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The Subadditive Ergodic Theorem and Recurrence Properties of Markovian Transformations

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1. INTRODUCTION

We prove a new maximal lemma in time for subadditive sequences with respect to a nonsingular transformation. This is used to give a new proof of the corresponding pointwise subadditive ergodic theorem. We also study some recurrence questions and the natural extension for these transformations.

Let $(X, \mathscr{B}, \mu)$ denote a σ-finite measure space. A *transformation* is a ($\mathscr{B}$-)measurable map of X into itself. Let $\omega: X \to \mathbf{R}$ be a nonnegative ($\mathscr{B}$-)measurable function; the transformation T is said to be *Markovian with respect to* (ω, μ) if it satisfies, for all nonnegative measurable functions f,

$$\text{(P1)} \qquad \int f \circ T \omega \, d\mu = \int f \, d\mu.$$

It readily follows that for all measurable functions f,

(P2) $f \geqslant 0$ a.e. if and ony if $f \circ T\omega \geqslant 0$ a.e.

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In this paper we prove a maximal inequality in time which has so far no counterpart in the theory of Markov operators. Lemma 2.2a is the key result in the proof of the subadditive ergodic theorem given in Section 3 (Theorem 3.4). This theorem can be deduced from the corresponding Akcoglu–Sucheston theorem [1, 2]; however, our proof is simple and self-contained. Sub-Markovian transformations are treated in Section 4 and some recurrence properties are proved in Section 5. Using the same idea as in Lemma 2.2, we prove a Kac's formula (Theorem 5.2) for Markovian transformations, and give several applications (cf. Corollary 5.4, Theorem 5.5, Theorem 5.6).

1.1. Definition. (a) If T is Markovian with respect to (ω, μ) we say that (ω, μ) is a *Markovian pair* for T, and when μ is understood from the contex, that ω is a *Markovian function* for T. Two pairs (ω, μ) and (ω', μ') are *cohomologous* if there exists a positive measurable function h (called the *transfer* function from (ω, μ) to (ω', μ')) such that

$$\mu' = h\mu \qquad \left(\text{i.e., } \frac{d\mu'}{d\mu} = h\right) \qquad \text{and} \qquad \omega' = \frac{h \circ T}{h}\,\omega.$$

(b) A *Hurewicz cocycle* for (ω, μ) is any sequence of nonnegative measurable functions $\{\omega_n\}_{n \geqslant 0}$ such that

$$\omega_0 = 1,\ \omega_1 = \omega, \text{ and } \omega_{n+m} = \omega_m \omega_n \circ T^m \qquad \mu\text{—a.e. for } n \geqslant 0, m \geqslant 0.$$

1.2. Lemma. (a) *If T is Markovian with respect to (ω, μ) then T is Markovian with respect to any cohomologous (ω', μ').*

(b) *If moreover $\{\omega_n\}_{n \geqslant 0}$ is a Hurewicz cocycle for (ω, μ), then for all $n \geqslant 0$, $\omega_n = \omega\, \omega \circ T \cdots \omega \circ T^{n-1}$ μ—a.e. and T^n is Markovian with respect to (ω_n, μ).*

1.3. Examples. (a) Suppose $\mu \circ T$ is σ-finite on $\mathscr{B}$. If T is a positive nonsingular (i.e., $\mu(T^{-1}A) = 0$ implies $\mu(A) = 0$) then T is Markovian with respect to (ω^μ, μ), where $\omega^\mu = (d\mu/d\mu T^{-1}) \circ T$ is the *Radon–Nikodym* derivative of T with respect to μ. Conversely, if T is Markovian with respect to (ω, μ), then T is positive nonsingular and $\omega^\mu = \mathbb{E}_\mu[\omega \mid T^{-1}\mathscr{B}]$ is the only $T^{-1}\mathscr{B}$-measurable Markovian function, where $\mathbb{E}_\mu$ denotes the usual conditional expectation function with respect to μ.

(b) If T is a C^1-diffeomorphism on a smooth manifold M, J is its Jacobian and λ a Lebesgue measure on M, then T is Markovian with respect to (J, λ).

(c) If $(X, \mathscr{B}, \mu)$ is a Lebesgue probability space and T is a n-to-1 nonsingular endomorphism, then it follows from a theorem of Rohlin that there exists a partition of X (mod 0) into n disjoint measurable sets $A_1, \dots, A_n$ such that the restriction T_i of T to each A_i is 1-to-1 from A_i onto a.e. X, and for any measurable set B in A_i, $\mu(B)=0$ implies $\mu(T_i B)=0$. Define $J_i = d\mu T_i/d\mu$ on A_i. Let $(p_1, \dots, p_n)$ be any probability vector and write $\omega = \sum p_i J_i \chi_{A_i}$; then T is Markovian with respect to (ω, μ). (In general, these Markovian functions are different from the Radon–Nikodym derivative of part (a)).

(d) If T is Markovian with respect to (ω, μ), we will see in Section 5 that the induced map T_A (for some set A) is also Markovian with respect to (ω_A, μ_A).

1.4. Definition. A *subadditive sequence* (with respect to (ω, μ)) is a sequence $\{f_n\}_{n \geq 0}$ of measurable functions satisfying:

(i) $f_0 = 0$ a.e.,

(ii) $f_{n+m} \leq f_m + f_n \circ T^m \omega_m$ a.e. for all $m, n \geq 0$,

where $\{\omega_n\}$ is any Hurewicz cocycle for (ω, μ). The sequence $\{f_n\}$ is called *superadditive* if $\{-f_n\}$ is subadditive.

We note that given a subadditive sequence $\{f_n\}_{n \geq 0}$ there exists a sequence of measurable functions $\{\hat{f}_n\}_{n \geq 0}$ such that

$$f_n(x) = \hat{f}_n(x) \qquad \text{a.e.},$$

and

$$\hat{f}_{n+m}(x) \leq \hat{f}_m(x) + \hat{f}_n(T^m x)\, \hat{\omega}_m(x) \qquad \text{for all} \quad x \in X,$$

where $\hat{\omega}_n = \omega_1 \cdot \omega_1 \circ T \cdots \omega_1 \circ T^{n-1}$.

