POSITIVE LYAPUNOV EXPONENT FOR GENERIC ONE-PARAMETER FAMILIES OF UNIMODAL MAPS

By

PH. THIEULLEN, C. TRESSER AND L. S. YOUNG*

Abstract. Let $\{f_a\}_{a\in\mathcal{A}}$ be a C^2 one-parameter family of non-flat unimodal maps of an interval into itself and a^* a parameter value such that

(a) f_{a^*} satisfies the Misiurewicz Condition,

(b) f_{a^*} satisfies a backward Collet–Eckmann-like condition,

(c) the partial derivatives with respect to x and a of $f_a^n(x)$, respectively at the critical value and at a^*, are comparable for large n.

Then a^* is a Lebesgue density point of the set of parameter values a such that the Lyapunov exponent of f_a at the critical value is positive, and f_a admits an invariant probability measure absolutely continuous with respect to the Lebesgue measure. We also show that given f_{a^*} satisfying (a) and (b), condition (c) is satisfied for an open dense set of one-parameter families passing through f_{a^*}.

I. Notations and main results

In [BC2] (see also [BC1]) Benedicks and Carleson proved the following theorem:

Theorem I.1. *Let $q_a = 1 - ax^2$, $0 \le a \le 2$, $-1 \le x \le 1$ be the real quadratic family. Then there exist $0 < \lambda < \log 2$ and a subset $\Omega_\lambda \subseteq [0,2]$ of positive Lebesgue measure such that for all $a \in \Omega_\lambda$:*

$$(*) \qquad \forall n \ge 0 \quad |Dq_a^n(1)| \ge \exp(n\lambda).$$

Property $(*)$ is useful for proving the existence of absolutely continuous invariant measures. The goal of this paper is to put Theorem I.1 into as general a context as possible. Leaving precise statements for later, we prove

(1) the quadratic family $\{q_a\}$ above can be replaced by any one-parameter family of C^2 unimodal maps passing through a *Misiurewicz point* a^* and satisfying certain tranversality conditions;

(2) every *Misiurewicz point* a^* is a Lebesgue density point for the set of parameter values with property $(*)$ for all generic one-parameter families passing through f_{a^*}.

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These results have been announced in [TTY]. Independently, M. Tsujii [Ts], S. van Strien and W. de Melo [MS] have obtained similar results for parameters a^* satisfying different hypotheses. Our line of proof follows very closely that of Benedicks and Carleson. In that sense a large part of this paper consists essentially of a detailed exposition of their work. The final manuscript has been written by the first author.

We now give a precise description of the one-parameter families we consider.

Definition I.2. Let $\mathcal{A}$ be an interval of parameters. A *regular family* $\{f_a\}_{a\in\mathcal{A}}$ is a C^2 one-parameter family of *unimodal* maps with *non-flat* critical point, that is a family of maps $[f_a : x \mapsto f_a(x)]$ of the interval $I = [-1, 1]$ which satisfy the following conditions:

(i) $[(x,a) \mapsto f_a(x)]$ is C^2 with respect to (x,a),

(ii) $c_0 = 0$ is the unique critical point of f_a, f_a is increasing on $[-1,0)$ and decreasing on $(0,1]$, $f_a^2(0) < 0 < f_a^1(0)$ and $f_a^2(0) \le f_a^3(0)$, for all $x \in (-1,0)$, $f_a(x) > x$,

(iii) there exist positive constants A_1^*, A_2^*, C^* and $\tau \ge 2$ such that for all $a \in \mathcal{A}$ and for all $(x,y) \in I$

$$A_1^*|x|^{\tau-1} \le |Df_a(x)| \le A_2^*|x|^{\tau-1} \quad\text{and}\quad \left|\frac{Df_a(x)}{Df_a(y)}\right| \le \exp C^*\left|\frac{x}{y} - 1\right|.$$

We denote by $c_n(a)$ the orbit of the critical point: $c_n(a) = f_a^n(0)$ where $f_a^n = f_a \circ \ldots \circ f_a$ n times. We also use the notation $f^n(x,a) = f_a^n(x)$ and write ∂_x, ∂_a for the derivatives with respect to x and a. A *stable periodic point* for a C^1-map $[f : I \to I]$ is a point x such that for some $p \ge 1$, $f^p(x) = x$ and $|Df^p(x)| \le 1$. We recall also that a unimodal map $[f : I \to I]$ is called *S-unimodal* or has *negative Schwarzian derivative* if it satisfies the following condition:

$$\forall x \neq 0 \quad Sf(x) = -2\sqrt{|f'(x)|}\Big(\frac{1}{\sqrt{|f'x)|}}\Big)'' = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\Big(\frac{f''(x)}{f'(x)}\Big)^2 < 0.$$

We begin by a simplified version of our main result.

Theorem I.3 (a special case) *Let a^* be a parameter value such that:*

(i) *f_{a^*} is a C^3-unimodal map, with negative Schwarzian derivative and $f''_{a^*}(0) \neq 0$,*

(ii) *there exists a constant $N \ge 1$ such that $c_N(a^*)$ is a nonstable periodic point x^* of period p,*

(iii) *if $[a \mapsto \chi(a)]$ denotes a local smooth continuation of x^* (i.e. $\chi(a^*) = x^*$ and $f_a^p(\chi(a)) = \chi(a)$ in a neighborhood of a^*), then*

$$\frac{d}{da}(\chi - c_N)(a^*) \neq 0.$$

Then a^* is a Lebesgue density point of a set of parameters $\mathcal{A}_{K,\lambda}$ of positive Lebesgue measure for some constants K, λ such that for all $a \in \mathcal{A}_{K,\lambda}$:

(a) *for all $n \geq 0$, $|Df_a^n(c_1(a))| \geq K \exp(n\lambda)$,*

(b) *f_a admits an invariant probability measure absolutely continuous with respect to the Lebesgue measure.*

We now introduce a new notion of perturbable parameter a^* which extends the three properties of Theorem I.3:

Definition I.4. Let $\{f_a\}_{a\in\mathcal{A}}$ be a regular family. A parameter a^* is said to be *perturbable*, if one can find a constant $\varepsilon^* > 0$ such that

(CE0)* for every $\delta \in (0, \varepsilon^*)$ and $n \geq 1$, if $x \in I$ satisfies $f_{a^*}^i(x) \notin (-\delta, \delta)$ and $f_{a^*}^n(x) \in (-\delta, \delta)$ for all $i = 0, \ldots, n-1$ then $|Df_{a^*}^n(x)| \geq \varepsilon^*$,

(M) for all $n \geq 1$, $|c_n(a^*)| \geq \varepsilon^*$ and f_{a^*} has no stable periodic point,

(T) $\lim_{n\to+\infty} \quad \partial_a f^n(c_0(a^*), a^*)/\partial_x f^{n-1}(c_1(a^*), a^*) \stackrel{def}{=} Q^* \neq 0$.

Our first result, which is the main one, is the following theorem:

Theorem I.5. *Let $\{f_a\}_{a\in\mathcal{A}}$ be a regular family. For every perturbable point a^*, there exist positive constants K, ε, α and λ such that a^* is a Lebesgue density point of the set Ω of parameter values a which verify the four conditions:*

(NS) *f_a has no stable periodic point,*

(ER) $\forall n \geq 1 \quad |f_a^n(0)| \geq \varepsilon \exp(-n\alpha)$,

(CE1) $\forall n \geq 0 \quad |Df_a^n(f_a(0))| \geq K \exp(n\lambda)$,

(CE2) *for all $n \geq 1$, if $x \in [-1, 1]$ satisfies $f_a^k(x) \neq 0$ for all $k = 0, \ldots, n-1$ and $f_a^n(x) = 0$, then $|Df_a^n(x)| \geq K \exp(n\lambda)$.*

The conditions (CE1) and (CE2) are referred to as the forward and backward Collet–Eckmann conditions; the condition (ER) is what Benedicks and Carleson call the exclusion rule. Before giving the main Corollary I.7, we recall two definitions.

A parameter a^* is said to be a *Lebesgue density point* of a Borel set $\Omega \subseteq \mathcal{A}$ if

$$\lim_{\varepsilon\to 0} \frac{|\Omega \cap (a^* - \varepsilon, a^* + \varepsilon)|}{|\mathcal{A} \cap (a^* - \varepsilon, a^* + \varepsilon)|} = 1,$$

* *Note added in proof*: For C^3-unimodal maps f satisfying the Misiurewicz condition (M) and the nonflatness condition $f''(0) \neq 0$, condition (CE0) is actually a consequence of the Koebe Distortion Principle [St;3.2] and the fact that $\sum_{i=0}^{n} |f^i(I_n)|$ is uniformly bounded in n for any n-homterval I_n [St;9.1]. Using moreover the fact that the length of $|I_n|$ decreases exponentially uniformly [St;11.1], we get to the conclusion: *there exist positive constants ε, K and λ such that, if $n \geq 1$, $x \in [-1, 1]$ satisfies $f^n(x) \in (-\varepsilon, \varepsilon)$ and $f^k(x) \neq 0$ for all $k = 0, 1, \ldots, n-1$, then $|Df^n(x)| \geq Ke^{n\lambda}$.*

where $|\Omega|$ denotes the Lebesgue measure of the set Ω.

A measure μ_a is said to be *invariant with respect to* f_a if $\mu_a(f_a^{-1}(B)) = \mu_a(B)$ for every Borel set $B \subseteq I$, and μ_a is said to be *absolutely continuous with respect to the Lebesgue measure* if $\mu_a(B) = 0$ for every Lebesgue-negligible Borel set B.

Using a result of T. Nowicki and S. van Strien [NS1; Theorem A], we obtain the celebrated Theorem of M. Jakobson [Ja1] and [Ja2] (see also [BC1], [Ry] and for the complex case [Re]).

Theorem I.6 [NS1] *Let $[f : I \to I]$ be a C^2 unimodal map with a unique non-flat critical point. If f satisfies the conditions (NS), (CE1) and (CE2), then f admits an invariant probability measure absolutely continuous with respect to the Lebesgue measure.*

Corollary I.7 [Ja1], [Ja2] *If $\{f_a\}_{a\in\mathcal{A}}$ is a regular family, any perturbable point is a Lebesgue density point of the set of parameter values a such that f_a admits an invariant probability measure absolutely continuous with respect to the Lebesgue measure.*

The definition of a perturbable point requires several remarks:

- The first condition (CE0) is technical and is only used to prove Theorem II.3 (essentially to prove that the exponent λ we obtain in Theorem II.1 is independent of the neighborhood the critical orbit avoids). We conjecture that this condition is always satisfied for maps satisfying the Misiurewicz condition. If f_{a^*} happens to have negative Schwarzian derivative, the condition is automatically true.

Lemma I.8 [CE; Appendix A] *Let $[f : I \to I]$ be a S-unimodal map satisfying condition (M). There exists a positive constant ε^* such that for all $n \geq 1$ and $x \in I$, if x satifies $f^n(x) \in (-\varepsilon^*, \varepsilon^*)$ and $f^k(x) \neq 0$ for all $k = 0, \dots, n-1$, then $|Df^n(x)| \geq \varepsilon^*$.*

- The second condition (M) is referred to as the *Misiurewicz condition* and says that the critical point is not recurrent. The fact that f_{a^*} do not possess stable periodic point (condition (NS)) ensures exponential growth for any sufficiently long orbits staying outside any neighborhood of the critical point (see Theorem II.1). This condition generalizes the *Strong Misiurewicz Condition* where an iterate of the critical point reaches a nonstable periodic point, (i.e. $c_N(a^*) = \chi(a^*)$ with $f_a^p(\chi(a)) = \chi(a)$ and $|Df_a^p(\chi(a))| > 1$ for all a in a neighborhood of a^*).

- The third condition (T) is a kind of transversality condition. It shows in which way the one-parameter family has to cross f_{a^*}. It should be noticed that condition (M) ensures the existence of the limit in (T) as is explained in Lemma VII.1. In the particular case where f_{a^*} satisfies the Strong Misiurewicz Condition, condition (T) is equivalent to the transversality of the two curves $[a \mapsto c_N(a)]$ and $[a \mapsto \chi(a)]$ (see Section VII.1 for a proof).

Proposition I.9 [Re], [Ru], [Ry] *If f_{a^*} satisfies the Strong Misiurewicz condition then*

$$Q(a^*) = \lim_{n\to+\infty} \frac{\partial_a f^n(c_0(a^*), a^*)}{\partial_x f^{n-1}(c_1(a^*), a^*)} = \left(\frac{dc_N}{da}(a^*) - \frac{d\chi}{da}(a^*)\right)/Df_a^{N-1}(c_1(a^*)).$$

For a map f_{a^*} which satisfies the more general Misiurewicz condition (M), the transversality condition is equivalent to the fact that $\{\frac{dc_n}{da}(a^*)\}_{n\geq 1}$ is not bounded (see Proposition VII.7). In the particular case of the quadratic family $q_a(x) = 1 - ax^2$, one can show [Do] that any parameter satisfying the Strong Misiurewicz condition also satisfies the transversality condition (T) (for $a^* = 2$, the computation shows $Q(a^*) = -1/3$), and consequently is a perturbable point ($\{q_a\}_{0\leq a\leq 2}$ is a regular family of S-unimodal maps). As the referee suggested, the transversality condition could be formulated as a nontangency of higher order in the case of higher smoothness.

The fact that the transversality condition can be checked after a finite number of iterations, enables us to prove that condition (T) is generic among all regular families passing through a Misiurewicz point. We first define the topology of such families.

Definition I.10 Let $a^* \in \mathcal{A}$ and f^* be a $\mathcal{C}^2$ unimodal map verifying the conditions (M), (NS) and (CE0). We denote by $\mathcal{R}(a^*,f^*)$ the set of regular families $\{f_a\}_{a\in\mathcal{A}}$ such that $f_{a^*} = f^*$. We consider $\mathcal{R}(a^*,f^*)$ as a subset of the space of continuous maps from $\mathcal{A}$ into the space $\mathcal{C}^2(I)$ and we define on $\mathcal{R}(a^*,f^*)$ the uniform $\mathcal{C}^2$-norm: for each $f = \{f_a\}_{a\in\mathcal{A}}$, $\| f \| = \sup_{a\in\mathcal{A},x\in I}\{|f_a(x)| + |Df_a(x)| + |D^2f_a(x)|\}$.

Our second result is the following proposition:

Proposition I.11 *The subset of regular families which satisfy condition (T) at the point a^* is dense and open in $\mathcal{R}(a^*,f^*)$.*

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II. Strategy of the proof

We choose once and for all a regular family $\{f_a\}_{a\in\mathcal{A}}$ and a perturbable point $a^* \in \mathcal{A}$. In order to simplify notation, we denote for all $n \geq 0$ and $a \in \mathcal{A}$

$$c_n(a) = f_a^n(0) \quad \text{and} \quad d_n(a) = Df_a^n(c_1(a)).$$

Before describing in detail the strategy of the proof, we summarize the main ideas in a nontechnical way. We start by choosing a small interval Ω_0 containing a^* and consider all the curves $\{c_n : \Omega_0 \to I\}$. If $c_n(a) = c_0$ for some parameter a, the critical point is a *super attractive* periodic point and we have to eliminate this parameter. Actually, we have to exclude also a small neighborhood V_a about all these parameters a; the size of V_a will depend on dc_n/da over V_a. The transversality condition shows that dc_n/da has the same magnitude as $d_{n-1}(a)$. In order to leave out some subset of Ω_0 of positive measure in the process of elimination, we show that $d_n(a)$ follows an exponential growth. Actually, polynomial growth would be enough (and easier to prove) to obtain Corollary I.7: it would be enough to use the Bound Return Theorem II.7. To obtain exponential growth, we study more carefully the statistic of the sequence $\{c_n\}_{n\geq 1}$ and show that a subsequence $\{c_{T_n}\}_{n\geq 1}$ behaves like a Markov chain.

We now return to a more precise formulation. As f_{a^*} is a $\mathcal{C}^2$ unimodal map without stable periodic point, f_{a^*} possesses a strongly hyperbolic dynamics outside any neighborhood of the critical point. More precisely, M. Misiurewicz, R. Mañé and essentially S. van Strien and W. de Melo have established

Theorem II.1 [Mi], [Mañ], [St], [Me; Theorem III.5.1] *Let $[f : I \to I]$ be a $\mathcal{C}^2$ unimodal map without stable periodic point. For every $\varepsilon > 0$ there exist positive constants $\lambda_{2.1}(\varepsilon)$ and $K_{2.1}(\varepsilon)$ such that, for every $x \in I$ and $n \geq 1$ satisfying $\{x, f(x), \ldots, f^{n-1}(x)\} \cap [-\varepsilon, \varepsilon] = \emptyset$,*

$$|Df^n(x)| \geq K_{2.1} \exp(n\lambda_{2.1}).$$

Using the above theorem and the fact that f_{a^*} satisfies the Misiurewicz condition (M), we can find positive constants ε^*, K^* and λ^* such that for all $n \geq 1$

$$|d_n(a^*)| \geq K^* \exp(n\lambda^*) \quad \text{and} \quad |c_n(a^*)| \geq \varepsilon^*.$$

In order to explain why a^* is a Lebesgue density point, we introduce the following notation.

Notation II.2. For every neighborhood V of a^* and $\delta \in (0, 1)$, we denote by $N_{2.2}(\delta, V)$ the smallest integer N such that $c_N(V) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$. By continuity of the family $\{f_a\}_{a\in\mathcal{A}}$, we obtain $\lim_{V\to\{a^*\}} N_{2.2}(\delta, V) = +\infty$. We also define $N_{2.2}(V) = N_{2.2}(\varepsilon^*, V)$.

Our main goal is to construct by induction a decreasing sequence of subsets $\{\Omega_k\}_{k\geq 0}$ such that Ω_0 is any interval containing a^* sufficiently small and for all $k \geq 1$ and $a \in \Omega_k$

$$\begin{aligned}
&(\mathrm{CE1}-k) && \forall\, 0 \leq i < 2^k N_{2.2}(\Omega_0) \quad |d_i(a)| \geq K^* \exp(i\lambda), \\
&(\mathrm{ER}-k) && \forall\, 1 \leq i \leq 2^k N_{2.2}(\Omega_0) \quad |c_i(a)| \geq \varepsilon^* \exp(-i\alpha), \\
& && |\Omega_{k+1}|/|\Omega_k| \geq 1 - \exp\{-2^{k-1}\alpha N_{2.2}(\Omega_0)\},
\end{aligned}$$

where $\lambda \in (0, \lambda^*)$ and $\alpha \in (0, \lambda)$ are constants which will be determined later. If the orbit has accumulated the expansion $|d_{n-1}(a)| \geq K^* \exp((n-1)\lambda)$ at time n, the exclusion rule (ER) allows us to keep some expansion at time $n+1$:

$$|d_n(a)| = |d_{n-1}(a)| \times |Df_a(c_n(a))| \geq A_1^* |d_{n-1}(a)| \times |c_n(a)|^{\tau-1} \geq K' e^{n(\lambda - \tau\alpha)},$$

for some constant $K' > 0$. We stress that all the constants chosen in the course of our proof will depend only on f_{a^*}. Following [BC1] and [BC2], we divide our proof into four main steps, which we now describe. Our proof for the last step differs slightly from the one in [BC2].

In the first step (see Section III. Perturbed Misiurewicz Theorem), we show that the orbit of any point x recovers an exponent $\lambda_{2.3} \in (0, \lambda^*)$ independent of δ as long as its orbit stays outside the neighborhood $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$ of c_0. More precisely, we prove:

Theorem II.3 *Let $\{f_a\}_{a \in \mathcal{A}}$ be a regular family and $a^* \in \mathcal{A}$ be a parameter satisfying (M) and (CE0). There exist positive constants $\delta_{2.3}$, $\lambda_{2.3}$, $K_{2.3}$ and for all $\delta \in (0, \delta_{2.3})$ a neighborhood $V_{2.3}(\delta)$ of a^* and a constant $K_{2.3}(\delta)$ such that, if $x \in I$, $a \in V_{2.3}(\delta)$, $n \geq 1$ and $\{x, f_a(x), \ldots, f_a^{n-1}(x)\} \cap (-\frac{1}{2}\delta, \frac{1}{2}\delta) = \emptyset$, then:*

(i) $|Df_a^n(x)| \geq K_{2.3}(\delta) \exp(n\lambda_{2.3})$,
(ii) *if $f_a^n(x) \in [-\delta_{2.3}, \delta_{2.3}]$, then* $|Df_a^n(x)| \geq K_{2.3} \exp(n\lambda_{2.3})$,
(iii) *if $(x, f_a^n(x)) \in [-\delta_{2.3}, \delta_{2.3}]$, then* $|Df_a^n(x)| \geq \exp(n\lambda_{2.3})$.

From now on we fix $\lambda \in (0, \lambda_{2.3})$, $\alpha \in (0, \alpha_{2.3}(\lambda))$ and $\tilde{\lambda} = \frac{1}{2}(\lambda + \lambda_{2.3}) + 10\alpha\tau$, where $\alpha_{2.3}(\lambda) = \lambda \min(\lambda, \lambda_{2.3} - \lambda)/100\tau\Lambda^*$ and $\Lambda^* = \log \sup_{x,a} |Df_a(x)|$.

Remark II.4 If f_{a^*} satisfies (M), Theorem II.3 is actually equivalent to the condition (CE0). ii) is a $\mathcal{C}^2$ version of a result already proved in [CE2; Appendix A].

In the second step (see Section IV. Bound Return Theorem), we prove a version (Theorem II.7) of Benedicks and Carleson's Bound Return Theorem which is more appropriate to our purpose. When the orbit of the critical point $\{c_i(a)\}_{i=1}^n$ returns to itself, $c_n(a)$ is close enough to c_0 so that the two orbits $\{c_{n+k}(a)\}_{k=0}^{p-1}$ and $\{c_k(a)\}_{k=1}^{p-1}$ become bound during a period p, where p is defined so that the distortion of the maps f_a^k, $k = 0, \ldots, p-1$, stays bounded. If $c_n(a)$ and c_0 are not too close, the period p is equal to a small proportion of n and the orbit $\{c_{n+k}(a)\}_{k=0}^{p}$ captures part of its past exponent:

$$|Df_a^p(c_n(a))| \geq \exp\left(p\frac{\lambda - 2\tau\alpha}{\tau}\right).$$

In order to define a *bound period p* globally for any $a \in \omega$ belonging to some interval of parameters ω, we begin to partition I into a countable number of subintervals:

Definition II.5 Let $\mathcal{M} = \{\mu_i\}_{i \geq 0}$ be the sequence defined by induction by $\mu_0 = 0$, $\mu_1 = 2$ and for all $i \geq 1$

$$\frac{1}{\mu_i} = \frac{\exp(-\mu_i) - \exp(-\mu_{i+1})}{\exp(-\mu_{i+1})}.$$

Let $\tilde{I}(\mu_i)$ denote the interval $(\exp(-\mu_{i+1}), \exp(-\mu_i)]$ and $\tilde{I}(-\mu_i)$ the symmetric of $\tilde{I}(\mu_i)$ about the origin.

It may happen that, for some ω and some integer n, $c_n(\omega)$ is not equal to a union of intervals $\tilde{I}(\mu_i)$. We define therefore a notion of *state*..

Definition II.6 A state is any interval $I(\mu_i)$ which satisfies

$$\tilde{I}(\mu_i) \subseteq I(\mu_i) \subseteq \tilde{I}(\mu_{i-1}) \cup \tilde{I}(\mu_i) \cup \tilde{I}(\mu_{i+1}),$$

and by the same convention: $I(-\mu_i) = -I(\mu_i)$.

