ERGODIC REDUCTION OF RANDOM PRODUCTS OF TWO-BY-TWO MATRICES

By

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Abstract. We consider a random product of two-by-two matrices of determinant one over an abstract dynamical system. When the two Lyapunov exponents are distinct, Oseledets' theorem asserts that the matrix cocycle is cohomologous to a diagonal matrix cocycle. When they are equal, we show that the cocycle is conjugate to one of three cases: a rotation matrix cocycle, an upper triangular matrix cocycle, or a diagonal matrix cocycle modulo a rotation by $\pi/2$.

1 Introduction and main results

Let us consider a smooth dynamical system (X, ϕ), where X is a compact oriented smooth Riemanniann manifold of dimension 2 and $\phi : X \to X$ is a smooth orientation preserving diffeomorphism acting on X. We want to understand the asymptotic behaviour of typical orbits $(\phi^n(x))_{n\in\mathbb{Z}}$. Since the number of degrees of freedom increases exponentially when we iterate, we assume in addition that the system admits some constant of motion. We assume precisely that the Lebesgue measure m of X, normalized to one, is preserved by ϕ, $\phi_*(m) = m$. Periodic orbits may be seen also as constants of motion but they are not usually typical with respect to the Lebesgue measure. We say that an orbit is typical if it returns infinitely often in any Borel set of positive Lebesgue measure with a frequency equal to the mass of the set. By invariance of m, Birkhoff's ergodic theorem asserts that almost all orbits are typical, provided the system is ergodic.

More modestly, we want to describe the asymptotic behaviour of infinitesimal perturbations of typical orbits. If $T\phi : TX \to TX$ denotes the tangent map and $x \in X$ is a periodic point of period $p \geq 1$, the asymptotic behaviour is determined by the operator $T_x\phi^p$ and infinitesimally the motion is either hyperbolic, parabolic or elliptic. We shall show that a similar classification exists for typical orbits (with respect to the Lebesgue measure).

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The case of hyperbolic non-periodic orbits has been studied since Oseledets [12]. These orbits are characterized by two distinct Lyapunov exponents $\lambda_+(x) > 0 > \lambda_-(x)$ and by one expanding direction $E_+(x)$ and one contracting direction $E_-(x)$. The two limits

$$\lambda_+(x) = \lim_{n \to +\infty} \frac{1}{n} \ln \|T_x\phi^n\| \quad \text{and} \quad \lambda_-(x) = \lim_{n \to +\infty} \frac{1}{n} \ln \|T_x\phi^{-n}\|$$

exist and satisfy $\lambda_+(x) + \lambda_-(x) = 0$ (we have assumed $\det(T_x\phi) = 1$). The tangent space can be decomposed into the sum of two invariant and measurable vector bundles of dimension one:

$$T_xX = E_+(x) \oplus E_-(x) \quad \text{and} \quad T_x\phi(E_\pm(x)) = E_\pm \circ \phi(x).$$

Each non-zero vector in $E_+(x)$ (resp. $E_-(x)$) is expanded (resp. contracted) exponentially. For all $v_\pm \in E_\pm(x)$,

$$\lim_{n \to +\infty} \frac{1}{n} \ln \|T_x\phi^n \cdot v_\pm\| = \lambda_\pm(x).$$

In the non-hyperbolic case the two Lyapunov exponents are equal to zero and there do not exist any more invariant sub-bundles. We now consider a more formal framework, which includes both the case of smooth dynamical systems and the case of random walk on $\mathrm{SL}(2,\mathbb{R})$. We have chosen the group of orientation preserving matrices in order to simplify the notations. We show in Section 2.5 how to extend the Main Theorem to the case of $\mathrm{GL}(2,\mathbb{R})$. From now on (X, m, ϕ) is an abstract dynamical system where X is a standard Borel space, $\phi : X \to X$ is a Borel invertible map and m is a ϕ-invariant probability measure on the Borel σ-algebra $\mathcal{B}_X$. We choose a Borel $\mathrm{SL}(2,\mathbb{R})$-valued function $M : X \to \mathrm{SL}(2,\mathbb{R})$ and define the associated random product or cocycle $M(n,x)$ for all $n \geq 0$ by

$$M(n,x) = M_{\phi^{n-1}(x)} \cdots M_{\phi(x)} M_x,$$

$$M(-n,x) = \big(M(n, \phi^{-n}(x))\big)^{-1} = M^{-1}_{\phi^{-n}(x)} \cdots M^{-1}_{\phi^{-1}(x)}.$$

Notice that M satisfies the cocycle identity, for all $m, n \in \mathbb{Z}$:

$$M(m+n, x) = M(n, \phi^m(x)) M(m, x).$$

Oseledets' theory tells us how to construct explicitly the invariant sub-bundles $E_\pm(x)$ in the hyperbolic case. By Kingman's ergodic theorem [9], the two Lyapunov exponents exist almost everywhere and can also be computed by using the polar decomposition, $M(n,x) = R(\alpha(n,x))|M(n,x)|$, where $R(\alpha)$ denotes the matrix rotation of angle α and $|M(n,x)|$ is the symmetric matrix $\big(M(n,x)^* M(n,x)\big)^{1/2}$ with eigenvalues $\chi_+(n,x) \geq 1 \geq \chi_-(n,x)$ and eigenspaces $E_\pm(n,x)$ (when the eigenvalues are distinct). Oseledets' theorem may be summarized as follows.

Theorem 1.1 (*Oseledets [12]*). *Assume that* $\ln\|M\|$ *is integrable; then the sequence* $\left(\frac{1}{n}\ln\chi_{\pm}(n,x)\right)_{n>0}$ *converges a.e. to* $\lambda_{\pm}(x)$, $\lambda_+(x)+\lambda_-(x)=0$ *and* $\lambda_{\pm}(x)=\lambda_{\pm}\circ\phi(x)$. *If* $\lambda_+(x)$ *is positive, then*

(a) $\lim_{n\to+\infty} M(n,\phi^{-n}(x))\cdot E_+(n,\phi^{-n}(x)) = E_+(x)$,

(b) $\lim_{n\to+\infty} E_-(n,x) = E_-(x)$,

(c) $\mathbb{R}^2 = E_+(x)\oplus E_-(x)$ *and* $M_x\cdot E_{\pm}(x) = E_{\pm}\circ\phi(x)$.

Oseledets' theorem is also true in higher dimensions, but the formulation of the statement is more complicated (see [12], [10], [14], [15], [11], [20] and [16]). By choosing a measurable basis adapted to this spectral decomposition, Oseledets' theorem may be restated as follows. There exist a measurable change of coordinates $K : X \to \mathrm{SL}(2,\mathbb{R})$ and a measurable diagonal matrix $D_x = \mathrm{diag}(v_x, v_x^{-1})$ such that

$$M_x = K_{\phi(x)}D_xK_x^{-1} \quad \text{and} \quad \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln|v\circ\phi^k(x)| = \lambda_+(x).$$

Such a conjugating matrix K is not unique; we shall show that there exists a unique "minimal" one up to a rotation modulo $\pi/2$ which corresponds to the choice of a basis of two vectors adapted to $E_{\pm}$ of equal length and area equal to one.

When we iterate a single matrix M in $\mathrm{SL}(2,\mathbb{R})$, the dynamics of the projective map PM acting on $\mathrm{P}\mathbb{R}^2$ is classified by the number of fixed points of PM and we obtain three cases: hyperbolic, parabolic or elliptic dynamics. The following theorem shows that the same classification remains true along most stationary orbits. A fourth case occurs when two lines are permuted globally in a non-cohomologous manner. We first recall two notions of recurrence for cocycles (a thorough analysis is given in [17]).

Definition 1.2. (i) The cocycle $M(n,x)$ is said to be recurrent if for any Borel set B of positive measure and any $\epsilon>0$ there exists $n\geq 1$ such that

$$m(B\cap\phi^{-n}B\cap\{x\in B \text{ s.t. } \|M(n,x)-\mathrm{Id}\|<\epsilon\})>0.$$

(ii) Infinity is said to be an essential value of $M(n,x)$ if for any Borel set B of positive measure and any real $R>0$ there exists $n\geq 1$ such that

$$m(B\cap\phi^{-n}B\cap\{x\in B \text{ s.t. } \|M(n,x)\|>R\})>0.$$

We first recall an easy dichotomy for general cocycles and then state the Main Theorem.

Lemma 1.3. *If $M : X \to \mathrm{SL}(2,\mathbb{R})$ is a measurable (not necessarily log-integrable) function and $X_t = \{x \in X : \lim_{n\to+\infty} \|M(n,x)\| = +\infty\}$, then on the complement set $X_r = X \setminus X_t$, $M(n,x)$ is a recurrent cocycle.*

The following theorem can be seen as a classification into four distinct types of dynamics. The log-integrability condition is used only to define the Lyapunov exponents and to prove various recurrence properties; it is not used in the classification result.

Main Theorem 1.4. *Assume $\int \ln \|M_x\|\, dm(x) < +\infty$. Then there exist a Borel function $K : X \to \mathrm{SL}(2,\mathbb{R})$ and a partition of X into four invariant Borel sets $X = X_H \cup X_P \cup X_E \cup X_{WH}$ such that $N_x = K^{-1}_{\phi(x)} M_x K_x$ takes one of the following forms:*

(i) *For a.e. $x \in X_H$,*

$$N_x = \begin{bmatrix} v_x & 0 \\ 0 & v_x^{-1} \end{bmatrix}$$

for some Borel $v : X_H \to \mathbb{R}$ satisfying $\lim_{n\to+\infty} \frac{1}{n} \ln |v \circ \phi^n(x)| = \lambda_+(x) > 0$.

(ii) *For a.e. $x \in X_P$,*

$$N_x = \begin{bmatrix} v_x & w_x \\ 0 & v_x^{-1} \end{bmatrix}$$

for some Borel $v, w : X_P \to \mathbb{R}$ satisfying $\lim_{n\to+\infty} \frac{1}{n} \ln |v \circ \phi^n(x)| = \lambda_+(x) = 0$.

(iii) *For a.e. $x \in X_E$,*

$$N_x = R(\omega_x) = \begin{bmatrix} \cos\omega_x & -\sin\omega_x \\ \sin\omega_x & \cos\omega_x \end{bmatrix}$$

for some Borel function $\omega : X_E \to \mathbb{R}$ not cohomologous to 0 modulo π on any invariant set. Restricted to X_E, $M(n,x)$ is recurrent and ∞ is not an essential value (on any invariant set). In particular $\lambda_+(x) = 0$.

(iv) *For a.e. $x \in X_{WH}$,*

$$N_x = R\Big(\mathbf{1}_A(x)\frac{\pi}{2}\Big) \begin{bmatrix} v_x & 0 \\ 0 & v_x^{-1} \end{bmatrix}$$

for some Borel function $v : X_{WH} \to \mathbb{R}$ and $A \subset X_{WH}$ such that $\mathbf{1}_A$ is not cohomologous to 0 modulo 2 on any invariant set. Restricted to X_{WH}, $M(n,x)$ is recurrent and ∞ is an essential value. In particular $\lambda_+(x) = 0$.

Moreover, we have

(a) *In all four cases,* $\|M_x\|^{-1} \leq \|K_x\|/\|K_{\phi(x)}\| \leq \|M_x\|$ *and* $\|N_x\| \leq \|M_x\|$ *a.e. More precisely, there exist Borel maps* $\gamma, u : X \to \mathbb{R}$ *such that*

$$\|K_{\phi(x)}K_x^{-1}\| \leq \|M_x\| \quad \textit{and} \quad K_x = R\Big(\gamma_x + \frac{\pi}{4}\Big) \begin{bmatrix} u_x & 0 \\ 0 & u_x^{-1} \end{bmatrix} R\Big(-\frac{\pi}{4}\Big).$$

(b) *If* $\tilde{K} : X \to \mathrm{SL}(2,\mathbb{R})$ *is a Borel function such that* $\tilde{N}_x = \tilde{K}_{\phi(x)}^{-1} M_x \tilde{K}_x$ *takes the same form as* N_x *on each set* X_H, X_P, X_E *and* X_{WH}, *then* $\|K_x\| \leq \|\tilde{K}_x\|$ *a.e. More precisely, there exist* $A \subset X_H \cup X_{WH}$, $A \cap X_H$ *invariant and* $D : X_H \cup X_{WH} \to \mathrm{SL}(2,\mathbb{R})$ *a diagonal matrix such that* $\tilde{K}_x = K_x R(\mathbf{1}_A(x)\frac{\pi}{2})D_x$ *a.e. on* $X_H \cup X_{WH}$ *and* $\|\tilde{K}\| = \|K\|$ *if and only if* $D_x = \pm\mathrm{Id}$. *There exists* $\alpha : X_E \to \mathbb{R}$ *such that* $\tilde{K}_x = K_x R(\alpha_x)$ *a.e. on* X_E. *In all four cases* $\|\tilde{K}\| = \|K\|$ *if and only if* $K_x^{-1}\tilde{K}_x$ *is a rotation a.e.*

We say that a Borel function $f : X \to \mathbb{R}$ is a coboundary modulo α if there exists $g : X \to \mathbb{R}$, Borel, such that $f - g \circ \phi + g \in \alpha\mathbb{Z}$. On the weakly hyperbolic set X_{WH}, the two lines $E_+(x) = K_x(\mathbb{R} \times \{0\})$ and $E_-(x) = K_x(\{0\} \times \mathbb{R})$ are globally M-invariant and are permuted in a non-cohomologous way. Conversely:

Proposition 1.5. *If* $\mathbf{1}_B$ *is not a coboundary modulo* 2 *on any invariant set,* $v : X \to \mathbb{R}^*$ *is log-integrable and* $M_x = R(\mathbf{1}_B(x)\frac{\pi}{2})\,\mathrm{diag}(v_x, v_x^{-1})$, *then*

(i) $M(n,x)$ *is recurrent (in particular, the Lyapunov exponents are zero).*

(ii) *If* X_∞ *denotes an invariant set of maximal measure on which* ∞ *is an essential value, then on* $X \backslash X_\infty$, M_x *is cohomologous to a rotation* $R(\omega_x)$, *where* $\omega_x = 0$ *modulo* $\frac{\pi}{2}$.

The next proposition gives a sufficient condition for a cocycle to possess an invariant line.

Proposition 1.6. *Let* $M : X \to \mathrm{SL}(2,\mathbb{R})$ *be a log-integrable cocycle such that the norm of* $M(n,x)$ *converges to* ∞ *a.e. Then*

(i) *There exists a measurable* M*-invariant line* $\xi : X \to P\mathbb{R}^2$, *that is a line* ξ_x *of* $\mathbb{R}^2$ *satisfying* $M_x(\xi_x) = \xi_{\phi(x)}$ *a.e. on* X.

(ii) *If there exist two measurable globally* M*-invariant lines* $\xi, \eta : X \to P\mathbb{R}^2$, $M_x\{\xi_x, \eta_x\} = \{\xi_{\phi(x)}, \eta_{\phi(x)}\}$, *then each line* ξ *and* η *is actually invariant with respect to* M *and* $\lim_{n\to+\infty} \frac{1}{n} \ln \|M(n,x)\| > 0$ *a.e. on* X.

Oseledets' theorem in the case of distinct Lyapunov exponents already shows that a log-integrable cocycle M is cohomologous to a diagonal cocycle $D_x = K_{\phi(x)}^{-1} M_x K_x$. The Main Theorem shows that D and K can be chosen so that D and $\|K \circ \phi\|/\|K\|$ are log-integrable and $(K^*K)^{1/2}$ is diagonal in the basis $\{\frac{1}{\sqrt{2}}(-1,1); \frac{1}{\sqrt{2}}(1,1)\}$ independently of x. Geometrically, K_x sends the canonical basis of $\mathbb{R}^2$ to a basis of two vectors of length $(\sin\theta_x)^{-1/2}$, where $\theta_x \in]0,\pi[$ is the angle between these two vectors and $\tan(\frac{1}{2}\theta_x) = \|K_x\|^{-2}$. In the category of log-integrable cocycles, the following proposition shows there cannot exist other general constraints.

Proposition 1.7. *If $K_x = R(\gamma_x + \frac{\pi}{4})\operatorname{diag}(u_x, u_x^{-1})R(-\frac{\pi}{4})$ for some Borel functions $\gamma, u : X \to \mathbb{R}$ such that $\ln|u \circ \phi/u|$ or $\ln\|K_{\phi(x)}K_x^{-1}\|$ is integrable, then there exists a log-integrable cocycle $N : X \to \mathrm{SL}(2,\mathbb{R})$ such that $M_x = K_{\phi(x)}N_xK_x^{-1}$ becomes log-integrable and has positive Lyapunov exponent.*

Similarly, we can reconstruct an elliptic cocycle from any such conjugating matrix K.

Proposition 1.8. *For any $K_x = R(\gamma_x + \frac{\pi}{4})\operatorname{diag}(u_x, u_x^{-1})R(-\frac{\pi}{4})$ such that $\|K_{\phi(x)}K_x^{-1}\|$ is log-integrable, there exists $\omega : X \to \mathbb{R}$ not a coboundary modulo π on any invariant set such that $M_x = K_{\phi(x)}R(\omega_x)K_x^{-1}$ is log-integrable.*

We now give two simple applications of the Main Theorem: one for random products of independent matrices and another one for random products of matrices with non-negative entries. We first recall the notion of independence:

Definition 1.9. Let (X, m, ϕ) be an ergodic invertible dynamical system, M a $\mathrm{SL}(2,\mathbb{R})$-valued measurable cocycle and $\mathcal{F}_0 \subset \mathcal{B}_X$ a sub-σ-algebra. We say that M is independent with respect to $\mathcal{F}_0$ if $M_x = M(0,x)$ is $\mathcal{F}_0$-measurable and the sequence of σ-algebras $\left(\mathcal{F}_n = \phi^n(\mathcal{F}_0)\right)_{n\in\mathbb{Z}}$ are independent.

Applied to the independent case, the Main Theorem gives

Proposition 1.10. *Let (X, m, ϕ) be an ergodic invertible dynamical system, $M : X \to \mathrm{SL}(2,\mathbb{R})$ a measurable function which is log-integrable and independent with respect to some σ-algebra $\mathcal{F}_0$. Then $\lim_{n\to+\infty} \frac{1}{n}\ln\|M(n,x)\| = \lambda_+$ exists and is constant a.e.*

(i) *If $\lambda_+ > 0$ then there exist $K : X \to \mathrm{SL}(2,\mathbb{R})$, $v : X \to \mathbb{R}^*$ measurable such that $M_x = K_{\phi(x)}\operatorname{diag}(v_x, v_x^{-1})K_x^{-1}$, $\int \ln|v(x)|\,dm(x) = \lambda_+$ and $\max(|v_x|, |v_x|^{-1}) \le \|M_x\|$ a.e.*

(ii) *If* $\lambda_+ = 0$ *then there exist* $K \in \mathrm{SL}(2,\mathbb{R})$ *(constant),* $\mathcal{F}_0$*-measurable functions* $v, w, \omega : X \to \mathbb{R}$ *such that* $\max(|v_x|, |v_x|^{-1}, |w_x|) \leq \|M_x\|$ *and a non-trivial set* $A \in \mathcal{F}_0$ *such that* $N_x = K^{-1}M_xK$ *is almost everywhere equal to either:*

(a) $N_x = \begin{bmatrix} v_x & w_x \\ 0 & v_x^{-1} \end{bmatrix}$ *where* $\int \ln |v(x)|\, dm(x) = 0$,

(b) $N_x = R(\omega_x)$ *a.e. on* X,

(c) $N_x = R(\mathbf{1}_A(x)\frac{\pi}{2})\,\mathrm{diag}(v_x, v_x^{-1})$ *where* $\int \ln |v(x)|\, dm(x) \in \mathbb{R}$.

The second application is a geometric proof of Wojtkowski's estimate.

Proposition 1.11 *([21]). Let* $(X, , m, \phi)$ *be an abstract dynamical system and* $M : X \to \mathrm{SL}(2,\mathbb{R})$ *a log-integrable function. If* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *where all the entries are non-negative, then for a.e.* $x \in X$

$$\lim_{n\to+\infty} \frac{1}{n} \ln \|M(n,x)\| \geq \mathbb{E}\Big[\ln(\sqrt{ad} + \sqrt{bc}) \mid \mathcal{I}_\phi\Big],$$

where $\mathcal{I}_\phi$ *denotes the Borel* σ*-algebra of* ϕ*-invariant sets.*

We mention for completeness that Wojtkowski has extended this lower bound to general symplectic matrices (see [22]).

We have added for the convenience of the reader three appendices. In Appendix A, we give a different and new proof of the main tool, Douady–Earle's theorem about the existence of a conformal barycenter. We use the Busemann function instead of an argument in degree theory. This method is borrowed from [3] and can be extended to higher dimensions. In Appendix B, we reprove the Dundford–Pettis theorem (see [4]). The fact that we have chosen $\phi : X \to X$ merely measurable introduces certain complications which are rarely explained. We introduce different topologies and prove that bounded sets in certain functional spaces are weakly compact. In Appendix C, we gather several results on recurrence of additive cocycles for finite or σ-finite abstract dynamical systems. Deeper results can be found in [17]. Finally, related works on the classification of $\mathrm{GL}(2,\mathbb{R})$-valued cocycles can be found in [5], [13].

