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Mather measures and the Bowen–Series transformation

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Abstract

We consider a specific example of a compact Riemannian surface M of genus 2 and constant negative curvature. We identify the boundary at infinity of M to the unit circle $\Sigma = \mathbb{S}^1$ and choose a particular Bowen–Series map $T : \Sigma \to \Sigma$. We first show that a suspension of the natural extension of (Σ, T) by a roof function cohomologous to $\ln T'$ is isomorphic to the geodesic flow on $T^1 M$. We choose a particular set of closed geodesics $(\delta_i)_{i=1}^4$ generating the fundamental group and a partition of Σ into disjoint intervals $(A_i)_{i=-4}^4$ naturally associated to (δ_i). We show that any ϕ^t-invariant probability measure μ minimizing $L = \frac{1}{2}\|v\|_x^2$ and with homology $h = \sum_{i=1}^4 h_i[\delta_i]$ corresponds by the previous isomorphism to a unique T-invariant probability measure m satisfying

$$h_i/\|h\|_s = \big[m(A_i) - m(A_{-i})\big] \Big/ \int \ln T' \,\mathrm{d}m \quad \forall i = 1, \ldots, 4.$$

We also show that any ϕ^t-invariant probability measure μ minimizing $\int (L - \omega)\,\mathrm{d}\mu$ for a fixed cohomology $[\omega]$ canonically corresponds to a T-invariant probability measure m minimizing

$$\int \Bigg(\|\omega\|_s \ln T' - \sum_{i=1}^{4} \omega_i [\mathbf{1}_{A_i} - \mathbf{1}_{A_{-i}}] \Bigg) \mathrm{d}m,$$

where $(\omega_i)_{i=1}^4$ are the coordinates of $[\omega]$ in the dual basis of $([\delta_i])$.

Résumé

Nous considérons un exemple spécifique de surface compacte M riemannienne de genre 2 et de courbure constante égale à 2. Nous identifions le bord à l'infini de M au cercle unité $\Sigma = \mathbb{S}^1$ et nous faisons le choix d'une application de Bowen–Series particulière $T : \Sigma \to \Sigma$. Nous montrons d'abord que le flot suspendu au dessus de (Σ, T) par une fonction plafond cohomologue à $\ln T'$ est isomorphe au flot géodésique sur $T^1 M$. Nous choisissons une famille de géodésiques fermées $(\delta_i)_{i=1}^4$ engendrant le groupe fondamental et une partition de Σ en intervalles disjoints $(A_i)_{i=-4}^4$ naturellement associés aux (δ_i). Nous montrons que toute

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mesure de probabilité ϕ^t-invariante μ minimisant $L = \frac{1}{2}\|v\|_x$ et d'homologie $h = \sum_{i=1}^{4} h_i[\delta_i]$ correspond par l'isomorphisme précédent à une unique mesure de probabilité T-invariante m vérifiant :

$$h_i/\|h\|_s = \big[m(A_i) - m(A_{-i})\big] \Big/ \int \ln T' \,\mathrm{d}m \quad \forall i = 1, \dots, 4.$$

Nous montrons aussi que les mesures de probabilité ϕ^t-invariantes μ minimisant $\int (L - \omega)\,\mathrm{d}\mu$ pour une cohomologie ω donnée, correspondent canoniquement aux mesures de probabilité T-invariantes m minimisant :

$$\int \Bigg(\|\omega\|_s \ln T' - \sum_{i=1}^{4} \omega_i [\mathbf{1}_{A_i} - \mathbf{1}_{A_{-i}}] \Bigg) \mathrm{d}m$$

où $(\omega_i)_{i=1}^4$ désignent les coordonnées de $[\omega]$ dans la base duale de $([\delta_i])$.

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1. Introduction

We consider a specific example of a two-dimensional compact Riemannian surface $M = \mathbb{D}/\Gamma$ of genus $g = 2$ generated by a torsion free discrete group Γ acting on the Poincaré disk $\mathbb{D}$, with a Riemannian metric $\|\ \|$ of constant negative curvature. The generators of the group are denoted by

$$\mathcal{G} = \{a_{-2g}, \dots, a_{-1}, a_1, \dots, a_{2g}\}$$

and we will use later the convention $a_{-i} = a_i^{-1}$.

The surface M is naturally identified with the fundamental domain D whose boundary ∂D is a union of $4g$ arcs $s_{-2g}, \dots, s_{-1}, s_1, \dots, s_{2g}$ which are pieces of isometric circles (as defined in Ford [16]) of the generators $a_{-2g}, \dots, a_{-1}, a_1, \dots, a_{2g}$. Each side s_i is sent by the generator a_i to the side s_{-i} and M is obtained by identifying opposite sides s_i and s_{-i} with the generator a_i. We also define a partition $(A_{-2g}, \dots, A_{-1}, A_1, \dots, A_{2g})$ of $\mathbb{S}^1$ in the following way: each A_i is an interval open to the left and closed to the right and a_i, considered as a map from $\mathbb{S}^1$ to itself, is expanding on the interior of A_i and indifferent at the right endpoint.

The geodesic flow ϕ^t on the tangent bundle TM can be studied by different methods. On the one hand the boundary of the fundamental domain can be used to defined a Poincaré section of the flow. The dynamics of the rays which bounces from one side to another one resembles the billiard dynamics. We call geodesic billiard such a Poincaré map (X, B). On the other hand C. Series [7,25] considers a one-dimensional expanding transformation $T : \Sigma \to \Sigma$, acting on the sphere at infinity $\Sigma = \mathbb{S}^1$ by isometries of Γ. Unfortunately this transformation is not canonically well defined because of the presence of overlapping isometric circles. One part of our work is to relate the two dynamics. We show that the geodesic billiard (X, B) is "arithmetically" conjugate to the natural extension $(\widehat{\Sigma}, \widehat{T})$ of a particular choice of (Σ, T). We call Bowen–Series transformation such a choice of (Σ, T). We think this result is new compared to the results of [1] and [25]. The only place where we use a specific example is in the proof of the existence of the isomorphism. The rest of the paper is independent of the example. We think the existence of such an isomorphism can be proved for any Fuchsian group.

A geodesic flow ϕ^t on TM has always a Lagrangian structure; it is moreover Anosov when the curvature is negative everywhere (M is compact). The Lagrangian function is simply given by $L(x, v) = \frac{1}{2}\|v\|_x^2$ where $(x, v) \in TM$. Periodic trajectories of a Lagrangian system can be obtained by minimizing the action $\int_0^T L(x, \dot{x})\,\mathrm{d}t$ over all closed trajectories in a fixed homotopy class. Mather introduced in [23] a notion of minimizing measures generalizing the notion of periodic orbits. This notion has been further developed in [2–4,8,9,14,15,21,22]. In the context of a Lagrangian given by a Riemannian metric, Mather minimizing measures have two equivalent formulations:

Definition 1. Let $\mathcal{M}_1^{\mathrm{comp}}(TM, \phi^t)$ be the set of probability measures which have a compact support and are invariant with respect to the geodesic flow ϕ^t acting on the tangent bundle TM. Given an homology $h \in \mathcal{H}_1(M, \mathbb{R})$, a cohomology $[\omega] \in \mathcal{H}^1(M, \mathbb{R})$ and a ϕ^t-invariant probability measure μ of compact support, we say

(i) μ is minimizing for an homology h if it minimizes the integral

$$\beta(h) \stackrel{\text{def}}{=} \inf\left\{\int \frac{1}{2}\|v\|_x^2 \, d\mu(x,v) \mid \mu \in \mathcal{M}_1^{\text{comp}}\big(TM, \phi^t\big),\ [\mu] = h\right\},$$

where $[\mu]$ denotes its real homology.

(ii) μ is minimizing for a cohomology $[\omega]$ if it minimizes the integral

$$-\alpha(\omega) \stackrel{\text{def}}{=} \inf\left\{\int (L-\omega)\, d\mu \mid \mu \in \mathcal{M}_1^{\text{comp}}\big(TM, \phi^t\big)\right\}.$$

The two variational problems are connected by Legendre transform as shown in Appendix A. It is known that the support of Mather minimizing measures are, in both cases, contained in a fixed energy level [8]. In the present case this level of energy is determined by the stable norm of h or ω and the Mather minimizing measures, renormalized on the unit tangent bundle, correspond to probability measures having an homology on the boundary of the unit ball of the stable norm.

The methods of the present paper are different. In the second part of our work we show that Mather minimizing measures are in one-to-one correspondence with minimizing measures sitting in an abstract dynamical systems. We call abstract dynamical systems, (Σ, T), a system which can be well coded, which is for example a system isomorphic to a sub-shift of finite type. We say

Definition 2. Let (Σ, T) be a measurable dynamical system and $A : \Sigma \to \mathbb{R}$ be a bounded measurable observable. We say that a T-invariant probability measure μ is minimizing for A if

$$\int A\, d\mu = \inf\left\{\int A\, dm \mid m\ T\text{-invariant probability measure}\right\}.$$

We show how to transform a continuous time problem to an equivalent discrete time problem by using a particular choice of a Bowen–Series transformation T and a particular suspension of $\widehat{T}$ isomorphic the geodesic flow. We give in Section 2 basic definitions and explain carefully the connection between the two transformations $\widehat{T}$ and B. We prove in Section 3 how to transform the three-dimensional problem (a ϕ^t-invariant probability measure μ with a fixed homology $[\mu] = h$) to a one-dimensional problem (a T-invariant probability measure m satisfying $(m(A_i) - m(A_{-i}))/\int \ln T'\, dm = h_i/\|h\|_s$). Existence of coboundaries is the main technical tool. We state in Appendix A the main properties of Mather's theory that we will use and also some notations on homology.

We point out that twist maps are discrete time versions of the dynamics of periodic Lagrangian flows on $\mathbb{S}^1$ (see [3]). The discretization described here is in some sense different because we consider an autonomous flow. In this case, the induced map B will be an Anosov diffeomorphism (with singularities due to the corners). We refer the reader to [18,19] for related results about subactions for Anosov flows and diffeomorphisms. An interesting relation of our main theorem with ground states of C^*-algebras appears in [12].

Our main results are the following theorem and its corollary. In Theorem 3 the measures are not necessarily probability measures. In Corollary 4 all measures are probability measures. We use the notation $[\omega]$ to denote the cohomology associated to the closed 1-form ω and $[\delta]$ to denote the homology associated to the closed curve δ.