Theorem II.7 (Bound Return Theorem) *Let $\{f_a\}_{a \in \mathcal{A}}$ be a regular family, $\lambda \in (0, \lambda_{2.3})$ and $\alpha \in (0, \alpha_{2.3}(\lambda))$. There exist constants $\delta_{2.7} = \exp(-\Delta_{2.7}) \in (0,1)$, $D_{2.7} \geq 1$ and a neighborhood $V_{2.7}$ of a^* such that, if $\omega \subseteq V_{2.7}$ is an interval satisfying:*

(i) *$c_n(\omega) \subseteq I(r)$ for some $|r| \in \mathcal{M}$ and $\Delta_{2.7} \leq |r| \leq n\alpha$,*

(ii) *$\forall a \in \omega\, \forall 1 \leq k \leq n\ |c_k(a)| \geq \varepsilon^* \exp(-k\alpha)$ and $|d_{k-1}(a)| \geq K^* \exp(k-1)\lambda/10$,*

(iii) *$\forall a \in \omega \quad \forall 1 \leq k \leq n/10 \quad |d_k(a)| \geq K^* \exp(k\lambda)$,*

then there exists an integer $p = p(n, r, \omega)$ such that for all $(a, s, t) \in \omega$ and $\omega' \subseteq \omega$:

(a) $(1/3\Lambda^*)|r| \leq p \leq (2\tau/\lambda)|r|$,

(b) $|Df_a^p(c_n(a))| \geq \exp\{p(\lambda - 2\tau\alpha)/\tau\}$,

(c) $\forall 1 \leq k \leq p \quad |Df_a^k(c_n(a))| \geq \exp(-\tau n\alpha + k\lambda)$,

(d) $\forall 1 \leq k < p+n \quad |c_k(a)| \geq 2\exp(-|r|) \geq \varepsilon^* \exp(-k\alpha)$,

(e) $\sum_{k=0}^{p-1} |(c_{n+k}(s) - c_{n+k}(t))/c_{n+k}(t)| \leq D_{2.7}|(c_n(s) - c_n(t))/c_n(t)|$,

(f) $|c_{n+p}(\omega')|/|c_n(\omega')| \geq \exp(-3\alpha p)/|I(r)|$,

(g) *for all $n \leq k < n+p$, if $c_k(\omega) \cap [-\frac{1}{2}\delta_{2.7}, \frac{1}{2}\delta_{2.7}] \neq \emptyset$, then $c_k(\omega) \subseteq I(r')$ for some $|r'| \geq \Delta_{2.7}$.*

In future, we will call $p(n,r,\omega)$ the *bound period associated* to (n,r,ω). If we combine Theorems II.3 and II.7, we obtain a simple criterion for the non-existence of stable periodic point and the backward Collet–Eckman condition:

Corollary II.8 *Let $\{f_a\}_{a\in\mathcal{A}}$ be a regular family, $a^* \in \mathcal{A}$ a parameter such that f_{a^*} satisfies the conditions (M) and (CE0), $\lambda \in (0,\lambda_{2.3})$ and $\alpha \in (0,\alpha_{2.3}(\lambda))$. There exists a neighborhood $V_{2.8}$ of a^* such that, if $a \in V_{2.8}$ and verifies:*

$$\text{(CE)} \qquad \forall k \geq 1 \quad |c_k(a)| \geq \varepsilon^* \exp(-k\alpha),$$

$$\text{(CE1)} \qquad \forall k \geq 0 \quad |d_k(a)| \geq K^* \exp(k\lambda),$$

then for all $x \in I$ and $n \geq 1$:

(i) *if $f_a^k(x) \neq 0$ for all $k \geq 1$, then* $\limsup_{n\to+\infty}(1/n)\log|Df_a^n(x)| \geq \lambda/2\tau$,

(ii) *if $f_a^k(x) \neq 0$ for $0 \leq k < n$ and $f_a^n(x) = 0$, then* $|Df_a^n(x)| \geq K^* \exp(n\lambda/2\tau)$.

In the third step (see Section V. Distortion of the tip), we show that the distortion of the map $[a \mapsto c_n(a)]$ is bounded from above uniformly with respect to n on any *adapted* interval ω.

Definition II.9. Let $\Delta \in \mathcal{M}$, $\delta = \exp(-\Delta)$ and $\omega \subseteq \mathcal{A}$ be an interval. We say that ω is (n,Δ)-adapted if for all $1 \leq k < n$, $c_k(\omega) \subseteq I(r_k)$ for some $|r_k| \geq \Delta$, $|r_k| \in \mathcal{M}$ whenever $c_k(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$ and if $c_n(\omega) \subseteq [-2\delta, 2\delta]$.

If ω is an interval of parameters such that c_n is injective on ω and $c_n(\omega)$ contains the critical point c_0, the proportion of parameters which are too close to c_0 at time n depends on the distortion of c_n on ω. The following lemma shows that, under the transversality condition (T), the velocity of the tip $\frac{d}{da}c_n(a)$ and the exponent of the critical orbit have the same magnitude. We choose, once for all, constants Q_1^* and Q_2^* such that:

$$Q_1^* < Q^* \stackrel{\text{def}}{=} \lim_{n\to+\infty} \frac{\partial_a f^n(c_0(a^*),a^*)}{\partial_x f^{n-1}(c_1(a^*),a^*)} < Q_2^*$$

Lemma II.10 *Let $\lambda > 0$. There exist $N_{2.10}(\lambda)$ and a neighborhood $V_{2.10}(\lambda)$ of a^* such that for every $n \geq N_{2.10}$ and $a \in V_{2.10}$, if $|d_k(a)| \geq K^* \exp(k\lambda)$ for all $0 \leq k < n$ then*

$$Q_1^* \leq \left|\frac{1}{d_{n-1}(a)} \frac{dc_n}{da}(a)\right| \leq Q_2^*.$$

Using properties (e), (f) and (g) of Theorem II.7, we can prove our main distortion theorem.

Theorem II.11 (Distortion Theorem) *Let $\lambda \in (0,\lambda_{2.3})$ and $\alpha \in (0,\alpha_{2.3}(\lambda))$. There exist $\delta_{2.11} \in (0,1)$, $D_{2.11} \geq 1$ and for every $\delta = \exp(-\Delta) \in (0,\delta_{2.11})$ a neighborhood $V_{2.11}(\delta)$ of a^* such that, if $n \geq 1$ and $\omega \subseteq V_{2.11}$ is an interval satisfying*

(i) $\forall a \in \omega \quad \forall 1 \leq k < n \quad |c_k(a)| \geq \varepsilon^* \exp(-k\alpha)$ *and* $|d_k(a)| \geq K^* \exp(k\lambda)$,
(ii) ω *is* (n, Δ)*-adapted,*
then for every $(s, t) \in \omega$:

$$\left|\frac{dc_n}{da}(s) \,/\, \frac{dc_n}{da}(t)\right| \leq D_{2.11}.$$

Finally in step four (see Section VI. A Markov-like dynamics), we prove our main induction step: Theorem I.5 is a direct consequence of Corollary II.8 and the following theorem:

Theorem II.12 (induction step) *Let* $\lambda \in (0, \lambda_{2.3})$ *and* $\alpha \in (0, \alpha_{2.3}(\lambda))$. *There exist* $\delta_{2.12} \in (0,1)$ *and for all* $\delta \in (0, \delta_{2.12})$ *a neighborhood* $V_{2.12}(\delta)$ *of* a^* *such that, if* $\omega \subseteq V_{2.12}(\delta)$ *is an interval satisfying:*
(i) $c_n(\omega) = I(r)$ *for some* $|r| \in \mathcal{M}$ *and* $\Delta \leq |r| \leq n\alpha$,
(ii) $\forall a \in \omega \quad \forall 1 \leq i \leq n \quad |c_i(a)| \geq \varepsilon^* \exp(-i\alpha)$ *and* $|d_{i-1}(a)| \geq K^* \exp(i-1)\lambda$,
(iii) $\forall a \in \omega \quad |d_{n-1}(a)| \geq \exp(n-1)\tilde{\lambda}$,
(iv) ω *is* (n, Δ)*-adapted,*
then ω *contains a disjoint union of intervals* ω' *and for each of these* ω'*, one can find* $n' \geq 2n$, $\Delta \leq |r'| \leq \alpha n'$ *such that* (i), (ii), (iii) *and* (iv) *are also verified by* ω', n', r' *and such that* $|\cup \omega'| \geq |\omega|\{1 - \exp(-n\alpha/2)\}$.

The proof of Theorem II.12 requires four main parts.

In the first part (see section VI.A), we show that $c_N(\Omega_0) \cap [-\delta, \delta]$ is a union of states $I(r)$ where $|r| \geq \Delta = -\log\delta$ and $N = N_{2.2}(\delta, \Omega_0)$. The remaining part $c_N(\Omega_0) \cap [-1, -\delta)$ or $c_N(\Omega_0) \cap (\delta, 1]$ is either included in one of the previous states or is equal to one of the intervals $J(\pm\Delta)$, where $J(\mu_i)$ is any interval verifying $I(\mu_{i-1}) \subseteq J(\mu_i) \subseteq (\exp(-\mu_i), 1]$ $(J(-\mu_i) = -J(\mu_i))$. Such intervals will be called *prestates* in the sequel.

In the second part (see section VI.B and VI.C), we assume that, for some interval ω and some integer $n \geq 1$, $c_n(\omega)$ is either equal to a prestate $J(\pm\Delta)$, or to one of the states $I(r)$ where $|r| \geq \Delta$.

If $c_n(\omega) = J(\pm\Delta)$, we call *free period* q, the smallest integer q such that $c_{n+q}(\omega)$ meets $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$. We then show that the length of $c_{n+q}(\omega)$ is at least 3δ and that, in particular, $c_{n+q}(\omega)$ contains again at least one of the two prestates $J(\pm\Delta)$.

If $c_n(\omega) = I(r)$ for some $|r| \geq \Delta$, we denote by p_0 the bounded period associated to (n, r, ω) and by q_0 the first time such that $c_{n+p_0+q_0}(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$. At time $n_1 = n + p_0 + q_0$, it may happen that $c_{n_1}(\omega)$ is included into some $I(r_1)$, $|r_1| \geq \Delta$; q_0 is then called a *partially free period.* We begin again the process: we denote by p_1 the bound period associated to (n_1, r_1, ω) which may be followed by a partially free period q_1. Let $n_2 = n_1 + p_1 + q_1$. We stop the process until $c_{n_{u+1}}(\omega)$ contains a state $I(r)$ with $|r| \geq \Delta$. We call *essential bound period* p associated to (n,r,ω), the total sum of bound and partialy free periods:

$$p = (p_0 + q_0) + \cdots + (p_{u-1} + q_{u-1}) + p_u .$$

We show that p satisfies all the properties stated in Theorem II.7 and we notice that, by construction, $c_{n+p+q}(\omega)$ contains at least one of the states $I(r)$, $|r| \geq \Delta$ and possibly one of the prestates $J(\pm\Delta)$, where q is the smallest integer such that $c_{n+p+q}(\omega)$ meets $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$.

In the third part (see section VI.D), we choose an interval ω which satisfies the assertions (i)...(iv) of Theorem II.12. We construct by induction on $0 \leq k \leq n$ an increasing sequence of partitions of ω. At stage k, ω is a disjoint union of intervals of the form $\omega(^{t_0 \cdots t_k}_{r_0 \cdots r_k})$ where $t_0 = n$, $r_0 = r$, $t_0 \leq \cdots \leq t_k$ and $|r_i| \geq \Delta$. By definition, $\omega(^{t_0}_{r_0}) = \omega$ and each $\omega' = \omega(^{t_0 \cdots t_k}_{r_0 \cdots r_k})$ is a disjoint union of intervals of the form $\omega'' = \omega(^{t_0 \cdots t_{k+1}}_{r_0 \cdots r_{k+1}})$.

The induction process stops as soon as the exclusion rule is violated or the time t_k exceeds $2n$. If $|r_k| > \alpha t_k$ or $t_k > 2n$ then $r_{k+1} = r_k$, $t_{k+1} = t_k$ and $\omega'' = \omega'$. If $|r_k| \leq \alpha t_k$ and $t_k \leq 2n$ then t_{k+1} and r_{k+1} are defined in the following way. Let p_k be the essential bound return associated to (t_k, r_k, ω'), where $\omega' = \omega(^{t_0 \cdots t_k}_{r_0 \cdots r_k})$ and q the smallest integer such that $c_{t_k+p_k+q}(\omega') \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$.

Either $c_{t_k+p_k+q}(\omega')$ is equal to a union of states $I(r)$, $|r| \geq \Delta$, then $q_k = q$, $t_{k+1} = t_k + p_k + q$ and ω' is equal to a disjoint union of intervals $\omega'' = \omega'(r_{k+1})$ corresponding to the part of ω' which is mapped by $c_{t_k+p_k+q}$ onto $I(r_{k+1})$.

Or $c_{t_k+p_k+q}(\omega')$ contains also one of the prestates $J(\pm\Delta)$. We note $t'_{k+1} = t_k + p_k + q$, by $\omega'(r_{k+1})$ the part of ω' which is mapped by $c_{t_k+p_k+q}$ onto $I(r_{k+1})$ and by $\omega'(\pm)$, the part of ω' mapped onto $J(\pm\Delta)$. Let $q(\pm)$ be the free period associated to $\omega'(\pm)$ and $t'_{k+1}(\pm) = t'_{k+1} + q(\pm)$. We already know from part two that $c_{t'_{k+1}(\pm)}(\omega'(\pm))$ contains again states and prestates. We note by $\omega'(\pm)(r_{k+1})$ the part of $\omega'(\pm)$ mapped by $c_{t'_{k+1}(\pm)}$ onto $I(r_{k+1})$ and by $\omega'(\pm,\pm)$ the part mapped onto $J(\pm\Delta)$. Let $q(\pm,\pm)$ be the free period associated to $\omega'(\pm,\pm)$ and $t'_{k+1}(\pm,\pm) = t'_{k+1}(\pm) + q(\pm,\pm)$ and so on.

We continue this process and obtain by induction a partition of ω' into a disjoint union of intervals of the form $\omega'' = \omega'(\pm,\ldots,\pm)(r_{k+1})$ where $t_{k+1} = t'_{k+1}(\pm,\ldots,\pm)$. By extention, we call also $q_k = q + q(\pm) + \cdots + q(\pm,\ldots,\pm)$ a free period; during such a period, the orbit of ω'' is disjoint from $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$.

We next prove that each $\omega' = \omega(^{t_0, \ldots, t_k}_{r_0, \ldots, r_k})$ is (t_k, Δ)-adapted. We already noticed that for each $1 \leq i \leq k$: $c_{t_i}(\omega') \subseteq I(r_i)$ and $c_{t_k}(\omega') = I(r_k)$. Given such interval ω', it may happen that the state $I(r_{k+1})$ is reached at infinitely many distinct times t_{k+1};

the main result of this part is the following estimate:

$$\sum_{t_{k+1}} \frac{|\omega\binom{t_0,\dots,t_{k+1}}{r_0,\dots,r_{k+1}}|}{|\omega\binom{t_0,\dots,t_k}{r_0,\dots,r_k}|} \le |I(r_{k+1})| \exp(2\Delta + 4\alpha p_k).$$

In the fourth part (see section VI.E), we begin to exclude all the intervals $\omega\binom{t_0,\dots,t_n}{r_0,\dots,r_n}$ which do not satisfy the exclusion rule $|r_n| > \alpha t_n$, and we denote by ω' the union of the remaining intervals. We then prove

$$\frac{|\omega \setminus \omega'|}{|\omega|} \le \frac{1}{2} \exp\Big(- n\frac{\alpha}{2}\Big).$$

Let us denote by $\mathbf{P}$ the normalized Lebesgue measure on ω'. Benedicks and Carleson's main idea in their second paper [BC2] is to consider $\{c_n\}_{n\ge 1}$ as a sequence of random variables on ω'. We define therefore $\{T_k\}_{k=0}^n$, $\{R_k\}_{k=0}^n$, $\{P_k\}_{k=0}^n$ and S by their values on each $\omega\binom{t_0,\dots,t_n}{r_0,\dots,r_n}$: $T_k(a) = t_k$, $R_k(a) = r_k$, $P_k(a) = p_k$ and $S(a) = \max\{s = 0,\dots,n : t_s \le 2n\}$. We define also $\{Q_k\}_{k=0}^n$ as usual by $T_k = P_k + Q_k$.

Let $\tilde{\Delta} = 100\Delta$. If $\Delta \le |R_i| \le \tilde{\Delta}$, we shall say that the corresponding essential bound period P_i is *short* and for $|R_i| > \tilde{\Delta}$, that the period is *long*. During a free period Q_k, the orbit gains an exponent $\lambda_{2.3}$ provided that $\omega \subseteq V_{2.3}(\delta)$ and the same exponent during a short bound period if we choose in addition $\omega \subseteq V_{2.3}(\tilde{\delta})$. On the other hand, the exponent during a long period depends on the exponent of the past orbit and is equal to $(\lambda - 2\tau\alpha)/\tau$ (Theorem II.7). We then show that the contribution of short bound periods is such that we recover an exponent $\tilde{\lambda}$ at time T_{S+1} and an exponent λ during the period $[T_0, T_{S+1}]$:

$$\mathbf{P}\Big(\sum_{i=0}^{S} P_i \mathbf{1}_{(|R_i|>\tilde{\Delta})} > n\frac{\lambda_{2.3} - \tilde{\lambda}}{\lambda_{2.3}}\Big) \le \frac{1}{2}\exp(-n\frac{\alpha}{2}),$$

where $\mathbf{1}_B$ denotes the characteristic function of a set B.

III. Perturbed Misiurewicz Theorem

The purpose of this section is to prove Theorem II.3. We start by recalling some notations; we then prove a "non-perturbed bound return lemma", in the sense of Benedicks and Carleson, for regular maps; and finally we prove Theorem II.3 for regular families.

We call a *regular map* $f : I \to I$, any C^2 unimodal map with a unique critical point $c_0 = 0$ satisfying the non-flatness condition:

$$A_1^* \le \frac{|Df(x)|}{|x|^{\tau-1}} \le A_2^* \quad \text{and} \quad \Big|\frac{Df(x)}{Df(y)}\Big| \le \exp\Big\{C^*\Big|\frac{x}{y} - 1\Big|\Big\}.$$

We recall also that f satisfies the Misiurewicz condition (M) if

$$\exists \varepsilon^* > 0 \quad \forall n \geq 1 \quad |c_n| \geq \varepsilon^* .$$

For such maps, Theorem II.1 tells us that the orbit of the critical point possesses strong hyperbolic properties; in particular, there exist constants $K^* > 0$ and $\lambda^* > 0$ such that for all $n \geq 1$ and $x \in I$

$$\{x, f(x), \dots, f^{n-1}(x)\} \cap [-\frac{1}{2}\varepsilon^*, \frac{1}{2}\varepsilon^*] \neq \emptyset \quad \Rightarrow \quad |Df^n(x)| \geq K^* \exp(n\lambda^*) .$$

We are now ready to state and prove a bound return lemma for regular maps satisfying the Misiurewicz condition. This lemma has been used previously by Collet–Eckmann ([CE; Appendix A]). If x is a given point close enough to the critical point, as long as the distance between x and c_0 stays bounded, x captures the hyperbolicity of c_0: in particular, after the bound period $p \sim \log(1/|x|)$, the derivative of f^p at x has recovered $|Df^p(x)| \sim |Df^{p-1}(c_1)|^{1/\tau} \sim |f^p((0,x))|/|(0,x)|$.

Lemma III.1 *Let $f : I \to I$ be a regular map satisfying the Misiurewicz condition. There exist $\delta_{3.1} \in (0,1)$, $D_{3.1} \geq 1$ and $K_{3.1} \in (0,1)$ such that for all $0 < |x| < \delta_{3.1}$, there exists an integer $p = p_{3.1}(x)$ which verifies the following properties:*

(i) $\forall j = 0, 1, \dots, p-1 \quad |f^j(0,x)| \leq \frac{1}{2}\varepsilon^*$ *and* $|f^p(0,x)| > \frac{1}{2}\varepsilon^*$,

(ii) $(\tau/2\Lambda^*) \log(1/|x|) \leq p \leq (2\tau/\lambda^*) \log(1/|x|)$,

(iii) $K_{3.1}|Df^{p-1}(c_1)|^{1/\tau} \leq |Df^p(x)| \leq K_{3.1}^{-1}|Df^{p-1}(c_1)|^{1/\tau}$,

(iv) $K_{3.1}|Df^{p-1}(c_1)|^{1/\tau} \leq |f^p(0,x)| \,/\, |(0,x)| \leq K_{3.1}^{-1}|Df^{p-1}(c_1)|^{1/\tau}$,

(v) $\forall j = 0, 1, \dots, p-1 \quad \forall (y,z) \in (0,x) \quad |Df^j(f(y))| \leq D_{3.1}|Df^j(f(z))|$.

Proof The first property is actually a definition of p. We notice in particular that f^j is monotone on $(0,x)$, since $f^{j-1}(0,x)$ never contains the critical point c_0 and $Df^j(x) \neq 0$ for all $j = 1, \dots, p$.

We start by proving that f^j has bounded distortion on $f(0,x)$ for all $j = 0, 1, \dots, p-1$. The second non-flatness condition implies

$$\forall (y,z) \in (0,x) \quad \left|\frac{Df^j(f(y))}{Df^j(f(z))}\right| \leq \exp\left\{C^* \sum_{i=1}^{j} \left|\frac{f^i(y) - f^i(z)}{f^i(z)}\right|\right\},$$

and Theorem II.1 implies that, for all $t \in f^i(y,z)$, $|Df^{p-i}(t)| \geq K^* \exp\{(p-i)\lambda^*\}$. In particular, we obtain

$$\begin{aligned} |I| = 2 \geq |f^p(y) - f^p(z)| &= \int_{f^i(y)}^{f^i(z)} |Df^{p-i}(t)|\, dt \\ &\geq \quad K^* |f^i(y) - f^i(z)| \exp\{(p-i)\lambda^*\} \end{aligned}$$

and we can chöose $D_{3.1} = \exp\{(2C^* / \varepsilon^* K^*)(1 - \exp - \lambda^*)^{-1}\}$.