2 Proof of the Main Theorem

Throughout the rest of this section, we choose an invertible abstract dynamical system (X, m, ϕ) and a log-integrable cocycle $M : X \to \mathrm{SL}(2,\mathbb{R})$. We denote by $\mathbb{D}$ the open unit disk of the complex plane $\mathbb{C}$ and by $\mathcal{M}ob^+(\mathbb{D})$ the group of

Möbius transformations which preserve $\mathbb{D}$. Using a standard isomorphism between $\mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$ and $\mathcal{M}ob^+(\mathbb{D})$, we first reduce the cocycle M to a cocycle M (we will use the same notation) taking values in $\mathcal{M}ob^+(\mathbb{D})$.

2.1 A quasi-conformal approach of SL$(2,\mathbb{R})$ The following is taken from the beginning of [1]. Consider a two-by-two matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of determinant one. When $\mathbb{R}^2$ is identified with $\mathbb{C}$, M becomes a linear operator in $\mathbb{C}$ which can be written in the form $M.z = (\partial_z M)z + (\partial_{\bar z} M)\bar z$, where $\partial_z M$ and $\partial_{\bar z} M$ are two complex numbers. A simple computation gives

$$\partial_z M = \tfrac{1}{2}[(a+d) + i(c-b)] \quad \text{and} \quad \partial_{\bar z} M = \tfrac{1}{2}[(a-d) + i(c+b)].$$

The Beltrami coefficient $\mu = \partial_{\bar z} M / \partial_z M$ measures the degree of non-conformality or distortion of M. If J denotes the determinant of M and $\chi_\pm$ the eigenvalues of $\sqrt{M^* M}$, then

$$J = ad - bc = |\partial_z M|^2 - |\partial_{\bar z} M|^2 = 1 \quad \text{and} \quad \chi_\pm = |\partial_z M| \pm |\partial_{\bar z} M|.$$

In particular, $\mu \in \mathbb{D}$ and

$$\ln \|M\| = \frac{1}{2} \ln \Big(\frac{\chi_+}{\chi_-}\Big) = \frac{1}{2} \ln \Big(\frac{1+|\mu|}{1-|\mu|}\Big) = d_{\mathbb{D}}(0,\mu),$$

where $d_{\mathbb{D}}(.,.)$ is the Poincaré metric of the unit disk (cf. Appendix A). Conversely, $(M^* M)^{1/2}$ can be rebuilt from μ as shown in the following lemma.

Lemma 2.1. *If* $M \in \mathrm{SL}(2,\mathbb{R})$, $e^{i\alpha} = \partial_z M / |\partial_z M|$, $\rho e^{i\theta} = \partial_{\bar z} M / \partial_z M$ *and* $\Delta = \mathrm{diag}(1,-1)$, *then* $M = (1-\rho^2)^{-1/2} R(\alpha) \big[\mathrm{Id} + \rho R(\tfrac{1}{2}\theta) \Delta R(-\tfrac{1}{2}\theta)\big]$.

Proof. $R(\alpha).z = e^{i\alpha} z$, $\Delta.z = \bar z$ and $|\partial_z M| = (1-\rho^2)^{-1/2}$. □

If μ_A, μ_B are the distortion coefficients of $A, B \in \mathrm{SL}(2,\mathbb{R})$, then the distortion coefficient μ_{AB} of the product AB is equal to $T_B^{-1}(\mu_A) = T_B^{-1} \circ T_A^{-1}(0)$ where for each $M \in \mathrm{SL}(2,\mathbb{R})$, $T_M \in \mathcal{M}ob^+(\mathbb{D})$ is the Möbius transformation of $\mathbb{D}$ defined by

$$T_M(t) = \frac{t\,\partial_z M - \partial_{\bar z} M}{\overline{\partial_z M} - t\,\overline{\partial_{\bar z} M}}.$$

This suggests associating to each linear operator $M \in \mathrm{SL}(2,\mathbb{R})$ a Möbius transformation which we denote by the same letter M. More precisely,

Proposition 2.2. *For each $M \in \mathrm{SL}(2,\mathbb{R})$, $\exp(i\alpha_M) = \partial_z M/|\partial_z M|$ and $\mu_M = \partial_{\bar z} M/\partial_z M$, we associate a Möbius transformation*

$$M(t) = \exp(i2\alpha_M)\frac{t+\mu_M}{1+t\bar\mu_M}.$$

This defines a surjective group endomorphism whose kernel is $\{\pm \mathrm{Id}\}$.

Proof. $\partial_z(AB) = \partial_z A\,\partial_z B + \partial_{\bar z} A\,\overline{\partial_{\bar z} B}$ and $\partial_{\bar z}(AB) = \partial_z A\,\partial_{\bar z} B + \partial_{\bar z} A\,\overline{\partial_z B}$. $\square$

The foregoing isomorphism between $\mathrm{PSL}(2,\mathbb{R})$ and $\mathcal{M}ob^+(\mathbb{D})$ has another geometric interpretation. If $M \in \mathrm{SL}(2,\mathbb{R})$, then M acts on the set of half lines identified with $\partial\mathbb{D}$ and acts also on $\partial\mathbb{D}$ when considered as a Möbius transformation. These two actions are conjugated by p as shown in

Lemma 2.3. *Let $p : t \in \partial\mathbb{D} \mapsto t^2 \in \partial\mathbb{D}$. Then $M \circ p = p \circ M$.*

Proof. $M(t)$ acts on $\partial\mathbb{D}$ by $M(t) = (t\,\partial_z M + \bar t\,\partial_{\bar z} M)\big/|t\,\partial_z M + \bar t\,\partial_{\bar z} M|$. $\square$

2.2 Proof of Oseledets' theorem when $\lambda_+ > 0 > \lambda_-$ As we have seen in the previous section, we may assume that the cocycle M takes values in $\mathcal{M}ob^+(\mathbb{D})$. As usual $M(n,x) = M_{\phi^{n-1}(x)} \circ \cdots \circ M_x$ and $M(-n,x) = M(n,\phi^{-n}(x))^{-1}$. The log-integrability condition is equivalent to the integrability of $d_{\mathbb{D}}(0, M_x(0))$ and the existence of Lyapunov exponents is given by

Lemma 2.4. *Let $M : X \to \mathcal{M}ob^+(\mathbb{D})$ be a log-integrable cocycle. Then the sequence $\left(\frac{1}{n}d(0, M(n,x)(0))\right)_{n>1}$ converges a.e. to a non-negative $\lambda_+(x)$.*

Proof. The sequence of functions $\delta_n(x) = d_{\mathbb{D}}(0, M(n,x)(0))$ is subadditive ($\delta_{m+n} \le \delta_m + \delta_n \circ \phi^m$); δ_1 is integrable; and by Kingman's ergodic theorem [9] (or [8]), $(\frac{1}{n}\delta_n)_{n\ge 1}$ converges a.e. If $\chi_\pm(n,x)$ denote the eigenvalues of $(M(n,x)^* M(n,x))^{1/2}$, then $d_{\mathbb{D}}(0, M(n,x)(0)) = \frac{1}{2}\ln[\chi_+(n,x)/\chi_-(n,x)]$; and by Oseledets' theorem, $\left(\frac{1}{n}\ln\chi_+(n,x)\right)_{n>0}$) converges a.e. to $\lambda_+(x)$. $\square$

The main difficulty in Oseledets' theorem is to show the existence of two measurable invariant bundles $E_\pm$ of dimension one. In the context of a cocycle M, this is equivalent to finding two measurable functions $\xi_\pm : X \to \partial\mathbb{D}$ which are M-invariant in the following sense: $M_x(\xi_\pm(x)) = \xi_\pm \circ \phi(x)$. The proof of part (i) of the Main Theorem is a consequence of the following proposition.

Proposition 2.5. *Let $M : X \to \mathcal{M}ob^+(\mathbb{D})$ be a measurable cocycle satisfying $\int d_{\mathbb{D}}(0, M_x(0))\,dm(x) < +\infty$. Then, on the set $\{\lambda_+ > 0\}$, the sequences $\left(M(n,\phi^{-n}(x))(0)\right)_{n\ge 0}$ and $\left(M(-n,\phi^{n}(x))(0)\right)_{n\ge 0}$ converge a.e. exponentially to two distinct M-invariant measurable functions $\xi_+(x)$ and $\xi_-(x)$.*

Proof. The proof is divided into two parts: in part one we prove the convergence of $\xi_n(x) = M(n, \phi^{-n}(x))(0)$ exponentially fast to the boundary of $\mathbb{D}$. In the second part, by choosing a conjugacy sending $\xi_+(x)$ to ∞, $M(n,x)$ becomes a Möbius transformation of the upper half plane; and we show that $M(-n, \phi(x))(0)$ converges to a real (and finite) point $\xi_-(x)$.

Part one. The log-integrability condition implies that $\left(\frac{1}{n} d_{\mathbb{D}}(0,\xi_n)\right)_{n\geq 1}$ converges to $\lambda_+ > 0$. In particular, $d_{\mathbb{D}}(0,\xi_n)$ converges to $+\infty$, which shows that ξ_n converges to the boundary of $\mathbb{D}$. If we establish that $|\xi_n - \xi_{n+1}|$ converges to 0 exponentially fast, then $(\xi_n)_{n\geq 1}$ is a Cauchy sequence which converges to a point $\xi_+(x) \in \partial\mathbb{D}$. Let us prove the assertion. By invariance of the Poincaré metric,

$$d_{\mathbb{D}}(\xi_n, \xi_{n+1}) = d_{\mathbb{D}}\left(0, M_{\phi^{-n-1}(x)}(0)\right).$$

Thanks to the integrability of M, given $\epsilon : X \to \mathbb{R}^+$ satisfying $\lambda_+ > 2\epsilon$, we have for sufficiently large n, $d_{\mathbb{D}}(\xi_n, \xi_{n+1}) < n\epsilon$ and $d_{\mathbb{D}}(\xi_n, 0) > n(\lambda_+ - \epsilon)$. Let $\gamma_n : [0,1] \to \mathbb{D}$ be a geodesic joining ξ_n and ξ_{n+1}; then

$$d_{\mathbb{D}}(\xi_n, \xi_{n+1}) = \int \frac{|\dot\gamma_n(t)|}{1 - |\gamma_n(t)|^2} dt \geq \inf_{[0,1]} \left(\frac{1}{1 - |\gamma_n|^2}\right) |\xi_n - \xi_{n+1}|.$$

We now estimate the euclidean distance of γ_n to the boundary of $\mathbb{D}$:

$$d_{\mathbb{D}}(\xi_n, \gamma_n(t)) \leq d_{\mathbb{D}}(\xi_n, \xi_{n+1}),$$

$$d_{\mathbb{D}}(0, \gamma_n(t)) \geq d_{\mathbb{D}}(0, \xi_n) - d_{\mathbb{D}}(\xi_n, \gamma_n(t)) \geq n(\lambda_+ - 2\epsilon),$$

$$d_{\mathbb{D}}(0, \gamma_n(t)) = \tfrac{1}{2} \ln\left((1 + |\gamma_n|)\big/(1 - |\gamma_n|)\right) \leq -\tfrac{1}{2} \ln \tfrac{1}{4}\left(1 - |\gamma_n(t)|^2\right).$$

We thus obtain $\inf_{[0,1]}(1 - |\gamma_n|^2)^{-1} \geq 4 \exp 2n(\lambda_+ - 2\epsilon)$ and that $(\xi_n - \xi_{n+1})_{n\geq 0}$ converges to 0 exponentially. A similar proof shows that $M(-n, \phi^n(0))$ converges to a point $\xi_- \in \partial\mathbb{D}$.

Before proving the second part ($\xi_+(x) \neq \xi_-(x)$ a.e.), we need the following technical lemma. We recall (see Lemma A.2) that G_ξ denotes the unique Möbius transformation sending $\partial\mathbb{D}$ to $\partial\mathbb{H}$, the boundary of the Poincaré upper half space, $\xi \in \partial\mathbb{D}$ to ∞ and 0 to i.

Lemma 2.6. *Let $M : X \to \mathcal{M}ob^+(\mathbb{D})$ a log-integrable cocycle, $\xi : X \to \partial\mathbb{D}$ a measurable M-invariant function and $T_x = G_{\xi\circ\phi(x)} M_x G_{\xi(x)}^{-1}$ the cohomologous Möbius transformation on the Poincaré upper half space. Then $T_x(z) = a(x)z - b(x)$ for some measurable $a, b : X \to \mathbb{R}$, where $a > 0$ a.e., $\ln(a)$ and $\ln^+|b|$ are integrable. If $T(n,x)(z) = a_n(x)z - b_n(x)$, then*

$$\lim_{n\to+\infty} \frac{1}{2n} |\ln a_n(x)| = \lim_{n\to+\infty} \frac{1}{n} d_{\mathbb{D}}(0, M(n,x)(0)) = \lambda_+(x) \quad a.e.$$

Proof. From Lemma A.5 we get

$$\ln\left[\max\left(a_n, \frac{1}{a_n}\right) + \frac{b_n^2}{a_n}\right] \le 2d_{\mathbb{D}}(0, M(n,x)(0)) \le \ln\left[1 + \max\left(a_n, \frac{1}{a_n}\right) + \frac{b_n^2}{a_n}\right],$$

which already implies that $\ln a$ and $\ln^+|b|$ are integrable. Moreover,

$$a_n(x) = \prod_{k=0}^{n-1} a \circ \phi^k(x) \quad \text{and} \quad b_n = \sum_{k=0}^{n-1} a_{n-k-1} \circ \phi^{k+1}(x) b \circ \phi^k(x)$$

and by Birkhoff's ergodic theorem $\left(\frac{1}{n}\ln a_n(x)\right)_n$ converges to γ a.e. The left inequality shows that $\frac{1}{2}|\ln a_n(x)|$ is bounded by $d_{\mathbb{D}}(0, M(n,x)(0))$ and thus $|\gamma| \le \lambda_+$ a.e. To prove the other inequality, for every $\epsilon > 0$ one can find a measurable function $c_\epsilon : X \to \mathbb{R}_*^+$ such that

$$c_\epsilon(x)^{-1}\exp(-n\epsilon - |k|\epsilon) \le a_n \circ \phi^k(x)\exp[-n\gamma(x)] \le c_\epsilon(x)\exp(n\epsilon + |k|\epsilon),$$

$$|b \circ \phi^k(x)| \le c_\epsilon(x)\exp(|k|\epsilon),$$

for all $n \ge 0$, $k \in \mathbb{Z}$ and a.e. $x \in X$. Then

$$|b_n(x)| \le c(x)^2 \sum_{k=0}^{n-1} \exp\left[k\epsilon + (n-k-1)(\gamma+\epsilon) + (k+1)\epsilon\right].$$

The bound from above of b_n^2/a_n is different depending on whether $\gamma \ge 0$ or $\gamma < 0$. If $\gamma \ge 0$ we get $|b_n(x)| \le e^{n(\gamma+3\epsilon)}c(x)^2/(e^{\gamma+\epsilon}-1)$. If $\gamma < 0$ we choose $\epsilon > 0$ such that $\gamma + \epsilon < 0$ and $|b_n(x)| \le e^{2n\epsilon}c(x)^2/(1-e^{\gamma+\epsilon})$. In both cases

$$\frac{b_n(x)^2}{a_n(x)} \le D(x)\exp n(\gamma + 7\epsilon) \quad \text{and} \quad \limsup_{n\to+\infty} \frac{1}{n}\frac{b_n(x)^2}{a_n(x)} \le \gamma. \qquad \square$$

Proof of Proposition 2.5. *Part two.* We have already proved the existence of $\xi_+(x)$ and $\xi_-(x)$ a.e.; we show here that $\xi_+(x) \ne \xi_-(x)$ a.e. As in the previous lemma, we rewrite the cocycle M in a new system of coordinates $T_x = G_{\xi_+\circ\phi(x)} \circ M_x \circ G_{\xi_+(x)}^{-1}$. Then $T(n,x)(z) = a_n(x)z - b_n(x)$, the sequence $\left(\frac{1}{2n}\ln a_n(x)\right)_{n>0}$ converges to some γ and $|\gamma| = \lambda_+$. If $\gamma < 0$, then

$$\lim_{n\to+\infty} T(n, \phi^{-n}(x))(i) = \sum_{k\ge 0} a_k \circ \phi^{-k}(x) b \circ \phi^{-k-1}(x)$$

exists a.e. and contradicts the convergence of $\Big(M(n,\phi^{-n}(x))(0)\Big)_{n\ge 0}$ to $\xi_+(x)$. Therefore $\gamma > 0$, $T(-n, \phi^n(x))(z) = z/a_n(x) + \sum_{k=0}^{n-1} b \circ \phi^k(x)/a_{k+1}(x)$ and $\lim_{n\to+\infty} T(-n,\phi^n(x))(i) = \sum_{k\ge 0} b\circ\phi^k(x)/a_{k+1}(x)$ exists a.e. on $\{\lambda_+ > 0\}$. $\square$

The log-integrability of $K_{\phi(x)}K_x^{-1}$ and the minimality of $\|K\|$ among all conjugating matrices will follow from the next two lemmas.

Lemma 2.7. *If* $(K, L, N) \in \mathcal{M}ob^+(\mathbb{D})$, $K^{-1}(0)$ *and* $L^{-1}(0)$ *are pure imaginary,* N *fixes* ± 1 *and* $M = LNK^{-1}$, *then* $d_{\mathbb{D}}(0, N(0)) \leq d_{\mathbb{D}}(0, M(0))$ *and* $d_{\mathbb{D}}(0, LK^{-1}(0)) \leq d_{\mathbb{D}}(0, M(0))$. *Moreover,* $d_{\mathbb{D}}(0, N(0)) = d_{\mathbb{D}}(0, M(0))$ *if and only if either* $N = \mathrm{Id}$ *and* $LK^{-1}(0) = 0$ *or* $K(0) = L(0) = 0$.

Proof. The assumptions on K, L, N imply

$$K(z) = e^{2i\gamma}\frac{z + ik}{1 - zik}, \qquad L(z) = e^{2i\lambda}\frac{z + il}{1 - zil}, \qquad N(z) = \frac{z + n}{1 + zn},$$

where k, l, n belong to $]-1, 1[$. Then

$$d_{\mathbb{D}}(0, M(0)) = d_{\mathbb{D}}(L^{-1}(0), NK^{-1}(0)) = d_{\mathbb{D}}(-il, (n - ik)/(1 - nik)).$$

If we define m by $d_{\mathbb{D}}(0, M(0)) = \frac{1}{2}\ln\Big((1 + m)\Big/(1 - m)\Big)$, we obtain

$$m^2 = \frac{n^2(1 + kl)^2 + (l - k)^2}{n^2(k + l)^2 + (1 - kl)^2}.$$

We first get $m^2 \geq n^2$, which is equivalent to $d_{\mathbb{D}}(0, M(0)) \geq d_{\mathbb{D}}(0, N(0))$. We also obtain $m^2 \geq (l - k)^2/(1 - kl)^2$ since the righthand side is increasing with respect to n, which is equivalent to $d_{\mathbb{D}}(0, M(0)) \geq d_{\mathbb{D}}(L^{-1}(0), K^{-1}(0))$. Moreover, $m^2 = n^2$ if and only if $(k + l)^2 n^2 + (k - l)^2 = 0$ if and only if $n = 0$ and $k = l$ or $k = l = 0$. □

Lemma 2.8. *Let* ξ_+, ξ_- *be two distinct points of* $\mathbb{D}$. *There exists a unique* $K \in \mathcal{M}ob^+(\mathbb{D})$ *such that* $K(1) = \xi_+$, $K(-1) = \xi_-$ *and* $K^{-1}(0) \in i\mathbb{R}$. *If* $\tilde{K}$ *is another Möbius transformation satisfying* $\tilde{K}(1) = \xi_+$ *and* $\tilde{K}(-1) = \xi_-$ *then* $d_{\mathbb{D}}(0, \tilde{K}(0)) \geq d_{\mathbb{D}}(0, K(0))$ *with equality if and only if* $\tilde{K} = K$.

Proof. (i) *Existence of* K. By hypothesis, $K(z) = e^{i\gamma}(z + ik)/(z - ik)$. If we denote by $\xi_+ = e^{i\alpha_+}$, $\xi_- = e^{i\alpha_-}$ the existence of K is equivalent to

$$(\alpha_+ - \gamma) + (\alpha_- - \gamma) = \pi \quad \text{ and } \quad \alpha_+ - \gamma \in]-\pi/2, \pi/2[\pmod{2\pi}.$$

The solutions are given by $\alpha_+ - \gamma = (\alpha_+ - \alpha_-)/2 \pm \pi/2$. Since $\xi_+ \neq \xi_-$, we have $\alpha_+ - \gamma \neq \pm\pi/2$ modulo 2π. Only one of these solutions belongs to $]-\pi/2, \pi/2[$, and we determine k by the equation $e^{i(\alpha_+ - \gamma)} = (1 + ik)/(1 - ik)$.