Theorem 3. *Let Σ be $\mathbb{S}^1$ minus the union of 4 periodic orbits of period 6 and by $(A_{-4}, \ldots, A_{-1}, A_1, \ldots, A_4)$ a partition of Σ by intervals associated to the generators. Let M^* be the surface M minus 2 self-intersecting closed geodesics passing through the same point and generating the fundamental group. We call $(\widehat{\Sigma}, \widehat{T})$ the natural extension of (Σ, T) and $\mathcal{M}(\Sigma, T)$ the set of T-invariant measures not necessarily of mass 1. Then*

(i) *There exists a roof function $\hat{\tau} : \widehat{\Sigma} \to \mathbb{R}^+$, cohomologous to $\ln T'$, such that the suspension of $(\widehat{\Sigma}, \widehat{T})$ by $\hat{\tau}$ is "arithmetically" isomorphic to the geodesic flow $\{\phi^t\}$ on M^*. Let $\hat{\Psi} : \widehat{\Sigma} \times \mathbb{R}/\hat{\tau} \to T^1M^*$ be the corresponding isomorphism. For any ϕ^t-invariant measure μ on T^1M^*, the corresponding measure $m = \hat{\Psi}^*(\mu)$ is the lift of a unique T-invariant measure satisfying $\mu(T^1M^*) = \int_\Sigma \ln T'\, dm$.*

(ii) *There exists a family of closed geodesics $(\delta_1, \ldots, \delta_4)$ which represents a basis of $\mathcal{H}_1(M, \mathbb{R})$, a family of closed 1-forms $([W_1], \ldots, [W_4])$ which represents a dual basis of $([\delta_1], \ldots, [\delta_4])$ such that, if $(h_1, \ldots, h_4)$ and $(\omega_1, \ldots, \omega_4)$ denote the coordinates of some homology h and cohomology $[\omega]$, then*

(a) *If μ is ϕ^t-invariant on T^1M^*, $(\mu_1,\dots,\mu_4)$ denotes its coordinates in the basis $([\delta_1],\dots,[\delta_4])$ and $m=\hat{\Psi}^*(\mu)$, then*

$$\mu_i = m(A_i) - m(A_{-i}) \quad \forall i = 1,\dots,4.$$

In particular, $\langle[\mu],[\omega]\rangle = \sum_{i=1}^4 \omega_i[m(A_i)-m(A_{-i})]$.

(b) *The stable norm $\|\omega\|_s$ of ω is equal to the supremum*

$$\sup\left\{\sum_{i=1}^4 \omega_i\big[m(A_i)-m(A_{-i})\big] \;\Big|\; m\in\mathcal{M}(\Sigma,T) \text{ and } \int \ln T'\,\mathrm{d}m = 1\right\}.$$

(c) *The unit ball in (h_i)-coordinates of the homological stable norm is*

$$\mathcal{B}_S = \left\{\big[m(A_i)-m(A_{-i})\big]_{i=1}^4 \;\Big|\; m\in\mathcal{M}(\Sigma,T) \text{ and } \int \ln T'\,\mathrm{d}m = 1\right\}.$$

Notice that we used the notion of a suspension of an invertible dynamical system (X,B) by a roof function $\tau : X\to\mathbb{R}$ which may become zero. By that we meant the standard suspension on the induced transformation on $\{\tau>0\}$.

We thus obtain that the problem of minimizing a Lagrangian L with a cohomological constraint ω is equivalent to an abstract problem of minimizing an observable for some piecewise one-dimensional expanding transformation T on the circle. We considered a similar problem in [11] for a smooth expanding transformation of the circle and a smooth observable.

Corollary 4. *There exist a basis $\{[W_i]\}_{i=1}^4$ of $\mathcal{H}^1$, a dual basis $\{[\delta_i]\}_{i=1}^4$ of $\mathcal{H}_1$ and a partition $\{A_i\}_{i=-4}^4$ such that for any $[\omega]\in\mathcal{H}^1$ and any $h\in\mathcal{H}_1$*

(i) *ϕ^t-invariant probability measures μ with compact support in TM^* minimizing $\int(L-\omega)\,\mathrm{d}\mu$ are in one-to-one correspondence with T-invariant probability measures minimizing*: $\|\omega\|_s \ln T' - \sum_{i=1}^4\omega_i[\mathbf{1}_{A_i}-\mathbf{1}_{A_{-i}}]$.

(ii) *ϕ^t-invariant probability measures μ of homology h, with compact support in TM^* and minimizing $\int L\,\mathrm{d}\mu$ are in one-to-one correspondence with T-invariant probability measures m satisfying for all $i=1,\dots,2g$:*

$$\int \big(h_i \ln T' - \|h\|_s[\mathbf{1}_{A_i}-\mathbf{1}_{A_{-i}}]\big)\,\mathrm{d}m = 0.$$

Where (ω_i) and (h_i) are the coordinates of $[\omega]$ and h in the basis $\{[W_i]\}_i$ and $\{[\delta_i]\}_i$.

The correspondence is given by the isomorphism $\hat{\Psi}$ in Theorem 3 and by normalizing the measure m to one using the following lemma [8–10,21]:

Lemma 5. *Let $h\in\mathcal{H}_1(M,\mathbb{R})$, $[\omega]\in\mathcal{H}^1(M,\mathbb{R})$ and μ be a ϕ^t-invariant probability measure of compact support on TM.*

(i) *μ is minimizing for h if and only if*

$$[\mu]=h \quad \textit{and} \quad \operatorname{supp}(\mu)\subset\big\{(x,v)\in TM \mid \|v\|_x = \|h\|_s\big\}.$$

(ii) *μ is minimizing for $[\omega]$ if and only if*

$$\int_{TM}\omega\,\mathrm{d}\mu = \sup\left\{\int_{TM}\omega\,\mathrm{d}\mu' \mid \mu'\in\mathcal{M}_1^{\mathrm{comp}}(TM,\phi^t)\right\},$$

$$\operatorname{supp}(\omega)\subset\big\{(x,v)\in TM \mid \|v\|_x = \|\omega\|_s\big\}.$$

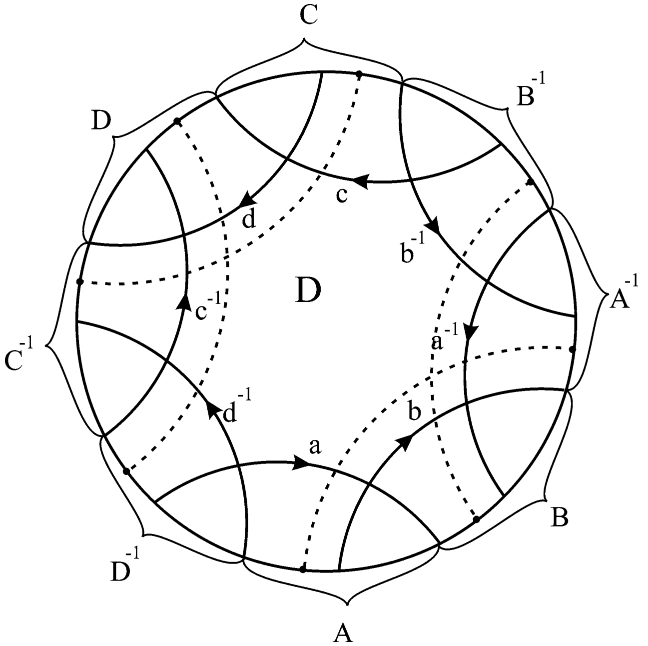

Fig. 1. Our particular example.

2. Basic definitions and isomorphism

In order to explain with the help of some pictures the main properties and the main difficulties in the analysis of the coding of the trajectories, we consider in this work a specific example of a compact Riemannian surface as shown in Fig. 1 (we refer the reader to [6,5,17,16] and [24] for general properties on Fuchsian groups and on geodesic flows on negative curvature surfaces). Nevertheless a great part of our analysis remains true for general compact Riemannian surfaces. The isomorphism Theorem 3(i) and the two coboundary equations (14) and (18) are true in general. Adler and Flatto [1] have obtained an isomorphism between the geodesic flow and a suspension of some symbolic dynamical systems. Our proof is geometric and gives an arithmetic conjugating map (Theorem 13) that is fundamental to prove Theorem 3(ii) and Corollary 4.

In this particular example the generators are a, b, c, d (the inverses are denoted respectively a^{-1}, b^{-1}, c^{-1}, d^{-1}) and the corresponding Dirichlet domain D is a regular octogon. The angle at each corner is equal to $\frac{1}{4}\pi$ and they are all identified in the surface M. The boundary of D corresponds to a union of two closed geodesics passing through this corner and self intersecting at this corner. We denote by M^* the surface M minus these two closed geodesics. We denote by Γ the group generated by a, b, c, d: they admit a unique relation

$$d^{-1}a^{-1}ba = c^{-1}d^{-1}cb \quad \Leftrightarrow \quad b^{-1}c^{-1}dcd^{-1}a^{-1}ba = 1.$$

The set of sides of ∂D are also denoted in the orientation preserving order:

$$\mathcal{S} = \{a, b, a_{-1}, b_{-1}, c, d, c_{-1}, d_{-1}\}$$

and the boundary at infinity $\partial\mathbb{D}$ is partitioned by corresponding semi-closed intervals as explained in Section 1 (see Fig. 1):

$$\mathcal{A} = \{A, B, A_{-1}, B_{-1}, C, D, C_{-1}, D_{-1}\}.$$

In the sequel we prefer to use the notations a_i, $a_{-i} = a_i^{-1}$ for the generators of Γ, s_i, s_{-i} for the corresponding sides of the fundamental domain D and A_i, A_{-i} for the corresponding partition at infinity. We begin by defining the geodesic billiard (X, B). We consider a trajectory of the geodesic flow starting at $(q_0, v_0) \in T^1\partial D$, $q_0 \in \partial D$ and v_0 pointing inward D, which hits in the forward direction ∂D at a point p_0. We denote by $\xi_0 \in \mathbb{S}^1$ and $\eta_0 \in \mathbb{S}^1$ the forward and backward prolongation of this trajectory. The point p_0 belongs to some side s_i and ξ_0 belongs to some interval

A_j, i and j may be distinct. In order to define B we use the generator a_i corresponding to the side s_i to obtain by reflection by a_i a new trajectory which again intersects the domain D. This new trajectory starts at $(q_1, v_1) \in T^1\partial D$ where $q_1 = a_i(p_0)$ belongs to the side s_{-i} and v_1 again points inward D. We just defined the standard return map to the section $T^1\partial D$ and call the geodesic billiard the map $B(q_0, v_0) = (q_1, v_1)$.

As it usually happens for billiards, the above definition of B is ambiguous at the corners of ∂D. We prefer to define B in a different way. We now choose the usual positive orientation for the sides s_i and, as for A_i, we assume that each s_i is semi-closed, open to the left and closed to the right. We also use the fact that a (non-oriented) geodesic which does not contain the origin $\mathcal{O}$ can be oriented so that $\mathcal{O}$ is on the left when moving positively along the geodesic. We finally introduce some notations. If (ξ, η) are two distinct points at infinity, we denote by $\overrightarrow{\eta\xi}$ the oriented geodesic starting at η and ending at ξ. If (p, q) are two distinct points in $\mathbb{D}$, we also denote by $\overrightarrow{qp}$ the oriented geodesic segment starting at q and ending at p. If q belongs to the geodesic $\overrightarrow{\eta\xi}$, we parametrize the geodesic by $\overrightarrow{\eta\xi}(q, t)$ starting at q when $t = 0$.

