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Néron models, Lie algebras, and reduction of curves of genus one

Qing Liu[1], **Dino Lorenzini**[2,⋆], **Michel Raynaud**[3]

[1] CNRS, Laboratoire A2X, Université de Bordeaux I, 351, cours de la Libération, 33405 Talence, France (e-mail: Qing.Liu@math.u-bordeaux.fr)
[2] Department of Mathematics, University of Georgia, Athens, GA 30602, USA (e-mail: lorenzini@math.uga.edu)
[3] Université de Paris-Sud, Bât. 425, 91405 Orsay Cedex, France (e-mail: michel.raynaud@math.u-psud.fr)

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Dedicated to John Tate

Let K be a discrete valuation field. Let $\mathcal{O}_K$ denote the ring of integers of K, and let k be the residue field of $\mathcal{O}_K$, of characteristic $p \geq 0$. Let $S := \operatorname{Spec} \mathcal{O}_K$. Let X_K be a smooth geometrically connected projective curve of genus 1 over K. Denote by E_K the Jacobian of X_K. Let X/S and E/S be the minimal regular models of X_K and E_K, respectively. In this article, we investigate the possible relationships between the special fibers X_k and E_k. In doing so, we are led to study the geometry of the Picard functor $\operatorname{Pic}_{X/S}$ when X/S is not necessarily cohomologically flat. As an application of this study, we are able to prove in full generality a theorem of Gordon on the equivalence between the Artin-Tate and Birch-Swinnerton-Dyer conjectures.

Recall that when k is algebraically closed, the special fibers of elliptic curves are classified according to their Kodaira type, which is denoted by a symbol $T \in \{\mathrm{I}_n, \mathrm{I}_n^*, n \in \mathbb{Z}_{\geq 0}, \mathrm{II}, \mathrm{II}^*, \mathrm{III}, \mathrm{III}^*, \mathrm{IV}, \mathrm{IV}^*\}$. Given a type T and a positive integer m, we denote by mT the new type obtained from T by multiplying all the multiplicities of T by m. When k is algebraically closed, the relationships between the type of a curve of genus 1 and the type of its Jacobian can be summarized as follows.

Theorem 6.6. *Assume that k is algebraically closed. Let X_K/K be a smooth, geometrically connected projective curve of genus 1 and let E_K/K be its*

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Jacobian. Let X/S and E/S be the minimal regular models of X_K and E_K, respectively. Let m denote the order of the element of $H^1(K, E_K)$ corresponding to the torsor X_K. If T denotes the type of E_k, then X_k is of type mT.

The most difficult part of this theorem is the case of additive reduction. As a corollary to Theorem 6.6 and of results of Bégueri [5] and Bertapelle [6] on the structure of $H^1(K, E_K)$ when K is complete, we prove in 6.7 the existence of torsors X_K having reduction of type mT, for any additive type T and integer $m = p^n$ and, in case the type T is semi-stable, for any integer $m > 0$.

To prove Theorem 6.6, we first show in 3.8 that there exists a canonical map of $\mathcal{O}_K$-modules $H^1(X, \mathcal{O}_X) \to H^1(E, \mathcal{O}_E)$ which extends the natural isomorphism $H^1(X_K, \mathcal{O}_{X_K}) \to H^1(E_K, \mathcal{O}_{E_K})$. The existence of this map is the main link between X and E, and is proved in the following general theorem on Néron models of Jacobians.

Theorem 3.1 (Raynaud, unpublished [47]). *Assume that k is algebraically closed. Let $f : X \to S$ be a proper flat curve, with X regular and with $f_*\mathcal{O}_X = \mathcal{O}_S$. Let J/S denote the Néron model of the Jacobian of X_K/K, and let* $\mathrm{Lie}(J)$ *denote its Lie algebra. Then the canonical morphism of $\mathcal{O}_K$-modules $H^1(X, \mathcal{O}_X) \to$* $\mathrm{Lie}(J)$*, which induces the canonical isomorphism $H^1(X_K, \mathcal{O}_{X_K}) \to$* $\mathrm{Lie}(J_K)$*, has a kernel and cokernel of same length.*

This theorem is a key ingredient in the proof of Theorem 6.6, and we provide here a complete proof. The statement and proof of Theorem 6.6 in the function field case was also known to Raynaud at the time he wrote [47]. Independently, Cossec and Dolgachev provided a proof of a slightly weaker version of Theorem 6.6 in the function field case in [11], Theorem 5.3.1, also using [47] as one of the main ingredient in their proof. The statement of 6.6 in the function field case is mentioned without proof in the second paragraph of [23].