(ii) *Extremality of* K. If $\tilde{K}$ sends 1 to ξ_+ and -1 to ξ_-, we show that $\tilde{K}(0)$ belongs to the geodesic Γ joining ξ_+ and ξ_- and that $K(0)$ realizes the minimum of $d_{\mathbb{D}}(0,\Gamma)$. Indeed, $\Gamma = K([-1,1])$, $\tilde{K}(0) = K(K^{-1}\circ\tilde{K}(0))$ and $K^{-1}\circ\tilde{K}$ fixes 1 and -1. Therefore, $K^{-1}\circ\tilde{K}([-1,1]) = [-1,1]$ and $\tilde{K}(0)\in\Gamma$. Moreover

$$d_{\mathbb{D}}(0,K(t)) = d_{\mathbb{D}}\Big(0,\frac{t+ik}{1-tik}\Big) = \frac{1}{2}\ln\Big(\frac{1+d}{1-d}\Big) \quad \text{with } d^2 = \frac{t^2+k^2}{1+t^2k^2}.$$

The minimum of $d_{\mathbb{D}}(0,K(t))$ is achieved if and only if $t=0$. □

Proof of the Main Theorem (part i). Since $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathcal{M}ob^+(\mathbb{D})$, we may assume that M is a cocycle taking values in $\mathcal{M}ob^+(\mathbb{D})$. By hypothesis, $\lambda_+(x)>0$ on an invariant set of positive measure and Proposition 2.5 implies the existence of two M-invariant functions ξ_+, ξ_-:

$$\xi_+(x) = \lim_{n\to+\infty} M(n,\phi^{-n}(x))(0) \quad \text{and} \quad \xi_-(x) = \lim_{n\to+\infty} M(-n,\phi^{n}(x))(0)$$

which satisfy $M_x(\xi_\pm) = \xi_\pm\circ\phi(x)$ and $\xi_+(x)\neq\xi_-(x)$ a.e. By Lemma 2.8, there exists $K_x\in\mathcal{M}ob^+(\mathbb{D})$ (depending measurably in x) such that $K_x(1)=\xi_+(x)$, $K_x(-1)=\xi_-(x)$ and $K_x^{-1}(0)\in i\mathbb{R}$. Then $N_x = K_{\phi(x)}^{-1}M_xK_x$ fixes ±1 each. As a matrix $N_x = \mathrm{diag}(u_x,u_x^{-1})$, $\ln\|N_x\| = |\ln|u_x|| = d_{\mathbb{D}}(0,N_x(0))$ and by Lemma 2.7

$$d_{\mathbb{D}}(0,N_x(0)) \le d_{\mathbb{D}}(0,M_x(0)),$$

$$|d_{\mathbb{D}}(0,K_x(0)) - d_{\mathbb{D}}(0,K_{\phi(x)}(0))| \le d_{\mathbb{D}}(0,M_x(0)).$$

If $\tilde{K}: X\to\mathcal{M}ob^+(\mathbb{D})$ is another conjugating matrix and $\tilde{N}_x = \tilde{K}_{\phi(x)}^{-1}M_x\tilde{K}_x$ leaves 1 and -1 invariant, then $\tilde{\xi}_+(x) = \tilde{K}_x(1)$ and $\tilde{\xi}_-(x) = \tilde{K}_x(-1)$ are invariant with respect to M and $\{\xi_+(x),\xi_-(x)\} = \{\tilde{\xi}_+(x),\tilde{\xi}_-(x)\}$ (this statement will be proved at the end). Let $A = \{x\in X_H : \tilde{K}_x(1)\neq K_x(1)\}$ and R the rotation $R_x(z) = e^{i\mathbf{1}_A(x)}z$. Then $\tilde{K}_xR_x$ and K_x send both ±1 to the same points. There thus exists $D_x\in\mathcal{M}ob^+(\mathbb{D})$ fixing ±1 such that $\tilde{K}_x = K_xR_xD_x$. By Lemma 2.8, $d_{\mathbb{D}}(0,\tilde{K}_xR_x(0)) \ge d_{\mathbb{D}}(0,K_x(0))$ with equality if and only if $\tilde{K}_x = K_xR_x$. Since $N_x = R_{\phi(x)}D_{\phi(x)}\tilde{N}_xD_x^{-1}R_x^{-1}$, necessarily $R_x = R_{\phi(x)}$ and A is invariant.

To prove the claim we show that any M-invariant $\xi: X\to\partial\mathbb{D}$ satisfies $\xi(x)\in\{\xi_(x),\xi_+(x)\}$. Suppose that $\xi(x)\neq\xi_+(x)$ on an invariant set of positive measure. After conjugation by $G_{\xi_+(x)}$, M_x becomes cohomologous to the cocycle $T_x = G_{\xi_+\circ\phi(x)}M_xG_{\xi_+(x)}^{-1}$, where $T_x(z) = a(x)z - b(x)$ and the sequence $\left(\frac{1}{n}\sum_{k=0}^{n-1}\ln a\circ\phi^k(x)\right)_{n>0}$ converges to $\lambda_+(x)$. Let $\zeta(x) = G_x(\xi(x))$ and $\zeta_-(x) = G_x(\xi_-(x))$. From the equality

$$T(-n,\phi^n(x))(z) - T(-n,\phi^n(x))(0) = z/a(n,x),$$

where $a(n,x) = \prod_{k=0}^{n-1} a\circ\phi^k(x)$, we obtain that $\Big(\zeta\circ\phi^n(x)/a(n,x)\Big)_{n>0}$ converges a.e. to $\zeta(x)-\zeta_-(x)$. Since the convergence also holds in measure and $(a(n,x))_{n>0}$

converges to $+\infty$, we obtain finally $\zeta(x) = \zeta_-(x)$ a.e. and therefore $\xi(x) = \xi_-(x)$ a.e. □

We end this section by proving that we cannot hope to obtain in the class of log-integrable cocycles additional properties on the conjugating matrix K.

Proof of Proposition 1.7. Let K be the corresponding Möbius transformation. By hypothesis, $K_x^{-1}(0)$ is pure imaginary and $d_{\mathbb{D}}(K_{\phi(x)}^{-1}(0), K_x^{-1}(0))$ is integrable. We show that there exists a Borel function $n : X \to]0,1[$ such that, if $N_x(z) = (z+n_x)/(1+zn_x)$ and $M_x = K_{\phi(x)} N_x K_x^{-1}$, then $d_{\mathbb{D}}(0, M_x(0))$ and $d_{\mathbb{D}}(0, N_x(0))$ are integrable. (We recall that the matrix cocycle is given by $N_x = \operatorname{diag}(u_x, u_x^{-1})$ where $u_x = ((1+n_x)\big/(1-n_x))^{1/2}$ and the corresponding top Lyapunov exponent is given by the limit of $\frac{1}{n}\sum_{k=0}^{n-1} \ln u \circ \phi^k$, which is positive a.e.) As in the proof of Lemma 2.7, we introduce the notation $K_x(z) = e^{2i\gamma_x}(z+ik_x)/(1-zik_x)$,

$$d_{\mathbb{D}}(0, M_x(0)) = \frac{1}{2}\ln\frac{1+m_x}{1-m_x} \quad \text{and} \quad d_{\mathbb{D}}(0, K_{\phi(x)}K_x^{-1}(0)) = \frac{1}{2}\ln\frac{1+\tilde{m}_x}{1-\tilde{m}_x}.$$

Since $d_{\mathbb{D}}(0, M_x(0)) \le \ln 2 - \frac{1}{2}\ln(1-m_x^2)$, it is enough to show that $-\ln(1-m_x^2)$ is integrable. One can show that $-\ln(1-m_x^2)$ is equal to $F(x, n_x)$, where

$$F(x,n) = \ln\left(\frac{n^2(k_x + k_{\phi(x)})^2 + (1 - k_x k_{\phi(x)})^2}{(1-n^2)(1-k_x^2)(1-k_{\phi(x)}^2)}\right).$$

When $n_x = 0$, $M_x = K_{\phi(x)}K_x^{-1}$ and $F(x,0) = -\ln(1-\tilde{m}_x^2)$ is integrable by assumption. We choose therefore any Borel function $n : X \to]0,1[$ such that $F(x, n_x)$ and $-\ln(1-n_x^2)$ are integrable. □

2.3 Proof of the Main Theorem when $\lambda_+ = 0 = \lambda_-$ In the case of positive Lyapunov exponents, the reduction of a log-integrable cocycle $M : X \to \mathcal{M}ob^+(\mathbb{D})$ to a hyperbolic cocycle N is a consequence of the existence of two M-invariant functions $\xi_\pm : X \to \partial\mathbb{D}$. In the same manner, the reduction of M to an elliptic cocycle is equivalent to finding an M-invariant function $\xi : X \to \mathbb{D}$; and *the reduction of M to an upper triangular matrix will follow from the existence of* just one M-invariant $\xi : X \to \partial\mathbb{D}$.

In the case of null Lyapunov exponents, these M-invariant functions cannot be obtained as limits. We first solve a weak equation $(M_x)_*(\nu_x) = \nu_{\phi(x)}$, where the unknown $\nu : X \to \mathcal{M}_1(\partial\mathbb{D})$ is a measurable function taking values in the space of probability measures on $\partial\mathbb{D}$ and $M_*(\nu)$ denotes the forward image by $M : \partial\mathbb{D} \to \partial\mathbb{D}$ of a probability measure ν. This first step is standard, the main technical part is the Dunford–Pettis theorem. We define precisely in Appendix B the weak topologies used in this theorem and give a condensed proof. The second step is new and is a substitute for Furstenberg's martingale argument. We use the Douady–Earle theorem [6], which associates a conformal barycenter $\mathrm{bar}(\nu) \in \mathbb{D}$ to any probability measure $\nu \in \mathcal{M}_1(\partial\mathbb{D})$ whose support does not contain masses greater than or equal to $\frac{1}{2}$. The main property satisfied by the conformal barycenter is that it is preserved by Möbius transformations of $\mathbb{D}$: if $M \in \mathcal{M}ob^+(\mathbb{D})$ then $M(\mathrm{bar}(\nu)) = \mathrm{bar}(M_*\nu)$. In Appendix A, we give a different proof of this theorem not using degree theory but rather using the convexity of the Busemann function. A more general statement which extends the Douady–Earle theorem can be found in [3].

We start by proving the existence of a weak solution $\nu : X \to \mathcal{M}_1(\partial\mathbb{D})$ of the equation $(M_x)_*(\nu_x) = \nu_{\phi(x)}$. We identify $\mathcal{M}_1(\partial\mathbb{D})$ and the set of positive linear forms ν on $E = \mathcal{C}_0(\partial\mathbb{D}, \mathbb{R})$ satisfying $\nu(\mathbf{1}) = \mathbf{1}$. We use the notations of Appendix B.

Proposition 2.9. *Let (X, m, ϕ) be an abstract dynamical system. For any measurable cocycle $M : X \to \mathcal{M}ob^+(\mathbb{D})$, there exists $\nu : X \to \mathcal{M}_1(\partial\mathbb{D})$ measurable such that $(M_x)_*(\nu_x) = \nu_{\phi(x)}$ a.e.*

Proof. We first choose a reference measure, the Lebesgue measure Leb and define a sequence of probability measures $\nu^n : X \to \mathcal{M}_1(\partial\mathbb{D})$ by

$$\nu_x^n = \frac{1}{n}\sum_{k=0}^{n-1} M(k, \phi^{-k}(x))_*(Leb).$$

By Corollary B.5, the unit ball $B_{E'}^{\infty}$ is compact metrizable. One can find $\nu : X \to \mathcal{M}_1(\partial\mathbb{D})$ measurable and a subsequence $(n_k)_{k>0}$ such that

$$\int \varphi(x)\nu_x^{n_k}(\psi)\,dm(x) \stackrel{k\to+\infty}{\longrightarrow} \int \varphi(x)\nu_x(\psi)\,dm(x)$$

for all $\varphi \in L^1_{\mathbb{R}}$, $\psi \in E$. By Lemma B.7, the weak convergence extends to essentially bounded measurable $\psi : X \to E$ and in particular to $\psi_x = \psi \circ M_x$,

$$\int \varphi(x)\nu_x^{n_k}(\psi \circ M_x)\,dm(x) \stackrel{k\to+\infty}{\longrightarrow} \int \varphi(x)\nu_x(\psi \circ M_x)\,dm(x).$$

By construction

$$(M_x)_*(\nu_x^n) = \frac{n+1}{n}\nu_{\phi(x)} + \frac{1}{n}[M(n, \phi^{-n+1}(x))_*Leb - Leb]$$

and we get finally $\int \varphi(x)\nu_x(\psi \circ M_x)\,dm(x) = \int \varphi(x)\nu_{\phi(x)}(\psi)\,dm(x)$ for all $\varphi \in L^1_{\mathbb{R}}$ and $\psi \in E$, which is equivalent to $(M_x)_*(\nu_x) = \nu_{\phi(x)}$. $\square$

Before giving the proof of the Main Theorem (parts ii–iv), we collect two lemmas about the conjugating matrix K in the elliptic case:

Lemma 2.10. *If $K, L, N \in \mathcal{M}ob^+(\mathbb{D})$, $K^{-1}(0)$ and $L^{-1}(0)$ are non-positive pure imaginary, N is a rotation and $M = LNK^{-1}$, then the conjugating Möbius tranformations satisfy $d_{\mathbb{D}}(L^{-1}(0), K^{-1}(0)) \leq d_{\mathbb{D}}(0, M(0))$.*

Proof. By hypothesis, we can write

$$K(z) = e^{2i\gamma}\frac{z+ik}{1-zik}, \quad L(z) = e^{2i\lambda}\frac{z+il}{1-zil}, \quad N(z) = e^{2i\omega}z$$

where $k, l \in [0,1[$. Moreover, $d_{\mathbb{D}}(0, M(0)) = d_{\mathbb{D}}(0, LNK^{-1}(0)) = d_{\mathbb{D}}(il, e^{2i\omega}ik)$; and if m is defined by $d_{\mathbb{D}}(0, M(0)) = \frac{1}{2}\ln\Big((1+m)\big/(1-m)\Big)$, we obtain the following relation between k, l, m and ω:

$$m^2 = \frac{(k-l)^2 + 4kl\sin^2\omega}{(1-kl)^2 + 4kl\sin^2\omega}.$$

Since the above function is increasing with respect to $\sin^2\omega$, we obtain

$$m^2 \geq \frac{(k-l)^2}{(1-kl)^2} = \tilde{m}^2 \quad \text{where } \frac{1}{2}\ln\left(\frac{1+\tilde{m}}{1-\tilde{m}}\right) = d_{\mathbb{D}}(0, LK^{-1}(0)).$$

We have just proved that $d_{\mathbb{D}}(L^{-1}(0), K^{-1}(0)) \leq d_{\mathbb{D}}(0, M(0))$. □

Contrary to the hyperbolic case, the conjugating matrix K is not unique in the elliptic case but they all differ from each other by an arbitrary rotation as the following lemma shows.

Lemma 2.11. *If a cocycle $M : X \to \mathrm{SL}(2,\mathbb{R})$ admits two decompositions $M_x = K_{\phi(x)}N_xK_x^{-1}$ and $M_x = \tilde{K}_{\phi(x)}\tilde{N}_x\tilde{K}_x^{-1}$, where N_x and $\tilde{N}_x$ are rotations of angle ω_x and $\tilde{\omega}_x$, and if ω_x is not a coboundary modulo π on any invariant set, then $K_x^{-1}\tilde{K}_x$ is a rotation and ω_x is cohomologous to $\tilde{\omega}_x$ modulo π.*

Proof. The cocycles N and $\tilde{N}$ are related by $N_x = K_{\phi(x)}^{-1}\tilde{K}_{\phi(x)}\tilde{N}_x\tilde{K}_x^{-1}K_x$. In conformal notation, $\xi(x) = K^{-1}\tilde{K}_x(0)$ is N-invariant. In polar coordinates, $\xi(x) = \rho_x e^{i2\gamma_x}$ and its norm is constant along the trajectories. On the set $\{\rho > 0\}$, $\omega = \gamma\circ\phi - \gamma$ modulo π, where $N_x(z) = e^{i2\omega_x}z$. On the set $\{\rho = 0\}$, $K_x^{-1}\tilde{K}_x(0) = 0$, there exists $\alpha : X \to \mathbb{R}$ such that $K_x^{-1}\tilde{K}_x(z) = e^{i2\alpha_x}z$ and $\omega = \tilde{\omega} + \alpha\circ\phi - \alpha$. By hypothesis, ω cannot be cohomologous to 0 modulo π on any invariant set, and $\rho = 0$ a.e. □

We also use in the main proof the following lemma on real coboundaries.

Lemma 2.12. *If $c : X \to \mathbb{R}$ is a measurable function such that $c \circ \phi - c$ is an integer a.e., then there exists a ϕ-invariant function $c^* : X \to \mathbb{R}$ such that $c(x) - c^*(x) \in \mathbb{Z}$ a.e.*

Proof. For each $x \in X$, we denote by $c^*(x)$ the unique real in $[0, 1[$ such that $c(x) - c^*(x) \in \mathbb{Z}$. Then $c \circ \phi - c \in \mathbb{Z}$ a.e. if and only if $c^* \circ \phi = c^*$ a.e. □

Proof of the Main Theorem (parts ii–iv). We do not assume in this section that M is log-integrable. Let X_P be an invariant set of maximal measure on which there exists an M-invariant function $\xi : X_P \to \partial\mathbb{D}$. If $\xi(x) = e^{i2\gamma_x}$ and if we choose $K_x(z) = e^{i2\gamma_x} z$ then $N_x = K_{\phi(x)}^{-1} M_x K_x$ leaves 1 invariant. In matrix notation,

$$N_x = \begin{bmatrix} v_x & w_x \\ 0 & v_x^{-1} \end{bmatrix}, \qquad K_x = R(\gamma_x),$$

$\|N_x\| = \|M_x\|$ and $\|K_x\| = 1$.

Let $X_E \subset X \setminus X_P$ be an invariant set of maximal measure on which there exists an M-invariant function $\xi : X_E \to \mathbb{D}$. For each x, we choose the unique $K_x \in \mathcal{M}ob^+(\mathbb{D})$ which satisfies $K_x(0) = \xi(x)$ and $K_x^{-1}(0) \in i\mathbb{R}^-$. After conjugation by K, $N_x = K_{\phi(x)}^{-1} M_x K_x$ becomes a rotation ($N_x(0) = 0$); and by Lemma 2.10, $\|K_{\phi(x)} K_x^{-1}\| \le \|M_x\|$, $\|N_x\| = 1 \le \|M_x\|$. In matrix notation,

$$N_x = R(\omega_x) \qquad K_x = R\Big(\gamma_x + \frac{\pi}{4}\Big) \begin{bmatrix} u_x & 0 \\ 0 & u_x^{-1} \end{bmatrix} R\Big(-\frac{\pi}{4}\Big)$$

for some $\omega : X_E \to \mathbb{R}$, $u : X_E \to \mathbb{R}^+$ measurable. If the angle ω is a coboundary on an invariant set of positive measure, all points in $\mathbb{D}$ would be M-invariant on that set, which would contradict the definition of X_P.

On $X_{WH} = X \setminus (X_P \cup X_E)$, Proposition 2.9 enables us to solve the equation $(M_x)_*(\nu_x) = \nu_{\phi(x)}$ where $\nu : X_{WH} \to \mathcal{M}_1(\partial\mathbb{D})$ is the unknown. Three cases may occur, depending on the cardinality of atoms of ν_x of mass not smaller than $\frac{1}{2}$. Let

$$c_\nu(x) = \operatorname{card}\{t \in \partial\mathbb{D} : \ \nu(t) \ge 1/2\}.$$

Since c_ν is an invariant function, X_{WH} may be partitioned into three invariant sets $\{c_\nu = i\}$ for $i = 0, 1, 2$.

On the set $\{c_\nu = 0\}$, the Douady–Earle Theorem A.11 implies that each ν_x admits a unique conformal barycenter $\xi(x) = \operatorname{bar}(\nu_x) \in \mathbb{D}$ measurable with respect to x (Proposition A.12). By conformality $M_x(\xi(x)) = \xi \circ \phi(x)$ on $\{c_\nu = 0\}$. By the maximality of X_E, the set $\{c_\nu = 0\}$ necessarily has measure 0.

On the set $\{c_\nu = 1\}$, each ν_x admits a unique atom $\xi(x)$ of mass not smaller than $\frac{1}{2}$ which is measurable with respect to x (Lemma A.14). The weak equation implies $M_x(\xi(x)) = \xi \circ \phi(x)$ a.e. on $\{c_\nu = 1\}$. By the maximality of X_P, the set $\{c_\nu = 1\}$ also has measure 0.

