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The index of an algebraic variety

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Abstract Let K be the field of fractions of a Henselian discrete valuation ring $\mathcal{O}_K$. Let X_K/K be a smooth proper geometrically connected scheme admitting a regular model $X/\mathcal{O}_K$. We show that the index $\delta(X_K/K)$ of X_K/K can be explicitly computed using data pertaining only to the special fiber X_k/k of the model X.

We give two proofs of this theorem, using two moving lemmas. One moving lemma pertains to horizontal 1-cycles on a regular projective scheme X over the spectrum of a semi-local Dedekind domain, and the second moving lemma can be applied to 0-cycles on an FA-scheme X which need not be regular.

The study of the local algebra needed to prove these moving lemmas led us to introduce an invariant $\gamma(A)$ of a singular local ring $(A, \mathfrak{m})$: the greatest common divisor of all the Hilbert-Samuel multiplicities $e(Q, A)$, over all $\mathfrak{m}$-primary ideals Q in $\mathfrak{m}$. We relate this invariant $\gamma(A)$ to the index of the exceptional divisor in a resolution of the singularity of Spec A, and we give a

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new way of computing the index of a smooth subvariety X/K of $\mathbb{P}^n_K$ over any field K, using the invariant γ of the local ring at the vertex of a cone over X.

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Contents

Let W be a non-empty scheme of finite type over a field F. Let $\mathcal{D}(W/F)$ denote the set of all degrees of closed points of W. The *index* $\delta(W/F)$ of W/F is the greatest common divisor of the elements of $\mathcal{D}(W/F)$. The index is also the smallest positive integer occurring as the degree of a 0-cycle on W. When W is integral, let W^{reg} denote the regular locus of W, open in W. We note in 6.8 that $\delta(W^{\text{reg}}/F)$ is a birational invariant of W/F.

Let now K be the field of fractions of a discrete valuation ring $\mathcal{O}_K$ with residue field k. Let $S := \operatorname{Spec} \mathcal{O}_K$. Let $X \to S$ be a proper flat morphism, with X regular and irreducible. Let X_K/K be the generic fiber of X/S. Write the special fiber X_k, viewed as a divisor on X, as $\sum_{i=1}^n r_i \Gamma_i$, where for each $i = 1, \dots, n$, Γ_i is irreducible, of multiplicity r_i in X_k. Using the intersection of Cartier divisors with 1-cycles on the regular scheme X, we easily find that $\gcd_i\{r_i \delta(\Gamma_i/k)\}$ divides $\delta(X_K/K)$ (see 8.1). Our theorem below strengthens this divisibility, and shows that when $\mathcal{O}_K$ is Henselian, the index of the generic fiber can be computed using only data pertaining to the special fiber.

Theorem 8.2 *Keep the above assumptions on X/S.*

(a) *Then* $\gcd_i\{r_i \delta(\Gamma_i^{\text{reg}}/k)\}$ *divides* $\delta(X_K/K)$.
(b) *When* $\mathcal{O}_K$ *is Henselian, then* $\delta(X_K/K) = \gcd_i\{r_i \delta(\Gamma_i^{\text{reg}}/k)\}$.

Theorem 8.2 answers positively a question of Clark ([10], Conj. 16). This theorem is known already when k is a finite field ([4], 1.6, see also [13], 3.1), or when k is algebraically closed (same proof as in [4]), or when X_K/K is a curve with semi-stable reduction ([10], Thm. 9).

We give two proofs of Theorem 8.2, using two different moving lemmas which may be of independent interest. The first proof uses the Moving Lemma 2.3 stated below. A slightly strengthened version is proved in the text. The definition and main properties of the notion of *rational equivalence of cycles* are recalled in Sect. 1.

Theorem 2.3 *Let R be a semi-local Dedekind domain, and let $S := \mathrm{Spec}(R)$. Let X/S be flat and quasi-projective, with X regular. Let C be a 1-cycle on X, closure in X of a closed point of the generic fiber of X. Let F be a closed subset of X such that for all $s \in S$, $F \cap X_s$ has codimension at least 1 in X_s. Then C is rationally equivalent to a cycle C' on X whose support does not meet F.*

Our second proof of Theorem 8.2 uses the Moving Lemma 6.5 below, which allows some moving of a multiple of a cycle on a scheme X which need not be regular. Recall (2.2) that X *is an* FA-*scheme* if every finite subset of X is contained in an affine open subset of X.

Theorem 6.5 *Let X be a Noetherian* FA-*scheme. Let F be a closed subset of X of positive codimension in X. Let $x_0 \in X$. Let Q be a $\mathfrak{m}_{X,x_0}$-primary ideal of $\mathcal{O}_{X,x_0}$, with Hilbert-Samuel multiplicity $e(Q, \mathcal{O}_{X,x_0})$. Then the cycle $e(Q)[\overline{\{x_0\}}]$ is rationally equivalent in X to a cycle Z such that no irreducible cycle occurring in Z is contained in F.*

Theorem 6.5 is a consequence of a local analysis of the Noetherian local ring $\mathcal{O}_{X,x_0}$ found in Sect. 4, and in particular in Theorem 4.5. Our investigation of the local algebra needed to prove Theorem 6.5 led us to introduce the following local invariant in 5.1. Let $(A, \mathfrak{m})$ be any Noetherian local ring. Let $\mathcal{E}(A)$ denote the set of all Hilbert-Samuel multiplicities $e(Q, A)$, for all $\mathfrak{m}$-primary ideals Q of A. Define $\gamma(A)$ *to be the greatest common divisor of the elements of $\mathcal{E}(A)$*. Theorem 6.5 and the definition of $\gamma(A)$ show that:

Corollary 6.7 *Let X/k be a reduced scheme of finite type over a field k and let $x_0 \in X$ be a closed point. Then $\delta(X^{\mathrm{reg}}/k)$ divides $\gamma(\mathcal{O}_{X,x_0}) \deg_k(x_0)$.*

This statement is slightly strengthened when $\mathcal{O}_{X,x_0}$ is not equidimensional in 7.13. Recall that the Hilbert-Samuel multiplicity $e(\mathfrak{m}, A)$ is the smallest element in the set $\mathcal{E}(A)$, and that it is a measure of the singularity of the ring A: if the completion of A is a domain (or, more generally, is unmixed [27], 6.8 and 6.9), then A is regular if and only if $e(\mathfrak{m}, A) = 1$. The invariant $\gamma(A)$ is also related to the singularity of the ring A: our next theorem shows that

$\gamma(A)$ is equal to the index of the exceptional divisor of a desingularization of Spec A.

Theorems 5.6 and 7.3 *Let A be an excellent Noetherian equidimensional local ring of positive dimension. Let $X :=$ Spec A, with closed point x_0. Let $f : Y \to X$ be a proper birational morphism such that Y is regular. Let $E := f^{-1}(x_0)$. Then $\gamma(A) = \delta(E/k(x_0))$.*

Note that in the above theorem, the set of degrees $\mathcal{D}(E)$ and the set of Hilbert-Samuel multiplicities $\mathcal{E}(A)$ need not be equal, and neither needs to contain the greatest common divisors of its elements. The proof that $\gamma(A) = \delta(E/k(x_0))$ involves a third set $\mathcal{N}$ of integers attached to A, which is an ideal in $\mathbb{Z}$, so that the greatest common divisor $n(A)$ of the elements of $\mathcal{N}$ belongs to $\mathcal{N}$ (5.4). The proof shows that $\gamma(A) = n(A)$ and $n(A) = \delta(E/k(x_0))$. The properties of the invariant $n(A)$ are further studied in Sect. 7.

As an application of Theorem 7.3, we obtain a new description of the index of a projective variety.

Theorem 7.4 *Let K be any field. Let V/K be a regular closed integral subscheme of $\mathbb{P}^n_K$. Denote by W a cone over V in $\mathbb{P}^{n+1}_K$. Let $w_0 \in W$ denote the vertex of the cone. Then $\delta(V/K) = \gamma(\mathcal{O}_{W,w_0})$.*

The variant of Hilbert's Tenth Problem, which asks whether there exists an algorithm which decides given a geometrically irreducible variety $V/\mathbb{Q}$, whether $V/\mathbb{Q}$ has a $\mathbb{Q}$-rational point, is an open question to this date ([46], p. 348). So is the possibly weaker question of the existence of an algorithm which decides, given $V/\mathbb{Q}$, whether $\delta(V/\mathbb{Q}) = 1$ (7.7). In view of Theorem 7.4, we may also ask whether there exists an algorithm which decides, given a $\mathbb{Q}$-rational point w_0 on a scheme of finite type $W/\mathbb{Q}$, whether $\gamma(\mathcal{O}_{W,w_o}) = 1$.

In the last section of this article, we settle a question of Lang and Tate [36], page 670, when the ground field K is imperfect, and prove:

Theorem 9.2 *Let X be a regular and generically smooth non-empty scheme of finite type over a field K. Then the index $\delta(X/K)$ is equal to the separable index $\delta_{sep}(X/K)$.*

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1 Rational equivalence

We review below the basic notation needed to state our moving lemmas. Let X be a Noetherian scheme. Let $\mathcal{Z}(X)$ denote the free Abelian group on the set of closed integral subschemes of X. An element of $\mathcal{Z}(X)$ is called a cycle, and if Y is an integral closed subscheme of X, we denote by $[Y]$ the associated element in $\mathcal{Z}(X)$.

Let $\mathcal{K}_X$ denote the sheaf of meromorphic functions on a Noetherian scheme X (see [32], top of page 204 or [39], Definition 7.1.13). Let $f \in \mathcal{K}_X^*(X)$. Its associated principal Cartier divisor is denoted by $\mathrm{div}(f)$ and defines a cycle on X:

$$\left[\mathrm{div}(f)\right] = \sum_x \mathrm{ord}_x(f_x)\left[\,\overline{\{x\}}\,\right]$$

where x ranges through the points of codimension 1 in X (i.e., the points x such that the closure $\overline{\{x\}}$ has codimension 1 in X; this latter condition is equivalent to the condition $\dim \mathcal{O}_{X,x} = 1$). The function $\mathrm{ord}_x : \mathcal{K}_{X,x}^* \to \mathbb{Z}$ is defined, for a regular element of $g \in \mathcal{O}_{X,x}$, to be the length of the $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X,x}/(g)$.

A cycle Z is *rationally equivalent to* 0 ([33], §2), or *rationally trivial*, if there are finitely many integral closed subschemes Y_i and principal Cartier divisors $\mathrm{div}(f_i)$ on Y_i, such that $Z = \sum_i [\mathrm{div}(f_i)]$. Two cycles Z and Z' are rationally equivalent in X if $Z - Z'$ is rationally equivalent to 0. We denote by $\mathcal{A}(X)$ the quotient of $\mathcal{Z}(X)$ by the subgroup of rationally trivial cycles.

1.1 We will need the following facts. Given a ring A and an A-module M, we denote by $\ell_A(M)$ the *length* of M. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be its minimal prime ideals. Let $\mathrm{Frac}(A)$ be the total ring of fractions of A (with $\mathrm{Frac}(A) = \mathcal{K}_X(X)$ for $X = \mathrm{Spec}\, A$) and denote by $\mathrm{ord}_A : \mathrm{Frac}(A)^* \to \mathbb{Z}$ the associated order function. Then

(1) Let $f \in \mathrm{Frac}(A)^*$, and let f_i denote the image of f in $\mathrm{Frac}(A/\mathfrak{p}_i)^*$. Using [5], Lemma 9.1/6, we obtain that

$$\mathrm{ord}_A(f) = \sum_{1 \le i \le t} \ell_{A_{\mathfrak{p}_i}}(A_{\mathfrak{p}_i})\, \mathrm{ord}_{A/\mathfrak{p}_i}(f_i).$$

(2) If A is reduced, then the canonical homomorphism

$$\mathrm{Frac}(A) \longrightarrow \bigoplus_i A_{\mathfrak{p}_i} = \bigoplus_i \mathrm{Frac}(A/\mathfrak{p}_i)$$

is an isomorphism.

(3) Let $A \subseteq B \subseteq \mathrm{Frac}(A)$ be a subring such that B/A is finite. Let $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ be the maximal ideals of B, and let $b \in \mathrm{Frac}(A)^*$. Then

$$\mathrm{ord}_A(b) = \sum_{1 \le i \le r} [B/\mathfrak{n}_i : A/\mathfrak{m}]\, \mathrm{ord}_{B_{\mathfrak{n}_i}}(b).$$

Indeed, our hypothesis implies that the A-module B/A has finite length, so for a regular element $b \in A$, $\ell_A(A/bA) = \ell_A(B/bB)$. Conclude using [21], A.1.3, and the isomorphism $B/bB \to \prod_{i=1}^{r} B_{\mathfrak{n}_i}/bB_{\mathfrak{n}_i}$.

Remark 1.2 Let $\Gamma_1, \ldots, \Gamma_r$ be the irreducible components of X, endowed with the reduced structure. Let $\xi_1, \ldots, \xi_r$, denote their generic points. Let $f \in \mathcal{K}_X^*(X)$, and let $f|_{\Gamma_i}$ be the meromorphic function restricted to Γ_i. If for all $i \le r$, $[\operatorname{div}(f|_{\Gamma_i})]$ only involves codimension 1 points of X, then $[\operatorname{div}(f)]$ is rationally equivalent to 0 on X, since $[\operatorname{div}(f)] = \sum_{i=1}^{r}[\operatorname{div}((f|_{\Gamma_i})^{\ell(\mathcal{O}_{X,\xi_i})})]$ (use 1.1 (1)).

However, in general $[\operatorname{div}(f)]$ *is not* rationally equivalent to 0. Consider for instance the projective variety X over a field k, union of $\mathbb{P}^1_k$ and $\mathbb{P}^2_k$ intersecting transversally at a single point $\infty \in \mathbb{P}^1_k$. Let x be a coordinate function on $\mathbb{P}^1_k$ with $[\operatorname{div}_{\mathbb{P}^1_k}(x)] = [0] - [\infty]$. Let f be a rational function on X which restricts to x on $\mathbb{P}^1_k$ and is equal to 1 on $\mathbb{P}^2_k$. Then $[\operatorname{div}_X(f)]$ is the 0-cycle [0], since the point ∞ does not have codimension 1 in X. It is clear however that [0] is not rationally trivial in X. This shows that the implication (1) $\Rightarrow$ (3) in the proposition in [20], §1.8, does not hold in general.

A proper morphism of schemes $\pi : Y \to X$ induces by *push forward of cycles* a group homomorphism $\pi_* : \mathcal{Z}(Y) \to \mathcal{Z}(X)$. If Z is any closed integral subscheme of Y, then $\pi_*([Z]) := [k(Z) : k(\pi(Z))][\pi(Z)]$, with the convention that $[k(Z) : k(\pi(Z))] = 0$ if the extension $k(Z)/k(\pi(Z))$ is not algebraic. It is known ([33], [54], and 1.5 below) that in general further assumptions are needed for a proper morphism π to induce a group homomorphism $\pi_* : \mathcal{A}(Y) \to \mathcal{A}(X)$. This is illustrated by our next example, also used later in 2.7, 7.2, and 7.17. (This example contradicts [21], Example 20.1.3.)

Example 1.3 We exhibit below a finite birational morphism $\pi : Y \to X$ of affine integral Noetherian schemes with Y regular, and a closed point $y_1 \in Y$ of codimension 1 with $[y_1]$ rationally equivalent to 0 on Y, but such that $\pi_*([y_1])$ is not rationally equivalent to 0 on X. The key feature in this example is that π maps the point y_1 of codimension 1 in Y to a point of codimension 2 in $X = \operatorname{Spec} A$. It turns out that A is not universally catenary. Our example is similar to that of [23], IV.5.6.11. The idea of the construction of a ring that is not universally catenary by gluing two closed points of distinct codimensions is due to Nagata (see [41], 14.E).

Let k_0 be any field. Let $k := k_0(t_\alpha)_{\alpha\in\mathbb{N}}$ be the field of rational functions with countably many variables. Consider the polynomial ring in one variable $k[S]$ and the discrete valuation ring $R := k[S]_{(Sk[S])}$. Let $Y := \operatorname{Spec} R[T]$. Let $P(T) \in k[T]$ be an irreducible polynomial of degree $d \ge 1$. Let $y_0 \in Y$ be the closed point corresponding to $\mathfrak{p} := (P(T), S)$ and let y_1 be the closed point corresponding to $\mathfrak{q} := (ST - 1)R[T]$. Then $\dim \overline{\{y_i\}} = 0$ and $\dim \mathcal{O}_{Y,y_i} = 2 - i$. The residue field $k(y_0) := k[T]/(P(T))$ is a finite extension of k of degree d, and $k(y_1) = \mathrm{Frac}(R) = k(S)$.

Choose a field isomorphism $\varphi : k \to k(S)$. Let $X := \mathrm{Spec}\, A$ be the scheme obtained by identifying y_1 and y_0 via φ (see [45], Teorema 1, [51], 3.4, or [16], 5.4):

$$A := \big\{ f \in R[T] \mid f(y_0) \in k,\ \varphi\big(f(y_0)\big) = f(y_1) \big\}.$$

By definition, A is the pre-image of the field $\{(\lambda, \varphi(\lambda)) \mid \lambda \in k\}$ under the canonical surjective homomorphism $R[T] \to k(y_0) \oplus k(y_1)$. The ideal $\mathfrak{m} := \mathfrak{p} \cap \mathfrak{q}$ of $R[T]$ is then a maximal ideal of A, defining a closed point $x_0 \in X$ whose residue field $k(x_0)$ is isomorphic to $\{(\lambda, \varphi(\lambda)) \mid \lambda \in k\}$. The inclusion $A \to R[T]$ induces a morphism $\pi : Y \to X$.

It is easy to see that $R[T]/\mathfrak{m}$ is finitely generated over $A/\mathfrak{m}$. Since $\mathfrak{m} \subseteq A$, we can thus produce a finite system of generators for the A-module $R[T]$. More precisely, we have $R[T] = A + T(TS-1)A + \cdots + T^{d-1}(TS-1)A + TSA$. Therefore, π is finite and, hence, A is Noetherian by Eakin-Nagata's theorem. The ring A has dimension 2 and, thus, is catenary. The induced morphism $\pi : Y \setminus \{y_0, y_1\} \to X \setminus \{x_0\}$ is an isomorphism. Indeed, for any special open subset $D(h) \subseteq X \setminus \{x_0\}$ (i.e., $h \in \mathfrak{m} \setminus \{0\}$), we have $hR[T] \subseteq \mathfrak{m} \subseteq A$. So any fraction g/h^n with $g \in R[T]$ is equal to gh/h^{n+1} with $gh \in A$, and $R[T]_h = A_h$.

Fix now $d \geq 2$. Let $f := ST - 1 \in \mathfrak{q}$. Then $[\mathrm{div}(f)] = [y_1]$. By construction, π induces an isomorphism $k(x_0) \simeq k(y_1)$, so that $\pi_*([y_1]) = [x_0]$. We claim that $[x_0]$ is not rationally trivial on X. Indeed, let C be a closed integral subscheme of X containing x_0 as a point of codimension 1. As $\mathcal{O}_{X,x_0}$ and X are of dimension 2, we must have $\dim C = 1$. Let $\tilde{C}$ be the schematic closure of $\pi^{-1}(C \setminus \{x_0\})$ in Y, and let $\rho : \tilde{C} \to C$ be the restriction of π. Then ρ is a finite birational morphism of integral Noetherian schemes of dimension 1. The point y_1 cannot belong to $\tilde{C}$, since otherwise the prime ideal $\mathfrak{q}$ would properly contain the prime ideal of height 1 corresponding to the generic point of $\tilde{C}$. Hence, $\rho^{-1}(x_0) = \{y_0\}$.

Now let $\mathrm{div}(g)$ be a principal Cartier divisor on C. Then, using 1.1 (3), $\mathrm{ord}_{x_0}(g) = [k(y_0) : k(x_0)]\, \mathrm{ord}_{y_0}(g) = d\, \mathrm{ord}_{y_0}(g)$. Therefore, if $n[x_0]$ is rationally equivalent to 0, then $d \mid n$. It follows that $[x_0]$ is not rationally equivalent to 0 when $d \geq 2$; in fact, $[x_0]$ has order d in the group $\mathcal{A}(X)$. The same proof shows that $[x_0]$ has order d in the group $\mathcal{A}(\mathrm{Spec}\, \mathcal{O}_{X,x_0})$.

1.4 For general Noetherian schemes, Thorup introduced a notion of rational equivalence depending on a grading δ_X on X, which turns the quotient $\mathcal{A}(X, \delta_X)$ of $\mathcal{Z}(X)$ by this equivalence into a covariant functor for proper morphisms and a contravariant functor for flat equitranscendental morphisms ([54], Proposition 6.5).

We briefly recall Thorup's theory below. A *grading* on a non-empty scheme X is a map $\delta_X : X \to \mathbb{Z}$ such that if $x \in \overline{\{y\}}$, then $\mathrm{ht}(x/y) \leq \delta_X(y) - \delta_X(x)$ ([54], 3.1). A grading δ_X is *catenary* if the above inequality is always an equality ([54], 3.6). An example of a grading on X is the canonical grading $\delta_{\mathrm{can}}(x) := -\dim \mathcal{O}_{X,x}$. This grading is catenary if and only if X is catenary and every local ring is equidimensional[1] ([54], p. 266) at every point.

Let Y be an integral closed subscheme of X with generic point η, and let $f \in k(Y)^*$. Denote by $[\mathrm{div}(f)]^{(1)}$ the cycle $[\mathrm{div}(f)]$ where we discount all components $\overline{\{x\}}$ such that $\delta_X(x) < \delta_X(\eta) - 1$. One defines the *graded rational equivalence* on $\mathcal{Z}(X)$ using the subgroup generated by the cycles $[\mathrm{div}(f)]^{(1)}$, for all closed integral subschemes of X. If δ_X is catenary, then the graded rational equivalence is the same as the usual (ungraded) one ([54], Note 6.6). Denote by $\mathcal{A}(X, \delta_X)$ the (graded) Chow group defined by the graded rational equivalence.

Let $f : Y \to X$ be a morphism essentially of finite type. Let δ_X be a grading on X. Then f induces a grading δ_f on Y defined in [54] (3.4), by

$$\delta_f(y) := \delta_X\big(f(y)\big) + \mathrm{trdeg}\big(k(y)/k\big(f(y)\big)\big).$$

If f is proper, then f induces a homomorphism $f_* : \mathcal{A}(Y, \delta_f) \to \mathcal{A}(X, \delta_X)$ ([54], Proposition 6.5). If X is universally catenary and equidimensional at every point, and $\delta_X = \delta_{\mathrm{can}}$, then δ_f is a catenary grading on Y ([54], 3.11). It is also true that if X is universally catenary and δ_X is a catenary grading, then δ_f is a catenary grading on Y ([54], p. 266, second paragraph).

1.5 In particular, assume that both X/S and Y/S are schemes of finite type over a Noetherian scheme S which is universally catenary and equidimensional at every point, and $f : Y \to X$ is a proper morphism of S-schemes. Let C and C' be two cycles on Y (classically) rationally equivalent. Then $f_*(C)$ and $f_*(C')$ are (classically) rationally equivalent on X.

In Example 1.3, endow X with the canonical grading, and Y with the grading δ_π. Then δ_X is catenary but δ_π is not, because y_1 has virtual codimension 2. Computations show that $\mathcal{A}(Y, \delta_\pi) = \mathbb{Z} \oplus \mathbb{Z}$, generated by the classes of $[y_1]$ and $[Y]$. The group $\mathcal{A}(Y)$ is isomorphic to $\mathbb{Z}$, generated by the class of $[Y]$. The group $\mathcal{A}(X)$ is isomorphic to $\mathcal{A}(X, \delta_{\mathrm{can}}) = (\mathbb{Z}/d\mathbb{Z}) \oplus \mathbb{Z}$, generated by the classes of $[x_0]$ and $[X]$, with the former of order d.

1.6 Let S be a separated integral Noetherian regular scheme of dimension at most 1. Let η denote its generic point. Endow S with the catenary grading

[1] Recall that a ring A of finite Krull dimension is equidimensional if $\dim A/\mathfrak{p} = \dim A$ for every minimal prime ideal $\mathfrak{p}$ of A. A point $x \in X$ is equidimensional if $\mathcal{O}_{X,x}$ is.

$1+\delta_{\mathrm{can}}$ (which is also the usual topological grading). Let $f: X \to S$ be a morphism of finite type, and endow X with the grading δ_f. This grading is catenary (1.4).

Let $n \geq 1$ and let $x \in X$ be such that $\delta_f(x) = n$. Then $[\overline{\{x\}}]$ is an *n-cycle* on (X, δ_f). If $\dim(S) = 1$ and $f(x)$ is a closed point $s \in S$, then $\overline{\{x\}}$ is a subscheme of dimension n of the fiber X_s. If $f(x) = \eta$, then $\overline{\{x\}} \to S$ is dominant and $\dim \overline{\{x\}}_\eta = n-1$. In the latter case, $\dim \overline{\{x\}} = n-1$ if and only if S is semi-local and $\overline{\{x\}}$ is contained in X_η. Otherwise, $\dim \overline{\{x\}} = n$.

In particular, the irreducible 1-cycles on (X, δ_f) are of two types: the integral closed subschemes C of X of dimension 1 such that C meets at least one closed fiber, and the closed points of X contained in X_η (in which case S must be semi-local). We say that a 1-cycle is *horizontal* if its support is quasi-finite over S, and that it is *vertical* if its support is not dominant over S.

2 Moving Lemma for 1-cycles on regular X/S with S semi-local

Let X be a quasi-projective scheme of pure dimension d a field k. Let X^{sing} denote the non-smooth locus of X. The classical Chow's Moving Lemma [49] and its generalization ([14], II.9, assuming k algebraically closed) immediately imply the following statement:

2.1 *Let* $0 \leq r \leq d$. *Let* Z *be a* r*-cycle on* X *with* $\mathrm{Supp}(Z) \cap X^{\mathrm{sing}} = \emptyset$. *Assume that* $\dim(X^{\mathrm{sing}}) < d - r$. *Let* F *be a closed subset of* X *of codimension at least* $r+1$ *in* X. *Then there exists an* r*-cycle* Z' *on* X*, rationally equivalent to* Z*, and such that* $\mathrm{Supp}(Z') \cap (F \cup X^{\mathrm{sing}}) = \emptyset$.

Our goal in this section is to prove a variant of this statement for a scheme X over a semi-local Dedekind base $S = \operatorname{Spec} R$. An application of such a relative moving lemma is given in Theorem 8.2.