The proof of Theorem 3.1 relies on Raynaud's results on Picard functors in [48], and on a theorem on morphisms of group schemes of finite type that is of independent interest: Let $u : G \to G''$ be a morphism of smooth group schemes of finite type over a complete discrete valuation ring $\mathcal{O}_K$ with algebraically closed residue field k. Assume that $u_K : G_K \to G''_K$ is surjective with smooth kernel. Then the length of the cokernel of $\mathrm{Lie}(u) : \mathrm{Lie}(G) \to \mathrm{Lie}(G'')$ is expressed in 2.1 in terms of two other invariants, the first one obtained using the group smoothening of $\mathrm{Ker}(u)$, and the second one defined as the dimension of a smooth group scheme D/k constructed so that $D(k) = \mathrm{Coker}(G(S) \to G''(S))$. Using 3.1, we show in Theorem 5.9 that the minimal regular models X and E have the same discriminant when k is perfect.

The second main ingredient in the proof of Theorem 6.6 is then the formula of T. Saito [51], which states that the Artin conductor is equal to the discriminant when k is perfect. Using this formula and 3.1, one easily shows in 6.5 that the curves $X_{\bar{k}}$ and $E_{\bar{k}}$ have the same number of irreducible components. To conclude the proof of Theorem 6.6, we use the description by Raynaud of the group of components Φ_J of the Néron model J/S of the Jacobian E_K/K: when k is algebraically closed, Φ_J can be computed using the intersection matrix of X_k.

It is likely that the complete relationship between the type of reduction of a curve of genus 1 and the type of reduction of its Jacobian may prove more difficult to express when k is imperfect. For instance, when the reduction is additive and K has characteristic zero, we show in 9.2 that the type of reduction of E_K depends not only on the type of reduction of X_K, but also, for instance, on $v(p)$. We are able, however, to completely describe the relationship between X_k and E_k when the reduction is semi-stable (8.1 and 8.3). In particular, we find that in this case already, the statement of 6.6 does not hold if k is not algebraically closed. When k is imperfect, the possible types of reduction of elliptic curves consist not only of the classical Kodaira types, but also of several new types. We give in Appendix A a list of these possible types of reduction, as well as of several additional types for curves of genus 1 without rational point.

As a corollary to Theorem 3.1, we provide an application to the conjectured equivalence between the Artin-Tate and Birch-Swinnerton-Dyer conjectures. Let k be a finite field of characteristic p. Let X/k be a smooth projective geometrically connected surface and denote by $\mathrm{Br}(X)$ its Brauer group. Let $f : X \to V$ be a proper and flat morphism, with V/k a smooth projective curve. Let K be the function field of V. Let us suppose that X_K/K is a smooth projective geometrically connected curve of genus $g \geq 1$. Let A_K denote the Jacobian of X_K and let $Ш(A_K)$ be its Shafarevich-Tate group. It is well-known that if either $Ш(A_K)$ or $\mathrm{Br}(X)$ is finite, then so is the other. Let δ and δ' denote respectively the index and the period of X_K. Similarly, for any place $v \in V$ with completion K_v, let δ_v and δ'_v denote the index and period of X_{K_v}/K_v, respectively.

Theorem 4.3. *Assume that* $Ш(A_K)$ *and* $\mathrm{Br}(X)$ *are finite. The equivalence of the Artin-Tate and Birch-Swinnerton-Dyer conjectures holds exactly when*

$$|Ш(A_K)| \prod_v \delta_v \delta'_v = \delta^2 |\mathrm{Br}(X)|.$$

This equality is satisfied if the periods δ'_v are pairwise coprime (4.6).