Definition 6. We first define the set X of all $(\xi, \eta) \in \mathbb{S}^1 \times \mathbb{S}^1$ such that the geodesic $\overrightarrow{\eta\xi}$ either intersects the interior of D or contains just one corner of ∂D and see $\mathcal{O}$ to the right (the complete geodesics corresponding to the sides of ∂D are not included in X). For each $(\xi, \eta) \in X$, we define two points $p = p(\xi, \eta)$ and $q = q(\xi, \eta)$ belonging to the geodesic so that $\overrightarrow{qp}$ corresponds to the intersection of $\overrightarrow{\eta\xi}$ with $\overline{D}$. In any case p and q belong to ∂D and $\overrightarrow{\eta\xi}$ contains a corner if and only if $p = q$. We also choose for each $(\xi, \eta) \in X$ the unique generator a_i which corresponds to the side s_i containing p and denote by $\gamma = \gamma(\xi, \eta)$ this unique element of $\mathcal{G}$. We are now able to define the geodesic billiard B by

$$B(\xi, \eta) = (\xi', \eta') \quad \text{where } \xi' = \big(\gamma(\xi, \eta)\big)(\xi) \text{ and } \eta' = \big(\gamma(\xi, \eta)\big)(\eta).$$

We note that $B : X \to X$ is an invertible transformation and almost corresponds to a Poincaré return map to $T^1\partial D$. We next define the return time in our more formal setting.

Definition 7. For any $(\xi, \eta) \in X$, we call return time the function

$$\tau(\xi, \eta) = d\big(p(\xi, \eta), q(\xi, \eta)\big).$$

It is now easy to see that the suspension $(X \times \mathbb{R}/\tau, \{B_\tau^t\})$ of (X, B) by the roof function τ (that is the suspension of the induced transformation on $\{\tau > 0\}$) is isomorphic to the geodesic flow $(T^1M^*, \{\phi^t\})$ where M^* denotes the surface M minus the union of the two previous closed geodesics corresponding to ∂D. The isomorphism Ψ is formally given by

$$\Psi(\xi, \eta, t) = \left(\overrightarrow{\eta\xi}\big(q(\xi, \eta), t\big), \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{\eta\xi}\big(q(\xi, \eta), t\big)\right) \quad \text{mod } \Gamma.$$

For any ϕ^t-measure μ in M^* there exists a unique B-invariant measure $\bar{\mu}$ on X such that

$$\Psi^*(\mu)(\mathrm{d}\xi, \mathrm{d}\eta, \mathrm{d}t) = \bar{\mu}(\mathrm{d}\xi, \mathrm{d}\eta) \times \mathrm{d}t \quad \text{mod } \tau.$$

One of the main property the map B possesses and that we will repeatedly use is formalized by the relation

$$\big(\gamma(\xi, \eta)\big)\big(p(\xi, \eta)\big) = q \circ B(\xi, \eta) \quad \forall(\xi, \eta) \in X.$$

In other words, if $(\xi_n, \eta_n)_{n\in\mathbb{Z}}$ denotes a B-trajectory, if p_n, q_n are the corresponding points in ∂D and γ_n is the corresponding generator in $\mathcal{G}$ then $\gamma_n(p_n) = q_{n+1}$ for all n.

We now define formally the Bowen–Series transformation T and its natural extension $\widehat{T}$. As it was explained at the beginning, the domains of the isometric circles overlap each other and a particular choice has to be done for T. The boundary at infinity is first partitioned into semi-closed intervals $A_{-2g}, \ldots, A_{-1}, A_1, \ldots, A_{2g}$, each A_i belongs to the unstable domain of the hyperbolic element a_i and the right endpoint of each A_i is neutral with respect to a_i, that is, the derivative of a_i is equals to 1. By definition the restriction of T to A_i equals a_i. More precisely:

Definition 8. Each interval A_i is naturally associated to a generator $a_i \in \Gamma$ (A_{-i} is associated to a_i^{-1}). Let $\hat{\gamma} : \mathbb{S}^1 \to \Gamma$ defined by $\hat{\gamma}(\xi) = a_i \Leftrightarrow \xi \in A_i$. Let $T : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $T(\xi) = (\hat{\gamma}(\xi))(\xi)$. The map T will be called the (right) Bowen–Series transformation. Note that T^2 is expanding.

In order to show that T is Markov with respect to some partition we introduce the set of all (complete) geodesics intersecting $\overline{D}$ which are sent by some element of Γ to an isometric circle of one of the generators. We denote by $\partial^2_\infty D \subset \mathbb{S}^1 \times \mathbb{S}^1$ this finite set of geodesics. In our example it corresponds to 4 periodic orbits of period 6 that is $\#\partial^2_\infty D = 24$.

The endpoints $\partial^1_\infty D$ of the geodesics in $\partial^2_\infty D$ determine uniquely a partition of $\mathbb{S}^1$. We denote by $\mathcal{P} = \{P_k\}_k$ this partition. Each P_k is a semi-closed interval, left open and right closed. It is easy to see that T becomes a one-dimensional Markov transformation with respect to $\mathcal{P}$. In order to define its natural extension, we introduce the left Bowen–Series transformation:

Definition 9. There exists a unique partition of $\mathbb{S}^1$ into intervals $(\tilde{A}_i)_{i=-4}^{4}$, left open and right closed where each $\tilde{A}_i$ belongs to the unstable domain of a_i and has a neutral left end point with respect to a_i. We call left Bowen–Series transformation the map $\widetilde{T}:\mathbb{S}^1 \to \mathbb{S}^1$ defined by:

$$\widetilde{T}(\eta) = \big(\tilde{\gamma}(\eta)\big)(\eta),$$

where $\tilde{\gamma}:\mathbb{S}^1 \to \Gamma$ and $\tilde{\gamma}(\eta) = a_{-i} \Leftrightarrow \eta \in \tilde{A}_{-i}$.

The natural extension of a dynamical system (Σ, T) is the space of all inverse branches for T. This space may have several isomorphic representations. We give in the following an "almost" smooth realization.

Definition 10. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, $\widehat{T}:\mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$\widehat{T}(\xi,\eta) = (\xi',\eta') \quad \text{where } \xi' = \big(\hat{\gamma}(\xi)\big)(\xi) \text{ and } \eta' = \big(\hat{\gamma}(\xi)\big)(\eta).$$

Let $\hat{p}:\mathbb{T}^2 \to \mathbb{S}^1$, $\hat{q}:\mathbb{T}^2 \to \mathbb{S}^1$ be the first and second projection. Let $\Sigma = \mathbb{S}^1 \setminus \partial^1_\infty D$ and $\widehat{\Sigma} = \{(\xi,\eta) \in \mathbb{T}^2 \setminus \partial^2_\infty D \mid \widehat{T}^n(\xi,\eta)$ is compatible for all $n\}$ where two endpoints (ξ,η) are said to be compatible if, whenever $\xi \in A_i$, $a_i(\eta) \in \tilde{A}_{-i}$.

We can then prove easily

Proposition 11. *For each P_k there exists a semi-closed interval Q_k, left open and right closed such that $\widehat{\Sigma} = \bigsqcup_k P_k \times Q_k \setminus \partial^2_\infty D$ (disjoint union).*

(i) *The map $\widehat{T}:\widehat{\Sigma} \to \widehat{\Sigma}$ becomes invertible with inverse map*

$$\widehat{T}^{-1}(\xi',\eta') = (\xi,\eta) \quad \textit{where } \xi = \big(\tilde{\gamma}(\eta')\big)(\xi') \textit{ and } \eta = \big(\tilde{\gamma}(\eta')\big)(\eta').$$

(ii) *$(\widehat{\Sigma}, \widehat{T})$ is the natural extension of (Σ, T). That is, the surjective map $\hat{p}:\widehat{\Sigma} \to \Sigma$ commutes with the two actions $\widehat{T}$ and T and the set of inverse branches for T of a point ξ is in one-to-one correspondence with the set of points in the fiber $\hat{p}^{-1}(\xi)$. Moreover $(\widehat{\Sigma}, \widehat{T}^{-1})$ is the natural extension of $(\Sigma, \widetilde{T})$.*

(iii) *$\widehat{T}$ is Markov in the sense of Bowen. If $\mathcal{P}(\xi)$ denotes one of the intervals P_k which contains ξ and $\mathcal{Q}(\xi)$ the corresponding Q_k, the family of rectangles $\{P_k \times Q_k\}_k$ possesses the following three properties*:
   (a) *$\mathcal{P}(T(\xi)) \subset (\hat{\gamma}(\xi))(\mathcal{P}(\xi))$ and $(\hat{\gamma}(\xi))(\mathcal{Q}(\xi)) \subset \mathcal{Q}(T(\xi))$.*
   (b) *$\mathcal{Q}(\xi) = \bigsqcup_{T(\xi')=\xi} (\hat{\gamma}(\xi'))(\mathcal{Q}(\xi'))$ (disjoint union).*
   (c) *For any inverse branch $\{\xi_{-n}\}_{n\geqslant 0}$ of ξ, that is $T(\xi_{-(n+1)}) = \xi_{-n}$ for all $n \geqslant 0$ and $\xi_0 = \xi$, the intersection $\bigcap_{n\geqslant 0} \hat{\gamma}^n_{-n}(\mathcal{Q}(\xi_{-n})) = \{\eta\}$ is a unique point where $\hat{\gamma}^n_{-n} = \hat{\gamma}(\xi_{-1}) \cdots \hat{\gamma}(\xi_{-n})$.*

The proof of this proposition is easy as soon as we know how to construct the intervals Q_k and how to prove their Markov properties. Figs. 2–7 give an explicit definition of Q_k for each P_k. We obtained this construction by trial and success and it seems interesting to find an arithmetic construction for any co-compact group Γ.

We just have defined two kinds of dynamical systems: the geodesic billiard which is a particular Poincaré section of the geodesic flow and the Bowen–Series transformation (and its associated natural extension) which has a more combinatorial nature and is easier to understand because of its Markov property. Each trajectory (ξ,η) generates two Γ-valued codings, $\{\gamma_n\}_{n\in\mathbb{Z}}$ and $\{\hat{\gamma}\}_{n\in\mathbb{Z}}$ where $\gamma_n = \gamma \circ B^n(\xi,\eta)$ is the coding associated to the billiard and $\hat{\gamma}_n = \hat{\gamma} \circ \widehat{T}^n(\xi,\eta)$ is the coding associated to the Bowen–Series transformation. The main lemma we are going to

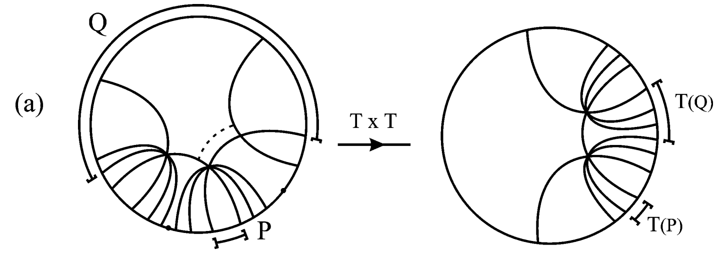

Fig. 2.