2.2 Let X be a scheme. We say that X *is an* FA-*scheme*, or simply that X is FA, if every finite subset of X is contained in an affine open subset of X. In particular, an FA-scheme is separated. The following examples of FA-schemes are well-known:

(1) Any affine scheme is FA. Any quasi-projective scheme over an affine scheme is FA ([39], Proposition 3.3.36). More generally, a scheme admitting an ample invertible sheaf is FA ([23], II.4.5.4).
(2) If X is FA, then any closed subscheme of X is clearly FA. The same holds for any open subset U of X. Indeed, let F be a finite subset of U, then F is contained in an affine open subset V of X. Hence, $F \subseteq U \cap V$ with $U \cap V$ quasi-affine. By (1), F is contained in an affine open subset of $U \cap V$.

(3) More generally, if Y is FA and $f : X \to Y$ is a morphism of finite type admitting a relatively ample invertible sheaf, then X is FA. Indeed, any finite subset of X has finite image in Y, so we can suppose that Y is affine. Then X admits an ample invertible sheaf [23], II.4.6.6, and we are reduced to the case (1).
(4) A Noetherian separated scheme of dimension 1 is FA ([48], Prop. VIII.1).

Suppose k is an algebraically closed field, and that X/k is a regular FA-scheme of finite type. Let S/k be a separated scheme of finite type. Then any proper k-morphism $X \to S$ is projective ([31], Cor. 2).

For the purpose of our next theorem, we will call a Noetherian integral domain R a *Dedekind domain* if it is integrally closed of dimension 0 or 1. A version of this theorem where R is not assumed to be semi-local is proved in [22], 7.2.

Theorem 2.3 *Let S be the spectrum of a semi-local Dedekind domain R. Let $f : X \to S$ be a separated morphism of finite type, with X regular and* FA. *Let C be a horizontal 1-cycle on X with* $\mathrm{Supp}(C)$ *finite over S. Let F be a closed subset of X such that for every $s \in S$, any irreducible component of $F \cap X_s$ that meets C is not an irreducible component of X_s. Then there exists a horizontal 1-cycle C' on X with $f_{|C'} : \mathrm{Supp}(C') \to S$ finite, rationally equivalent to C, and such that* $\mathrm{Supp}(C') \cap F = \emptyset$.

In addition, since S is semi-local, C consists of finitely many points, and since X is FA, *there exists an affine open subset V of X which contains C. Then, for any such open subset V, the horizontal 1-cycle C' can be chosen to be contained in V, and to be such that if $g : Y \to S$ is any separated morphism of finite type with an open embedding $V \to Y$ over S, then C and C' are closed and rationally equivalent on Y.*

Proof It suffices to prove the theorem when C is irreducible and $\mathrm{Supp}(C) \cap F \neq \emptyset$. Choose an affine open subset V of X containing C. Since C is closed in V, it is affine.

Proposition 3.2 shows the existence of a finite birational morphism $D \to C$ such that the composition $D \to C \to S$ is a local complete intersection morphism (l.c.i.). Clearly, when S is excellent, we can take $D \to C$ to be the normalization morphism, in which case D is even regular, and 3.2 is not needed. Since C is affine and $D \to C$ is finite, there exists for some $N \in \mathbb{N}$ a closed immersion $D \to C \times_S \mathbb{A}^N_S \subseteq V \times_S \mathbb{A}^N_S$.

Let $U := V \times_S \mathbb{A}^N_S$. We claim that it suffices to prove the theorem for the 1-cycle D and the closed subset $\mathbf{F} := F \times_S \mathbb{A}^N_S$ in the affine scheme $f' : U \to S$. Indeed, let D' be a horizontal 1-cycle whose existence is asserted by the theorem in this case. In particular, $\mathrm{Supp}(D') \cap \mathbf{F} = \emptyset$. Let $V \to Y$ be any open immersion over S. Consider the associated open immersion $U \to Y \times_S \mathbb{P}^N_S$

and the projection $p : Y \times_S \mathbb{P}^N_S \to Y$. By hypothesis, D and D' are closed and rationally equivalent in $Y \times_S \mathbb{P}^N_S$. One easily checks that $p_*(D) = C$ because $D \to C$ is birational. It follows from 1.5 that $p_*(D) = C$ is rationally equivalent to $C' := p_*(D')$ on Y. Moreover, $\mathrm{Supp}(C') \cap F = \emptyset$.

The existence of D' with the required properties follows from Proposition 2.4 below. Indeed, first note that since D/S is l.c.i., each local ring $\mathcal{O}_{D,x}$, $x \in D$, is an absolute complete intersection ring ([23], IV.19.3.2). It follows that the closed immersion $D \to U$ is a regular immersion ([23], IV.19.3.2).

Let $d := \mathrm{codim}(D, U)$. We note that $d > 0$ since $\mathrm{Supp}(C) \cap F \neq \emptyset$ and for each point in $\mathrm{Supp}(C) \cap F$ over $s \in S$, $F \cap V_s$ is not an irreducible component of V_s. Let $x \in D$ be a closed point, and let $s := f'(x)$. Then $\dim \mathcal{O}_{D,x} = \dim(S)$, $\dim \mathcal{O}_{U,x} = d + \dim(S)$, and $\dim \mathcal{O}_{U_s,x} = d$. Our assumption on F implies that the irreducible components of $\mathbf{F} \cap U_s$ passing through x have dimension at most $d - 1$. We can thus apply 2.4 below to conclude the proof of 2.3. □

Proposition 2.4 *Let S be any semi-local affine Noetherian scheme. Let $U \to S$ be a morphism of finite type with U affine. Let C be an integral closed subscheme of U, of codimension $d \geq 1$, and finite over S. Suppose that the closed immersion $C \to U$ is regular. Let F be a closed subset of U such that for all closed points $s \in S$, the irreducible components of $F \cap U_s$ that intersect C all have dimension at most $d - 1$. Then there exists a cycle C' on U rationally equivalent to C and such that*:

(1) *The support of C' is finite over S and does not meet $F \cup C$. Moreover, for any closed point $s \in S$,* $\mathrm{Supp}(C')$ *does not contain any irreducible component of U_s.*
(2) *Suppose that S is universally catenary. Let $Y \to S$ be any separated morphism of finite type and let $h : U \to Y$ be any S-morphism. Then $h_*(C)$ is rationally equivalent to $h_*(C')$ on Y.*

Proof (Reduction to the case $d = 1$) Write $U := \mathrm{Spec}\, A$ and $C := V(J)$. Suppose $d \geq 2$. By hypothesis, the (A/J)-module J/J^2 is locally free, hence free of rank d. Now lift a basis of J/J^2 to elements $f_1, \ldots, f_d \in J$. For all $\mathfrak{p} \in C$, we have $J_\mathfrak{p} = f_1 A_\mathfrak{p} + \cdots + f_d A_\mathfrak{p}$ by Nakayama's Lemma. As $J_\mathfrak{p}$ is generated by a regular sequence by hypothesis, Lemma 2.6 implies that $f_1, \ldots, f_d$ is a regular sequence in $A_\mathfrak{p}$.

Let $\Gamma_1, \ldots, \Gamma_n$ denote the irreducible components of $F \cap U_s$ that intersect C, with s ranging through the finitely many closed points of S. By hypothesis, $\dim \Gamma_i \leq d - 1$ for all i. Apply Lemma 2.5 to A, J, $\Gamma_1, \ldots, \Gamma_n$, and $f_1, \ldots, f_{d-1}$ as above. We obtain the existence of $g_1, \ldots, g_{d-1}$, such that $C \subseteq V(g_1, \ldots, g_{d-1})$, and such that every irreducible component of $V(g_1, \ldots, g_{d-1}) \cap \Gamma_i$ either has dimension 0 or is contained in C.

Lemma 2.6 implies that $g_1, \dots, g_{d-1}, f_d$ is a regular sequence at the points of C, so the immersion $C \to V(g_1, \dots, g_{d-1})$ is regular. It is easy to check that the proposition is proved if it can be proved for the closed subsets C and $F \cap V(g_1, \dots, g_{d-1})$ inside the affine scheme $V(g_1, \dots, g_{d-1})$. Moreover, we note that now

(a) at any point $\mathfrak{p}$ of C, C is defined in $V(g_1, \dots, g_{d-1})$ by the regular element f_d, and
(b) any irreducible component of $F \cap V(g_1, \dots, g_{d-1})_s$ is either contained in C or disjoint from C.

We make one further reduction if $V := V(g_1, \dots, g_{d-1})$ is not integral. For every $x \in C$, f_d is a regular element of $\mathcal{O}_{V,x}$, and $\mathcal{O}_{C,x} \simeq \mathcal{O}_{V,x}/(f_d)$. This easily implies that $\mathcal{O}_{V,x}$ is a domain. Thus, there is a unique irreducible component W of V through x. Clearly, W is independent of x, and in a neighborhood of C, V coincides with W endowed with the reduced subscheme structure. The morphism $C \to W$ is still regular. We may thus replace V with the integral subscheme W.

We assume henceforth that U is integral and that $d = 1$. Fix now an open S-immersion $U \to X$, with X/S projective and X integral. Let $\overline{F}$ be the Zariski closure of F in X. Let Z denote the closed subset of X consisting in the finite union of the following closed sets:

(a) $X \setminus U$;
(b) All irreducible components of $\overline{F} \cap X_s$ which do not intersect C, for each closed point $s \in S$; and
(c) One closed point of Γ which does not belong to C, for each irreducible component Γ of U_s which is not contained in C, and for each closed point $s \in S$.

By construction, $Z \cap C = \emptyset$.

Since C is proper over S, it is closed in X. Since $d = 1$, C is in fact a Cartier divisor on U, and since it is closed in X, we can extend it to a Cartier divisor on X. Let $\mathcal{J}$ be the sheaf of ideals on X defining C. This is then an invertible sheaf, and as usual we let $\mathcal{O}_X(nC) := \mathcal{J}^{-n}$.

Let $\mathcal{I}$ be the sheaf of ideals on X defining the reduced induced structure on Z. Let $\mathcal{I}(nC)$ denote the image of $\mathcal{I} \otimes \mathcal{O}_X(nC)$ in $\mathcal{O}_X(nC)$. For all $n \geq 2$, we have a natural exact sequence

$$0 \to \mathcal{I}\big((n-1)C\big) \to \mathcal{I}(nC) \to \mathcal{F}_n \to 0,$$

where $\mathcal{F}_n$ is a coherent sheaf annihilated by $\mathcal{J}$ and, hence, supported on C. Applying $H^1(X, -)$ to the above exact sequence, we get

$$H^1\big(X, \mathcal{I}\big((n-1)C\big)\big) \to H^1\big(X, \mathcal{I}(nC)\big) \to 0$$

because C is affine. Since X/S is projective, we have obtained a system of finitely generated $\mathcal{O}_S(S)$-modules with surjective transition maps. Since $\mathcal{O}_S(S)$ is Noetherian, the transition maps are eventually isomorphisms. Hence, there exists $n_0 \geq 0$ such that for all $n \geq n_0$, $H^0(X, \mathcal{I}(nC)) \to H^0(C, \mathcal{F}_n)$ is surjective.

By hypothesis, the stalk $\mathcal{J}_x$ at each $x \in C$ is generated by a regular element. Since C is semi-local and closed in the affine scheme U, we can find an affine open subset V in $X \setminus Z$ containing C such that $\mathcal{J}_{|V}$ is principal, say generated by a function φ. For all $n \geq 0$, φ^{-n} induces a generator $\overline{\varphi^{-n}}$ of $\mathcal{O}(nC)_{|C}$. Since $Z \cap C = \emptyset$, we find for all $n \geq 0$ that $(\mathcal{F}_n)_{|C}$ is isomorphic to $\mathcal{O}(nC)_{|C}$. Thus, for all $n \geq n_0$, we can find a global section f_n of $\mathcal{I}(nC)$ which lifts the generator $\overline{\varphi^{-n}}$ of $\mathcal{O}_X(nC)_{|C}$.

Fix $n \geq n_0$, and consider $g := (1 + f_{n+1})/(1 + f_n) \in \mathcal{K}_X(X)$. Let

$$C' := C + \big[\mathrm{div}_X(g)\big].$$

Then $\mathrm{Supp}(C')$ is projective over S because it is closed in X. By construction, $[\mathrm{div}_V(g)] = -C$, and $[\mathrm{div}_X(g)]$ has support disjoint from Z. Thus, $C \cap \mathrm{Supp}(C') = \emptyset$, and since Z contains $X \setminus U$, we find that $\mathrm{Supp}(C') \subseteq U$. Since U is affine, $\mathrm{Supp}(C')$ is finite over S.

Recall that for each closed point $s \in S$, Z contains all irreducible components of $\bar{F} \cap X_s$ which do not intersect C. Recall also that by hypothesis (in the case $d = 1$), for all closed points $s \in S$, the irreducible components of $F \cap U_s$ that intersect C have dimension 0 and are thus contained in C. It follows that $F \cap \mathrm{Supp}(C') = \emptyset$.

Finally, recall that for each closed point $s \in S$ and for each irreducible component Γ of U_s which is not contained in C, then Z contains a closed point of Γ which does not belong to C. Then $\mathrm{Supp}(C')$ does not contain any irreducible component of U_s which is not contained in C. This shows (1).

(2) Now suppose that S is universally catenary. We start with the following reduction. Recall from the beginning of the proof the existence of a closed integral subscheme W of U such that $C \subset W$ is a regular embedding and C has codimension 1 in W. Let T denote the schematic closure of the image of W in S. It suffices to prove (2) for $W \to T$, and the morphism $h' : W \to Y \times_S T$. We are thus reduced to the case where both U and S are integral. In particular, S is equidimensional at every point and universally catenary, and the theory recalled in 1.4 applies.

Fix as in (1) an open S-immersion $U \to X$, with X/S projective and X integral. Let g be as in (1). Let $\Gamma \subseteq X \times_S Y$ be the schematic closure of the graph of the rational map $X \dashrightarrow Y$ induced by $h : U \to Y$. Let $p : \Gamma \to X$ and $q : \Gamma \to Y$ be the associated projection maps over S. Since Y/S is separated, the graph of $h : U \to Y$ is closed in $U \times_S Y$. Hence, $p : p^{-1}(U) \to U$ is an isomorphism. Since Γ is integral and its generic point maps to the generic point of X, the rational function g on X induces a rational function, again denoted by g, on Γ. As $p : p^{-1}(U) \to U$ is an isomorphism, we let $p^*(C)$ and $p^*(C')$ denote the preimages of C and C' in $p^{-1}(U)$; they are closed subschemes of Γ. Since g is an invertible function in a neighborhood of $X \setminus U$, $[\mathrm{div}_\Gamma(g)] = p^*(C) - p^*(C')$, and $p^*(C)$ and $p^*(C')$ are rationally equivalent on Γ. Fix a catenary grading on S and define gradings on schemes of finite type over S accordingly (1.4). Then, as q is proper, q_*p^*C and q_*p^*C' are rationally equivalent in Y. Since $h_*C = q_*p^*C$ and $h_*C' = q_*p^*C'$, (2) follows. □

Lemma 2.5 *Let $U = \mathrm{Spec}\, A$ be a Noetherian affine scheme. Let $C := V(J)$ be a closed subset of U. Let $\Gamma_1, \ldots, \Gamma_n$ be irreducible closed subsets of U. Let $f_1, \ldots, f_\delta \in J$. Then there exist $g_1, \ldots, g_\delta \in J$ such that $g_i \in f_i + J^2$ for all $i = 1, \ldots, \delta$, and such that the following property holds. Let $i \le \delta$ and $j \le n$. Then any irreducible component of $\Gamma_j \cap V(g_1, \ldots, g_i)$ not contained in C has codimension i in Γ_j and, hence, dimension at most* $\dim \Gamma_j - i$.

Proof For $j = 1, \ldots, n$, let $\mathfrak{q}_j$ be the prime ideal of A such that $\Gamma_j = V(\mathfrak{q}_j)$. If $\mathfrak{q}_j$ contains J for all $j \le n$, we set $g_i := f_i$ for all $i = 1, \ldots, \delta$, and the lemma is proved. Suppose that for some j, $\mathfrak{q}_j$ does not contain J. Upon renumbering if necessary, assume that the ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ do not contain J, and $\mathfrak{q}_{m+1}, \ldots, \mathfrak{q}_n$ contain J. The lemma is proved if we can prove it for the sets $\Gamma_1, \ldots, \Gamma_m$. We may thus assume that none of the Γ_j's is contained in C or, in other words, that none of the $\mathfrak{q}_j$'s contain J.

We proceed by induction on δ. When $\delta = 1$, we find that $f_1A + J^2 \not\subseteq \mathfrak{q}_j$ for all $j \le n$. Then there exists $a_1 \in J^2$ such that $g_1 := f_1 + a_1 \notin \bigcup_{1 \le j \le n} \mathfrak{q}_j$ ([9], Lemma 1.2.2 or [29], Theorem 124, page 90). Suppose that Θ is an irreducible component of $\Gamma_j \cap V(g_1)$. Then Θ has codimension 1 in Γ_j and $\dim \Theta \le \dim \Gamma_j \cap V(g_1) \le \dim \Gamma_j - 1$, since $A/\mathfrak{q}_j$ is a domain, and $g_1 \notin \mathfrak{q}_j$.

If $\delta \ge 2$, we apply the induction hypothesis to the sequence $f_1, \ldots, f_{\delta-1}$ to obtain the desired $g_1, \ldots, g_{\delta-1}$. We then apply the case $\delta = 1$ to the ring $A/(g_1, \ldots, g_{\delta-1})$, the ideal $J/(g_1, \ldots, g_{\delta-1})$, the image of f_δ in $A/(g_1, \ldots, g_{\delta-1})$, and to the irreducible components of the $\Gamma_j \cap V(g_1, \ldots, g_{\delta-1})$'s which are not contained in C. We find then an element $\bar{g}_\delta$ in $\bar{f}_\delta + (J/(g_1, \ldots, g_{\delta-1}))^2$, which we lift to $g'_\delta = f_\delta + g_1a_1 + \cdots + g_{\delta-1}a_{\delta-1} + j$ with $j \in J^2$ and $a_i \in A$. Since the desired property is now achieved for the irreducible components of $\Gamma_j \cap V(g_1, \ldots, g_{\delta-1}, g'_\delta)$ not contained in C, we find that the sequence $g_1, \ldots, g_{\delta-1}, g_\delta$, with $g_\delta := f_\delta + j$, satisfies the conclusion of the lemma. □

Lemma 2.6 *Let A be a Noetherian local ring. Let I be a proper ideal of A generated by a regular sequence $f_1, \ldots, f_d$. Let $g_1, \ldots, g_d \in I$. If the image of $\{g_1, \ldots, g_d\}$ in I/I^2 is a basis of I/I^2 over A/I, then $g_1, \ldots, g_d$ is a regular sequence.*

Proof This is well-known, and follows from the equivalence between quasi-regular sequences and regular sequences in Noetherian local rings ([41], 15.B, Theorem 27). □

Remark 2.7 We note that the cycle C' in Proposition 2.4 (1) is in the same graded component of $\mathcal{Z}(U)$ as C, for every catenary grading on U. Note also that it may happen in 2.4 (1) that the cycle C' is the trivial cycle, with empty support. Indeed, consider the case where $U \to S$ is a finite morphism. According to 2.4 (1), the support of C' does not meet C, and for any closed point $s \in S$, $\mathrm{Supp}(C')$ does not contain any point in $U_s \setminus C$. It follows that $\mathrm{Supp}(C')$ does not contain any closed point of U. Since U is affine, we find that $\mathrm{Supp}(C')$ is empty.

We now modify Example 1.3 to produce an example to show that the statement of Proposition 2.4 (2) does not hold in general if S is not assumed to be universally catenary. Indeed, with the notation as in Example 1.3, let $d > 1$, let Y' be the semi-localization of Y at $\{y_0, y_1\}$, and let X' the pinching of Y' (that is, the scheme obtained by identifying y_0 and y_1 in Y'). Let again π denote the natural finite morphism $Y' \to X'$. Then the cycle $[y_1]$ is rationally trivial on Y', but the cycle $\pi_*([y_1])$ is not rationally trivial on X'. For the desired example where the statement of Proposition 2.4 (2) does not hold, we take S and Y to be X', U to be Y', and C to be $\{y_1\}$.

3 Inductive limits of l.c.i. algebras

We prove in this section a technical statement needed when considering base schemes S which are not excellent, as in the proof of 2.3, or of 6.2 in [22].

Lemma 3.1 *Let $(R, (\pi))$ be a discrete valuation ring with field of fractions K of characteristic $p > 0$. Let L/K be a purely inseparable extension of degree p, and let R_L be the integral closure of R in L. Then there exists a sequence $(\alpha_n)_{n\in\mathbb{N}^*}$ with $\alpha_n \in R_L$ and $L = K(\alpha_n)$, such that the R-algebras $B_n := R[\alpha_n]$ are finite and local complete intersections over R, and $R_L = \bigcup_n B_n$.*

Proof Recall that since $[L : K] = p$, R_L is also a discrete valuation ring, with maximal ideal $\mathfrak{m}$, and either $e_{\mathfrak{m}/(\pi)} f_{\mathfrak{m}/(\pi)} = 1$ and R_L is not a finitely generated R-module, or $e_{\mathfrak{m}/(\pi)} f_{\mathfrak{m}/(\pi)} = p$ and R_L is a finitely generated R-module (see [6], VI.8.5, Theorem 2). Moreover, if $e_{\mathfrak{m}/(\pi)} = p$ and $s \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $R_L = R[s]$. If $f_{\mathfrak{m}/(\pi)} = p$ and $s \in R_L$ reduces modulo $\mathfrak{m}$ to a t such that $R_L/\mathfrak{m} = (R/(\pi))(t)$, then $R_L = R[s]$.

The lemma is thus completely proved in the case where R_L is a finitely generated R-module by setting $B_n = R_L$ for all n. When $e_{\mathfrak{m}/(\pi)} f_{\mathfrak{m}/(\pi)} = 1$, we can embed R_L into the completion of $\widehat{R}$ of R with respect to (π). Since $[L : K] = p$, we can choose $\alpha \in R_L$ such that $L = K(\alpha)$. For any $n \geq 1$, we can approximate the element α by an element $r_n \in R$, such that $\alpha = r_n + \pi^n \alpha_n$ for some $\alpha_n \in R_L$. Let $B_n := R[\alpha_n]$. Then $(B_n)_{n \geq 1}$ is an increasing sequence of finite l.c.i. algebras over R, and $L = K(\alpha_n)$ for all n.

It remains to show that $R_L = \bigcup_n B_n$. Let $b \in R_L$. Then $b = \sum_{j=0}^{p-1} t_j \alpha^j$, with $t_j \in K$ for $j = 0, \ldots, p-1$. Replacing α by $r_n + \pi^n \alpha_n$, we see that for $n > \max_{j \geq 1} |\operatorname{ord}_\pi(t_j)|$, we can write $b = c_n + \beta_n$ with $c_n = \sum_j t_j r_n^j \in K$ and $\beta_n \in R[\alpha_n]$. It follows that $c_n \in R_L$ and, hence, $c_n \in R = K \cap R_L$. Therefore, $b \in R[\alpha_n]$. □

Proposition 3.2 *Let A be a Dedekind domain, with field of fractions K. Let B be an integral domain containing A, and with field of fractions L. Assume that B is finite over A. Then there exists a domain C with $B \subseteq C \subseteq L$ such that C is finite over A, and a local complete intersection over A.*

Proof Let K_{sep} be the separable closure of K in L and let B_{sep} be the integral closure of A in K_{sep}. Then B_{sep} is a Dedekind domain finite over A. Let B' be the sub-A-algebra of L generated by B and B_{sep}. Then it is finite over B_{sep}. If we can find C containing B', finite and l.c.i. over B_{sep}, then C contains B and is finite and l.c.i. over A. Therefore, it is enough to treat the case when $K = K_{\text{sep}}$. We thus assume now that L is purely inseparable over K, and we prove the proposition by induction on $[L : K]$. The case $[L : K] = 1$ is trivial. Suppose $[L : K] > 1$ and that the proposition is true for any Dedekind domain A' and for any purely inseparable extension L' of $\operatorname{Frac}(A')$ of degree strictly less than $[L : K]$.

We will construct C as the global sections of a coherent sheaf of A-algebras $\mathcal{C}$ over $\operatorname{Spec} A$. Let L' be a subextension of L of index p. Consider the sub-A-algebra $B \cap L'$ of L'. It is finite over A because B is finite over A. By induction hypothesis, it is contained in a finite l.c.i. A-algebra $B' \subseteq L'$. Let $\alpha \in B$ be such that $L = L'[\alpha]$. We claim that $B'[\alpha]$ is l.c.i. over B', so that $B'[\alpha]$ is a finite l.c.i. A-algebra. Indeed, the minimal polynomial of α over L' is $x^p - \alpha^p \in B'[x]$, since by hypothesis, $\alpha^p \in L' \cap B \subseteq B'$. Therefore, the map $B'[x]/(x^p - \alpha^p) \to B'[\alpha]$ is an isomorphism.

As $B'[\alpha]$ and B are both finite over A and have the same field of fractions, there exists $f \in A \setminus \{0\}$ such that $B'[\alpha]_f = B_f$ (where as usual B_f denotes the localization of B with respect to the multiplicative set $\{1, f, f^2, \ldots\}$). The algebra B_f is finite and l.c.i. over A_f, and we define $\mathcal{C}(D(f))$ to be B_f.

Let now $\mathfrak{p}$ be any maximal ideal of A, and let $\mathfrak{q}$ be the unique maximal ideal of B lying over $\mathfrak{p}$. Let us show that $B_\mathfrak{q}$ is contained in some finite l.c.i. $A_\mathfrak{p}$-algebra $\mathcal{C}_\mathfrak{p}$ contained in L. First, note that the localization of B at the multiplicative set $A \setminus \mathfrak{p}$ is equal in L to $B_\mathfrak{q}$, because $x^{[L:K]} \in A \setminus \mathfrak{p}$ for all $x \in B \setminus \mathfrak{q}$. In particular, $B_\mathfrak{q}$ is finite over $A_\mathfrak{p}$. Write $B_\mathfrak{q} = \sum_{1 \leq i \leq r} b_i A_\mathfrak{p}$, with

$b_i \in B$. Let R be the integral closure of $A_{\mathfrak{p}}$ in L'. It follows from Lemma 3.1 that there exists $\alpha \in L$, integral over R with $L = L'(\alpha)$, and such that $b_i \in R[\alpha]$ for all $i \leq r$. Then there also exists a finite sub-$A_{\mathfrak{p}}$-algebra D of L', with $\alpha^p \in D$ and $b_i \in D[\alpha]$ for all $i \leq r$, and such that $\mathrm{Frac}(D) = L'$. Apply now the induction hypothesis to the extension $A_{\mathfrak{p}} \subseteq D$: there exists $D \subseteq E \subseteq R$ such that E is finite and l.c.i. over $A_{\mathfrak{p}}$. Set $\mathcal{C}_{\mathfrak{p}} := E[\alpha]$. We have $\mathrm{Frac}(\mathcal{C}_{\mathfrak{p}}) = L'[\alpha] = L$. Since $\alpha^p \in D$, the minimal polynomial of α over L' is in $D[x]$. As earlier, we find that $E[\alpha]$ is l.c.i. over E and, therefore, also over $A_{\mathfrak{p}}$.