Notation. Throughout this paper, with the exception of Sect. 4, K denotes a discrete valuation field, $\mathcal{O}_K$ is the ring of integers of K, k is the residue field of $\mathcal{O}_K$, and π is a uniformizing element. Starting in Sect. 2, the letter S will be reserved to denote $\operatorname{Spec} \mathcal{O}_K$.

Contents

1. Review of Lie algebras

We review in this section several basic facts about Lie algebras needed in the next sections. We first recall the general definition of the Lie algebra of a group functor, and then consider the special case of the functor $\mathrm{Pic}_{X/S}$. All group functors considered are assumed to be commutative.

For any scheme T, let

$$T_\varepsilon := T \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}\,(\mathbb{Z}[\varepsilon]/\varepsilon^2).$$

The canonical projection $p : T_\varepsilon \to T$ is a finite faithfully flat morphism of finite presentation. It admits a canonical section $i : T \to T_\varepsilon$ corresponding to the homomorphism $\mathcal{O}_{T_\varepsilon} = \mathcal{O}_T \oplus \varepsilon\mathcal{O}_T \to \mathcal{O}_T, \quad b_1 + \varepsilon b_2 \mapsto b_1$.

Let S be any scheme, and let $\mathcal{F}$ be a contravariant commutative group functor on the category of S-schemes. By definition, $\mathcal{L}\mathrm{ie}\,(\mathcal{F})$ is the functor

$$T \mapsto \mathrm{Ker}\,(\mathcal{F}(T_\varepsilon) \to \mathcal{F}(T)),$$

where $\mathcal{F}(T_\varepsilon) \to \mathcal{F}(T)$ is defined by the immersion $i : T \to T_\varepsilon$. Any homomorphism of group functors $h : \mathcal{F} \to \mathcal{G}$ induces canonically a homomorphism $\mathrm{Lie}\,(h) : \mathcal{L}\mathrm{ie}\,(\mathcal{F}) \to \mathcal{L}\mathrm{ie}\,(\mathcal{G})$. If $\mathcal{F}$ is a sheaf for some topology on S, then so is $\mathcal{L}\mathrm{ie}\,(\mathcal{F})$.

Recall that $\mathcal{L}\mathrm{ie}\,(\mathcal{F})$ is equipped with a structure of $\mathcal{O}_S$-module as follows. Suppose that S is affine and let T be any S-scheme. Let $a \in \mathcal{O}_S(S)$. Then the $\mathcal{O}_S(S)$-algebra homomorphism $\mathcal{O}_T \oplus \varepsilon\mathcal{O}_T \to \mathcal{O}_T \oplus \varepsilon\mathcal{O}_T$ defined by

$$(1) \qquad \psi_a : b_1 + \varepsilon b_2 \mapsto b_1 + \varepsilon a b_2$$

induces a morphism $u_a : T_\varepsilon \to T_\varepsilon$ such that $u_a \circ i = i$ and $p \circ u_a = p$. Hence, u_a acts on $\mathcal{L}\mathrm{ie}\,(\mathcal{F})(T)$. This is the multiplication by a in $\mathcal{L}\mathrm{ie}\,(\mathcal{F})(T)$. See [13], II, §4, 1.2, or [SGA3], Tome I, Exposé 2, for more details.

Let us call an algebraic space G over an algebraic space S a *group space* if it is a group object in the category of algebraic spaces over S (see [25], and also [7], p. 96). Let G be a group scheme or group space over a scheme S.

We denote by $\mathcal{L}\mathrm{ie}\,(G)$ the Lie algebra of the sheaf associated to G, and by $\mathrm{Lie}\,(G)$ the $\mathcal{O}_S(S)$-module $\mathrm{Lie}\,(G) := \mathrm{Ker}\,(G(S_\varepsilon) \to G(S)) = \mathcal{L}\mathrm{ie}\,(G)(S)$.