On the set $\{c_\nu = 2\}$, ν_x consists of two Dirac measures of mass $\frac{1}{2}$ each $\nu_x = \frac{1}{2}(\delta_{\zeta(x)} + \delta_{\eta(x)})$, where $\zeta(x)$ and $\eta(x)$ depend measurably on x. Since ν is M-invariant, $\{\zeta(x), \eta(x)\}$ is globally preserved:

$$M_x\{\zeta(x), \eta(x)\} = \{\zeta \circ \phi(x), \eta \circ \phi(x)\}.$$

By Lemma 2.8, there exists a unique $K_x \in \mathcal{M}ob^+(\mathbb{D})$ such that $K_x(1) = \zeta(x)$, $K_x(-1) = \eta(x)$ and $K_x^{-1}(0) \in i\mathbb{R}$. The cocycle $N_x = K_{\phi(x)}^{-1} M_x K_x$ preserves or permutes the points $\{\pm 1\}$ and in matrix notation

$$N_x = R\Big(\mathbf{1}_B(x)\frac{\pi}{2}\Big)\begin{bmatrix} v_x & 0 \\ 0 & v_x^{-1} \end{bmatrix}, \quad K_x = R\Big(\gamma_x + \frac{\pi}{4}\Big)\begin{bmatrix} u_x & 0 \\ 0 & u_x^{-1} \end{bmatrix} R\Big(-\frac{\pi}{4}\Big),$$

where $B = \{x \in \{c_\nu = 2\} : M_x(\zeta(x)) = \eta \circ \phi(x)\}$. The estimates of Lemma 2.7 give $\|N_x\| \le \|M_x\|$ and $\|K_{\phi(x)} K_x^{-1}\| \le \|M_x\|$. Let us show that $\mathbf{1}_B$ cannot be cohomologous to 0 on an invariant set of positive measure. Otherwise, there would exist $\alpha : \{c_\nu = 2\} \to \mathbb{R}$ satisfying $\mathbf{1}_B = \alpha \circ \phi - \alpha$ modulo 2, which could be chosen by Lemma 2.12 such that $\alpha(x) \in \mathbb{Z}$ a.e. Then

$$R\Big(\mathbf{1}_B(x)\frac{\pi}{2}\Big)\begin{bmatrix} v_x & 0 \\ 0 & v_x^{-1} \end{bmatrix} = R\Big(\alpha_{\phi(x)}\frac{\pi}{2}\Big)\begin{bmatrix} \tilde{v}_x & 0 \\ 0 & \tilde{v}_x^{-1} \end{bmatrix} R\Big(-\alpha_x\frac{\pi}{2}\Big),$$

where $\tilde{v}(x) = v(x)$ if and only if $\alpha(x) \in 2\mathbb{Z}$ and $\tilde{v}(x) = v(x)^{-1}$ otherwise. In matrix notation, $M_x = \tilde{K}_{\phi(x)} \tilde{N}_x \tilde{K}_x^{-1}$,

$$\tilde{N}_x = \begin{bmatrix} \tilde{v}_x & 0 \\ 0 & \tilde{v}_x^{-1} \end{bmatrix}, \quad \tilde{K}_x = R\Big(\gamma_x + \alpha_x\frac{\pi}{2} + \frac{\pi}{4}\Big)\begin{bmatrix} \tilde{u}_x & 0 \\ 0 & \tilde{u}_x^{-1} \end{bmatrix} R\Big(-\frac{\pi}{4}\Big),$$

where $\tilde{u}(x) = u(x)$ if and only if $\alpha(x) \in 2\mathbb{Z}$ and $\tilde{u}(x) = u(x)^{-1}$ otherwise. The cocycle M would then admit two invariant lines, which contradicts the maximality of X_P.

Let us now prove the minimality of the norm of the conjugating matrix K. If the cocycle M admits on X_E another decomposition $M_x = \tilde{K}_{\phi(x)} \tilde{N}_x \tilde{K}_x^{-1}$, where $\tilde{N}_x$ is a rotation of angle $\tilde{\omega}_x$ not cohomologous to 0 modulo π, then $K_x^{-1}\tilde{K}_x$ is a rotation by Lemma 2.11 and satisfies $\|\tilde{K}_x\| = \|K_x\|$ a.e. In the same manner, if the cocycle admits on X_{WH} a decomposition of the form $M_x = \tilde{K}_{\phi(x)} \tilde{N}_x \tilde{K}_x^{-1}$, where $\tilde{N}_x = R(\mathbf{1}_{\tilde{B}}(x)\frac{\pi}{2}) \operatorname{diag}(\tilde{u}_x, \tilde{u}_x^{-1})$, then $\tilde{\zeta}(x) = \tilde{K}_x(1)$ and $\tilde{\eta}(x) = \tilde{K}_x(-1)$ are globally

M-invariant and necessarily $\{\tilde{\zeta}(x), \tilde{\eta}(x)\}$ coincides with $\{\zeta(x), \eta(x)\}$. (Otherwise, M would admit 3 or 4 globally invariant points on $\partial\mathbb{D}$; and putting equal masses on these points, we would obtain by the Douady–Earle theorem another M-invariant point in $\mathbb{D}$, which would contradict the maximality of X_E.) Let

$$\bar{B} = \{x \in X_{WH} : \tilde{\zeta}_x = \eta_x\};$$

then $\bar{K}_x = \tilde{K}_x R(\mathbf{1}_{\bar{B}}(x)\frac{\pi}{2})$ satisfies $\bar{K}_x(1) = \zeta(x)$, $\bar{K}_x(-1) = \eta(x)$ and by Lemma 2.8, $\|\bar{K}_x\| = \|\tilde{K}_x\| \geq \|K_x\|$. As in the proof in Oseledets' case, there exists $D : X \to \mathrm{SL}(2,\mathbb{R})$ diagonal such that $\bar{K}_x = K_x R(\mathbf{1}_{\bar{B}}(x)\frac{\pi}{2})D_x$. □

We have actually given a partial proof of the Main Theorem: we have shown how to conjugate a general cocycle to a cocycle which is either parabolic, elliptic or weak-hyperbolic. We have not yet proved the properties of recurrence of these basic cocycles, and we postpone the complete proof of the Main Theorem to Section 3.

We prefer to close this section by giving the proof of Proposition 1.8. We first establish an abstract lemma for additive real cocycles.

Lemma 2.13. *Let (X, m, ϕ) be an ergodic (not necessarily invertible) dynamical system and $\delta : X \to \mathbb{R}^+$ a non-negative function such that $\int \delta\, dm > 0$. Then any integrable $\omega : X \to \mathbb{R}$ satisfying $0 < \int \omega\, dm < \int \delta\, dm$ is cohomologous to some integrable $\tilde{\omega} : X \to \mathbb{R}$ satisfying $0 \leq \tilde{\omega}(x) \leq \delta(x)$ a.e.*

Proof. *Part one.* We first find a conjugating function $c_1 : X \to \mathbb{R}$ such that $\omega + c_1 \circ \phi - c_1 \geq 0$ a.e. Let $c_1 = \inf_{k\geq 0} S_k\omega$ (where $S_k\omega = \sum_{i=0}^{k-1} \omega \circ \phi^i$). Since $(\frac{1}{k}S_k\omega)_{k>0}$ converges to $\int \omega\, dm > 0$, c_1 is non-positive and finite a.e. Moreover, $S_{k+1}\omega = S_k\omega \circ \phi + \omega$ for all $k \geq 0$ so by taking the infimum on both sides, we obtain $c_1 \leq c_1 \circ \phi + \omega$ a.e.

Part two. Let $\omega_1 = \omega + c_1 \circ \phi - c_1$. Since $(c_1 - c_1 \circ \phi)^+$ is integrable, $c_1 - c_1 \circ \phi$ is integrable too. We now find $c_2 : X \to \mathbb{R}$ such that $\tilde{\omega} = \omega_1 + c_2 \circ \phi - c_2$ is the desired cocycle. Let $c_2 = \sup_{k\geq 0} S_k(\omega_1 - \delta)$. As before, c_2 is finite since $\int(\omega_1 - \delta)dm = \int(\omega - \delta)dm < 0$, non-negative and

$$c_2 \geq \sup_{k\geq 1} S_k(\omega_1 - \delta) = c_2 \circ \phi + \omega_1 - \delta \quad \text{a.e.}$$

We first observe that $\tilde{\omega} \leq \delta$. Either for all $k \geq 1$, $S_k(\omega_1 - \delta) < 0$, in which case $c_2 = 0$ and $\tilde{\omega} = \omega_1 + c_2 \circ \phi \geq \omega_1 \geq 0$, or there exists $k \geq 1$ such that $S_k(\omega_1 - \delta) \geq 0$, in which case $c_2 = c_2 \circ \phi + \omega_1 - \delta$ and $\tilde{\omega} = \delta \geq 0$. In both cases $\tilde{\omega} \geq 0$ a.e. □

Proof of Proposition 1.8. In conformal notation, the conjugating matrix $K_x(z) = e^{i2\gamma_x}(z + ik_x)/(1 - zik_x)$ and $k_x > 0$ if and only if $|u_x| > 1$. We may assume that $k_x > 0$ a.e. by choosing $\alpha : X \to \{0, \pi/2\}$ so that the new conjugating matrix $\tilde{K}_x = R(\alpha_{\phi(x)})K_x R(-\alpha_x)$ satisfies

$$\tilde{K}_x(z) = e^{i2\tilde{\gamma}_x}(z + i|k_x|)/(1 - zi|k_x|).$$

We choose $\alpha(x) = 0$ if and only if $k_x \geq 0$ and $\alpha(x) = \pi/2$ otherwise. By Lemma 2.10, for any $\tilde{\omega} : X \to \mathbb{R}$ the cocycle $\tilde{M}_x = \tilde{K}_{\phi(x)}R(\tilde{\omega}_x)\tilde{K}_x^{-1}$ satisfies

$$\ln \|\tilde{M}_x\| = \frac{1}{2}\ln\left(\frac{1 + m(x, \tilde{\omega}_x)}{1 - m(x, \tilde{\omega}_x)}\right) = F(x, \tilde{\omega}_x),$$

$$m^2(x, \omega) = \frac{(|k_x| - |k_{\phi(x)}|)^2 + 4|k_x k_{\phi(x)}| \sin^2 \omega}{(1 - |k_x k_{\phi(x)}|)^2 + 4|k_x k_{\phi(x)}| \sin^2 \omega}.$$

Set $\omega = 0$; then $F(x, 0) = \ln \|\tilde{K}_{\phi(x)}\tilde{K}_x^{-1}\| = \Big| |\ln |u \circ \phi(x)|| - |\ln |u(x)|| \Big|$ is integrable by hypothesis. Since $F(x, \omega)$ is increasing with respect to $\omega \in [0, \pi/2]$ for x fixed, we construct $\delta : X \to]0, \pi/2[$ such that $F(x, \delta(x))$ is integrable too. From the theory of weak orbit equivalence (see [17]), one can find a cocycle $\bar{\omega} : X \to [0, \pi[$ not cohomologous to 0 modulo π. For any integer $N \geq 1$, $\frac{1}{N}\bar{\omega}$ is not cohomologous to 0 modulo π; and we may therefore assume that $\bar{\omega}$ satisfies in addition $0 < \int \bar{\omega}\, dm < \int \delta\, dm$. By the previous lemma, $\bar{\omega}$ is cohomologous (in $\mathbb{R}$) to some $\tilde{\omega}$ satisfying $\tilde{\omega}(x) \in]0, \delta(x)[$ a.e. Finally, let $\omega = \tilde{\omega} + \alpha \circ \phi - \alpha$; then $M_x = R(-\alpha_{\phi(x)})\tilde{M}_x R(\alpha_x)$ is log-integrable and ω is not cohomologous to 0 modulo π. □

2.4 An application to Wojtkowski's cone theory In this section we give a geometric proof of Wojtkowski's estimate (Proposition 1.11) on random products of two-by-two matrices with non-negative entries. For such products $M(n, x)$, the top Lyapunov exponent is positive, and Wojtkowski's estimate [21] gives a lower bound of that exponent by a computable formula taking into account only M_x itself and not its iterates. Technically, Wojtkowski exhibits a superadditive cocycle and can apply Birkhoff's ergodic theorem. For any $M \in \mathrm{SL}(2, \mathbb{R})$ with non-negative entries we define a real number $\rho(M)$ as follows:

$$\text{if} \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad \rho(M) = \ln(\sqrt{ad} + \sqrt{bc}).$$

Lemma 2.14. *For any $M, N \in \mathrm{SL}(2, \mathbb{R})$, $\rho(M)$ is a positive real number, $\ln \|M\| \geq \rho(M)$ and $\rho(MN) \geq \rho(M) + \rho(N)$.*

Using Birkhoff's ergodic lemma, the proof of Proposition 1.11 is a direct consequence of Lemma 2.14. Our purpose in this section is to give a geometric proof of that lemma. Let us first recall the definition of the cross ratio of four points $(z_1, \dots, z_4)$ of the plane:

$$[z_1, z_2, z_3, z_4] = \frac{z_3 - z_1}{z_2 - z_1} \frac{z_2 - z_4}{z_3 - z_4}.$$

We need the following estimate on cross-ratios:

Lemma 2.15. *For all* $\alpha, \beta \in [0, \pi/4]$, $[1, e^{i2\alpha}, e^{i(\pi - 2\beta)}), -1] = (\tan\alpha \tan\beta)^{-1}$.

A matrix preserves the cone $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ if it has non-negative entries. Such a matrix acts on half lines of $\mathcal{C}$ identified to the first quadrant of $\partial\mathbb{D}$. If we conjugate by $t \in \partial\mathbb{D} \mapsto t^2 \in \partial\mathbb{D}$, the corresponding Möbius transformation of the disk sends the upper half-space $\mathbb{H}$ into itself.

Lemma 2.16. *If* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$ *has non-negative entries then*

$$d_{\mathbb{D}}(\mathbb{R}, M(\mathbb{R})) = \tfrac{1}{2} \ln\left((\sqrt{\zeta} + 1)\big/(\sqrt{\zeta} - 1)\right) = \rho(M),$$

where $\zeta = [1, M(1), M(-1), -1]$ *and* $d_{\mathbb{D}}(\mathbb{R}, M(\mathbb{R}))$ *denotes the hyperbolic distance between the two geodesics* $\mathbb{R}$ *and* $M(\mathbb{R})$.

Proof. (i) *The first equality.* Since the cross-ratio and the hyperbolic distance are invariant with respect to a Möbius tranformation, we may assume that $M(1)$ and $M(-1)$ are symmetric with respect to the imaginary axis. Let $M(1) = e^{i2\theta}$, $M(-1) = e^{i(\pi - 2\theta)}$. The cross-ratio is then given according to the previous lemma by $[1, M(1), M(-1), -1] = (\tan\theta)^{-2}$. On the other hand, the geodesic passing through $M(1)$ and $M(-1)$ intersects the imaginary axis at $\mu = i\tan\theta$. The hyperbolic distance between $\mathbb{R}$ and $M(\mathbb{R})$ is then given by

$$d_{\mathbb{D}}(\mathbb{R}, M(\mathbb{R})) = \tfrac{1}{2} \ln\left((1 + |\mu|)\big/(1 - |\mu|)\right) = \tfrac{1}{2} \ln\left((\sqrt{\zeta} + 1)\big/(\sqrt{\zeta} - 1)\right).$$

(ii) *The second equality.* If we consider M as an element of $\mathrm{SL}(2, \mathbb{R})$ and denote by α (resp. β) the angle of the horizontal axis (resp. the vertical axis) with its image, we have $\alpha = c/a$ and $\beta = b/d$. If M is considered as a Möbius transformation, because of the conjugacy $t \in \partial\mathbb{D} \mapsto t^2 \in \partial\mathbb{D}$, we have on the other hand $M(1) = e^{i2\alpha}$ and $M(-1) = e^{i(\pi - 2\beta)}$. The first part shows

$$d_{\mathbb{D}}(\mathbb{R}, M(\mathbb{R})) = \frac{1}{2} \ln \frac{1 + \sqrt{\tan\alpha \tan\beta}}{1 - \sqrt{\tan\alpha \tan\beta}} = \ln(\sqrt{ad} + \sqrt{bc}). \qquad \square$$

Proof of Lemma 2.14. Using conformal notation, by the definition of the distance between two lines, we have $d_{\mathbb{D}}(0, M(0)) \geq d_{\mathbb{D}}(\mathbb{R}, M(\mathbb{R}))$. If N is another Möbius transformation sending the upper half space into itself, the geodesic realizing the minimum of the distance between $\mathbb{R}$ and $MN(\mathbb{R})$ intersects $M(\mathbb{R})$ and we thus obtain

$$\begin{aligned} d_{\mathbb{D}}(\mathbb{R}, MN(\mathbb{R})) &\geq d_{\mathbb{D}}(\mathbb{R}, M(\mathbb{R})) + d_{\mathbb{D}}(M(\mathbb{R}), MN(\mathbb{R})) \\ &= d_{\mathbb{D}}(\mathbb{R}, M(\mathbb{R})) + d_{\mathbb{D}}(\mathbb{R}, N(\mathbb{R})). \end{aligned} \qquad \square$$

2.5 Extension of the Main Theorem to GL$(2,\mathbb{R})$. We denote by $\mathcal{M}ob(\mathbb{D})$ the group of isometries of the hyperbolic disk $\mathbb{D}$. Then $\mathcal{M}ob(\mathbb{D})$ is equal to the disjoint union of $\mathcal{M}ob^+(\mathbb{D})$, the set of isometries preserving orientation and $\mathcal{M}ob^-(\mathbb{D})$ those which reverse orientation. Any map in $\mathcal{M}ob^-(\mathbb{D})$ can be written in a unique way as MI, where $M \in \mathcal{M}ob^+(\mathbb{D})$ and $I(z) = \bar{z}$. In the same way $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathcal{M}ob^+(\mathbb{D})$. We have

Proposition 2.17. $\mathrm{PGL}(2,\mathbb{R})$ *is isomorphic to* $\mathcal{M}ob(\mathbb{D})$.

Proof. For every $M \in \mathrm{GL}(2,\mathbb{R})$, $M(z) = az + b\bar{z}$, we associate a matrix $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$ and its projective action on $\hat{\mathbb{C}}$, $M(z) = (az+b)/(\bar{b}z+\bar{a})$. This defines a group homomorphism onto the group of complex Möbius tranformations preserving S^1. We denote by $J(z) = 1/\bar{z}$ the inversion about S^1; then $JM = MJ$ for all $M \in \mathrm{GL}(2,\mathbb{R})$. If $\det M = |a|^2 - |b|^2 > 0$, $\tilde{M} = M$ belongs to $\mathcal{M}ob^+(\mathbb{D})$; if $\det M < 0$, $\tilde{M} = JM$ belongs to $\mathcal{M}ob^-(\mathbb{D})$. We thus obtain a surjective group homomorphism $M \in \mathrm{GL}(2,\mathbb{R}) \mapsto \tilde{M} \in \mathcal{M}ob(\mathbb{D})$ whose kernel is $\{\lambda \mathrm{Id} : \lambda \in \mathbb{R}^*\}$. $\square$

The fact that Douady–Earle's barycenter is preserved by the whole group $\mathcal{M}ob(\mathbb{D})$ enables us to extend the Main Theorem to $\mathrm{GL}(2,\mathbb{R})$. In order to avoid repeating the same notation, we stress the points of difference.

Main Theorem (revisited) 2.18. *Let $M(n,x)$ be a* $\mathrm{GL}(2,\mathbb{R})$*-valued log-integrable cocycle. Then one can construct a measurable conjugating matrix*

$$K_x = R\Big(\gamma_x + \frac{\pi}{4}\Big) \begin{bmatrix} u_x & 0 \\ 0 & u_x^{-1} \end{bmatrix} R\Big(-\frac{\pi}{4}\Big) \in \mathrm{SL}(2,\mathbb{R})$$

such that $N_x = K_{\phi(x)}^{-1} M_x K_x$ takes one of the four following forms:

(i) $N_x = \det M_x \begin{bmatrix} v_x & 0 \\ 0 & \epsilon_x v_x^{-1} \end{bmatrix}$, $\lambda_+(x) = \mathbb{E}[\ln|v| \mid \mathcal{I}_\phi] > 0$,

(ii) $N_x = \det M_x \begin{bmatrix} v_x & w_x \\ 0 & \epsilon_x v_x^{-1} \end{bmatrix}$, $\lambda_+(x) = \mathbb{E}[\ln|v| \mid \mathcal{I}_\phi] = 0$,

(iii) $N_x = \det M_x \begin{bmatrix} \cos\omega_x & -\epsilon_x \sin\omega_x \\ \sin\omega_x & \epsilon_x \cos\omega_x \end{bmatrix}$,

(iv) $N_x = \det M_x \; R(\boldsymbol{I}_A(x)\frac{\pi}{2}) \begin{bmatrix} v_x & 0 \\ 0 & \epsilon_x v_x^{-1} \end{bmatrix}$, $\boldsymbol{I}_A$ *is not a coboundary modulo* 2,

where $\epsilon_x = \mathrm{sgn}(\det M_x)$. *In all four cases*

$$\|N_x\| \leq \|M_x\| \quad \textit{and} \quad \|K_{\phi(x)} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_x \end{bmatrix} K_x^{-1}\| \leq \|M_x\|.$$

In particular $(\|K_x\|/\|K_{\phi(x)}\|)^{\pm 1} \leq \|M_x\|$.

3 Recurrence properties

We study in this section the recurrence properties of a cocycle $M(n, x)$ in each case (hyperbolic, parabolic, elliptic and weak hyperbolic) and finish the proof of the Main Theorem. We begin with general definitions and properties.