Write $\mathrm{Spec}\, A = D(f) \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$. For each $i = 1, \dots, s$, there exists an open neighborhood U_i of $\mathfrak{p}_i$ with $U_i \setminus \{\mathfrak{p}_i\} \subseteq D(f)$, and a finite sub-$\mathcal{O}(U_i)$-algebra C_i of L such that $C_i \otimes_{\mathcal{O}(U_i)} A_{\mathfrak{p}_i} = \mathcal{C}_{\mathfrak{p}_i}$ and

$$C_i \otimes_{\mathcal{O}(U_i)} \mathcal{O}\big(U_i \setminus \{\mathfrak{p}_i\}\big) = B_f \otimes_{A_f} \mathcal{O}\big(U_i \setminus \{\mathfrak{p}_i\}\big).$$

We then can glue the sheaves $(C_i)^{\sim}{}_{/U_i}$ and the sheaf $(B_f)^{\sim}{}_{/D(f)}$ into a coherent sheaf $\mathcal{C}$ of A-algebras. The global sections $C := \mathcal{C}(\mathrm{Spec}\, A)$ is a finite sub-A-algebra of L which contains B and is l.c.i. by construction. □

4 Hilbert-Samuel multiplicities

Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let Q be an $\mathfrak{m}$-primary ideal of A (equivalently, Q is a proper ideal of A containing some power of $\mathfrak{m}$). Let M be a non-zero finitely generated A-module. Recall that there exists a polynomial $f_Q(x) \in \mathbb{Q}[x]$ such that for all n large enough, $f_Q(n) = \ell_A(M/Q^n M)$. This polynomial has degree $d = \dim M := \dim V(\mathrm{Ann}(M))$.

The *Hilbert-Samuel multiplicity* $e(Q, M)$ is the coefficient of x^d in $f_Q(x)$ multiplied by $d!$. When there is no need to specify the ring A, we may write $e(Q, A)$ simply as $e(Q)$. The integer $e(A) := e(\mathfrak{m}, A)$ is called the Hilbert-Samuel multiplicity of A. If A is regular, then $e(A) = 1$.

We prove in Proposition 4.9 below that the Hilbert-Samuel multiplicity $e(Q, M)$ can be expressed in terms of Hilbert-Samuel multiplicities $e((\mathbf{f}), M)$ of ideals generated by strict systems of parameters. This result is a key ingredient in the proof of the main result of this section, Theorem 4.5. A more global geometric version of Theorem 4.5 is given in the Generic Moving Lemma 6.5.

4.1 Let $(A, \mathfrak{m})$ be a Noetherian local ring. An ordered sequence $\mathbf{f} = \{f_1, \dots, f_r\}$ of $r \geq 1$ elements of $\mathfrak{m}$ is said to be *strictly secant* if f_1 does not belong to any minimal prime ideal of A and if, for all $i \in \{2, \dots, r\}$, f_i does not belong to any minimal prime ideal over $(f_1, \dots, f_{i-1})$. The sequence $\mathbf{f}$ is called a *strict system of parameters* of A if $r = \dim A$. In general, the property of being strictly secant depends on the order of the f_i's.

A sequence $\{f_1, \ldots, f_r\}$ of $\mathfrak{m}$ is called *secant* if $\dim A/(f_1, \ldots, f_r) = \dim A - r$ ([7], VIII.26, Definition 1). The sequence $\{f_1\}$ is secant if and only f_1 does not belong to any minimal prime ideal $\mathfrak{p}$ of A such that $\dim A/\mathfrak{p} = \dim A$ ([7], VIII.27, Proposition 3). By induction, one sees that a strictly secant sequence is secant.

In a Noetherian local ring $(A, \mathfrak{m})$ of dimension d, a *system of parameters* of A is a system $f_1, \ldots, f_d$ of elements of $\mathfrak{m}$ such that the ideal $(f_1, \ldots, f_d)$ contains a power of $\mathfrak{m}$ or, equivalently, such that the A-module $A/(f_1, \ldots, f_d)$ has finite length. Since a strict system of parameters $\mathbf{f} = \{f_1, \ldots, f_d\}$ is a secant sequence, we find that $\dim A/(f_1, \ldots, f_d) = \dim A - d = 0$, and $\mathbf{f}$ is a system of parameters.

4.2 Recall that any ideal in $\mathfrak{m}$ generated by i elements has height at most i. It follows that a sequence $\{f_1, \ldots, f_r\}$ is strictly secant if and only if for all $i \leq r$, all minimal prime ideals over $(f_1, \ldots, f_i)$ have height i.

Lemma 4.3 *Let* $(A, \mathfrak{m})$ *be a Noetherian catenary equidimensional local ring. Then a secant sequence* (*resp. a system of parameters*) *is strictly secant* (*resp. a strict system of parameters*).

Proof Let $f \in A$ be such that $\dim A/fA = \dim A - 1$. Since A is equidimensional, f does not belong to any minimal prime ideal of A and, thus, the sequence $\{f\}$ is strictly secant. The quotient $A/(f)$ is catenary since A is. We claim that A/fA is also equidimensional. Indeed, let $\mathfrak{q}$ be any minimal prime ideal of A over (f) and let $\mathfrak{p}$ be a minimal prime ideal of A contained in $\mathfrak{q}$. By Krull's principal ideal theorem, $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = 1$. Hence, since A is catenary, $\mathrm{ht}(\mathfrak{m}/\mathfrak{q}) = \mathrm{ht}(\mathfrak{m}/\mathfrak{p}) - \mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = \dim A - 1$ for any minimal prime $\mathfrak{q}/(f)$ of $A/(f)$. Therefore, $A/(f)$ is equidimensional.

Let now $\mathbf{f} := \{f_1, \ldots, f_r\}$ be a secant sequence for A. If $r = 1$, then $\mathbf{f}$ is strictly secant. If $r > 1$, then $\{f_2, \ldots, f_r\}$ is secant for $A/(f_1)$ and, by induction, strictly secant for $A/(f_1)$. Then $\mathbf{f}$ is strictly secant for A. □

Lemma 4.4 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring. Let* Q *be a* $\mathfrak{m}$*-primary ideal, and let* I *be a proper ideal of* A. *Let* $\mathbf{g} := \{g_1, \ldots, g_r\}$ *be a strictly secant sequence of elements of* $(Q + I)/I$. *Then there exists a strict system of parameters* $\{f_1, \ldots, f_d\}$ *in* Q *such that for* $i \leq r$, *the map* $A \to A/I$ *sends* f_i *to* g_i, *and for* $i > \dim A/I$, $f_i \in I \cap Q$.

Proof When $I = (0)$, the lemma states that any strictly secant sequence of elements of Q can always be completed into a strict system of parameters contained in Q. We leave the proof of this fact to the reader.

Assume that $I \neq (0)$. We first show by induction on r that $\mathbf{g}$ can be lifted to a strictly secant sequence in Q. Let $f_1' \in Q$ be a lift of g_1. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$

be the minimal prime ideals of A. If $f_1' \notin \mathfrak{p}_i$ for all $i = 1, \dots, m$, then by definition $\{f_1'\}$ is a strictly secant sequence.

Suppose now that f_1' belongs to some $\mathfrak{p}_i$, and renumber these minimal primes so that there exists $m_0 > 0$ such that $f_1' \in \mathfrak{p}_i$ if and only if $i \le m_0$. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be the minimal prime ideals of A over I. If $i \le m_0$, then $\mathfrak{q}_j \not\subseteq \mathfrak{p}_i$ (otherwise they would be equal, but $f_1' \notin \mathfrak{q}_j$) and $Q \not\subseteq \mathfrak{p}_i$ because $\dim A \ge r \ge 1$. So for $i \le m_0$, $Q \cap (\bigcap_{1 \le j \le n} \mathfrak{q}_j) \cap (\bigcap_{k \ge m_0+1} \mathfrak{p}_k)$ is not contained in $\mathfrak{p}_i$ and, therefore, there exists

$$\alpha \in Q \cap I \cap \left(\bigcap_{k \ge m_0+1} \mathfrak{p}_k \right) \Big\backslash \bigcup_{1 \le i \le m_0} \mathfrak{p}_i .$$

Let $f_1 := f_1' + \alpha \in Q$. Then $\{f_1\}$ is a strictly secant sequence contained in Q, and f_1 maps to g_1 in A/I, as desired.

Let us assume by induction that we can lift $\{g_1, \dots, g_{r-1}\}$ to a strictly secant sequence $\{f_1, \dots, f_{r-1}\}$ in Q. Let $J := (f_1, \dots, f_{r-1})$. Apply the case $r = 1$ to the ring $A' := A/J$, with the ideals $I' := I + J/J$ and $Q' := Q + J/J$: the strictly secant sequence $\{\bar{g}_r\}$ in A'/I' lifts to a strictly secant sequence $\{\bar{f}_r\}$ in Q', and we let f_r denote a lift in Q of $\bar{f}_r$. Then $\{f_1, \dots, f_{r-1}, f_r\}$ in Q is the desired strictly secant sequence lifting $\mathbf{g}$.

Now we complete $\mathbf{g}$ into a strict system of parameters in $(Q + I)/I$ and lift it to a strictly secant sequence $f_1, \dots, f_n$ in Q (with $n := \dim A/I \ge r$). It is easy to check that $\mathfrak{m} = \sqrt{I \cap Q + (f_1, \dots, f_n)}$. Then the image of $I \cap Q$ in $A/(f_1, \dots, f_n)$ contains a power of the maximal ideal, and we can use the case '$I = (0)$' applied to the ring $A/(f_1, \dots, f_n)$ to find that there exist $f_{n+1}, \dots, f_d \in I \cap Q$ whose images in $A/(f_1, \dots, f_n)$ form a strict system of parameters. Then $\{f_1, \dots, f_d\}$ is a strict system of parameters of A as desired. □

The following theorem, whose proof relies heavily on Proposition 4.9, is the main result of this section. We use Theorem 4.5 to prove the Generic Moving Lemma 6.5, which in turn is used to prove Theorem 8.2.

Theorem 4.5 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \ge 1$, and let Q be a $\mathfrak{m}$-primary ideal of A. Let F be a closed subset of* Spec A *with* $\dim F < d$. *Then there exist integral closed subschemes $C_1, \dots, C_n$ of* Spec A, *of dimension* 1, *and invertible rational functions $\varphi_i \in k(C_i)^*$ such that*:

(i) *If $F \ne \emptyset$, then for all $i \le n$, $C_i \cap F = \{\mathfrak{m}\}$.*
(ii) $e(Q) = \sum_{1 \le i \le n} \operatorname{ord}_{C_i}(\varphi_i)$.

In particular, $e(Q)[\mathfrak{m}]$ is rationally trivial on Spec A.

We will need the following two facts.

4.6 Let $(A, \mathfrak{m})$ be a Noetherian local ring, let Q be a $\mathfrak{m}$-primary ideal of A. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the minimal prime ideals of A such that $\dim A/\mathfrak{p}_i = \dim A$. Then

$$e(Q, A) = \sum_{1 \le i \le t} \ell(A_{\mathfrak{p}_i}) e\big((Q + \mathfrak{p}_i)/\mathfrak{p}_i, A/\mathfrak{p}_i\big)$$

([7], VIII, §7, n° 1, Proposition 3). In particular, $e(A) = \sum_{1 \le i \le t} \ell(A_{\mathfrak{p}_i}) \cdot e(A/\mathfrak{p}_i)$.

4.7 (Associative Law for Multiplicities) Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $f_1, \dots, f_d$ be a system of parameters. Denote by $\mathbf{f}$ the sequence $f_1, \dots, f_d$ and by $(\mathbf{f})$ the ideal generated by the f_i's. Fix $s \le d$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the minimal prime ideals of A over $(f_1, \dots, f_s)$ such that $\mathrm{ht}(\mathfrak{p}_i) = s$ and $\dim A/\mathfrak{p}_i = d - s$. (When $s = 0$, we interpret $(f_1, \dots, f_s)$ to be the ideal (0).) Then

$$e\big((\mathbf{f}), A\big) = \sum_{i=1}^{t} e\big((f_1, \dots, f_s)A_{\mathfrak{p}_i}, A_{\mathfrak{p}_i}\big) e\big((f_{s+1}, \dots, f_d) + \mathfrak{p}_i/\mathfrak{p}_i, A/\mathfrak{p}_i\big)$$

(see [38], or [44], Theorem 18, page 342, or [27] (1.8), or [42], exer. 14.6, page 115).

4.8 *Proof of 4.5* It suffices to prove the theorem when $F \neq \emptyset$. Let I be a proper ideal of A such that $F = V(I)$. By hypothesis, $r := \dim F < d := \dim A$.

Proposition 4.9 (1) shows that the multiplicity $e(Q)$ can be expressed in terms of the multiplicity of ideals generated by (strict) systems of parameters. Thus, we are reduced to proving the theorem in the case Q is generated by a system of parameters.

Proposition 4.9 (2) shows that it is enough to consider the case where Q is generated by a strict system of parameters $\mathbf{f} = \{f_1, \dots, f_d\}$ of A such that, when $r \ge 1$, the image of $\{f_1, \dots, f_r\}$ in A/I is a strict system of parameters of A/I. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the minimal prime ideals of A over $(f_1, \dots, f_{d-1})$ of height $d - 1$ with $\dim A/\mathfrak{p}_i = 1$. The formula in 4.7 gives

$$e(Q, A) = \sum_{1 \le i \le t} d_i e(f_d A, A/\mathfrak{p}_i),$$

with $d_i := e((f_1, \dots, f_{d-1})A_{\mathfrak{p}_i}, A_{\mathfrak{p}_i})$. As $A/\mathfrak{p}_i$ is integral, $e(f_d A, A/\mathfrak{p}_i) = \mathrm{ord}_{A/\mathfrak{p}_i}(f_d)$ where $\mathrm{ord}_{A/\mathfrak{p}_i}(f_d)$ denotes the order of the image of f_d in $A/\mathfrak{p}_i$. Let $C_i := V(\mathfrak{p}_i)$, $1 \le i \le t$, and let $\varphi_i := f_d^{d_i}|_{C_i}$. Then

$$e(Q, A) = \sum_{1 \le i \le t} \mathrm{ord}_{C_i}(\varphi_i),$$

as desired. When $\dim F = r > 0$, $\dim A/(I, f_1, \ldots, f_r) = 0$ by construction. Thus, $\dim(C_i \cap F) = \dim V(I + \mathfrak{p}_i) = 0$ for all $i = 1, \ldots, t$. That $e(Q)[\mathfrak{m}]$ is rationally equivalent to 0 follows immediately from the definition recalled at the beginning of Sect. 1. □

Part (3) of our next proposition is a slight strengthening of a well-known theorem ([9], Corollary 4.5.10, or [56], VIII, §10, Theorem 22 when $M = A$). Parts (1) and (2) are used in 4.5, and Part (4) will be used in 5.2.

Proposition 4.9 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and let* M *be a nonzero finitely generated* A*-module. Let* Q *be an* $\mathfrak{m}$*-primary ideal of* A.

(1) *There exist finitely many strict systems of parameters* $\mathbf{f}^\alpha, \alpha \in \mathcal{A}$, *contained in* Q, *such that* $e(Q, M) = \sum_{\alpha \in \mathcal{A}} \pm e((\mathbf{f}^\alpha), M)$.
(2) *Fix a proper ideal* I *of* A *with* $\mathrm{Ann}_A(M) \subseteq I$. *Let* $r := \dim V(I)$. *When* $r > 0$, *we can choose the strict system of parameters* $\{f_1^\alpha, \ldots, f_{\dim A}^\alpha\}$ *in* (1) *such that for each* $\alpha \in \mathcal{A}$, *the image of the sequence* $\{f_1^\alpha, \ldots, f_r^\alpha\}$ *in* A/I *forms a strict system of parameters in* A/I.
(3) *If* $A/\mathfrak{m}$ *is infinite, then* $e(Q, M) = e((\mathbf{f}), M)$ *for some strict system of parameters* $\mathbf{f}$ *contained in* Q, *satisfying, when applicable, the property in* (2).
(4) *Let* $\mathfrak{p}$ *be a prime ideal of* A *with* $\mathrm{ht}(\mathfrak{p}) \geq 1$. *Let* Q_0 *be a* $\mathfrak{p}A_\mathfrak{p}$*-primary ideal of* $A_\mathfrak{p}$. *Then there exist finitely many strictly secant sequences* $\mathbf{f}^\alpha$ *in* $\mathfrak{m}$, $\alpha \in \mathcal{A}$, *whose images in* $A_\mathfrak{p}$ *are contained in* Q_0 *and such that* $e(Q_0, A_\mathfrak{p}) = \sum_{\alpha \in \mathcal{A}} \pm e((\mathbf{f}^\alpha), A_\mathfrak{p})$.

Proof (1) and (2). Let $J := \mathrm{Ann}_A(M)$. By definition, $\dim M := \dim A/J$. We can consider M as an A/J-module. We have $e_A(Q, M) = e_{A/J}((Q + J)/J, M)$ because $JM = 0$. Suppose that we can construct strict systems of parameters $\mathbf{g}^\alpha$ in $(Q + J)/J$ such that $e_{A/J}((Q + J)/J, M)$ is a combination of the $e_{A/J}((\mathbf{g}^\alpha), M)$'s and such that the image of $\{g_1^\alpha, \ldots, g_r^\alpha\}$ in A/I is a strict system of parameters for all α. By Lemma 4.4, we can lift each $\mathbf{g}^\alpha$ and complete it into a strict system of parameters $\mathbf{f}^\alpha = \{f_1^\alpha, \ldots, f_{\dim A}^\alpha\}$ in Q with $f_i^\alpha \in J$ for all $i > \dim M$. Then $e_{A/J}((\mathbf{g}^\alpha), M) = e_A((\mathbf{f}^\alpha), M)$, and (1) and (2) hold. Therefore, it remains to prove (1) and (2) when $\dim M = \dim A$.

Assume that $d := \dim M = \dim A$. We proceed by induction on d. If $d = 0$, then a strict system of parameters $\mathbf{f}$ for A is empty, and we set $\mathbf{f} = (0)$. Then $e(Q, M) = \ell(M) = e((0), M)$, and no additional condition is required in (2) since $\dim A/I = 0$.

Assume that $d \geq 1$. Let P be the set consisting of the minimal prime ideals of A, of the associated primes of M, and if $\dim A/I > 0$, of the minimal prime ideals of A over I. Our hypotheses imply that if $\mathfrak{m} \notin \mathrm{Ass}(M)$, then $\mathfrak{m} \notin P$. Suppose that $\mathfrak{m} \in \mathrm{Ass}(M)$. Then there exists an exact sequence $(0) \to M' \to M \to M'' \to (0)$ with M' isomorphic to $A/\mathfrak{m}$. Since $\dim(M) > 0$, we find that for any $\mathfrak{m}$-primary ideal Q_0 of A, $e(Q_0, M) = e(Q_0, M'')$ ([7], VIII.46, Prop. 5, or use the proof[2] of [42], 14.6). It follows that it suffices to prove our statement for modules M with $\mathfrak{m} \notin \mathrm{Ass}(M)$. In particular, we can assume that $\mathfrak{m} \notin P$.

We apply Lemma 4.10 with this set P of primes. Pick some $s \geq s_0$ and $x_s \in Q^s$, $x_{s+1} \in Q^{s+1}$ as in 4.10. By our choice of P, both $\{x_s\}$ and $\{x_{s+1}\}$ are strictly secant sequences in Q (4.10(c)). Suppose $d = 1$. Then $e(Q, M) = e((x_{s+1}), M) - e((x_s), M)$ (4.10(b)). When $r = 1$, our choice of P shows that the images of $\{x_s\}$ and $\{x_{s+1}\}$ in A/I are again strict systems of parameters in A/I. Hence, (1) and (2) are true when $d = 1$.

Suppose now that $d \geq 2$ and that (1) and (2) hold for $d - 1$. Lemma 4.10(b) shows that

$$e(Q, M) = e(Q, M/x_{s+1}M) - e(Q, M/x_s M).$$

Let $B := A/x_s A$. Then $\dim B = d - 1$ and $M/x_s M$ is a B-module of dimension $d - 1$ (4.10(a)). Let IB denote the ideal $(I + x_s A)/x_s A$ of B. By our choice of primes in P, we find that $\dim V(IB) = \max\{0, r - 1\}$. We also have $e_A(Q, M/x_s M) = e_B(QB, M/x_s M)$. By induction hypothesis, there exist finitely many strict systems of parameters $\mathbf{g}^\alpha$ in QB with

$$e(QB, M/x_s M) = \sum \pm e\big((\mathbf{g}^\alpha), M/x_s M\big)$$

and such that, if $\dim V(IB) \geq 1$, the image of $g_1^\alpha, \dots, g_{r-1}^\alpha$ in B/IB is a strict system of parameters. Let $\mathbf{f}^\alpha$ be any lifting of $\mathbf{g}^\alpha$ in Q. Then $\{x_s, \mathbf{f}^\alpha\}$ is a strict system of parameters in Q, and the image in A/I of the first r elements of $\{x_s, \mathbf{f}^\alpha\}$ is a strict system of parameters in A/I. By our choice of primes in P, x_s is not a zero divisor in M. It follows from [42], 14.11 that[3]

$$e\big((\mathbf{g}^\alpha), M/x_s M\big) = e\big((\mathbf{f}^\alpha), M/x_s M\big) = e\big((x_s, \mathbf{f}^\alpha), M\big).$$

Repeating the same argument with x_{s+1} instead of x_s allows us to express the multiplicity $e(Q, M/x_{s+1}M)$ in a similar way. Since $e(Q, M) = e(Q, M/x_{s+1}M) - e(Q, M/x_s M)$, (1) and (2) follow.

(3) When $A/\mathfrak{m}$ is infinite, instead of choosing two elements x_s and x_{s+1} as we did above, Part (3) is proved by modifying the proof of (1) and (2), using only the element x_1 whose existence is asserted in 4.10(d), with $e(Q, M) = e(Q, M/x_1 M)$. We leave the details to the reader.

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[2] It should be noted that the definition of $e(Q, M)$ taken in [42], page 107, (or in [53], 11.1.5) is different from the definition taken at the beginning of this section and in [7], VIII.72. Thus the statement of [42], 14.6, cannot be applied directly.

[3] Due to the two different definitions of $e(Q, M)$, the proof of [42], 14.11, needs to be slightly adjusted to prove the second equality.

(4) Let R be the preimage of Q_0 in A under the natural map $A \to A_\mathfrak{p}$. Clearly, $R \subseteq \mathfrak{p}$. Consider the graded rings

$$\mathrm{Gr}(A) := \bigoplus_{n\geq 0} R^n/R^{n+1} \quad \text{and} \quad \mathrm{Gr}(A_\mathfrak{p}) := \bigoplus_{n\geq 0} Q_0^n/Q_0^{n+1},$$

and the natural homomorphism of graded rings $\rho : \mathrm{Gr}(A) \to \mathrm{Gr}(A_\mathfrak{p})$. Let $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_k$ be the associated prime ideals of $\mathrm{Gr}(A_\mathfrak{p})$ not containing $\mathrm{Gr}_+(A_\mathfrak{p})$. Then $\{\rho^{-1}(\mathfrak{q}'_j), j = 1, \ldots, k\}$ is a set of homogeneous prime ideals of $\mathrm{Gr}(A)$ not containing $\mathrm{Gr}_+(A)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the minimal prime ideals of A. We are going to associate below to each $\mathfrak{q}_i$ a homogeneous prime ideal $\widetilde{\mathfrak{q}_i}$ of $\mathrm{Gr}(A)$ which does not contain $\mathrm{Gr}_+(A)$. We will then apply the Prime Avoidance Lemma 4.11 to the ideal $I := \mathrm{Gr}_+(A)$ in the ring $\mathrm{Gr}(A)$ with the set of homogeneous ideals $\{\rho^{-1}(\mathfrak{q}'_j), j = 1, \ldots, k\} \cup \{\widetilde{\mathfrak{q}_i}, i = 1, \ldots, m\}$.

For any prime ideal $\mathfrak{q}$ of A, let $\mathfrak{q}^{\mathrm{hom}} := \bigoplus_{n\geq 0}(\mathfrak{q} \cap R^n)/(\mathfrak{q} \cap R^{n+1})$, which we view in a natural way as an ideal in $\mathrm{Gr}(A)$. This ideal is homogeneous, but may not be prime. We claim that when $\mathfrak{p} \not\subseteq \mathfrak{q}$, then $\sqrt{\mathfrak{q}^{\mathrm{hom}}}$ does not contain $\mathrm{Gr}_+(A)$. Indeed, if $\sqrt{\mathfrak{q}^{\mathrm{hom}}}$ contains $\mathrm{Gr}_+(A)$, then $\mathfrak{q}^{\mathrm{hom}}$ contains a power of $\mathrm{Gr}_+(A)$. So for some $n > 0$, $R^n = R^{n+1} + (\mathfrak{q} \cap R^n)$. Hence, modulo $\mathfrak{q}$, $\overline{R}^n = \overline{R}^{n+1}$. Since $\bigcap_{m\geq n} \overline{R}^m = (0) = \overline{R}^n$, we find that $R^n \subseteq \mathfrak{q}$, and passing to radicals, we find that $\mathfrak{p} \subseteq \mathfrak{q}$, a contradiction. Now by standard results, $\sqrt{\mathfrak{q}^{\mathrm{hom}}}$ is homogeneous, and it is the intersection of all the prime ideals minimal above it, which are also homogeneous. For each $i = 1, \ldots, m$, we let $\widetilde{\mathfrak{q}_i}$ denote one of the minimal prime ideals of $\sqrt{\mathfrak{q}_i^{\mathrm{hom}}}$ that does not contain $\mathrm{Gr}_+(A)$.