Recall the definition of the subfunctor G^0/S of a group functor G/S with representable fibers (see [48], 3.2 d), or [7], p. 233):

$$G^0(T) := \left\{ a \in G(T) \mid a_t \in G_t^0(\mathrm{Spec}\,(k(t)),\ \text{for all } t \in T \right\},$$

where G_t^0 is the connected component of 0 in G_t, and $k(t)$ denotes the residue field of the point t, with natural morphism $\mathrm{Spec}\,(k(t)) \to T$. The following facts are known but we have been unable to find a reference for some of them in the literature.

Proposition 1.1. *Let S be a scheme.*

(a) *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be an exact sequence of (pre)sheaves on the category of S-schemes. Then*

$$0 \to \mathcal{L}\mathrm{ie}\,(\mathcal{F}') \to \mathcal{L}\mathrm{ie}\,(\mathcal{F}) \to \mathcal{L}\mathrm{ie}\,(\mathcal{F}'')$$

is an exact sequence of (pre)sheaves.

(b) *Let $p_G : G \to S$ be a group scheme. Let $\epsilon_G : S \to G$ be the zero section, and set $\omega_{G/S} := \epsilon_G^* \Omega^1_{G/S}$. Then there is a natural isomorphism $\rho_G : p_G^* \omega_{G/S} \to \Omega^1_{G/S}$.*
Given a morphism of S-group schemes $f : F \to G$, the exact sequence $f^*\Omega^1_{G/S} \to \Omega^1_{F/S} \to \Omega^1_{F/G} \to 0$ induces an exact sequence of $\mathcal{O}_S$-modules

$$\omega_{G/S} \xrightarrow{\alpha} \omega_{F/S} \longrightarrow \epsilon_F^* \Omega^1_{F/G} \to 0.$$

Let $\omega^\vee_{G/S}$ denote the dual of $\omega_{G/S}$. There is a canonical isomorphism of $\mathcal{O}_S$-modules $\mu_G : \mathcal{L}\mathrm{ie}\,(G) \to \omega^\vee_{G/S}$, functorial in G.

(c) *Let $f : F \to G$ be a smooth (resp. étale) morphism of group spaces over S. Then $\mathcal{L}\mathrm{ie}\,(F)(T) \to \mathcal{L}\mathrm{ie}\,(G)(T)$ is surjective (resp. bijective) for any S-scheme T which is affine.*

(d) *Let G/S be a group functor with representable fibers. Then $\mathcal{L}\mathrm{ie}\,(G^0) \to \mathcal{L}\mathrm{ie}\,(G)$ is an isomorphism.*

(e) *Let $f : F \to G$ be a morphism of smooth group schemes of finite type over S. Assume that S is affine and that* $\mathrm{Lie}\,(F) \to \mathrm{Lie}\,(G)$ *is surjective. Then f is smooth.*

Proof. (a) follows easily from the definitions. For (b), see [SGA3], tome 1, II.4.11.

(c) After making the base change $T \to S$ if necessary, we may assume that S is affine and prove the assertions for $\mathrm{Lie}\,(F) \to \mathrm{Lie}\,(G)$. Let

$$\begin{tikzcd} F \arrow[r, "f"] & G \\ S \arrow[u, "\epsilon_F"] \arrow[r, hook, "i"] & S_\varepsilon \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

be a commutative diagram of S-morphisms. We have to complete the diagram with a morphism $S_\varepsilon \to F$, unique if f is étale. When F and G are both schemes, then the assertion follows from classical results for schemes (see for instance [7], 2.2/6). In the case of algebraic spaces, we proceed as follows. Consider $H = S_\varepsilon \times_G F$ as a S_ε-smooth algebraic space, and S as a S_ε-scheme via the canonical immersion i. We have to extend $(i, \epsilon_F) : S \to H$ to a morphism of algebraic spaces $S_\varepsilon \to H$. To do so, represent H as a quotient $R \rightrightarrows U \to H$ of an S_ε-scheme U by an étale equivalence relation R. In particular, for any scheme T, $H(T)$ is the quotient of the set $U(T)$ by the equivalence relation $R(T) \subset U(T) \times U(T)$. When f is smooth (resp. étale), we may choose such an U with $U \to S_\varepsilon$ smooth (resp. étale) (see [25], II.3.2). The lifting assertions to be proved follow then from the analogue results for schemes.