We use now this distortion result to compare the derivative of f^{p-1} at $f(0)$ and at $f(t)$, $t \in (0,x)$ and the non-flatness condition to estimate the "bad" derivative $Df(t)$:

$$A_1^* |t|^{\tau-1} D_{3.1}^{-1} |Df^{p-1}(c_1)| \leq |Df^p(t)| \leq A_2^* |t|^{\tau-1} D_{3.1} |Df^{p-1}(c_1)|\,. \tag{1}$$

The length $|f^p((0,x))| = \int_{(0,x)} |Df^p(t)|\, dt$ is computed in the same manner:

$$\frac{A_1^*}{\tau D_{3.1}} |Df^{p-1}(c_1)|.|x|^\tau \leq |f^p(0,x)| \leq \frac{A_1^* D_{3.1}}{\tau} |Df^{p-1}(c_1)|.|x|^\tau\,.$$

On the other hand, $\frac{1}{2}\varepsilon^* \leq |f^p(0,x)| \leq |Df| \frac{1}{2}\varepsilon^*$, and we obtain

$$\frac{\tau\varepsilon^*}{2A_2^* D_{3.1}} \leq |Df^{p-1}(c_1)|.|x|^\tau \leq \frac{\tau D_{3.1}|Df|\varepsilon^*}{2A_1^*}. \tag{2}$$

In order to get rid of constants, we choose $\delta_{3.1}$ such that $\delta_{3.1} \leq (\tau\varepsilon^*/2A_2^* D_{3.1})^{2/\tau}$ and $\delta_{3.1} \leq (2A_1^*/\tau D_{3.1}|Df|\varepsilon^*)^{1/\tau}$, which proves assertion (ii):

$$\exp(p\lambda^*) \leq |Df^{p-1}(c_1)| \leq \exp(p\Lambda^*) \quad \text{and} \quad |x|^{-\tau/2} \leq |Df^{p-1}(c_1)| \leq |x|^{-2\tau}\,.$$

If we combine the inequalities (1) and (2), we obtain

$$\begin{aligned} \frac{A_1^*}{D_{3.1}} \left[\frac{\tau\varepsilon^*}{2A_2^* D_{3.1}}\right]^{(\tau-1)/\tau} |Df^{p-1}(c_1)|^{1/\tau} &\leq |Df^p(x)| \\ &\leq A_2^* D_{3.1} \left[\frac{\tau D_{3.1}|Df|\varepsilon^*}{2A_1^*}\right]^{(\tau-1)/\tau} |Df^{p-1}(c_1)|^{\frac{1}{\tau}}\,, \end{aligned}$$

which proves assertion (iii) if we choose $K_{3.1}$ such that

$$K_{3.1} \leq \frac{A_1^*}{\tau D_{3.1}} \left[\frac{\tau\varepsilon^*}{2A_2^* D_{3.1}}\right]^{(\tau-1)/\tau} \quad \text{and} \quad K_{3.1}^{-1} \geq \frac{A_2^* D_{3.1}}{\tau} \left[\frac{\tau D_{3.1}|Df|\varepsilon^*}{2A_1^*}\right]^{(\tau-1)/\tau}.$$

Assertion (iv) is proven in the same manner as (iii) using

$$\frac{A_1^*}{\tau}|x|^{\tau-1} \leq \frac{|f((0,x))|}{|(0,x)|} \leq \frac{A_2^*}{\tau}|x|^{\tau-1}\,,$$

$$D_{3.1}^{-1}|Df^{p-1}(c_1)| \leq \frac{|f^p((0,x))|}{|f((0,x))|} \leq D_{3.1}|Df^{p-1}(c_1)|.$$

□

We consider now a regular family $\{f_a\}_{a\in\mathcal{A}}$ and a^* a parameter in $\mathcal{A}$. We assume in the sequel of this section that f_{a^*} satisfies the Misiurewicz condition, the non-flatness condition and the technical Collet–Eckmann condition. We apply Theorem II.1 for f_{a^*}: for every $\varepsilon > 0$ there exist constants $K_{2.1}(\varepsilon)$ and $\lambda_{2.1}(\varepsilon)$ such that, for all $x \in I$ and $n \geq 1$,

$$\{x, f_{a^*}(x), \ldots, f_{a^*}^{n-1}(x)\} \cap (-\varepsilon, \varepsilon) = \emptyset \quad \Longrightarrow \quad |Df_{a^*}^n(x)| \geq K_{2.1}(\varepsilon)\exp\{n\lambda_{2.1}(\varepsilon)\}.$$

We want to show that f_a satisfies a stronger hyperbolic condition, where the above constants $K_{2.1}(\varepsilon)$ and $\lambda_{2.1}(\varepsilon)$ can be chosen independently of ε and a in a neighborhood of a^*.

Proof of Theorem II.3 Let $\delta < \delta_{2.3}$; $\delta_{2.3}$ is a constant we shall define later. For any point $x \in I$ whose orbit $\{x, f_a(x), \ldots, f_a^{n-1}(x)\}$ is disjoint from $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$, either the orbit is already disjoint from $(-\delta_{2.3}, \delta_{2.3})$, in which case we shall choose $K_{2.3} = K_{2.1}(\delta_{2.3})$, $\lambda_{2.3} = \lambda_{2.1}(\delta_{2.3})$, or we can define a sequence of increasing times

$$t_0 = 0 \leq t_1 < \cdots < t_{s-1} < t_s \leq n = t_{s+1},$$

where t_1 is the first time $k \geq 0$ such that $f_a^k(x) \in (-\delta_{2.3}, \delta_{2.3})$, t_2 is the second time $k > t_1$ such that $f_a^k(x) \in (-\delta_{2.3}, \delta_{2.3})$ and so forth. By the chain rule, the derivative at x of f_a^n is equal to

$$Df_a^n(x) = \prod_{k=0}^{s} Df_a^{t_{k+1}-t_k}(f_a^{t_k}(x)).$$

We now notice that both Lemma III.1 and Theorem II.1, applied to f_{a^*}, can be perturbed with respect to a: if a is close enough to a^*, we can choose the constants $\lambda_{2.1}$, $K_{2.1}$, $D_{3.1}$ and $K_{3.1}$ uniformly with respect to a. We choose once and for all

$$\delta_{2.3} = \min\left(\frac{1}{2}\varepsilon^*, \delta_{3.1}, \left(\frac{K_{3.1}K^*}{8}\right)^{4\Lambda^*/\lambda^*}\right), \quad m_{2.3} = \frac{2}{\lambda_{2.1}(\delta_{2.3}/2)}\log\frac{2}{K_{2.1}(\delta_{2.3}/2)},$$

$$\lambda_{2.3} = \min\left(\frac{\lambda^*}{2T}, \frac{1}{2}\lambda_{2.1}\left(\frac{1}{2}\delta_{2.3}\right), \frac{1}{m_{2.3}}\log 2\right).$$

We choose $V_{2.3}(\delta)$ sufficiently close to a^* so that for all $\delta < |y| < \delta_{2.3}$ and $a \in V_{2.3}(\delta)$:

(i) there exists $p = p_{3.1}(y,a) \geq 1$ such that $\{f_a(y), \ldots, f_a^{p-1}(y)\} \cap (-\delta_{2.3}, \delta_{2.3}) = \emptyset$ and $|Df_a^p(y)| \geq (4/K^*)\exp(p\lambda_{2.3})$,

(ii) if for some $q \geq 1$, $\{y, f_a(y), \dots, f_a^{q-1}(y)\} \cap (-\delta_{2.3}, \delta_{2.3}) = \emptyset$ and $f_a^q(y) \in (-\delta_{2.3}, \delta_{2.3})$ then: $|Df_a^q(y)| \geq (K^*/4)\exp(q\lambda_{2.3})$.

The proof of the first assertion comes from the fact that, in Lemma III.1, $p = p_{3.1}(y, a^*)$ for any $\delta < |y| < \delta_{2.3}$, goes to infinity when $\delta_{2.3}$ goes to 0 but is bounded from above by a constant depending on δ:

$$\frac{\tau}{2\Lambda^*}\log\frac{1}{\delta_{2.3}} \leq p \leq \frac{2\tau}{\lambda^*}\log\frac{1}{\delta},$$

$$|Df_a^p(y)| \geq \frac{1}{2}|Df_{a^*}^p| \geq \frac{1}{2}K_{3.1}\exp\left(p\frac{\lambda^*}{\tau}\right) \geq \frac{4}{K^*}\exp(p\lambda_{2.3}).$$

For the proof of the second assertion, we decompose q into periods of length $m_{2.3}$ (except maybe the last one). During a period of length $m_{2.3}$

$$\begin{aligned}|Df_a^{m_{2.3}}(y)| \geq \frac{1}{2}|Df_{a^*}^{m_{2.3}}(y)| &\geq \frac{1}{2}K_{2.1}\left(\frac{\delta_{2.3}}{2}\right)\exp\left\{m_{2.3}\lambda_{2.1}\left(\frac{\delta_{2.3}}{2}\right)\right\}\\ &\geq \exp\left\{\frac{1}{2}m_{2.3}\lambda_{2.1}\left(\frac{\delta_{2.3}}{2}\right)\right\}.\end{aligned}$$

During a period of length r less than $m_{2.3}$, we use the condition (CE0)

$$|Df_a^r(y)| \geq \frac{1}{2}|Df_{a^*}^r(y)| \geq \frac{1}{2}K^* \geq \frac{1}{4}K^*\exp(m_{2.3}\lambda_{2.3}) \geq \frac{1}{4}K^*\exp(r\lambda_{2.3})$$

and the fact that $V_{2.3}(\delta)$ is chosen sufficiently close to a^* such that, for all $a \in V_{2.3}(\delta)$, $0 \leq k < m_{2.3}$ and y satisfying $\{y, f_a(y), \dots, f_a^k(y)\} \cap (-\delta_{2.3}, \delta_{2.3}) = \emptyset$,

$$|f_a^k(y) - f_{a^*}^k(y)| < \frac{1}{2}\delta_{2.3} \quad \text{and} \quad |Df_a^k(y)| \geq \frac{1}{2}|Df_{a^*}^k(y)|.$$

Using these two assertions, we can finish the proof. During the period $t_{k+1} - t_k$, $1 \geq k \geq s-1$, the point $y = f_a^{t_k}(x)$ follows the orbit of the critical point $\{c_0(a^*), \dots, c_{p-1}(a^*)\}$ and then stays outside $(-\delta_{2.3}, \delta_{2.3})$ during a period q until it enters at time t_{k+1} the interval $(-\delta_{2.3}, \delta_{2.3})$: $t_{k+1} - t_k = p_k + q_k$ and

$$|Df_a^{t_{k+1}-t_k}(f_a^{t_k}(x))| \geq \exp\{(t_{k+1} - t_k)\lambda_{2.3}\}.$$

During the last period $t_{s+1} - t_s$, either $f_a^{t_{s+1}}(x) \in (-\delta_{2.3}, \delta_{2.3})$ and we can choose $K_{2.3} = \frac{1}{4}K^*$ if $x \notin (-\delta_{2.3}, \delta_{2.3})$ and $K_{2.3} = 1$ if $x \in (-\delta_{2.3}, \delta_{2.3})$, or $f_a^{t_{s+1}}(x)$ is still outside $(-\delta_{2.3}, \delta_{2.3})$. Let us set $y = f_a^{t_s}(x)$. If $p_s \leq t_{s+1} - t_s$, then

$$|Df_a^{t_{s+1}-t_s}(y)| \geq \frac{4K_{2.1}(\delta_{2.3}/2)}{K^*}\exp\{(t_{s+1} - t_s)\lambda_{2.3}\}$$

and we can choose $K_{2.3} = 4K_{2.1}(\delta_{2.3}/2)/K^*$. If $p > t_{s+1} - t_s$, using property (v) of Lemma III.1, we have

$$|Df_a^{t_{s+1}-t_s}(y)| = |Df_a(y)|\,|Df_a^{t_{s+1}-t_s-1}(f_a(y))| \geq \frac{A_1}{2D_{3.1}}\delta^{\tau-1}\exp\{(t_{s+1}-t_s)\lambda_{3.1}\}$$

and we can choose $K_{2.3}(\delta) = A_1\delta^{\tau-1}/2\delta_{3.1}$. □

Remark III.2 The exponent λ of the main Theorem I.4 may be as close as we want to $\lambda_{2.3}$. Unfortunately, $\lambda_{2.3}$ depends on the constants $K_{2.1}(\varepsilon)$ and $\lambda_{2.1}(\varepsilon)$ of Theorem II.1 which are not easily computable. We could also get rid of the condition (CE0) by allowing $K_{2.1}(\varepsilon)$ to depend on ε in the following way:

$$\lim_{\varepsilon\to 0}\frac{\log K_{2.1}(\varepsilon)}{\log\varepsilon} = 0\,.$$

IV. Bound Return Theorem

In this section, we prove Theorem II.7 which is the main step in the proof in Benedicks–Carleson's Theorem I.1. It says roughly that, if the orbit of the critical point is hyperbolic and does not come too fast to itself, uniformly on some interval ω of parameters,

$$\forall\, 1 \leq k \leq n \quad \forall\, a \in \omega \quad |c_k(a)| \geq \varepsilon^* e^{-k\alpha} \quad \text{and} \quad |d_{k-1}(a)| \geq K^* e^{(k-1)\lambda}\,,$$

the orbit recovers some exponential growth, approximately equal to $-\log|c_n(a)|$, after a period p called the *bound period*:

$$\forall\, a \in \omega \quad |Df_a^p(c_n(a))| \geq \exp p\Big(\frac{\lambda - 2\alpha\tau}{\tau}\Big)\,.$$

IV.A. Intermediate lemmas

We start by stating and proving several lemmas on the distortion of $f_a^n(x)$ with respect to both x and a. We define, for all n, $\|\partial_a f^n\| = \sup_{(x,a)}|\partial_a f^n(x,a)|$ and $\|\partial_x f^n\| = \sup_{(x,a)}|\partial_x f^n(x,a)|$. We define also the quotient

$$Q_n(a) = |\partial_a f^n(c_0,a)|/\partial_x f^{n-1}(c_1,a)|\,.$$

Proof of Lemma II.10 For every $(x,a) \in I \times \mathcal{A}$, we have by definition $f^{n+1}(x,a) = f(f^n(x,a),a)$. If we differentiate this equality with respect to x and a, we get

$$\partial_x f^n(c_1(a),a) = \partial_x f(c_n(a),a)\partial_x f^{n-1}(c_1(a),a)\,,$$

$$\partial_a f^{n+1}(c_0, a) = \partial_x f(c_n(a), a)\partial_a f^n(c_0, a) + \partial_a f(c_n(a), a).$$

In particular, we obtain a recurrence formula for Q_n:

$$Q_{n+1}(a) = Q_n(a) + \frac{\partial_a f(c_n(a), a)}{\partial_x f^n(c_1(a), a)}.$$

If $n > N$ by induction we get

$$|Q_n(a) - Q_N(a)| \leq \sum_{k=N}^{n-1} \frac{\| \partial_a f \|}{|\partial_x f^n(c_1(a), a)|} \leq \frac{\| \partial_a f \| e^{-N\lambda}}{K^*(1 - e^{-\lambda})} = R_N.$$

We first choose $N = N_{2.10}$ large enough such that $Q_1^* < |Q_N(a^*)| - R_N$, $|Q_N(a^*)| + R_N < Q_2^*$ and then choose a neighborhood $V_{2.10}$ about a^* such that these two inequalities still hold for all $a \in V_{2.10}$. □

Lemma IV.1 *For every $n \geq 1$ we have the uniform bound*

$$\frac{\|\partial_a f^n\|}{\|\partial_x f\|^n} \leq \frac{\|\partial_a f\|}{\|\partial_x f\| - 1}.$$

Proof We use once more the formula $\partial_a f^{n+1}(x, a) = \partial_x f(x_n, a)\partial_a f^n(x, a) + \partial_a f(x_n, a)$ where $x_n = f^n(x, a)$. By induction we get

$$\partial_a f^{n+1}(x, a) = \sum_{k=0}^{n} \partial_a f(x_k, a)\partial_x f^{n-k}(x_{k+1}, a).$$

□

Corollary IV.2 *With the same notations as in Lemma II.10, for all $\omega \subsetneq V_{2.10}(\lambda)$ and for all $n > m > N_{2.10}(\lambda)$,*

$$\frac{|c_n(\omega)|}{|c_m(\omega)|} \geq \frac{Q_1^*}{Q_2^*} \inf\{Df_a^{n-m}(c_m(a)) : a \in \omega\}.$$

Proof

$$|c_n(\omega)| = \int_\omega |\partial_a f^n(c_0, s)| ds \geq Q_1^* \int_\omega |Df_s^{n-1}(c_1(s))| ds,$$
$$Df_s^{n-1}(c_1) = Df_s^{n-m}(c_m) Df_s^{m-1}(c_1),$$
$$|c_m(\omega)| = \int_\omega |\partial_a f^m(c_0, s)| ds \leq Q_2^* \int_\omega |Df_s^{m-1}(c_1)| ds.$$

□

Lemma IV.3 *Let (α, λ) be such that $\alpha < \Lambda^*/\tau$. There exist constants $N_{4.3} \geq 1$, $D_{4.3} \geq 1$ and a neighborhood $V_{4.3}$ of a^* such that, if $\omega \subseteq V_{4.3}$, $n \geq N_{4.3}$, $1 \leq p \leq n\lambda/20\Lambda^*$ and $x \in I$ satisfy,*

(i) $\forall a \in \omega\ \forall 0 \leq k < n \quad |d_k(a)| \geq K^* \exp(k\lambda/10)$,

(ii) $\forall a \in \omega\ \forall 1 \leq k < p \quad |f_a^k(x)| \geq \frac{1}{2}\varepsilon^*(1-e^{-\alpha})\exp(-k\alpha)$,

then, for all $s, t \in \omega$, $|Df_s^p(x)| \leq D_{4.3}|Df_t^p(x)|$.

Proof Using the non-flatness condition, we compute the distortion as usual:

$$\left|\frac{Df_s^p(x)}{Df_t^p(x)}\right| = \left|\frac{Df_s(x)}{Df_t(x)}\right| \prod_{k=1}^{p-1} \left|\frac{Df_s(f_s^k(x))}{Df_t(f_t^k(x))}\right|$$

$$\leq \frac{A_2^*}{A_1^*} \exp \sum_{k=1}^{p-1} \left\{ C^* \frac{|f_s^k(x) - f_t^k(x)|}{|f_t^k(x)|} + \frac{\|\partial_{xa}^2 f\| \ |s-t|}{A_1^* |f_t^k(x)|^{\tau-1}} \right\}.$$

Moreover, using the previous Lemma IV.1 and the hypothesis (i) on ω, we get

$$|f_s^k(x) - f_t^k(x)| \leq \frac{\|\partial_a f\|}{\|\partial_x f\| - 1} \|\partial_x f\|^k \, |s-t|,$$

$$1 \geq |c_n(\omega)| = \int_\omega |\partial_a f^n(c_0, s)| \, ds \geq Q_1^* \int_\omega |d_{n-1}(s)| \, ds \geq Q_1^* K^* |\omega| \exp\{(n-1)\lambda/10\}.$$

The total distortion is therefore bounded by

$$\left|\frac{Df_s^p(x)}{Df_t^p(x)}\right| \leq \frac{A_2^*}{A_1^*} \exp\left\{ C' \sum_{k=0}^{p-1} \exp\left\{ k(\tau\alpha + \Lambda^*) - n\frac{\lambda}{10} \right\} \right\} \leq \frac{A_2^*}{A_1^*} \exp\left\{ \frac{C'}{\exp(2\Lambda^*) - 1} \right\},$$

where $C' = \{(1-e^{-\alpha})C^* \|\partial_a f\| / (\|\partial_x f\| - 1) + \|\partial_{xa}^2 f\| e^\lambda / A_1^* \varepsilon^{*\tau}\} / Q_1^* K^*$ is a uniform constant. By hypothesis on α and p, the expression on the right hand side is bounded uniformly in n by a constant depending only on α. □

IV.B. Proof of Theorem II.7

Step 1. Definition of the bound period

To simplify the notations, we assume that $r > 0$, that is $c_n(\omega) \subseteq (0, 1]$. We denote by $(a, b) = \omega$ (or $(b, a) = \omega$) the endpoints of ω such that $0 < c_n(b) < c_n(a)$. We define the bound period $p = p(n, r, \omega)$, in a unique way, by the following inequalities:

$$\forall 0 \leq k \leq p-1 \quad |c_k(a) - c_{n+k}(a)| < \varepsilon^* \exp(-2k\alpha),$$

$$|c_p(a) - c_{n+p}(a)| \geq \varepsilon^* \exp(-2p\alpha).$$

In particular, $f_a^k(0, c_n(a))$ never intersects c_0 whenever $k < n$.

Step 2. Distortion of f_a^k on (c_1, c_{n+1})

We show the existence of a constant $D'(\alpha)$ such that

$$\left|\frac{Df_a^k(f_a(x))}{Df_a^k(f_a(y))}\right| \le D'(\alpha)$$

for every $x, y \in (c_0, c_n(a))$, $0 \le k \le p-1$. Let us call $x_k = f_a^k(x)$; then

$$\left|\frac{Df_a^k(x_1)}{Df_a^k(y_1)}\right| \le \exp\left\{C^* \sum_{i=1}^{k} \left|\frac{x_i - y_i}{y_i}\right|\right\}.$$

Since f_a^k is strictly monotone on $(c_0, c_n(a))$,

$$|x_i - y_i| \le |c_i(a) - c_{n+i}(a)| \le \varepsilon^* \exp(-2i\alpha).$$

We get, by definition of p,

$$|y_i| \ge \varepsilon^*(1 - \exp(-i\alpha))\exp(-i\alpha) \ge \varepsilon^*(1 - e^{-\alpha})\exp(-i\alpha)$$

and the first constant of distortion: $D'(\alpha) = \exp\{C^*/(1 - \exp(-\alpha))^2\}$.

Step 3. Bound from above of p

Using the fact that f_a^{p-1} behaves almost linearly on (c_1, c_{n+1}), we get

$$1 \ge |f_a^p(c_n) - f_a^p(c_0)| = \int_0^{c_n} |Df_a^p(x)|\, dx \ge \frac{A_1^*}{D'(\alpha)} |Df_a^{p-1}(c_1)| \int_0^{c_n} x^{\tau-1}\, dx,$$

$$1 \ge \frac{A_1^* K^*}{\tau D'(\alpha)} |c_n|^\tau \exp\{(p-1)\lambda\}$$

(provided that $p \le n/10$). Since $c_n(\omega)$ belongs to some $I(r)$, $|c_n(a)| \ge \frac{1}{4}\exp(-r)$ and $\exp(p\lambda) \le 4^\tau e^\lambda \tau D'(\alpha) e^{r\tau}/A_1^* K^*$. We get rid of the constant by choosing $\Delta_{2.7}$ sufficiently big so that $\tau 4^\tau e^\lambda D'(\alpha)/A_1^* K^*$ is smaller than $\exp(\tau\Delta_{2.7})$. We still have to prove that p satisfies the induction hypothesis $p \le n/10$. By the above inequality $p \le (2\tau/\lambda)r$ and, by hypothesis, $\varepsilon^* e^{-n\alpha} \le |c_n(a)| \le 2e^{-r}$. If we choose $\Delta_{2.7}$ big enough, $2/\varepsilon^* \le \exp(\frac{1}{2}\Delta_{2.7})$, we get $r \le 2n\alpha$ and $p \le (4\tau\alpha/\lambda)n \le n/10$, by the choice of $\alpha < \lambda/40\tau$.

Step 4. Bound from below of p

We use the same inequalities but in reverse order. By definition of p

$$|f_a^p(c_n) - f_a^p(0)| \ge \varepsilon^* \exp(-2p\alpha).$$

On the other hand, we bound uniformly $Df_a(x)$ by $\|\partial_x f\|$ and we get

$$\int_0^{c_n} |Df_a^p(x)|\, dx \leq \frac{A_2^*}{\tau} |c_n|^\tau \exp(p\Lambda^*).$$

Once more, we use the fact that $c_n(a)$ belongs to $I(r)$: $|c_n(a)| \leq 2r^{-r}$, and in order to get rid of the constant, we choose $\Delta_{2.7}$ such that $A_2^* 2^\tau/\varepsilon^*\tau \leq \exp(\Delta_{2.7}(\tau - 1))$. Finally, combining these three inequalities, we obtain $r \leq p(\Lambda^* + 2\alpha) \leq 3p\Lambda^*$.