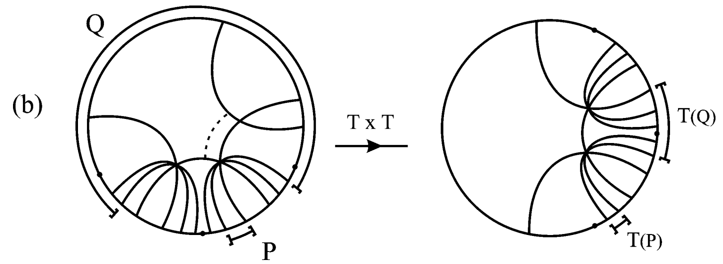

Fig. 3.

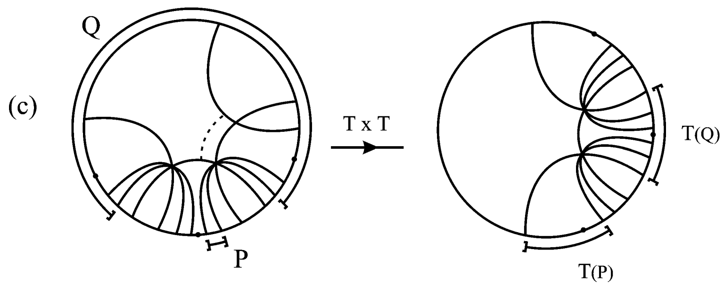

Fig. 4.

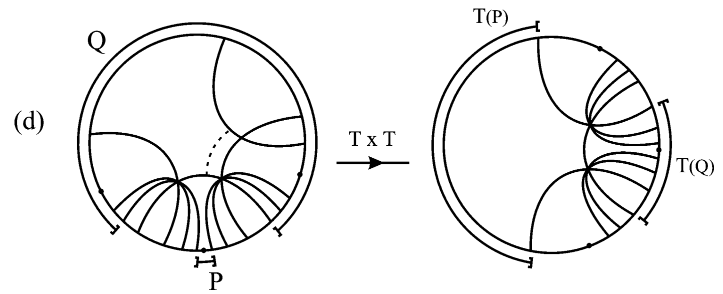

Fig. 5.

prove shows that the two codings are "equivariant". The fact that the codings may be different arises when ξ belongs to two overlapping isometric circles say $\xi \in A_i$ and when $p = p(\xi, \eta)$ belongs to a distinct side, say $p \in s_j$, $i \neq j$. In Fig. 8(0) we show a trajectory in $\widehat{\Sigma} \cap X$ which ends in two overlapping circles a and b. In Figs. 8(1)–8(3) we then show the iteration of the two parallel trajectories, the real one under B and the virtual one under $\widehat{T}$. At the fourth iteration, Fig. 8(4), the two trajectories coincide again unless we come back to the initial case 8(0). A more precise analysis will be done later.

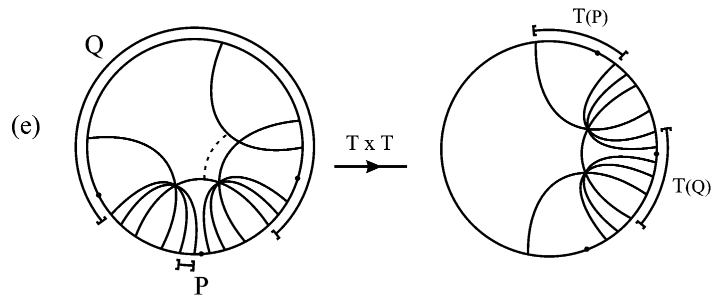

Fig. 6.

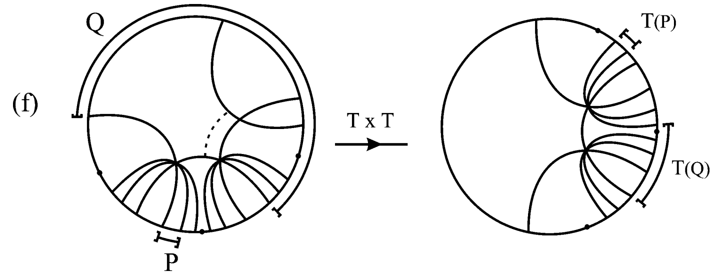

Fig. 7.

The two spaces $\widehat{\Sigma}$ and X are distinct; neither of the two is included in the other. There are trajectories for instance in X nearly tangent to ∂D and seeing $\mathcal{O}$ to the right which do not belong to $\widehat{\Sigma}$. There are trajectories in $\widehat{\Sigma}$ which do not intersect the fundamental domain D.

We will now explain the fundamental lemma which will enable us to give a very precise description of the isomorphism between $(\widehat{\Sigma}, \widehat{T})$ and (X, B).

Lemma 12 *((Fundamental)). Let $K = \widehat{\Sigma} \cap X$, then*

(i) *The orbit of any (ξ, η) of $\widehat{\Sigma}$ (or of X) eventually hits K under the action of $\{\widehat{T}^{-n}\}_{n \geqslant 0}$ (or $\{B^{-n}\}_{n \geqslant 0}$).*

(ii) *If $(\xi, \eta) \in K$ and if $n = n(\xi, \eta) \geqslant 1$ denotes the first return time to K under the action of $\{\widehat{T}^n\}_{n \geqslant 0}$ (n can be infinite) then $n(\xi, \eta)$ is also the first return time to K under $\{B^n\}_{n \geqslant 0}$ and*

$$\prod_{i=0}^{n-1} \hat{\gamma} \circ \widehat{T}^i(\xi, \eta) = \prod_{i=0}^{n-1} \gamma \circ B^i(\xi, \eta).$$

Proof. We only prove the second statement of this lemma. The proof is mainly based on the sequence of figures starting at Fig. 9. In each case we have drawn two complete isometric circles (and a third one incomplete). We have also drawn two other circles of $\partial^2_\infty D$; they all go through the same point which is one of the corner of the fundamental domain. Each figure represents locally a part of the partition $\mathcal{P}$. On the left-hand side we iterate $\widehat{T}^n(\xi, \eta)$, on the right-hand side we iterate $B^n(\xi, \eta)$ where $(\xi, \eta) \in K$ initially. Moreover we mark the geodesic (ξ, η) by a point I that we follow under the iteration of $\widehat{T}$ or B. We assume initially that $\widehat{T}(\xi, \eta) \notin K$ which is equivalent to $\hat{\gamma}(\xi, \eta) \neq \gamma(\xi, \eta)$ or $B(\xi, \eta) \notin K$. After one iteration by $\widehat{T}$ or B we are in case 1. After two iterations (there is no corresponding figure) we also get $\widehat{T}^2(\xi, \eta) \notin K$ and $B^2(\xi, \eta) \notin K$. After three iterations we discuss 4 different cases: case 2, case 3, case 5 and case 7. In cases 2 and 3, simultaneously after one iteration by $\widehat{T}$ or B, we come to the case 4 where the two geodesics $\widehat{T}^4(\xi, \eta) \in K$ and $B^4(\xi, \eta) \in K$ intersect the interior of D. At this time the element of the group Γ

$$\big[\hat{\gamma} \circ T^4(\xi, \eta) \cdots \hat{\gamma}(\xi, \eta)\big]\big[\gamma \circ B^4(\xi, \eta) \cdots \gamma(\xi, \eta)\big]^{-1}$$

sends the point $I \in \partial D$ to the point $\hat{I} \in \partial D$ and the re-entering geodesic $B^4(\xi, \eta)$ to the re-entering geodesic $\widehat{T}^4(\xi, \eta)$.

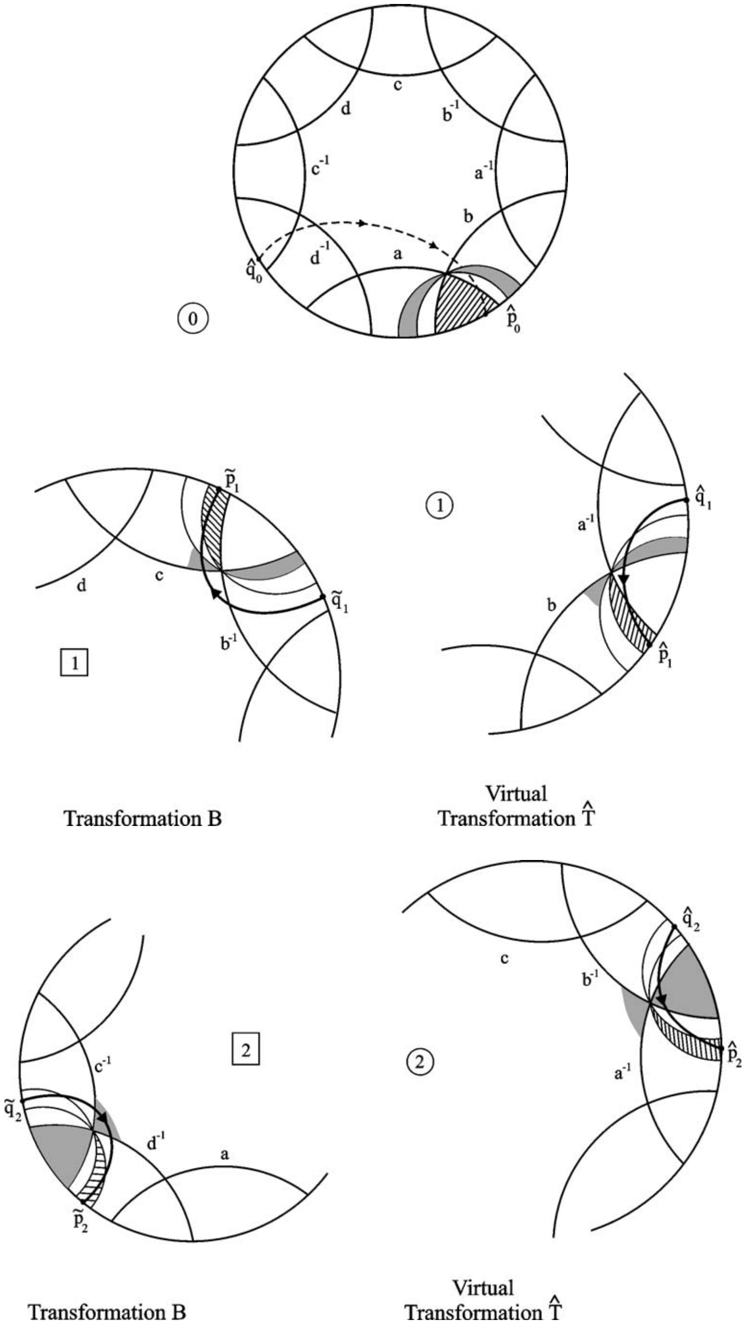

Fig. 8.