3.1 Cocycles and coboundaries We denote as usual by (X, m, ϕ) an abstract finite or σ-finite dynamical system, not necessarily ergodic or invertible, and by G a locally compact second countable (in particular Polish) group. A G-valued cocycle is a measurable function $a : \mathbb{N} \times G \to G$ satisfying the cocycle property

$$a(m+n, x) = a(n, \phi^m(x))\, a(m, x) \text{ a.e.}$$

For such $\mathbb{N}$-actions, a cocycle is actually given by a unique $\tilde{a} : X \to G$ and the associated cocycle is given by

$$a(n, x) = \tilde{a}_{\phi^{n-1}(x)} \cdots \tilde{a}_{\phi(x)} \tilde{a}_x.$$

In the sequel, we identify the two notations. A cocycle $a(n, x)$ is called a coboundary if there exists $c : X \to G$ measurable such that

$$a(n, x) = c_{\phi^n(x)}\, c_x^{-1},$$

and two cocycles $a, b : X \to G$ are said to be cohomologous if there exists $c : X \to G$ measurable (called the conjugating function) such that

$$a(n, x) = c_{\phi^n(x)}\, b(n, x)\, c_x^{-1}.$$

If B is a Borel set of positive measure and τ_B denotes the first return time to B, we define the induced cocycle a_B: for a.e. $x \in B$

$$\tau_B(x) = \inf\{k \geq 1 : \phi^k(x) \in B\} \quad \text{and} \quad a_B(x) = a(\tau_B(x), x).$$

In order to classify cohomologous cocycles, it is standard to introduce two important notions, recurrence and essential value.

Definition 3.1. (i) A cocycle $a(n,x)$ is said to be recurrent if for every Borel set B of positive measure and every neighborhood V of the neutral element of G there exists $n \geq 1$ such that

$$m\left(B \cap \phi^{-n}B \cap \{x \in B \mid a(n,x) \in V\}\right) > 0.$$

(ii) Infinity is said to be an essential value if for every Borel set B of positive measure and every compact $\mathcal{K}$ of G there exists $n \geq 1$ such that

$$m\left(B \cap \phi^{-n}B \cap \{x \in B \mid a(n,x) \notin \mathcal{K}\}\right) > 0.$$

Lemma 3.2. *Let (X,m,ϕ) be a σ-finite conservative dynamical system, G a locally compact second countable group and $a(n,x) : \mathbb{N} \times X :\rightarrow G$ a G-valued cocycle. Let $(\hat{X},\hat{m},\hat{\phi})$ be the group extension $\hat{X} = X \times G$, where $\hat{m} = m \otimes m_G$ (m_G is the Haar measure) and $\hat{\phi}(x,g) = (\phi(x), a(x)g)$. Then $a(n,x)$ is a recurrent cocycle if and only if $(\hat{X},\hat{m},\hat{\phi})$ is conservative.*

Proof. (i)$\Rightarrow$(ii). Let $\theta : X \rightarrow \mathbb{R}^+$ be integrable and $\psi : G \rightarrow \mathbb{R}^+$ continuous and integrable. For every $g \in G$ and a.e. $x \in X$

$$\sum_{n \geq 0} (\theta \otimes \psi) \circ \hat{\phi}^n(x,g) = +\infty \text{ a.e.}$$

Indeed, choose $\epsilon > 0$ such that $A_\epsilon = \{\theta > \epsilon\}$ has positive measure and $B_\epsilon = \{\psi > \epsilon\}$ is nonempty. Since B_ϵ is an open set it has positive measure and, for every $g \in B_\epsilon$, $B_\epsilon g^{-1}$ is a neighborhood of the neutral element $e \in G$. By hypothesis, for a.e. $x \in A_\epsilon$, there exists an infinite number of n's such that $\phi^n(x) \in A_\epsilon$ and $a(n,x) \in B_\epsilon g^{-1}$, that is $\theta \circ \phi^n(x) > \epsilon$ and $\psi(a(n,x)g) > \epsilon$. Using Lemma C.5, we see that $\hat{\phi}$ is conservative.

(ii)$\Rightarrow$(i). If V is a neighborhood of $e \in G$, $m_G(V) > 0$; and if B is a Borel set of positive measure, then for a.e. $x \in B$ and $g \in V$, there exists $n \geq 1$ such that $\phi^n(x) \in B$ and $a(n,x)g \in V$. In particular, $a(n,x) \in VV^{-1}$. □

Corollary 3.3. *If G is a compact second-countable group and (X,m,ϕ) is a conservative σ-finite dynamical system, then any G-valued cocycle $a(n,x)$ is recurrent.*

Proof. It is enough to show that the group extension $(\hat{X}, \hat{m}, \hat{\phi})$ introduced in Lemma 3.2 is conservative. Let $\theta : X \to \mathbb{R}^+$ be a positive and integrable function; then considered as a function of (x, g), θ is again integrable (since m_G has finite measure) and satisfies

$$\sum_{n \geq 0} \theta \circ \hat{\phi}^n(x, g) = \sum_{n \geq 0} \theta \circ \phi^n(x) = +\infty \qquad \text{a.e. on } \hat{X}.$$

By Lemma C.5, $\hat{\phi}$ is conservative. □

Corollary 3.4. *Let G be a locally compact second-countable group, H a normal and compact subgroup of G and $a(n, x)$ a G-valued cocycle. Let $\bar{G} = G/H$ and $\bar{a}(n, x)$ be the $\bar{G}$-valued corresponding cocycle. Then* (i) *$a(n, x)$ is recurrent if and only if* (ii) *$\bar{a}(n, x)$ is recurrent.*

Proof. We only prove (ii)⇒(i). Denote by $(\hat{X}, \hat{m}, \hat{\phi})$ the $\bar{G}$-group extension associated to $\bar{a}(n, x)$ as defined in Lemma 3.2. The transformation $\hat{\phi}$ is conservative since $\bar{a}(n, x)$ is recurrent. We now consider $a(n, x)$ as a cocycle over $\hat{X}$ and show that it is cohomologous to a H-cocycle. Let $\sigma : \bar{G} \to G$ be a measurable section. Then for every $x \in X$, $\bar{g} \in \bar{G}$, $\overline{a(x)\sigma(\bar{g})} = \bar{a}(x)\bar{g}$. There exists a unique $b(x, \bar{g}) \in H$ such that $a(x)\sigma(\bar{g}) = \sigma(\bar{a}(x)\bar{g})b(x, \bar{g})$. Considered as a function over $\hat{X}$, the cocycle a satisfies

$$a(x) = \sigma \circ \hat{\phi}(x) b(x) \sigma(x)^{-1}, \qquad x \in \hat{X}.$$

By the previous corollary, $b(n, x)$ and, in particular, $a(n, x)$ are recurrent over $\hat{X}$ and therefore over X. □

Corollary 3.5. *Let (X, m, ϕ) be a finite dynamical system, G a locally compact second-countable group, $a(n, x)$ a G-cocycle and X_t the transient part $X_t = \{x \in X : \lim_{n \to +\infty} a(n, x) = \infty\}$. Then $a(n, x)$ is a recurrent cocycle on $X \setminus X_t$.*

Proof. Let $(\mathcal{K}_i)_{i \geq 0}$ be compact subsets of G such that $\mathcal{K}_i \subset \text{int}(\mathcal{K}_{i+1})$. To show that $a(n, x) \to \infty$, it is enough to show that

$$\forall\, i \geq 0 \;\exists\, n_0 \;\forall\, n \geq n_0, \qquad a(n, x) \notin \mathcal{K}_i \qquad \text{a.e.}$$

We introduce the complementary set X_r of points x such that, for some $\mathcal{K}_i$, $a(n, x)$ returns to $\mathcal{K}_i$ infinitely often. The set X_r is invariant ($\phi^{-1}(X_r) = X_r$), and we can introduce the group extension $(\hat{X}_r, \hat{m}_r, \hat{\phi}_r)$ as in Lemma 3.2. Let $\psi : G \to \mathbb{R}^+$ be a continuous, positive and integrable function. Considered as an integrable function over $\hat{X}$, since ψ is uniformly bounded from below on each $\mathcal{K}_i$ by a positive constant

$$\sum_{n \geq 0} \psi \circ \hat{\phi}^n(x, g) = +\infty \qquad \text{a.e. } x \in X_r,\ g \in G.$$

We use the fact that every $\mathcal{K}_i g^{-1}$ is included in some int$(\mathcal{K}_j)$. Thanks to Lemma C.5, $\hat{\phi}$ is conservative and $a(n,x)$ is recurrent. □

The previous corollary is an abstract form of Lemma 1.3.

3.2 The weak hyperbolic case The main purpose of this section is to prove a converse of part (iv) of the Main Theorem. To prove Proposition 1.5, we introduce a subgroup G of affine transformations of $\mathbb{R}$, $a(t) = \alpha(t+\beta)$, $t \in \mathbb{R}$ where $\alpha \in \{\pm 1\}$ and $\beta \in \mathbb{R}$ are constants and we shall consider G-valued cocycles $a_x(t) = \alpha_x(t+\beta_x)$, where $\alpha : X \to \{\pm 1\}$ and $\beta : X \to \mathbb{R}$ are measurable functions. We first notice that $a(n,x)$ can be written as

$$a(n,x)(t) = \alpha(n,x)(t+\beta(n,x)),$$

where $\alpha(n,x) = \alpha \circ \phi^{n-1}(x) \cdots \alpha \circ \phi(x)\alpha(x)$ and $\beta(n,x)$ is an α-cocycle defined by

$$\beta(n,x) = \sum_{k=0}^{n-1} \alpha(k,x)\beta \circ \phi^k(x).$$

To study double recurrence of $\alpha(n,x)$ and $\beta(n,x)$, we introduce a two point extension $(\hat{X}, \hat{m}, \hat{\phi})$, where $\hat{X} = X \times \{\pm 1\}$, $\hat{m} = m \otimes \frac{1}{2}(\delta_{+1} + \delta_{-1})$ and $\hat{\phi}(x,\epsilon) = (\phi(x), \alpha(x)\epsilon)$. Let $\hat{\beta}(x,\epsilon) = \epsilon\beta(x)$; then an easy computation shows that $\hat{\beta}(n,x,\epsilon) = \epsilon\beta(n,x)$ and $\hat{\phi}^n(x,\epsilon) = (\phi^n(x), \alpha(n,x)\epsilon)$.

Lemma 3.6. *If α is not equal to a (multiplicative) coboundary on any ϕ-invariant set, then any $\hat{\phi}$-invariant set $\hat{B}$ is of the form $\hat{B} = B \times \{\pm 1\}$, where B is ϕ-invariant.*

Proof. We first prove that -1 is an essential value for $\alpha(n,x)$: that is, for any B of positive measure there exists $n \geq 1$ such that

$$m\left(B \cap \phi^{-n}B \cap \{x \in B : \ \alpha(n,x) = -1\}\right) > 0.$$

By contradiction, there exists B of positive measure such that for every $n \geq 1$ and a.e. $x \in B$, $\alpha_B(n,x) = 1$ where α_B denotes the induced cocycle on B. On $\tilde{B} = \bigcup_{n\geq 0} \phi^{-n}(B)$ we can extend τ_B and α_B by the same formula:

$$\tau_B(x) = \inf\{k \geq 1 : \ \phi^n(x) \in B\} \quad \text{and} \quad \alpha_B(x) = \prod_{k=0}^{\tau_B(x)-1} \alpha \circ \phi^k(x).$$

An easy computation shows that $\alpha = \alpha_B / \alpha_B \circ \phi$ on the ϕ-invariant set $\tilde{B}$, which is the desired contradiction.

Now let $\hat{B}$ be a $\hat{\phi}$-invariant set $\hat{B} = B_+ \times \{+1\} \cup B_- \times \{-1\}$. We prove by contradiction that $B_+ \subset B_-$ a.e. Otherwise, $B = B_+ \setminus B_-$ would have positive measure and for a.e. $x \in B$ there would exist $n \geq 1$ such that $\phi^n(x) \in B$ and $\alpha(n,x) = -1$. In other words, $B \times \{+1\} \subset \bigcup_{n\geq 1} \hat{\phi}^{-n} B \times \{-1\}$ and $B \times \{+1\} \subset \bigcup_{n\geq 1} \phi^{-n}(X \setminus \hat{B}) = \hat{B}$, which is a contradiction. □

Proposition 3.7. *Let $a(n,x)$ be a G-valued cocycle, $a(x)(t) = \alpha_x(t + \beta_x)$ where $\alpha : X \to \{\pm 1\}$ is not a coboundary on any invariant set and β is integrable. Then $a(n,x)$ is a recurrent cocycle. If X_∞ denotes a ϕ-invariant set of maximal measure on which $+\infty$ is an essential value with respect to $a(n,x)$, then on the complement there exists $\theta : X \setminus X_\infty \to \mathbb{R}$ measurable such that $a(x)(\theta_x) = \theta_{\phi(x)}$ a.e. on $X \setminus X_\infty$.*

Proof. *$a(n,x)$ is recurrent.* Let B be a Borel set of positive measure, ϵ a positive number and $\hat{B} = B \times \{+1\}$. For any $\hat{\phi}$-invariant set $\hat{C}$, $\hat{C} = C \times \{\pm 1\}$ for some C and $\int_{\hat{C}} \hat{\beta}\, d\hat{m} = 0$. By Atkinson's Theorem C.2, $\hat{\beta}(n,x,\epsilon)$ is a recurrent cocycle: there exists $n \geq 1$ such that

$$\hat{m}\left(\hat{B} \cap \hat{\phi}^{-n}\hat{B} \cap \{(x,\epsilon) \in \hat{B} :\ |\hat{\beta}(n,x,\epsilon)| < \epsilon\}\right) > 0$$

or, in other words, there exists $n \geq 1$ such that

$$m\left(B \cap \phi^{-n} \cap \{x \in B :\ \alpha(n,x) = 1 \text{ and } |\beta(n,x)| < \epsilon\}\right) > 0.$$

Since $V_\epsilon = \{a \in G : (a(t) = t + \beta\ \forall t \in \mathbb{R}) :\ |\beta| < \epsilon\}$ defines a neighborhood basis of Id, we have proved that $a(n,x)$ is a recurrent cocycle.

Existence of θ. It is enough to show, for any ϕ-invariant set $X' \subset X \setminus X_\infty$ of positive measure, that there exist $C \subset X'$ ϕ-invariant of positive measure and $\theta : C \to \mathbb{R}$ such that $a(x)(\theta_x) = \theta_{\phi(x)}$ a.e. on C. By the definition of X_∞, there exist $B \subset X'$ of positive measure and $R > 0$ such that $|\beta_B(n,x)| \leq R$ for a.e. in B and every $n \geq 0$, where

$$\beta_B(n,x) = \sum_{k=0}^{n-1} \alpha_B(k,x)\beta_B \circ \phi_B^k(x) \quad \text{and} \quad \beta_B(x) = \sum_{k=0}^{\tau_B(x)-1} \alpha(k,x)\beta \circ \phi^k(x),$$

$$\alpha_B(k,x) = \prod_{i=0}^{k-1} \alpha_B \circ \phi_B^k(x) \quad \text{and} \quad \alpha_B(x) = \prod_{i=0}^{\tau_B(x)-1} \alpha \circ \phi^i(x).$$

Let $\hat{B} = B \times \{\pm 1\}$; then $\hat{\beta}_{\hat{B}}(n,x,\epsilon) = \epsilon\beta_B(n,x)$ and $|\hat{\beta}_{\hat{B}}(n,x,\epsilon)| \leq R$. By Lemma C.1, $\hat{\beta}$ is equal to a coboundary $\hat{\gamma}$ on an invariant set $\hat{C}$ (in fact, $\hat{C} =$

$\bigcup_{n\geq 0} \hat{\phi}^{-n}(\hat{B})$); and by the previous lemma, $\hat{C} = C \times \{\pm 1\}$ for some ϕ-invariant C: $\epsilon\beta(x) = \hat{\gamma} \circ \hat{\phi}(x,\epsilon) - \hat{\gamma}(x,\epsilon)$ for a.e. $x \in C$ and every $\epsilon = \pm 1$. By taking $\epsilon = 1$ and $\epsilon = -1$, we obtain

$$\begin{aligned}\hat{\gamma}(x,1) + \hat{\gamma}(x,-1) &= \hat{\gamma} \circ \hat{\phi}(x,1) + \hat{\gamma} \circ \hat{\phi}(x,-1)\\ &= \hat{\gamma}(\phi(x),1) + \hat{\gamma}(\phi(x),-1).\end{aligned}$$

Without changing $\hat{\beta}$, we modify $\hat{\gamma}$ by $\tilde{\gamma}(x,\epsilon) = \hat{\gamma}(x,\epsilon) - \frac{1}{2}(\hat{\gamma}(x,1) + \hat{\gamma}(x,-1))$ so that $\tilde{\gamma}$ satisfies $\tilde{\gamma}(x,1) + \tilde{\gamma}(x,-1) = 0$ a.e. $x \in C$. In particular, we obtain $\tilde{\gamma}(\phi(x),\alpha(x)) = \alpha(x)\tilde{\gamma}(\phi(x),1)$ for a.e. x on C. If $\theta(x) = \tilde{\gamma}(x,1)$, then

$$\alpha(x)\theta \circ \phi(x) = \tilde{\gamma} \circ \hat{\phi}(x,1) = \beta(x) + \theta(x).$$

This shows the existence of $\theta : C \to \mathbb{R}$ satisfying $a(x)(\theta_x) = \theta_{\phi(x)}$ a.e. on C. □

Proof of Proposition 1.5. We first notice that, according to Lemma 3.4, $M(n,x)$ is a recurrent cocycle if and only if the corresponding projective cocycle is. Moreover, M_x is cohomologous to a rotation $R(\omega_x)$, where $\omega_x = 0$ modulo $\pi/2$, if and only if the projective cocycle is cohomologous to a rotation $R(\bar{\omega}_x)$, where $\bar{\omega}_x = 0$ modulo π. We can therefore choose the conformal notation. On the Poincaré disk,

$$M_x(z) = \exp(i\pi\mathbf{1}_B(x))\frac{z+\mu_x}{1+z\mu_x} \quad \text{where} \quad \left(\frac{1+\mu_x}{1-\mu_x}\right)^{1/2} = |v_x|.$$

After conjugation by $G_1(z) = i(1+z)/(1-z)$ or $G_1^{-1}(z) = (iz+1)/(iz-1)$, the action of M_x on the Poincaré upper half plane becomes

$$G_1 \circ M_x \circ G_1^{-1} = \begin{cases} v_x^2 z & \text{for } x \notin B, \\ -(v_x^2 z)^{-1} & \text{for } x \in B. \end{cases}$$

The imaginary line is invariant and, conjugating by $L(z) = \ln(-iz)$ on $\mathbb{R}^+$, we finally obtain an affine map $a(x) : \mathbb{R} \to \mathbb{R}$,

$$L \circ G_1 \circ M_x \circ G_1^{-1} \circ L^{-1}(t) = a_x(t) = \alpha_x(t + \beta_x),$$

where $\alpha_x = \exp(i\pi\mathbf{1}_B)$ and $\beta_x = 2\ln|v_x|$. By hypothesis on $\mathbf{1}_B$, α is not a coboundary on any invariant set. Thanks to Proposition 3.7, $a(n,x)$ or $M(n,x)$ is recurrent. On $X \setminus X_\infty$, there exists thus an a-invariant function $\theta : X \setminus X_\infty \to \mathbb{R}$ which satisfies $a_x(\theta_x) = \theta_{\phi(x)}$ a.e. on $X \setminus X_\infty$. By conjugating, $\xi_x = (L \circ G)^{-1}(\theta(x)) \in \mathbb{R}$ becomes an M-invariant function $M_x(\xi_x) = \xi_{\phi(x)}$. Conjugating by $K_x(z) = (z+\xi_x)/(1+z\xi_x)$, M_x becomes cohomologous to

$$K_{\phi(x)}^{-1} \circ M_x \circ K_x(z) = z\exp(i\pi\mathbf{1}_B(x)). \qquad \square$$

Proof of Proposition 1.6. (i) Since $M(n,x)$ has to be recurrent on X_E and X_{WH} according to the Main Theorem, these two sets have measure zero; and on each remaining invariant set X_H or X_P, $M(n,x)$ is cohomologous to an upper triangular matrix $N(n,x)$: $M_x = K_{\phi(x)} N_x K_x^{-1}$. If Δ denotes the horizontal axis on $\mathbb{R}^2$, $K_x(\Delta)$ is an M-invariant line.

(ii) In conformal notation, if $\{\xi_x, \eta_x\}$ are globally M-invariant, as in the main proof, M_x is cohomologous to $N_x = R(\mathbf{1}_B(x)\frac{\pi}{2})\,\mathrm{diag}(v_x, v_x^{-1})$ for some log-integrable $v : X \to \mathbb{R}^*$. Thanks to Proposition 1.5, $\mathbf{1}_B$ is cohomologous to 0, $\mathbf{1}_B = \gamma \circ \phi - \gamma \pmod 2$ where the range of γ can be chosen in $\mathbb{Z}$. Then

$$M_x = R\Big(\frac{\pi}{2}\gamma \circ \phi(x)\Big)\,\mathrm{diag}(\tilde{v}_x, \tilde{v}_x^{-1}) R\Big(\frac{\pi}{2}\gamma(x)\Big)$$

where $\tilde{v}_x = v_x$ if $\gamma(x) \in 2\mathbb{Z}$ and $\tilde{v}_x = v_x^{-1}$ otherwise. By hypothesis

$$\ln \|M(n,x)\| = \Big|\ln |\tilde{v}(n,x)|\Big| = \Big|\sum_{k=0}^{n-1} \ln |\tilde{v} \circ \phi^k(x)|\Big|$$

converges a.e. to $+\infty$; by Atkinson's Theorem, $\mathbb{E}[\ln|\tilde{v}| \mid \mathcal{I}] \neq 0$ a.e.; and by Birkhoff's ergodic theorem, $\left(\frac{1}{n}\ln|\tilde{v}(n,x)|\right)_{n>0}$ converges to $+\infty$ a.e. $\square$

3.3 The elliptic case We give in this section a necessary and sufficient condition for a cocycle $M : \mathbb{N} \times X :\to \mathrm{SL}(d,\mathbb{R})$ to be cohomologous to a rotation R.