Step 5. Exclusion rule during the bound period

We want to show that, for all $s \in \omega$ and $1 \leq k \leq p$,

$$|c_{n+k}(s)| \geq \varepsilon^* \exp\{-\alpha(n+k)\}.$$

As f_a^k is monotone on $(0, c_n)$, $|f_a^k(c_n(s))| \geq \varepsilon^*(1 - e^{-\alpha})e^{-k\alpha}$, for every $s \in \omega$. Using Lemmas IV.1 and IV.2, we obtain

$$|f_s^k(c_n(s)) - f_a^k(c_n(s))| \leq \frac{\|\partial_a f\|}{\|\partial_x f\| - 1} \exp(k\Lambda^*)\, |s - a|,$$

$$|s - a| \leq \frac{1}{Q_1^* K^*} \exp\Big(-\frac{n}{10}\lambda\Big).$$

We choose $V_{2.7}$ sufficiently small such that $(1 - e^{-\alpha})e^{-k\alpha} \geq 2e^{-(k+n)\alpha}$, and $\|\partial_a f\| \leq Q_1^* K^*(\|\partial_x f\| - 1)\varepsilon^* e^{n\alpha}$, for all $n \geq N_{2.2}(\delta, V_{2.7})$. After simplification, we have

$$|f_s^k(c_n(s)) - f_a^k(c_n(a))| \leq \varepsilon^* \exp\Big(k\Lambda^* - \frac{n}{10}\lambda + n\alpha\Big) \leq \varepsilon^* \exp\{-(n+k)\alpha\},$$

because of the bound from above of p: $p \leq (4\alpha\tau/\lambda)n$, and the hypothesis on α: $\alpha < \lambda^2/100\tau\Lambda^*$.

Step 6. Uniform distortion of $f_s^p(x)$ for all $s \in \omega$, $x \in c_n(\omega)$

We claim there exists a constant $D''(\alpha)$ depending only on α such that

$$\left|\frac{Df_s^k(x)}{Df_t^k(y)}\right| \leq D''(\alpha),$$

for all $0 \leq k \leq p$, $(s,t) \in \omega$ and $(x,y) \in c_n(\omega)$. Using step 1 and the fact $I(r) \subseteq (\frac{1}{4}e^{-r}, 2e^{-r})$, we see that

$$\left|\frac{Df_a^k(x)}{Df_a^k(y)}\right| \leq \frac{8^\tau A_2^*}{A_1^*} D'(\alpha).$$

It is therefore enough to compute $|Df_s^k(x)/Df_t^k(x)|$ for all $(s,t) \in \omega$, $x \in c_n(\omega)$ and, by Lemma IV.3, to check that $|f_s^k(x)| \geq \frac{1}{2}(1-e^{-\alpha})\varepsilon^* e^{-k\alpha}$ for all $s \in \omega$, $x \in c_n(\omega)$ and $1 \leq k \leq p-1$. Using the techniques in step 5, we have

$$|f_s^k(x) - f_a^k(x)| \leq \frac{\|\partial_a f\|}{Q_1^* K^*(\|\partial_x f\| - 1)} \exp\left(k\Lambda^* - \frac{n}{10}\lambda\right),$$

$$|f_a^k(x)| \geq \varepsilon^*(1-e^{-\alpha})e^{-k\alpha}.$$

Using the assumption $\alpha < \lambda^2/100\tau\Lambda^*$ and the bound from below $k \leq (4\alpha\tau/\lambda)n$, we have

$$\exp(k\Lambda^* - \frac{n}{10}\lambda) \leq \exp\left(-k\alpha - \frac{n}{50}\lambda\right)$$

and the claim is proven if we choose $\Delta_{2.7}$ sufficiently large so that $\|\partial_a f\|/Q_1^* K^* \|\partial_x f\|$ is smaller than $\frac{1}{2}\varepsilon^*(1-e^{-\alpha})\exp(n\lambda/50)$.

Step 7. Growth after the bound period

We show

$$\forall s \in \omega \quad |Df_s^p(c_n(s))| \geq \exp\left\{p\frac{\lambda - 2\tau\alpha}{\tau}\right\}.$$

By the chain rule, the non-flatness condition and the distortion inequalities obtained in steps 2 and 6, we have

$$|Df_s^p(c_n(s))| \geq \frac{1}{D''(\alpha)}|Df_a^p(c_n(a))| \geq \frac{A_1^*}{D'(\alpha)D''(\alpha)}|c_n(a)|^{\tau-1}\,|d_{p-1}(a)|.$$

On the other hand, by definition of p

$$\varepsilon^* \exp(-2p\alpha) \leq \int_0^{c_n(a)} |Df_a^p(x)|\,dx \leq \frac{A_2^* D'(\alpha)}{\tau}|c_n(a)|^{\tau}\,|d_{p-1}(a)|.$$

Combining these two inequalities, we eliminate $|c_n(a)|$:

$$|Df_a^p(c_n(a))| \geq \frac{A_1^*}{D'(\alpha)D''(\alpha)}\left[\frac{\varepsilon^*\tau}{A_2^* D'(\alpha)}\right]^{(\tau-1)/\tau} \left|d_{p-1}(a)\right|^{1/\tau} \exp\left(-2\frac{\tau-1}{\tau}\alpha p\right).$$

We finally eliminate the constant by choosing $\Delta_{2.7}$ such that

$$\exp\left(\frac{2\alpha r}{3\tau\Lambda^*}\right)\frac{A_1^*}{D'(\alpha)D''(\alpha)}\left[\frac{\varepsilon^*\tau}{A_2^* D'(\alpha)}\right]^{\frac{\tau-1}{\tau}} \geq 1.$$

Step 8. Growth during the bound period

The techniques are the same as in the previous step:

$$|Df_s^k(c_n(s))| \geq \frac{1}{D''(\alpha)}|Df_a^k(c_n(a))| \geq \frac{A_1^*}{D'(\alpha)D''(\alpha)}|c_n(a)|^{\tau-1}|d_{k-1}(a)|,$$

$$|Df_s^k(c_n(s))| \geq \frac{A_1^*(\varepsilon^*)^{\tau-1}}{D'(\alpha)D''(\alpha)} \exp\{-n(\tau-1)\alpha + k\lambda\} \geq \exp(k\lambda - n\alpha\tau).$$

We choose $V_{2.7}$ so that $(A_1^*(\varepsilon^*)^{\tau-1}/D'(\alpha)D''(\alpha))\exp\{N_{2.2}(\delta, V_{2.7})\alpha\} \geq 1$.

Step 9. Distortion of c_n on ω

We claim the existence of a constant $D'''(\alpha)$ depending only on α such that

$$\frac{1}{D'''(\alpha)}\left|\frac{c_{n+k}(a) - c_k(a)}{c_n(a)}\right| \leq \frac{|c_{n+k}(\omega')|}{|c_n(\omega')|} \leq D'''(\alpha)\left|\frac{c_{n+k}(a) - c_k(a)}{c_n(a)}\right|$$

for all $\omega' \subseteq \omega$ and $0 \leq k \leq p$. We apply Lemma IV.2 to obtain

$$\frac{|c_{n+k}(\omega')|}{|c_n(\omega')|} \leq \frac{Q_2^*}{Q_1^*} \sup_{s\in\omega} |Df_s^k(c_n(s))|,$$

provided that $n/10 \geq N_{2.10}(\lambda)$ and $\omega \subseteq V_{2.10}(\lambda)$. We can prove similarly

$$\frac{|c_{n+k}(\omega')|}{|c_n(\omega')|} \geq \frac{Q_1^*}{Q_2^*} \inf_{s\in\omega} |Df_s^k(c_n(s))|.$$

Using the distortion inequalities of steps 2 and 6,

$$\frac{1}{D''(\alpha)}|Df_a^k(c_n(a))| \leq |Df_s^k(c_n(s))| \leq D''(\alpha)|Df_a^k(c_n(a))|,$$

$$|c_{n+k}(a) - c_k(a)| = \int_0^{c_n(a)} |Df_a^k(x)|\,dx$$

and

$$\frac{A_1^*}{A_2^*\tau D'(\alpha)}|Df_a^k(c_n(a))| \leq \frac{|c_{n+k}(a) - c_n(a)|}{|c_n(a)|} \leq \frac{A_2^* D'(\alpha)}{A_1^*\tau}|Df_a^k(c_n(a))|.$$

The claim is proven if we choose $D'''(\alpha) = \tau(Q_2^*/Q_1^*)(A_2^*/A_1^*)D'(\alpha)D''(\alpha)$.

As a corollary of that distortion estimate (recalling $|c_n(a)| \geq |c_n(t)|$), we get

$$\left|\frac{c_{n+k}(s) - c_{n+k}(t)}{c_{n+k}(t)}\right| \leq D'''(\alpha)\left|\frac{c_n(s) - c_n(t)}{c_n(t)}\right| \left|\frac{c_{n+k}(a) - c_k(a)}{c_{n+k}(t)}\right|.$$

The second part of the proof of step 6 gives $|c_{n+k}(t)| \geq \frac{1}{2}\varepsilon^*(1 - e^{-\alpha})e^{-\alpha k}$, which can be combined with the previous inequality:

$$\sum_{k=0}^{p-1}\left|\frac{c_{n+k}(s) - c_{n+k}(t)}{c_{n+k}(t)}\right| \leq \frac{2D'''(\alpha)}{1 - e^{-\alpha}} \sum_{k=0}^{p-1} \exp(-k\alpha)\left|\frac{c_n(s) - c_n(t)}{c_n(t)}\right|.$$

This gives the definition of $D_{2.7}$: $D_{2.7} = 2D'''(\alpha)/(1-e^{-\alpha})^2$.

Step 10. Exponential growth of $c_n(\omega)$ after the bound period

We use the other inequality in the previous distortion estimate

$$\frac{|c_{n+p}(\omega')|}{|c_n(\omega')|} \geq \frac{1}{D'''(\alpha)} \frac{|c_{n+p}(a) - c_n(a)|}{|c_n(a)|} \geq \frac{\varepsilon^*}{4rD'''(\alpha)} \frac{\exp(-2p\alpha)}{|I(r)|}.$$

(We recall: $|c_n(a)| \leq 2e^{-r} \leq 4r|I(r)|$.) We thus obtain

$$\frac{|c_{n+p}(\omega')|}{|c_n(\omega')|} \geq \frac{\exp(-3p\alpha)}{|I(r)|}$$

if we choose $\Delta_{2.7}$ sufficiently large so that for all $r \geq \Delta_{2.7}$

$$\frac{\varepsilon^*}{4D'''(\alpha)} \exp(p\alpha) \geq \frac{\varepsilon^*}{4D'''(\alpha)} \exp(r\alpha/3\Lambda^*) \geq r.$$

Step 11. Controlled return of the orbit of ω.

We show that, whenever $c_{n+k}(\omega)$ intersects $[-\frac{1}{2}\delta_{2.7}, \frac{1}{2}\delta_{2.7}]$, $\delta_{2.7} = \exp(-\Delta_{2.7})$, $c_{n+k}(\omega)$ is contained in an interval $I(r_k)$ for some $|r_k| \geq \Delta_{2.7}$.

Let s be a parameter in ω for which $|c_{n+k}(s)| < \frac{1}{2}\delta_{2.7}$; there exists then $r = \mu_i \in \mathcal{M}$, $r \geq \Delta_{2.7}$ such that

$$\exp(-\mu_{i+1}) < |c_{n+k}(s)| \leq \exp(-\mu_i).$$

Using the distortion inequality 9 and the second part of step 6, we get

$$|c_{n+k}(\omega)| \leq D'''(\alpha)|c_{n+k}(a) - c_k(a)| \leq \varepsilon^* D'''(\alpha)\exp(-2k\alpha),$$

$$e^{-r} \geq |c_{n+k}(s)| \geq \frac{1}{2}\varepsilon^*(1-e^{-\alpha})e^{-k\alpha}.$$

Combining these two inequalities, we eliminate $e^{-k\alpha}$,

$$|c_{n+k}(\omega)| \leq \varepsilon^* D'''(\alpha)\Big[\frac{2}{\varepsilon^*(1-e^{-\alpha})}\Big]^2 e^{-2r}.$$

On the other hand,

$$|I(\mu_i)| \geq \exp(-\mu_i) - \exp(-\mu_{i+1}) \geq \frac{1}{2r}\exp(-r).$$

The claim is proven if we choose $\Delta_{2.7}$ such that

$$\frac{4D'''(\alpha)}{\varepsilon^*(1-e^{-\alpha})^2} \leq \frac{1}{r}e^r \quad (\forall r \geq \Delta_{2.7}).$$

□

IV.C. Existence of absolutely continuous invariant measures

Let $\{f_a\}_{a\in\mathcal{A}}$ be a regular family and $a^* \in \mathcal{A}$ such that f_{a^*} has no stable periodic point, satisfies the Misiurewicz condition (M) and the technical Collet–Eckmann condition (CE0). We show that both the non-existence of a stable periodic point and the backward Collet–Eckmann condition (CE2) are preserved for all parameters in a neighborhood of a^* which satisfy the exclusion rule (ER) and the forward Collet–Eckman condition (CE1).

Proof of Corollary II.8 We begin to prove an estimate about the growth of $|Df^p(x)|$ for any point close to 0. The proof is very similar to that of Theorem II.7 and the details will be skipped. We first define a bound period $p = p(x)$ by

$$|f_a^k(x) - c_k(a)| < \varepsilon^* \exp(-2k\alpha) \quad (\forall 0 \le k < p),$$

$$|f_a^p(x) - c_p(a)| \ge \varepsilon^* \exp(-2p\alpha).$$

Then the growth after the bound period p is given by

$$|Df_a^p(x)| \ge K_{2.8} \exp p\frac{\lambda - 2(\tau-1)\alpha}{\tau}$$

for some constant $K_{2.8} = (A_1^*/D'(\alpha))(\varepsilon^*\tau/A_2^*D'(\alpha))^{(\tau-1)/\tau}$ and the bound period is related to x by $\varepsilon^* \le |x|\exp(2p\Lambda^*)$. We choose $\delta_{2.8} > 0$ such that for all $|x| < \delta_{2.8}$, $K_{2.8}K_{2.3}\exp p(2\alpha/\tau) \ge 1$.

(i) Either $|f_a^n(x)| \ge \delta_{2.8}$ for all n sufficiently large and we are finished by Theorem II.3, or $|f_a^n(x)| < \delta_{2.8}$ for infinitely many n. We define by induction a sequence $q_0, p_1, q_1, \ldots$ by

$q_0 = \inf\{n \ge 0 : f_a^n(x) \in (-\delta_{2.8}, \delta_{2.8})\}$,
$p_1 = p(f_a^{q_0}(x))$,
$q_1 = \inf\{n \ge 0 : f_a^n(f_a^{q_0+p_1}(x)) \in (-\delta_{2.8}, \delta_{2.8})\}$,
$p_2 = p(f_a^{q_0+p_1+q_1}(x)) \ldots$

Using Theorem II.3 and the assumption on $\delta_{2.8}$, we have, for all $a \in V_{2.8} = V_{2.3}(\delta_{2.8})$ and $n = q_0 + p_1 + q_1 + \cdots + p_r + q_r$,

$$|Df_a^n(x)| \ge K_{2.3} \exp\left(n\frac{\lambda - 2\tau\alpha}{\tau}\right).$$

(ii) If $f^n(x) = 0$ and $f^k(x) \ne 0$ for all $0 \le k < n$, then n has the following decomposition: $n = q_0 + p_1 + q_1 + \cdots + p_r + q_r$ since, during a bound period, the orbit of x is disjoint from 0. □

V. Distortion of the tip

The purpose of this section is to compute the distortion of the map $[a \mapsto c_n(a)]$ on an interval of parameters ω which returns to a neighborhood of 0 in a controlled manner. We assume essentially that ω is (n, Δ)-adapted for some large Δ depending on (α, λ) and that $\{c_n(a)\}_{n\geq 1}$ satisfies the exclusion rule and the forward Collet–Eckman condition. The main dificulty is to bound from above the sum

$$\sum_{k=1}^{n} \frac{|c_k(\omega)|}{d(c_0, c_k(\omega))},$$

where $|c_k(\omega)|$ denotes the Lebesgue measure of $c_k(\omega)$ and $d(c_0, c_k(\omega))$ the smallest distance between a point of $c_k(\omega)$ and the critical point. The choice of the partition $\{I(\mu_n)\}_{n\geq 1}$ is not arbitrary; we need endpoints of exponential type in order to get a bound return $p(x)$ of magnitude r for all $x \in I(r)$ and, in order to bound the distortion, we need also the following estimate:

$$\lim_{n\longrightarrow+\infty} \frac{|I(\mu_n)|}{d(c_0, I(\mu_n))} = 0.$$

Proof of Theorem II.11 Let ω be an interval of parameters satisfying the hypotheses (i) and (ii) of Theorem II.11. The proof is divided into two steps.

Step 1. We show there exist two constants $0 < \beta < 1$, $D \geq 1$, an increasing sequence of times $t_0 = 1 < t_1 < \cdots < t_{s+1} = n$ and a sequence of returns $(r_1, \ldots, r_s) \in \mathcal{M}^s$, $r_k \geq 2$, such that, for all $k = 1, \ldots, s$,

(a) $c_{t_k}(\omega) \subseteq I(\pm r_k)$,

(b) $|c_{t_{k+1}}(\omega)| \geq \exp(-\beta r_k)\,\dfrac{|c_{t_k}(\omega)|}{|I(r_k)|}$ (except maybe for $k = s$),

(c) $\displaystyle\sum_{t_k\leq i<t_{k+1}} \frac{|c_i(\omega)|}{d(c_0, c_i(\omega))} \leq D\Big\{\frac{|c_{t_k}(\omega)|}{d(c_0, c_{t_k}(\omega))} + \frac{|c_{t_{k+1}}(\omega)|}{d(c_0, c_{t_{k+1}}(\omega))}\Big\}$,

(d) $\forall\,(a, b) \in \omega \quad \left|\dfrac{Df_a^{t_1-1}(c_1(a))}{Df_b^{t_1-1}(c_1(b))}\right| \leq D$.

We first choose $\Delta_{2.11} \geq \Delta_{2.7}$ and $V_{2.11} \subseteq V_{2.7}$ such that, for all $1 \leq k < N_{2.10}$, $c_k(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] = \emptyset$. We then define by induction, an increasing sequence of times: $t_0 = 1$, t_1 is the first $k > t_0$ such that $c_k(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$. By assumption, $c_{t_1}(\omega) \subseteq I(\pm r_1)$ for some $r_1 \in \mathcal{M}$. By the choice of $V_{2.11}$ we can use the Bound Return Theorem II.7: we call p_1 the bound period associated to (r_1, t_1, ω). Either $t_1 + p_1 > n$ and we stop the construction, or $t_1 + p_1 \leq n$ and we wait for the first time $k \geq t_1 + p_1$ such that $c_k(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$. We call q_1 this escape period and $t_2 = t_1 + p_1 + q_1$. By assumption, $c_{t_2}(\omega) \subseteq I(\pm r_2)$ and so on. We thus define an increasing sequence: $t_0 = 1 < t_1 < \cdots < t_s < t_{s+1} = n$.

For each $1 \le k < s$, $t_{k+1} - t_k = p_k + q_k$, $c_{t_k}(\omega) \subseteq I(\pm r_k)$ and $t_{s+1} - t_s$ may be smaller than p_s. We first notice that (e) and (f) of the Bound Return Theorem imply

$$\sum_{i=t_k}^{t_k+p_k-1} \frac{|c_i(\omega)|}{d(c_0, c_i(\omega))} \le D_{2.7} \frac{|c_{t_k}(\omega)|}{d(c_0, c_{t_k}(\omega))} \quad \text{and} \quad \frac{|c_{t_k+p_k}(\omega)|}{|c_{t_k}(\omega)|} \ge \frac{\exp(-3\alpha\tau r_k)}{|I(r_k)|}.$$

From Theorem II.3 we know that there exist two constants $K_{2.3}$ and $\lambda_{2.3}$ independent of δ such that, for all $a \in \omega$ and $0 < i \le q_k$,

$$|Df_a^{q_k-i}(c_{t_k+p_k+i}(a))| \ge K_{2.3} e^{(q_k-i)\lambda_{2.3}}$$

provided that $\delta < \delta_{2.3}$. By Corollary IV.2 we obtain

$$\frac{|c_{t_{k+1}}(\omega)|}{|c_{t_k+p_k+i}(\omega)|} \ge \frac{Q_1^* K_{2.3}}{Q_2^*} e^{(q_k-i)\lambda_{2.3}}.$$

Moreover, $c_{t_k+p_k+i}(\omega)$ is disjoint from $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$ for all $0 \le i < q_k$,

$$\begin{aligned}\sum_{i=t_k+p_k}^{t_{k+1}-1} \frac{|c_i(\omega)|}{d(c_0, c_i(\omega))} &\le \frac{2Q_2^*}{Q_1^* K_{2.3}} \sum_{i=1}^{q_k} e^{-i\lambda_{2.3}} \frac{|c_{t_{k+1}}(\omega)|}{\delta} \\ &\le \frac{4Q_2^*}{Q_1^* K_{2.3}(\exp\lambda_{2.3} - 1)} \frac{|c_{t_{k+1}}(\omega)|}{|I(r_{k+1})|}\end{aligned}$$

(we have used the assumption $c_{t_{k+1}}(\omega) \subseteq I(\pm r_{k+1})$ for some $r_{k+1} \ge \Delta$ and the estimate $\delta = \exp(-\Delta) \ge \exp(-r_{k+1}) \ge \frac{1}{2}|I(r_{k+1})|$).

The proof of the last period $t_{s+1} - t_s$ is almost identical, except in the case $n > t_s + p_s$, where we use the assumption $c_n(\omega) \subseteq [-2\delta, 2\delta]$ to be able to apply once more property (ii) of Theorem II.3 and the estimate $|c_n(\omega)|/\delta \le 2$. To prove condition (b) we use the estimate from above: $|c_{t_{k+1}}(\omega)|/|c_{t_k+p_k}(\omega)| \ge Q_1^* K_{2.3}/Q_2^*$. We choose $\Delta_{2.11}$ sufficiently large so that $Q_1^* K_{2.3}/Q_2^* \ge \exp(-\alpha\tau\Delta_{2.11})$ and condition (b) follows with $\beta \le 4\alpha\tau$.

Condition (d) says that the distortion of c_{t_1} on ω is uniformly bounded. We recall that the constant $N_{2.10}$ depends only on λ. Before the period $N = N_{2.10}$ we have an a priori upper bound

$$\left|\frac{d_{N-1}(a)}{d_{N-1}(b)}\right| \le \frac{1}{K^*} \exp(N-1)(\Lambda^* - \lambda).$$

The total distortion is thus bounded by

$$\left|\frac{d_{t_1-1}(a)}{d_{t_1-1}(b)}\right| \le \frac{1}{K^*} \exp\left\{N(\Lambda^* - \lambda) + C^* \sum_{k=N}^{t_1-1} \left|\frac{c_k(a) - c_k(b)}{c_k(b)}\right| + \frac{\|\partial_{xa}^2 f\| \, |a-b|}{A_1^*(\varepsilon^*)^{\tau-1}\exp(-k\tau\alpha)}\right\},$$

with

$$\sum_{k=N}^{t_1-1} \left| \frac{c_k(a) - c_k(b)}{c_k(b)} \right| \leq \frac{4Q_2^*}{Q_1^* K_{2.3}(\exp \lambda_{2.3} - 1)},$$

$$\sum_{k=N}^{t_1-1} \frac{|a-b|}{\exp(-k\tau\alpha)} \leq \frac{\exp \lambda}{K^* Q_1^* \{\exp(\tau\alpha) - 1\}}.$$

Step 2. We finish the proof of Theorem II.11. As soon as $n \geq N_{2.10}$, we can compare the two derivatives ∂_x and ∂_a:

$$\left| \frac{dc_n}{da}(a) / \frac{dc_n}{da}(b) \right| \leq \frac{Q_2^*}{Q_1^*} |d_{n-1}(a)/d_{n-1}(b)|.$$

Using the nonflatness condition and property (d) of step 1, we obtain

$$\left| \frac{d_{n-1}(a)}{d_{n-1}(b)} \right| \leq D \exp \left\{ C^* \sum_{k=t_1}^{n-1} \left| \frac{c_k(a) - c_k(b)}{c_k(b)} \right| + \sum_{k=1}^{n-1} \frac{\|\partial_{xa}^2 f\| \, |a-b|}{A_1^* |c_k(b)|^{\tau-1}} \right\}.$$

We first show that the second sum is uniformly bounded. By assumption, $|c_k(t)| \geq \varepsilon^* \exp(-k\alpha)$ for all $1 \leq k < n$. On the other hand, because of the exponential growth, the length of ω is small as the following inequalities show:

$$1 \geq |c_n(\omega)| = \int_\omega |\partial_a f^n(0,a)| \, da \geq \int_\omega Q_1^* |d_{n-1}(a)| \, da \geq Q_1^* K^* e^{(n-1)\lambda} |\omega|.$$

If we combine these two inequalities we get

$$\exp \left\{ \frac{\|\partial_{xa}^2 f\|}{A_1^*} \sum_{k=1}^{n-1} \frac{|a-b|}{|c_k(b)|^{\tau-1}} \right\} \leq \exp \left\{ \frac{\|\partial_{ax}^2 f\| e^\lambda}{A_1^* Q_1^* K^* \varepsilon^\tau} \frac{\exp n(\tau\alpha - \lambda)}{\exp(\tau\alpha) - 1} . Q \right\},$$

which is uniformly bounded provided that $\alpha < \lambda/\tau$. We now show that the first sum is bounded.