Necessarily the two products of γ's coincide:

$$\hat{\gamma} \circ T^4(\xi, \eta) \cdots \hat{\gamma}(\xi, \eta) = \gamma \circ B^4(\xi, \eta) \cdots \gamma(\xi, \eta).$$

The cases 5 and 7 are similar to the initial case since after one iteration we come back to case 1. The cases 6 and 8 are indeed identical to case 1 but in case 8 the two geodesics are marked by two new point $\hat{I}'$ and I' which are related by

$$I' = \big[\gamma \circ B^4(\xi, \eta) \cdots \gamma(\xi, \eta)\big]\big[\hat{\gamma} \circ T^4(\xi, \eta) \cdots \hat{\gamma}(\xi, \eta)\big]^{-1} \hat{I}'.$$

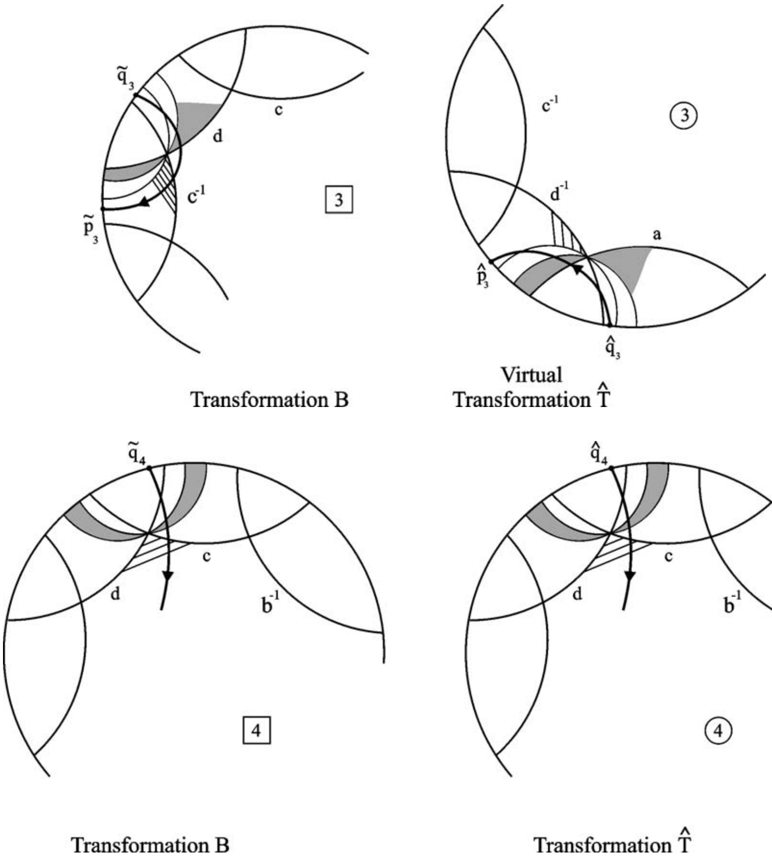

Fig. 8 (*continued*).

We start again the whole discussion and wait until we are in case 4 where we get an element g of the form $[\prod_{i=0}^{n}\hat{\gamma}_i][\prod_{i=0}^{n}\gamma_i]^{-1}$ which send some point $I \in \partial D$ to some point $\hat{I} \in \partial D$ and a re-entering geodesic to another re-entering geodesic. That element g has to be the identity. $\square$

Notice that there exist orbits both in $\widehat{\Sigma}$ and in X which hit K infinitely often in the past but never return to K in the future: these orbits belong to the stable manifold of some "hidden" periodic orbits.

Theorem 13. *There exists a Γ-valued map $\rho : \widehat{\Sigma} \to \Gamma$ such that*:

(i) *If $\pi : \widehat{\Sigma} \to X$ is defined by $\pi(\xi,\eta) = (\rho(\xi,\eta)\xi, \rho(\xi,\eta)\eta)$ then π is an isomorphism between $\widehat{\Sigma}$ and X which conjugates the two actions $\widehat{T}$ and B, that is $\pi \circ \widehat{T} = B \circ \pi$.*
(ii) *Moreover $\hat{\gamma}$ and $\gamma \circ \pi$ are equivariant in the sense $\rho \circ \widehat{T}\hat{\gamma} = \gamma \circ \pi\rho$.*

Proof. If (ξ,η) belongs to $K = \widehat{\Sigma} \cap X$ we define $\rho(\xi,\eta) = e$ (the neutral element in Γ). Otherwise we consider the first return time $n = n(\xi,\eta)$ such that $\widehat{T}^{-n}(\xi,\eta) \in K$ and we define:

$$\rho(\xi,\eta) = \left[\prod_{i=0}^{n-1} \gamma \circ B^i \circ \widehat{T}^{-n}(\xi,\eta)\right]\left[\prod_{i=0}^{n-1} \hat{\gamma} \circ \widehat{T}^i \circ \widehat{T}^{-n}(\xi,\eta)\right]^{-1}$$

(where $\prod_{i=0}^{n-1}$ denotes a non-commutative product beginning to the right by the index $i=0$). The fundamental lemma tells us we can choose any return time to K in the definition of ρ. We now define π as in the theorem:

$$\pi(\xi,\eta) = \big(\rho(\xi,\eta)\xi, \rho(\xi,\eta)\eta\big).$$

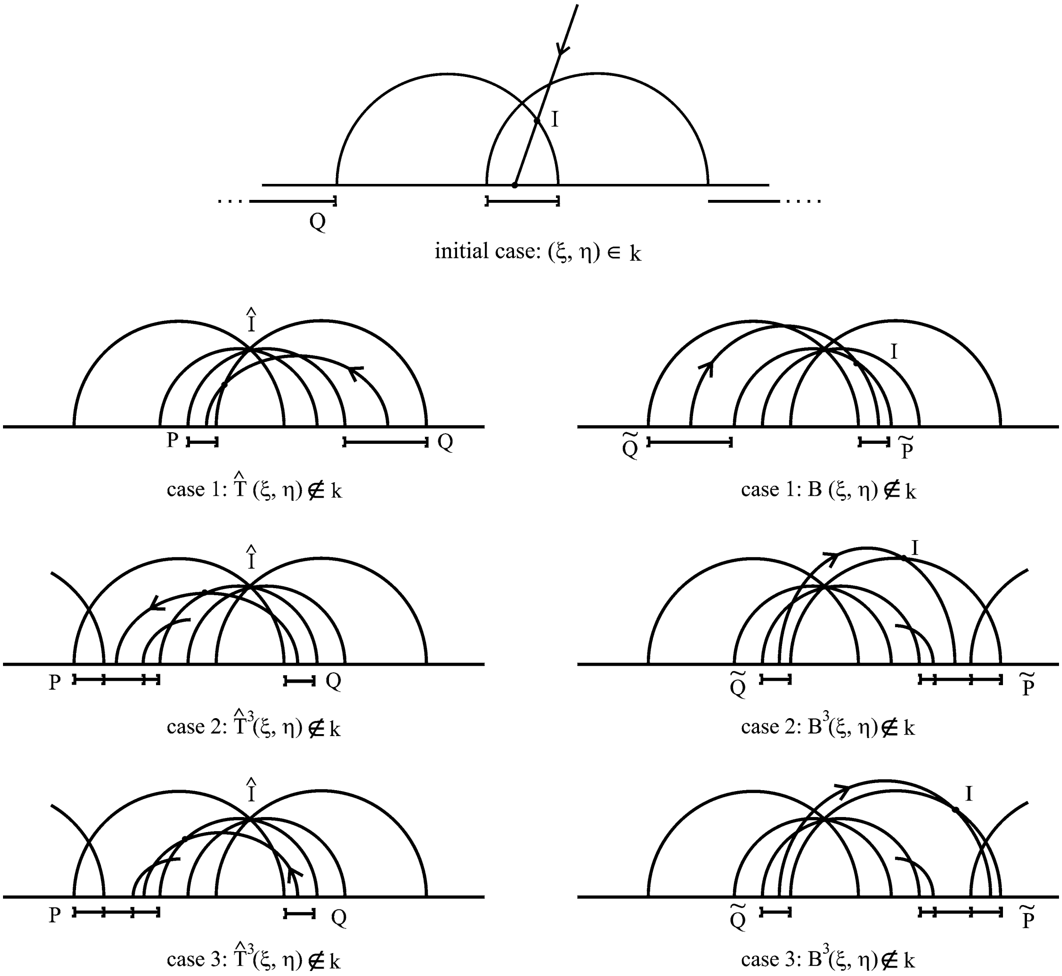

Fig. 9. Fundamental lemma.

By definition of π, $\pi(\xi,\eta) = B^n \circ \widehat{T}^{-n}(\xi,\eta)$ for any return time (possibly 0) of (ξ,η) to K under the action of $\{\widehat{T}^{-n}\}_{n\geqslant 0}$. In particular, if n is a return time, then $n+1$ is a return time of $\widehat{T}(\xi,\eta)$ and

$$\pi \circ \widehat{T}(\xi,\eta) = B^{n+1} \circ \widehat{T}^{-n}(\xi,\eta) = B \circ \pi(\xi,\eta).$$

Moreover

$$\begin{aligned}
\rho \circ \widehat{T}(\xi,\eta) &= \left[\prod_{i=0}^{n} \gamma \circ B^i \circ \widehat{T}^{-n}(\xi,\eta)\right]\left[\prod_{i=0}^{n} \hat{\gamma} \circ \widehat{T}^i \circ \widehat{T}^{-n}(\xi,\eta)\right]^{-1} \\
&= \gamma \circ \pi(\xi,\eta)\left[\prod_{i=0}^{n-1} \gamma \circ B^i \circ \widehat{T}^{-n}(\xi,\eta)\right]\left[\prod_{i=0}^{n-1} \hat{\gamma} \circ \widehat{T}^i \circ \widehat{T}^{-n}(\xi,\eta)\right]^{-1} \hat{\gamma}(\xi,\eta)^{-1} \\
&= \gamma \circ \pi(\xi,\eta)\rho(\xi,\eta)\hat{\gamma}(\xi,\eta)^{-1}. \qquad \square
\end{aligned}$$