We conclude from Lemma 4.11, applied to the ideal $I := \mathrm{Gr}_+(A)$ in the ring $\mathrm{Gr}(A)$ with the set of homogeneous ideals $\{\rho^{-1}(\mathfrak{q}'_j), j = 1, \ldots, k\} \cup \{\widetilde{\mathfrak{q}_i}, i = 1, \ldots, m\}$, that there exists $s_0 \geq 0$ such that, for all $s \geq s_0$, there exists $x_s \in R^s$ whose class in R^s/R^{s+1} does not belong to $\bigcup_j \rho^{-1}(\mathfrak{q}'_j) \cup (\bigcup_i \widetilde{\mathfrak{q}_i})$. In particular, $x_s \notin \mathfrak{q}_i$ for all $i = 1, \ldots, m$, which by definition implies that $\{x_s\}$ is strictly secant in A. Moreover, let ξ denote the image of x_s in Q_0^s/Q_0^{s+1}. Then, by construction, $\xi \notin \mathfrak{q}'_j$, for all $j = 1, \ldots, k$.

As in 4.10, we use this latter fact to be able to apply [7], VIII.79, Prop. 9 and Lemma 3, to verify a hypothesis needed in [7], VIII.77, Prop. 8.b). It follows immediately from VIII.77, Prop. 8.b), (i) and (ii), because $\mathrm{ht}(\mathfrak{p}) \geq 1$, that

$$se(Q_0, A_\mathfrak{p}) = \begin{cases} e(Q_0, A_\mathfrak{p}/x_s A_\mathfrak{p}) & \text{if } \dim A_\mathfrak{p} \geq 2, \\ e((x_s), A_\mathfrak{p}) & \text{if } \dim A_\mathfrak{p} = 1. \end{cases}$$

We can now prove (4) when $\mathrm{ht}(\mathfrak{p})=1$. Fix $s \geq s_0$ as above, and consider x_s and x_{s+1}. We find that

$$e(Q_0, A_{\mathfrak{p}}) = e\big((x_{s+1}), A_{\mathfrak{p}}\big) - e\big((x_s), A_{\mathfrak{p}}\big),$$

with both $\{x_s\}$ and $\{x_{s+1}\}$ strictly secant in A, as desired. When $\mathrm{ht}(\mathfrak{p}) \geq 2$, we proceed by induction on $\mathrm{ht}(\mathfrak{p})$. Since $\{x_s\}$ is strictly secant in A, it is secant in $A_{\mathfrak{p}}$, so that $\dim A_{\mathfrak{p}}/(x_s) = \mathrm{ht}(\mathfrak{p}/(x_s)) = \mathrm{ht}(\mathfrak{p}) - 1$. Let $B := A/(x_s)$. We apply the induction hypothesis to the ring B, prime ideal $\mathfrak{p}B$ and $\mathfrak{m}B$-primary ideal $Q_0 B$. The details are left to the reader. □

Lemma 4.10 *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let Q be a proper ideal of A. Let M be a finitely generated A-module with $\dim M \geq 1$ and M/QM of finite length. Let P be a finite set of prime ideals of A with $\mathfrak{m} \notin P$. Then there exists $s_0 \geq 1$ such that for all $s \geq s_0$, there exists $x_s \in Q^s$ with the following properties*:

(a) $\dim(M/x_s M) = \dim M - 1$.
(b)

$$se(Q, M) = \begin{cases} e(Q, M/x_s M) & \textit{if } \dim M \geq 2, \textit{ and} \\ e(x_s A, M) & \textit{if } \dim M = 1. \end{cases}$$

(c) *$x_s \notin \mathfrak{p}$ for every $\mathfrak{p} \in P$.*
(d) *If $A/\mathfrak{m}$ is infinite, then there exists an element $x_1 \in Q \setminus Q^2$ with the above properties.*

Proof Consider the graded ring $\mathrm{Gr}(A) := \bigoplus_{n \geq 0} Q^n/Q^{n+1}$. For any A-module N, the group $\mathrm{Gr}(N) := \bigoplus_{n \geq 0} Q^n N/Q^{n+1} N$ is a $\mathrm{Gr}(A)$-module in a natural way. Consider then the $\mathrm{Gr}(A)$-graded module $L := \mathrm{Gr}(M) \oplus (\bigoplus_{\mathfrak{p} \in P} \mathrm{Gr}(A/\mathfrak{p}))$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the associated prime ideals of L not containing $\mathrm{Gr}_+(A)$. These are homogeneous ideals of $\mathrm{Gr}(A)$ ([41], (10.B)). By Lemma 4.11 applied to $I = \mathrm{Gr}_+(A)$, there exists $s_0 \geq 1$ such that for all $s \geq s_0$, $\mathrm{Gr}_s(A)$ is not contained in $\bigcup_i \mathfrak{p}_i$.

For $s \geq s_0$, let $x_s \in Q^s$ whose class ξ in $\mathrm{Gr}_s(A)$ does not belong to $\bigcup_i \mathfrak{p}_i$. Let $\varphi : L \to L$ be the multiplication-by-ξ map. By [7], VIII.79, Prop. 9 and Lemma 3, $\mathrm{Ker}\,\varphi$ has finite length over $\mathrm{Gr}(A)$ and, hence, over A/Q. This is one of the hypotheses needed to now apply [7], VIII.77, Prop. 8.b). Moreover, our hypothesis that $\mathfrak{m} \notin P$ shows that $\dim A/\mathfrak{p} \geq 1$ for all $\mathfrak{p} \in P$. Since we also assumed that $\dim M \geq 1$, we can now use [7], VIII.77, Prop. 8.b), to obtain that $\dim M/x_s M = \dim M - 1$ (proving (a)), and for each $\mathfrak{p} \in P$, $\dim(A/(x_s, \mathfrak{p})) = \dim(A/\mathfrak{p}) - 1$. In particular, it follows that $x_s \notin \mathfrak{p}$, proving (c). Part (b) follows immediately from VIII.77, Prop. 8.b), (i) and (ii). Remarque 4) on [7], VIII.79, shows that when $A/\mathfrak{m}$ is infinite, we can apply the above proof to an element $x_1 \in Q \setminus Q^2$. □

The following Prime Avoidance Lemma for graded rings is needed in the proofs of 4.9 and 4.10. (For related statements, see [53], Theorem A.1.2, or [6], III, 1.4, Prop. 8, page 161.)

Lemma 4.11 *Let $B = \bigoplus_{s\geq 0} B_s$ be a graded ring. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be homogeneous prime ideals of B not containing B_1. Let $I = \bigoplus_{s\geq 0} I_s$ be a homogeneous ideal such that $I \not\subseteq \mathfrak{p}_i$ for all $i \leq r$. Then there exists an integer $s_0 \geq 1$ such that for all $s \geq s_0$, $I_s \not\subseteq \bigcup_{1\leq i\leq r} \mathfrak{p}_i$.*

Proof We proceed by induction on r. If $r = 1$, choose $t \in B_1 \setminus \mathfrak{p}_1$ and a homogeneous element $\alpha \in I \setminus \mathfrak{p}_1$, say of degree s_0. Then $t^{s-s_0}\alpha \in I_s \setminus \mathfrak{p}_1$ for all $s \geq s_0$, as desired. Let $r \geq 2$ and suppose that the lemma is true for $r - 1$. We can suppose that $\mathfrak{p}_i$ is not contained in $\mathfrak{p}_r$ for all $i \neq r$, so that $I\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1} \not\subseteq \mathfrak{p}_r$. Similarly, we can suppose that $\mathfrak{p}_r$ is not contained in $\mathfrak{p}_i$ for all $i \neq r$, so that $I\mathfrak{p}_r \not\subseteq (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{r-1})$. Hence, we can apply the case $r = 1$ and the induction hypothesis to obtain that there exists s_0 such that for all $s \geq s_0$, there are homogeneous elements $f_s \in I\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1} \setminus \mathfrak{p}_r$ and $g_s \in I\mathfrak{p}_r \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{r-1})$ of degree s. It is easy to check that $f_s + g_s \in I_s \setminus \bigcup_{1\leq i\leq r} \mathfrak{p}_i$, as desired. □

Example 4.12 Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let Q be an $\mathfrak{m}$-primary ideal of A. Proposition 4.9 (3), states that when $A/\mathfrak{m}$ is infinite, there exists a system of parameters **f** contained in Q such that $e(Q, A) = e((\mathbf{f}), A))$. We show below that when $A/\mathfrak{m}$ is finite, such a system of parameters **f** contained in Q may not exist.

Consider the ring $A := \mathbb{F}_2[[x, y]]/(xy(x + y))$, with $\mathfrak{m} = (x, y)$. It is clear that $e(\mathfrak{m}, A) = 3$. Since this ring has dimension 1, the multiplicity $e((f), A)$ is equal to the length of $A/(f)$. It is shown in [25], 3.2, that there exists no regular element $f \in \mathfrak{m}$ such that $\mathfrak{m}^3 \subseteq (f)$. This implies that $A/(f)$ cannot have length 3.

Note that in the ring $B := \mathbb{F}_4[[x, y]]/(xy(x + y))$, the element $z := x - ty$, with $t \in \mathbb{F}_4 \setminus \mathbb{F}_2$, is such that $B/(z)$ has length 3.

5 Two local invariants

Let $(A, \mathfrak{m})$ be a Noetherian local ring of positive dimension. We introduce in this section two new invariants of A, $\gamma(A)$ in 5.1, and $n(A)$ in 5.4. We show in Theorem 5.6 below that $n(A) = \gamma(A)$ when A satisfies some natural hypotheses such as being reduced, excellent, and equidimensional. In 7.3, we will relate $n(A)$ to a resolution of singularities $Y \to \operatorname{Spec} A$ when A is universally catenary.

5.1 Let $(A, \mathfrak{m})$ be a Noetherian local ring. Consider the set $\mathcal{E}$ of all Hilbert-Samuel multiplicities $e(Q, A)$, for all $\mathfrak{m}$-primary ideals Q of A. Let $\gamma(A)$ denote *the greatest common divisor of the elements of* $\mathcal{E}$. Clearly, $\gamma(A)$ divides $e(\mathfrak{m})$.

Proposition 4.9 (1) shows that the greatest common divisor of the integers $e(Q)$, taken only over the subset of all ideals Q generated by strict systems of parameters, is equal to $\gamma(A)$. Theorem 4.5 implies that $\gamma(A)[\mathfrak{m}]$ is rationally equivalent to zero in $\operatorname{Spec} A$. It is obvious that $\gamma(A) = 1$ when A is regular. When $\dim A = 0$, it follows from the definitions that $\gamma(A) = \ell_A(A)$.

Let $d := \dim A \geq 1$, and let $U \subseteq \operatorname{Spec} A$ be any dense open subset. Fix $d' \in [0, d-1]$, and let

$$U\big(d'\big) := \big\{\mathfrak{p} \in U \mid \operatorname{ht}(\mathfrak{p}) = d',\ \dim(A/\mathfrak{p}) = d - d'\big\}.$$

Let

$$g(U) := \gcd_{\mathfrak{p} \in U(d')} \big\{\gamma(A_{\mathfrak{p}})\gamma(A/\mathfrak{p})\big\}.$$

If $V \subseteq U$ is another dense open subset, then $g(U)$ divides $g(V)$. When $d' = 0$, it follows immediately from 4.6 and the definitions that $g(U)$ divides $\gamma(A)$. We generalize this statement in our next proposition; the case $d' = d - 1$ will be used in the proof of Theorem 5.6.

Proposition 5.2 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$, *and let* $U \subseteq \operatorname{Spec} A$ *be any dense open subset. Fix* $d' \in [0, d-1]$. *Then* $U(d') \neq \emptyset$ *and*

$$\gamma(A) = \gcd_{\mathfrak{p} \in U(d')} \big\{\gamma(A/\mathfrak{p})\gamma(A_{\mathfrak{p}})\big\}.$$

Proof Let us first prove that $g(U)$ divides $\gamma(A)$. As mentioned above, the case $d' = 0$ follows from 4.6, so we now assume that $d' \geq 1$. It suffices to prove the divisibility for any open dense $V \subseteq U$, so shrinking U if necessary, we can suppose that the complement $V(I)$ of U in $\operatorname{Spec} A$ has dimension $d - 1$. Moreover, $\operatorname{ht}(I) > 0$ since U is dense.

Let $\mathbf{f} = f_1, \ldots, f_d$ be any strict system of parameters of A such that $f_1, \ldots, f_{d-1}$ induces a strict system of parameters of A/I (4.4). In particular, the image of $\{f_1, \ldots, f_{d'}\}$ in A/I is strictly secant. Therefore, any minimal prime ideal of A over $(f_1, \ldots, f_{d'})$ has height d', and any minimal prime ideal of A/I over the ideal $I + (f_1, \ldots, f_{d'})/I$ has height d' (4.2). Let $\mathfrak{p}$ be a minimal prime ideal of A over $(f_1, \ldots, f_{d'})$. Then $\mathfrak{p} \in U$, since otherwise, $I \subseteq \mathfrak{p}$ and $d' = \operatorname{ht}(\mathfrak{p}) \geq \operatorname{ht}(\mathfrak{p}/I) + \operatorname{ht}(I) = d' + \operatorname{ht}(I) > d'$. Since $\dim A/(f_1, \ldots, f_{d'}) = d - d'$, there exists such a $\mathfrak{p}$ with $\dim A/\mathfrak{p} = d - d'$, and then $\mathfrak{p} \in U(d')$. Using 4.7, we find that $g(U)$ divides $e((\mathbf{f}), A)$. Hence, Proposition 4.9 (2) implies that $g(U)$ divides $\gamma(A)$.

Let us now prove that $\gamma(A)$ divides $g(U)$. It suffices to prove this divisibility when $U = \operatorname{Spec} A$. We start with the case $d' = d - 1$. Fix $\mathfrak{p} \in U(d-1)$, so that $\operatorname{ht}(\mathfrak{p}) = d - 1$, and $\dim A/\mathfrak{p} = 1$. We need to show that $\gamma(A)$ divides $\gamma(A_{\mathfrak{p}})\gamma(A/\mathfrak{p})$. Proposition 4.9 (1) shows that $\gamma(A/\mathfrak{p})$ is equal to the greatest common divisor of the integers $\operatorname{ord}_{A/\mathfrak{p}}(\phi)$, where $\phi \in \operatorname{Frac}(A/\mathfrak{p})^*$. When $d = 1$, then $\gamma(A_{\mathfrak{p}}) = \ell(A_{\mathfrak{p}})$. (Recall the convention in 4.7 that if $d = 1$, then $e((f_1, \dots, f_{d-1}), A_{\mathfrak{p}}) = \ell(A_{\mathfrak{p}})$.) Then when $d \geq 1$, Proposition 4.9(4) shows that $\gamma(A_{\mathfrak{p}})$ can be computed as the greatest common divisor of the integers $e((f_1, \dots, f_{d-1}), A_{\mathfrak{p}})$, where $f_1, \dots, f_{d-1}$ is a strictly secant sequence in $\mathfrak{p}$. Therefore, it is enough to show that $\gamma(A)$ divides $\operatorname{ord}_{A/\mathfrak{p}}(\phi)e((\mathbf{f}'), A_{\mathfrak{p}})$ for all $\phi \in \operatorname{Frac}(A/\mathfrak{p})^*$ and for all strictly secant sequences $\mathbf{f}' = f_1, \dots, f_{d-1}$ contained in $\mathfrak{p}$.

Fix $\phi \in \operatorname{Frac}(A/\mathfrak{p})^*$ and a strictly secant sequence $\mathbf{f}' = f_1, \dots, f_{d-1}$ contained in $\mathfrak{p}$. By construction, $\mathfrak{p}_1 := \mathfrak{p}$ is a minimal prime ideal of A over $(\mathbf{f}')$. Let $\mathfrak{p}_2, \dots, \mathfrak{p}_m$ denote the other minimal primes over $(\mathbf{f}')$. Use the isomorphism $\operatorname{Frac}(A/\sqrt{(\mathbf{f}')}) \to \bigoplus_i \operatorname{Frac}(A/\mathfrak{p}_i)$ to find an invertible rational function $\phi' \in \operatorname{Frac}(A/\sqrt{(\mathbf{f}')})$ which restricts to ϕ in $\operatorname{Frac}(A/\mathfrak{p})$ and to 1 in $\operatorname{Frac}(A/\mathfrak{p}_i)$ for all $i \geq 2$. Let $a, b \in \mathfrak{m} \setminus \bigcup_{i \geq 1} \mathfrak{p}_i$ be such that a/b maps to ϕ' in $\operatorname{Frac}(A/\sqrt{(\mathbf{f}')})$. The sequences $\{\mathbf{f}', a\}$ and $\{\mathbf{f}', b\}$ are strict systems of parameters in $\mathfrak{m}$. Then 4.7 gives

$$e\big((\mathbf{f}', a), A\big) = \sum e\big((\mathbf{f}'), A_{\mathfrak{p}_i}\big) \operatorname{ord}_{A/\mathfrak{p}_i}(a),$$

where the sum is over the indices i such that $\dim A/\mathfrak{p}_i = 1$. We proceed similarly for b to find that

$$e\big((\mathbf{f}', a), A\big) - e\big((\mathbf{f}', b), A\big) = \operatorname{ord}_{A/\mathfrak{p}}(\phi)e\big((\mathbf{f}'), A_{\mathfrak{p}}\big),$$

because $a = b$ in $A/\mathfrak{p}_i$ if $i \geq 2$. Hence, $\gamma(A)$ divides $\operatorname{ord}_{A/\mathfrak{p}}(\phi)e((\mathbf{f}'), A_{\mathfrak{p}})$ and the proposition is proved when $d' = d - 1$.

The proposition is now true when $d = 1$. Suppose that $d \geq 2$, and proceed by induction on d. Fix $d' \leq d - 2$. We showed above that $(\operatorname{Spec} A)(d') \neq \emptyset$, so let $\mathfrak{p} \in (\operatorname{Spec} A)(d')$, with $\operatorname{ht}(\mathfrak{p}) = d'$, and $\dim A/\mathfrak{p} = d - d'$. Since $(\operatorname{Spec} A/\mathfrak{p})(d - d' - 1) \neq \emptyset$, let $\mathfrak{q} \supseteq \mathfrak{p}$ be any prime ideal of A such that $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = d - d' - 1$ and $\dim A/\mathfrak{q} = 1$. It follows that $\operatorname{ht}(\mathfrak{q}) = d - 1$, so that we obtain from the above considerations that

$$\gamma(A) \quad \text{divides} \quad \gamma(A/\mathfrak{q})\gamma(A_{\mathfrak{q}}).$$

Since $\dim A_{\mathfrak{q}} = d - 1$, we can apply the induction hypothesis to $A_{\mathfrak{q}}$; consider the ideal $\mathfrak{p}A_{\mathfrak{q}}$, with $\operatorname{ht}(\mathfrak{p}A_{\mathfrak{q}}) = d'$ and $\dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) = \dim(A_{\mathfrak{q}}) - d'$. It follows that

$$\gamma(A_{\mathfrak{q}}) \quad \text{divides} \quad \gamma(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}})\gamma\big((A_{\mathfrak{q}})_{\mathfrak{p}A_{\mathfrak{q}}}\big).$$

Let $B := A/\mathfrak{p}$ and $\bar{\mathfrak{q}} := \mathfrak{q}B$. With this notation, the above two divisibility conditions give

$$\gamma(A) \quad \text{divides} \quad \gamma(B/\bar{\mathfrak{q}})\gamma(B_{\bar{\mathfrak{q}}})\gamma(A_{\mathfrak{p}}).$$

Varying $\bar{\mathfrak{q}}$ in $(\operatorname{Spec} B)(\dim B - 1)$, and using the case $d' = d - 1$ established above, we get that $\gamma(B) = \gcd_{\bar{\mathfrak{q}}} \gamma(B/\bar{\mathfrak{q}})\gamma(B_{\bar{\mathfrak{q}}})$, so that $\gamma(A)$ divides $\gamma(B)\gamma(A_{\mathfrak{p}})$. Since $B = A/\mathfrak{p}$, the result follows. □

5.3 Recall that the generic point of an irreducible component of a closed subset F of a scheme X is called a *maximal point* of F. For convenience in the statement of our next definition, we denote by $\max(F)$ the set of the maximal points of F.

Let $(A, \mathfrak{m})$ be a Noetherian local ring of positive dimension. Let U be any dense open subset of $\operatorname{Spec} A$. Denote by x_0 the unique closed point of $\operatorname{Spec} A$. Let $\mathcal{N}_U$ denote the set of all integers n occurring as the order $\operatorname{ord}_{x_0}(f)$ of some rational function $f \in \mathcal{K}_C^*(C)$ on any reduced closed curve C in $\operatorname{Spec} A$ with $\max(C) \subseteq U$.

We claim that $\mathcal{N}_U$ is in fact an ideal in $\mathbb{Z}$. Indeed, suppose that for a given open subset U in $\operatorname{Spec} A$, there exist reduced closed one-dimensional subschemes C and C' in $\operatorname{Spec} A$, and $f \in \mathcal{K}_C^*(C)$, $f' \in \mathcal{K}_{C'}^*(C')$, such that $\max(C) \subseteq U$, $\max(C') \subseteq U$, and $n = \operatorname{ord}_{x_0}(f)$, $n' = \operatorname{ord}_{x_0}(f')$. Let us use 1.1, (1) and (2), to build functions on $C \cup C'$. First extend f to a function F on $C \cup C'$, with $F = 1$ on every component of C' which is not a component of C. Similarly, extend g to G on $C \cup C'$, with $G = 1$ on every component of C which is not a component of C'. Then $F^a G^b$ is such that $\operatorname{ord}_{x_0}(F^a G^b) = an + bn' \in \mathcal{N}_U$.

Denote by $n(U, A)$, or $n(U, \operatorname{Spec} A)$, *the greatest common divisor of the positive elements of $\mathcal{N}_U$*. Clearly, if $V \subseteq U$ is dense and open in $\operatorname{Spec} A$, then $n(U, A)$ divides $n(V, A)$.

5.4 Let A be a Noetherian local ring of positive dimension as above. Consider the ideal $\mathcal{N} := \bigcap_U \mathcal{N}_U$, where U runs through all dense open subsets of $\operatorname{Spec} A$. Theorem 4.5 immediately implies that $\mathcal{N} \neq (0)$. Let $n(A)$ denote *the greatest common divisor of the positive elements of $\mathcal{N}$*. The integer $n(A)$ is the positive generator of the ideal $\mathcal{N}$. When $\dim A = 0$, we let $n(A) := 1$.

By definition, $n(U, A)$ divides $n(A)$ for any dense open subset U of $\operatorname{Spec} A$. In fact, $n(A)$ is the smallest positive integer n such that, for every dense open set U of $\operatorname{Spec} A$, there exists a reduced closed curve C in $\operatorname{Spec} A$ with $\max(C) \subseteq U$ and a rational function $f \in \mathcal{K}_C^*(C)$ such that $n = \operatorname{ord}_{x_0}(f)$. In view of Theorems 4.5 and 6.4, we call $n(A)$ the *moving multiplicity* of A.

Lemma 5.5 *Let A be a Noetherian local ring of positive dimension. There exists a dense open subset U_0 of* $\operatorname{Spec} A$ *such that $n(U_0, A) = n(A)$. In particular, for all dense open subsets V contained in U_0, $n(V, A) = n(A)$.*

Proof Since the ring $\mathbb{Z}/\mathcal{N}$ is Artinian, we can find finitely many dense open sets U_i such that $\mathcal{N} = \bigcap_i \mathcal{N}_{U_i}$. We let $U_0 := \bigcap_i U_i$. □

Theorem 4.5 motivated our definition of $n(A)$. It follows from this theorem that $n(A)$ divides $e(Q)$ for all $\mathfrak{m}$-primary ideals Q of A. Hence,

$$n(A) \quad \text{divides} \quad \gamma(A).$$

In particular, if A is regular, then $n(A) = 1$, but the converse is false. We now show that under some natural hypotheses, such as A being reduced, excellent, and equidimensional of positive dimension, then $n(A) = \gamma(A)$ (see also 7.12).

Theorem 5.6 *Let A be a Noetherian equidimensional local ring of positive dimension. Suppose that A is catenary and the regular locus of* $\operatorname{Spec} A$ *contains a dense open subset of* $\operatorname{Spec} A$. *Then $n(A) = \gamma(A)$.*

Proof It suffices to show that $\gamma(A)$ divides $n(A)$. Let U be any dense open subset of the regular locus of $\operatorname{Spec} A$. By hypothesis, there exist a reduced curve C in $\operatorname{Spec} A$ with $\max(C) \subseteq U$ and $f \in \mathcal{K}_C^*(C)$ such that $n(A) = \operatorname{ord}(f)$. Let $C_1, \dots, C_n$ be the irreducible components of C. Each C_i corresponds to a prime ideal $\mathfrak{p}_i$ of A. Since the generic point of C_i belongs to U, $A_{\mathfrak{p}_i}$ is regular, so that $\gamma(A_{\mathfrak{p}_i}) = 1$. Because A is catenary and equidimensional, we find that $\operatorname{ht}(\mathfrak{p}_i) = \dim A - 1$. Proposition 5.2 implies then that $\gamma(A)$ divides $\gcd_i(\gamma(A/\mathfrak{p}_i))$. Using 1.1 (1) applied to $\mathcal{O}_C(C)$, we find that $\gamma(A)$ divides $\operatorname{ord}(f)$ if $\gamma(A)$ divides $\operatorname{ord}(f|_{C_i})$ for each i. Since $A/\mathfrak{p}_i$ is an integral domain, $\operatorname{ord}(g) := \ell((A/\mathfrak{p}_i)/(g)) = e((g), A/\mathfrak{p}_i)$ for any non-zero $g \in A/\mathfrak{p}_i$. It follows from its definition that $\gamma(A/\mathfrak{p}_i)$ divides $\operatorname{ord}(g)$. Hence, $\gamma(A)$ divides $n(A)$. □

By construction, $n(A) = n(A/\sqrt{0})$. Let $\widehat{A}$ denote the completion of A with respect to its maximal ideal. Then it is clear from the definition that $\gamma(A) = \gamma(\widehat{A})$. Note however that in general, $\gamma(A) \neq \gamma(A/\sqrt{0})$ and, as we show in our next example, $n(A) \neq n(\widehat{A})$ in general.