To simplify the notation, we call $\sigma_k = |c_{t_k}(\omega)|$. We recall also the following estimate: $1 \leq \exp(\mu_{i+1} - \mu_i) \leq 2$. Using $\pm c_{t_k}(\omega) \subseteq I(r_k) \subseteq [\frac{1}{4}\exp(-r_k), 2\exp(-r_k)]$ and estimate (c) of step 1, we get

$$\sum_{k=t_1}^{n-1} \left| \frac{c_k(a) - c_k(b)}{c_k(b)} \right| = \sum_{k=1}^{s} \sum_{i=t_k}^{t_{k+1}-1} \frac{|c_i(\omega)|}{d(c_0, c_i(\omega))} \leq 8D \sum_{k=1}^{s} \sigma_k \exp(r_k),$$

with the following a priori estimate: $\sigma_k \exp(r_k) \leq 7/r_k$. We now show that the last sum is uniformly bounded from above and divide the proof into three parts.

Part 1. We claim that $\sum_{r\in\mathcal{M}} e^{-\beta r}$ is finite. Indeed, if n is an integer and $(\mu_1,\dots,\mu_{t+1})$ denotes the elements of $\mathcal{M}$ by increasing order between $n-1$ and n, by definition of the sequence we have

$$e^{-\mu_k} - e^{-\mu_{k+1}} \geq \frac{1}{n} e^{-n},$$

$$e^{-(n-1)} \geq e^{-\mu_1} - e^{-\mu_{t+1}} \geq \frac{t}{n} e^{-n},$$

which shows that $\mathrm{card}\{\mu \in \mathcal{M} : n-1 \leq \mu \leq n\} \leq ne$ and prove the claim.

Part 2. We show it is enough to assume that $(r_0,\dots,r_s)$ are pairewise distinct. We notice first that the estimate (b) of step 1 implies that $(\sigma_0,\dots,\sigma_s)$ is exponentialy increasing,

$$\sigma_k \leq 2e^{-(1-\beta)r_k}\sigma_{k+1} \leq \frac{2}{e}\sigma_{k+1},$$

in particular for each $r \in \{r_0,\dots,r_s\}$, if $k(r)$ denotes the largest index k such that $r_k = r$,

$$\sum_{k:r_k=r} \sigma_k e^{r_k} \leq \sum_{i=0}^{\infty} \left(\frac{2}{e}\right)^i \sigma_{k(r)} e^r.$$

We may arrange by increasing order the set $\{k(r_i) : i = 0,\dots,s\} = \{k_0,\dots,k_u\}$ and denote $\tilde{r}_i = r_{k_i}$ and $\tilde{\sigma}_i = \sigma_{k_i}$. We verify that $(\tilde{r}_i, \tilde{\sigma}_i)$ satisfies the assumption (b):

$$\tilde{\sigma}_{i+1} \geq \exp(-\beta\tilde{r}_i)\frac{\tilde{\sigma}_i}{|I(\tilde{r}_i)|}$$

(we use the fact that $|I(r)| \leq \exp(-\beta r)$ for all $r \geq 2$ and β sufficiently small).

Part 3. We assume now that $(r_0,\dots,r_s)$ are pairwise disjoint and divide the sum $\sum_k \sigma_k \exp(r_k)$ into two sums. Either, for some indexes k, $\sigma_k \exp(r_k) < 4\exp(-\beta r_k)$ and the total sum is bounded by $4\sum_{r\in\mathcal{M}} \exp(-\beta r)$. Or, for some indexes k, $\sigma_k \exp(r_k) \geq 4\exp(-\beta r_k)$. If $k < l$ are such indexes, using property (b) we have $2\exp(-r_l) \geq \sigma_l \geq \frac{1}{2}\sigma_k \exp(1-\beta)r_k \geq 2\exp(-2\beta r_k)$, in particular $r_l \leq 2\beta r_k$ and the total sum over these indexes is bounded by $\sum_k r_k \leq (49/2)\sum_{i\geq 0}(2\beta)^i$. □

VI. A Markov-like dynamics

VI.A. Beginning the induction

We denote by Ω an interval containing a^*, and by $[-\delta,\delta]$ a neighborhood of the critical point we shall define later. We recall that $N = N_{2.2}(\delta,\Omega)$ denotes the first time N such that $c_N(\Omega)$ meets $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$. We prove in the next lemma the first step of the induction, (ER-0), (CE1-0) and that $c_N(\omega)$ is equal to a disjoint union of states $I(r)$, $|r| \geq \Delta$ and possibly prestates $J(\pm\Delta)$.

Lemma VI.1 *Let $\alpha > 0$ and $\lambda \in (0, \lambda_{2.3})$. For every $\delta \in (0, \delta_{2.3})$ there exists a neighborhood $V_{6.1}(\delta)$ of a^* such that:*

(i) *for every Ω containing a^* and $N = N_{2.2}(\delta, \Omega)$, $c_N(\Omega)$ contains one of the states $I(\pm\Delta)$,*

(ii) *for all $a \in V_{6.1}(\delta)$ and $n \geq 1$ such that $\{c_k(a)\}_{k=1}^n$ is disjoint from $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$, $|d_n(a)| \geq K^* \exp(n\tilde{\lambda})$ and $|c_n(a)| \geq \varepsilon^* \exp(-n\alpha)$.*

Proof We recall that a^* satisfies the Misiurewicz condition and that f_{a^*} satisfies Theorem II.1: for all $n \geq 1$, $|d_n(a^*)| \geq K^* \exp(n\lambda^*)$ and $|c_n(a^*)| \geq \varepsilon^*$. If we choose $2\delta < \varepsilon^*$, since $c_N(\Omega)$ contains $c_N(a^*)$, $c_N(\Omega)$ contains either $[\frac{1}{2}\delta, 2\delta]$ or $[-2\delta, -\frac{1}{2}\delta]$ and consequently one of the states $I(\pm\Delta)$. We then choose N such that $K_{2.3}(\delta) \exp N(\lambda_{2.3} - \tilde{\lambda}) \geq K^*$, $2\varepsilon^* \exp(-N\alpha) \leq \delta$ and $V_{6.1}$ small enough such that for all $a \in V_{6.1}$ and $1 \leq k \leq N$, $|d_k(a)| \geq K^* \exp(k\tilde{\lambda})$ and $|c_k(a)| \geq \varepsilon^* \exp(-k\alpha)$. The second assertion follows from Theorem II.3: if $n > N$ and $\{c_k(a)\}_{N+1}^n$ is disjoint from $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$, then

$$|d_n(a)| \geq K_{2.3}(\delta) \exp(n\lambda_{2.3}) \geq K^* \exp(n\tilde{\lambda}),$$

$$|c_n(a)| \geq \frac{1}{2}\delta \geq \varepsilon^* \exp(-N\alpha) \geq \varepsilon^* \exp(-n\alpha).$$

□

VI.B. Bound and free periods

As we have seen in the previous section (VI.A), $c_N(\Omega)$ may contain a prestate $J(\pm\Delta)$. Let $\omega = \Omega \cap c_N^{-1}(J(\pm\Delta))$ be the subinterval of Ω which is mapped by c_N to one of the prestates $J(\pm\Delta)$. We denote by q the first time such that $c_{N+q}(\omega)$ meets $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$; q is called the free period associated to $(N, \pm\Delta, \omega)$. We show that $c_{N+q}(\omega)$ contains again a state $I(\pm\Delta)$. If $c_{N+q}(\omega)$ contains a prestate, we continue this process. More generally we have the following lemma:

Lemma VI.2. *Let $\lambda \in (0, \lambda_{2.3})$. There exist $\delta_{6.2} \in (0,1)$ and, for every $\delta \in (0, \delta_{6.2})$, a neighborhood $V_{6.2} \subseteq V_{6.1}$ of a^* such that, if $\omega \subseteq V_{6.2}$ is an interval which satisfies for some integer $n \geq 1$,*

(i) $c_n(\omega) = J(\pm\Delta)$,

(ii) $|d_k(a)| \geq K^* \exp(k\lambda)$ *for all $a \in \omega$ and $0 \leq k < n$,*

then $c_{n+q}(\omega)$ contains either $[-2\delta, -\frac{1}{2}\delta]$ or $[\frac{1}{2}\delta, 2\delta]$ (where q is the smallest integer such that $c_{n+q}(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$).

Proof Since f_{a^*} verifies properties (i) and (ii) of Lemma III.1, for every $\delta < \delta_{3.1}$ one can find a neighborhood $V_{6.2}$ of a^* such that for all $a \in V_{6.2}$, $1 \leq k < (\tau/2\Lambda^*)\Delta = p(\delta)$, $|f_a^k(\pm\delta)| > \frac{1}{4}\varepsilon^*$. Either $q < p(\delta)$, for $\delta < \frac{1}{8}\varepsilon^*$, $c_{n+q}(\omega)$ contains an interval of length $\frac{1}{4}\varepsilon^* - \frac{1}{2}\delta > 2\delta - \frac{1}{2}\delta$. Or $q \geq p(\delta)$, then for all $a \in V_{6.2} \subseteq V_{2.3}(\delta)$ and $n \leq k \leq n+q$

$$|d_{k-1}(a)| \geq K^* K_{2.3}(\delta) \exp\{(n-1)\lambda + (k-n)\lambda_{2.3}\} \geq K^* \exp\left\{(k-1)\frac{\lambda}{2}\right\},$$

provided we choose $V_{6.2}$ small enough such that $K_{2.3}(\delta)\exp\{N_{2.2}(V_{6.2})\lambda/2\} \geq 1$. If, moreover, we choose $V_{6.2} \subseteq V_{2.10}(\lambda/2)$ such that $N_{2.2}(V_{6.2}) \geq N_{2.10}(\lambda/2)$, Corollary IV.2 applied to any $\omega' \subseteq \omega$ verifying $c_{n+q}(\omega') \subseteq [-\delta_{2.3}, \delta_{2.3}]$ gives

$$\frac{|c_{n+q}(\omega')|}{|c_n(\omega')|} \geq \frac{Q_1^*}{Q_2^*} K_{2.3} \exp(q\lambda_{2.3}).$$

Either $c_{n+q}(\omega) \not\subseteq [-\delta_{2.3}, \delta_{2.3}]$ and, for $\delta < \frac{1}{2}\delta_{2.3}$, $c_{n+q}(\omega)$ contains an interval of length $\delta_{2.3} - \frac{1}{2}\delta > 2\delta - \frac{1}{2}\delta$, or $c_{n+q}(\omega) \subseteq [-\delta_{2.3}, \delta_{2.3}]$ and $|c_{n+q}(\omega)| \geq 3\delta$ provided δ is chosen such that $(Q_1^*/Q_2^*)K_{2.3}\exp(\tau\Delta\lambda_{2.3}/2\Lambda^*) \geq 6\Delta$ (remember $|c_n(\omega)| \geq \delta/2\Delta$). □

We now combine these two lemmas to prove the first step of the induction.

Lemma VI.3 (First step of the induction) *Let $\lambda \in (0, \lambda_{2.3})$, $\alpha \in (0, \alpha_{2.3}(\lambda))$ and $\tilde{\lambda} = \frac{1}{2}(\lambda + \lambda_{2.3}) + 10\alpha\tau$. There exist $\delta_{6.3} \in (0,1)$ and, for any $\delta \in (0, \delta_{6.3})$, a neighborhood $V_{6.3}(\delta)$ of a^* such that any $\Omega_0 \subseteq V_{6.3}$ contains a subset Ω_1 equal to a disjoint union of intervals of the form $\omega = \omega(^t_r)$, where $t \geq 2N_{2.2}(\Omega_0)$ and $|r| \geq \Delta$, which satisfy the following properties:*

- $c_k(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] = \emptyset \quad (\forall\, 1 \leq k < t)$,
- $c_t(\omega) = I(r)$,
- $|d_k(a)| \geq K^* \exp(k\tilde{\lambda}) \quad (\forall\, 0 \leq k < t,\ \forall a \in \omega)$,
- $|c_k(a)| \geq \varepsilon^* \exp(-k\alpha) \quad (\forall\, 1 \leq k < t,\ \forall a \in \omega)$,
- $|\Omega_0 \setminus \Omega_1|/|\Omega_0| \leq \exp\{-\frac{1}{2}N_{2.2}(\Omega_0)\alpha\}$.

Proof We begin to construct a partition of Ω_0 into intervals of the form $\tilde{\omega}(t)$ satisfying the properties:

- t is an integer and $t \geq N_{2.2}(\Omega)$,
- $c_k(\tilde{\omega}(t)) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] = \emptyset \quad (\forall\, 1 \leq k < t)$,
- $|d_k(a)| \geq K^* \exp(k\tilde{\lambda}) \quad (\forall a \in \tilde{\omega}(t),\ \forall\, 0 \leq k < t)$,
- $c_t(\tilde{\omega}(t))$ contains one of the states $I(\pm\Delta)$ and is equal to a disjoint union of states $I(r)$, $|r| \geq \Delta$.

The construction is done by induction. By Lemma VI.1, $c_N(\Omega_0)$ ($N = N_{2.2}(\delta, \Omega_0)$) contain $I(\pm\Delta)$; let $\tilde{\omega}(N)$ be the part of Ω_0 which is mapped by c_N onto $c_N(\Omega_0) \cap (\bigcup_{|r|\geq\Delta} I(r))$. $c_N(\Omega_0)$ may also contain a prestate $J(\pm\Delta)$; let $\tilde{\omega}(\pm 1)$ be the part which is mapped onto $J(\pm\Delta)$. We wait until $c_{N+q}(\tilde{\omega}(\pm 1))$ meets $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$ and repeat the process at time $N+q$: $\tilde{\omega}(\pm 1)(N+q)$ is the part which is mapped onto $c_{N+q}(\tilde{\omega}(\pm 1)) \cap (\bigcup_{|r|\geq\Delta} I(r))$, and so on.

Lemma VI.1 tells us that the exclusion rule is also satisfied during the period $[1,t]$ on $\tilde{\omega}(t)$. The interval $\tilde{\omega}(t)$ is therefore (t,Δ)-adapted and the distortion of c_t is uniformly bounded by a constant $D_{2.11}$ independent of t. On each $\tilde{\omega}(t)$, we eliminate the intervals $\omega(^t_r)$ which are mapped by c_t into $I(r)$ for $|r|>\alpha t$:

$$\frac{|\omega(^t_r)|}{|\tilde{\omega}(t)|} \leq D_{2.11}\frac{|I(r)|}{|I(\Delta)|}.$$

($c_t(\tilde{\omega}(t))$ contains one of the states $I(\pm\Delta)$ and then $|c_t(\tilde{\omega}(t))| > |I(\Delta)|$.) The total proportion we eliminate is therefore bounded by

$$\sum_{|r|>t\alpha} \frac{|\omega(^t_r)|}{|\tilde{\omega}(t)|} \leq \frac{4D_{2.11}}{|I(\Delta)|}\exp(-t\alpha)\,.$$

We choose once more $V_{6.3}$ sufficiently small so that

$$4D_{2.11} \leq |I(\Delta)| \exp\{\frac{1}{2}N_{2.2}(V_{6.3})\alpha\}.$$

Finally, Ω_1 is the remaining part of Ω_0:

$$\Omega_1 = \bigcup_t \bigcup_{|r|\leq\alpha t} \omega(^t_r) \quad \text{and} \quad \frac{|\Omega_0 \setminus \Omega_1|}{|\Omega_0|} \leq \exp\Big\{-\frac{1}{2}\alpha N_{2.2}(\Omega_0)\Big\}.$$

□

VI.C. Essential bound returns

Let $\omega \subseteq \Omega_0$ be an interval of parameters such that, for some $n \geq N_{2.2}(\Omega_0)$ and some state $I(r)$, $|r|\geq\Delta$, $c_n(\omega)=I(r)$. We know from Lemma II.7 that we recover the exponent $(\lambda-2\alpha\tau)/\tau$ after a short period p_0, compared to the period n. At time $n+p_0$, $c_{n+p_0}(\omega)$ may be disjoint from $[-\frac{1}{2}\delta,\frac{1}{2}\delta]$; we denote by q_0 the first time such that $c_{n+p_0+q_0}(\omega)$ meets $[-\frac{1}{2}\delta,\frac{1}{2}\delta]$. At that time $n_1=n+p_0+q_0$, it may happen that $c_{n_1}(\omega)$ no longer contains a state $I(r)$, $|r|\geq\Delta$; q_0 is then called a partially free period, $c_{n_1}(\omega)$ is included in some $I(r_1)$, $|r_1|\geq\Delta$ and we repeat the process. We denote by p_1 the bound period associated to (n_1,r_1,ω) which may be followed by a partially free period q_1. Let $n_2=n_1+p_1+q_1$ and so on. We stop the process until $c_{n_{u+1}}(\omega)$ contains a state $I(r)$ with $|r|\geq\Delta$; $p=n_u-n_0+p_u$ is called the essential bound return associated to (n,r,ω). More precisely we have

Lemma VI.4 *Let $\lambda\in(0,\lambda_{2.3})$, $\alpha\in(0,\alpha_{2.3}(\lambda))$. There exist $\delta_{6.4}\in(0,1)$ and, for any $\delta\in(0,\delta_{6.4})$, a neighborhood $V_{6.4}(\delta)$ of a^* such that, for any interval $\omega\subseteq V_{6.4}$, any integer $n\geq 1$ satisfying for all $a\in\omega$:*

(i) $c_n(\omega) = I(r)$ *for some* $n\alpha \geq |r| \geq \Delta$,

(ii) $\forall\, 1 \leq k \leq n \quad |c_k(a)| \geq \varepsilon^* \exp(-k\alpha)$ *and* $|d_{k-1}(a)| \geq K^* \exp\{(k-1)\lambda/4\}$,

(iii) $\forall\, 1 \leq k \leq n/4 \quad |d_k(a)| \geq K^* \exp(k\lambda)$,

one can find integers $p = p_{6.4}$ *and* $q = q_{6.4}$ *having the following properties*:

(a) $(1/3\Lambda^*)|r| \leq p \leq (2\tau/\lambda)|r|$,

(b) $\forall\, a \in \omega \quad \forall\, 1 \leq k \leq p \quad |Df_a^k(c_n(a))| \geq \exp\{-2n\tau\alpha + k(\lambda - 3\tau\alpha)/\tau\}$,

(c) $\forall\, a \in \omega \quad |Df_a^p(c_n(a))| \geq \exp p(\lambda - 3\tau\alpha)/\tau$,

(d) $\forall\, a \in \omega \quad \forall\, 1 \leq k < p \quad |c_{n+k}(a)| \geq 2\exp(-|r|) \geq \varepsilon^* \exp(-(n+k)\alpha)$,

(e) ω *is* $(n+p, \Delta)$*–adapted (except that* $c_{n+p}(\omega)$ *may not be included into* $[-2\delta, 2\delta]$),

(f) $\forall\, 0 \leq k < q \;\; c_{n+p+k}(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] = \emptyset$ *and* $c_{n+p+q}(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$,

(g) $c_{n+p+q}(\omega)$ *contains a state* $I(r)$ *for some* $|r| \geq \Delta$,

(h) *if* $c_{n+p+q}(\omega) \subseteq [-2\delta, 2\delta]$ *then* $|c_{n+p+q}(\omega)| \geq \exp\{-4\alpha p\}$.

Proof We construct by induction a sequence of inessential return times: $n = n_0 < n_1 < \cdots < n_{u+1}$, $n_{i+1} - n_i = p_i + q_i$, where p_i is the bound period associated to (n_i, r_i, ω) and q_i is the partially free period which follows p_i. For each $0 \leq i \leq u$, $c_{n_i}(\omega) \subseteq I(r_i)$; in particular, ω is (n_{u+1}, Δ)-adapted (except that $c_{n_{u+1}}(\omega)$ may not be included into $[-2\delta, 2\delta]$). Let $p = n_u + p_u - n_0$ be the essential bound return associated to (n, r, ω) and $q = q_u$ the free period. The main problem is to prove that the hypotheses of Lemma II.7 are satisfied at each stage of the construction. We assume that we can apply (i)...(iii) of Lemma II.7 for (n_i, r_i, ω) and that we have already proven the following properties:

$$|Df_a^{n_{k+1}-n_k}(c_{n_k}(a))| \geq \exp\left\{(n_{k+1} - n_k)\frac{\lambda - 3\alpha\tau}{\tau}\right\}$$

for all $0 \leq k < i$, $\Delta \leq |r_i| \leq |r_0|$ and $n_i - n_0 \leq (2\tau/\lambda)|r_0|$.

Step one. We compute the exponent during each period $[n_i, n_{i+1}]$ with $i < u$ (the proof for the period $[n_u, n_u + p_u]$ is similar):

$$|Df_a^{n_{i+1}-n_i}(c_{n_i}(a))| \geq K_{2.3} \exp\left\{q_i\lambda_{2.3} + p_i\frac{\lambda - 2\alpha\tau}{\tau}\right\} \geq \exp\left\{(n_{i+1} - n_i)\frac{\lambda - 3\alpha\tau}{\tau}\right\}$$

(we have chosen $\Delta_{6.4}$ large enough so that $K_{2.3}\exp\{2\alpha\Delta_{6.4}/3\Lambda^*\} \geq 1$). We point out that we used property (ii) of Theorem II.3 since $c_{n_{i+1}}(\omega) \subseteq [-2\delta, 2\delta]$.

Step two. We compute the exponent at time $n_i + k$ with $0 \le k < p_i$:

$$|d_{n_{i+k}}(a)| \ge K^* \exp\left\{n_0\frac{\lambda}{4} + (n_i - n_0)\frac{\lambda - 3\alpha\tau}{\tau} + k\lambda - n_i\alpha\tau\right\}$$

$$\ge K^* \exp\left\{n_0\frac{\lambda}{4} + k\frac{\lambda}{10} - 2n_0\alpha\tau\right\} \ge \exp\left\{(n_i + k)\frac{\lambda}{10}\right\}.$$

We have used the estimates $n_i \le 2n_0$ and $n_0(\frac{1}{4}\lambda - 2\alpha\tau) \ge \frac{1}{5}\lambda n_0 \ge \frac{1}{10}n_i\lambda$.