In fact, let

$$\hat{f}_n = \inf_{0 = k_0 \leq k_1 \leq \cdots \leq k_p = n} \left\{ \sum_{i=0}^{p-1} f_{k_{i+1} - k_i} \circ T^{k_i} \omega_{k_i} \right\}.$$

1.5. *Remark.* If $\{f_n\}_{n \geq 0}$ is subadditive with respect to (ω, μ), and (ω', μ') is cohomologous to (ω, μ) with transfer function h, then $\{f_n/h\}_{n \geq 0}$ is subadditive with respect to (ω', μ'). This remark allows us, by standard methods, to reduce the proofs of many of the results below to the case of finite measure. (We assume finite measure to define the conditional expectation with respect to any sub-σ-algebra, in particular, the sub-σ-algebra of invariant sets.)

1.6. Definition. The σ-algebra of (T-)invariant sets with respect to (ω, μ) is defined to be $\mathscr{I} = \{A \in \mathscr{B} : \chi_A \circ T\omega = \chi_A \omega \ \mu\text{—a.e.}\}$. It is easy to see that f is $\mathscr{I}$-measurable if and only if $f \circ T^n \omega_n = f\omega_n$ a.e. for all $n \geqslant 0$. Furthermore, if f is nonnegative, and μ has finite mass,

$$\mathbb{E}_\mu[f \mid \mathscr{I}] = \mathbb{E}_\mu[f \circ T^n \omega_n \mid \mathscr{I}] \qquad \text{a.e. for all} \quad n \geqslant 0.$$

1.7. LEMMA. *Assume μ is a finite measure.*

(a) *For any nonnegative function g, $\mathbb{E}_\mu[g \mid \mathscr{I}]$ is a positive function a.e. on the set $\{x \in X : \sum_{i=0}^{\infty} g(T^i(x))\, \omega_i(x) = \infty\}$.*

(b) *For any superadditive sequence of nonnegative functions $\{g_n\}$,*

$$\mathbb{E}_\mu\left[\sup_{n \geqslant 1} \frac{1}{n} \mathbb{E}[g_n \mid \mathscr{I}]\right] = \sup_{n \geqslant 1} \frac{1}{n} \mathbb{E}_\mu[g_n].$$

Proof. (a) Write $A = \{x \in X : \mathbb{E}_\mu[g \mid \mathscr{I}] = 0\}$, then A is invariant, and $\int_A \sum_{i=0}^{n-1} g(T^i(x))\, \omega_i(x)\, d\mu = n \int_A \mathbb{E}_\mu[g \mid \mathscr{I}]\, d\mu = 0$.

(b) For any integers $1 \leqslant q \leqslant p \leqslant N$, and n in the integer part of N/q, we have $\sum_{k=0}^{n-1} g_q \circ T^{kq} \omega_{kq} + g_{N-nq} \circ T^{nq} \omega_{nq} \leqslant g_N$ a.e. So $\mathbb{E}_\mu[\sup_{1 \leqslant q \leqslant p} (1/q)\, \mathbb{E}_\mu[g_q \mid \mathscr{I}] \leqslant (N/N-p) \sup_{n \geqslant 1} (1/n)\, \mathbb{E}_\mu[g_n]$. ∎

2. MAXIMAL LEMMAS AND THE RECURRENT PART

2.1. LEMMA. *Let $\{f_n\}_{n \geqslant 0}$ be a subadditive sequence and $p \geqslant 1$ an integer. If there exists a measurable function $\tau : X \to \{1, 2, ..., p\}$ such that $f_\tau \leqslant 0$ a.e. then, for all $n \geqslant p$,*

$$f_n(x) \leqslant \sum_{i=n-p}^{n-1} |f_1(T^i(x))|\, \omega_i(x) \qquad a.e.$$

Proof. We define an increasing sequence of times by induction: $\tau_0 = 0$, $\tau_{m+1}(x) = \tau(T^{\tau_m(x)}(x)) + \tau_m(x)$. By subadditivity, for every $n \geqslant 1$, for a.e. $x \in X$, and for $m \geqslant 1$ such that $\tau_m(x) < n \leqslant \tau_{m+1}(x)$,

$$f_n(x) \leqslant \sum_{i=0}^{m-1} f_{\tau_{i+1}-\tau_i}(T^{\tau_i}(x))\, \omega_{\tau_i}(x) + f_{n-\tau_m}(T^{\tau_m}(x))\, \omega_{\tau_m}(x).$$

Since $f_\tau \leqslant 0$ a.e. we have,

$$f_{\tau(T^m x)}(T^m(x))\, \omega_m(x) \leqslant 0 \text{ a.e.}$$

for all $m \geqslant$. So

$$f_n(x) \leqslant f_{n-\tau_m}(T^{\tau_m}(x))\, \omega_{\tau_m}(x) \leqslant \sum_{i=\tau_m}^{n-1} f_1(T^i(x))\, \omega_i(x) \qquad \text{a.e.} \quad ∎$$

2.2. MAXIMAL LEMMAS FOR TIME. *Let* $\{f_n\}_{n\geqslant 0}$ *be a subadditive sequence,* $p\geqslant 1$ *an integer. Define*

$$A=\{x\in X: f_k(x)\geqslant 0 \quad \textit{for all} \quad 1\leqslant k\leqslant p\},$$

and

$$B=\{x\in X: f_k(x)>0 \quad \textit{for some} \quad 1\leqslant k\leqslant p\}.$$

Then for all integers $n\geqslant p$ *and for a.e.* $x\in X$,

$$\text{(a)} \qquad f_n\leqslant \sum_{i=0}^{n-1} f_1\circ T^i\chi_A\circ T^i\omega_i+\sum_{i=n-p}^{n-1} |f_1\circ T^i|\,\omega_i,$$

$$\text{(b)} \qquad f_n^+\leqslant \sum_{i=0}^{n-1} f_1\circ T^i\chi_B\circ T^i\omega_i+\sum_{i=n-p}^{n-1} |f_1\circ T^i|\,\omega_i.$$

Proof. (a) Define a function τ as follows:

$$\tau(x)=1 \text{ if } x\in A, \text{ and } \tau(x)=\min\{k\in\{1,2,...,p\}: f_k(x)<0\}$$

if $x\notin A$. We claim that

$$f_{\tau(x)}(x)\leqslant \sum_{i=0}^{\tau(x)-1} f_1(T^i(x))\,\chi_A(T^i(x))\,\omega_i(x) \qquad \text{a.e.}$$

To show the claim first assume $x\in A$, then $\tau(x)=1$ and $f_{\tau(x)}(x)=f_1(x)\,\chi_A(x)\,\omega_0(x)$; if $x\notin A$, then $f_{\tau(x)}(x)<0$ and it suffices to show that each term in the above sum is nonnegative. This is clear if $T^i(x)\notin A$; if $T^i(x)\in A$ we have in particular that $f_1(T^i(x))\geqslant 0$.