Proposition 3.8. *If $M(n,x)$ is a measurable $\mathrm{SL}(d,\mathbb{R})$-valued cocycle (not necessarily log-integrable) and if ∞ is not an essential value on any invariant set, then $M(n,x)$ is cohomologous to a $\mathrm{SO}(d,\mathbb{R})$-valued cocycle.*

Proof. Let X' be a ϕ-invariant set of maximal measure on which $M(n,x)$ is cohomologous to a rotation. We want to prove that $X' = X$ a.e. By contradiction, since ∞ is not an essential value on $X \setminus X'$, there exist B in $X \setminus X'$ of positive measure and a constant $K > 0$ such that $\|M_B(n,x)\| \leq K$ for every $n \geq 1$ and a.e. $x \in B$. We claim that M_B is cohomologous to a rotation R, that is, $M_B(x) = K_{\phi_B(x)} R_x K_x^{-1}$ for some measurable function $K : B \to \mathrm{SL}(d,\mathbb{R})$. We extend the conjugating matrix K and the rotation R on $\tilde{B} = \bigcup_{n \geq 0} \phi^{-n} B$ by

$$R_x = \mathrm{Id}, \qquad K_x = M(\tau_B(x), x)^{-1} K \circ \phi^{\tau_B(x)}(x)$$

for every $x \in \tilde{B} \setminus B$ and notice that $M_x = K_{\phi(x)} R_x K_x^{-1}$ a.e. on $\tilde{B}$, which is a contradiction.

To prove the claim, we first simplify notation by assuming that $\|M(n,x)\| \leq K$ for every $n \geq 1$ and for a.e. $x \in X$. In conformal notation,

$$\begin{aligned} M(n,\phi^{-n}(x))(z) &= e^{i2\alpha_{n,x}}\,(z+\mu_{n,x})\,/\,(1+z\bar{\mu}_{n,x}), \\ M(n,\phi^{-n}(x))_* Leb &= J_{n,x} Leb, \\ J_{n,x}(t) &= (1-|\mu_{n,x}|^2)/|t - M(n,\phi^{-n}(x))(0)|^2. \end{aligned}$$

By hypothesis, $J_{n,x} \leq (1+|\mu_{n,x}|)/(1-|\mu_{n,x}|) = e^{2K}$ is uniformly bounded. By Proposition A.14, we obtain an M-invariant $\mathcal{M}_1(\partial\mathbb{D})$-valued function ν as a weak limit point of $(\nu(n,x))_{n>0}$:

$$\nu(n,x) = \frac{1}{n}\sum_{k=0}^{n-1} M(k,\phi^{-k}(x))_* Leb.$$

For some subsequence $(n_k)_{k>0}$, for every $\varphi \in L^1(\mathbb{R})$ and ψ continuous on $\partial\mathbb{D}$

$$\lim_{k\to+\infty} \int \varphi(x)\nu(n_k,x)(\psi)\,dm(x) = \int \varphi(x)\nu_x(\psi).$$

Since $\nu(n_k,x) \leq e^{2K} Leb$, $\nu_x \leq e^{2K} Leb$ so ν_x is absolutely continuous with respect to Leb. By the Douady–Earle theorem and Lemma A.12, there exists $\xi : X \to \mathbb{D}$, M-invariant measurable. As in the proof of the Main Theorem (parts ii–iv), we conclude that M is cohomologous to a rotation. □

3.4 An extension of Furstenberg's theorem This section is devoted to proving Proposition 1.10. We assume that (X,m,ϕ) is an ergodic invertible dynamical system and $M(n,x)$ is a log-integrable cocycle independent with respect to some σ-algebra $\mathcal{F}_0$ (see Definition 1.9). By ergodicity, the Lyapunov exponent λ_+ is constant a.e. If it is equal to zero, the cocycle $M(n,x)$ is cohomologous to either a parabolic or an elliptic or a weak hyperbolic cocycle. Since, in each case, the conjugating matrix appears as an M-invariant function, the main ingredient of the proof is the following lemma.

Lemma 3.9. *Let (X,m,ϕ) be an ergodic invertible dynamical system, $\mathcal{F}_0$ a sub-σ-algebra of $\mathcal{B}_X$ such that $\left(\phi^{-n}(\mathcal{F}_0)\right)_{n\in\mathbb{Z}}$ are independent and generate $\mathcal{B}_X$ and $M : X \to \mathcal{M}ob^+(\mathbb{D})$ a $\mathcal{F}_0$-measurable cocycle. If $M(n,x)$ is recurrent, then any $\mathcal{B}_X$-measurable M-invariant function $\xi : X \to \bar{\mathbb{D}}$ is constant a.e.*

Proof. As in Lemma 3.2, we introduce the group extension $(\hat{X},\hat{m},\hat{\phi})$ defined by $\hat{X} = X \times \mathcal{H}$, where $\mathcal{H} = \mathcal{M}ob^+(\mathbb{D})$, $\hat{m} = m \otimes m_{\mathcal{H}}$, $\hat{\phi}(x,h) = (\phi(x), M_x h)$. We also extend ξ to $\hat{X}$ by $\hat{\xi}(x,h) = h^{-1}(\xi_x)$. An algebraic computation shows that $\hat{\xi}$ is $\hat{\phi}$-invariant. We want to apply Lemma C.6 to the two sub-σ-algebras $\hat{\mathcal{D}}_0 = \mathcal{D}_0 \otimes \mathcal{B}_{\mathcal{H}}$

and $\hat{\mathcal{E}}_0 = \mathcal{E}_0 \otimes \mathcal{B}_{\mathcal{H}}$, where $\mathcal{D}_0 = \bigvee_{k\geq 0} \phi^{-k}\mathcal{F}_0$ and $\mathcal{E}_0 = \bigvee_{k<0} \phi^{-k}\mathcal{F}_0$. Since M is $\mathcal{F}_0$-measurable, $\hat{\phi}^{-1}(\hat{\mathcal{D}}_0) \subset \hat{\mathcal{D}}_0$ and $\hat{\phi}(\hat{\mathcal{E}}_0) \subset \hat{\mathcal{E}}_0$. If we assume for a while that $\hat{\mathcal{D}}_0$ and $\hat{\mathcal{E}}_0$ generate $\hat{\mathcal{B}}_{\hat{X}}$, since $(\hat{X}, \hat{m}, \hat{\phi})$ is conservative ($M(n,x)$ is recurrent), we would obtain that ξ is measurable with respect to $\hat{\mathcal{D}}_0 \cap \hat{\mathcal{E}}_0$. For a.e. $h \in \mathcal{H}$, the map $x \mapsto h^{-1}(\xi_x)$ would be $\mathcal{D}_0 \cap \mathcal{E}_0$-measurable, that is, constant a.e. and ξ would be constant a.e.

Since $\{\phi^k \mathcal{F}_0\}_{k\geq 0}$ generates $\mathcal{B}_X$, to prove that $\bigvee_{k\geq 0} \hat{\phi}^k(\mathcal{D}_0)$ generates $\hat{\mathcal{B}}_{\hat{X}}$ it is enough to prove that $f \circ \phi^{-k} \otimes g$ belongs to $\hat{\phi}^k(\hat{\mathcal{D}}_0)$ for any $f : X \to \mathbb{R}$ $\mathcal{D}_0$-measurable and $g : \mathcal{H} \to \mathbb{R}$ $\mathcal{B}_{\mathcal{H}}$-measurable. Indeed, let $\hat{f}$ be such that $\hat{f}(x,h) = f(x)g(M(n,x)h)$; then $\hat{f}$ is $\hat{\mathcal{D}}_0$-measurable and $f \circ \phi^{-k} \otimes g = \hat{f} \circ \hat{\phi}^{-k}$ (everywhere). □

We first obtain the following corollary which generalizes Lemma C.6 to some non-abelian groups.

Corollary 3.10. *Let (X, m, ϕ) be an ergodic invertible dynamical system, let $\{\phi^n \mathcal{F}\}_{n\in\mathbb{Z}}$ be a generating sequence of independent σ-algebras and K a* $\mathrm{SL}(2,\mathbb{R})$-*valued measurable function. If the coboundary $M_x = K_{\phi(x)} K_x^{-1}$ is measurable with respect to $\mathcal{F}_0$, then K is constant a.e.*

Proof. In conformal notation, for each $\xi \in \mathbb{D}$, $\xi(x) = K_x(\xi)$ is M-invariant and therefore constant a.e. Since this is true for all ξ, K is constant a.e. □

We now show how this last lemma applies to the proof of Proposition 1.10.

Proof of Proposition 1.10. Let λ_+ be the top Lyapunov exponent constant a.e. If $\lambda_+ > 0$, nothing new is said. If $\lambda_+ = 0$, then three cases may occur, as the Main Theorem shows.

The elliptic case: There exist $K : X \to \mathrm{SL}(2,\mathbb{R})$ and $\omega : X \to \mathbb{R}$ such that $M_x = K_{\phi(x)} R(\omega_x) K_x^{-1}$ a.e. on X. The cocycle $M(n,x)$ is recurrent and $\xi(x) = K_x(0)$ is M-invariant, therefore constant a.e. With the normalization $K_x^{-1}(0) \in i\mathbb{R}^-$, K has to be constant a.e.

The weak-hyperbolic case: There exist $K : X \to \mathrm{SL}(2,\mathbb{R})$, $A \in \mathcal{B}_X$ and $v : X \to \mathbb{R}^*$ such that $M_x = K_{\phi(x)} R(\mathbf{1}_A(x)\frac{\pi}{2}) \operatorname{diag}(v_x, v_x^{-1}) K_x^{-1}$ and $\mathbf{1}_A$ is not cohomologous to 0. Proposition 1.5 shows that $M(n,x)$ is recurrent. Let $\xi_+(x) = K_x(1)$ and $\xi_-(x) = K_x(-1)$; then $\{\xi_+(x), \xi_-(x)\}$ is globally M-invariant. We then introduce the quotient space $\hat{\mathbb{D}} = \partial\mathbb{D} \times \partial\mathbb{D} \setminus \sim$, compact metrizable, where $\sim$ is the equivalence relation $(\xi, \eta) \sim (\eta, \xi)$. Each Möbius transformation acts on $\hat{\mathbb{D}}$ in the trivial way, and $\hat{\xi}(x) = (\xi_+(x), \xi_-(x)) \in \hat{\mathbb{D}}$ becomes M-invariant. As in the proof of the previous lemma, $\hat{\xi}$ has to be constant a.e. and K can be chosen constant a.e.

The general case: As in the proof of the Main Theorem, we consider a weak limit point ν_x^+ of the sequence

$$\frac{1}{n}\sum_{k=1}^{n} M(k, \phi^{-k}(x))_* Leb.$$

Since $c_+(x) = \mathrm{card}\{t \in \partial\mathbb{D} : \nu_x^+(t) \geq \frac{1}{2}\}$ is constant along the trajectories, $c_+(x)$ is constant a.e. If $c_+ = 0$, M is cohomologous to a rotation and we have seen that the conjugating matrix is constant a.e. If $c_+ = 2$, M is cohomologous to $N_x = R(\mathbf{1}_A(x)\frac{\pi}{2})\,\mathrm{diag}(v_x, v_x^{-1})$, $M(n,x)$ is recurrent (otherwise, $\|M(n,x)\|$ would go to $+\infty$ and λ_+ would be positive) and we have seen that the conjugating matrix can be chosen constant a.e.

If $c_+ = 1$, we also choose a weak limit point ν_x^- of

$$\frac{1}{n}\sum_{k=1}^{n} M(-k, \phi^{k}(x))_* Leb.$$

As before, ν^- is M-invariant and $c_-(x) = \mathrm{card}\{t \in \partial\mathbb{D}$ s.t. $\nu_x^-(t) \geq \frac{1}{2}\}$ is constant a.e. The same conclusions hold when $c_- = 0$ and $c_- = 2$. Let us assume $c_- = 1$. We have thus proved the existence of two M-invariant functions $\xi^+, \xi^- : X \to \partial\mathbb{D}$. The function ξ^+ is measurable with respect to $\bigvee_{n\geq 1} \phi^n \mathcal{F}_0$, and ξ^- is measurable with respect to $\bigvee_{n\geq 0} \phi^{-n}\mathcal{F}_0$. If ξ_x^+ is not equal to ξ_x^- a.e., then $M(n,x)$ is cohomologous to a diagonal matrix and has to be recurrent ($\lambda_+ = 0$). By Lemma 3.9, ξ^+ and ξ^- are constant a.e. Otherwise, $\xi_x^+ = \xi_x^-$ a.e. and ξ^+ (for instance) is measurable with respect to $\bigvee_{n\geq 0} \phi^{-n}\mathcal{F}_0 \cap \bigvee_{n\geq 0} \phi^{n}\mathcal{F}_0$. Thanks to the independence of $(\phi^n \mathcal{F}_0)_{n\in\mathbb{Z}}$, $\xi^+ = \xi^-$ is constant a.e. so $M(n,x)$ is cohomologous to an upper triangular matrix with constant conjugating matrix. $\square$

Appendixes

A Conformal dynamics

The purpose of this appendix is to gather elementary facts on the hyperbolic geometry of the unit disk and to give a new proof of the main tool, namely, the existence of Douady–Earle's conformal barycenter.

A.1 Möbius group on the disk and the halfplane We denote by $\mathbb{D}$ the unit open disk of $\mathbb{C}$. A Möbius transformation is a map $M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $M(z) = (az+b)/(cz+d)$ with $ad - bc$ non-zero. A Möbius transformation M preserves $\mathbb{D}$ if and only if M has the form $M_{\alpha,\mu}(z) = e^{i2\alpha}(z+\mu)/(1+z\bar{\mu})$ where $\alpha \in \mathbb{R}$ and $\mu \in \mathbb{D}$. The set of Möbius transformations of $\mathbb{D}$ is denoted by $\mathcal{M}ob^+(\mathbb{D})$. Apart from the identity, the dynamics of Möbius transformations can be classified into three distinct classes depending on the number of fixed points inside $\bar{\mathbb{D}}$.

The first tool we use in the study of general $\mathcal{M}ob^+(\mathbb{D})$-valued cocycles is the Poincaré metric.

Proposition A.1. *There is a unique (up to a multiplicative constant) Riemannian metric $d_{\mathbb{D}}$ on $\mathbb{R}$ invariant by the group of Möbius transformations. Infinitesimally it is given by $|dz|/(1-|z|^2)$ and satisfies*

(i) $d_{\mathbb{D}}(M(\eta), M(\xi)) = d_{\mathbb{D}}(\eta,\xi) \quad \forall M \in \mathcal{M}ob^+(\mathbb{D}),\ \eta,\xi \in \mathbb{D}$,

(ii) $d_{\mathbb{D}}(\eta,\xi) = \frac{1}{2}\ln\Big(|1-\eta\bar{\xi}| + |\eta-\xi| \Big/ |1-\eta\bar{\xi}| - |\eta-\xi|\Big)$.

It is convenient sometimes to introduce new coordinates where a point $\xi \in \partial\mathbb{D}$ is seen at infinity. More precisely, we have

Lemma A.2. *Let $\mathbb{H} = \{z \in \mathbb{C} : \Im m(z) > 0\}$ be the upper halfplane. For any $\xi \in \partial\mathbb{D}$, there exists a unique Mobius transformation $G_\xi : \mathbb{D} \to \mathbb{H}$ sending $\partial\mathbb{D} \setminus \{\xi\}$ onto $\{\Im m(z) = 0\}$, ξ to ∞ and 0 to i: $G_\xi(z) = i(\xi+z)/(\xi-z)$. If $d_{\mathbb{H}} = (G_\xi)_* d_{\mathbb{D}}$ denotes the new Poincaré metric on $\mathbb{H}$, infinitesimally $d_{\mathbb{H}}$ is given by $\frac{1}{2}|dz|/\Im m(z)$.*

Any Möbius transformations preserving $\mathbb{H}$ and ∞ is equal to some map $T_{a,b}(z) = az - b$ where $a > 0$ and $b \in \mathbb{R}$. Conjugating M by G_ξ, we obtain

Lemma A.3. *For any $\xi \in \partial\mathbb{D}$, $M \in \mathcal{M}ob^+(\mathbb{D})$ and $T_{a,b} = G_{M(\xi)} M G_\xi^{-1}$,*

$$a = 1 - |M(0)|^2 \Big/ |M(\xi) - M(0)|^2, \quad b = 2\Im m\Big(\overline{M(\xi)}M(0)\Big) \Big/ |M(\xi) - M(0)|^2.$$

As we have seen in Proposition 2.2, $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathcal{M}ob^+(\mathbb{D})$. Conjugating by G_1 we obtain another isomorphism between $\mathrm{PSL}(2,\mathbb{R})$ and the group of Möbius transformations of $\mathbb{H}$. The choice we have adopted in Proposition 2.2 implies

Lemma A.4. *Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R})$, $M_{\alpha,\mu}$ the corresponding Möbius transformation on $\mathbb{D}$ and $T = G_1 M_{\alpha,\mu} G_1^{-1}$; then $T(z) = -(az-b)/(cz-d)$.*

We conclude this section with an estimate.

Lemma A.5. *For any $a > 0$ and $b \in \mathbb{R}$*

$$d_{\mathbb{H}}(i, ai - b) = \frac{1}{2}\ln\left[\frac{1}{2}\left(a + \frac{1}{a} + \frac{b^2}{a}\right) + \frac{1}{2}\sqrt{\left(a + \frac{1}{a} + \frac{b^2}{a}\right)^2 - 4}\right],$$

$$\frac{1}{2}\ln\left[\max\left(a, \frac{1}{a}\right) + \frac{b^2}{a}\right] \leq d_{\mathbb{H}}(i, ai - b) \leq \frac{1}{2}\ln\left[\max\left(a, \frac{1}{a}\right) + \frac{b^2}{a} + 1\right].$$

A.2 The Douady–Earle theorem This section is devoted to the proof of the Douady–Earle theorem A.11 using ideas of convex analysis in conformal geometry. The main tool is the Busemann function, which is geodesically convex for the Poincaré metric.

Definition A.6. Let ξ be a point of $\partial\mathbb{D}$. For every $z \in \mathbb{D}$, we denote by $\mathcal{C}_z$ the circle tangent to $\partial\mathbb{D}$ containing z and ξ, by $\mathcal{D}_z$ the disk bounded by $\mathcal{C}_z$, and by $\pi(z)$ the intersection point of $\mathcal{C}_z$ and the line joining ξ and 0. We define the Busemann function at the point ξ by

$$b_\xi(z, z') = -d_{\mathbb{D}}(\pi(z), \pi(z')) \quad \text{if } \mathcal{D}_z \subset \mathcal{D}_{z'},$$
$$b_\xi(z, z') = d_{\mathbb{D}}(\pi(z), \pi(z')) \quad \text{if } \mathcal{D}_z \supset \mathcal{D}_{z'}.$$

For any $\xi \in \partial\mathbb{D}$, a hyperbolic cone Γ_ξ is a convex open set in $\mathbb{D}$ delimited by one connected arc of $\partial\mathbb{D} \setminus \{\xi\}$ and by two geodesic lines passing through ξ. The Busemann function is characterized by the following property invariant under Möbius transformations.

Proposition A.7. *For every $\xi \in \partial\mathbb{D}$, $(z, z') \in \mathbb{D}$ and every hyperbolic cone Γ_ξ containing z and z',*

$$b_\xi(z, z') = \lim_{w \to \xi \ w \in \Gamma_\xi} d_{\mathbb{D}}(z, w) - d_{\mathbb{D}}(z', w).$$

Proof. By conformal transformation $G_\xi : \mathbb{D} \to \mathbb{H}$, the Busemann function at ∞ takes the form $b_\infty(z, z') = d_{\mathbb{H}}(\Im m(z), \Im m(z'))$ if $\Im m(z) < \Im m(z')$, and the form $b_\infty(z, z') = -d_{\mathbb{H}}(\Im m(z), \Im m(z'))$ if $\Im m(z) > \Im m(z')$. In this geometry, a hyperbolic cone is delimited by vertical lines: $\Gamma_\infty = \{z \in \mathbb{H} : u < \Re e(z) < v\}$ for some $u, v \in \mathbb{R}$. For every $z, w \in \Gamma_\infty$, we define $t = t(z, w) = \Re e(z) + i\Im m(w)$. By the triangle inequality we have $|d_{\mathbb{H}}(z, w) - d_{\mathbb{H}}(z, t)| \leq d_{\mathbb{H}}(w, t)$ and

$$\begin{aligned} d_{\mathbb{H}}(w, t) &= d_{\mathbb{H}}\Big(\frac{w}{\Im m(w)}), \frac{t}{\Im m(t)}\Big) \\ &\leq \sup\Big\{d_{\mathbb{H}}(x + i, y + i) : \frac{u}{\Im m(w)} < x, y < \frac{v}{\Im m(w)}\Big\}. \end{aligned}$$

Letting w go to ∞ in Γ_∞, we obtain

$$\lim_{w\to\infty} d_{\mathbb{H}}(z,w) - d_{\mathbb{H}}(z,t(z,w)) = 0,$$
$$\lim_{w\to\infty} d_{\mathbb{H}}(z,w) - d_{\mathbb{H}}(i\Im m(z), i\Im m(w)) = 0.$$

We conclude the proof by noting that $i\Im m(z)$, $i\Im m(z')$ and $i\Im m(w)$ are all on the same geodesic line. □

We could have used a more analytic approach to define the Busemann function using the Poisson kernel.