Step three. We compute the exponent at time $n_i + p_i + k$ with $0 \le k < q_i$:

$$|d_{n_i+p_i+k}(a)| \ge K_{2.3}(\delta)K^* \exp\left\{n_0\frac{\lambda}{4} + (n_i + p_i - n_0)\frac{\lambda - 3\alpha\tau}{\tau} + k\lambda_{2.3}\right\}$$

$$\ge K_{2.3}(\delta)K^* \exp\left\{n_0\frac{\lambda}{4} + k\frac{\lambda}{10}\right\} \ge K^* \exp\left\{(n_i + p_i + k)\frac{\lambda}{10}\right\}.$$

We have used the estimate $n_i + p_i \le n_0 + (4\tau/\lambda)|r_0| \le 2n_0$ and chosen $V_{6.4}$ sufficiently small so that $K_{2.3}(\delta)\exp\{2\alpha N_{2.2}(V_{6.4})\tau\} \ge 1$.

Step four. We show that $n_{i+1} - n_0 \le (2\tau/\lambda)|r_0|$ (the case $n_u + p_u - n_0 \le (2\tau/\lambda)|r_0|$ is similar). We choose $V_{6.4} \subseteq V_{2.10}(\lambda/10)$ such that $N_{2.2}(V_{6.4}) \ge N_{2.10}(\lambda/10)$ and apply Lemma II.10 and Corollary IV.2:

$$|c_{n_{i+1}}(\omega)| \ge \frac{Q_1^*}{Q_2^*}|c_{n_i}(\omega)| \exp\left\{(n_{i+1} - n_i)\frac{\lambda - 3\alpha\tau}{\tau}\right\}.$$

We recall that $|c_{n_0}(\omega)| \ge |2r_0|^{-1}\exp(-|r_0|)$ and that $n_{i+1} - n_0 \ge p_0 \ge |r_0|/3\Lambda^*$. We choose $\Delta_{6.4}$ large so that $(Q_1^*/Q_2^*)|2r|^{-1}\exp\{4\alpha|r|/3\Lambda^*\} \ge 1$ for all $|r| \ge \Delta_{6.4}$ and obtain

$$1 \ge |c_{n_{i+1}}(\omega)| \ge \exp\left\{ - |r_0| + (n_{i+1} - n_0)\frac{\lambda - 4\alpha\tau}{\tau}\right\}.$$

In particular we get $n_{i+1} - n_0 \le (2\tau/\lambda)|r_0|$.

Step five. We show that $|r_{i+1}| \le |r_0|$ for $i < u$; property (d) will follow:

$$4\exp\{-|r_{i+1}|\} \ge |c_{n_{i+1}}(\omega)| \ge \frac{Q_1^*}{Q_2^*}\frac{1}{2|r_i|}\exp\left\{ - |r_i| + (n_{i+1} - n_i)\frac{\lambda - 3\alpha\tau}{\tau}\right\}.$$

We choose once more $\Delta_{6.4}$ large so that $(Q_1^*/Q_2^*)|4r|^{-1}\exp\{|r|(\lambda - 3\alpha\tau)/3\tau\Lambda^*\} \ge 1$ for all $|r| \ge \Delta_{6.4}$ and obtain $|r_{i+1}| \le |r_i|$.

Step six. We prove the property (b). For $0 \le k \le p_i$ and $0 \le l \le q_i$

$$|Df_a^{n_i - n_0 + k}(c_{n_0}(a))| \ge \exp\left\{(n_i - n_0)\frac{\lambda - 3\alpha\tau}{\tau} + k\lambda - n_i\alpha\tau\right\}$$

$$\ge \exp\left\{(n_i - n_0 + k)\frac{\lambda - 3\alpha\tau}{\tau} - 2n_0\alpha\tau\right\},$$

$$|Df_a^{n_i - n_0 + p_i + l}(c_{n_0}(a))| \ge K_{2.3}(\delta)\exp\left\{(n_i - n_0)\frac{\lambda - 3\alpha\tau}{\tau} + p_i\frac{\lambda - 2\alpha\tau}{\tau} + l\lambda_{2.3}\right\}$$

$$\ge \exp\left\{(n_i - n_0 + p_i + l)\frac{\lambda - 3\alpha\tau}{\tau} - 2n_0\alpha\tau\right\}$$

provided we choose $V_{6.4}$ such that $K_{2.3}(\delta)\exp\{2N_{2.2}(V_{6.4})\alpha\tau\} \geq 1$.

Step seven. We prove the property (h); we assume that $c_{n_{u+1}}(\omega) \subseteq [-2\delta, 2\delta]$:

$$|c_{n_{u+1}}(\omega)| \geq \frac{Q_1^*}{Q_2^*}K_{2.3}^2|c_{n_0+p_0}(\omega)|\exp\left\{(q_0+q_u)\lambda_{2.3}+(n_u-n_1)\frac{\lambda-3\alpha\tau}{\tau}+p_u\frac{\lambda-2\alpha\tau}{\tau}\right\}.$$

We choose $\Delta_{6.4}$ such that $(Q_1^*/Q_2^*)K_{2.3}\exp\{\alpha\Delta_{6.4}/3\Lambda^*\} \geq 1$ and obtain

$$|c_{n_u+p_u}(\omega)| \geq |c_{n_0+p_0}(\omega)|\exp(-\alpha p_0) \geq \exp(-4\alpha p_0).$$

□

VI.D. Dynamics of the tip between n and $2n$

At stage k, Ω_k is a union of adapted intervals ω; for each of them, there exist an integer n and a state $I(r)$ such that $c_n(\omega) = I(r)$ and $|r| \leq n\alpha$. We study in this section the dynamics of the tip during the period $[n; 2n]$.

Lemma VI.5 *Let $\lambda \in (0, \lambda_{2.3})$ and $\alpha \in (0, \alpha_{2.3}(\lambda))$. There exist $\delta_{6.5} \in (0, 1)$ and for all $\delta \in (0, \delta_{6.5})$ a neighborhood $V_{6.5}$ of a^* such that, for any interval $\omega \subseteq V_{6.5}$ and $n \geq 1$,*

(i) *$c_n(\omega) = I(r)$ for some $\Delta \leq |r| \leq n\alpha$,*

(ii) *$\forall a \in \omega \quad \forall 1 \leq i \leq n$ $|d_{i-1}(a)| \geq K^*\exp\{(i-1)\lambda\}$ and $|c_i(a)| \geq \varepsilon^*\exp(-i\alpha)$,*

(iii) *ω is (n, Δ)-adapted,*

(iv) *$\forall a \in \omega$ $|d_n(a)| \geq K^*\exp(n\tilde{\lambda})$.*

One can construct an increasing sequence of partitions of ω in the following way:

(a) *For each $0 \leq s \leq n$, ω is equal to a disjoint union of intervals $\omega\binom{t_0,\dots,t_s}{r_0,\dots,r_s}$ where $t_0 = n \leq t_1 \leq \cdots \leq t_s$, $r_0 = r$, $|r_i| \geq \Delta$, and two distinct sequences $\binom{t_0,\dots,t_s}{r_0,\dots,r_s} \neq \binom{t_0',\dots,t_s'}{r_0',\dots,r_s'}$ give two disjoint intervals: $\omega\binom{t_0,\dots,t_s}{r_0,\dots,r_s} \cap \omega\binom{t_0',\dots,t_s'}{r_0',\dots,r_s'} = \emptyset$,*

(b) *$\omega\binom{t_0}{r_0} = \omega$; for each $0 \leq s < n$, $\omega\binom{t_0,\dots,t_s}{r_0,\dots,r_s}$ is equal to a disjoint union of intervals $\omega\binom{t_0,\dots,t_{s+1}}{r_0,\dots,r_{s+1}}$,*

(c) *for each $0 \leq s \leq n$ and $\tilde{\omega} = \omega\binom{t_0,\dots,t_s}{r_0,\dots,r_s}$*

- $\tilde{\omega}$ is (t_s, Δ)-adapted,

- $\forall 0 \leq j \leq s \quad c_{t_j}(\tilde{\omega}) \subseteq I(r_j)$ and $c_{t_s}(\tilde{\omega}) = I(r_s)$,

- $\forall a \in \tilde{\omega} \quad \forall 1 \leq i \leq t_s \quad |d_{i-1}(a)| \geq K^*\exp\{(i-1)\lambda/4\}$ and $|c_i(a)| \geq \varepsilon^*\exp(-i\alpha)$,

(d) *for each* $0 \leq s \leq n$ *and* $\tilde{\omega} = \omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}$, *either* $|r_s| > \alpha t_s$ *or* $t_s > 2n$, *then* $r_{s+1} = r_s$ *and* $t_{s+1} = t_s$: *the process is stopped, or* $|r_s| \leq \alpha t_s$ *and* $t_s \leq 2n$, *then* $\tilde{\omega}$ *is partitioned into a countable number of intervals of the form* $\tilde{\tilde{\omega}} = \omega\binom{t_0,\ldots,t_{s+1}}{r_0,\ldots,r_{s+1}}$; $t_{s+1} > t_s$, $|r_{s+1}| \geq \Delta$ *and each period* $t_{s+1} - t_s$ *is the sum of two periods:* $t_{s+1} - t_s = p_s + q_s$ *verifying*

- $(1/3\Lambda^*)|r_s| \leq p_s \leq (2\tau/\lambda)|r_s|$,
- $\forall\, 1 \leq i < p_s \quad \forall a \in \tilde{\tilde{\omega}} \quad |c_{t_s+i}(a)| \geq 2\exp(-|r_s|)$,
- $\forall\, 0 \leq i < q_s \quad \forall a \in \tilde{\tilde{\omega}} \quad |c_{t_s+p_s+i}(a)| \geq \frac{1}{2}\exp(-\Delta)$,

(e) *since the state* $I(r_{s+1})$ *may happen at different times* t_{s+1}, *the proportion of obtaining this state is given by*

$$\sum_{t_{s+1}} \frac{|\omega\binom{t_0,\ldots,t_{s+1}}{r_0,\ldots,r_{s+1}}|}{|\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}|} \leq |I(r_{s+1})|\exp(2\Delta + 4\alpha p_s)\,,$$

where the summation is taken over all possible $\tilde{\tilde{\omega}} \subseteq \tilde{\omega}$, r_{s+1} *being fixed;*

moreover

$$\sum_{|r_n|>\alpha t_n} \frac{|\omega\binom{t_0,\ldots,t_n}{r_0,\ldots,r_n}|}{|\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}|} \leq \frac{1}{2}\exp\Big(-\frac{\alpha}{2}t_s\Big)\,,$$

where the summation is taken over all $\omega\binom{t_0,\ldots,t_n}{r_0,\ldots,r_n} \subseteq \omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}$ *satisfying* $|r_n| > \alpha t_n$.

Proof *Step one:* the construction. We construct by induction on s a partition of ω into disjoint intervals $\tilde{\omega} = \omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}$. By definition, $\omega = \omega\binom{t_0}{r_0}$. We assume we have already constructed $\tilde{\omega}$ with $|r_s| \leq \alpha t_s$ and $t_s \leq 2n$. For all $i = 0,\ldots,s$, p_i denotes the essential bound period associated to $(t_i, r_i, \tilde{\omega})$ and q_i the free period. The properties (i), (ii) and (iii) of Lemma VI.4 are satisfied. Let p_s be the essential bound period associated to $(t_s, r_s, \tilde{\omega})$ and q the smallest integer such that $c_{t_s+p_s+q}(\omega) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$. We know that p_s satisfies

- $(1/3\Lambda^*)|r_s| \leq p_s \leq (2\tau/\lambda)|r_s|$,
- $\forall a \in \tilde{\omega} \quad \forall\, 1 \leq i < p_s \quad |c_{t_s+i}(a)| \geq 2\exp(-|r_s|)$,
- $\tilde{\omega}$ is $(t_s + p_s, \Delta)$-adapted (except that $c_{t_s+p_s}(\tilde{\omega})$ may not be in $[-2\delta, 2\delta]$),
- $c_{t_s+p_s+q}(\tilde{\omega})$ contains a state $I(r)$ for some $|r| \geq \Delta$ and meets $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$.

In order to apply Lemma VI.2 we compute the exponent during the period $[t_s, t_s + p_s]$. Let $0 \leq k < p_s$, then

$$|d_{t_s+k}(a)| \geq K^*\exp(n\tilde{\lambda})\prod_{i=0}^{s-1}\Big\{K_{2.3}\exp(p_i\frac{\lambda - 3\tau\alpha}{\tau} + q_i\lambda_{2.3})\Big\}\exp\Big(k\frac{\lambda - 3\tau\alpha}{\tau} - 2t_s\tau\alpha\Big),$$

$$|d_{t_s+k}(a)| \geq K^* \exp\left\{(t_s+k)\frac{\lambda}{2\tau}\right\}$$

(we have chosen $\Delta_{6.5}$ sufficiently large so that $K_{2.3}\exp(\Delta_{6.5}\alpha/3\Lambda^*) \geq 1$ and used the estimates $n(\tilde{\lambda}-\lambda) \geq 4\tau\alpha n \geq 2t_s\alpha\tau$).

Either $c_{t_s+p_s+q}(\tilde{\omega})$ is equal to a union of states $I(r)$, $|r| \geq \Delta$; then, by definition, $q_s = q$, $t_{s+1} = t_s + p_s + q$ and $\tilde{\omega}$ is equal to a disjoint union of intervals $\tilde{\omega}(r_{s+1}) = \omega\binom{t_0,\dots,t_{s+1}}{r_0,\dots,r_{s+1}}$ corresponding to the part of $\tilde{\omega}$ which is mapped by $c_{t_s+p_s+q}$ onto $I(r_{s+1})$.

Or $c_{t_s+p_s+q}(\tilde{\omega})$ contains also one of the prestates $J(\pm\Delta)$. We denote by $\tilde{\omega}(0)$ the part of $\tilde{\omega}$ which is mapped by $c_{t_s+p_s+q}$ onto $\bigcup_{|r|\geq\Delta} I(r)$ and by $\tilde{\omega}(\pm 1)$ the part of $\tilde{\omega}$ which is mapped onto $J(\pm\Delta)$. We denote also $t'_{s+1} = t_s + p_s + q$. Let $q(\pm 1)$ be the first time such that $c_{t'_{s+1}+q(\pm 1)}(\tilde{\omega}(\pm 1)) \cap [-\frac{1}{2}\delta, \frac{1}{2}\delta] \neq \emptyset$, $t'_{s+1}(\pm 1) = t'_{s+1} + q(\pm 1)$ and $q_s(\pm 1) = q + q(\pm 1)$. The exponent during the period $[t_s+p_s, t_s+p_s+q_s]$ is equal to $\lambda/2\tau$. For all $0 \leq k < q_s$

$$\begin{aligned}|d_{t_s+p_s+k}(a)| \geq & K^* K_{2.3}(\delta)\exp\left\{n\tilde{\lambda} + p_s\frac{\lambda-3\alpha\tau}{\tau} + k\lambda_{2.3}\right\} \\ & \times \prod_{i=0}^{s-1}\left\{K_{2.3}\exp(p_i\frac{\lambda-3\tau\alpha}{\tau} + q_i\lambda_{2.3})\right\},\end{aligned}$$

$$|d_{t_s+p_s+k}(a)| \geq K^* \exp\left\{(t_s+p_s+k)\frac{\lambda}{2\tau}\right\},$$

provided that $V_{6.5}$ is chosen such that $K_{2.3}(\delta)\exp\{N_{2.2}(V_{6.5})(\tilde{\lambda}-\lambda)\} \geq 1$.

Lemma VI.2 tells us that $c_{t'_{s+1}(\pm 1)}(\tilde{\omega}(\pm 1))$ contains one of the intervals $[-\frac{1}{2}\delta, -2\delta)$ or $(2\delta, \frac{1}{2}\delta]$. We denote by $\tilde{\omega}(\pm 1)(0)$ the part which is mapped by $c_{t'_{s+1}(\pm 1)}$ onto $\bigcup_{|r|\geq\Delta} I(r)$ and by $\tilde{\omega}(\pm 1, \pm 1)$ the part which is mapped onto $J(\pm\Delta)$. We continue this process and obtain a partition of $\tilde{\omega}$ into a disjoint union of intervals parametrized by $(\varepsilon_1,\dots,\varepsilon_u)$ in $\{\pm 1\}^u$, $u \geq 0$, $\tilde{\omega} = \bigcup_{u,\varepsilon_i}\tilde{\omega}(\varepsilon_1,\dots,\varepsilon_u)(0)$, where for each $\omega' = \tilde{\omega}(\varepsilon_1,\dots,\varepsilon_u)(0)$, there exists an integer $t' = t'_{s+1}(\varepsilon_1,\dots,\varepsilon_u) = t_s+p_s+q_s(\varepsilon_1,\dots,\varepsilon_u)$, $q_s(\varepsilon_1,\dots,\varepsilon_u) = q + q(\varepsilon_1) + \dots + q(\varepsilon_1,\dots,\varepsilon_u)$ such that:

- $c_{t'}(\omega')$ is a union of states $I(r)$, $|r| \geq \Delta$ and has a length bigger than $\delta/2$,
- $\forall a \in \omega' \quad \forall 0 \leq i < q_s(\varepsilon_1,\dots,\varepsilon_u) \quad |c_{t_s+p_s+i}(a)| \geq \frac{1}{2}\exp(-\Delta)$,
- $\forall a \in \omega' \quad \forall 0 \leq i < t' \; |d_i(a)| \geq K^*\exp(i\lambda/2\tau)$.

We denote by $\omega(\varepsilon_1,\dots,\varepsilon_u)(r')$ the part of ω' which is mapped by $c_{t'}$ onto $I(r')$. Since ω' is (t',Δ)-adapted, provided $V_{6.5} \subseteq V_{2.11}(\alpha,\lambda/2\tau,\delta)$, we know that the distortion of $c_{t'}$ is uniformly bounded by a constant $D_{2.11}$ independent of Δ. If $c_{t_s+p_s+q}(\tilde{\omega}) = \bigcup_{|r'|\geq\Delta} I(r')$ then

$$\frac{|\tilde{\omega}(r')|}{|\tilde{\omega}|} \leq D_{2.11}\frac{|I(r')|}{\exp(-4\alpha p_s)};$$

otherwise, for all $u \geq 0$,

$$\frac{|\tilde{\omega}(\varepsilon_1, \ldots, \varepsilon_u)(r')|}{|\tilde{\omega}(\varepsilon_1, \ldots, \varepsilon_u)(0)|} \leq 2D_{2.11} \frac{|I(r)|}{\exp(-\Delta)}.$$

Finally, we denote by $\omega\binom{t_0,\ldots,t_s,t'}{r_0,\ldots,r_s,r'}$ any interval of the form $\tilde{\omega}(\varepsilon_1, \ldots, \varepsilon_u)(r')$. We just have constructed the partition at stage $s+1$ and proved

$$\sum_{t'} \frac{|\omega\binom{t_0,\ldots,t_s,t'}{r_0,\ldots,r_s,r'}|}{|\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}|} \leq 2D_{2.11} \frac{|I(r')|}{\min\{\exp(-4\alpha p_s), \delta\}} \leq \frac{1}{4}|I(r')| \exp\{4\alpha p_s + 2\Delta\}$$

(provided we choose $\Delta_{6.5}$ such that $4D_{2.11}\exp(\Delta_{6.5}) \geq 1$).

Step two: proof of part (e). We prove by decreasing induction on s that, for each $\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}$ satisfying $|r_s| \leq \alpha t_s$ and $t_s \leq 2n$,

$$\sum_{|r_n| > \alpha t_n} \frac{|\omega\binom{t_0,\ldots,t_n}{r_0,\ldots,r_n}|}{|\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}|} \leq \frac{1}{2} \exp\left(-t_s \frac{\alpha}{2}\right),$$

where the summation is taken over all intervals $\omega\binom{t_0,\ldots,t_n}{r_0,\ldots,r_n}$ included in $\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}$.

For $s = n$, the sum of the left side is empty and the assertion is obvious. Suppose the assertion is true for $s+1, s+2, \ldots, n$. If $\tilde{\omega} = \omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}$ is chosen such that $t_s \leq 2n$ and $|r_s| \leq \alpha t_s$, using the above inequality we have

$$\sum_{t', |r'| > \alpha t'} \frac{|\omega\binom{t_0,\ldots,t_s,t'}{r_0,\ldots,r_s,r'}|}{|\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}|} \leq \frac{1}{2} \exp\{4\alpha p_s + 2\Delta - \alpha t_s\},$$

where the summation is taken over all intervals $\omega\binom{t_0,\ldots,t_s,t'}{r_0,\ldots,r_s,r'}$ included in $\tilde{\omega}$ with $|r'| > \alpha t'$.

We use the estimate $p_s \leq (2\tau\alpha/\lambda)t_s$, $64\tau\alpha \leq \lambda$ and choose $V_{6.5}$ small enough so that $\alpha N_{2.2}(V_{6.5}) \geq 16\Delta$, to obtain

$$\sum_{t', |r'| > \alpha t'} \frac{|\omega\binom{t_0,\ldots,t_s,t'}{r_0,\ldots,r_s,r'}|}{|\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}|} \leq \frac{1}{2} \exp\left(-\frac{3\alpha t_s}{4}\right).$$

On each $\omega' = \omega\binom{t_0,\ldots,t_s,t'}{r_0,\ldots,r_s,r'}$ remaining (i.e. satisfying $|r'| \leq \alpha t'$); either $t' > 2n$ and ω' contains no other interval of the form $\omega\binom{t_0,\ldots,t_n}{r_0,\ldots,r_n}$ with $|r_n| > \alpha t_n$; or $t' \leq 2n$ and by induction we have

$$\sum_{|r_n| > \alpha t_n} \frac{|\omega\binom{t_0,\ldots,t_n}{r_0,\ldots,r_n}|}{|\omega\binom{t_0,\ldots,t_s}{r_0,\ldots,r_s}|} \leq \frac{1}{2} \exp\left(-\frac{\alpha t'}{2}\right) + \frac{1}{2} \exp\left(-\frac{3\alpha t_s}{4}\right) \leq \frac{1}{2} \exp\left(-\frac{\alpha t_s}{2}\right)$$

(we have used the estimate $t' - t_s \geq p_s \geq (\Delta_{6.5}/3\Lambda^*)$ and chosen $\Delta_{6.5}$, $V_{6.5}$ such that $2\exp(-\alpha\Delta_{6.5}/6\Lambda^*) \leq 1$). □

VI.E. A Markov chain type argument

We use the notations of the previous lemma. We fix an interval of parameters $\omega \subseteq V_{6.5}$ such that:

- $c_n(\omega) = I(r)$ for some $n\alpha \geq |r| \geq \Delta$,
- $\forall a \in \omega \quad \forall 1 \leq i \leq n \quad |c_i(a)| \geq \varepsilon^* \exp(-i\alpha)$ and $|d_{i-1}(a)| \geq K^* \exp\{(i-1)\lambda\}$,
- $\forall a \in \omega \quad |d_{n-1}(a)| \geq K^* \exp\{(n-1)\tilde{\lambda}\}$,
- ω is (n, Δ)-adapted.