Example 5.7 There exists a local Noetherian domain A of dimension 1 such that its completion $\widehat{A}$ has a single non-zero minimal prime ideal $\mathfrak{P} = \sqrt{(0)}$, and such that $\widehat{A}^{\text{red}} := \widehat{A}/\mathfrak{P}$ is a discrete valuation ring (see, e.g., [3], (3.0.1)). The ring A satisfies the hypotheses of Theorem 5.6, so

$$n(A) = \gamma(A) = \gamma(\widehat{A}) = \ell(\widehat{A}_{\mathfrak{P}})\gamma\left(\widehat{A}^{\text{red}}\right) = \ell(\widehat{A}_{\mathfrak{P}}),$$

while $n(\widehat{A}) = n(\widehat{A}^{\mathrm{red}}) = 1$. Hence, $n(A) > n(\widehat{A})$. We provide in 7.17 an example of a Noetherian local ring A with Henselization A^h such that $n(A) > n(A^h)$.

6 A generic Moving Lemma

6.1 We extend to schemes the definition of the moving multiplicity in 5.4 as follows. Let X be a Noetherian scheme and let $x_0 \in X$ be a point with $\dim \mathcal{O}_{X,x_0} \geq 1$. Define

$$n_X(x_0) := n(\mathcal{O}_{X,x_0}).$$

The main result in this section is Theorem 6.4 below, which details one of the most useful uses of the invariant $n_X(x_0)$.

We start with two preliminary propositions. The first one is proved in [8], II.9.3, for affine Noetherian schemes. Recall that an FA-scheme X is a scheme such that every finite subset of X is contained in an affine open subset of X (2.2).

Proposition 6.2 (Moving Cartier divisors) *Let X be a Noetherian* FA-*scheme. Let D be a Cartier divisor on X and let $\mathcal{F}$ be a finite subset of X. Then there exists a Cartier divisor D' on X, linearly equivalent to D and such that* $\mathrm{Supp}(D') \cap \mathcal{F} = \emptyset$.

Proof Let $x_1, \ldots, x_m$ be closed points of X such that every point of $\mathcal{F}$ and of $\mathrm{Ass}(X)$ specializes to some of the x_i's. In particular, for any open subset V of X containing $\{x_1, \ldots, x_m\}$, the natural restriction map $\mathcal{K}_X(X) \to \mathcal{K}_X(V)$ is an isomorphism (see [39], 7.1.15).

Let U be an affine open subset containing $x_1, \ldots, x_m$. Recall that by definition, $\mathcal{O}_X(D)$ is an invertible subsheaf of $\mathcal{K}_X$. The canonical homomorphism

$$\mathcal{O}_X(D)(U) \to \bigoplus_{1 \leq i \leq m} \mathcal{O}_X(D) \otimes k(x_i)$$

is surjective (Chinese Remainder Theorem), so there exists $f \in H^0(U, \mathcal{O}_X(D)|_U) \subseteq \mathcal{K}_X(U)$ such that for all $i = 1, \ldots, m$, the image of f in $\mathcal{O}_X(D) \otimes k(x_i)$ is a basis. Then f_{x_i} is a basis of $\mathcal{O}_X(D)_{x_i}$ for all $i = 1, \ldots, m$. This implies that there exists an open subset $V \subseteq U$ containing $\mathcal{F}$ and $\mathrm{Ass}(X)$ such that f_x is a basis of $\mathcal{O}_X(D)_x$ for all $x \in V$. It follows that $f \in \mathcal{K}_X^*(V)$. Extend f to $g \in \mathcal{K}_X^*(X)$ using the isomorphism $\mathcal{K}_X(X) \to \mathcal{K}_X(V)$. Then $D - \mathrm{div}(g)$ has support disjoint from $\mathcal{F}$. □

Let X be a Noetherian scheme, and let F be a closed subset of X. For convenience, we will say that a cycle Z on X *generically avoids* F if no irreducible cycle occurring in Z is contained in F. In particular, no irreducible component of $\operatorname{Supp} Z$ is contained in F.

Proposition 6.3 *Let X be a Noetherian* FA*-scheme. Let F be a closed subset of X of positive codimension in X. Let $x_0 \in X$, and $j : \operatorname{Spec} \mathcal{O}_{X,x_0} \to X$ be the canonical injection. Denote again by x_0 the closed point of* $\operatorname{Spec} \mathcal{O}_{X,x_0}$*, and let $F' := j^{-1}(F)$. Suppose that there exist integral closed subschemes $C_1, \ldots, C_r$ of* $\operatorname{Spec} \mathcal{O}_{X,x_0}$ *of dimension* 1*, elements $f_i \in k(C_i)^*$ for $i = 1, \ldots, r$, and an integer $n \geq 1$, such that $C_i \cap F' = \{x_0\}$ and $n[x_0] = \sum_i [\operatorname{div}(f_i)]$ in $\mathcal{Z}(\operatorname{Spec} \mathcal{O}_{X,x_0})$.*

Then the cycle $n[\overline{\{x_0\}}]$ in $\mathcal{Z}(X)$ is rationally equivalent in X to a cycle which generically avoids F. More precisely, when $x_0 \in F$, let $\overline{C}_i$ be the scheme-theoretic closure of $j(C_i)$ in X. Then $n[\overline{\{x_0\}}]$ is rationally equivalent in $\bigcup_i \overline{C_i}$ to a cycle Z which generically avoids $F \cap (\bigcup_i \overline{C_i})$, and such that each irreducible cycle occurring in Z is of codimension 1 *in some $\overline{C}_i$.*

Proof Since $\overline{C}_i$ is the scheme-theoretic closure of $j(C_i)$ in X and $C_i \cap F' = \{x_0\}$, the closed subset $\overline{C}_i$ is not contained in F. For each $i = 1, \ldots, r$, there exists by hypothesis a function $g_i \in k(\overline{C}_i)^*$ defined on a dense open subset of $\overline{C}_i$, whose stalk in $\mathcal{O}_{\overline{C}_i, x_0}$ is f_i, and such that $[\operatorname{div}_{\overline{C}_i}(g_i)] = \operatorname{ord}_{x_0}(f_i)[\overline{\{x_0\}}] + Z_i$ for some cycle Z_i on $\overline{C}_i$ whose support does not contain $\overline{\{x_0\}}$. To conclude the proof, it is enough to show that for all $i = 1, \ldots, r$, Z_i is rationally equivalent on $\overline{C}_i$ to a cycle which generically avoids $F \cap \overline{C}_i$. Then $n[\overline{\{x_0\}}]$ is rationally equivalent on X to a cycle which generically avoids F, as desired.

Dropping the subscript i for ease of notation, we are in the following situation. On the Noetherian integral FA-scheme $\overline{C}$, we have a cycle Z, all of whose irreducible components are of codimension 1 in $\overline{C}$ and whose generic points are all contained in a dense open subset U of $\overline{C}$. Moreover, $Z|_U = [\operatorname{div}(g)]$. We claim that Z is rationally equivalent on $\overline{C}$ to a cycle which generically avoids a proper closed subset $F \cap \overline{C}$ of $\overline{C}$.

Indeed, let $\mathcal{F}$ be the set of points of $F \cap \overline{C}$ of codimension 1 in $\overline{C}$. Let $V := \overline{C} \setminus \operatorname{Supp} Z$ and $W := U \cup V$. Then W is open, and $Z|_W = [D]$, where D is the Cartier divisor on W given by the charts $\{(U, g), (V, 1)\}$. By construction, W contains all codimension 1 points of $\overline{C}$, and so contains $\mathcal{F}$. Since W is an FA-scheme (2.2 (2)), we can use Proposition 6.2 to find an invertible rational function $f \in k(W)^*$ such that $\operatorname{Supp}[D + \operatorname{div}(f)]$ does not meet $\mathcal{F}$. Let $h \in k(\overline{C})^*$ be the unique rational function on $\overline{C}$ extending f, and consider $Z' := Z + [\operatorname{div}(h)]$. Then $Z'|_W = [D + \operatorname{div}(f)]$ and, hence, $\operatorname{Supp}(Z')$ does not meet $\mathcal{F}$ either. In other words, the irreducible components of $\operatorname{Supp}(Z')$ are not contained in F. As Z' only involves irreducible cycles of codimension 1, it generically avoids F. □

Theorem 6.4 (Generic Moving Lemma) *Let X be a Noetherian* FA-*scheme. Let F be a closed subset of X of positive codimension in X. Let $x_0 \in X$, and consider the cycle $[\overline{\{x_0\}}]$ in $\mathcal{Z}(X)$. Then $n_X(x_0)[\overline{\{x_0\}}]$ is rationally equivalent in X to a cycle which generically avoids F.*

Proof The theorem is obvious when $x_0 \notin F$. So assume now that $x_0 \in F$. Since $\operatorname{codim}(F, X) > 0$, none of the irreducible components of F are irreducible components of X. Hence, the preimage F' of F under the natural map $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$ has dimension smaller than $\dim \mathcal{O}_{X,x_0}$. In particular, $\dim \mathcal{O}_{X,x_0} \geq 1$, and we can apply Lemma 5.5. Denote again by x_0 the closed point of $\operatorname{Spec} \mathcal{O}_{X,x_0}$. Let then U_0 be a dense open set of $\operatorname{Spec} \mathcal{O}_{X,x_0}$ contained in $\operatorname{Spec} \mathcal{O}_{X,x_0} \setminus F'$, and such that $n(U_0, \mathcal{O}_{X,x_0}) = n(\mathcal{O}_{X,x_0}) = n_X(x_0)$ (5.5).

We can thus find integral closed subschemes $C_1, \ldots, C_r$ of $\operatorname{Spec} \mathcal{O}_{X,x_0}$ of dimension 1, and $f_i \in k(C_i)^*$, such that $C_i \cap F' = \{x_0\}$ and $n[\overline{\{x_0\}}] = \sum_i [\operatorname{div}(f_i)]$ in $\mathcal{Z}(\operatorname{Spec} \mathcal{O}_{X,x_0})$, with $n = n_X(x_0)$. The theorem follows from Proposition 6.3. □

The proof of Theorem 8.2 uses only the following version of Theorem 6.4, whose proof does not require the definition and main properties of the invariant $n_X(x_0)$ discussed in the previous section.

Theorem 6.5 *Let X be a Noetherian* FA-*scheme. Let F be a closed subset of X of positive codimension in X. Let $x_0 \in X$, and consider the cycle $[\overline{\{x_0\}}]$ in $\mathcal{Z}(X)$. Let Q be a $\mathfrak{m}_{X,x_0}$-primary ideal of $\mathcal{O}_{X,x_0}$. Then $e(Q)[\overline{\{x_0\}}]$ is rationally equivalent in X to a cycle which generically avoids F.*

Proof The proof of this version is completely analogous to the proof of 6.4. The theorem is obvious when $x_0 \notin F$. So assume now that $x_0 \in F$. Since $\operatorname{codim}(F, X) > 0$, none of the irreducible components of F are irreducible components of X. Hence, the preimage F' of F under the natural map $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$ has dimension smaller than $\dim \mathcal{O}_{X,x_0}$. In particular, $\dim \mathcal{O}_{X,x_0} \geq 1$, and we can apply Theorem 4.5. Denote again by x_0 the closed point of $\operatorname{Spec} \mathcal{O}_{X,x_0}$. We can thus find integral closed subschemes $C_1, \ldots, C_r$ of $\operatorname{Spec} \mathcal{O}_{X,x_0}$ of dimension 1, and $f_i \in k(C_i)^*$, such that $C_i \cap F' = \{x_0\}$ and $n[\overline{\{x_0\}}] = \sum_i [\operatorname{div}(f_i)]$ in $\mathcal{Z}(\operatorname{Spec} \mathcal{O}_{X,x_0})$, with $n = e(Q)$. The theorem follows from Proposition 6.3. □

Remark 6.6 In Theorem 6.4, a multiple of the cycle $[\overline{\{x_0\}}]$ can be moved, but in general the irreducible cycle $[\overline{\{x_0\}}]$ itself cannot be moved. Indeed, consider for instance the singular projective curve X over $\mathbb{R}$ defined by the equation $x^2 + y^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. The singular point $x_0 := (0:0:1)$ is the unique rational point of X, and all closed points of $X \setminus \{x_0\}$ are smooth and have degree $2 = [\mathbb{C} : \mathbb{R}]$. Therefore the 0-cycle $[x_0]$ cannot be rationally equivalent to a 0-cycle with support in X^{reg}.

Corollary 6.7 *Let X be a scheme of finite type over a field k. Let U be any dense open subset of X. Let $x_0 \in X$ be a closed point. Let Q be a $\mathfrak{m}_{X,x_0}$-primary ideal of $\mathcal{O}_{X,x_0}$. Then $e(Q)\deg_k(x_0)$ is the degree of some 0-cycle Z with support in U. If in addition X is separated, then Z can be chosen to be rationally equivalent in X to $e(Q)[x_0]$. It follows that when X is reduced, then $\delta(X^{\mathrm{reg}}/k)$ divides $\gamma(\mathcal{O}_{X,x_0})\deg_k(x_0)$.*

Proof Let V be an affine neighborhood of x_0 in X. We let $V \to \overline{V}$ be an open dense immersion of V into a projective variety $\overline{V}$. We apply Theorem 6.5 to $\overline{V}$ and the closed subset $\overline{V} \setminus (U \cap V)$ of positive codimension. Then $e(Q)[x_0]$ is rationally equivalent in $\overline{V}$ to a 0-cycle Z with support in $U \cap V$. Since $\overline{V}$ is projective, we find that $\deg(e(Q)[x_0]) = \deg(Z)$ because any principal divisor on a projective curve over k has degree 0 (see, e.g., [39], Corollary 7.3.18).

Assume now that X is separated. Apply Theorem 6.5 to find a closed curve C in X, a function $f \in k(C)^*$, and a 0-cycle Z on $C \setminus \{x_0\}$, such that on C, $e(Q)[x_0] - Z = [\mathrm{div}(f)]$. Let $U_1 := C \setminus \mathrm{Supp}\, Z$, and $U_2 := C \setminus \{x_0\}$. Then let D be the Cartier divisor given by the pairs (U_1, f) and $(U_2, 1)$. By construction, $[D] = e(Q)[x_0]$.

Since C is separated, Nagata's Theorem lets us find an open embedding of C into a proper curve C' over k, and such a curve is known to be also projective. Extend in a natural way D to a Cartier divisor D' on C'. Let $F := X \setminus U$. Using 6.2, we can find a Cartier divisor D'' linearly equivalent to D' on C' and such that $\mathrm{Supp}(D'')$ does not intersect $(C' \setminus C) \cup (C \cap F)$. As above, $\deg(D'') = \deg(D')$ since C' is a projective curve. Since $C \to C'$ is an open immersion, we find that D'' restricted to C is equivalent to D on C. □

Let k be any field. An *algebraic variety X over k* is a scheme of finite type over k (not necessarily separated). Let $\mathcal{D}$ denote the set of all degrees of closed points of X. When X/k is not empty, the *index* $\delta(X/k)$ of X/k is the greatest common divisor of the elements of $\mathcal{D}$.

The following proposition is only slightly more general than [12], page 599, or [10], Lemma 12. See also [15], 1.12.

Proposition 6.8 *Let X be a (non necessarily proper) regular non-empty algebraic variety over a field k. Then $\delta(U/k) = \delta(X/k)$ for any dense open subset U of X. In particular, if X_1 and X_2 are two integral regular algebraic varieties over k which are birational, then $\delta(X_1/k) = \delta(X_2/k)$.*

Proof Clearly, $\delta(X/k)$ divides $\delta(U/k)$. Let $x_0 \in X$ be a closed point. Applying Corollary 6.7, x_0 has same degree as some 0-cycle with support in U. Hence $\delta(U/K)$ divides $\deg(x_0)$, so $\delta(U/K)$ divides $\delta(X/K)$. □

Remark 6.9 Let X/k be an integral normal algebraic variety. Let k' be the algebraic closure of k in the field of rational functions $k(X)$. As k'/k is finite (hence integral), we have $k' \subseteq \mathcal{O}_X(X)$. Therefore, $X \to \mathrm{Spec}(k)$ factors as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. We will write X/k' when we regard X as a variety over k' through the morphism $X \to \mathrm{Spec}(k')$. We have $\delta(X/k) = [k' : k]\delta(X/k')$. Note that as a variety over k', X is geometrically irreducible.

Let X be any Noetherian scheme. Let $U \subseteq X$ be an open subset. Consider the natural map $\mathcal{Z}(U) \to \mathcal{A}(X)$, which sends an irreducible cycle on U to its Zariski closure in X modulo rational equivalence. Denote by $\mathcal{A}(X, U)$ the cokernel of $\mathcal{Z}(U) \to \mathcal{A}(X)$. Our next proposition generalizes [50], Proposition 7.1, where X is assumed to be integral and regular, and of finite type over the spectrum of a discrete valuation ring.

Proposition 6.10 *Let U be a dense open subset of a Noetherian* FA-*scheme X.*

(1) *If X is regular, then $\mathcal{A}(X, U) = (0)$.*
(2) *$\mathcal{A}(X, U)$ is a torsion group. Let $x_0 \in X$ be a point with* $\dim \mathcal{O}_{X,x_0} \geq 1$. *Then the class of $[\overline{\{x_0\}}]$ in the cokernel $\mathcal{A}(X, U)$ has order dividing $n_X(x_0)$, and there exists one such open subset U where the order is exactly $n_X(x_0)$.*
(3) *$\mathcal{A}(X, U)$ has finite exponent in the following situations*:
    (a) *X is (quasi-)excellent.*
    (b) *There exists a morphism of finite type $f : X \to S$ with S Noetherian regular and f either open or equidimensional.*

Proof (1) and (2) are immediate consequences of Theorem 6.4. Part (3) results from the fact that the set $\{e(\mathcal{O}_{X,x}) \mid x \in X\}$ is bounded, as discussed in Proposition 6.12 (3) and Remark 6.13 below. □

Remark 6.11 If $V \subset U$ are two dense open subsets of an integral scheme X, then there is a natural surjective group homomorphism $f_{V,U} : \mathcal{A}(X, V) \to \mathcal{A}(X, U)$. Let $\mathcal{O}$ denote the set of all dense open subsets U of X, with the relation $V \leq U$ if $V \supseteq U$. Define $\mathcal{S}(X)$ to be the inverse limit of the projective system $\{\mathcal{A}(X, U)\}_{U \in \mathcal{O}}$. Let $x_0 \in X$ be a point with $\dim \mathcal{O}_{X,x_0} \geq 1$. Then Proposition 6.10 implies that the class of $[\overline{\{x_0\}}]$ in $\mathcal{S}(X)$ has order $n_X(x_0)$, and that when X is quasi-excellent, $\mathcal{S}(X)$ is a torsion group of finite exponent. In particular, since $n_X(x_0) = 1$ when x_0 is regular, the group $\mathcal{S}(X)$ is generated by the classes of certain irreducible cycles $[\overline{\{x\}}]$ with x singular on X.

Proposition 6.12 *Let $f : X \to S$ be a morphism of finite type over a Noetherian scheme S. Then*

(1) *X is the union of finitely many open subsets X_i, each endowed with a quasi-finite S-morphism $f_i : X_i \to W_i := \mathbb{A}_S^{d_i}$, such that every $x \in X$ belongs to some X_i with $\dim_x X_{f(x)} = d_i$.*
(2) *Assume that S is irreducible. Suppose in addition that either f is locally*[4] *equidimensional and S is universally catenary, or that f is open. Then, with $x \in X_i$ as in* (1), $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{W_i, f_i(x)}$.
(3) *Suppose that S is regular and that f is either open or locally equidimensional. Then the set of multiplicities $\{e(\mathcal{O}_{X,x}) \mid x \in X\}$ is bounded.*

Proof (1) By [23], IV.13.3.1.1, for each $x \in X$, there exists an open neighborhood U_x of x and a quasi-finite S-morphism $U_x \to \mathbb{A}_S^d$, where $d = \dim_x X_{f(x)}$. For any integer $n \geq 0$, let $F_n := \{x \in X \mid \dim_x X_{f(x)} \geq n\}$ and $G_n = F_n \setminus F_{n+1}$. By a theorem of Chevalley, F_n is closed ([23], IV.13.1.3), and $F_n = \emptyset$ if $n \geq n_0$ for some n_0 ([23], IV.13.1.7). For all $n \leq n_0$, G_n is quasi-compact and, hence, can be covered by finitely many open subsets $\{U_{n,j}\}_j$ belonging to the covering $\{U_x, x \in X\}$. Since $x \in X$ belongs to $x \in G_d$ for $d = \dim_x X_{f(x)}$, x belongs to some $U_{d,j}$.

(2) Suppose that $x \in X_i$, with $f_i : X_i \to W_i := \mathbb{A}_S^{d_i}$ such that $\dim_x X_{f(x)} = d_i$. Clearly, $\dim \mathcal{O}_{W_i, f_i(x)} = \dim \mathcal{O}_{X_{f(x)},x} + \dim \mathcal{O}_{S, f(x)}$. We thus need to show that

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{X_{f(x)},x} + \dim \mathcal{O}_{S, f(x)}.$$

That this equality holds when f is flat is remarked in [23], IV.13.2.12 (i), and proved for f open in [23], IV.14.2.1. The other case follows from [23], IV.13.3.6.

(3) Since S is regular, it is the disjoint union of finitely many integral regular schemes, and we are reduced to consider the case where S is integral. Let f_i be as in (1). For every i, there exists an open immersion of $u : X_i \to Z_i$ and a finite morphism $g_i : Z_i \to W_i$ with $g_i \circ u = f_i$ (Zariski's Main Theorem [23], IV.8.12.6). Our hypotheses allow us to apply (1) and (2), and every $x \in X$ belongs to some X_i with $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{W_i, f_i(x)}$. For such an x, we find that there exists an affine open neighborhood of $f_i(x)$ such that $\mathcal{O}_{W_i}(U)$ injects into $\mathcal{O}_{Z_i}(f_i^{-1}(U))$, since both of these rings have the same dimensions. Since g_i is finite, there exists a surjective homomorphism $\mathcal{O}_{W_i}^{m_i} \to g_{i*}(\mathcal{O}_{Z_i})$. Then, for all $x \in X_i$ such that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{W_i, f_i(x)}$,

[4] See [23] IV.13.2.2, with a correction in [23] (**Err**$_{\text{IV}}$, **34**) on pp. 356–357 of no. **32**.

we have $e(\mathcal{O}_{X,x}) \leq m_i e(\mathcal{O}_{W_i, f_i(x)})$, and since W_i is regular by hypothesis, $e(\mathcal{O}_{W_i, f_i(x)}) = 1$ (use [56], VIII, §10, Corollary 1 to Theorem 24, along with the discussion (3) following Corollary 1). Therefore, the set of the multiplicities $e(\mathcal{O}_{X,x})$ is bounded by $\max_i\{m_i\}$. □

Remark 6.13 Let X be a Noetherian excellent scheme of finite dimension. Then only a finite number of polynomials appear as Hilbert-Samuel polynomials of $\mathcal{O}_{X,x}$ when x varies in X. In particular, the set of the multiplicities $\{e(\mathcal{O}_{X,x}) \mid x \in X\}$ is bounded. This statement is found in [2], Remark III.1.3, page 83. Let us explain now how the latter conclusion still holds if one only assumes that X is a Noetherian quasi-excellent scheme.

Recall that an excellent scheme is a quasi-excellent and universally catenary scheme. This latter hypothesis is not needed in [2], Remark III.1.3: One finds in the discussion following [2], II (2.2.1), page 34, that in the use of normal flatness, the needed hypothesis is that the regular locus of a closed subset Y of X is open in Y. This hypothesis is satisfied when X is quasi-excellent.

Let X be a Noetherian quasi-excellent scheme. The results of [2] show that the set of the multiplicities $\{e(\mathcal{O}_{X,x}) \mid x \in X\}$ is bounded, even when X is not assumed to have finite dimension. Indeed, first note that there is a finite decreasing sequence of closed subsets $X = F_0 \supset F_1 \supset \cdots \supset F_n = \emptyset$ such that $F_i \setminus F_{i+1}$ with the reduced structure is regular and X is normally flat along $F_i \setminus F_{i+1}$. This uses "Noetherian induction" and one does not need to argue using the finite dimensionality of X as in [2], Remark III.1.3. The relevant statement occurs in [2], page 28, just before Theorem (2): For a local Noetherian ring $\mathcal{O}$ with Hilbert-Samuel function $H^{(1)}_{\mathcal{O}}$ (see page 26) and with P a prime ideal of coheight c and $\mathcal{O}/P$ regular, $H^{(1)}_{\mathcal{O}} = H^{(1+c)}_{\mathcal{O}_P}$ if and only if $\operatorname{Spec}\mathcal{O}$ normally flat along $\operatorname{Spec}\mathcal{O}/P$ at the closed point. The direction that we need is stated in [2], (2.1.2) Chapter 0, page 33. This implies that if a Noetherian scheme X is normally flat along a connected locally closed regular scheme Y, then the multiplicity of $\mathcal{O}_{X,y}$ $(y \in Y)$ is constant on Y.

Example 6.14 A Noetherian domain A of dimension 1 with the following properties is exhibited in [26], 3.2: $\operatorname{Spec} A$ has infinitely many singular maximal ideal $\mathfrak{m}_i$, $i \in I$, and such a domain can be found even when, for each i, the efg-numbers at $\mathfrak{m}_i$ are specified in advance (where $\mathfrak{m}_i B = \prod_{j=1}^{g(i)} \mathfrak{n}_{ij}^{e(\mathfrak{n}_{ij}/\mathfrak{m}_i)}$ in the integral closure B of A, and $f(\mathfrak{n}_{ij}/\mathfrak{m}_i) := [B/\mathfrak{n}_{ij} : A/\mathfrak{m}_i]$). Moreover, for each i, $B_{\mathfrak{m}_i}$ is a finitely generated $A_{\mathfrak{m}_i}$-module ([26], 3.3). Choosing such an example with $g(i) = 1$ for all i and $\lim_i f(\mathfrak{n}_{i1}/\mathfrak{m}_i) = \infty$ produces an example of a Noetherian scheme X of dimension 1 such that the set $\{e(\mathcal{O}_{X,x}) \mid x \in X\}$ is not bounded. Indeed, since A is a domain of dimension 1, there exist nonzero elements $a_i, b_i \in A$ such that $e(\mathfrak{m}_i, A) = e((a_i), A) - e((b_i), A)$ (4.5). It follows from 1.1 (3) that both $e((a_i), A)$ and $e((b_i), A)$ are divisible by $f(\mathfrak{n}_{i1}/\mathfrak{m}_i)$.