Definition A.8. Let $\xi \in \partial\mathbb{D}$ be given. The Poisson kernel of the unit disk $\mathbb{D}$ is the function $p(z,\xi) = (1-|z|^2)/|z-\xi|^2$.

The Buseman function is related to the Poissson kernel by

Lemma A.9. *For all $\xi \in \partial\mathbb{D}$, $z \in \mathbb{D}$, $b_\xi(0,z) = \frac{1}{2}\ln p(z,\xi)$.*

Proof. We first assume $\xi = 1$. For any $z = re^{i\theta} \in \mathbb{D}$, the equation of the circle C_z tangent to $\partial\mathbb{D}$ and containing z and ξ is given by

$$(x-\omega)^2 + y^2 = (1-\omega)^2 = (r\cos\theta - \omega)^2 + (r\sin\theta)^2,$$

where $(\omega, 0)$ is the center of C_z and ω is given by $2\omega = (1-r^2)/(1-r\cos\theta)$. If $\pi(z)$ denotes the intersection point of C_z and the real axis, we obtain $\pi(z) = 2\omega - 1$ and

$$b_\xi(0,z) = \frac{1}{2}\ln\frac{1+\pi(z)}{1-\pi(z)} = \frac{1}{2}\ln\frac{1-r^2}{1-2r\cos\theta + r^2}.$$

For a general ξ, $b_\xi(0,z) = b_1(0,z/\xi)$ and $p(z,\xi) = p(z/\xi, 1)$. □

We say that a function $\psi : \mathbb{D} \to \mathbb{R}$ is geodesically convex (or geodesically strictly convex or affine) on a geodesic $\gamma : \mathbb{R} \to \mathbb{D}$, if $\psi \circ \gamma$ is convex (or strictly convex or affine) in the usual sense.

Proposition A.10. *Let $\xi \in \partial\mathbb{D}$ and $w \in \mathbb{D}$ be given. Then the function $z \in \mathbb{D} \mapsto b_\xi(z,w) \in \mathbb{R}$ is geodesically strictly convex on any geodesic not containing ξ and geodesically affine on any geodesic containing ξ.*

Proof. Let γ be a geodesic.

γ does not contain ξ. By the invariance of the Busemann function under conformal transformations, we may assume that $\xi = 1$, γ is the imaginary axis and is parametrized by $\gamma(t) = i\tanh(t)$. The circle C_t tangent to $\partial\mathbb{D}$ containing $\gamma(t)$ and ξ intersects the real axis at $\gamma^2(t)$ and

$$b_\xi(\gamma(t),0) = d_{\mathbb{D}}(0,\gamma(t)^2) = \tfrac{1}{2}\ln\left(1+\tanh^2(t)\big/1-\tanh^2(t)\right),$$

$$\frac{d}{dt} b_\xi(\gamma(t),0) = \tanh(2t), \qquad \frac{d^2}{dt^2} b_\xi(\gamma(t),0) = 2/\cosh^2(2t) > 0.$$

γ contains ξ. We may assume that $\xi = 1$ and γ is the real axis. The geodesic γ is then parametrized by $\gamma(t) = \tanh(t)$ and either $t < 0$ and $b_\xi(\gamma(t),0) = d_{\mathbb{D}}(0,\gamma(t)) = -t$ or $t > 0$ and $b_\xi(\gamma(t),0) = -d_{\mathbb{D}}(0,\gamma(t)) = -t$. □

We are now able to prove the main tool we need in Section 2. The proof does not rely on degree theory as in [6] but uses the convexity of the Busemann function to obtain a unique minimum. This idea is borrowed from [3].

Theorem A.11 ***(Douady–Earle [6]).*** *Let ν be a probability measure on $\partial\mathbb{D}$ such that $\nu(\{t\}) < \frac{1}{2}$ for all $t \in \partial\mathbb{D}$. Then there exists a unique point* $\mathrm{bar}(\nu)$ *in $\mathbb{D}$, called the conformal barycenter of ν, which realises for any $w \in \mathbb{D}$ the minimum of the function $z \in \mathbb{D} \mapsto \int_{\partial\mathbb{D}} b_\xi(z,w)\,d\nu(\xi)$.*

Proof. We write $\psi(z) = \int b_\xi(z,0)\,d\nu(\xi)$ and observe that $\int b_\xi(z,w)\,d\nu(\xi)$ is equal to $\psi(z) - \psi(w)$. It is therefore enough to prove that ψ has a unique minimum in $\mathbb{D}$. We actually show that ψ is strictly geodesically convex and that $\psi(z)$ converges to $+\infty$ uniformly when $z \to \partial\mathbb{D}$. The proof is divided into two parts.

Existence. We use Lemma A.9 to obtain, for all $z \in \mathbb{D}$ and $\xi \in \partial\mathbb{D}$,

$$b_\xi(z,0) = d_{\mathbb{D}}(z,0) + \ln\frac{|z-\xi|}{1+|z|}.$$

Since $b_\xi(z,0) \geq -d_{\mathbb{D}}(z,0)$ always holds, we obtain for all $z \in \mathbb{D}$ and $\epsilon \in]0,1[$

$$\begin{aligned}\psi(z) &\geq -\nu\big(B(z,\epsilon)\cap\partial\mathbb{D}\big)\,d_{\mathbb{D}}(z,0) + \nu\big(\partial\mathbb{D}\setminus B(z,\epsilon)\big)\big[d_{\mathbb{D}}(z,0) + \ln\tfrac{1}{2}\epsilon\big]\\ &\geq \big[1 - 2\nu\big(B(z,\epsilon)\cap\partial\mathbb{D}\big)\big]d_{\mathbb{D}}(z,0) + \ln\tfrac{1}{2}\epsilon,\end{aligned}$$

where $B(z,\epsilon)$ is the euclidean ball of radius ϵ. By the hypothesis on ν, there exist constants $\epsilon > 0$ and $\nu^* < \frac{1}{2}$ such that, for every arc A of $\partial\mathbb{D}$ of length less than 2ϵ, $\nu(A) < \nu^*$. Since $B(z,\epsilon)\cap\partial\mathbb{D}$ is an arc of $\partial\mathbb{D}$ of length at most 2ϵ, we finally obtain, for all $z \in \mathbb{D}$, $\psi(z) \geq (1-2\nu^*)d_{\mathbb{D}}(z,0) + \ln\frac{1}{2}\epsilon$, which shows that $\psi(z)$ converges to $+\infty$ when $|z| \to 1$.

Uniqueness. Suppose to the contrary that z_1 and z_2 realize the minimum of ψ. Let γ be the arc-length geodesic joining $z_1 = \gamma(t_1)$ and $z_2 = \gamma(t_2)$. By the convexity of $\psi\circ\gamma$, $\psi\circ\gamma$ has to be constant on $[t_1,t_2]$ and therefore the second derivative of $\psi\circ\gamma$,

$$\int \frac{d^2}{dt^2} b_\xi(\gamma(t),0)\,d\nu(\xi) = 0 \qquad \text{for all } t \in [t_1,t_2].$$

Since the second derivative of a convex function is non-negative, $b_\xi(\gamma(t),0)$ is affine in t, ν a.e.; and ν is therefore supported by $\{\gamma^+,\gamma^-\}$, the endpoints of γ in $\partial\mathbb{D}$. This last statement is in contradiction with $\nu(t) < \frac{1}{2}$ for all $t \in \partial\mathbb{D}$. □

We next prove that $\mathrm{bar}(\nu)$ is continuous with respect to ν.

Proposition A.12. *The set $\mathcal{U}_0$ of probability measures ν such that $\nu(t) < \frac{1}{2}$ for every $t \in \partial\mathbb{D}$ is open. The map $\nu \in \mathcal{U}_0 \mapsto \mathrm{bar}(\nu) \in \mathbb{D}$ is continuous and $M(\mathrm{bar}(\nu)) = \mathrm{bar}(M_*\nu)$ for any $M \in \mathcal{M}ob^+(\mathbb{D})$.*

Proof. To prove that $\mathcal{U}_0$ is open, we fix $\nu_0 \in \mathcal{U}_0$ and construct a finite covering of $\partial\mathbb{D}$ by compact sets $K_1, \dots, K_p$ such that $\nu_0(K_i) < \frac{1}{2}$. There then exists a neighborhood $\mathcal{V}_0$ of ν_0 such that for every $\nu \in \mathcal{V}_0$, $\nu(K_i) < \frac{1}{2}$ and therefore $\nu(t) \le \nu(K_i) < \frac{1}{2}$ for all $t \in K_i$, which proves $\mathcal{V}_0 \subset \mathcal{U}_0$.

To prove that $\mathrm{bar}(\nu)$ is continuous, we fix $\nu_0 \in \mathcal{U}_0$, $\epsilon_0 > 0$ and write $z_0 = \mathrm{bar}(\nu_0)$ and $b_0 = \int b_\xi(z_0, 0)\, d\nu_0(\xi)$. We first show that for some neighborhood $\mathcal{V}_0$ of ν_0 and $r_0 \in]0,1[$

$$\forall \nu \in \mathcal{V}_0,\ \ \forall |z| > r_0, \qquad \int b_\xi(z,0)\, d\nu(\xi) \ge b_0 + 1. \tag{1}$$

The estimations in the proof of the Douady–Earle theorem give for any probability measure ν, $z \in \mathbb{D}$ and $\epsilon > 0$,

$$\int b_\xi(z,0)\, d\nu(\xi) \ge d_{\mathbb{D}}(z,0)\big[1 - 2\nu\big(B(z,\epsilon) \cap \partial\mathbb{D}\big)\big] + \ln \tfrac{1}{2}\epsilon. \tag{2}$$

We construct a covering of $\bar{\mathbb{D}}$ by open sets $(V_i)_{i=1}^p$ so that $\nu_0(\bar{V}_i \cap \partial\mathbb{D}) < \frac{1}{2}$. There then exists a neighborhood $\mathcal{V}_0$ of ν_0 and $\epsilon > 0$ such that for every $\nu \in \mathcal{V}_0$ and $z \in \mathbb{D}$, $\nu(\bar{V}_i \cap \partial\mathbb{D}) \le \frac{1}{2} - \epsilon$ and $B(z,\epsilon)$ belongs to some V_i. Inequality (2) becomes $\int b_\xi(z,0)\, d\nu(\xi) \ge 2\epsilon d_{\mathbb{D}}(z,0) + \ln \frac{1}{2}\epsilon$; and for $|z| > r_0$ sufficiently close to 1, inequality (1) is satisfied. We now define a compact set

$$K_0 = \{z \in \mathbb{D} :\ |z| \le r_0 \ \text{ and } \ |z - z_0| \ge \epsilon_0\}$$

and by the uniqueness of $\mathrm{bar}(\nu_0)$ choose $\eta > 0$ such that $\int b_\xi(z,0)\, d\nu_0(\xi) \ge b_0 + 4\eta$ for every $z \in K_0$. We construct an η-net of K_0, $\{z_1, \dots, z_p\}$, and choose a neighborhood $\mathcal{V}_0$ of ν_0 such that $|\int b_\xi(z_i,0)\, d\nu(\xi) - \int b_\xi(z_i,0)\, d\nu_0(\xi)| < \eta$ for every $\nu \in \mathcal{V}_0$ and $i = 0, \dots, p$. Using the fact that any $z \in K_0$ is η-close to some z_i and the inequality $|b_\xi(z,0) - b_\xi(z_i,0)| \le d_{\mathbb{D}}(z, z_i)$, we obtain $\int b_\xi(z,0)\, d\nu(\xi) \ge b_0 + \eta$ and $\int b_\xi(z_0,0)\, d\nu(\xi) < b_0 + \eta$ for all $z \in K_0$. These two last inequalities imply that $\mathrm{bar}(\nu) \in B(z_0, \epsilon_0)$ for all $\nu \in \mathcal{V}_0$, and the proof is complete. □

The proof of Douady–Earle uses an argument of degree theory to prove that a certain vector field necessarily has a zero. We show that the barycenter, obtained in Theorem A.11 as a solution of a variational problem, is also a zero of the vector field $\mathrm{grad}_z B$ defined in the following proposition.

Proposition A.13. *Let ν be a probability measure on $\partial\mathbb{D}$ satisfying $\nu(t) < \frac{1}{2}$ for every $t \in \partial\mathbb{D}$. For any $w \in \mathbb{D}$, the gradient with respect to z (for the hyperbolic metric) of the function $B(z,w) = \int b_\xi(z,w)\, d\nu(\xi)$ is given by*

$$(\mathrm{grad}_z B)(z,w) = \int \frac{z-\xi}{1-\xi\bar{z}}\, d\nu(\xi).$$

Proof. By Lemma A.9 we have $B(z,0) = -\frac{1}{2}\int \ln\left(1 - |z|^2/|z-\xi|^2\right) d\nu(\xi)$. Differentiating with respect to z, we obtain for every $h \in \mathbb{C}$

$$\frac{d}{dz}\left(-\frac{1}{2}\ln\left(\frac{1-|z|^2}{|z-\xi|^2}\right)\right).h = \frac{1}{1-|z|^2}\Re e\left(\bar{h}\Big(z + \frac{1-|z|^2}{|z-\xi|^2}(z-\xi)\Big)\right),$$

$$\mathrm{grad}_z b_\xi(z,0) = z + \frac{1-|z|^2}{|z-\xi|^2}(z-\xi) = \frac{z-\xi}{1-\xi\bar{z}}. \qquad \square$$

We end this section by proving that the set of measures on $\partial\mathbb{D}$ with one or two atoms of mass not smaller than $\frac{1}{2}$ is a Borel set.

Lemma A.14. *The set $\mathcal{U}_1$ of probability measures $\nu \in \mathcal{M}ob^+(\mathbb{D})$ having a unique atom $\delta_\nu \in \partial\mathbb{D}$ of mass not smaller than $\frac{1}{2}$ is a Borel set and the map $\nu \in \mathcal{U}_1 \mapsto \delta_\nu \in \partial\mathbb{D}$ is continuous.*

Proof. We first introduce three sets:

$$F = \{\nu \in \mathcal{M}_1(\partial\mathbb{D}) : \#\{t \in \partial\mathbb{D} : \nu(t) \geq \tfrac{1}{2}\} \geq 1\},$$
$$G = \{\nu \in \mathcal{M}_1(\partial\mathbb{D}) : \#\{t \in \partial\mathbb{D} : \nu(t) \geq \tfrac{1}{2}\} \leq 1\},$$
$$S = \{\nu \in \mathcal{M}_1(\partial\mathbb{D}) : \nu(t) = 1 \text{ for some } t \in \partial\mathbb{D}\}.$$

By Proposition A.12, F is closed.

We first show that $G \setminus S$ is open. Let $\nu_0 \in G \setminus S$; then either $\nu_0(t) < \frac{1}{2}$ for every $t \in \partial\mathbb{D}$ and ν_0 belongs to the previous open set $\mathcal{U}_0 \subset G$, or $\nu_0(t_0) \in [\frac{1}{2}, 1[$ for some $t_0 \in \partial\mathbb{D}$ and $\nu_0(t) < \frac{1}{2}$ for every $t \neq t_0$. As before, we construct a covering of $\partial\mathbb{D}$ by compact sets $K_0, \dots, K_N$, where $\nu_0(K_i) < \frac{1}{2}$ for $i = 1, \dots, N$ and $\nu_0(K_0) < 1$. For ν sufficiently close to ν_0, $\nu(K_i) < \frac{1}{2}$ for $i = 1, \dots, N$, $\nu(K_0) < 1$ and ν belongs to $G \setminus S$. We have just proved that $G \setminus S$ is open.

To prove that S is Borel, identify $[0,1[$ to $\partial\mathbb{D}$ and observe that S can be written in the form

$$S = \bigcap_{n \geq 1} \bigcup_{0 \leq p,q < 1, p-q < 1/n} \{\nu \in \mathcal{M}_1(\partial\mathbb{D}) : \nu([q,p]) = 1\}.$$

To prove the continuity of $\nu \in \mathcal{U}_1 \mapsto \delta_\nu \in \partial\mathbb{D}$, choose a sequence $(\nu_n)_{n\geq 0}$ of measures of $\mathcal{U}_1$ converging to $\nu_\infty \in \mathcal{U}_1$ and let t_n, t_∞ be the corresponding unique atom of mass not smaller than $\frac{1}{2}$. By compactness, some subsequence $(t'_n)_{n\geq 0}$ converges to a limit point t^*. If $\psi : \partial\mathbb{D} \to [0,1]$ is a continuous test function satisfying $\psi = 1$ on a neighborhood of t^*, then $\psi(t'_n) = 1$ for large n and $\nu_\infty(\psi) = \lim_{n\to+\infty} \nu_n(\psi) \geq \frac{1}{2}$. By letting ψ converge to the Dirac function at t^*, we obtain $\nu_\infty(t^*) \geq \frac{1}{2}$; and by uniqueness $t^* = t_\infty$. Thus the sequence $(t_n)_{n>0}$ admits a unique limit point and $\nu \mapsto \delta_\nu$ is continuous. □

The set $\mathcal{U}_2$ of probability measures $\nu \in \mathcal{M}_1(\partial\mathbb{D})$ having two atoms of mass $\frac{1}{2}$ each is Borel since $\mathcal{U}_2 = \mathcal{M}_1(\partial\mathbb{D}) \setminus (\mathcal{U}_0 \cup \mathcal{U}_1)$ and, if we denote by the two atoms of ν, $\xi(\nu), \eta(\nu) \in [0,1[$, on each set

$$\mathcal{U}_2(\epsilon) = \{\nu \in \mathcal{U}_2 : \ \xi(\nu), \eta(\nu) \in [0, 1-\epsilon], |\xi(\nu) - \eta(\nu)| \geq \epsilon\},$$

the function $\nu \in \mathcal{U}_2(\epsilon) \mapsto (\xi(\nu), \eta(\nu)) \in \mathbb{R}^2$ is continuous.

B The Dunford–Pettis theorem

Let E be a separable Banach space, E' the dual space equipped with the weak topology and $(X, \mathcal{B}, m)$ a standard measurable space, where X is metrizable, complete and separable, $\mathcal{B}$ is its Borel σ-algebra and m is a σ-finite measure on $\mathcal{B}$. The Dunford–Pettis theorem [4, Chap. VI, §2, n° 5] tells us that the dual of L^1_E can be identified with $L^\infty_{E'}$. In particular, this theorem shows that the unit ball $B^\infty_{E'}$ of $L^\infty_{E'}$ is weakly compact.

Definition B.1. Let $\mathcal{L}^1_E$ be the space of measurable functions $\psi : X \to E$ such that $\int \|\psi_x\|\, dm(x) < +\infty$ and $\mathcal{L}^\infty_{E'}$ be the space of measurable functions $\nu : X \to E'$ such that $\operatorname{ess\,sup}_{x\in X} |\nu_x(\psi)| < +\infty$. We define two equivalence relations:

$$\begin{aligned} \psi \sim \psi' \quad &\Longleftrightarrow \quad m\big(\{x \in X : \psi_x \neq \psi'_x\}\big) = 0, \\ \nu \sim \nu' \quad &\Longleftrightarrow \quad \forall \psi \in E \quad m\big(\{x \in X : \ \nu_x(\psi) \neq \nu'_x(\psi)\}\big) = 0. \end{aligned}$$

Denote the quotient spaces $L^1_E = \mathcal{L}^1_E / \sim$ and $L^\infty_{E'} = \mathcal{L}^\infty_{E'} / \sim$. They become, respectively, a Banach space where $\|\psi\| = \int \|\psi_x\|\, dm(x)$ and a Frechet space where the weak topology is defined by the family of semi-norms

$$p_{\varphi,\psi}(\nu) = \int |\varphi(x)\nu_x(\psi)|\, dm(x), \quad \varphi \in L_{\mathbb{R}}, \ \psi \in E.$$

If $(\varphi_i)_{i\geq 0}$ is dense in $L^1_{\mathbb{R}}$ and $(\psi_j)_{j\geq 0}$ is dense in E, then the vector space generated by $(\varphi_i\psi_j)_{i,j\geq 0}$ is dense in L^1_E. A continuous linear form T on L^1_E gives rise to a continuous bilinear form on $L^1_{\mathbb{R}} \times E$ by $B(\varphi, \psi) = T(\varphi\psi)$. The main part of the Dunford–Pettis proof consists in showing that such a bilinear form can be represented by an element $\nu \in L^\infty_{E'}$.