We begin to exclude the parameters a which do not satisfy the exclusion rule. Let ω' be the union of all intervals $\omega\binom{t_0,\dots,t_n}{r_0,\dots,r_n}$ satisfying $|r_n| \leq \alpha t_n$ (in particular on such intervals, $|r_i| \leq \alpha t_i$ for all $0 \leq i \leq n$). We know from Lemma VI.5 that

$$\frac{|\omega \setminus \omega'|}{|\omega|} \leq \frac{1}{2}\exp\Big(-\frac{\alpha}{2}n\Big).$$

We consider ω' as a probability space. Let $\mathbf{P}$ be the normalized induced Lebesgue measure on ω'. We define on ω' sequences of random variables, P_i, Q_i, T_i, R_i and S constant on each interval $\omega\binom{t_0,\dots,t_n}{r_0,\dots,r_n}$ included in ω':

$$\forall a \in \omega\binom{t_0,\dots,t_n}{r_0,\dots,r_n} \quad P_i(a) = p_i, \quad Q_i(a) = q_i, \quad T_i(a) = t_i, \quad R_i(a) = r_i,$$

and $S(a)$ is the largest integer s such that $t_s \leq 2n$. The following corollary shows that the sequence $\{R_i\}_{i=0}^S$ behaves like a Markov chain.

Corollary VI.6 *Using the notations of Lemma VI.5, there exists $\delta_{6.6} \in (0,1)$ such that, for all $\delta \in (0, \delta_{6.6})$, for all $(r_0, r_1, \dots, r_{s+1})$ verifying $|r_i| \geq \Delta$,*

$$\frac{\mathbf{P}(R_0 = r_0, \dots, R_{s+1} = r_{s+1}, s \leq S)}{\mathbf{P}(R_0 = r_0, \dots, R_s = r_s, s \leq S)} \leq \exp\Big\{3\Delta - |r_{s+1}| + \frac{8\tau\alpha}{\lambda}|r_s|\Big\}.$$

Proof As $(R_0 = r_0, \dots, R_s = r_s, s \leq S)$ is equal to the union of intervals $\omega\binom{t_0,\dots,t_s}{r_0,\dots,r_s}$, $t_0 = n < t_1 < \dots < t_s \leq 2n$, $|r_i| \leq \alpha t_i$, where we have subtracted all the subintervals $\omega\binom{t_0,\dots,t_n}{r_0,\dots,r_n}$ verifying $|r_n| > \alpha t_n$. Using part (e) of Lemma VI.5 we have, for each of these intervals $\omega\binom{t_0,\dots,t_s}{r_0,\dots,r_s}$,

$$\sum_{t_{s+1}} \frac{|\omega\binom{t_0,\dots,t_{s+1}}{r_0,\dots,r_{s+1}}|}{|\omega\binom{t_0,\dots,t_s}{r_0,\dots,r_s}|} \leq 2\exp\Big\{2\Delta - |r_{s+1}| + \frac{8\tau\alpha}{\lambda}|r_s|\Big\},$$

where the summation is taken over all subintervals $\omega(\begin{smallmatrix} t_0,\ldots,t_{s+1} \\ r_0,\ldots,r_{s+1}\end{smallmatrix})$, r_{s+1} fixed. Since

$$\frac{|\omega(\begin{smallmatrix} t_0,\ldots,t_s \\ r_0,\ldots,r_s\end{smallmatrix}) \cap \omega'|}{|\omega(\begin{smallmatrix} t_0,\ldots,t_s \\ r_0,\ldots,r_s\end{smallmatrix})|} \geq 1 - \exp(-\frac{\alpha t_s}{2}) \geq \frac{1}{2},$$

$$\sum_{t_{s+1}} \frac{|\omega(\begin{smallmatrix} t_0,\ldots,t_{s+1} \\ r_0,\ldots,r_{s+1}\end{smallmatrix}) \cap \omega'|}{|\omega(\begin{smallmatrix} t_0,\ldots,t_s \\ r_0,\ldots,r_s\end{smallmatrix}) \cap \omega'|} \leq 4 \exp\left\{2\Delta - |r_{s+1}| + \frac{8\tau\alpha}{\lambda}|r_s|\right\}.$$

We choose $\Delta_{6.6}$ sufficiently large so that $4 \leq \exp(\Delta_{6.6})$. □

In order to recover the exponent $\tilde{\lambda}$ at time T_{S+1} and an exponent λ during the period $[T_0, T_{S+1}]$, we exclude the orbits $\{c_i(a)\}_{i=1}^{T_{S+1}}$ which come to the critical point too often. For $\tilde{\Delta} > \Delta$ we denote by $\omega'(\tilde{\Delta})$ the subset of ω':

$$\omega'(\tilde{\Delta}) = \left\{a \in \omega' : \sum_{k=0}^{S} P_k \mathbf{1}_{(|R_k| > \tilde{\Delta})} \leq T_0 \frac{\lambda_{2.3} - \tilde{\lambda}}{\lambda_{2.3}}\right\},$$

where $\mathbf{1}_B$ represents the characteristic function of the subset B.

Lemma VI.7 *We use the notation of Lemma VI.5. There exist $\delta_{6.7} \in (0,1)$ and, for all $\tilde{\delta} < \delta < \delta_{6.7}$, a neighborhood $V_{6.7}(\tilde{\delta})$ of a^* such that, for all interval $\omega \subseteq V_{6.7}(\tilde{\delta})$ and for all $a \in \omega'(\tilde{\Delta})$,*
- $\forall\, 1 \leq i \leq T_{S+1}(a)$ $\quad |c_i(a)| \geq \varepsilon^* \exp(-i\alpha)$ and $|d_{i-1}(a)| \geq K^* \exp\{(i-1)\lambda\}$,
- $|d_{T_{S+1}-1}(a)| \geq K^* \exp\{(T_{S+1} - 1)\tilde{\lambda}\}$.

Proof We choose $V_{6.7} \subseteq V_{2.3}(\tilde{\delta})$; then the exponent at T_{S+1} is equal to $\tilde{\lambda}$:

$$|Df_a^{T_{S+1}-T_0}(c_{T_0}(a))| \geq \prod_{|R_s| \leq \tilde{\Delta}} \exp\{\lambda_{2.3}(T_{s+1} - T_s)\} \prod_{|R_s| > \tilde{\Delta}} \left\{K_{2.3} \exp\left(P_s \frac{\lambda}{2\tau} + Q_s \lambda_{2.3}\right)\right\}$$
$$\geq \exp\left\{\lambda_{2.3} \sum_{s=0}^{S} Q_s + P_s \mathbf{1}_{(|R_s| \leq \tilde{\Delta})}\right\}$$

(we have used the estimate $K_{2.3}\exp(\Delta_{6.7}\lambda/6\tau\Lambda^*) \geq 1$ and the property (iii) of Theorem II.3 for the case $|R_s| \leq \tilde{\Delta}$). If we choose in addition $a \in \omega'(\tilde{\Delta})$, we obtain (notice that $T_{S+1} - T_0 \geq T_0$)

$$\lambda_{2.3} \sum_{s=0}^{S} Q_s + P_s \mathbf{1}_{(|R_s| \leq \tilde{\Delta})} = \lambda_{2.3}(T_{S+1} - T_0) - \lambda_{2.3} \sum_{s=0}^{S} P_s \mathbf{1}_{(|R_s| > \tilde{\Delta})}$$

$$\geq \lambda_{2.3}(T_{S+1} - T_0) - (\lambda_{2.3} - \tilde{\lambda}) T_0 \geq \tilde{\lambda}(T_{S+1} - T_0).$$

We compute now the exponent during the period $[T_u, T_u + P_u]$, $0 \leq u \leq S$. Let $0 \leq k < P_u$; then $|d_{T_u+k}(a)|$ is bigger than

$$K^* \exp\{T_0(\tilde{\lambda} - 4\alpha\tau)\} \prod_{|R_s| \leq \tilde{\Delta}}^{s<u} \exp\left\{\lambda_{2.3}(T_{s+1} - T_s)\right\} \prod_{|R_s| > \tilde{\Delta}}^{s<u} K_{2.3} \exp\left\{\frac{\lambda}{2\tau} P_s + \lambda_{2.3} Q_s\right\}$$

$$\geq K^* \exp\left\{T_0(\tilde{\lambda} - 4\alpha\tau) + \lambda_{2.3}(T_u - T_0) - \lambda_{2.3} \sum_{s=0}^{S} P_s \mathbf{1}_{(|R_s| > \tilde{\Delta})}\right\}$$

$$\geq K^* \exp\{T_u \tilde{\lambda} - 4\alpha\tau T_0 - (\lambda_{2.3} - \tilde{\lambda}) T_u\} \geq K^* \exp\{\lambda(T_u + k)\}$$

(we have used the estimate $k \leq P_u \leq (2\tau/\lambda)|R_u| \leq (2\tau\alpha/\lambda)T_u$ and the definition of $\tilde{\lambda}$: $\tilde{\lambda} \geq \frac{1}{2}(\lambda_{2.3} + \lambda) + 3\alpha\tau$). The exponent during the period $[T_u + P_u, T_{u+1}]$ is computed in the same manner. Let $0 \leq k < Q_u$; then $|d_{T_u+P_u+k}(a)|$ is bigger than

$$K^* K_{2.3}(\tilde{\delta}) \exp\left\{T_0 \tilde{\lambda} + \lambda_{2.3}(T_u - T_0 - \sum_{s=0}^{S} P_s \mathbf{1}_{(|R_s| > \tilde{\Delta})} + k)\right\} \geq K^* \exp\{(T_u + P_u + k)\lambda\}$$

(provided we choose $V_{6.7}$ such that $K_{2.3}(\tilde{\delta}) \exp\{N_{2.2}(V_{6.7}) 4\alpha\tau\} \geq 1$). □

Before we prove Proposition II.12 we need two combinatorial lemmas.

Lemma VI.8 *There exists a constant* Γ_1 *such that for all integers* $1 \leq p \leq n$

$$C_n^p \leq \Gamma_1 \exp\left\{nH\left(\frac{p}{n}\right)\right\},$$

where C_n^p *denotes the binomial coefficient and* $H(x)$ *the function defined for* $x \in [0, 1]$ *by* $H(x) = -x \log x - (1 - x) \log(1 - x)$.

Proof Using Stirling's formula, one can find constants Γ and Γ' such that, for all $n \geq 1$, $\Gamma \leq n!/\sqrt{2\pi n}\, e^{-n} n^n \leq \Gamma'$. In particular

$$C_n^p \leq (\Gamma'/\Gamma^2) \sqrt{n/(2\pi p(n-p)}\, \exp\{nH(p/n)\}.$$

In both cases ($1 \leq p \leq n/2$, or $n/2 \leq p \leq n$), we have $n \leq 2p(n - p)$. We choose $\Gamma_1 = \Gamma'/(\sqrt{\pi}\Gamma^2)$. □

Lemma VI.9 *There exists* $\Delta_{6.9} \in \mathcal{M}$, $\Delta_{6.9} \geq \Delta_{6.7}$ *such that, for all integers* $R > \Delta_{6.9}$,

$$\operatorname{card}\left\{(r_1, \ldots, r_u) : u \geq 1, |r_i| \in \mathcal{M}, |r_i| \geq \Delta_{6.9} \text{ and } R \leq r_1 + \cdots + r_u < R + 1\right\}$$
$$\leq \exp\left(\frac{R}{8}\right).$$

Proof We notice first that $u \leq [(R+1)/\Delta_{6.9}]$, where $[x]$ denotes the integer part of x. We define for all $u \geq 1$ the following two sets and a map $\theta : B_u \to B'_u$ by

$$B_u = \{(r_1, \ldots, r_u) : r_i \geq \Delta_{6.9}, r_i \in \mathcal{M}, R \leq r_1 + \cdots + r_u < R + 1\},$$

$$B'_u = \{(\rho_1, \ldots, \rho_u) : \rho_i \text{ is an integer, } \rho_i \geq \Delta_{6.9} \text{ and } \rho_1 + \cdots + \rho_u = R\},$$

$$\theta(r_1, \ldots, r_u) := ([r_1], [r_1 + r_2] - [r_1], \ldots, [r_1 + r_2 + \cdots + r_u] - [r_1 + \cdots + r_{u-1}]).$$

If $(\rho_1, \ldots, \rho_u) \in B'_u$ and $(r_1, \ldots, r_u)$ belongs to the fiber $\theta^{-1}(\rho_1, \ldots, \rho_u)$, then $\rho_i - 1 \leq r_i \leq \rho_i + 1$. Given $\rho \geq 1$, we denote by increasing order the parameters $\mu_i \in \mathcal{M}$ which belong to $[\rho - 1, \rho + 1]$:

$$\rho - 1 \leq \mu_k < \mu_{k+1} < \cdots < \mu_{k+l} \leq \rho + 1.$$

The sequence $\{\mu_i\}$ verifies, for all $k \leq i < l$, $\exp(-\mu_i) - \exp(-\mu_{i+1}) \geq (\rho+1)^{-1}\exp\{-(\rho+1)\}$. Adding these inequalities, we obtain

$$\frac{l}{\rho+1}\exp\{-(\rho+1)\} \leq \exp(-\mu_k) - \exp(-\mu_{k+l}) \leq \exp\{-(\rho-1)\}$$

and finally $l \leq (\rho+1)e^2$. One can find a constant Γ such that $(\rho+1)e^2 + 1 \leq \Gamma\exp(\rho/16)$ for all $\rho \geq 1$. The cardinal of each fiber $\theta^{-1}(\rho_1, \ldots, \rho_u)$ is thus bounded by $\Gamma^u \exp(\rho_1 + \ldots + \rho_u)/16 \leq \Gamma^u \exp(R/16)$. The cardinal of B'_u is bounded by C_R^u and the cardinal of $\bigcup_u B_u$ is therefore bounded by

$$\sum_{u=1}^{[(R+1)/\Delta_{6.9}]} \Gamma_1 \exp\left\{RH\left(\frac{u}{R}\right) + \frac{R}{16} + \frac{R+1}{\Delta_{6.9}}\log 2\Gamma\right\}.$$

Since $H(x)$ is an increasing function over $[0, 1/e]$, $([(R+1)/\Delta_{6.9}]/R \leq (2/\Delta_{6.9}) \leq (1/e))$, we simplify the above expression by $\Gamma_1((R+1)/\Delta_{6.9})\exp R(H(2/\Delta_{6.9}) + (1/16) + (2/\Delta_{6.9})\log 2\Gamma) \leq \exp(R/8)$, if we choose $\Delta_{6.9}$ sufficiently large. □

VI.F. Proof of the main Proposition

We are now able to prove the main induction step:

Proof of Proposition II.12 Using Lemma VI.7, the fact that $P_s \leq 2\tau|R_s|/\lambda$ and the estimate $|\omega \setminus \omega'|/|\omega| \leq \frac{1}{2}\exp(-\frac{1}{2}\alpha n)$, it is enough to prove that, for $\tilde{\Delta}$ sufficiently large, $\tilde{\Delta} = 100\Delta$:

$$\frac{|\omega' \setminus \omega'(\tilde{\Delta})|}{|\omega'|} = \mathbf{P}\Big(\sum_{s=0}^{S} |R_s|\mathbf{1}_{(|R_s|>\tilde{\Delta})} > n\frac{\lambda(\lambda_{2.3} - \tilde{\lambda})}{2\tau\lambda_{2.3}}\Big) \leq \frac{1}{2}\exp\Big(-\frac{1}{2}\alpha n\Big).$$

We begin to intersect the set $\{\sum_{s=0}^{S}|R_s|\mathbf{1}_{(|R_s|>\tilde{\Delta})} > n\lambda(\lambda_{2.3}-\tilde{\lambda}/2\tau\lambda_{2.3}\}$ with all the possible sets $(R_0 = r_0,\dots,R_s = r_s, S = s)$, $|r_i| \geq \Delta$.The intersection is not empty if and only if there exists a sequence $0 \leq i_1 < \cdots < i_u \leq s$ such that:

- $|r_{i_1}| + \cdots + |r_{i_u}| > n\lambda(\lambda_{2.3}-\tilde{\lambda})/2\tau\lambda_{2.3}$,
- $\forall i \in \{i_1,\dots,i_u\} \quad |r_i| > \tilde{\Delta}$,
- $\forall i \in \{0,1,\dots,s\}\setminus\{i_1,\dots,i_u\} \quad |r_i| \leq \tilde{\Delta}$.

For all $i = i_1,\dots,i_u$, $\tilde{\Delta} \leq |R_i| \leq 3\Lambda^* P_i$. If we add these inequalities, we obtain $\tilde{\Delta}u \leq 3\Lambda^*\sum_{i=1}^{S} P_i \leq 3\Lambda^*(T_S - T_0 + p_s) \leq 9\Lambda^* n$. The number of all possible sequences $(i_1,\dots,i_u)$ where $1 \leq u \leq 9\Lambda^* n/\tilde{\Delta}$ and $0 \leq i_1 < \cdots < i_u < n$ is thus bounded by

$$\sum_{u=1}^{n} C_n^u \leq \sum_{u=1}^{n} \Gamma_1 \exp nH\Big(\frac{u}{n}\Big) \leq \Gamma_1 n \exp nH\Big(\frac{9\Lambda^*}{\Delta_{2.12}}\Big)$$

(we have chosen $\Delta_{2.12} \geq 9\Lambda^* e$).

If such a sequence $(i_1,\dots,i_u)$ is fixed and $R \geq [n\lambda(\lambda_{2.3}-\tilde{\lambda})/2\tau\lambda_{2.3}]$ is any integer, the number of all possible sequences $(r_1,\dots,r_u)$ such that $|r_i| \geq \tilde{\Delta}$ and $R \leq |r_1| + \cdots + |r_u| < R+1$ is bounded by $\exp(R/8)$.

From now on we fix a sequence $(i_1,\dots,i_u)$ satisfying $0 \leq i_1 < \cdots < i_u < n$ and a sequence $(\tilde{r}_1,\dots,\tilde{r}_u)$ satisfying $|\tilde{r}_j| > \tilde{\Delta}$ and $R \leq |\tilde{r}_1| + \cdots + |\tilde{r}_u| < R+1$. We show that

$$\begin{aligned}
&\sum \mathbf{P}(R_0 = r_0,\dots,R_s = r_s, s = S)\\
&\leq \exp(3u\Delta)\frac{\exp\{-(|\tilde{r}_1| + \cdots + |\tilde{r}_u|)\}}{\exp\{-(|r_0| + |\tilde{r}_1| + \cdots + |\tilde{r}_{u-1}|)8\tau\alpha/\lambda\}},
\end{aligned}$$

where the summation is taken over $s \geq i_u$ and all possible sequences $(r_1,\dots,r_s)$ such that $|r_i| \geq \Delta$, $|r_i| \in \mathcal{M}$, $r_{i_j} = \tilde{r}_j$ for $j = 1,\dots,u$ and $|r_i| \leq \tilde{\Delta}$ for $i \notin \{i_1,\dots,i_u\}$.

Indeed $\mathbf{P}(R_0 = r_0,\dots,R_s = r_s, s = S) \leq \mathbf{P}(R_0 = r_0,\dots,R_{i_u} = r_{i_u}, i_u \leq S)$ and using Corollary VI.6 and the fact that $|r_{i_{u-1}}| \geq \tilde{\Delta} \geq |r_{i_u-1}|$ if $i_{u-1} \neq i_u - 1$, the above sum is bounded by

$$\sum \mathbf{P}(R_0 = r_0,\dots,R_{i_{u-1}} = r_{i_{u-1}}, i_{u-1} \leq S)\exp\Big\{3\Delta - |\tilde{r}_u| + \frac{8\tau\alpha}{\lambda}|\tilde{r}_{u-1}|\Big\}$$

where the summation is taken over all $(r_0,\dots,r_{i_{u-1}})$, $|r_i| \geq \Delta$, $r_{i_j} = \tilde{r}_j$ for $j = 1,\dots,u-1$ and $|r_i| \leq \tilde{\Delta}$ for $i \neq i_1,\dots,i_{u-1}$. We repeat this process and the claim is proved.

Since $|r_0| \leq \alpha n \leq (4\tau\lambda_{2.3}\alpha/\lambda(\lambda_{2.3}-\tilde{\lambda}))R \leq R$ and $u \leq 2R/\tilde{\Delta}$,

$$\begin{aligned}
\frac{\exp\{-(|\tilde{r}_1| + \cdots + |\tilde{r}_u|) + 3u\Delta\}}{\exp\{-(|r_0| + |\tilde{r}_1| + \cdots + |\tilde{r}_{u-1}|)8\tau\alpha/\lambda\}} &\leq \exp\Big\{R\big(\frac{16\alpha\tau}{\lambda} - 1 + 6\frac{\Delta}{\tilde{\Delta}}\big)\Big\}\\
&\leq \exp\Big\{-\frac{1}{2}R\Big\}
\end{aligned}$$

(we have used the estimate $64\alpha\tau \leq \lambda$ and $24\Delta \leq \tilde{\Delta}$).

We finally sum over all possible sequences $(i_1, \dots, i_u)$ and $(\tilde{r}_1, \dots, \tilde{r}_u)$ and obtain

$$\mathbf{P}\Big(\sum_{s=0}^{S} |R_s| \mathbf{1}_{(|R_s|>\tilde{\Delta})} > n\frac{\lambda(\lambda_{2.3}-\tilde{\lambda})}{2\tau\lambda_{2.3}}\Big) \leq \sum_{R>n\frac{\lambda(\lambda_{2.3}-\tilde{\lambda})}{4\tau\lambda_{2.3}}} \Gamma_1 n \exp\Big\{nH\Big(\frac{9\Lambda^*}{\Delta_{2.12}}\Big)+\frac{R}{8}-\frac{R}{2}\Big\}$$

$$\leq \frac{\Gamma_1}{1-\exp(-1/4)} n \exp\Big\{nH\Big(\frac{9\Lambda^*}{\Delta_{2.12}}\Big) - n\frac{\lambda(\lambda_{2.3}-\tilde{\lambda})}{16\tau\lambda_{2.3}}\Big\}$$

(we have used the estimates $\lambda(\lambda_{2.3}-\tilde{\lambda}) \leq 16\alpha\tau\lambda_{2.3}$, chosen $\Delta_{2.12} \geq \Delta_{6.9}$ such that $4H(9\Lambda^*/\Delta_{2.12}) \leq \alpha$ and $V_{2.12} \subseteq V_{6.7}(\tilde{\delta})$ such that

$$(2\Gamma_1/1-\exp(-1/4))n\exp(-n\alpha/4) \leq 1$$

for all $n \geq N_{2.2}(V_{2.12})$). □

Proof of Theorem I.5 We fix once and for all constants

$$\lambda \in (0, \lambda_{2.3}), \quad \alpha \in (0, \alpha_{2.3}(\lambda)), \quad \delta \in (0, \delta_{2.12}(\lambda, \alpha))$$

and

$$\tilde{\lambda} = \frac{1}{2}(\lambda + \lambda_{2.3}) + 10\alpha\tau.$$

We recall that $N_{2.2}(V)$ denotes the first integer N such that $c_N(V) \cap [-\frac{1}{2}\varepsilon^*, \frac{1}{2}\varepsilon^*] \neq \emptyset$.

We construct by induction a decreasing sequence of subsets of $V_{2.12} \cap V_{6.3} \cap V_{2.8}$, $\{\Omega_k\}_{k\geq 0}$, where Ω_0 is any interval containing a^* and

$$\frac{|\Omega_k \setminus \Omega_{k+1}|}{|\Omega_k|} \leq \exp\{-2^{k-1}\alpha N_{2.2}(\Omega_0)\}.$$

Each Ω_k, $k \geq 1$, is a disjoint union of intervals ω satisfying the following properties:

- $c_n(\omega) = I(r)$ for some $n \geq 2^k N_{2.2}(\Omega_0)$ and $\Delta \leq |r| \leq \alpha n$,
- $\forall a \in \omega \quad \forall 1 \leq i \leq n \quad |c_i(a)| \geq \varepsilon^* \exp(-i\alpha)$ and $|d_{i-1}(a)| \geq K^* \exp\{(i-1)\lambda\}$,
- $\forall a \in \omega \quad |d_{n-1}(a)| \geq K^* \exp\{(n-1)\tilde{\lambda}\}$,
- ω is (n, Δ)-adapted.