(d) The inclusion $\mathrm{Ker}(G(T_\varepsilon) \to G(T)) \subseteq \mathrm{Ker}(G^0(T_\varepsilon) \to G^0(T))$ follows from the definition. The reverse inclusion is obvious.

(e) The natural isomorphism $\rho_G : p_G^*\omega_{G/S} \to \Omega^1_{G/S}$ recalled in (b) induces a commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
p_F^*\epsilon_F^*(f^*\Omega^1_{G/S}) \simeq f^*(p_G^*\epsilon_G^*(\Omega_{G/S})) & \longrightarrow & p_F^*\epsilon_F^*\Omega^1_{F/S} & \longrightarrow & p_F^*\epsilon_F^*\Omega^1_{F/G} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle f^*(\rho_G)} & & \Big\downarrow{\scriptstyle \rho_F} & & \Big\downarrow & & \\
f^*\Omega^1_{G/S} & \longrightarrow & \Omega^1_{F/S} & \longrightarrow & \Omega^1_{F/G} & \longrightarrow & 0.
\end{array}$$

It follows that $p_F^*(\epsilon_F^*\Omega^1_{F/G}) \simeq \Omega^1_{F/G}$. Proposition 2.2/8 in [7] implies that $F \to G$ is smooth if $\Omega^1_{F/G}$ is locally free. Let us show then that $\epsilon_F^*\Omega^1_{F/G}$ is locally free. Consider the exact sequence of $\mathcal{O}_S$-modules

$$\omega_{G/S} \xrightarrow{\alpha} \omega_{F/S} \longrightarrow \epsilon_F^*\Omega^1_{F/G} \to 0.$$

Since S is affine and $\mathrm{Lie}(F) \to \mathrm{Lie}(G)$ is surjective, we find using (b) that $\omega^\vee_{F/S} \to \omega^\vee_{G/S}$ is surjective. Since $\omega_{G/S}$ and $\omega_{F/S}$ are locally free, we find that $\omega_{G/S} \xrightarrow{\alpha} \omega_{F/S}$ is injective. Let $\mathcal{F}$ be any coherent $\mathcal{O}_S$-module. The surjectivity of $\omega^\vee_{F/S} \to \omega^\vee_{G/S}$ implies that of $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\omega_{F/S}, \mathcal{F}) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\omega_{G/S}, \mathcal{F}) = \omega^\vee_{G/S} \otimes \mathcal{F}$. Considering then the long exact sequence of cohomology associated with $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\cdot, \mathcal{F})$ and the exact sequence $0 \to \omega_{G/S} \to \omega_{F/S} \to \epsilon_F^*\Omega^1_{F/G} \to 0$, we deduce that $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_S}(\epsilon_F^*\Omega^1_{F/G}, \mathcal{F}) = 0$. Hence $\epsilon_F^*\Omega^1_{F/G}$ is locally free. $\square$

1.2 The Lie algebra of $\mathrm{Pic}_{X/S}$. Let S be a scheme and let $f : X \to S$ be an S-scheme. We denote by $\mathrm{Pic}_{X/S}$ the *relative Picard functor of X over S*. It is the fppf (faithfully flat and finite presentation) sheaf associated with the presheaf

$$P_{X/S} : (\mathrm{Sch}/S)^0 \to (\mathrm{Sets}), \quad T \mapsto \mathrm{Pic}(X \times_S T).$$

If f is proper, the relative Picard functor is also the étale-sheaf associated with $P_{X/S}$ ([7], p. 203).