3. Homology in (Σ, T)

Let $\hat{\tau} = \tau \circ \pi$, then the suspension $(\widehat{\Sigma} \times \mathbb{R}/\hat{\tau}, \{\widehat{T}^t_{\hat{\tau}}\})$ of $(\widehat{\Sigma}, \widehat{T})$ by the roof function $\hat{\tau}$ is isomorphic to $(T^1M^*, \{\phi^t\})$ and the isomorphism is given by:

$$\hat{\Psi}(\xi,\eta,t) = \Psi\big(\pi(\xi,\eta),t\big).$$

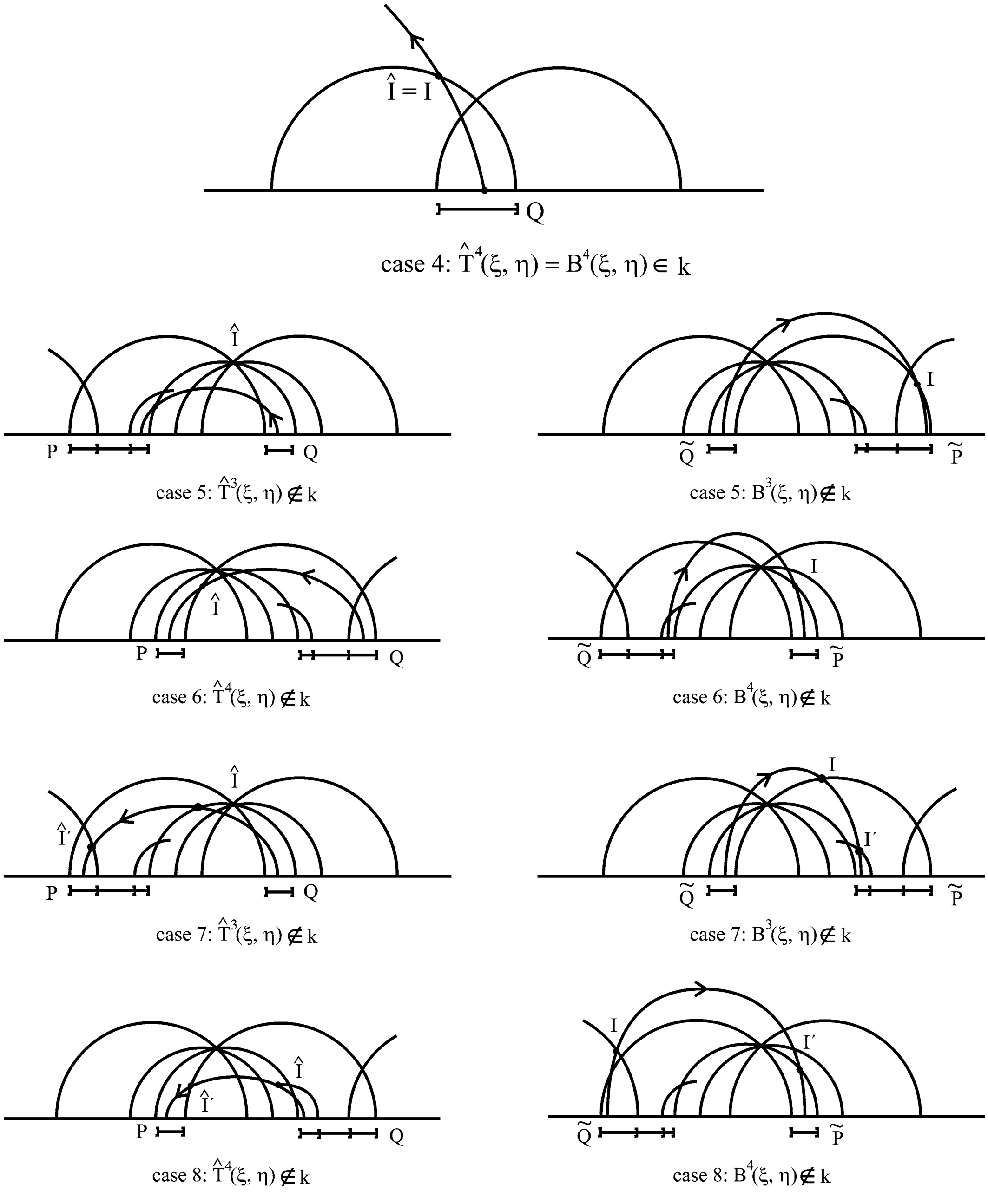

Fig. 9 (*continued*).

For any ϕ^t-invariant measure μ in M^*, there exists a unique $\widehat{T}$-invariant measure $\hat{\mu}$ in $\widehat{\Sigma}$ such that,

$$\Psi^*(\mu)(\mathrm{d}\xi, \mathrm{d}\eta, \mathrm{d}t) = \hat{\mu}(\mathrm{d}\xi, \mathrm{d}\eta) \times \mathrm{d}t \quad \mod \hat{\tau}.$$

Since $\widehat{T}$ is the natural extension of T, any $\widehat{T}$-invariant measure $\hat{\mu}$ corresponds to a unique T-invariant measure m such that

$$\pi^*(m) = \hat{\mu}.$$

As we do not normalize $\hat{\mu}$ we obtain the relation:

$$\int_{T^1M^*} \mathrm{d}\mu = \int_{\widehat{\Sigma}} \hat{\tau}\,\mathrm{d}\hat{\mu}.$$

We now show that the return time $\hat{\tau} = \tau \circ \pi$ as a function on $\widehat{\Sigma}$ is cohomologous to a function which depends only on the first variable. The main tool we will use is given by the Busemann function $b_\xi(p,q)$. For any $\xi \in \mathbb{S}^1$, for any $p,q \in D$, we call Busemann function $b_\xi(p,q)$, the oriented distance between the two horocycles tangent to $\mathbb{S}^1$ at ξ and passing through p and q. The value of $b_\xi(p,q)$ can be positive or negative and in order to fix the sign we define formally:

$$b_\xi(p,q) = \text{“}d(p,\xi) - d(q,\xi)\text{”} = \lim_{k\to+\infty}\big(d(p,z_k) - d(q,z_k)\big),$$

where $\{z_k\}_k$ converges in $\overline{\mathbb{D}}$ to ξ. We recall that two real valued functions α, β defined on $\widehat{\Sigma}$ are said to be cohomologous if there exists $c : \widehat{\Sigma} \to \mathbb{R}$ such that $\alpha = \beta + c - c \circ \widehat{T}$. The following proposition proves part (i) of the main Theorem 3.

Proposition 14. *The return time $\hat{\tau}(\xi,\eta) = \tau \circ \pi(\xi,\eta)$ is cohomologous to* $\ln T'(\xi) = b_\xi(\mathcal{O}, \hat{\gamma}(\xi)^{-1}(\mathcal{O}))$.

Proof. The proof is divided into two parts. In the first part we show that $\tau(\xi,\eta)$ is cohomologous to $b_\xi(\mathcal{O},\gamma^{-1}\mathcal{O})$ where $\gamma = \gamma(\xi,\eta)$. In the second part we use the equivariance between $\gamma \circ \pi$ and $\hat{\gamma}$ by ρ to conclude.

Part one. In order to simplify the notations, we use $p = p(\xi,\eta)$ and $q = q(\xi,\eta)$. Then

$$\begin{aligned}\tau(\xi,\eta) &= b_\xi(q,p) = b_\xi(q,\mathcal{O}) + b_\xi\big(\mathcal{O},\gamma^{-1}\mathcal{O}\big) + b_\xi\big(\gamma^{-1}\mathcal{O},p\big)\\ &= c(\xi,\eta) - c\circ B(\xi,\eta) + b_\xi\big(\mathcal{O},\gamma^{-1}\mathcal{O}\big),\end{aligned}$$

where $c(\xi,\eta) = b_\xi(q(\xi,\eta),\mathcal{O})$. We have used the fact that the Busemann function is γ-invariant and the identity $\gamma p = q \circ B$.

Part two. Let $(\hat{\xi}_0,\hat{\eta}_0) \in \widehat{\Sigma}$ and to simplify $\pi(\hat{\xi}_0,\hat{\eta}_0) = (\xi_0,\eta_0)$, $\gamma_0 = \gamma(\xi_0,\eta_0)$, $\rho_0 = \rho(\hat{\xi}_0,\hat{\eta}_0)$, $\rho_1 = \rho \circ \widehat{T}(\hat{\xi}_0,\hat{\eta}_0)$, $(\xi_1,\eta_1) = B(\xi_0,\eta_0) = (\gamma_0\xi_0,\gamma_0\eta_0)$. We recall the equivariance relation $\gamma_0 = \rho_1\hat{\gamma}_0\rho_0^{-1}$. Then

$$\begin{aligned}b_{\xi_0}\big(\mathcal{O},\gamma_0^{-1}\mathcal{O}\big) &= b_{\xi_0}\big(\mathcal{O},\rho_0\hat{\gamma}_0^{-1}\rho_1^{-1}\mathcal{O}\big)\\ &= b_{\xi_0}(\mathcal{O},\rho_0\mathcal{O}) + b_{\rho_0^{-1}\xi_0}\big(\mathcal{O},\hat{\gamma}_0^{-1}\mathcal{O}\big) - b_{\rho_1\hat{\gamma}_0\rho_0^{-1}\xi_0}(\mathcal{O},\rho_1\mathcal{O})\\ &= \hat{c}\big(\hat{\xi}_0,\hat{\eta}_0\big) - \hat{c}\circ\widehat{T}\big(\hat{\xi}_0,\hat{\eta}_0\big) + b_{\hat{\xi}_0}\big(\mathcal{O},\hat{\gamma}_0^{-1}\mathcal{O}\big),\end{aligned}$$

where $\hat{c}(\xi,\eta) = b_\xi(\rho(\xi,\eta)^{-1}\mathcal{O},\mathcal{O})$. Notice that the total coboundary is

$$\hat{c}(\xi,\eta) + c\circ\pi(\xi,\eta) = b_\xi\big(\rho(\xi,\eta)^{-1}q\circ\pi(\xi,\eta),\mathcal{O}\big).$$

The equality $\ln T'(\xi) = \ln \frac{\mathrm{d}}{\mathrm{d}\xi}\hat{\gamma}_0(\xi) = b_\xi(\mathcal{O},\hat{\gamma}_0^{-1}(\mathcal{O}))$ is standard. □

We now want to transport the notion of homology of a measure μ in M^* to a similar notion for the corresponding measure m in Σ. We refer the reader to the appendix for definitions, main properties and notations on Mather's theory that will be used in this section. An homology of a measure is known as soon as its action $\int_{T^1M}\omega\,\mathrm{d}\mu$ on any closed 1-form ω is given. More generally, let us introduce some notations.