An application of Lemma 2.1 now completes the proof.

(b) Define a function τ by

$$\tau(x)=1 \text{ if } x\notin B, \text{ and } \tau(x)=\min\{k\in\{1,2,...,p\}: f_k(x)>0\}$$

if $x\in B$. Once more it is enough to show

$$f_{\tau(x)}^+(x)\leqslant \sum_{i=0}^{\tau(x)-1} f_1(T^i(x))\,\chi_B(T^i(x))\,\omega_i(x) \qquad \text{a.e.}$$

Since $\{f_n^+\}$ is also a subadditive sequence. Now if $x\in B$ then $0<f_\tau\leqslant \sum_{i=0}^{\tau-1} f_1\circ T^i\omega_i$, and the proof is complete if we show $T^i(x)\in B$ for each $0\leqslant i\leqslant \tau-1$. In fact, $f_\tau\leqslant f_i+f_{\tau-i}\circ T^i\omega_i$ and $f_i(x)\leqslant 0$, so $f_{\tau-i}(T^ix)>0$. ∎

2.3. *Remark.* (a) A version of Lemma 2.2 for finite measure preserving transformations is contained in the proof of the Birkhoff ergodic theorem given in [7, 6].

(b) It is clear from the proof that if the subadditive sequence and the Hurewicz cocycle satisfy the properties of their definition for all $x \in X$ (not just a.e.) then Lemma 2.2 also holds for all $x \in X$.

(c) The only property of $\{\omega_n\}$ used in the proof is that it is a cocycle.

2.4. INTEGRATED MAXIMAL LEMMAS. *Let $\{f_n\}_{n \geqslant 0}$ be a subadditive sequence, $p \geqslant 1$ an integer, and A, B as defined in Lemma 2.2. Then*

(a) $\inf_{n \geqslant 1} (1/n) \int f_n \, d\mu = \lim_{n \to \infty} (1/n) \int f_n \, d\mu \leqslant \int_A f_1 \, d\mu$,

(b) $0 \leqslant \lim_{n \to \infty} (1/n) \int f_n^+ \, d\mu \leqslant \int_B f_1 \, d\mu$.

Furthermore, if $h \in L_+^\infty(X, \mu)$ then $\{hf_n\}_n$ is a subadditive sequence of integrable functions, and if μ is a finite measure,

(c) $\lim_{n \to \infty} (1/n) \mathbb{E}_\mu[f_n \mid \mathscr{I}] \leqslant \mathbb{E}_\mu[f_1 \chi_A \mid \mathscr{I}]$

(d) $\lim_{n \to \infty} (1/n) \mathbb{E}_\mu[f_n^+ \mid \mathscr{I}] \leqslant \mathbb{E}_\mu[f_1 \chi_B \mid \mathscr{I}]$.

Proof. Apply Lemma 2.2. ∎

We note that if $\{f_n\}$ is a subadditive sequence of integrable functions then $\{\mathbb{E}[f_n \mid \mathscr{I}]\}$ is a subadditive sequence of real numbers and so $\lim_{n \to \infty} (1/n) \mathbb{E}[f_n \mid \mathscr{I}] = \inf_{n \geqslant 1} (1/n) \mathbb{E}[f_n \mid \mathscr{I}]$ a.e.

2.5. *Notation.* Write

$$\mathscr{R}(f, \omega) = \left\{ x \in X \colon \sum_{i=0}^{\infty} f(T^i(x)) \, \omega_i(x) = \infty \right\}$$

and

$$\mathscr{T}(f, \omega) = \left\{ x \in X \colon \sum_{i=0}^{\infty} f(T^i(x)) \, \omega_i(x) < \infty \right\}.$$

It follows that for any positive function h, $\mathscr{R}(f, \omega) = \mathscr{R}(f/h, (h \circ T/h)\omega)$.

2.6. PROPOSITION. *Let f, g be two nonnegative integrable functions. Then almost everywhere on the set $\mathscr{R}(f, \omega)$, $\sum_{i=0}^{\infty} g(T^i(x)) \, \omega_i(x)$ takes only the values 0 and ∞.*

Proof. Write $C = \mathscr{R}(f, \omega)$ and $D = \mathscr{T}(g, \omega)$. Then C and D are invariant sets. For all integers $p \geqslant 1$ and real numbers $r > 0$ define $B_p^r = \{x \in X \colon f_k(x) - r g_k(x) > 0 \text{ for some } 1 \leqslant k \leqslant p\}$.

Then by Lemma 2.4b, $\int_{C \cap D \cap B_p^r} (f - rg) \, d\mu \geqslant 0$. Since $C \cap D \subseteq \bigcup_{p \geqslant 1} \uparrow B_p^r$, $\int_{C \cap D} g \, d\mu \leqslant (1/r) \int f \, d\mu$ for all $r \geqslant 0$. Thus, $\int_{C \cap D} g \circ T^i \omega_i \, d\mu = 0$ for $i \geqslant 1$ and $\int_{C \cap D} \sum_{i=0}^{\infty} g(T^i(x)) \, \omega_i(x) \, d\mu = 0$. ∎

Proposition 2.6. remains true when g is only a nonnegative function (by a standard approximation argument), and allows us to define the recurrent part of T with respect to (ω, μ), $\mathscr{R}(\omega)$, by $\mathscr{R}(\omega) = \mathscr{R}(f, \omega)$ for any positive integrable function f. ($\mathscr{R}(\omega)$ is defined μ—a.e.)