Let us introduce some notation for the statement of the next proposition. Let $f : X \to S$, $g : Y \to S$ be S-schemes and $h : X \to Y$ be a morphism of S-schemes. Consider $\mathcal{O}_X$ as a sheaf for the fppf, or étale, or Zariski topology. The map $h^\# : \mathcal{O}_Y \to h_*\mathcal{O}_X$ induces $R^1(h^\#) : R^1 g_*\mathcal{O}_Y \to R^1 g_*(h_*\mathcal{O}_X)$. We denote by $R^1(h) : R^1 g_*\mathcal{O}_Y \to R^1 f_*\mathcal{O}_X$ the canonical homomorphism which is the composition of $R^1(h^\#)$, and the canonical homomorphism $R^1 g_*(h_*\mathcal{O}_X) \to R^1 f_*\mathcal{O}_X$. When S is affine, we may also denote $R^1(h)$ by $H^1(h)$. We define similarly $R^1(h)$ for the sheaf $\mathcal{O}_X^*$.

Proposition 1.3. *Let S be a scheme and let $f : X \to S$ be a quasi-compact separated morphism of schemes.*

(a) *Let $\underline{R}^1 f_*\mathcal{O}_X$ denote the fppf-sheaf on S associated to the presheaf*

$$T \longmapsto H^1(X_T, \mathcal{O}_{X_T}).$$

If T is affine, then $\Gamma(T, \underline{R}^1 f_*\mathcal{O}_X) = H^1(X_T, \mathcal{O}_{X_T})$.

(b) *There exists a canonical isomorphism of fppf-sheaves of $\mathcal{O}_S$-modules*

$$\theta_X : \underline{R}^1 f_*\mathcal{O}_X \longrightarrow \mathcal{L}\mathrm{ie}\,(\mathrm{Pic}_{X/S}).$$

(c) *Let $g : Y \to S$ be quasi-compact and separated, and let $h : X \to Y$ be a morphism of S-schemes. Let $\hat{h} : \mathrm{Pic}_{Y/S} \to \mathrm{Pic}_{X/S}$ be the canonical morphism induced by h. Then the diagram*

$$\begin{tikzcd} \underline{R}^1 g_*\mathcal{O}_Y \arrow[r, "\theta_Y"] \arrow[d, "R^1(h)"'] & \mathcal{L}\mathrm{ie}\,(\mathrm{Pic}_{Y/S}) \arrow[d, "\mathrm{Lie}\,(\hat{h})"] \\ \underline{R}^1 f_*\mathcal{O}_X \arrow[r, "\theta_X"] & \mathcal{L}\mathrm{ie}\,(\mathrm{Pic}_{X/S}) \end{tikzcd}$$

is commutative. We shall abbreviate (c) *by saying that* the isomorphism θ_X is functorial on X.

Proof. (a) Under the hypothesis on f, the formation of $R^1 f_*\mathcal{O}_X$ commutes with flat base change. Hence, it is easy to see that $\underline{R}^1 f_*\mathcal{O}_X$ is nothing but the sheaf $W(R^1 f_*\mathcal{O}_X)$ for the fppf topology (see [37], II.1.2 (d)).

(b) Let T be a S-scheme. We have a *split* exact sequence of (Zariski) sheaves on X_T

$$0 \longrightarrow \mathcal{O}_{X_T} \xrightarrow{\alpha} \mathcal{O}^*_{X_{T_\varepsilon}} \underset{\gamma}{\overset{\beta}{\rightleftarrows}} \mathcal{O}^*_{X_T} \longrightarrow 1 \tag{2}$$

defined in an obvious way by $\alpha(b) = 1+b\varepsilon$, $\beta(b_1+b_2\varepsilon) = b_1$ and $\gamma(b) = b$. Hence, we have a split exact sequence of Zariski-sheaves on T

$$0 \to R^1 f_{T*}\mathcal{O}_{X_T} \to R^1 f_{T*}\mathcal{O}^*_{X_{T_\varepsilon}} \to R^1 f_{T*}\mathcal{O}^*_{X_T} \to 1.$$