Proposition II.12 shows how to construct Ω_{k+1} from Ω_k for $k \geq 1$. The proportion of the remaining subset Ω_∞ is thus bounded from below by

$$\frac{|\Omega_\infty|}{|\Omega_0|} \geq \prod_{k=1}^{\infty} \{1 - \exp(-2^k \alpha N_{2.2}(\Omega_0))\}.$$

The fact that the right-hand side of this inequality goes to one when $|\Omega_0|$ goes to zero proves that a^* is a Lebesgue density point.

Finally, Corollary II.8 shows that, for any $a \in \Omega_\infty$, f_a cannot have stable periodic points and satisfies the two Collet–Eckmann conditions (CE1), (CE2). □

VII. Transversality and genericity

Let $\{f_a\}_{a \in \mathcal{A}}$ be a regular one-parameter family. In the first part of this section we show that, if f_{a^*} is a Misiurewicz map, condition (T) is equivalent to the transversality of the curves $[a \mapsto c_N(a)]$ and $[a \mapsto \chi(c_N(a^*), a)]$ for any $N \geq 1$ where $[(x,a) \mapsto \chi(x,a)]$ is a *smooth continuation* of the invariant compact set $\Lambda^* = \overline{\{c_n(a) : n \geq 1\}}$. In the second part of this section we prove that condition (T) is generic among all regular one-parameter families passing through a Misiurewicz map.

VII.A. Transversality

We begin with a simpler example where some iterate of the critical point is equal to a nonstable periodic point. We recall that, whenever the limit exists, $Q(a)$ denotes

$$Q(a) \stackrel{\text{def}}{=} \lim_{n \to +\infty} \frac{\partial_a f^n(c_0, a)}{\partial_x f^{n-1}(c_1, a)} .$$

Lemma VII.1 *Let $\{f_a\}_{a \in \mathcal{A}}$ be a one-parameter family and $a \in \mathcal{A}$ any parameter. Then*

$$\sum_{n=0}^{+\infty} \frac{1}{|Df_a^n(c_1(a))|} < +\infty \implies Q(a) = \sum_{k=0}^{+\infty} \frac{\partial_a f(c_k(a), a)}{\partial_x f^k(c_1(a), a)} .$$

Proof For every $(x,a) \in I \times \mathcal{A}$ we have by definition $f^{n+1}(x,a) = f(f^n(x,a),a)$. To simplify the notation we write $c_n = c_n(a)$. If we differentiate this equality with respect to x and a we obtain

$$\partial_x f^n(c_1, a) = \partial_x f(c_n, a) \partial_x f^{n-1}(c_1, a) ,$$

$$\partial_a f^{n+1}(c_0, a) = \partial f_x(c_n, a) \partial_a f^n(c_0, a) + \partial_a f(c_n, a) .$$

Let $Q_n(a)$ denote the quotient $Q_n(a) = \partial_a f^n(c_0, a) \,/\, \partial_x f^{n-1}(c_1, a)$. We have

$$Q_n(a) = \sum_{k=0}^{n-1} \frac{\partial_a f(c_k, a)}{\partial_x f^k(c_1, a)} .$$

By hypothesis the series converges absolutely. □

Let us now denote x^* a nonstable periodic point for f_{a^*} of period p. For any a sufficiently close to a^*, the equation $x = f^p(x,a)$ has a unique solution close to x^*, $\chi(a) = f^p(\chi(a), a)$, where χ is a C^2 function, and has a derivative at a^* equal to

$$\frac{d\chi}{da}(a^*) = \frac{\partial_a f^p(x^*, a^*)}{1 - \partial_x f^p(x^*, a^*)} .$$

We assume now that c_0 is a preimage of x^*: $c_N(a^*) = x^*$ for some $N \geq 1$. Using the above lemma we can compute exactly the limit $Q(a^*)$.

Proof of Proposition I.9. We repeat the same arguments as in the proof of Lemma VII.1,

$$f^{N+(n+1)p}(x,a) = f^p(f^{N+pn}(x,a),a),$$

$$Q_{N+(n+1)p}(a) = Q_{N+pn}(a) + \frac{\partial_a f^p(c_{N+pn},a)}{\partial_x f^{N-1}(c_1,a)\partial_x f^{p(n+1)}(c_N,a)}.$$

By induction we obtain

$$Q_{N+np}(a) = Q_N(a) + \sum_{k=1}^{n-1} \frac{\partial_a f^p(c_{N+kp},a)}{\partial_x f^{N-1}(c_1,a)\partial_x f^{kp}(c_N,a)}.$$

If we now choose

$$a = a^*, \quad c_{N+kp}(a^*) = c_N(a^*) = x^*, \quad \partial_x f^{kp}(c_N,a) = (\partial_x f^p(c_N,a))^k$$

then

$$Q(a^*) = Q_N(a^*) - \frac{\partial_a f^p(x^*,a^*)}{\partial_x f^{N-1}(c_1,a)(1-\partial_x f^p(x^*,a^*))}.$$

□

In the remainder of this section, we assume that f_{a^*} satisfies the Misiurewicz condition and show that $Q(a^*)$ can be computed in the same manner. In order to do so , we introduce the notion of *smooth continuation* of a compact invariant set.

Definition VII.2 Let $\{f_a\}_{\in A}$ be a $\mathcal{C}^2$ one-parameter family, $a^* \in \mathcal{A}$ and $\Lambda^* \subseteq I$ a compact f_{a^*}-invariant set (i.e. $f_{a^*}(\Lambda^*) \subseteq \Lambda^*$). We call a smooth continuation of Λ^* a map $\chi : \Lambda^* \times V^* \to I$, where V^* is a neighborhood of a^* which satisfies:

(i) for each $a \in V^*$, $[x \in \Lambda^* \mapsto \chi(x,a)]$ is injective,

(ii) for each $x \in \Lambda^*$, $[a \in V^* \mapsto \chi(x,a)]$ is differentiable,

(iii) $[(x,a) \in \Lambda^* \times V^* \mapsto \chi(x,a)]$ and $[(x,a) \in \Lambda^* \times V^* \mapsto \partial_a\chi(x,a)]$ are continuous,

(iv) for all $x \in \Lambda^*$, $\chi(x,a^*) = x$,

(v) for all $(x,a) \in \Lambda^* \times V^*, f(\chi(x,a),a) = \chi(f(x,a^*),a)$.

If we use the notation $\chi_a(x) = \chi(x,a)$, we notice that $\Lambda_a = \chi_a(\Lambda^*)$ is a compact f_a-invariant set and that the dynamical system (Λ_a, f_a) is topologically conjugate to (Λ_{a^*}, f_{a^*}): $f_a \circ \chi_a = \chi_a \circ f_{a^*}$ on $\Lambda_{a^*} = \Lambda^*$. The following proposition shows that a hyperbolic compact invariant set possesses a smooth continuation.

Proposition VII.3 *Let $\{f_a\}_{a\in\mathcal{A}}$ be a C^2 one-parameter family of unimodal maps and $a^* \in \mathcal{A}$. If f_{a^*} has no stable periodic point, any compact f_{a^*}-invariant set disjoint from the critical point possesses a smooth continuation.*

Before we prove the above proposition, we explain how to parametrize a hyperbolic invariant set by a subshift of finite type, and extend Theorem II.1 for a one-parameter family.

Notation VII.4 Let $f : I \to I$ be a C^2 unimodal map without stable periodic point and $q \neq c_2$ be a periodic point of period $r \geq 3$. We may assume that $\{f(q), \dots, f^{r-1}(q)\}$ is disjoint from (q', q'') (where q' and q'' denote the negative and positive preimage of $f(q)$). We notice first that $f(q) \in (c_0, c_1)$. Actually, if c''_{-1} denotes the positive preimage of c_0, there must exist a point p of the orbit of q in (c_0, c_1). Either $p \in [q, c''_{-1})$ and $f(q) \geq f(p) > 0$, or $p \in (c''_{-1}, c_1]$ and any preimage of p belongs to (c'_{-1}, c''_{-1}). We also notice that $f^2(q) < 0$ using the assumption $r \geq 3$. In particular,

$$\Xi = \{x \in [f^2(q), q'] \cup [q'', f(q)] : f^n(x) \notin (q', q'') \forall n \geq 0\}$$

is a compact f-invariant set which contains the orbit of q. We denote by increasing order the points of $\{f(q), \dots, f^{r-1}(q), q', q''\}$ in the following way: $c_2 < \tilde{q}_0 < \cdots < \tilde{q}_{s-1} < c_0 < \tilde{q}_s < \cdots < \tilde{q}_r$ and define open intervals $[i] = (\tilde{q}_{i-1}, \tilde{q}_i)$ for $i = 1, \dots, r$. $\{[1], \dots, [r]\}$ determines a Markov partition with associate matrix of transition M: $M_{is} = 0$ for all $i = 1, \dots, r$ and $M_{ij} = 1$ if and only if $[j] \subseteq f([i])$ for all $j \neq s$ and $i = 1, \dots, r$. Let $\Sigma_M(q)$ denote the compact set of all admissible sequences

$$\Sigma_M(q) = \{\underline{x} = (x_n)_{n\geq 0} : x_n = 1, \dots, r \quad M_{x_n, x_{n+1}} = 1 \quad \forall n \geq 0\}.$$

$\Sigma_M(q)$ is called a subshift of finite type; we note $[\sigma : \Sigma_M(q) \to \Sigma_M(q)]$ the leftshift. If $\underline{x} = (x_0, \dots, x_n)$ is an admissible sequence of length $n+1$ (i.e. $M_{x_k, x_{k+1}} = 1$ for $k = 0, \dots, n-1$) we denote by $[x_0, \dots, x_n]$ the interval $[x_0, \dots, x_n] = \bigcap_{k=0}^n f^{-k}([x_k])$. By the Markov property we obtain $f^k([x_0, \dots, x_n]) = [x_k, \dots, x_n]$ for all $0 \leq k \leq n$.

Lemma VII.5 *Let $\{f_a\}_{a\in\mathcal{A}}$ be a C^2 one-parameter family of unimodal maps and $a^* \in \mathcal{A}$ such that f_{a^*} has no stable periodic point. For every $\varepsilon > 0$, there exist $\lambda_{7.5}(\varepsilon) > 0$, $N_{7.5}(\varepsilon) \geq 1$ and $V_{7.5}(\varepsilon)$ a neighborhood of a^* such that, for every $x \in I$, $a \in V_{7.5}$, $n \geq N_{7.5}$, if $\{x, f_a(x), \dots, f_a^{n-1}(x)\}$ is disjoint from $[-\varepsilon, \varepsilon]$, then $|Df_a^n(x)| \geq \exp(n\lambda_{7.5})$.*

Proof Let $\lambda_{7.5}$ be any positive real such that $\lambda_{7.5} < \lambda_{2.1}$. We then choose $p \geq 2/(\lambda_{2.1} - \lambda_{7.5}) \log(2/K_{2.1})$ and $N_{7.5} \geq (2(\lambda_{7.5} - \lambda)/(\lambda_{2.1} - \lambda_{7.5}) + 1)p$, where $\lambda = \inf\{\log|Df_a(x)| : a \in \mathcal{A} \text{ and } |x| \geq \varepsilon\}$. We decompose the orbit into blocs of length p, $n = kp + q$, $0 \leq q < p$:

$$Df_a^n(x) = \prod_{i=0}^{k-1} Df_a^p(f_a^{ip}(x) Df_a^q(f_a^{kp}(x)) .$$

We choose now $V_{7.5}$ such that, for all $a \in V_{7.5}$,

$$|f_a^i(x) - f_a^i(x)| < \frac{\varepsilon}{2} \quad (\forall 0 \leq i < p \text{ and } x \in I),$$

$$|Df_a^p(x)| \geq \frac{1}{2}|Df_{a^*}^p(x)| \quad (\forall x \in I \quad s.t. \quad |f_{a^*}^i(x)| \geq \frac{\varepsilon}{2} \quad i = 0, \ldots, p-1).$$

Using Theorem II.1, for every $x \in I$ satisfying $\{x, f_a(x), \ldots, f_a^{n-1}\} \cap [-\varepsilon, \varepsilon] = \emptyset$, we have for every $n \geq N_{7.5}$ and $a \in V_{7.5}$

$$|Df_a^n(x)| \geq \left(\frac{K_{2.1}}{2}\right)^k \exp(q\lambda + kp\lambda_{2.1}) \geq \exp(n\lambda_{7.5}) .$$

□

Proof of Proposition VII.3 Let Λ^* be a f_{a^*}-invariant compact set disjoint from the critical point c_0. Since c_0 is in the closure of the set of periodic points for f_{a^*}, we can find a periodic point $q \notin \Lambda^*$ of period $r \geq 3$ close to c_0 such that $\{f_{a^*}(q), \ldots, f_{a^*}^{r-1}(q)\}$ is disjoint from (q', q'') and $\Lambda^* \subseteq \Xi$ (cf. notation VII.4). The itinerate of each point $x \in \Lambda^*$ is disjoint from $\{q', q'', f_{a^*}(q), \ldots, f_{a^*}^{r-1}(q)\}$; x determines a unique admissible sequence $\underline{x} = (x_n)_{n \geq 0}$ where $x_n \in \{1, \ldots, r\}$ is defined by $f^n(x) \in [x_n]$. Conversely, for each $\underline{x} \in \Sigma_M$, by hyperbolicity of f_{a^*} (Theorem II.1), $\bigcap_{n \geq 0} f_{a^*}^{-n}(\overline{[x_n]})$ is reduced to a point $\theta(\underline{x})$. We have just defined a map $\theta : \Sigma_M \to \Xi$ which is bijective from $\theta^{-1}(\Lambda^*)$ into Λ^* and which conjugate the shift to f_{a^*}: $\theta \circ \sigma = f_{a^*} \circ \theta$.

Let us denote by $q : V \to I$ a $\mathcal{C}^1$ continuation of the periodic point q in a neighborhood of a^* ($f^r(q(a), a) = q(a)$ $\forall a \in V^*$ and $q(a^*) = q$). We choose V^* sufficiently small so that $\{q'(V^*), q''(V^*), q_1(V^*), \ldots, q_{r-1}(V^*)\}$ are pairwise disjoint, disjoint from Λ^* and from $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Let us denote by $\{[1]_a, \ldots, [r]_a\}$ the Markov partition associated to $q(a)$. We notice that the transition matrix is independent of $a \in V^*$. Using the same reasons, for each $\underline{x} \in \Sigma_M$, $a \in V^*$, $\bigcap_{n \geq 0} f_a^{-n}(\overline{[x_n]}_a)$ is reduced to a point $\theta(\underline{x}, a) = \theta_a(\underline{x})$. For each $\underline{x} \in \Sigma_M$ and $a \in V^*$, we denote by $f_{\underline{x}}^{-n} : \bigcup_{a \in V^*} n_a \times \{a\} \to I$ the inverse branch of f^n defined by $f^n(f_{\underline{x}}^{-n}(z, a), a) = z$ and $f_{\underline{x}}^{-n}(z, a) \in [x_0, \ldots, x_n]_a$ for all $z \in [x_n]_a$.

We claim that $\theta(\underline{x}, a)$ is continuous with respect to $(\underline{x}, a)$, differentiable with respect to a and that $\partial_a \theta(\underline{x}, a)$ is continuous with respect to $(\underline{x}, a)$. We define

$\theta_n(\underline{x}, a) = f_{\underline{x}}^{-n}(z_n(\underline{x}, a), a)$ where $z_n(\underline{x}, a)$ is the middle point of $[x_n]_a$. We first notice that $(\theta_n)_{n\geq 0}$ converges uniformly to $\theta(\underline{x}, a)$ since

$$|\theta(\underline{x}, a) - \theta_n(\underline{x}, a)| \leq K_{7.5}(\varepsilon)\exp\{-n\lambda_{7.5}(\varepsilon)\}$$

and that θ_n is continuous with respect to $(\underline{x}, a)$ (we use the fact that $x_n = y_n$ whenever $\underline{x}$ is close to $\underline{y}$). For fixed z, the derivative with respect to a of $f_{\underline{x}}^{-n}(z, a)$ is given by

$$\partial_a f_{\underline{x}}^{-n}(z, a) = -\frac{\partial_a f^n(f_{\underline{x}}^{-n}(z, a), a)}{\partial_x f^n(f_{\underline{x}}^{-n}(z, a)a)} = -\sum_{k=1}^{n} \frac{\partial_a f(f_{\underline{x}}^{-n+k-1}(z, a), a)}{\partial_x f^k(f_{\underline{x}}^{-n}(z, a), a)},$$

and by the chain rule we obtain

$$\partial_a \theta_n(\underline{x}, a) = \frac{\partial_a z_n(\underline{x}, a)}{\partial_x f^n(\theta_n(\underline{x}, a), a)} - \sum_{k=1}^{n} \frac{\partial_a f(f_a^{k-1}(\theta_n(\underline{x}, a), a)}{\partial_x f^k(\theta_n(\underline{x}, a), a)}.$$

Since $(\partial_a z_n)_{n\geq 0}$ is uniformly bounded and f^n is uniformly expanding,

$$|\partial_x f^n(\theta_n(\underline{x}, a), a)| \geq K_{7.5}(\varepsilon)\exp\{n\lambda_{7.5}(\varepsilon)\},$$

the first term tends uniformly to zero and the summation converges to

$$\partial_a \theta(\underline{x}, a) = \lim_{n\to+\infty} \partial_a \theta_n(\underline{x}, a) = -\sum_{k=1}^{\infty} \frac{\partial_a f(f_a^{k-1} \circ \theta(\underline{x}, a), a)}{\partial_x f^k(\theta(\underline{x}, a), a)},$$

which proves the claim.

Finally, we prove that θ_a is injective on $\theta^{-1}(\Lambda^*)$ for all a close to a^*. Suppose

$$x = \theta_a(\underline{x}) = \theta_a(\underline{y}) \quad \text{for some } \underline{x} \neq \underline{y}.$$

Then there exists $n \geq 0$ such that $x_n \neq y_n$. Since $f_a^n(x) \in [x_n]_a \cap [y_n]_a$, x is a preimage of $q(a)$. By uniform continuity of θ, we can choose a neighborhood of a^* such that $\theta_a \circ \theta^{-1}(\Lambda^*)$ do not contain $q(a)$, for all $a \in V^*$. In particular, $\underline{x}$ or $\underline{y}$ cannot be in $\theta^{-1}(\Lambda^*)$. □

Remark VII.6 Let $\{f_a\}_{a\in\mathcal{A}}$ be a $\mathcal{C}^2$ one-parameter family of unimodal maps and Λ^* be a f_{a^*}-invariant compact set. If $\chi : \Lambda^* \times V^* \to I$ is a smooth continuation of Λ^*, then for all $x \in \Lambda^*$

$$\lim_{n\to+\infty} \frac{\partial_a f^n(x, a^*)}{\partial_x f^n(x, a^*)} = -\partial_a \chi(x, a^*) = \sum_{k=1}^{+\infty} \frac{\partial_a f(f_{a^*}^{k-1}(x, a^*)}{\partial_x f^k(x, a^*)}.$$

Proposition VII.7 *Let $\{f_a\}_{a\in\mathcal{A}}$ be a C^2 one-parameter family of unimodal maps, $a^* \in \mathcal{A}$ such that f_{a^*} satisfies the Misiurewicz condition and has no stable periodic point and $\Lambda^* = \overline{\{c_n(a^*) : n \geq 1\}}$. If $\chi : \Lambda^* \times V^* \to I$ is a smooth continuation of Λ^* then, for every $N \geq 1$,*

$$\lim_{n\to+\infty} \frac{\partial_a f^n(c_0, a^*)}{\partial_x f^{n-1}(c_1, a^*)} = \frac{1}{\partial_x f^{N-1}(c_1, a^*)}\Big\{\frac{dc_N}{da}(a^*) - \partial_a \chi(c_N(a^*), a^*)\Big\}.$$

In particular, the sequence $(\frac{d}{da}c_n(a^*))_{n\geq 0}$ is either uniformly bounded or grows exponentially.

Proof We apply Remark VII.6 to $x = c_N(a^*)$ and use the identity

$$Q_n(a) = Q_N(a) + \frac{1}{\partial_x f^{N-1}(c_1, a)} \frac{\partial_a f^{n-N}(c_N, a)}{\partial_x f^{n-N}(c_N, a)}.$$

□

VII.B. Genericity

We fix f_* a C^2 unimodal map satisfying the Misiurewicz condition and without stable periodic point, and denote by $\mathcal{T}(a^*,f_*)$ the subset of $\mathcal{R}(a^*,f_*)$ of regular families $f = \{f_a\}_{a\in\mathcal{A}}$ which satisfies condition (T) at a^*.

Proof of Proposition I.11

Part one. We show that $\mathcal{T}(a^*,f_*)$ is open in $\mathcal{R}(a^*,f_*)$. Indeed, by Lemma VII.1, we have for any regular family $f \in \mathcal{R}(a^*,f_*)$

$$Q(a^*,f) = \sum_{k=0}^{+\infty} \frac{\partial_a f(f_*^k(0), a^*)}{Df_*^k(f_*(0))}.$$

By convergence of the series $\sum_{k=0}^{+\infty} |Df_*^k(f_*(0))|^{-1}$, $[f \mapsto Q(a^*,f)]$ is continuous.

Part two. We show that $\mathcal{T}(a^*,f_*)$ is dense in $\mathcal{R}(a^*,f_*)$. Let $f \in \mathcal{R}(a^*,f_*)$ be such that $Q(a^*,f) = 0$ and assume there exists $n \geq 0$ such that $f_*^{n+1}(0) \in (-1,1)$ and $f_*^n(0)$ is an isolated point in $\{f_*^k(0) : k \geq 0\}$ (otherwise $f_*^2(0) = f_*^3(0)$ and a^* has to be an endpoint of $\mathcal{A}$). We now construct a small perturbation g which agrees with f on $\overline{\{f_*^k(0) : k \geq 0\}} \setminus \{f_*^n(0)\} \times \mathcal{A}$ and such that $\partial_a f(f_*^n(0), a^*) \neq \partial_a g(f_*^n(0), a^*)$. For example, we take

$$g(x,a) = f(x,a) + \varepsilon\eta\phi\Big(\frac{x - f_*^n(0)}{\varepsilon}\Big)\psi\Big(\frac{a - a^*}{\eta}\Big)$$

(where $\phi : \mathbf{R} \to \mathbf{R}$ is equal to 1 on $(-1,1)$ and equal to 0 on $\mathbf{R} \setminus [-2,2]$ and $\psi : \mathbf{R} \to \mathbf{R}$ is equal to the identity on $(-1,1)$ and equal to 0 on $\mathbf{R} \setminus [-2,2]$). Using once more Lemma VII.1, we obtain

$$Q(a^*,g) = Q(a^*,g) - Q(a^*,f) = \frac{\partial_a f(f_*^n(0),a^*) - \partial_a g(f_*^n(0),a^*)}{Df_*^n(f_*0))} \neq 0 .$$

□

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Ph. Thieullen
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ PARIS SUD
91405 ORSAY CEDEX, FRANCE

C. Tresser
I.B.M.
P.O. BOX 210
YORKTOWN HEIGHTS, NY 10958, USA

L. S. Young
DEPARTMENT OF MATHEMATICS
U.C.L.A.
LOS ANGELES, CA 90024, USA
AND
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
TUCSON, AZ 85721, USA

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