2.7. COROLLARY. *Let g be a nonnegative integrable function, and h be a measurable function. Then $h \circ T \geqslant h$ a.e. on $\mathscr{R}(g, \omega)$ if and only if $h \circ T = h$ a.e. on $\mathscr{R}(g, \omega)$.*

Proof. Write $F = \{x \in X : h(x) \leqslant r\}$, where r is any real number. Suppose that $h \circ T \geqslant h$ a.e. on $\mathscr{R}(g, \omega)$, then $\chi_F \circ T \leqslant \chi_F$ a.e. on $\mathscr{R}(g, \omega)$. Define $G = \{x \in X : \chi_F - \chi_F \circ T \geqslant c\}$, for some $c > 0$. Proposition 2.6 implies

$$\sum_{i=0}^{\infty} \chi_G(T^i(x))\, \omega_i(x) \in \{0, \infty\} \qquad \text{a.e. on } \mathscr{R}(g, \omega),$$

so

$$\sum_{i=0}^{\infty} \chi_G(T^i(x)) \in \{0, \infty\} \qquad \text{a.e. on } \mathscr{R}(g, \omega).$$

Since $\chi_G \leqslant (1/c)\ (\chi_F - \chi_F \circ T)$ and $\sum_{i=0}^{\infty} (\chi_F - \chi_F \circ T) \circ T^i < 2$ everywhere, $\mu(G \cap \mathscr{R}(g, \omega)) = 0$. ∎

3. THE SUBADDITIVE ERGODIC THEOREM

The following lemma is the main step in the proof of the subadditive theorem. The Hurewicz ergodic theorem is a particular case of this lemma. This theorem is used to prove a corollary that yields the invariance of some functions.

3.1. LEMMA. *Assume that μ is a finite measure. Let $\{f_n\}$ be a subadditive sequence of integrable functions, and g a nonnegative integrable function. Define*

$$F = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_\mu[f_n \mid \mathscr{I}], \qquad G = \mathbb{E}_\mu[g \mid \mathscr{I}].$$

Then a.e. on $\mathscr{R}(g, \omega)$,

$$\frac{F}{G} \leqslant \liminf_{n \to \infty} \frac{f_n}{\sum_{i=0}^{n-1} g \circ T^i \omega_i}.$$

Proof. Write $g_n = \sum_{i=0}^{n-1} g \circ T^i \omega_i$, and $h = \liminf_{n\to\infty} f_n/g_n$. Using the formula

$$\frac{f_{n+1}}{g_{n+1}} \leqslant \frac{f_1 + f_n \circ T\omega}{g_1 + g_n \circ T\omega}$$

it is easy to see that $h \circ T \geqslant h$ on $\mathscr{R}(g, \omega)$ a.e. and hence by Corollary 2.7, $h \circ T = h$ a.e. on $\mathscr{R}(g, \omega)$. So $C_r = \{x \in \mathscr{R}(g, \omega) : h < r\}$ is invariant for any r. Define

$$F_n^r = f_n - r g_n$$

and

$$A_p = \{x \in X : F_n^r(x) \geqslant 0 \qquad \text{for all} \quad 1 \leqslant n \leqslant p\}.$$

Since $\{F_n^r\}_{n \geqslant 0}$ is a subadditive sequence, by Lemma 2.4(a),

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}[F_n^r \mid \mathscr{I}] \leqslant \mathbb{E}[F_1^r \chi_{A_p} \mid \mathscr{I}].$$

Since $\bigcap_{p \geqslant 0} A_p \cap C_r = \varnothing$, $\lim_{n\to\infty} (1/n)\, \mathbb{E}[F_n^r \mid \mathscr{I}] \leqslant 0$ a.e. on C_r and so $F - rG \leqslant 0$ a.e. on C_r. ∎

3.2. Hurewicz Ergodic Theorem [3]. *Let f and g be integrable functions, g assumed to be nonnegative. Then a.e. on the set $\mathscr{R}(g, \omega)$,*

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} f \circ T^i \omega_i}{\sum_{i=0}^{n-1} g \circ T^i \omega_i} = \frac{\mathbb{E}_{h\mu}[f/h \mid \mathscr{I}]}{\mathbb{E}_{h\mu}[g/h \mid \mathscr{I}]},$$

for any positive μ-integrable function h.

Proof. Let $\mu' = h\mu$ and $\omega' = ((h \circ T)/h)\omega$. Then

$$\frac{\sum_{i=0}^{n-1} f \circ T^i \omega_i}{\sum_{i=0}^{n-1} g \circ T^i \omega_i} = \frac{\sum_{i=0}^{n-1} (f/h) \circ T^i \omega_i'}{\sum_{i=0}^{n-1} (g/h) \circ T^i \omega_i'} \qquad \text{a.e.}$$

Apply Lemma 3.1 (with respect to (ω', μ')) to $f_n' = \sum_{i=0}^{n-1} (f/h) \circ T^i \omega_i'$ and $g_n' = \sum_{i=0}^{n-1} (g/h) \circ T^i \omega_i'$. ∎

3.3. Corollary. *Let f and g be integrable functions, g assumed to be nonnegative. Then a.e. on the set $\mathscr{R}(g, \omega)$,*

$$\lim_{n\to\infty} \frac{f \circ T^n \omega_n}{\sum_{i=0}^{n-1} g(T^i(x))\, \omega_i(x)} = 0.$$

Proof. We may assume that μ is a finite measure. Let $h = f \circ T\omega - f$. Then $h \circ T^i\omega_i = f \circ T^{i+1}\omega_{i+1} - f \circ T^i\omega_i$, so $(f \circ T^n\omega_n - f)/g_n = h_n/g_n$, where $g_n = \sum_{i=0}^{n-1} h \circ T^i\omega_i$. Using Theorem 3.2 we obtain $\lim_{n\to\infty} h_n/g_n = \lim_{n\to\infty} (f \circ T^n\omega_n)/g_n = \mathbb{E}_\mu[h \mid \mathscr{I}]/\mathbb{E}_\mu[g \mid \mathscr{I}] = 0$ a.e. ∎

3.4. THEOREM. *Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space and T a Markovian transformation with respect to (ω, μ). Let $\{f_n\}$ be a subadditive sequence of integrable functions, and $\{g_n\}$ a superadditive sequence of non-negative integrable functions. Then a.e. on $\mathscr{R}(g_1, \omega)$,*

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \frac{\lim_{n\to\infty} (1/n)\, \mathbb{E}_{h\mu}[f_n/h \mid \mathscr{I}]}{\lim_{n\to\infty} (1/n)\, \mathbb{E}_{h\mu}[g_n/h \mid \mathscr{I}]},$$

for any positive μ-integrable function h.