7 A different perspective on the index

Let X be a Noetherian scheme. Let $x_0 \in X$ be a point with $\dim \mathcal{O}_{X,x_0} \geq 1$. We study in this section the invariant $n_X(x_0)$ introduced in 6.1. In particular, let $f : Y \to X$ be a morphism of finite type, and set $E := Y \times_X \operatorname{Spec} k(x_0)$. We relate in 7.1 the invariant $n_X(x_0)$ with the index of the scheme $E/k(x_0)$. This leads us in Corollary 7.4 to provide a new way of computing the index of a regular closed subvariety X of a projective space using data pertaining only to the singular vertex of a cone over X.

Theorem 7.1 *Let X be a Noetherian scheme. Let $f : Y \to X$ be a morphism of finite type such that the generic point of every irreducible component of Y maps to the generic point of an irreducible component of X. Let $x_0 \in X$ be a point with* $\dim \mathcal{O}_{X,x_0} \geq 1$, *and set* $E := Y \times_X \operatorname{Spec} k(x_0)$. *Assume that* $E \neq \emptyset$.

(a) *Assume that f is birational and proper. Then $\delta(E/k(x_0))$ divides $n_X(x_0)$.*
(b) *Assume that f is birational and finite. Then*

$$\gcd\big\{n_Y(y)\big[k(y) : k(x_0)\big] \mid y \in f^{-1}(x_0), y \text{ closed}\big\} \quad \textit{divides} \quad n_X(x_0).$$

(c) *Assume that $\mathcal{O}_{X,x_0}$ is universally catenary. Then*

$$n_X(x_0) \quad \textit{divides} \quad \gcd\big\{n_Y(y)\big[k(y) : k(x_0)\big] \mid y \in f^{-1}(x_0), y \text{ closed}\big\}.$$

Proof After the base change $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$, we can suppose that X is local with closed point x_0.

(a) Let us show that $\delta(E/k(x_0))$ divides $n_X(x_0)$. Let U be any dense open subset of X such that $f^{-1}(U) \to U$ is an isomorphism. Since U is dense, there exists, by definition of $n(\mathcal{O}_{X,x_0})$, a one-dimensional reduced closed subscheme C of X such that $\max(C) \subseteq U$ and a rational function $g \in \mathcal{K}_C^*(C)$ such that $n_X(x_0)[x_0] = [\operatorname{div}(g)]$. Let $\tilde{C}$ be the strict transform of C in Y. Then we have a finite birational morphism $\pi : \tilde{C} \to C$, and

$$n_X(x_0)[x_0] = \sum_{y \in \pi^{-1}(x_0)} \operatorname{ord}_y(g)\big[k(y) : k(x_0)\big][x_0]$$

(use 1.1 (3)). As $\pi^{-1}(x_0) \subseteq E$, we conclude that $\delta(E/k(x_0)) \mid n_X(x_0)$.

(b) We will proceed as in the proof of (a), after carefully choosing the initial dense open subset U of X. First, we may assume that X and Y are affine. Let $y_1, \ldots, y_n$ denote the preimages of x_0. For each $i = 1, \ldots, n$, choose a dense open subset V_i of Y such that $n(V_i \cap \operatorname{Spec} \mathcal{O}_{Y,y_i}, \mathcal{O}_{Y,y_i}) = n_Y(y_i)$ (see 5.5). Since f is birational, let $V \subseteq \operatorname{Spec} B$ be a dense open subset such that $f_{|V}$ is an isomorphism. Then $W := (\bigcap_i V_i) \cap V$ is a dense open subset of Y, contained in each V_i, and such that $n(W \cap \operatorname{Spec} \mathcal{O}_{Y,y_i}, \mathcal{O}_{Y,y_i}) = n_Y(y_i)$. We let $U := f(W)$, a dense open subset in X with $f^{-1}(U) = W$ by construction. As in the proof of (a), we find that

$$n_X(x_0)[x_0] = \sum_{y \in \pi^{-1}(x_0)} \operatorname{ord}_y(g)\big[k(y) : k(x_0)\big][x_0].$$

By our choice of W, we find that for each $y \in \pi^{-1}(x_0)$, $n_Y(y)$ divides $\operatorname{ord}_y(g)$, and the result follows.

(c) Recall that we assume $X = \operatorname{Spec} \mathcal{O}_{X,x_0}$. Let y_0 be a closed point in E. We claim that $n_X(x_0)$ divides $n_Y(y_0)[k(y_0) : k(x_0)]$. Let W be an open affine neighborhood of y_0 in Y. Since W is of finite type over the affine scheme X, we can find a projective scheme $g : Z \to X$ and an open embedding $i : W \to Z$ such that $g \circ i = f_{|W}$. Since $n_Y(y_0) = n_W(y_0) = n_Z(y_0)$, it suffices to prove our claim in the case where f is projective.

Using 5.5, we may find a dense open subset U of X which does not contain x_0, and such that $n_X(x_0) = n(U, \mathcal{O}_{X,x_0})$. Let $V := f^{-1}(U)$. Then V is dense in Y. Let $s : \operatorname{Spec} \mathcal{O}_{Y,y_0} \to Y$ denote the natural morphism, and let $V' := s^{-1}(V)$. Since V' is dense in $\operatorname{Spec} \mathcal{O}_{Y,y_0}$, we can find finitely many closed integral subschemes C_i of $\operatorname{Spec} \mathcal{O}_{Y,y_0}$ with generic points in V', and an invertible rational function g on $\bigcup_i C_i$ such that $\operatorname{ord}_{y_0}(g) = n_Y(y_0)$. Since Y is Noetherian and FA, we can apply Proposition 6.3 and find that $n_Y(y_0)[y_0]$ is rationally equivalent to a cycle Z on Y such that $\max(\operatorname{Supp} Z) \subseteq V$. More precisely, there exist finitely many closed integral subschemes $\overline{C}_i$ in Y such that y_0 has codimension 1 in $\overline{C}_i$, such that the generic point of $\overline{C}_i$ is in V for all i, and such that $n_Y(y_0)[y_0]$ is rationally equivalent on $S := \bigcup_i \overline{C}_i$ to a cycle Z with $\max(\operatorname{Supp} Z) \subseteq V$, and such that each irreducible cycle occurring in Z is of codimension 1 in some $\overline{C}_i$.

Let $\overline{D}_i$ denote the schematic image of $\overline{C}_i$ in X. Then $\overline{D}_i$ is universally catenary. We can thus use the dimension formula (see [23], IV.5.6.5.1) for the morphism $\overline{C}_i \to \overline{D}_i$,

$$\dim \mathcal{O}_{\overline{C}_i,z_0} + \operatorname{trdeg}\big(k(z_0)/k(x_0)\big) = \dim \mathcal{O}_{\overline{D}_i,x_0} + \operatorname{trdeg}\big(k(\eta_i)/k(\xi_i)\big),$$

where z_0 is any closed point of $\overline{C}_i$, and η_i and ξ_i denote respectively the generic point of $\overline{C}_i$ and $\overline{D}_i$. Since $\dim \mathcal{O}_{\overline{C}_i,y_0} = 1$ by construction, and $\operatorname{trdeg}(k(y_0)/k(x_0)) = 0$ by hypothesis, we find that $\dim \mathcal{O}_{\overline{D}_i,x_0} = 1$ and $\operatorname{trdeg}(k(\eta_i)/k(\xi_i)) = 0$. It follows that $\dim \mathcal{O}_{\overline{C}_i,z_0} = 1$ for all z_0 closed in $\overline{C}_i$, so that $\dim(\overline{C}_i) = 1$. Since x_0 is the only closed point of $\overline{D}_i$, we find that $\dim(\overline{D}_i) = 1$. Since $\dim(\overline{C}_i) = 1$, we find that $\max(\operatorname{Supp} Z)$ consists in a set of closed points of Y. Since V does not contain any closed point of Y by construction, we find that $n_Y(y_0)[y_0]$ is rationally trivial on Y.

The morphism $f_{|S} : S \to f(S)$ is proper, with $f(S)$ universally catenary. Hence, the cycle $f_*(n_Y(y_0)[y_0])$ is rationally trivial on $f(S)$ and, thus, on X (use [54], 6.5 and 6.7). Since the irreducible components of S have dimension 1, since the generic points of $f(S)$ belong to U by construction, and since $f_*(n_Y(y_0)[y_0]) = n_Y(y_0)[k(y_0) : k(x_0)][x_0]$ is rationally trivial on $f(S)$, we find that by definition, $n(U, \mathcal{O}_{X,x_0})$ divides $n_Y(y_0)[k(y_0) : k(x_0)]$, and the statement of (c) follows. □

Remark 7.2 The hypothesis that X is universally catenary is needed in 7.1 (c). Indeed, consider the finite birational morphism $\pi : Y \to X$ described in Example 1.3. The scheme Y is regular, and X is catenary but not universally catenary. The preimage of the point $x_0 \in X$ consists in the two regular points y_0 and y_1 in Y, with $[k(y_0) : k(x_0)] = d$, and $[k(y_1) : k(x_0)] = 1$. It follows from the discussion in 1.3 that $n_X(x_0)$ is divisible by d. Thus, when $d > 1$, the morphism π provides an example where

$$\gcd\big\{n_Y(y)\big[k(y) : k(x_0)\big] \mid y \in f^{-1}(x_0), y \text{ closed}\big\} = \delta\big(E/k(x_0)\big) < n_X(x_0).$$

Corollary 7.3 *Let X be a universally catenary Noetherian scheme. Let $x_0 \in X$ be a point with $\dim \mathcal{O}_{X,x_0} \geq 1$. Let $f : Y \to X$ be a proper birational morphism such that $f^{-1}(x_0)$ is contained in the regular locus of Y. Let $E := Y \times_X \operatorname{Spec} k(x_0)$. Then $n_X(x_0) = \delta(E/k(x_0))$.*

In particular, if X is an integral excellent scheme of dimension 1, and if $f : Y \to X$ is the normalization morphism, then $n_X(x_0) = \gcd\{[k(y) : k(x_0)] \mid y \in f^{-1}(x_0)\}$.

Proof The statement follows immediately from the previous theorem, since $n_Y(y) = 1$ for any $y \in f^{-1}(x_0)$ because y is regular on Y by hypothesis. □

Corollary 7.4 *Let K be any field. Let V/K be a regular closed integral subscheme of $\mathbb{P}^n/K$, and denote by W/K a cone over V/K in $\mathbb{P}^{n+1}/K$. Let w_0 denote the vertex of W. Then $\delta(V/K) = n(\mathcal{O}_{W,w_0}) = \gamma(\mathcal{O}_{W,w_0})$.*

Proof Let $Z \to W$ denote the blow-up of the vertex w_0 of the cone W. It is well-known that the exceptional divisor E of $Z \to W$ is isomorphic to V/K. When V/K is regular, the points of E are regular on Z. Theorem 5.6, along with 7.3, shows that $\delta(V/K) = n(\mathcal{O}_{W,w_0}) = \gamma(\mathcal{O}_{W,w_0})$. □

Keep the notation of Corollary 7.4. Consider the sets

$$\mathcal{D}(V/K) := \big\{\deg_K(P), P \text{ closed point of } V\big\}$$

and

$$\mathcal{E}(\mathcal{O}_{W,w_0}) := \big\{e(Q, \mathcal{O}_{W,w_0}), Q \text{ is primary}\big\}.$$

Corollary 7.4 shows that $\gcd(d, d \in \mathcal{D}) = \gcd(e, e \in \mathcal{E})$. Let us note here that the sets $\mathcal{D}$ and $\mathcal{E}$ can be very different. For instance, it is known that $e(\mathfrak{m}_{W,w_0}) = \deg(V)$, and this integer is the minimal element in $\mathcal{E}$. At the same time, it may happen that $V(K) \neq \emptyset$, so that $1 \in \mathcal{D}(V)$.

If Q is a primary ideal of $\mathcal{O}_{W,w_0}$ generated by a system of parameters, then any positive multiple of $e(Q, \mathcal{O}_{W,w_0})$ belongs to $\mathcal{E}$ ([53], 11.2.9). When K is infinite, this statement remains true for all $\mathfrak{m}_{w_0}$-primary ideals of $\mathcal{O}_{W,w_0}$ (use [56], Thm. 22). In view of the property of the set $\mathcal{D}$ in Proposition 7.5, one may wonder whether an analogous property holds for the set $\mathcal{E}$.

In the next proposition, we call a field K *Hilbertian* if every separable Hilbert set of K is non-empty. This is the definition adopted in [17], 12.1, but not the one used in [35], page 225, for instance. Both definitions agree when $\mathrm{char}(K) = 0$, such as in the case of number fields, and the reader will see that the proof of 7.5 is simpler in this case.[5] A scheme V/K of finite type is called *generically smooth* if it contains a dense open subset U/K which is smooth over K.

Proposition 7.5 *Let K be a Hilbertian field. Let V/K be an irreducible regular generically smooth algebraic variety of positive dimension. Then there exists $n_0 > 0$ such that*

$$\{n\delta(V/K), n \geq n_0\} \subseteq \mathcal{D}(V/K).$$

Proof Let K' denote the subfield of elements of $K(V)$ algebraic over K. Then V/K' is geometrically irreducible and $\delta(V/K) = [K' : K]\delta(V/K')$ (see 6.9). It is thus sufficient to prove the proposition when V/K is geometrically irreducible. Choose an affine open subset U of V. Then $\delta(U/K) = \delta(V/K)$ (6.8). We can thus find finitely many closed points $P_1, \ldots, P_r$ in U such that $\gcd(\deg_K(P_i), i = 1, \ldots, r) = \delta(V/K)$. In case $\mathrm{char}(K) > 0$, we use 9.2 to show that we can assume that each point P_i has its residue field $K(P_i)$ separable over K. Since U/K is quasi-projective, we can use [43], 2.3, and find a geometrically integral curve C/K in U which contains $P_1, \ldots, P_r$ in its smooth locus C^{sm} (the proof of [43], 2.3, uses Bertini's Theorem in [28], where the only hypothesis on K is that it is infinite). Let C'/K denote a regular projective curve containing C^{sm} as a dense open subset. Clearly, $\delta(C'/K) = \delta(C^{sm}/K)$ and $\delta(C^{sm}/K)$ divides $\gcd(\deg_K(P_i), i = 1, \ldots, r) = \delta(V/K)$.

Since $C' \setminus C^{sm}$ is a finite set, it suffices to prove the proposition in the case where V/K is the smooth locus of a regular projective geometrically integral curve $\overline{V}$, of arithmetical genus $g(\overline{V}) = \dim H^1(\overline{V}, \mathcal{O}_{\overline{V}})$.

[5] The case of inseparable extensions requires careful consideration (see for instance the Notes on page 230 of [17]). We also note that Proposition 5.2 in Chapter 9 of [35], page 240, requires further hypotheses to ensure that the element y in its proof exists.

Let again $P_1, \dots, P_r$ in V be such that $\gcd(\deg_K(P_i), i = 1, \dots, r) = \delta(V/K)$. Clearly, every large enough integer multiple j of $\delta(V/K)$ can be written as $j = \sum_{i=1}^r x_i \deg_K(P_i)$ with $x_i \geq 0$. Suppose that $j \geq 2g(\overline{V}) - 1 + \max_i(\deg_K(P_i))$. Then by the Riemann-Roch Theorem there exists a function $f \in H^0(\overline{V}, \sum_{i=1}^r x_i P_i)$ which does not belong to $H^0(\overline{V}, \mathcal{O}(D))$ for any effective $D < (\sum_{i=1}^r x_i P_i)$. It follows that f defines a morphism $f : \overline{V} \to \mathbb{P}^1$ over K of degree equal to j. If the induced extension $K(\overline{V})/K(\mathbb{P}^1)$ is separable, then our assumption that K is Hilbertian implies that f has irreducible fibers, and so, that points of degree j on V exist.

If $\mathrm{char}(K) > 0$, we can modify the argument as follows. Consider a closed point P_0 of $\overline{V}$ with residue field $K(P_0)$ separable over K, and such that P_0 is distinct from $P_1, \dots, P_r$. Then every large enough integer multiple j of $\delta(V/K)$ can be written as $j = \deg_K(P_0) + \sum_{i=1}^r x_i \deg_K(P_i)$ with $x_i \geq 0$. As before, when $j \geq 2g(\overline{V}) - 1 + \max(\deg_K(P_i), i = 0, \dots, r)$, we use the Riemann-Roch theorem to define a morphism $f : \overline{V} \to \mathbb{P}^1$ over K of degree equal to j. This morphism now has the property that the induced extension $K(\overline{V})/K(\mathbb{P}^1)$ is separable, and we can conclude using the fact that K is Hilbertian, as above. □

Remark 7.6 The statement 7.5 does not hold if V/K is not assumed to be regular. Indeed, consider the curve $X/\mathbb{Q}$ defined by $x^2 + y^2 = 0$. Then $\delta(X/\mathbb{Q}) = 1$, and $(0, 0)$ is the only point of odd degree.

Remark 7.7 When X/F is a smooth proper geometrically connected curve of genus $g > 1$ over any field F, slightly more can be said about the set $\mathcal{D}(X/F)$. Since the canonical class can be represented by an effective divisor, the curve X/F has at least one point of degree at most $2g - 2$. Let Q be a point of minimal degree on X.

We claim that there exists a divisor $\sum_{i=1}^s a_i P_i$ of degree $\delta(X/F)$ such that for all $i = 1, \dots, s$,

$$\deg(P_i) < g + \deg(Q) \leq 3g - 2.$$

Indeed, let $D := \sum_{i=1}^s a_i P_i$ be a divisor of degree $\delta(X/F)$ such that $\max_i(\deg(P_i))$ is minimum among all such divisors. We may clearly choose such a divisor D such that the degrees of the closed points in the support of D are pairwise distinct. After reordering if necessary, we may assume that $\deg(P_1) < \cdots < \deg(P_s)$. If $\deg(P_s) \geq \deg(Q) + g$, then by the Riemann-Roch theorem, the divisor $P_s - Q$ is equivalent to an effective divisor $\sum_{j=1}^k b_j Q_j$, with $b_j > 0$ and $\deg(Q_j) < \deg(P_s)$. Replacing P_s by $Q + \sum_{j=1}^k b_j Q_j$ in D contradicts the minimality of $\max_i(\deg(P_i))$.

Suppose now that there exists an algorithm which determines, given an irreducible variety V/F, whether V/F has an F-rational point. Then there *exists an algorithm which determines the index $\delta(X/F)$ of a smooth proper geometrically irreducible curve X/F of genus g.* Indeed, for a given $d \geq 1$, consider the quotient $X^{(d)}/F$ of the d-fold product X^d/F by the action of the symmetric group S_d acting by permutation on X^d. Then a F-rational point on $X^{(d)}$ corresponds to a point of X defined over an extension L/F of degree $[L:F]$ dividing d. To compute $\delta(X/F)$, it suffices to determine whether $X^{(d)}/F$ has a F-rational point for $d = 1, \dots, 3g-3$.

Example 7.8 It goes without saying that most often, the explicit determination of the set $\mathcal{D}(X/F)$ is a very difficult problem. Consider for instance the Fermat curve $X_p/\mathbb{Q}$ given by the equation $x^p + y^p = z^p$, with $p > 3$ prime. This curve has obvious points of degrees 1, $p-1$, and p. Points of degree 2 on the line $x + y = z$ were noted already by Cauchy and Liouville ([55], Introduction). Indeed, a point $(x:y:1)$ on the intersection of X_p with the line $x + y = z$ satisfies:

$$x^p + (1-x)^p - 1 = x(x-1)\left(x^2 - x + 1\right)^b E_p(x),$$

with $E_p(x) \in \mathbb{Z}[x]$, and $b = 1$ if $p \equiv 2 \pmod 3$ and $b = 2$ if $p \equiv 1 \pmod 3$. Mirimanoff conjectured in 1903 that $E_p(x)$ is irreducible over $\mathbb{Q}$ for all primes $p \geq 11$. Klassen and Tzermias conjecture in [30] that any point P on $X_p/\mathbb{Q}$ of degree at most $p-2$ lies on the line $x + y = z$. Putting these two conjectures together when $p \geq 11$, we would find that

$$\mathcal{D}(X_p/\mathbb{Q}) = \left\{1, 2, \deg\left(E_p(x)\right), p-1, p, \dots\right\}.$$

Example 7.9 Let A be any Noetherian local ring of dimension $d > 0$. Set $X = \operatorname{Spec} A$, with closed point x_0. Our next example shows that $n_X(x_0)$ is not bounded when d is fixed.

Let k be a field which has finite extensions of any given degree. Let $Y = \operatorname{Spec} B$ be an affine normal integral algebraic variety of dimension $d > 0$ over k. Let $y_0 \in Y$ be a closed point corresponding to a maximal ideal $\mathfrak{m}$ of B. Let $r \geq 1$, let $A := k + \mathfrak{m}^r$, and let $X = \operatorname{Spec}(A)$. Let $\pi : Y \to X$ be the induced morphism. Then π is finite birational, $x_0 := \pi(y_0) \in X(k)$, and $\pi : Y \setminus \{y_0\} \to X \setminus \{x_0\}$ is an isomorphism. Theorem 7.1 shows that $n_X(x_0) = n_Y(y_0)[k(y_0):k]$. A straightforward computation shows that $e(\mathcal{O}_{X,x_0}) = e(\mathcal{O}_{Y,y_0})[k(y_0):k]r^{\dim B}$.

Remark 7.10 Let $(A, \mathfrak{m})$ be a Noetherian local ring of positive dimension. Let $s(A)$ denote the smallest positive integer s such that there exist a reduced closed one-dimensional subscheme C in $\operatorname{Spec} A$, and $f \in \mathcal{K}_C^*(C)$, with $s = \operatorname{ord}_{\mathfrak{m}}(f)$. In other words, $s(A)$ is the order of the class of $[\mathfrak{m}]$ in the Chow group $\mathcal{A}(\operatorname{Spec} A)$, and in the notation of 5.4, $s(A) = n(\operatorname{Spec} A, A)$. It is clear that $s(A)$ divides $n(A)$, and we note in the example below that it may happen that $s(A) < n(A)$.

Let $k := \mathbb{R}$. Consider the projective plane curve C/k given by the equation $y^2z + x^2z + x^3 = 0$. Let X/k denote the affine cone over C in $\mathbb{A}^3/k$. Let $c_0 \in C$ denote the singular point corresponding to $(0:0:1)$. The preimage of c_0 in the normalization of C consists of a single point with residue field $\mathbb{C}$. Hence, it follows from 7.3 that $n_C(c_0) = 2$. Let now $x_0 := (0, 0, 1) \in X(k)$. Using a desingularization of x_0 and 7.3, we find that $n_X(x_0) = 2$. We claim that $s(\mathcal{O}_{X,x_0}) = 1$. Indeed, the ring $\mathcal{O}_{X,x_0}$ is the localization at $(x, y, z-1)$ of the ring $k[x, y, z]/(y^2z + x^2z + x^3)$. The ideal (x, y) defines a closed subscheme of $\operatorname{Spec} \mathcal{O}_{X,x_0}$ on which $\operatorname{ord}_{x_0}(z-1) = 1$.

We conclude this section with a further study of the integer $n_X(x_0)$ which will not be used in the remainder of this article.

Proposition 7.11 *Let X be a Noetherian scheme and let $x_0 \in X$ be a point with* $\dim \mathcal{O}_{X,x_0} \geq 1$. *Let $\Gamma_1, \ldots, \Gamma_r$ be the irreducible components of X passing through x_0, each endowed with the structure of integral subscheme. Then*

$$n_X(x_0) = \gcd\{n_{\Gamma_i}(x_0) \mid 1 \leq i \leq r\}.$$

Proof We can suppose that X is local with closed point x_0. Let U be a dense open subset of X and let $i \leq r$. Then $U \cap \Gamma_i$ is dense in Γ_i. So $n_{\Gamma_i}(x_0) = \operatorname{ord}(f)$ for some $f \in \mathcal{K}_C^*(C)$ where C is a reduced curve in Γ_i, with $\max(C) \subseteq U \cap \Gamma_i \subseteq U$. Hence. $n_X(x_0)$ divides $n_{\Gamma_i}(x_0)$ for all $i \leq r$.

For each i, fix a dense open subset U_i of $\Gamma_i \setminus \bigcup_{j \neq i} \Gamma_j$ such that $n_{\Gamma_i}(x_0) = n(U_i, \Gamma_i)$ (see 5.5). Let U be a dense open subset of X such that $n_X(x_0) = n(U, X)$. Replacing each U_i by $U_i \cap U$ and U by $\bigcup_i (U_i \cap U)$ if necessary, we can suppose that $U := \bigcup_i U_i$ satisfies $n_X(x_0) = n(U, X)$. Let C be a reduced curve in X such that $\max(C) \subseteq U$ and $n_X(x_0) = \operatorname{ord}(f)$ for some $f \in \mathcal{K}_C^*(C)$. Let $C_i = \Gamma_i \cap C$. Then $n_X(x_0) = \sum_i \operatorname{ord}(f_i)$ where f_i is the image of f in $\mathcal{K}_{C_i}^*(C_i)$. As $\max(C_i) \subseteq U_i$, we have $n_{\Gamma_i}(x_0) = n(U_i, \Gamma_i) \mid \operatorname{ord}(f_i)$. Therefore $n_X(x_0)$ is divisible by the gcd of the $n_{\Gamma_i}(x_0)$'s. □

Remark 7.12 Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of A, and assume that $\dim A/\mathfrak{p}_i = d$ if and only if $i \leq s$. Proposition 5.2 implies that

$$\gamma(A) = \gcd\{\ell(A_{\mathfrak{p}_i})\gamma(A/\mathfrak{p}_i) \mid 1 \leq i \leq s\}.$$

Proposition 7.11 implies that

$$n(A) = \gcd\{n(A/\mathfrak{p}_i) \mid 1 \leq i \leq r\}.$$

We produced in 5.6 a class of rings where $n(A) = \gamma(A)$. The above formulas indicate that the hypotheses in 5.6 that A be equidimensional ($s = r$) and that $\mathfrak{p}_i \in \mathrm{Reg}(A)$ for all $i = 1, \dots, r$, are optimal.

Our next proposition strengthens Corollary 6.7 when $n_X(x_0) < \gamma(\mathcal{O}_{X,x_0})$.