Definition 15. For any function $Y : T^1M \to \mathbb{R}$, $Y = Y(q,v)$, we denote by $\widetilde{Y}$ the corresponding function $\widetilde{Y} = Y \circ \Psi$ on the suspension $X \times \mathbb{R}/\tau$ and by $\overline{Y}$ the function defined on the base X:

$$\overline{Y}(\xi,\eta) = \int_{t=0}^{\tau(\xi,\eta)} \widetilde{Y}(\xi,\eta,t)\,\mathrm{d}t = \int_{t=0}^{\tau(\xi,\eta)} Y\left(\overrightarrow{\eta\xi}\big(q(\xi,\eta),t\big), \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{\eta\xi}\big(q(\xi,\eta),t\big)\right)\mathrm{d}t.$$

By convention $\overline{Y}(\xi,\eta) = 0$ on $\{\tau = 0\}$ (we recall that the suspension is only defined for the induced map on $\{\tau > 0\}$).

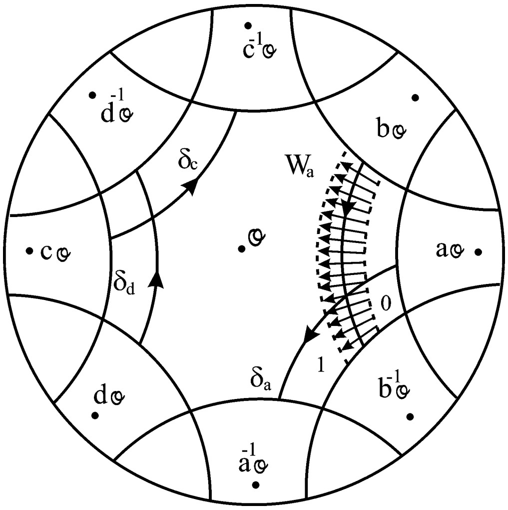

Fig. 10.

By the previous isomorphism $\Psi : X \times \mathbb{R}/\tau \to T^1M^*$, if μ is ϕ^t-invariant and $\bar{\mu}$ is the corresponding B-invariant measure, we obtain

$$\int_X \overline{Y}\,\mathrm{d}\bar{\mu} = \int_{T^1M} Y\,\mathrm{d}\mu.$$

In particular if ω is a closed 1-form on M and if we denote by the same letter ω the corresponding Γ-invariant (exact) 1-form defined on $\mathbb{D}$, the previous definition gives

$$\overline{\omega}(\xi,\eta) = \int_{\overrightarrow{q(\xi,\eta)p(\xi,\eta)}} \omega.$$

We choose a basis of $\mathcal{H}_1(M,\mathbb{Z})$ in the following way (see Fig. 10). Each generator a_i is an hyperbolic element in Γ and admits a unique axis (a globally invariant geodesic). This geodesic cuts the two corresponding sides s_i and s_{-i} and the resulting segment gives a closed periodic orbit in M that we call δ_i.

We orientate δ_i so that the positive endpoint coincides with the unstable fixed point of a_i. We thus obtain a basis $([\delta_1],\dots,[\delta_{2g}])$ of $\mathcal{H}_1(M,\mathbb{Z})$: the homology of any closed curved is equal to a linear combination of $[\delta_i]$ with integer coefficients. We denote by $([W_1],\dots,[W_{2g}])$ the dual basis in $\mathcal{H}^1(M,\mathbb{R})$. For instance in Fig. 10, W_a is equal to the differential of a jump function, locally defined in a strip about the closed curve δ_b, null on one side of the strip and equal to 1 on the other side. Since the closed curve $\overrightarrow{\mathcal{O}a_i^{-1}\mathcal{O}}$ is homologue to δ_i, we obtain

$$\int_{\overrightarrow{\mathcal{O}a_i^{-1}\mathcal{O}}} W_j = 0, \quad \text{if } i \neq j, \quad \text{and} \quad \int_{\overrightarrow{\mathcal{O}a_i^{-1}\mathcal{O}}} W_j = 1, \quad \text{if } i = j.$$

In order to simplify the notations we introduce

Definition 16. For any element γ in the group Γ we define its homology $H(\gamma)$ by the formula

$$\big\langle H(\gamma), [\omega]\big\rangle = \int_{\overrightarrow{\mathcal{O}\gamma^{-1}\mathcal{O}}} \omega = \int_{\mathcal{O}}^{\gamma^{-1}\mathcal{O}} \omega.$$

We can then prove:

Lemma 17. *The map $H : \Gamma \to \mathcal{H}_1(M,\mathbb{R})$ is a surjective group homomorphism.*

Proof. We first show that H is an homomorphism. Let α and β two elements of the group Γ. Then

$$\langle H(\alpha\beta),[\omega]\rangle=\int_{\mathcal{O}}^{\beta^{-1}\mathcal{O}}\omega+\int_{\beta^{-1}\mathcal{O}}^{\beta^{-1}\alpha^{-1}\mathcal{O}}\omega=\langle H(\beta),[\omega]\rangle+\langle H(\alpha),[\omega]\rangle.$$

We have used the fact that ω is Γ-equivariant on $\mathbb{D}$. Moreover H is surjective since $H(a_i)=[\delta_i]$. $\square$

The following proposition proves part (ii) of the main Theorem 3.

Proposition 18. *If ω is a closed 1-form on M and $[\omega]=\sum_{i=1}^{2g}\omega_i[W_i]$, then $\widehat{\omega}(\xi,\eta)=\overline{\omega}\circ\pi(\xi,\eta)$ is cohomologous to a function depending only on ξ namely the function*

$$\sum_{i=1}^{2g}\omega_i[\mathbf{1}_{A_i}-\mathbf{1}_{A_{-i}}](\xi)$$

which is constant and equal to ω_i on A_i and equal to $-\omega_i$ on A_{-i}.

Proof. The proof is done in two parts like in Proposition 14.

Part one. We first show that $\overline{\omega}(\xi,\eta)$ is cohomologous to $\langle H(\gamma(\xi,\eta)),[\omega]\rangle$. To simplify the notations, we introduce $\gamma_0=\gamma(\xi,\eta)$, $p_0=p(\xi,\eta)$, $q_0=q(\xi,\eta)$ and $q_1=q\circ B(\xi,\eta)=\gamma_0p_0$. Then

$$\begin{aligned}\overline{\omega}(\xi,\eta)&=\int_{\overrightarrow{q_0p_0}}\omega=\int_{\overrightarrow{q_0\mathcal{O}}}\omega+\int_{\overrightarrow{\mathcal{O}\gamma_0^{-1}\mathcal{O}}}\omega+\int_{\overrightarrow{\gamma_0^{-1}\mathcal{O}p_0}}\omega\\&=c(\xi,\eta)-c\circ B(\xi,\eta)+\langle H(\gamma(\xi,\eta)),[\omega]\rangle,\end{aligned}$$

where $c(\xi,\eta)=\int_{\overrightarrow{q(\xi,\eta)\mathcal{O}}}\omega$.

Part two. We use now the equivariance between $\gamma\circ\pi(\xi,\eta)$ and $\hat{\gamma}(\xi)$, namely $\gamma\circ\pi=\rho\circ\widehat{T}\hat{\gamma}\rho^{-1}$, to get

$$H(\gamma\circ\pi)=H(\rho\circ\widehat{T})-H(\rho)+H(\hat{\gamma}).$$

Since $\hat{\gamma}=a_i$ on the set A_i and $\hat{\gamma}=a_i^{-1}$ on the set A_{-i}, we obtain

$$\widehat{\omega}(\xi,\eta)=\hat{c}(\xi,\eta)-\hat{c}\circ\widehat{T}(\xi,\eta)+\sum_{i=1}^{2g}\langle[\delta_i],[\omega]\rangle[\mathbf{1}_{A_i}-\mathbf{1}_{A_{-i}}],$$

where as in Proposition 14 the total coboundary is:

$$\hat{c}(\xi,\eta)=\int_{\overrightarrow{\rho(\xi,\eta)^{-1}q\circ\pi(\xi,\eta)\mathcal{O}}}\omega.\qquad\square$$

Appendix A. Mather theory and stable norm

In order to understand the minimizing geodesic problem in the perspective of Aubry–Mather theory, we consider briefly the case of a general Lagrangian $L(x,v)$, $(x,v)\in TM$.

We denote by $h\in\mathcal{H}_1(M,\mathbb{R})=\mathbb{R}^{2n}$ the real 1-homology class on the compact Riemann surface M, that is, the set of linear forms acting on the vector space of closed differential 1-forms on M modulo the subspace of exact forms. We denote by $\mathcal{M}_1^{\mathrm{comp}}(TM,\phi^t)$ the set of probability measures μ which have compact support in TM and are invariant by the flow $\{\phi^t\}$ generated by the Euler–Lagrange equation for $L(x,v)$.

Definition 19. Given any invariant probability measure μ with compact support in TM, we define its homological position $[\mu]\in\mathcal{H}_1(M,\mathbb{R})$, as a linear operator acting on 1-differential closed forms $[\omega]\in\mathcal{H}^1(M,\mathbb{R})$ by means of

$$\langle[\mu],[\omega]\rangle=\int\omega(x,v)\,\mathrm{d}\mu(x,v).$$

We refer the reader to Contreras–Iturriaga [9,14] and Paternain [24] for all technical details and assumptions about the Lagrangians, the measures on the Aubry–Mather theory, etc. We will just recall the main definitions.

We say an homology class $h \in \mathcal{H}_1(M,\mathbb{R})$ is rational, if there exists $r \in \mathbb{R}^*$ such that rh is an homology class with integer coefficients. We are considering orientable two-dimensional surfaces and therefore there is no torsion. Let

$$\widetilde{\mathcal{M}}_h = \left\{\mu \in \mathcal{M}_1^{\mathrm{comp}}\left(TM, \phi^t\right) \,\middle|\, [\mu] = h\right\}.$$

Consider a convex super-linear Lagrangian $L(x,v)$ on the tangent bundle TM of M (see [9,10,14,15] for definitions), then the action of a measure $\mu \in \widetilde{\mathcal{M}}_h$ is by definition

$$A(\mu) = \int L(x,v)\,\mathrm{d}\mu(x,v).$$

We define Mather's β function $\beta : \mathcal{H}_1(M,\mathbb{R}) \to \mathbb{R}$ by:

$$\beta(h) = \min_{\mu \in \widetilde{\mathcal{M}}_h} A(\mu).$$

The function β is convex in the variable $h \in \mathcal{H}_1(M,\mathbb{R})$. Under mild conditions (convexity, superlinearity, etc.) there exist minimizing measures μ_h and we call such probability measures, Mather minimizing measures of homology h. A Mather minimizing measure may not be unique. For a generic Lagrangian such a measure μ_h is unique [20]. The minimizing measure is not always ergodic for a given L and h as above. Given h, the set of h-minimizing measures is the convex closure of minimizing ergodic measures (perhaps for a different h). From the Mather non-crossing property [9,14] and Haeflinger argument (see [22]), it follows that, if a minimizing probability $\mu \in \widetilde{\mathcal{M}}_h$ has homological position h rational, then the support of μ is contained on a finite union of closed geodesic trajectories. We would like to understand the homological position $[\mu]$ in a more geometrical way. This will be done next.