Proof. Using Remark 1.5 and arguing as in Theorem 3.2 we may assume that μ is finite. In addition, it suffices to show the theorem when $\{g_n\}$ is an additive sequence.

Define

$$F = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_\mu[f_n \mid \mathscr{I}] \qquad \text{and} \qquad G = \mathbb{E}_\mu[g_1 \mid \mathscr{I}].$$

Because of Lemma 3.1 it suffices to show $\limsup_{n\to\infty} (f_n/g_n) \leqslant F/G$. Let $C_r = \{x \in \mathscr{R}(g_1, \omega) : \limsup_{n\to\infty} f_n/g_n > r\}$. As in the proof of Lemma 3.1, C_r is an invariant set. We claim that for any integer $N \geqslant 1$,

$$\limsup_{n\to\infty} \frac{f_n}{g_n} = \limsup_{n\to\infty} \frac{f_{nN}}{g_{nN}} \qquad \text{a.e. on } \mathscr{R}(g_1, \omega).$$

In fact, if n is given and k the integer part of n/N then $f_n - rg_n \geqslant 0$ implies $f_{kN} - rg_{kN} \geqslant H \circ T^{kN}\omega_{kN}$, where $H = \min_{0 \leqslant k \leqslant N-1} (rg_k - f_k)$. Using Corollary 3.3 we have $\lim_{n\to\infty} (|H| \circ T^{kN}\omega_{kN})/g_{kN} = 0$, which completes the proof of the claim.

Now T^N is Markovian with respect to (ω_N, μ), and $F_n^N = \sum_{k=0}^{n-1} (f_N - rg_N) \circ T^{kN}\omega_{kN}$ is an additive sequence. Write $B_p^N = \{x \in X : F_k^N(x) > 0$ for some $1 \leqslant k \leqslant p\}$. Then by Lemma 2.4b $\mathbb{E}_\mu[\chi_{B_p^N} F_1^N \mid \mathscr{I}^N] \geqslant 0$ a.e., where $\mathscr{I}^N$ is the σ-algebra of T^N-invariant sets with respect to (ω_N, μ). By Definition 1.6 we have $\mathscr{I}^1 \subseteq \mathscr{I}^N$. Then $\mathbb{E}_\mu[\chi_{B_p^N}(f_N - rg_N) \mid \mathscr{I}] \geqslant 0$ a.e. Since $f_{nN} - rg_{nN} \leqslant F_n^N$, using the claim we obtain $C_r \subseteq \bigcup_{p \geqslant 0} B_p^N$. Therefore $\mathbb{E}_\mu[f_N - rg_N) \mid \mathscr{I}] \geqslant 0$ a.e. on C_r for every N, and so $\lim_{N\to\infty} (1/N)\{\mathbb{E}_\mu[f_N \mid \mathscr{I}] - r\mathbb{E}_\mu[g_N \mid \mathscr{I}]\} \geqslant 0$ a.e. on C_r. ∎

4. Sub-Markovian Transformations

In this section we prove the subadditive theorem for sub-Markovian transformations; in fact, we define the recurrent part of T and on this set T is shown to be Markovian. The general theorem then is obtained as a consequence of Theorem 3.4.

4.1. Definition. The transformation T is said to be *sub-Markovian with respect to* (ω, μ) if for any measurable function f,

(P1′) $f \geqslant 0$ a.e. implies $\int f \circ T\omega \, d\mu \leqslant \int f \, d\mu$, and

(P2′) $f \geqslant 0$ a.e. implies $f \circ T\omega \geqslant 0$ a.e.

4.2. Integrated Maximal Lemma. *Let* $\{f_n\}_{n \geqslant 0}$ *be a subadditive sequence,* $p \geqslant 1$ *an integer, and* B *as defined in Lemma* 2.2. *Then*

$$\int_B f_1 \, d\mu \geqslant 0.$$

Proof. We note first that for any integrable function f, the sequence $\int f \circ T^n \omega_n \, d\mu$ converges. Define

$$F_n = \sum_{i=0}^{n-1} f_1 \circ T^i \chi_B \circ T^i \omega_i + \sum_{i=n-p}^{n-1} |f_1 \circ T^i| \, \omega_i, \qquad \text{for all} \quad n \geqslant p.$$

Then by Lemma 2.2(b) F_n is a nonnegative function. Since

$$\begin{aligned} \chi_B f_1 = F_n - F_n \circ T\omega + (\chi_B f_1) \circ T^n \omega_n \\ - |f_1 \circ T^{n-p}| \, \omega_{n-p} + |f_1 \circ T^n| \, \omega_n, \end{aligned}$$

then

$$\begin{aligned} \int_B f_1 \, d\mu &\geqslant \lim_{n \to \infty} \int f_1 \circ T^n \chi_B \circ T^n \omega_n \, d\mu \\ &\geqslant \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f_1 \circ T^i \chi_B \circ T^i \omega_i \\ &\geqslant \lim_{n \to \infty} \frac{1}{n} \int F_n \geqslant 0. \quad \blacksquare \end{aligned}$$

We remark that Proposition 2.6 is also true for sub-Markovian transformations (the main part is Lemma 2.4(b), which shows that also in this case the recurrent part of T, $\mathscr{R}(\omega) = \mathscr{R}(\omega, f)$, does not depend μ—a.e. on the positive integrable function f.

4.3. THEOREM. *If T is sub-Markovian with respect to (ω, μ) then T is Markovian with respect to $(\omega, \chi_{\mathscr{R}(\omega)}\, \mu)$.*

Proof. $[A \in \mathscr{B} \to \int_{T^{-1}A} \omega \, d\mu]$ defines a σ-finite measure on $\mathscr{B}$, absolutely continuous with respect to μ. Thus there exists a measurable function h such that $0 \leqslant h \leqslant 1$ and $\int f \circ T\omega \, d\mu = \int fh \, d\mu$, for any nonnegative f. Thus

$$0 \leqslant \int_{\mathscr{R}(\omega, f)} (1-h) \sum_{i=0}^{n-1} f \circ T^i \omega_i \, d\mu \leqslant \int_{\mathscr{R}(\omega, f)} f \, d\mu,$$

which implies $h = 1$ a.e. on $\mathscr{R}(\omega)$. ∎

This theorem shows that the subadditive theorem also holds for sub-Markovian transformations.

