beta < \tilde{\beta} = \alpha \frac{\ln(1/\lambda_s)}{\ln(\Lambda_u/\lambda_s)}.$$

According to Theorem 1, there exists V, $\tilde{\beta}$-Hölder, satisfying :

$$A - m \leq V \circ f - V \qquad (\forall\, x \in M).$$

Define $R = V \circ f - V - A + m$, $\phi_n = \min(R, 1/n)$ and $B_n = A + \phi_n$. Then ϕ_n is $\tilde{\beta}$-Hölder with $\text{Höld}_{\tilde{\beta}}(\phi_n) \leq \text{Höld}_{\tilde{\beta}}(R)$ and

$$\begin{gathered} A - m \leq B_n - m \leq V \circ f - V \qquad (\forall\, x \in M) \\ B_n - m = V \circ f - V \qquad (\forall\, x \in \{R < 1/n\}). \end{gathered}$$

In particular $m(B_n, f) = m(A, f)$ and the V-action set of B_n contains a neighborhood $\{R < 1/n\}$ of $\Omega(A, f)$. Using the shadowing lemma, we construct a periodic orbit $\text{orb}(p)$ inside $\{R < 1/n\}$ and we just have proved a perturbation B_n of A satisfies

$$\text{orb}(p) \cup \Omega(A, f) \subset \Omega(B_n, f).$$

Let ψ_n be any $\tilde{\beta}$-Hölder function with small $\tilde{\beta}$-Hölder norm satisfying:

$$\begin{gathered} \psi_n(x) = 0 \qquad (\forall\, x \in \text{orb}(p)) \\ \psi_n(x) > 0 \qquad (\forall\, x \in M \setminus \text{orb}(p)). \end{gathered}$$

Then $A_n = B_n - \psi_n = A + \phi_n - \psi_n$ satisfies $\Omega(A_n, f) = \text{orb}(p)$, has small $\mathcal{C}^0$ norm and (possibly large) uniform $\tilde{\beta}$-Hölder norm. Therefore (A_n) converges to A in the $\mathcal{C}^\beta$-topology and each A_n has a unique maximizing measure which is supported on a periodic orbit. ■

References

[1] G. Atkinson, Recurrence of cocycles and random walks, J. London Math. Soc. (2), **13** (1976) 486–488.

[2] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics, vol. **470**. Springer Verlag, Berlin, Heidelberg, New York, 1975.

[3] G. Contreras, A.O. Lopes, Ph. Thieullen. Lyapunov minimizing measures for expanding maps of the circle. Ergod. Th. & and Dynam. Sys. (2001), **21**, 1–31.

[4] A. Livciz. Some homology properties of Y-systems. *Mathematical Notes of the USSR Academy of Sciences*, **10** (1971), 758–763.