Proposition B.2. (i) *If B is a continuous bilinear form on $L^1_{\mathbb{R}} \times E$, then there exists a unique $\nu \in L^\infty_{E'}$ such that $B(\varphi, \psi) = \int \varphi(x)\nu_x(\psi)\,dm(x)$.*

(ii) *Conversely, for any $\nu \in L^\infty_{E'}$, the bilinear form defined above is continuous and satisfies $\|B\| = \text{ess sup}_{x\in X} \|\nu_x\| < +\infty$.*

Proof. (i) For every $\psi \in E$, the map $\varphi \in L^1_{\mathbb{R}} \mapsto B(\varphi, \psi)$ is a continuous linear form on $L^1_{\mathbb{R}}$ which can be represented by some $\tilde{\nu}(\cdot, \psi)$ in $L^\infty_{\mathbb{R}}$, so that $B(\varphi, \psi) = \int \varphi(x)\tilde{\nu}(x, \psi)\,dm(x)$ for all $\varphi, \psi \in L^1_{\mathbb{R}} \times E$. As a function of $\psi \in E$, $\tilde{\nu}(\cdot, \psi) \in L^\infty_{\mathbb{R}}$ is linear, continuous and $\text{ess sup}_{x\in X} |\tilde{\nu}(x, \psi)| \leq \|B\|\,\|\psi\|$. The next lemma shows such a continuous linear map can be lifted to $\mathcal{L}^\infty_{\mathbb{R}}$. Let $\nu(\cdot, \psi) \in \mathcal{L}^\infty_{\mathbb{R}}$ be a linear lift satisfying $\text{ess sup}_{x\in X} |\nu(x, \psi)| \leq \|B\|\,\|\psi\|$. Taking the evaluation at any point $x \in X$, we obtain a continuous linear form $\nu_x : \psi \in E \mapsto \nu(x, \psi) \in \mathbb{R}$ on E that is an element $\nu \in L^\infty_{E'}$. Uniqueness of such a ν requires the separability of E.

(ii) To prove that $\|B\| < \infty$, we use the separability of E and the Banach–Steinhaus theorem. □

Lemma B.3. *If $\psi \in E \mapsto \tilde{\nu}(\psi) \in L^\infty_{\mathbb{R}}$ is a continuous linear map, then there exists a continuous linear map $\psi \in E \mapsto \nu(\psi) \in \mathcal{L}^\infty_{\mathbb{R}}$ such that for all $\psi \in E$, $\nu(\psi) = \tilde{\nu}(\psi)$ in $L^\infty_{\mathbb{R}}$ and $\sup_{x\in X} |\nu(\psi)| = \text{ess sup}_{x\in X} |\tilde{\nu}(\psi)|$.*

Proof. Let $(\psi_i)_{i\geq 0}$ be a sequence of linearly independent vectors in E such that the closure of the span of $(\psi_i)_{i\geq 0}$ is E itself. For each ψ_i we choose a lift $\nu_i \in \mathcal{L}^\infty_{\mathbb{R}}$ of $\tilde{\nu}(\psi_i)$. We define ν on all rational linear combinations $F = \{\sum_{i=1}^n \lambda_i\psi_i \mid \lambda_i \in \mathbb{Q}\}$ by $\nu(\sum_{i=1}^n \lambda_i\psi_i) = \sum_{i=1}^n \lambda_i\nu_i$. Since F is countable, there exists N of measure zero such that $\sup_{X\setminus N} |\nu(\psi)| = \|\tilde{\nu}(\psi)\|_\infty$ for all $\psi \in F$. We may assume now that $\nu_i = 0$ on N. Since $\|\nu(\psi)\|_\infty \leq \|\tilde{\nu}\|\,\|\psi\|_E$ for a dense set of ψ's, we can extend ν continuously to E. □

We are now able to state and prove the canonical isometry between the dual $(L^1_E)'$ and $L^\infty_{E'}$.

Theorem B.4. *If $T \in (L^1_E)'$, then there exists a unique $\nu \in L^\infty_{E'}$ such that $T(\varphi\psi) = \int \varphi(x)\nu_x(\psi)\,dm(x)$ for all $\varphi, \psi \in L^1_{\mathbb{R}} \times E$. The transformation $T \mapsto \nu$ defines a bijective linear map and $\|T\| = \text{ess sup}_{x\in X} \|\nu_x\|$.*

Proof. As we have already seen, T defines a continuous bilinear form B on $L^1_{\mathbb{R}} \times E$ which can be represented by a unique $\nu \in L^\infty_{E'}$. Conversely, let L be the vector space generated by $\{\varphi\psi \mid \varphi \in L^1_{\mathbb{R}}, \psi \in E\}$ and T defined on L by $T(\sum_{i=1}^n \varphi_i\psi_i) = \sum_{i=1}^n \int \varphi_i(x)\nu_x(\psi_i)\,dm(x)$. This is well defined since if $\sum_{i=1}^n \varphi_i\psi_i = 0$, then $\int \nu_x(\sum_{i=1}^n \varphi_i(x)\psi_i)\,dm(x) = 0$. Moreover, $|T(\sum_{i=1}^n \varphi_i\psi_i)| \leq$

$\operatorname{ess\,sup}_{x\in X} \|\nu_x\| \, \|\sum_{i=1}^{n} \varphi_i \psi_i\|_{L^1_E}$ and can therefore be extended continuously to L^1_E with $\|T\| = \operatorname{ess\,sup}_{x\in X} \|\nu_x\|$. □

Corollary B.5. *The unit ball of $L^\infty_{E'}$, $B^\infty_{E'} = \{\nu \in L^\infty_{E'} : \operatorname{ess\,sup}_{x\in X} \|\nu_x\| \leq 1\}$, is compact metrizable with respect to the weak topology given by the semi-norms $p_{\varphi,\psi}(\nu) = \int |\varphi(x)\nu_x(\psi)| \, dm(x)$.*

We now apply this approach to identify the disintegration of a measure on a product space to an element of $B^\infty_{E'}$. Let Y be a locally compact separable space and $E = \mathcal{C}_0(Y, \mathbb{R})$ the space of continuous functions with compact support on Y. If $\nu \in E'$ is positive ($\nu(\psi) \geq 0$ for any $\psi \geq 0$) then, by Riesz's theorem, ν is a Borel measure finite on any compact subset of Y. If in addition $\nu(\mathbf{1}) = 1$, then ν is a probability measure. We denote by $\pi : X \times Y \to X$ the projection onto X.

Proposition B.6. *If $\hat{m}$ is a Borel measure on $X \times Y$ such that $\pi_*(\hat{m}) = m$, then there exists a unique $\nu \in L^\infty_{E'}$ such that for all $\varphi \in L^1_{\mathbb{R}}$, $\psi \in E$*

$$\begin{cases} \iint \varphi(x)\psi(y) \, d\hat{m}(x,y) = \int \varphi(x)\nu_x(\psi) \, dm(x), \\ \nu_x \text{ is positive and } \nu_x(\mathbf{1}) = 1 \text{ a.e.} \end{cases}$$

In other words, $(\nu_x)_{x\in X}$ is a family of Borel probability measures on Y such that for all Borel sets A, B, $\hat{m}(A \times B) = \int_A \nu_x(B) \, dm(x)$ and $\nu_x(B)$ is measurable with respect to x.

Proof. The bilinear form on $L^1_{\mathbb{R}} \times E$, $B(\varphi, \psi) = \iint \varphi(x)\psi(y) \, d\hat{m}(x,y)$ is continuous. By the Dunford–Pettis theorem, there exists $\nu \in L^\infty_{E'}$ such that $B(\varphi, \psi) = \int \varphi(x)\nu_x(\psi) \, dm(x)$ for all $\varphi, \psi \in L^1_{\mathbb{R}} \times E$. If $\varphi, \psi \geq 0$, then $B(\varphi, \psi) \geq 0$, ν_x is a positive linear form on E and therefore a Borel measure on Y. By taking an increasing sequence of positive ψ_n converging pointwise to $\mathbf{1}$, we obtain $\nu_x(\mathbf{1}) = 1$ a.e. Given $A \in \mathcal{B}_X$, the set of $B \in \mathcal{B}_Y$ such that $x \mapsto \nu_x(B)$ is measurable and $\hat{m}(A \times B) = \int_A \nu_x(B) \, dm(x)$ is a monotone class containing the open sets (as increasing limits of positive ψ_n) and therefore equal to $\mathcal{B}_Y$. □

We end this section by proving a technical lemma on convergence.

Lemma B.7. *If $\nu^n \in B^\infty_{E'}$ converges weakly to ν, then for every $\varphi \in L^1_{\mathbb{R}}$ and every $\psi : X \to E$ measurable and essentially bounded we have*

$$\lim_{n\to+\infty} \int \varphi(x)\nu^n_x(\psi_x) \, dm(x) = \int \varphi(x)\nu_x(\psi_x) \, dm(x).$$

Proof. Let $(\psi_i)_{i\geq 0}$ be a dense subset of the ball of radius $\operatorname{ess\,sup}_{x\in X} \|\psi_x\|_\infty$. For each $\epsilon > 0$ we construct a partition $(A^\epsilon_i)_{i\geq 0}$ of X such that A^ϵ_i is a subset of

$\{x \in X : \|\psi_x - \psi_i\|_\infty < \epsilon\}$. Let $\psi_x^\epsilon = \sum_{i\geq 0} \mathbf{1}_{A_i^\epsilon}(x)\psi_i$. Then by Lebesgue's theorem and the weak convergence of ν^n to ν we have

$$\lim_{n\to+\infty} \sum_{i\geq 0} \int_{A_i^\epsilon} \varphi(x)\nu_x^n(\psi_i)\,dm(x) = \sum_{i\geq 0} \int_{A_i^\epsilon} \varphi(x)\nu_x(\psi_i)\,dm(x),$$

which shows the lemma is true for ψ_x^ϵ. Since $|\int \varphi(x)\nu_x^n(\psi_x - \psi_x^\epsilon)\,dm(x)|$ is uniformly bounded by $\|\psi_x - \psi_x^\epsilon\|_\infty \|\varphi\|_{L^1_{\mathbb{R}}}$, we can permute the two limits and the lemma is proved. □

C Conservative dynamics

C.1 Finite measure preserving case We recall in this section basic facts about recurrence of $\mathbb{R}$-valued cocycles and give short proofs for the sake of completeness.

In the sequel, (X, m, ϕ) denotes an abstract dynamical system which is not necessarily ergodic nor invertible. The notions of recurrence and essential values have been introduced in Section 3. The following lemma is a characterization for a cocycle to be a coboundary.

Lemma C.1. *If $a(n,x)$ is a measurable cocycle, then* (i) *a is not equal to a coboundary (on any invariant Borel set of positive measure) if and only if* (ii) *∞ is an essential value for a.*

Proof. (i)⇒(ii). Assume, to the contrary, that there exists $B \in \mathcal{B}_X$ of positive measure and $R > 0$ such that $|a_B(n,x)| \leq R$ for all $n \geq 0$ and almost all $x \in B$. Define

$$S(x) = \sup_{n\geq 0} a_B(n,x) \quad \text{and} \quad I(x) = \inf_{n\geq 0} a_B(n,x).$$

The cocycle property $a_B(n+1,x) = a_B(n,\phi_B(x)) + a_B(1,x)$ implies

$$S(x) \geq S\circ\phi_B(x) + a_B(x) \quad \text{and} \quad I(x) \leq I\circ\phi_B(x) + a_B(x).$$

Then $S - I \geq (S-I)\circ\phi_B$, $S - I$ is ϕ_B-invariant and actually $S - S\circ\phi_B = a_B$ a.e. on B. We extend S to $\tilde{B} = \bigcup_{n\geq 0}\phi^{-n}(B)$ by

$$S(x) = S\circ\phi^{\tau_B(x)} + \textstyle\sum_{k=0}^{\tau_B(x)-1} a\circ\phi^k$$

and verify that $S - S\circ\phi = a$ a.e. on $\tilde{B}$. This is a contradiction.

(ii)⇒(i). Indeed, if $a = c - c\circ\phi$ on a ϕ-invariant set B and $R > 0$ is chosen so that $\tilde{B} = \{x \in B : |c(x)| \leq \frac{1}{2}R\}$ has positive measure, for every $x \in B$ which returns to B ($\phi^n(x) \in B$ for some $n \geq 1$), we have $|a(n,x)| \leq R$, which is a contradiction. □

For integrable $\mathbb{R}$-valued cocycles, there is a simple characterization for a cocycle to be recurrent. Our proof is shorter than the original.

Theorem C.2 ***(Atkinson [2]).*** *If $a : X \to \mathbb{R}$ is an integrable function then* (i) $\int_B a\,dm = 0$ *(for every Borel ϕ-invariant set B) if and only if* (ii) $a(n,x)$ *is a recurrent cocycle.*

Proof. Denote by $\mathcal{I}_\phi$ the σ-algebra of Borel ϕ-invariant sets. Condition (i) is equivalent to $\mathbb{E}\left[a \mid \mathcal{I}_\phi\right] = 0$ a.e., where $\mathbb{E}\left[\cdot \mid \mathcal{I}_\phi\right]$ denotes the conditional expectation. We only prove (i)$\Rightarrow$(ii).

By contradiction, there exist $B \in \mathcal{B}_X$ of positive measure and $\epsilon > 0$ such that for every $n \geq 1$ and a.e. $x \in B$, the induced cocycle $|a_B(n,x)| \geq \epsilon$. Then for all $p > n \geq 0$ and a.e. $x \in B$

$$|a_B(p,x) - a_B(n,x)| = |a_B(p-n, \phi_B^n(x))| \geq \epsilon.$$

On any interval $[-R, R]$, there cannot exist more than $N(R)$ (the integer part of $1 + 2R/\epsilon$) distinct times $n \geq 0$ for which $a_B(n,x) \in [-R, R]$. In other words, there exists at least one $n \in [0, N(R)]$ such that $|a_B(n,x)| > R$. We then construct by induction two increasing sequences of integers $(n_i)_{i\geq 0}$ and $(R_i)_{\geq 0}$ where n_i is the smallest n such that $|a_B(n_i,x)| > R_i$ and

$$R_{i+1} = 1 + \sup\{|a_B(n,x)| :\ n \leq N(R_i)\}$$

(the choice of R_{i+1} implies $n_{i+1} > N(R_i) \geq n_i$). Then for a.e. $x \in B$

$$\liminf_{i\to+\infty} \frac{1}{n_i}|a_B(n_i,x)| \geq \liminf_{i\to+\infty} \frac{R_i}{n_i} \geq \liminf_{i\to+\infty} \frac{(N(R_i)-2)\epsilon}{2n_i} \geq \frac{\epsilon}{2}.$$

If $(\tau_B^n)_{n\geq 0}$ denotes the sequence of successive return times to B, then for every $n \geq 1$, $a_B(n,x) = a(\tau_B^n(x), x)$. Since $(\frac{1}{n}\tau_B^n(x))_{n>0}$ converges a.e. to $\mathbb{E}[\mathbb{1}_B \mid \mathcal{I}_\phi] \leq 1$, the sequence $|\frac{1}{n}a(n,x)|$ converges a.e. to $\left|\mathbb{E}[a \mid \mathcal{I}_\phi]\right| \geq \epsilon/2$ on $\tilde{B} = \bigcup_{n\geq 0} \phi^{-n}(B)$. □

C.2 σ-finite measure preserving case The recurrence of a cocycle is often related to the fact that some skew product extensions are conservative. If the range of a cocycle is not finite (more generally, not compact), the extension is usually σ-finite. We give in this section some properties of conservative dynamical systems.

Definition C.3. A σ-finite abstract dynamical system (X, m, ϕ) is said to be conservative if for any $B \in \mathcal{B}_X$ of positive measure there exists $n \geq 1$ such that $m(B \cap \phi^{-n}(B)) > 0$.

We observe that (X, m, ϕ) is conservative if and only if (X, m, ϕ^{-1}) is conservative too. Ergodic theory can be done for σ-finite conservative dynamical systems. The starting point is the Hurewicz theorem. We shall not use it, but will use instead the main ingredient of its proof, the maximal lemma which we now recall.

Lemma C.4. *If $a : X \to \mathbb{R}$ is an integrable function and B is the Borel set $B = \{x \in X : \exists n \geq 1$ s.t. $a(n,x) > 0\}$, then $\int_B a\, dm \geq 0$.*

The maximal lemma will help us to prove a characterization for a dynamical system to be conservative. We give here a different proof than the one in [18].

Lemma C.5. *Let $a : X \to \mathbb{R}^+$ be a positive and integrable function. Then* (i) *(X, m, ϕ) is conservative if and only if* (ii) *$\sum_{n\geq 0} a \circ \phi^n(x)$ diverges a.e.*

Proof. (ii)$\Rightarrow$(i). Let B be a Borel set of positive measure, b the function $\mathbf{1}_B$ and suppose that $\tilde{X} = \{x \in X : \lim_{n\to+\infty} b(n,x) < +\infty\}$ has positive measure. Since $\tilde{X}$ is ϕ-invariant and $(a(n,x) - b(n,x))_{n\geq 0}$ converges to $+\infty$, by the maximal lemma $\int_{\tilde{X}} (a-b)\, dm \geq 0$. The same equality is also true for $\frac{1}{N}a$ instead of a. By letting N go to $+\infty$, we obtain that $\tilde{X}$ is disjoint from B and that a.e. point in B returns infinitely often into B.

(i)$\Rightarrow$(ii). This part is obvious. □

The following lemma shows that under some conditions related the notion of K-system, if a coboundary $f \circ \phi - f$ is $\mathcal{F}$-measurable for some sub-σ-algebra, then f is $\mathcal{F}$-measurable itself. A more elaborate version is given in [19, Lemma 4.3].

Lemma C.6. *Let (X, m, ϕ) be a conservative, σ-finite and invertible dynamical system and $f : X \to \mathbb{R}$ a $\mathcal{B}_X$-measurable function. If there exists a σ-finite sub-σ-algebra $\mathcal{F} \subset \mathcal{B}_X$ invariant ($\phi^{-1}\mathcal{F} \subset \mathcal{F}$) and generating ($\bigvee_{n\geq 0} \phi^n \mathcal{F} = \mathcal{B}_X$) such that $f \circ \phi - f$ is measurable with respect to $\mathcal{F}$, then f itself is measurable with respect to $\mathcal{F}$.*

Proof. *Case $m(X) < +\infty$.* For every $\epsilon > 0$, there exist $n \geq 0$, a function $\bar{f}_\epsilon$ measurable with respect to $\mathcal{F}$ and a set A_ϵ of measure $m(A_\epsilon) > 1 - \epsilon$ such that for every x in A_ϵ, $|f(x) - \bar{f}_\epsilon \circ \phi^{-n}(x)| < \epsilon$. Then $|f \circ \phi^n - \bar{f}_\epsilon| < \epsilon$ on $B_\epsilon = \phi^{-n}(A_\epsilon)$ and $|f - g_\epsilon| < \epsilon$ on B_ϵ, where $g_\epsilon = f \circ \phi^n - f - \bar{f}_\epsilon$ is $\mathcal{F}$-measurable. On the set $\bigcup_m \bigcap_{n\geq m} B_{1/2^n}$ of full measure, $g_{1/2^n}$ converges pointwise to f, which is thus $\mathcal{F}$-measurable.

The general case. Since $\mathcal{F}$ is σ-finite, it is enough to show that the restriction f_F of f to any set of finite measure $F \in \mathcal{F}$ is $\mathcal{F}$-measurable. Let $F \in \mathcal{F}$ be of finite positive measure, (F, m_F, ϕ_F) the induced map on F and $\mathcal{F}_F$ the σ-algebra $\{B \cap F : B \in \mathcal{F}\}$. We are going to show that $f_F \circ \phi_F - f_F$ is $\mathcal{F}_F$-measurable,

$\mathcal{F}_F$ is ϕ_F-invariant and generating. The first part of the proof will imply that f_F is $\mathcal{F}_F$-measurable.

To prove the assertion, denote by τ the first return time to F and observe that for all $B \in \mathcal{F}$ and $n \geq 1$

$$\{\tau = n\} = F \cap \phi^{-1}(F^c) \cap \cdots \cap \phi^{-n+1}(F^c) \cap \phi^{-n}(F) \in \mathcal{F},$$

$$f_F \circ \phi_F - f_F = \sum_{n \geq 1} \mathbf{1}_{\{\tau = n\}}(f \circ \phi^n - f) \quad \text{is } \mathcal{F}_F\text{-measurable},$$

$$\phi^{-1}(B) = \bigcup_{n \geq 1} F \cap \phi^{-1}(F^c) \cap \cdots \cap \phi^{-n+1}(F^c) \cap \phi^{-n}(B) \in \mathcal{F}_F.$$

To prove that $\mathcal{F}_F$ is generating, we consider for every $n \geq 0$ and $f : X \to \mathbb{R}$ measurable with respect to $\mathcal{F}$ the function $\bar{f}(x) = f \circ \phi^{-n} \circ \phi_F^n(x)$. Since

$$\bar{f} = \sum_{k \geq n} \mathbf{1}_{\{\tau_n = k\}} f \circ \phi^{k-n}(x) \quad \forall x \in F,$$

where τ_n is the nth return time to F, $\bar{f}$ is measurable with respect to $\mathcal{F}$, $f \circ \phi^{-n} = \bar{f} \circ \phi_F^{-n}$ is measurable with respect to $\phi_F^n(\mathcal{F})$ and we have proved that $\left(\phi^n(\mathcal{F})\right)_F \subset \phi_F^n(\mathcal{F}_F)$. Since $(\phi^n(\mathcal{F}))_{n \geq 0}$ generates $\mathcal{B}_X$, $(\phi_F^n(\mathcal{F}_F))_{n \geq 0}$ generates $\mathcal{B}_F$ too. $\square$

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