Proposition 7.13 *Let X/k be a reduced scheme of finite type over a field k and let $x_0 \in X$ be a closed point with* $\dim \mathcal{O}_{X,x_0} \geq 1$. *Then $\delta(X^{\mathrm{reg}}/k)$ divides $n_X(x_0)\deg_k(x_0)$.*

Proof Let $\Gamma_1, \dots, \Gamma_r$ be the irreducible components of X passing through x_0, each endowed with the structure of integral subscheme. Since $\delta(X^{\mathrm{reg}}/k)$ divides $\delta(\Gamma_i^{\mathrm{reg}}/k)$ for all $i = 1, \dots, r$, we find from 7.11 that it suffices to prove the statement when X is irreducible. When X is irreducible, $n_X(x_0) = \gamma(\mathcal{O}_{X,x_0})$ (5.6), and the results follows immediately from Corollary 6.7. □

Proposition 7.14 *Let A be Noetherian excellent local ring* (*e.g., the local ring at some point of a scheme of finite type over a field*) *with* $\dim(A) \geq 1$. *If the residue field k of A is algebraically closed, then $n(A) = 1$.*

Proof Let $X := \operatorname{Spec} A$, with closed point x_0. Proposition 7.11 shows that it suffices to prove the statement when X is integral. Let U be a dense open subset of $\operatorname{Spec} \mathcal{O}_{X,x_0}$. There exists an integral curve C in X that contains x_0 and some point of U (Theorem 4.5). Consider the normalization map $\tilde{C} \to C$. Applying 7.3 to this map and using the fact that k is algebraically closed, we find that $n_C(x_0) = 1$. By definition, there exists $f \in \mathcal{K}_C(C)^*$ such that $\mathrm{ord}(f) = 1$. It follows from the definitions that $n_X(x_0) = 1$. □

Proposition 7.15 *Let X/k be a scheme of finite type over a field k. Let $x_0 \in X$ be a k-rational closed point with* $\dim(\mathcal{O}_{X,x_0}) \geq 1$. *For any extension F/k, denote by $x_{0,F}$ the unique preimage of x_0 under the natural base change map $X_F \to X$. Then there exists a finite extension F/k such that $n_{X_F}(x_{0,F}) = 1$. Moreover, any such extension F/k has degree divisible by $n_X(x_0)$.*

Proof Assume first that X/k is geometrically integral. We find then that for any extension F/k, $n_{X_F}(x_{0,F}) = \gamma(\mathcal{O}_{X_F,x_{0,F}})$ (5.6). Let $\overline{k}/k$ denote an algebraic closure of k. Since 7.14 shows that $n_{X_{\overline{k}}}(x_{0,\overline{k}}) = 1$, we can find ideals of definition $Q_1, \dots, Q_r \subset \mathcal{O}_{X_{\overline{k}},x_{0,\overline{k}}}$ such that $\gcd_i(e(Q_i, \mathcal{O}_{X_{\overline{k}},x_{0,\overline{k}}})) = 1$. For each $i = 1, \dots, r$, choose a system of generators for Q_i, and denote by F/k the extension of k generated by the coefficients of the rational functions needed to define all these generators. Thus, the generators of Q_i are elements of $\mathcal{O}_{X_F,x_{0,F}}$, and generate in this ring an ideal of definition that we shall denote by P_i. To conclude the proof, it suffices to note that $e(Q_i, \mathcal{O}_{X_{\overline{k}},x_{0,\overline{k}}}) = e(P_i, \mathcal{O}_{X_F,x_{0,F}})$.

Assume now that X/k is not geometrically integral. First make a finite extension k'/k such that each irreducible component of $X_{k'}/k'$ is geometrically integral. Let x'_0 be the preimage of x_0 under $X_{k'} \to X$. For each irreducible component Γ/k' of $X_{k'}/k'$ passing through x'_0, we can find using our earlier argument an extension F_Γ/k' such that the moving multiplicity of the unique preimage of x'_0 under $\Gamma_{F_\Gamma} \to \Gamma$ is equal to 1. Let F/k be one of the extensions F_Γ. It follows then from Proposition 7.11 that the moving multiplicity of the preimage of x'_0 under $X_F \to X_{k'}$ is equal to 1.

Let now F/k such that $n_{X_F}(x_{0,F}) = 1$. Consider the finite flat base change $f : X_F \to X$, with $f(x_{0,F}) = x_0$ and $k(x_0) = k$. Theorem 7.1 (c) shows that $n_X(x_0)$ divides $[F:k]n_{X_F}(x_{0,F})$. It follows that the degree of the extension F/k is divisible by $n_X(x_0)$. □

As the last paragraph of the above proof indicates, when $n_X(x_0) > 1$ and F/k is separable and non-trivial, the étale morphism $f : X_F \to X$, with $f(x_{0,F}) = x_0$, is such that $n_{X_F}(x_{0,F}) < n_X(x_0)$. This example shows that the hypothesis on the residue fields of the points x_0 and y_0 is necessary in (b) of our next proposition.

Proposition 7.16 *Let X be a Noetherian scheme, and let $f : Y \to X$ be a smooth morphism. Let $x_0 \in X$ with* $\dim \mathcal{O}_{X,x_0} \geq 1$ *and let y_0 be any point in the fiber $f^{-1}(x_0)$. Then*

(a) *$n_Y(y_0)$ divides $n_X(x_0)$.*
(b) *Assume that X is universally catenary. If $k(x_0) \to k(y_0)$ is an isomorphism, then $n_Y(y_0) = n_X(x_0)$.*
(c) *Assume that X is universally catenary. Let $g : Z \to X$ be any smooth morphism with $z_0 \in g^{-1}(x_0)$ such that $k(z_0)$ is isomorphic over $k(x_0)$ to $k(y_0)$. Then $n_Z(z_0) = n_Y(y_0)$.*

Proof (a) Since by definition $n_X(x_0) = n(\mathcal{O}_{X,x_0})$, we can assume that X is local with closed point x_0 by replacing f, if necessary, by $Y \times_X \operatorname{Spec} \mathcal{O}_{X,x_0} \to \operatorname{Spec} \mathcal{O}_{X,x_0}$. We can also assume that $Y \to X$ is of finite type.

Since f is smooth at y_0, replacing Y by an open neighborhood of y_0 in Y if necessary, we can suppose that f factors as the composition of an étale map $Y \to \mathbb{A}^d_X$ followed by the canonical projection $\mathbb{A}^d_X \to X$. Thus it is enough to prove (a) separately for étale morphisms and for the projections $\mathbb{A}^d_X \to X$ for all $d \geq 1$.

We now make one further reduction, to the case where X is local integral of dimension 1. This paragraph and the next only assume that f is flat. Consider the natural morphism $i : \operatorname{Spec} \mathcal{O}_{Y,y_0} \to Y$. Using Lemma 5.5, we can find a dense open set U' of Y such that $n(\mathcal{O}_{Y,y_0}) = n(i^{-1}(U'), \mathcal{O}_{Y,y_0})$. Note as in 5.5 that for any dense open subset $U_1 \subset U'$, we have $n(\mathcal{O}_{Y,y_0}) =$

$n(i^{-1}(U_1), \mathcal{O}_{Y,y_0})$. We now choose a dense open subset $U \subseteq U'$ such that U is *fiberwise empty or dense in Y over X*, that is, such that for all $x \in X$, U_x is either empty or dense in Y_x. Indeed, first use [23], IV, 9.7.8 (applied to each irreducible component of X), to find a dense open subset X_1 of X such that for each $x \in X_1$, say belonging to an irreducible component $\overline{\{\eta\}}$ of X, the number of geometric irreducible components of Y_x is equal to the number of geometric irreducible components of Y_η. Note now that f is flat, so the image $f(U')$ of U' in X is open. Apply then again [23], IV, 9.7.8, to the morphism $f_{|U'} : U' \to f(U')$ and find an appropriate dense open subset V_1 of $f(U')$ with similar conditions on the fibers. It follows that $W := V_1 \cap X_1$ is a dense open subset of X, and we take $U := f_{|U'}^{-1}(V_1 \cap X_1) \subset U'$. Note that $f(U) = W$.

Recall that $n(U, \mathcal{O}_{Y,y_0}) = n_Y(y_0)$, and that $n(W, \mathcal{O}_{X,x_0})$ divides $n_X(x_0)$. To prove that $n_Y(y_0)$ divides $n_X(x_0)$, it suffices to prove that $n(U, \mathcal{O}_{Y,y_0})$ divides $n(W, \mathcal{O}_{X,x_0})$. Thus, it is enough, given any integral curve C in X containing x_0 and whose generic point η belongs to W, and given any non-zero regular function $g \in \mathcal{O}(C)$, to find a one dimensional closed subscheme D of $\operatorname{Spec} \mathcal{O}_{Y,y_0}$, contained in $Y \times_X C$ with maximal points in U, and an element $g_1 \in \mathcal{K}_D^*(D)$ such that $\operatorname{ord}_{y_0}(g_1)$ divides $\operatorname{ord}_{x_0}(g)$. As $U \times_X \eta$ is dense in $Y \times_X \eta$, we find that $U \times_X C$ is a dense open subscheme of $Y \times_X C$. Therefore, to prove that $n_Y(y_0)$ divides $n_X(x_0)$, it is enough to prove that $n_{Y \times_X C}(y_0)$ divides $n_C(x_0)$; indeed, then there exists such a D and g_1 with $\operatorname{ord}_{y_0}(g_1) = n_{Y \times_X C}(y_0)$, and this latter integer divides $n_C(x_0)$, which by definition itself divides $\operatorname{ord}_{x_0}(g)$.

Returning to the case where f is smooth as in the proposition, we find that it suffices to prove (a) when X is local integral of dimension 1 with generic point η. Now suppose that f is étale. We keep the notation U and W of the previous paragraphs. Suppose given a nonzero regular function g on X (X can now play the role of C). Let $D := \operatorname{Spec} \mathcal{O}_{Y,y_0}$. As U_η is dense in the discrete set Y_η, it is equal to Y_η. Hence, the maximal points of D belong to U. Let $g_1 := f^*(g)$. Then $\operatorname{ord}_{y_0}(g_1) = \operatorname{ord}_{x_0}(g)$ because f is étale, and (a) is proved in this case.

Suppose now that f is a projection morphism $\mathbb{A}_X^d \to X$. An easy induction in d reduces the problem to the case $d = 1$. Consider now $f : Y := \mathbb{A}_X^1 \to X$. Assume first that y_0 is a closed point of Y. Write $X = \operatorname{Spec} R$, with R a local domain with maximal ideal $\mathfrak{m}$. Fix $g \in R \setminus \{0\}$. Let $F(T) \in R[T]$ be a monic polynomial which lifts a generator of the maximal ideal of $k(x_0)[T]$ corresponding to $y_0 \in Y_{x_0}$. Let $S := R[T]/(F(T))$, and $D := \operatorname{Spec} S$. Then S is finite and flat over R, with $S/\mathfrak{m}S = k(y_0)$ a field, so that S is local. Let $g_1 = g \in S$. The element g_1 is regular by flatness and we have

$$\operatorname{ord}_{y_0}(g_1) = \ell_S\big(S/(g_1)\big) = \ell_S(S \otimes_R R/gR) = \ell_R(R/gR) = \operatorname{ord}_{x_0}(g),$$

where the third equality holds by flatness and because $S/\mathfrak{m}S = k(y_0)$ is a field (see [21], A.4.1, or [39], Exercise 7.1.8(b)). To conclude the proof, we need to further specify the element $F(T)$, so that D is such that $D_\eta \subset U$. For this we modify a given $F(T)$ by an element $\lambda \in \mathfrak{m}$ as follows. Write $Y_\eta \setminus U_\eta$ as $V(\phi(T))$ for some non-zero $\phi(T) \in \mathrm{Frac}(R)[T]$. Since $\mathfrak{m}$ is infinite, we can find $\lambda \in \mathfrak{m}$ such that $\gcd(F(T)+\lambda, \phi(T)) = 1$ in $\mathrm{Frac}(R)[T]$. Then we take the curve D defined by $F(T)+\lambda$.

Let us consider now the case where y_0 is the generic point of the fiber of $\mathbb{A}^1_X \to X$ above x_0. As before, let $R := \mathcal{O}_{X,x_0}$, with maximal ideal $\mathfrak{m}$. Then $\mathcal{O}_{Y,y_0} = R[T]_{\mathfrak{m}R[T]} =: R(T)$. Let I be an ideal in R. Then I is $\mathfrak{m}$-primary if and only if $IR(T)$ is $\mathfrak{m}R(T)$-primary, and when I is $\mathfrak{m}$-primary, then $e(I,R) = e(IR(T), \mathfrak{m}R(T))$ (as stated for instance in [53], 8.4.2). It follows that $\gamma(R(T))$ divides $\gamma(R)$. Since $n(R(T))$ divides $\gamma(R(T))$, we find that $n(R(T))$ divides $n(R)$ when $n(R) = \gamma(R)$. When R is a local domain of dimension 1, the hypotheses of 5.6 are satisfied and $n(R) = \gamma(R)$.

(b) When X is universally catenary, we can apply Theorem 7.1 (c) (since f is open, the hypothesis on generic points is satisfied) and obtain that $n_X(x_0)$ divides $n_Y(y_0)[k(y_0):k(x_0)]$. This gives the divisibility relation $n_X(x_0) \mid n_Y(y_0)$ when $[k(y_0):k(x_0)] = 1$.

(c) Consider the Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{\beta} & Y \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ Z & \longrightarrow & X. \end{array}$$

Let w denote a point of $W := Z \times_X Y$ above z_0 and y_0 inducing isomorphisms $k(z_0) \to k(w)$ and $k(y_0) \to k(w)$. The morphisms α and β are smooth, and it follows from (b) applied to α and β that $n_Z(z_0) = n_W(w) = n_Y(y_0)$. □

Example 7.17 We show in this example that there exist a local ring $\mathcal{O}$ and an ind-étale local extension $\mathcal{O}^h$ such that $\mathrm{Spec}\,\mathcal{O}^h \to \mathrm{Spec}\,\mathcal{O}$ induces an isomorphism on closed points, and such that $n(\mathcal{O}^h) < n(\mathcal{O})$. In our example, the local ring $\mathcal{O}$ is not universally catenary, while the ring $\mathcal{O}^h$ is the Henselization of $\mathcal{O}$, and is universally catenary.

Consider again the finite birational morphism $\pi : Y \to X$ described in Example 1.3. The scheme Y is regular, and X is catenary but not universally catenary. The preimage of the point $x_0 \in X$ consists in the two regular points y_0 and y_1 in Y, with $[k(y_0):k(x_0)] = d$, and $[k(y_1):k(x_0)] = 1$. It follows from the discussion in 1.3 that $n_X(x_0)$ is divisible by d. Let $\mathcal{O} := \mathcal{O}_{X,x_0}$, and consider the Henselization $\mathcal{O}^h$ of $\mathcal{O}_{X,x_0}$. It is known that a Noetherian local Henselian ring of dimension 2 is universally catenary (see, e.g, [47], 2.23 (i), with for instance, [53], B.5.1). We let x_0' denote the closed point of Spec $\mathcal{O}^h$. We have the following diagram with Cartesian squares:

$$\begin{array}{ccccc} Y_{\mathcal{O}^h} & \longrightarrow & Y_{\mathcal{O}} & \longrightarrow & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} \mathcal{O}^h & \longrightarrow & \operatorname{Spec} \mathcal{O} & \longrightarrow & X. \end{array}$$

We use 7.1, (b) and (c), on π' to conclude that

$$\gcd\{n_{Y_{\mathcal{O}^h}}(y)[k(y):k(x_0')] \mid y \in \pi'^{-1}(x_0'), y \text{ closed}\} = n(\mathcal{O}^h) = 1.$$

Thus, when $d > 1$, we find that $1 = n(\mathcal{O}^h) < n(\mathcal{O})$.

Remark 7.18 It is easy to construct examples of flat morphisms $f : Y \to X$ with $y_0 \in Y$ such that $n_Y(y_0) > n_X(f(y_0))$. Indeed, start with an integral scheme Y of finite type over a field k with a closed point $y_0 \in Y$ such that $n_Y(y_0) > 1$. Choose any non-constant function $f : Y \to \mathbb{P}^1_k$. This morphism is then flat, and since $\mathbb{P}^1_k$ is regular, $n_{\mathbb{P}^1_k}(f(y_0)) = 1$.

8 Index of varieties over a discrete valuation field

8.1 Let K be the field of fractions of a discrete valuation ring $\mathcal{O}_K$, with maximal ideal (π) and residue field k. Let $S := \operatorname{Spec}(\mathcal{O}_K)$. Let $\mathcal{X}$ be an integral scheme, and let $f : \mathcal{X} \to S$ be a flat, separated, surjective morphism of finite type.

Since f is flat, $\operatorname{div}(\pi)$ is a Cartier divisor on $\mathcal{X}$, and we denote its associated cycle by $[\operatorname{div}(\pi)] = \sum_{i=1}^n r_i \Gamma_i$. Each Γ_i is an integral variety over k, of multiplicity r_i in $\mathcal{X}_k$. The generic fiber of $\mathcal{X}/S$ is denoted by X/K.

For any irreducible 1-cycle C (endowed with the reduced induced structure) on $\mathcal{X}$ and for any Cartier divisor D on $\mathcal{X}$ whose support does not contain C, the restriction $D|_C$ is again a Cartier divisor on C. The associated cycle $[D|_C]$ is supported on the special fiber of $\mathcal{X} \to S$. Writing $[D|_C] = \sum_{x \text{ closed}} (D.C)_x[x]$, then its degree over k is $\deg_k[D|_C] = \sum_x (D.C)_x \deg_k(x)$. Note that if D is effective, then for any closed point $x \in C \cap \operatorname{Supp} D$, $(D.C)_x = \ell(\mathcal{O}_{C,x}/\mathcal{O}_C(-D)_x) \geq 1$.

Assume that $C \to S$ is finite. Then C is the closure in $\mathcal{X}$ of a closed point $P \in X$, and we have

$$\deg_K(P) = \deg_k(\operatorname{div}(\pi)|_C).$$

Assume now that $\mathcal{X}$ is locally factorial (e.g., regular). Then the Weil divisors Γ_i are Cartier divisors (again denoted by Γ_i) and we can write that $\mathrm{div}(\pi) = \sum_{i=1}^n r_i \Gamma_i$ as Cartier divisors. It follows that

$$\deg_K(P) = \sum_{x \in \mathcal{X}_k \cap C} \left(\sum_{\Gamma_i \ni x} r_i (\Gamma_i . C)_x \deg_k(x) \right). \tag{1}$$

In particular,

$$\gcd_i \{ r_i \delta(\Gamma_i / k) \} \quad \text{divides} \quad \overline{\delta}(X/K),$$

where $\overline{\delta}(X/K)$ denotes the greatest common divisor of the integers $\deg_K(P)$, with $P \in X$ closed, and whose closure in $\mathcal{X}$ is finite over S. This statement is sharpened in our next theorem.

When $f : \mathcal{X} \to S$ is proper, the closure of any closed point $P \in X$ is finite and flat over S and, thus, in this case, $\gcd_i\{r_i \delta(\Gamma_i/k)\}$ divides $\delta(X/K) = \overline{\delta}(X/K)$.

Theorem 8.2 *Let $f : \mathcal{X} \to S$ be as above, with $\mathcal{X}$ regular. Let X/K denote the generic fiber of $\mathcal{X}/S$.*

(a) *Then* $\gcd_i\{r_i \delta(\Gamma_i^{\mathrm{reg}}/k)\}$ *divides* $\overline{\delta}(X/K)$.
(b) *When $\mathcal{O}_K$ is Henselian, then* $\overline{\delta}(X/K) = \gcd_i\{r_i \delta(\Gamma_i^{\mathrm{reg}}/k)\}$.

Proof For ease of notation, we will write $\gcd(\mathcal{X}_k) := \gcd_i\{r_i \delta(\Gamma_i^{\mathrm{reg}}/k)\}$.

(a) Let P be a closed point of X whose closure in $\mathcal{X}$ is finite over S, and let us show that $\gcd(\mathcal{X}_k)$ divides $\deg_K(P)$. If $\overline{\{P\}} \cap \Gamma_i \subseteq \Gamma_i^{\mathrm{reg}}$ for all $i \le n$, then Formula (1) above shows that $\gcd(\mathcal{X}_k)$ divides $\deg_K(P)$. In general, though, $\overline{\{P\}}$ may intersect the singular locus of some Γ_i. If that is the case, then we can end the proof in two different ways.

The first method relies on Theorem 2.3, and assumes in addition that $\mathcal{X}$ is FA when $\mathcal{O}_K$ is not Henselian. Indeed, 2.3 shows that there exists an affine open subset V of $\mathcal{X}$ which contains the 1-cycle $\overline{\{P\}}$ and a 1-cycle C rationally equivalent to $\overline{\{P\}}$ in V, and whose support is proper over S and does not intersect the singular locus F of $(\mathcal{X}_k)_{\mathrm{red}}$. Then P is rationally equivalent on V_K to $C_{|V_K}$, whose support is a union of closed points of X. We claim that $\deg_K(P) = \deg_K C_{|X}$. Indeed, since V is affine, we can consider an open embedding $V \to \mathcal{Y}$ over S where $\mathcal{Y}/S$ is projective. Theorem 2.3 shows that $\overline{\{P\}}$ and C are closed and rationally equivalent in $\mathcal{Y}$. Then $\deg_K(P) = \deg_K C_{|\mathcal{Y}_K} = \deg_K C_{|V_K} = \deg_K C_{|X}$. The above discussion shows that each point in $\mathrm{Supp}\, C_{|X}$ has degree divisible by $\gcd(\mathcal{X}_k)$, so that $\gcd(\mathcal{X}_k)$ divides $\deg_K(P)$, as desired.

The second method relies in the end on Theorem 6.5 and its Corollary 6.7. It consists in a succession of blowing-ups, starting with the blowing-up of specializations of P in $\mathcal{X}$ as in Lemma 8.3 (3), to produce a new regular model $\mathcal{Y}$, where the specializations of P belong to the regular locus of $(\mathcal{Y}_k)_{\mathrm{red}}$. Our initial discussion above implies that $\gcd(\mathcal{Y}_k)$ divides $\deg_K(P)$. Then Lemma 8.3 (2) shows that $\gcd(\mathcal{Y}_k) = \gcd(\mathcal{X}_k)$, and (a) is proved.

(b) Let $\Gamma_i^0 := \Gamma_i^{\mathrm{reg}} \setminus \bigcup_{j \neq i} \Gamma_j$. Proposition 6.8 shows that $\delta(\Gamma_i^{\mathrm{reg}}/k) = \delta(\Gamma_i^0/k)$. We use then Proposition 8.4 (3) on each closed point of Γ_i^0 to find that $\overline{\delta}(X/K)$ divides $\gcd(\mathcal{X}_k)$. □

Lemma 8.3 *Let $f : \mathcal{X} \to S$ be as in 8.1, and assume $\mathcal{X}$ regular. Let $d := \dim X$. Let $x_0 \in \mathcal{X}_k$ be a closed point and let $\Gamma_1, \dots, \Gamma_s$ denote the irreducible components of $\mathcal{X}_k$ which contain x_0. Let e_i denote the Hilbert-Samuel multiplicity of x_0 on Γ_i. Consider the blowing-up $\tilde{\mathcal{X}} \to \mathcal{X}$ of $\mathcal{X}$ along the reduced closed subscheme $\{x_0\}$.*

(1) *The scheme $\tilde{\mathcal{X}}$ is regular, and the exceptional divisor E_1 in $\tilde{\mathcal{X}}$ is isomorphic to $\mathbb{P}^d_{k(x_0)}$. The multiplicity $r(E_1)$ of E_1 in $\tilde{\mathcal{X}}_k$ is $\sum_{i=1}^s r_i e_i$.*
(2) *We have $\gcd(\tilde{\mathcal{X}}_k) = \gcd(\mathcal{X}_k)$.*
(3) *Let P be a closed point of X. Then there exists a finite sequence*

$$\mathcal{X}_m \to \mathcal{X}_{m-1} \to \cdots \to \mathcal{X}_0 = \mathcal{X}$$

such that each morphism $\mathcal{X}_i \to \mathcal{X}_{i-1}$ is the blowing-up of a closed point in the special fiber, and such that the closure of P in $\mathcal{X}_m$ intersects $(\mathcal{X}_m)_k$ only in regular points of $((\mathcal{X}_m)_k)_{\mathrm{red}}$.

Proof (1) As we blow-up a regular scheme along the regular center $\operatorname{Spec} k(x_0)$, we have $\tilde{\mathcal{X}}$ regular and $E_1 \simeq \mathbb{P}^d_{k(x_0)}$ ([39], 8.1.19). In the regular local ring $\mathcal{O}_{\mathcal{X},x_0}$, factor a uniformizing element π of $\mathcal{O}_K$ as $\pi = u g_1^{r_1} \cdot \dots \cdot g_s^{r_s}$, where g_i is a local equation in $\mathcal{X}$ of the component Γ_i at x_0, and u is a unit. It is not hard to see that the Hilbert-Samuel multiplicity of x_0 on Γ_i is the positive integer e_i such that $g_i \in (\mathfrak{m}_{\mathcal{X},x_0})^{e_i} \setminus (\mathfrak{m}_{\mathcal{X},x_0})^{e_i+1}$. Since the associated graded ring $\bigoplus_{q \geq 0} (\mathfrak{m}^q_{\mathcal{X},x_0} / \mathfrak{m}^{q+1}_{\mathcal{X},x_0})$ is a polynomial ring over $k(x_0)$, we find that $\pi \in (\mathfrak{m}_{\mathcal{X},x_0})^{\sum_i r_i e_i} \setminus (\mathfrak{m}_{\mathcal{X},x_0})^{\sum_i r_i e_i + 1}$. Let ξ denote the generic point of E_1. Since $\tilde{\mathcal{X}}$ is regular, the local ring $\mathcal{O}_{\tilde{\mathcal{X}},\xi}$ is a discrete valuation ring with normalized valuation v_ξ, and the multiplicity of E_1 in $\tilde{\mathcal{X}}_k$ is $v_\xi(\pi)$. We leave it to the reader to check that if $f \in (\mathfrak{m}_{\mathcal{X},x_0})^r \setminus (\mathfrak{m}_{\mathcal{X},x_0})^{r+1}$, then $v_\xi(f) = r$.

(2) Let $\tilde{\Gamma}_i$ denote the strict transform of Γ_i in $\tilde{\mathcal{X}}$. The blowing-up $\tilde{\mathcal{X}} \to \mathcal{X}$ restricts to a morphism $\tilde{\Gamma}_i \to \Gamma_i$ which is nothing but the blowing-up of Γ_i along x_0. Hence, the varieties $\tilde{\Gamma}_i/k$ and Γ_i/k are birational and $\delta(\tilde{\Gamma}_i^{\mathrm{reg}}/k) = \delta(\Gamma_i^{\mathrm{reg}}/k)$ (6.8). Recall that

$$\gcd(\tilde{\mathcal{X}}_k) = \gcd_{1 \leq i \leq n} \left\{ r_i \delta\left(\tilde{\Gamma}_i^{\mathrm{reg}}/k\right), r(E_1)\delta(E_1/k) \right\}.$$

We thus find that $\gcd(\tilde{\mathcal{X}}_k)$ divides $\gcd(\mathcal{X}_k)$. Clearly, $\delta(E_1/k) = [k(x_0) : k] = \deg_k(x_0)$. Corollary 6.7 implies that for every Γ_i passing through x_0, $e_i \deg_k(x_0)$ is the degree of some 0-cycle supported in Γ_i^{reg} and, thus, $e_i \deg_k(x_0)$ is divisible by $\delta(\Gamma_i^{\text{reg}}/k)$. Therefore $\gcd(\mathcal{X}_k)$ divides $\gcd(\tilde{\mathcal{X}}_k)$.