5. RECURRENCE AND NATURAL EXTENSION

If $\mathscr{R}(\omega) = X$ we say that T is (ω, μ)-*recurrent*. If T is a nonsingular endomorphism, then its Radon–Nikodym derivative ω^μ is a Markovian function for μ, and as in [9], (in the case that μ is σ-finite on $T^{-1}\mathscr{B}$) T is called μ-recurrent if T is (ω^μ, μ)-recurrent. If T is invertible, μ-recurrence of T is equivalent to conservativity of T (with respect to μ) (cf. 5.4). However, a conservative ergodic endomorphism may have Markovian pairs (ω, μ) with T not (ω, μ)-recurrent [4].

5.1. DEFINITION. For any set A write $A^* = \bigcup_{i \geqslant 1} T^{-i}A$. If $A \subseteq A^*$ then we can define the return time to A by $\tau_A(x) = \inf\{k \geqslant 1 \colon T^k(x) \in A\}$ if $x \in A^*$, and $\tau_A = 1$ if $x \notin A^*$.

Then put $T_A(x) = T^{\tau_A}(x)$ for $x \in X$. The restriction of T_A to A is called the *induced map* on A. The following theorem generalizes the well-known Kac's formula to the context of Markovian transformations.

5.2. THEOREM. *Let T be Markovian with respect to (ω, μ) and define $\omega_A = \omega_{\tau_A}$, $\mu_A = \chi_A \mu$, $f_A = \sum_{i=0}^{\tau_A - 1} f \circ T^i \omega_i$, for any nonnegative function f and any set A such that $A \subset A^*$. Then T_A is sub-Markovian with respect to (ω_A, μ_A) and*

(a) $\int_A f_A \, d\mu \leqslant \int_{A^*} f \, d\mu$,

(b) $\mathscr{R}(\omega_A) = A \cap \mathscr{R}(\omega)$,

(c) $\int_{A \cap \mathscr{R}(\omega)} f_A \, d\mu = \int_{A^* \cap \mathscr{R}(\omega)} f \, d\mu$.

Proof. Let $p > 0$ be an integer and $\tau \colon X \to \mathbb{N}$ be defined by $\tau(x) =$

$\min(\tau_A(x), p)$ if $x \in A$ and $\tau(x) = 1$ if $x \notin A$. If $g = \sum_{i=0}^{\tau-1} f \circ T^i \omega_i$, then for all $x \in X$,

$$\sum_{i=0}^{\tau-1} (\chi_A g) \circ T^i \omega_i \leqslant \sum_{i=0}^{\tau-1} (\chi_{A^*} f) \circ T^i \omega_i,$$

and so $\int_A g \, d\mu \leqslant \int_{A^*} f \, d\mu$ (Lemma 2.1), which proves (a), if we let p go to ∞.

If we apply (a) to $(\chi_A f) \circ T\omega$ instead of f we obtain that T_A is sub-Markovian with respect to (ω_A, μ_A). Moreover, if f is integrable then f_A is integrable and so $A \cap \mathscr{R}(\omega, f) = \mathscr{R}(\omega_A, f_A)$. To prove (c) we may assume T is (ω, μ)-recurrent. Then a.e. in $A \cup (X - A^*)$,

$$\sum_{i=0}^{\tau_A - 1} (\chi_A g) \circ T^i \omega_i = \sum_{i=0}^{\tau_A - 1} (\chi_{A^*} f) \circ T^i \omega_i.$$

Since $T_A(x) \in A \cup (X - A^*)$, for a.e. $x \in X$,

$$\sum_{i=\tau_A}^{\tau_A \circ T_A + \tau_A - 1} (\chi_A g) \circ T^i \omega_i = \sum_{i=\tau_A}^{\tau_A \circ T_A + \tau_A - 1} (\chi_{A^*} f) \circ T^i \omega_i.$$

If we define an increasing sequence of times by

$$\tau_p = \sum_{i=0}^{\tau_A - 1} \tau_A \circ T_A^i,$$

then for any $p \geqslant 1$,

$$\sum_{\tau_p}^{\tau_{p+1} - 1} (\chi_A g) \circ T^i \omega_i = \sum_{\tau_p}^{\tau_{p+1} - 1} (\chi_{A^*} f) \circ T^i \omega_i,$$

and

$$\sum_{\tau_1}^{\tau_{p+1} - 1} (\chi_A g) \circ T^i \omega_i = \sum_{\tau_1}^{\tau_{p+1} - 1} (\chi_{A^*} f) \circ T^i \omega_i \qquad \text{a.e.}$$

Part (c) follows from the last equality using the Hurewicz theorem. ∎

5.3. COROLLARY. *The induced map of any σ-finite measure preserving transformation on a set A which has finite measure and satisfies $A \subset A^*$ is also measure preserving.*

Proof. The induced map is sub-Markovian with respect to $(1, \mu_A)$ and its recurrent part is equal to A (since χ_A is integrable). Using Theorem 4.3 we obtain that T_A is Markovian with respect to $(1, \mu_A)$. ∎

5.4. COROLLARY (Halmos). *If T is a nonsingular automorphism on (X, μ) then it is conservative if and only if T is μ-recurrent.*

Proof. Suppose T is conservative. Write $\omega = \omega^{\mu}$. Let f be a positive integrable function. Since $\{x \in X : \sum_{i=0}^{\infty} f(T^i(x))\, \omega_i(x) < \infty\}$ is invariant, if it has positive measure we may assume it is all of X. Define $F = \sum_{i=0}^{\infty} f \circ T^i \omega_i$. Then $F = F \circ T\omega + f$ and T^{-1} is Markovian with respect to

$$\left(\frac{F \circ T^{-1}}{F} \cdot \frac{1}{\omega \circ T^{-1}}, F\mu\right) = (\omega', \mu').$$

Since T^{-1} is conservative and $\omega' > 1$, T^{-1} is μ-recurrent. Then, using Kac's formula, for any positive set A we have, $\int_A \omega'_A \, d\mu = \mu'(A)$, and so $\omega' = 1$ a.e. on A, which contradicts $\omega' > 1$. ∎

The following theorem was shown in [9, 5] for the case when ω is the Radon–Nikodym derivative (cf. [9] for references to the original results in the invertible case). We obtain one implication as a consequence of Theorem 5.2, the others are as in [9, 5].