We consider oriented closed geodesic orbits δ_i in M, which connect each pair of corresponding sides s_i and s_{-i}. For example, in our genus 2 example, δ_a connects a and a^{-1}, δ_b connects b and b^{-1}, and so on... (see Fig. 10). The homologies $[\delta_i]$, $i \in \{1,2,\dots,2g\}$, form a basis of $\mathcal{H}_1(M,\mathbb{Z})$. We show in Fig. 10 all the oriented closed geodesics and the first form W_a of the dual basis. To construct this basis W_i, $i \in \{1,2,\dots,2g\}$, for the first cohomological group of M (see [13], Section II 3, page 39), we consider strips Ω_i (or tubular neighborhoods) in M around each δ_i. Suppose each strip is parametrized by $g_i(x,y)$, $x \in (-1,1)$ and $y \in \mathbb{R}$, 1-periodic in y, in such way that $y \in [0,1] \mapsto g_i(0,y)$ is a parametrization of the curve δ_i. We also consider a C^∞ function $f : (-1,1) \times \mathbb{R} \to \mathbb{R}$, independent of the second variable, monotonous with respect to the first variable, such that $f(x,y) = 0$ for $x < -\frac{1}{2}$ and $f(x,y) = 1$ for $x > \frac{1}{2}$. We finally define a family of local functions, $f_i = f \circ g_i^{-1}$ and differential forms $W_i = \mathrm{d}f_i$ on M. We have just obtained, up to a reordering, a dual basis of $[\delta_i]$, $i = 1,\dots,2g$.

Note that, for an oriented closed curve α which intersects transversally each δ_i, $i = 1,\dots,2g$, the integer value $\int_\alpha W_i = d_i$, $i = 1,\dots,2g$, counts the algebraic number of intersections (with $+$ or $-$ signs according to orientation) of α with each δ_i.

Consider a simple closed curve α in M parametrized by $\alpha(t)$ with period t_0 (then $(\alpha(t),\alpha'(t))$ is a closed curve in TM). Define the measure $\mu^\alpha(\mathrm{d}x,\mathrm{d}v)$ by normalizing $\mathrm{d}t$ on the closed curve α (or (α,α')) (see [9,14]). More precisely if $\alpha : [0,t_0] \to M$, then for any continuous function $f(x,v)$,

$$\int f(x,v)\,\mathrm{d}\mu^\alpha(x,v) = \frac{1}{t_0}\int_0^{t_0} f\big(\alpha(t),\alpha'(t)\big)\,\mathrm{d}t.$$

We say μ^α is the measure associated to the curve α. Its homology is given by:

$$\begin{aligned}
\left[\mu^\alpha\right] &= \left(\int W_1(x,v)\,\mathrm{d}\mu^\alpha(x,v),\dots,\int W_{2g}(x,v)\,\mathrm{d}\mu^\alpha(x,v)\right) \\
&= \left(\frac{1}{t_0}\int_\alpha W_1,\dots,\frac{1}{t_0}\int_\alpha W_{2g}\right) = \left(\frac{1}{t_0}d_1,\dots,\frac{1}{t_0}d_{2g}\right) = (h_1,h_2,\dots,h_{2g}).
\end{aligned}$$

The measure μ^α represents an homology class $h = d/t_0$ with rational coefficients. Note that the homological position of the curve α (the geometric object without any parametrization) is an element $d = (d_1,\dots,d_{2g})$ of $\mathcal{H}_1(M,\mathbb{Z}) \subset \mathcal{H}_1(M,\mathbb{R})$ with integer coefficients.

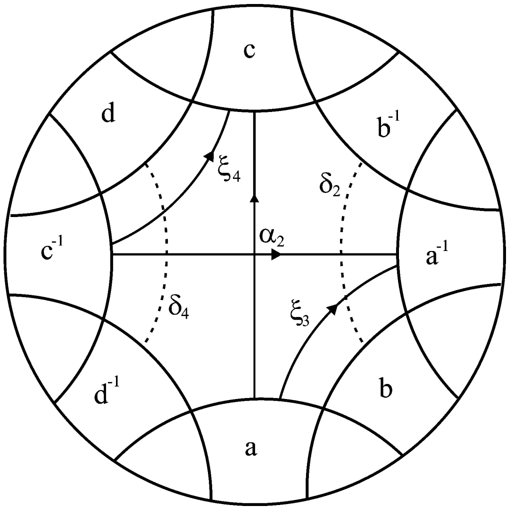

Fig. 11. Example 2.

We will consider from now on the special case where $L(x, v) = \frac{1}{2}\|v\|^2$ (that is, the square of the Riemannian length) for the constant negative curvature surface M.

Definition 20. Consider the Riemannian–Lagrangian $L(x, v) = \frac{1}{2}\|v\|^2$. Given the homological position h, a Mather measure for h is a measure minimizing

$$\beta(h) = \min_{\mu\in\widetilde{\mathcal{M}}_h} \int \frac{1}{2}\|v\|^2 \,\mathrm{d}\mu(x, v).$$

The purpose of our work is to analyze such probability measures.

Example 1. We consider a simple example which is useful to understand. We choose two closed geodesics γ_1 and γ_2 with the same homological position $h = d/T$ ($d \in \mathcal{H}_1(M, \mathbb{Z})$) and different lengths $l(\gamma_1) < l(\gamma_2)$ and we consider respectively two parameterizations $\alpha_1(t)$ and $\alpha_2(t)$ on $t \in [0, T]$. Then $\|\alpha_1'(t)\| = l(\alpha_1)/T$ and $\|\alpha_2'(t)\| = l(\alpha_2)/T$ and we obtain two invariant measures $\mu_1 = \mu^{\alpha_1}$ and $\mu_2 = \mu^{\alpha_2}$ (according to the above notation), associated respectively to α_1 and α_2 such that

$$\frac{1}{2T^2} l(\gamma_1)^2 = \int L(x, v)\,\mathrm{d}\mu_1(x, v) < \int L(x, v)\,\mathrm{d}\mu_2(x, v) = \frac{1}{2T^2} l(\gamma_2)^2.$$

This computation shows which of the two measures has smaller action.

Example 2. Another interesting example is the following. In Fig. 11 we construct two sets of closed geodesics: on the one hand α_2 is a unique closed geodesic passing through $\mathcal{O}$ and having a self intersection at $\mathcal{O}$ and on the other hand α_1 is equal to the union of two closed geodesics ζ_3 and ζ_4 representing the axis of a and b. Depending on the parametrization of α_2 (with period T_2), ζ_3 (with period T_3) and ζ_4 (with period T_4), via dt we obtain respectively different measures μ^{α_2} and

$$\mu^{\alpha_1} = T_3/(T_3 + T_4)\mu^{\zeta_3} + T_4/(T_3 + T_4)\mu^{\zeta_4}.$$

We can choose parameterizations such that they determine measures (as above) with the same homological position

$$h = \frac{d}{T} = \left(0, \frac{1}{T_2}, 0, \frac{1}{T_2}\right) = \left(0, \frac{1}{T_3 + T_4}, 0, \frac{1}{T_3 + T_4}\right).$$

Where $T_3 + T_4 = T = T_2$. The curve α_2 cannot be the support of a minimal measure because it does not satisfy the non-crossing property. If we choose a parametrization $\alpha_1(t)$ of $\zeta_3 \cup \zeta_4$ (continuous by part) on the interval $[0, T] =$

$[0, T_3 + T_4]$ with fixed velocity $\|v\| = (l(\zeta_3) + l(\zeta_4))/T$ we get the minimizing measure (of the form μ^{α_1}) for h. An h-minimizing measure is not always ergodic. The existence of two invariant curves ζ_3 and ζ_4 contained in the set α_1 shows that the h-minimal measure μ^{α_1} is not ergodic.

After these two examples we return to the general case we are interested in. Consider now a general measure $\mu \in \widetilde{\mathcal{M}}_h$. Given $h = [\mu]$, we can express the h_i, $i = 1, \ldots, 2g$, such that $h = h_1[\delta_1] + h_2[\delta_2] + \cdots + h_{2g}[\delta_{2g}] \in \mathcal{H}_1(M, \mathbb{R})$ in a similar way ($h = (h_1, \ldots, h_{2g}) \in \mathbb{R}^{2g} = \mathcal{H}_1(M, \mathbb{R})$). One can show that $h_i = \int W_i(x, v)\,\mathrm{d}\mu(x, v)$, for all $i = 1, \ldots, 2g$.

For an ergodic $\mu \in \widetilde{\mathcal{M}}_h$, one can choose (x, v) with probability 1 according to μ and a geodesic $x(t) \in M$ determined by $x(0) = x$ and $x'(0) = v$. For each $i = 1, \ldots, 2g$ consider the oriented intersection denoted by $b^i_N = \int_{-N}^{N} W_i(x(t), x'(t))\,\mathrm{d}t$ of $x(t)$, $t \in (-N, N)$, with the generator δ_i. One can show from the above and Birkhoff Theorem that

$$\lim_{N\to\infty} \frac{b^i_N}{2N} = \int W_i\,\mathrm{d}\mu = h_i, \quad i = 1, \ldots, 2g.$$

In this case we say that $x(t)$ has asymptotic homological position h.

Consider a general h-minimizing measure not necessarily ergodic. From [8] (Theorem 1) one gets that for a minimizing measure all of the vectors in its support have the same energy (in the example mentioned in last remark above for the set α_1 – the union of two connected curves ζ_3 and ζ_4 – the energy $\frac{1}{2}\|v\|^2$ have to be the same in each of the curves ζ_3 and ζ_4.

There is a difference between the curves that support the measure and the measure itself. A non-ergodic minimizing measure is a convex linear combination of ergodic minimizing measures (all supported on geodesics parametrized with the same speed). Its homological position is the convex linear combination of the homological positions of its ergodic components. If they are different, since the ergodic components must be minimizing (in their homology class), the non-ergodic measure must be in a flat domain of the beta function. In this sense such a non-ergodic minimizing measure is a kind of "fake" minimizing object: there is no vector in its support with the prescribed asymptotic homology.

The integral of the asymptotic homology positions of different points (x, v) gives the homological position h of such non-ergodic measure. This is the geometrical way to compute the homological position of a general measure $\mu \in \widetilde{\mathcal{M}}_h$. Therefore, one can obtain the homological position $[\mu] = (h_1, \ldots, h_{2g})$, in terms of $\mu(\mathrm{d}x, \mathrm{d}v)$-integrals of forms $W_i(x, v)$ as above.

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