(3) Let P be a closed point on X. We describe below how to obtain a new regular model using a sequence of blowing-ups along closed points

$$\mathcal{X}_m \to \mathcal{X}_{m-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}$$

such that the specializations of P in $\mathcal{X}_m$ are regular points in $((\mathcal{X}_m)_k)_{\text{red}}$.

Suppose $\overline{\{P\}}$ meets the singular locus F of $(\mathcal{X}_k)_{\text{red}}$. Let $\mathcal{Y} \to \mathcal{X}$ be the blowing-up of $\mathcal{X}$ along $\overline{\{P\}} \cap F$. Then $\mathcal{Y} \to \mathcal{X}$ is also obtained by successively blowing-up points of $\overline{\{P\}} \cap F$. If P specializes only to regular points of $\mathcal{Y}_k$, then the process stops. Otherwise, let y be a specialization of P in $\mathcal{Y}_k$ belonging to the singular locus of $(\mathcal{Y}_k)_{\text{red}}$. Then y belongs to an exceptional divisor E_1, of multiplicity $r(E_1)$, and to at least one strict transform $\tilde{\Gamma}_i$. Using these two facts and Formula (1), we find that

$$\deg_K(P) > r(E_1).$$

Consider the blowing-up $\mathcal{Z} \to \mathcal{Y}$ of $\mathcal{Y}$ along $\overline{\{P\}}$ intersected with the singular locus of $(\mathcal{Y}_k)_{\text{red}}$. The exceptional divisor E_2 above y has multiplicity $r(E_2)$ in $\mathcal{Z}_k$, and since $y \in E_1 \cap \tilde{\Gamma}_i$, Part (1) of this lemma implies that $r(E_2) > r(E_1)$. Repeating the above argument on E_2 shows that $\deg_K(P) > r(E_2)$. Therefore, the process must stop after at most $1 + \deg_K(P)$ steps. □

Let F be any field, and let W/F be a scheme of finite type. Recall that a closed point $P \in W$ is called *separable* if the residue field extension $F(P)/F$ is a separable extension.

Proposition 8.4 *Let $f : \mathcal{X} \to S$ be as in 8.1. Let $x_0 \in \mathcal{X}_k$ be a closed point, regular in $\mathcal{X}$. Let $\Gamma_1, \ldots, \Gamma_s$ be the irreducible components of $\mathcal{X}_k$ passing through x_0. Let $e_i := e(\mathcal{O}_{\Gamma_i, x_0})$ denote the Hilbert-Samuel multiplicity of x_0 on Γ_i. Then*

(1) *The Hilbert-Samuel multiplicity $e(\mathcal{O}_{\mathcal{X}_k, x_0})$ of x_0 on $\mathcal{X}_k$ is equal to $\sum_{i=1}^{s} r_i e_i$.*
(2) *Any closed point $P \in X$ such that $x_0 \in \overline{\{P\}}$ has $\deg_K(P) \geq e(\mathcal{O}_{\mathcal{X}_k, x_0}) \times \deg_k(x_0)$.*
(3) *Assume that $\mathcal{O}_K$ is Henselian. If k is infinite, or if x_0 is a regular point of $(\mathcal{X}_k)_{\text{red}}$, then there exists a closed point $P \in X$ such that $x_0 \in \overline{\{P\}}$ and*

$$\deg_K(P) = e(\mathcal{O}_{\mathcal{X}_k, x_0}) \deg_k(x_0).$$

If k is finite, then there exists a 0*-cycle on X of degree $e(\mathcal{O}_{\mathcal{X}_k,x_0})\deg_k(x_0)$ and such that each closed point in its support specializes to x_0 in $\mathcal{X}$.*

If X/K is generically smooth, then the point P and the support of the 0*-cycle can be chosen to be separable over K.*

Proof (1) This is the same computation as in Lemma 8.3 (1), or use 4.6.

(2) Let $\tilde{\mathcal{X}} \to \mathcal{X}$ be the blowing-up along x_0 with exceptional divisor E_1. We saw in 8.3 (1) that E_1 is isomorphic to $\mathbb{P}^d_{k(x_0)}$ and has multiplicity $e(\mathcal{O}_{\mathcal{X}_k,x_0})$ in $\tilde{\mathcal{X}}_k$. As P has a specialization in $\tilde{\mathcal{X}}$ belonging to E_1, Formula (1) before Theorem 8.2 shows that $\deg_K(P) \geq e(\mathcal{O}_{\mathcal{X}_k,x_0})\deg_k(x_0)$.

(3) Let us suppose first that x_0 is a regular point of $(\mathcal{X}_k)_{\mathrm{red}}$. Since $\mathcal{X}^{\mathrm{reg}}$ is open in $\mathcal{X}$ ([23] IV.6.12.6 (ii)), we may without loss of generality assume that $\mathcal{X} = \operatorname{Spec} A$ is affine, irreducible, and regular, with irreducible special fiber. Since $\mathcal{O}_{X,x_0}$ is factorial, we may if necessary replace $\mathcal{X}$ by an open dense subset and assume that the uniformizer π of $\mathcal{O}_K$ factors as $\pi = ut^e$ in A, with $t \in A$, $u \in A^*$ and $e := e(\mathcal{O}_{\mathcal{X}_k,x_0})$. By hypothesis, there exists a system of generators $\{f_1, \dots, f_d\}$ of the maximal ideal $\mathfrak{m}$ of $A/(t)$ corresponding to x_0 with $d = \dim(A/(t))$.

For $i = 1, \dots, d$, let $g_i \in A$ be any lift of f_i. Let T_0 be the closed subscheme of $\mathcal{X}$ defined by $T_0 := \operatorname{Spec} A/(g_1, \dots, g_d)$. Consider the induced morphism $\varphi : T_0 \to S$. Clearly, $\varphi^{-1}((\pi)) = \{x_0\}$. Let $T_1 \subseteq T_0$ denote the open subset (see [23], IV.13.1.4) consisting of all the points of T_0 where φ is quasi-finite. It is clear that $x_0 \in T_1$, and x_0 is in fact a regular point of T_1 since $A/(g_1, \dots, g_d, t)$ is a field. Using [23] IV.6.12.6 (ii) again, we find that the regular locus of T_1 is open, and we can thus if necessary replace T_1 by a open subset containing x_0 and assume that T_1 is regular. Since $\mathcal{O}_K$ is Henselian, we may use [5], 2.3, Proposition 4 (e), and obtain that there exists an open neighborhood T of x_0 in T_1 such that $T \to S$ is finite. Since both T and S are regular and $T \to S$ is finite, we find that $T \to S$ is also flat. The generic point P of $T = \overline{\{P\}}$ has thus degree $e\deg_k(k(x_0))$ over K, as desired.

Let us now assume in addition that X/K is generically smooth. We claim that we can find a lift $P \in X$ of x_0 with $K(P)/K$ separable of degree $e\deg_k(k(x_0))$ over K. Keep the assumptions in the first paragraph of the proof of (3) above. If $d = 0$, then the generic smoothness of X/K implies that the fraction field of A is separable over K, and our claim is true.

Assume now that $d \geq 1$, and let us show that the lifts g_i can be chosen such that the generic point of the associated T is separable over K. To start, we show that there exists a lift g_1 of f_1 such that the closed irreducible subscheme $V(g_1)$ of $\mathcal{X}$ is flat of finite type over S, with generic fiber generically smooth. We then conclude the proof by induction on the dimension.

Note that there always exists $h \in A$ such that the differential dh in $\Omega^1_{A\otimes K/K}$ is not zero. Start with any lift $g \in A$ of f_1. If the differential dg is zero, replace g by the lift $g + \pi h$ and assume now that $dg \neq 0$. Let Z be the proper closed subset of $\mathcal{X}$ where $\mathcal{X} \to S$ is not smooth. Then on $\mathcal{X} \setminus Z$, the sheaf of differentials is locally free, and we let Z' be the zero locus of dg in $\mathcal{X} \setminus Z$. By construction, Z' is closed in $\mathcal{X} \setminus Z$, and $Z \cup Z'$ contains only finitely many irreducible closed subsets of codimension 1 in $\mathcal{X}$.

The ideals $(g + \pi^s)$ of $A_{\mathfrak{m}}$, $s \in \mathbb{N}$, are infinitely many pairwise distinct prime ideals (recall that by construction (g) is a prime ideal of $A_{\mathfrak{m}}$; and the maximal ideal of $A_{\mathfrak{m}}$ is also generated by $t, g + \pi^s, g_2, \dots, g_d$). Therefore, we can choose $g_1 := g + \pi^s$ for some s such that in $\mathcal{X}$, $V(g_1)$ is not contained in $Z \cup Z'$. Since $V(g_1)$ is not contained in Z, we find that $V(g_1)$ intersects the smooth locus of $\mathcal{X} \to S$. The generic fiber of the morphism $V(g_1) \to S$ is then smooth at a point outside $Z \cup Z'$ since $d(g_1)$ is not zero at such a point. We conclude the proof of the existence of P with $K(P)/K$ separable when X/K is generically smooth by induction on the dimension.

In the general case where x_0 need not be regular in $(\mathcal{X}_k)_{\mathrm{red}}$, consider the blowing-up $\tilde{\mathcal{X}} \to \mathcal{X}$ of x_0 as in (2). If k is infinite, then there exists a $k(x_0)$-rational point x_1 in the interior of E_1, and so regular in $(\tilde{\mathcal{X}}_k)_{\mathrm{red}}$. By the above, there exists a closed point P of X such that $\deg_K(P) = e(\mathcal{O}_{\mathcal{X}_k, x_0}) \deg_k(x_0)$ and which specializes to x_1 in $\tilde{\mathcal{X}}$. Then P specializes to x_0 in $\mathcal{X}$, as desired.

If k is finite, then $\delta(Y/k) = 1$ for any geometrically irreducible algebraic variety Y over k ([37], Corollary 3. See also [15], 3.11). Therefore, there exists a 0-cycle supported in the interior of E_1, of degree 1 over $k(x_0)$. Then lift this 0-cycle to X as above. □

Variations on the statement of 8.4 (3) when x_0 is a regular point of $(\mathcal{X}_k)_{\mathrm{red}}$ can be found in [13], Lemma 2.3, or [5], Corollary 9.1/9. When we relax the hypothesis that $\mathcal{X}$ is regular in Theorem 8.2 (b), we obtain:

Proposition 8.5 *Let $\mathcal{O}_K$ be Henselian. Let $f : \mathcal{X} \to S$ be as in 8.1, with $\mathcal{X}$ regular in codimension one. Then $\delta(X^{\mathrm{reg}}/K)$ divides* $\gcd_i\{r_i \delta(\Gamma_i^{\mathrm{reg}}/k)\}$.

Proof Indeed, $\mathcal{X}^{\mathrm{reg}}$ is open in $\mathcal{X}$ ([23] IV.6.12.6 (ii)). Since $\mathcal{X}$ is regular in codimension one, $\mathcal{X} \setminus \mathcal{X}^{\mathrm{reg}}$ is of codimension ≥ 2. Thus, $\Gamma_i^{\mathrm{reg}} \cap \mathcal{X}^{\mathrm{reg}}$ is not empty, and $\delta(\Gamma_i^{\mathrm{reg}} \cap \mathcal{X}^{\mathrm{reg}}/k) = \delta(\Gamma_i^{\mathrm{reg}}/k)$ for all i (6.8). Then lift closed points of $\Gamma_i^{\mathrm{reg}} \cap \mathcal{X}^{\mathrm{reg}}$ to X^{reg} as in 8.4 (3). □

Remark 8.6 Let Γ/k be an integral normal algebraic variety. Let k' be the algebraic closure of k in the field of rational functions $k(\Gamma)$. Let $e(\Gamma/k')$ be the geometric multiplicity of Γ as k'-scheme ([5], 9.1/3). We claim that when Γ is regular,

$$e\big(\Gamma/k'\big) \quad \text{divides} \quad \delta\big(\Gamma/k'\big).$$

Indeed, let L/k' be a separable closure of k'. Then $e(\Gamma \times_{k'} L/L) = e(\Gamma/k')$, and $\delta(\Gamma \times_{k'} L/L)$ divides $\delta(\Gamma/k')$. Let $x_0 \in \Gamma \times_{k'} L$ be a Cohen-

Macaulay closed point, and let $f_1, \dots, f_d$ be a maximal regular sequence in $\mathfrak{m}_{x_0}\mathcal{O}_{\Gamma\times_{k'}L,x_0}$ of length $\mu := \ell(\mathcal{O}_{\Gamma\times_{k'}L,x_0}/(f_1,\dots,f_d))$. By [5], 9.1/7 (b), $e(\Gamma/k')$ divides $\mu \deg_L(x_0)$. Hence, when Γ is regular, so is $\Gamma \times_{k'} L$ and $e(\Gamma/k')$ divides $\delta(\Gamma/k')$. Note that $e(\Gamma/k') = e(\Gamma/k)$ if k'/k is separable.

Let now $\mathcal{X} \to S$ be as in Theorem 8.2, with f proper and $\mathcal{X}$ regular. Let k_i denote the algebraic closure of k in the function field $k(\Gamma_i)$ of the component Γ_i of $\mathcal{X}_k$. As noted in 6.9, the scheme Γ_i^{reg} is defined over k_i, and we have $\delta(\Gamma_i^{\mathrm{reg}}/k) = [k_i : k]\delta(\Gamma_i^{\mathrm{reg}}/k_i)$. The geometric multiplicity $e(\Gamma_i/k_i)$ of Γ_i/k_i is also the geometric multiplicity of $\Gamma_i^{\mathrm{reg}}/k_i$. It follows then from above that $e(\Gamma_i/k_i)$ divides $\delta(\Gamma_i^{\mathrm{reg}}/k_i)$. Hence, Theorem 8.2 (a) implies that

$$\gcd_i\bigl\{r_i[k_i : k]e(\Gamma_i/k_i)\bigr\} \quad \text{divides} \quad \delta(X/K),$$

answering a question in [4], 1.6.

Remark 8.7 (1) In general, if $\mathcal{O}_K$ is not Henselian, $\gcd_i\{r_i\delta(\Gamma_i^{\mathrm{reg}}/k)\}$ is not equal to $\delta(X/K)$. This can be seen easily when $\mathcal{X}/\mathcal{O}_K$ is of relative dimension 0. We can also consider a smooth projective conic X over $\mathbb{Q}$ without rational point, and with a regular proper model $\mathcal{X}$ over $\mathbb{Z}$. If p is a prime of good reduction of X, then $\gcd(\mathcal{X}_{\mathbb{F}_p}) = 1$ because every smooth conic over $\mathbb{F}_p$ has an $\mathbb{F}_p$-rational point, while $\delta(X/\mathbb{Q}) = 2$.

(2) In general we cannot replace $\delta(\Gamma_i^{\mathrm{reg}}/k)$ by $\delta(\Gamma_i/k)$ in 8.2(b). For example, let $\mathcal{O}_K = \mathbb{R}[[t]]$ and let $\mathcal{X} = \mathrm{Proj}(\mathcal{O}_K[x,y,z]/(x^2+y^2+tz^2))$. Then $\mathcal{X}$ is regular, flat and projective over $\mathcal{O}_K$. The special fiber $\Gamma := \mathcal{X}_k$ is integral, with a singular rational point, and its regular locus is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. In this example, $\delta(X/K) = 2$, but $r(\Gamma)\delta(\Gamma/\mathbb{R}) = 1$.

Example 8.8 Let A/K be a central simple K-algebra of dimension n^2. The square root of the degree over K of the skew-field D such that A is isomorphic to $M_r(D)$ for some $r \geq 1$ is called the index $\mathrm{ind}(A)$ of A. Associated with A is a Severi-Brauer variety X/K, a twisted form of $\mathbb{P}^{n-1}/K$, with $\delta(X/K) = \mathrm{ind}(A)$.

Suppose that K is a complete discrete valuation field with perfect residue field k. Let $\Lambda/\mathcal{O}_K$ be a hereditary order of A. Associated with Λ is a model $\mathcal{X}/\mathcal{O}_K$ of X/K called an Artin model ([19], 2.1). Artin ([1], 1.4) shows that when Λ is a maximal order, the model $\mathcal{X}$ is regular. The special fiber $\mathcal{X}_k$ is described in some cases in 2.4 and 2.5 of [19]. In particular, the special fiber in the model described in [19] 2.5 contains only irreducible components Γ of multiplicity 1 such that $\delta(\Gamma/k) = 1$, and $\delta(\Gamma^{\mathrm{reg}}/k) = \delta(X/K)$.

8.9 Let W be a non-empty scheme of finite type over a field F. Let $\mathcal{D}$ be the set of all degrees of closed points of W. Denote by $\nu(W/F)$ the smallest integer in $\mathcal{D}$. Clearly, $\delta(W/F)$ divides $\nu(W/F)$.

Let $f: \mathcal{X} \to S$ be as in 8.1, and assume $\mathcal{X}$ regular and f proper. Then Proposition 8.4 (2) shows that

$$\nu(X/K) \geq \min_{x_0 \text{ closed in } \mathcal{X}_k} \left\{ e(\mathcal{O}_{\mathcal{X}_k, x_0}) \deg_k(x_0) \right\},$$

with equality if in addition $\mathcal{O}_K$ is Henselian and k is infinite (8.4 (3)).

Lang and Tate asked in [36], page 670, whether $\nu(W/F) = \delta(W/F)$ when W is a homogeneous space for an Abelian variety A/F. Recall that non-empty scheme W/F is a homogeneous space for A/F if W/F is endowed with a transitive action of A/F, in the sense that the natural morphism $A \times_F W \to W \times_F W$, which sends (a, w) to $(a \cdot w, w)$, is surjective as a morphism of fppf-sheaves.

Let us say that a field k is WC(0) if every homogeneous space X/k for any Abelian variety B/k has a k-rational point ([11], 1.2). A finite field is WC(0) ([34], Theorem 2), and it follows from the definitions that a pseudo-algebraically closed field k is WC(0).

Proposition 8.10 *Let $\mathcal{O}_K$ be Henselian with a* WC(0) *perfect residue field. Let A be an Abelian variety over K having good reduction. Let X/K be a homogeneous space for A. Then $\delta(X/K) = \nu(X/K)$.*

Proof As A is an Abelian variety, X is a principal homogeneous space for a quotient B of A. Let $\mathcal{B}/\mathcal{O}_K$ be the Néron model of B over $\mathcal{O}_K$. Then $\mathcal{B}$ is an Abelian scheme ([52], Corollary 2 to Theorem 1). We know ([40], Proposition 8.1) that X has a regular proper model $\mathcal{X}/\mathcal{O}_K$ such that $\mathcal{X}_k = rV$ with V proper, smooth over k (because k is perfect), and V/k is a homogeneous space for $\mathcal{B}_k$. Hence, $V(k) \neq \emptyset$, so that $\delta(V/k) = \nu(V/k) = 1$. Using Proposition 8.4, we find that $\nu(X/K) = r\nu(V/k) = r$. Theorem 8.2 shows that $\delta(X/K) = r\delta(V/k) = r$. □

9 The separable index

Let k be any field, and let X/k be a scheme. The set of separable closed points of X can be empty, even when X/k is regular. This is the case for instance for $X = \operatorname{Spec} L$, where L/k is a non-trivial purely inseparable extension. On the other hand, when X/k is smooth, the set of separable closed points of X is dense in X ([5], 2.2/13).

When X/k is generically smooth and non-empty, define the *separable index* $\delta_{\text{sep}}(X/k)$ of X/k to be the greatest common divisor of the degrees of the separable closed points of X. Clearly, $\delta(X/k)$ divides $\delta_{\text{sep}}(X/k)$, and the question of whether $\delta(X/k)$ and $\delta_{\text{sep}}(X/k)$ are always equal was raised by

Lang and Tate in [36], page 670. In this section, we answer this question positively. The case where X/k is a smooth projective curve was treated already in [24], Theorem 3.

Let us note first that our work in the previous section lets us answer the question positively under the following additional hypotheses.

Corollary 9.1 *Let K be the field of fractions of a Henselian discrete valuation ring $\mathcal{O}_K$. Let $f : \mathcal{X} \to S$ be as in 8.1, with f proper and flat, and $\mathcal{X}$ regular. Let X/K denote the generic fiber of $\mathcal{X}/S$, assumed to be generically smooth. Then*

$$\delta(X/K) = \delta_{\mathrm{sep}}(X/K) = \gcd_i\left\{r_i \delta\left(\Gamma_i^{\mathrm{reg}}/k\right)\right\}.$$

Proof Follows immediately from Theorem 8.2 (b) and Proposition 8.4 (3). □

The proof of the next theorem is independent of the results in the previous sections of the paper.

Theorem 9.2 *Let X be a regular and generically smooth non-empty scheme of finite type over a field k. Then $\delta(X/k) = \delta_{\mathrm{sep}}(X/k)$.*

Proof Let X^{sm} denote the dense open subset of X where X/k is smooth. It follows immediately from Proposition 6.8 and the fact that X is regular that $\delta(X/k) = \delta(X^{sm}/k)$. Thus, it suffices to prove the theorem when X/k is smooth.

Assume that X/k is smooth. It suffices to prove that for any closed point $x_0 \in X$, there exists a separable 0-cycle on X of degree $\deg_k(x_0)$. We proceed by induction on $d := \dim_{k(x_0)} \Omega^1_{k(x_0)/k}$. If $d = 0$, then x_0 is already separable. If $d = 1$, then by Lemma 9.3, x_0 belongs to a closed curve C in X which is smooth at x_0, and we can conclude using Lemma 9.4.

Now fix $d \geq 2$, and suppose that the statement holds for any closed point y in any smooth variety over k such that $\dim_{k(y)} \Omega^1_{k(y)/k} \leq d - 1$. Replacing X by the smooth locus of the closed subvariety Y whose existence is proved in Lemma 9.3, we can suppose that $\dim_{x_0} X = d$. Since x_0 is smooth in X, we can replace X by an affine open subset if necessary, and assume that there exists an étale morphism $f : X \to \mathbb{A}^d_k$. Let $\pi : \mathbb{A}^d_k \to \mathbb{A}^{d-1}_k$ be the projection to the first $d - 1$ coordinates. Let $z_0 := \pi(f(x_0))$. Then $C := X \times_{\mathbb{A}^{d-1}_k} \operatorname{Spec} k(z_0)$ is a closed subscheme of X containing x_0. It is a smooth curve over $k(z_0)$ because it is étale over the affine line $\pi^{-1}(z_0) = \mathbb{A}^1_{k(z_0)}$. Therefore, we can use Lemma 9.4, and obtain that there exists a 0-cycle D on C, separable over $k(z_0)$, and of degree $\deg_{k(z_0)}(x_0)$. For any point y in the support of D, we have

$$\dim_{k(y)} \Omega^1_{k(y)/k} = \dim_{k(z_0)} \Omega^1_{k(z_0)/k} \leq \dim \mathbb{A}_k^{d-1} = d - 1.$$

Thus, we can apply the induction hypothesis to each such point y. It follows that there exists a separable 0-cycle on X/k of degree $\deg_k(x_0)$. □

Lemma 9.3 *Let X be a smooth algebraic variety over a field k, and let $x_0 \in X$. Then x_0 is contained in a closed subvariety Y of X of pure dimension $\dim_{k(x_0)} \Omega^1_{k(x_0)/k}$ which is smooth at x_0.*

Proof Let $\mathfrak{m}_0$ be the maximal ideal of $\mathcal{O}_{X,x_0}$. When $f \in \mathfrak{m}_0$, denote by $\bar{f}$ its class in $\mathfrak{m}_0/\mathfrak{m}_0^2$. The closed immersion $\operatorname{Spec} k(x_0) \to \operatorname{Spec} \mathcal{O}_{X,x_0}$ induces the canonical second fundamental exact sequence

$$\mathfrak{m}_0/\mathfrak{m}_0^2 \xrightarrow{\delta} \Omega^1_{X,x_0} \otimes k(x_0) \xrightarrow{\mu} \Omega^1_{k(x_0)/k} \longrightarrow 0$$

([41], Theorem 58, page 187). Let $f_1, \ldots, f_m \in \mathfrak{m}_0$ be such that $\delta(\bar{f}_1), \ldots, \delta(\bar{f}_m)$ form a basis of $\operatorname{Ker}(\mu)$. Choose an open neighborhood U of x_0 and $g_i \in \mathcal{O}_X(U)$ such that for all $i = 1, \ldots, m$, f_i is the stalk of g_i at x_0. Let Y be a closed subscheme of X, equal to $V(g_1, \ldots, g_m)$ in an open neighborhood of x_0. As $\{f_1, \ldots, f_m\}$ is part of a regular system of parameters of $\mathcal{O}_{X,x_0}$, we find that

$$\dim_{x_0} Y = \dim_{x_0} X - m = \dim_{k(x_0)} \Omega^1_{k(x_0)/k}.$$

Since $(\Omega^1_{Y/k})_{x_0} \otimes k(x_0) \simeq \Omega^1_{k(x_0)/k}$, we find that $(\Omega^1_{Y/k})_{x_0}$ is generated by $\dim_{x_0} Y$ elements, and this implies that Y is smooth at x_0 (see, e.g., [5], 2.2/15). We may thus replace Y by its irreducible component which contains x_0. □

Lemma 9.4 *Let C be a smooth connected curve over k. Let $\overline{C}$ be a regular scheme, separated and of finite type over k, such that there exists an open immersion $C \to \overline{C}$. Let $z \in C$ be a closed point. Then there exists a separable 0-cycle on C which is rationally equivalent to $[z]$ in $\overline{C}$.*

Proof The connected component of $\overline{C}$ containing C embeds as an open subscheme of a regular compactification C' of C. Lemma 9.4 follows from a stronger statement ([18], Chapter 3, Lemma 3.16), in the proof of which the smooth compactification (which may not exist in general) should be replaced by a regular compactification. We thank O. Wittenberg for the reference to [18]. □

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