5.5. THEOREM. *Let T be Markovian with respect to (ω, μ) and assume $\omega > 0$ a.e. The following are equivalent.*

(a) *T is (ω, μ)-recurrent.*

(b) *The skew product T_ω is conservative.*

(c) *1 belongs to the ratio set of T with respect to (ω, μ).*

Proof. (a) $\Rightarrow$ (b) We first show $\limsup_{n \to \infty} \omega_n \geq 1$ a.e. Let $\alpha > 0$ and $N \geq 1$ be an integer such that $A = (\sup_{n \geq N} \omega_n \leq \alpha)$ has positive measure. Since T is recurrent, $A \subset A^*$ and if $\omega_A(n, x) = \prod_{i=0}^{n-1} \omega_A \circ T_A^i(x)$ then

$$\alpha\mu(A) \geq \int_A \omega_A(N, x)\, d\mu(x) = \mu(A),$$

and thus $\alpha \geq 1$.

Now let f be a nonnegative integrable function, write $f^*(x, y) = f(x)\, e^{-y}$, and let $B \in \mathscr{B}$ be such that $\inf_B f > 0$ and has positive measure. Since the induced map T_B is recurrent, then $\limsup_{n \to \infty} \omega_B(n, x) \geq 1$ a.e. on B, and so

$$\sum_{i=0}^{\infty} f^* \circ T_\omega^i(x, y)$$

$$= \sum_{i=0}^{\infty} f \circ T_B^i(x)\, e^{-y/\omega_A(i, y)} = \infty \qquad \text{a.e. on} \quad B \times \mathbf{R}^+. \quad ∎$$

5.6. THEOREM. *Let T be a nonsingular endomorphism with Markovian pair (ω, μ). If (ω, μ) is recurrent, then T admits a σ-finite invariant measure equivalent to μ if and only if ω is a coboundary.*

Proof. First assume that the invariant measure ν is finite; so $(1, \nu)$ is a Markovian pair for T. We may assume $\nu(X) = 1$, and by changing to a cohomologous pair that $\mu = \nu$. For $\varepsilon > 0$ define $\omega^{\varepsilon} = \sup\{\omega, \varepsilon\}$. Birkhoff's ergodic theorem applied to the integrable function $\log(\omega^{\varepsilon})$ gives

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\omega^{\varepsilon} \circ T^i x) = \mathbb{E}_{\mu}[\log \omega^{\varepsilon} \mid \mathscr{I}] \qquad \mu\text{—a.e.}$$

Since ω is recurrent, 1 belongs to the ratio set of T with respect to (ω, μ) and so

$$\lim_{n \to \infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \omega^{\varepsilon} \circ T^i(x)$$

$$\geqslant \limsup_{n \to \infty} \frac{1}{n} \log \omega_n \geqslant 0 \qquad \mu\text{–a.e.}$$

Using the concavity of log and Jensen's inequality we obtain

$$0 \leqslant \mathbb{E}_{\mu}[\log \omega^{\varepsilon} \mid \mathscr{I}] \leqslant \log \mathbb{E}_{\mu}[\omega^{\varepsilon} \mid \mathscr{I}].$$

Taking limits as $\varepsilon \to 0$ gives

$$0 \leqslant \mathbb{E}_{\mu}[\log \omega \mid \mathscr{I}] \leqslant \log \mathbb{E}_{\mu}[\omega \mid \mathscr{I}] \leqslant 0.$$

The strict concavity of log forces $\omega = \mathbb{E}_{\mu}[\omega \mid \mathscr{I}] = 1$.

If T admits an infinite invariant measures ν, we may again assume $\mu = \nu$, and $(1, \nu)$ is Markovian for T. Let A be a set of positive finite measure. Since ω is recurrent, the induced map T_A is recurrent with respect to (ω_A, μ_A) and preserves the measure μ_A. By the first part, $\omega_A = 1$ μ—a.e., and since this is for any finite set A then $\omega = 1$ μ—a.e. ∎

5.7. DEFINITION. Let $\mathscr{X} = (X, \mathscr{B}, T, \omega, \mu)$ and $\mathscr{X}' = (X', \mathscr{B}', T', \omega', \mu')$ be two dynamical systems. We say that $\mathscr{X}'$ is a (*Markovian*) extension of $\mathscr{X}$ if there is a measurable map $P: X' \to X$ and a positive measurable function $\pi: X' \to \mathbf{R}^+$ such that:

(i) $P \circ T' = T \circ P$ a.e.,

(ii) $f \geqslant 0$ a.e. implies $\int f \circ P\pi \, d\mu' = \int f \, d\mu$,

(iii) $\pi \circ T'\omega' = \omega \circ P\pi$ a.e.

P is called a *conjugate projection* and Markovian with respect to (μ', π, μ).

5.8. DEFINITION. Let $(X, \mathscr{B})$ be a standard Borel space, and T a $\mathscr{B}$-measurable map. Define $X' = \{(x_i)_{i \geqslant 0}: Tx_{i+1} = x_i, i \geqslant 0\}$ and $T': X' \to X'$ by $T'(x_i) = (Tx_0, x_0, x_1, ...)$. Given $A \in \mathscr{B}$ define cylinder sets in X' by $A^{(k)} = \{(x_i) \in X' : x_k \in A\}$. Let $\mathscr{A}^{(k)}$ consist of all sets of the form $A^{(k)}$ for $A \in \mathscr{B}$ and $k \geqslant 0$, and $\mathscr{B}'$ be the Borel σ-algebra generated by $\bigvee_{k \geqslant 0} \mathscr{A}^{(k)}$. If X' is nonempty, $(X', \mathscr{B}')$ is a standard Borel space, T' is bijective, T' and T'^{-1} are $\mathscr{B}$-measurable. The first projection $P: X' \to X$ is called the natural projection. $(X', \mathscr{B}', T')$ is the inverse limit of $(X, \mathscr{B}, T)$.

The following theorem constructs an invertible extension for Markovian transformations. Maharam [8] has obtained before the authors a different construction of an invertible extension (cf. [9]). When T is onto everywhere and $\omega > 0$ a.e. is the Radon–Nikodym derivative a proof of the following theorem is given in [9].

5.9. THEOREM. *Let $(X, \mathscr{B})$ be a standard Borel space, T be a $\mathscr{B}$-measurable function, and $(X', \mathscr{B}', T')$ be the inverse limit of $(X, \mathscr{B}, T)$. If T is Markovian with respect to (ω, μ) then X' is nonempty and there exists a unique σ-finite measure μ' on $\mathscr{B}'$ such that*

(i) $\mu = \mu' \circ P^{-1}$,

(ii) *T' is Markovian with respect to (ω', μ'), where $\omega' = \omega \circ P$, where $P: X' \to X$ is the natural projection.*

Proof. Now we first prove that P is onto a.e. For any measurable set A of finite mass $\alpha, 0 < \alpha < 1$, we construct a sequence of compact sets $\{K_n\}_{n \geqslant 0}$ such that:

(i) the restriction of T^i to K_n is continuous for all $0 \leqslant i \leqslant n$,

(ii) $T(K_{n+1}) \subset K_n$, $K_0 \subset A$,

(iii) $\mu\{\bigcap_{n \geqslant 0} T^n(K_n)\} \geqslant \alpha\mu(A)$.

In fact, by induction, using Egoroff's and Lusin's theorems, construct compact sets $\{K_n\}_{n \geqslant 0}$ satisfying (i), (ii), and

(iv) $\mu(K_0) \geqslant \sqrt{\alpha}\, \mu(A)$ and

$$\int_{K_{n+1}} \omega_{n+1}\, d\mu \geqslant \alpha^{1/2^{n+2}} \int_{K_n} \omega_n\, d\mu.$$

If K_n has been constructed, choose L_n such that the restriction of T to L_n is continuous, L_n is compact, and

$$\int_{L_n \cap T^{-1}(K_n)} \omega_{n+1}\, d\mu \geqslant \alpha^{1/2^{n+2}} \int_{K_n} \omega_n\, d\mu.$$

Then $K_{n+1} = L_n \cap T^{-1}(K_n)$ satisfies (i), (ii), (iv), and

$$\begin{aligned}\mu(T^{n+1}(K_{n+1})) &= \int \chi_{T^{n+1}(K_{n+1})} \circ T^{n+1} \omega_{n+1}\, d\mu \\ &\geqslant \int_{K_{n+1}} \omega_{n+1}\, d\mu \\ &\geqslant \exp[(1/2 + 1/4 + \cdots + 1/2^{n+2}) \ln \alpha]\, \mu(A) \\ &\geqslant \alpha\mu(A).\end{aligned}$$

Now if $x \in \bigcap_{n \geqslant 0} T^n(K_n)$, then there exists $x_n \in K_n$ such that $T^n(x_n) = x$. Using a diagonal procedure for each sequence $\{T^{n-i}(x_n)\}_{n \geqslant i}$ belonging to K_i, one constructs a sequence of limit points $y_i \in K_i$ such that $y_0 = x$ and $T(y_{i+1}) = y_i$.

(We note that the construction in [9] of the extension μ' on $\mathscr{B}'$ of the measure μ satisfies (by the same calculation as in [9, Lemma 9])

$$\mu'(A^{(k)}) = \int_A \omega_k\, d\mu.$$

Now define measures μ'_k on $\mathscr{A}^{(k)}$ by

$$\mu'_k(A^{(k)}) = \int_A \omega_k\, d\mu \qquad \text{for} \quad A^{(k)} \in \mathscr{A}^{(k)}.$$

It is readily checked that $\{\mu'_k\}$ is a consistent family of measures, and thus by the Kolmogoroff consistency theorem it extends to a unique σ-finite measure μ' on $\mathscr{B}'$. Furthermore, for any $\mathscr{A}^{(k)}$-measurable function $f' = f \circ P_k$, where $P_k = P \circ T'^k$, $k \geqslant 1$, if $\omega' = \omega \circ P$,

$$\begin{aligned}\int f' \circ T'\omega'\, d\mu' &= \int f \circ P_{k-1} \circ T\omega\, d\mu \\ &= \int f \circ P_{k-1}\, d\mu = \int f \circ P_k\, d\mu'.\end{aligned}$$

Finally, if T' is Markovian with respect to (ω', ν), where $\nu \circ P^{-1} = \mu$, then

$$\nu(A^{(k)}) = \int_{P^{-1}(A)} \omega'_k\, d\nu = \int_A \omega_k\, d\mu = \mu'(A^{(k)}). \quad \blacksquare$$

5.10. *Remark.* (a) The properties of Theorem 5.5, and the natural extension described in Theorem 5.8 depend only on the cohomology class of (ω, μ). If (ω, μ) and $(\bar{\omega}, \bar{\mu})$ are two cohomologous pairs, their respective natural extensions are isomorphic. The natural extension of $(X, \mathscr{B}, T)$ with respect to (ω, μ) is conservative if and only of T is (ω, μ)-recurrent. If T is not (ω, μ)-recurrent, then the natural extension with respect to (ω, μ) is not conservative.

(b) The natural extension described in Theorem 5.8 is canonical in the following sense. Any Y extension of X is also an extension of X'. If $Q\colon Y \to X$ is a conjugate projection, Markovian with respect to π, then there exists a unique $Q'\colon Y \to X'$ conjugate projection Markovian with respect to π and satisfying $P \circ Q' = Q$.

5.11. EXAMPLE. Consider the modified Boole transformation $Tx = 1/2(x - 1/x)$ defined on the real line. Let λ denote Lebesque measure and μ denote Cauchy distribution. These measures are equivalent and T is a conservative nonsingular endomorphism with respect to them. (In fact, one can show that $(\mathbf{R}, T, \mu)$ is isomorphic to $x \to 2x \pmod 1$ on the unit interval.) One can calculate that $\omega^\lambda = \frac{1}{2}$ and $\omega^\mu = 1$ a.e. It follows that T is (ω^μ, μ)-recurrent and is not $(\omega^\lambda, \lambda)$-recurrent. λ is an infinite measure but if h is any positive function of integral 1, then if we define $\nu = h\lambda$ and $\theta = ((h \circ T)/h)\, \omega^\lambda$, then ν is a probability measure, $(\omega^\lambda, \lambda)$ is cohomologous to (θ, ν), (θ, ν) is a Markovian pair for T, and T is not (θ, ν)-recurrent. The natural extension of T with respect to (ω^μ, μ) is conservative ergodic [9], while the natural extension of T with respect to (θ, ν) is dissipative.

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