Let $p : S_\varepsilon \to S$ be the canonical projection. It is a fppf morphism. For any flat finite presentation S-scheme T (resp. S_ε-scheme T'), we have

$$(3) \qquad p_* P_{X_\varepsilon/S_\varepsilon}(T) = P_{X/S}(T_\varepsilon), \quad P_{X_\varepsilon/S_\varepsilon}(T') = p^{-1} P_{X/S}(T').$$

We have a split exact sequence of fppf-presheaves

$$0 \to R^1 f_* \mathcal{O}_X \to p_* P_{X_\varepsilon/S_\varepsilon} \to P_{X/S} \to 0.$$

Therefore, we have a split exact sequence of fppf-sheaves

$$0 \to \underline{R}^1 f_* \mathcal{O}_X \to p_* \mathrm{Pic}_{X_\varepsilon/S_\varepsilon} \to \mathrm{Pic}_{X/S} \to 0.$$

From (3) we see that $p^{-1}\mathrm{Pic}_{X/S} = \mathrm{Pic}_{X_\varepsilon/S_\varepsilon}$ and, hence,

$$p_* \mathrm{Pic}_{X_\varepsilon/S_\varepsilon}(T) = \mathrm{Pic}_{X_\varepsilon/S_\varepsilon}(T_\varepsilon) = \mathrm{Pic}_{X/S}(T_\varepsilon).$$

This implies canonically an isomorphism of fppf-sheaves on S

$$\theta_X : \underline{R}^1 f_* \mathcal{O}_X \simeq \mathcal{L}\mathrm{ie}(\mathrm{Pic}_{X/S}).$$

Let us show that the above isomorphism is compatible with the structure of $\mathcal{O}_S$-modules. To simplify the notation, we can suppose that S is affine and consider only S-sections. Let $a \in \mathcal{O}_S(S)$. Then we can 'multiply' the exact sequence (2) by a, i.e., we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\alpha"] \arrow[d, "\cdot a"'] & \mathcal{O}^*_{X_\varepsilon} \arrow[r, "\beta"] \arrow[d, "\psi_a"'] & \mathcal{O}^*_X \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\alpha"] & \mathcal{O}^*_{X_\varepsilon} \arrow[r, "\beta"] & \mathcal{O}^*_X \arrow[r] & 1
\end{tikzcd}$$

where $\psi_a = u_a^\#$ is defined by (1). Hence, ψ_a acts on $\mathrm{Pic}_{X/S}$ as $R^1(u_a)$. Since the multiplication by a on $\mathrm{Lie}(\mathrm{Pic}_{X/S})$ is also induced by u_a, we see immediately that $R^1 f_* \mathcal{O}_X \to \mathrm{Lie}(\mathrm{Pic}_{X/S})$ is a homomorphism of $\mathcal{O}_S$-modules.

(c) The commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}^*_{Y_\varepsilon} \arrow[d] \\
h_* \mathcal{O}_X \arrow[r, "h_*\alpha"] & h_* \mathcal{O}^*_{X_\varepsilon}
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
R^1 g_* \mathcal{O}_Y \arrow[r, "R^1(\alpha)"] \arrow[d] & R^1 g_* \mathcal{O}^*_{Y_\varepsilon} \arrow[d] \\
R^1 g_* (h_* \mathcal{O}_X) \arrow[r, "R^1(h_*\alpha)"] & R^1 g_* \big(h_* \mathcal{O}^*_{X_\varepsilon}\big).
\end{tikzcd}$$

Since the construction of $R^1 g_*(h_*\mathcal{F}) \to R^1 f_*\mathcal{F}$ is functorial on $\mathcal{F}$,

$$\begin{array}{ccc} R^1 g_*\mathcal{O}_Y & \longrightarrow & R^1 g_*\mathcal{O}^*_{Y_\varepsilon} \\ {\scriptstyle R^1(h)}\downarrow & & \downarrow{\scriptstyle R^1(h)} \\ R^1 f_*\mathcal{O}_X & \longrightarrow & R^1 f_*\mathcal{O}^*_{X_\varepsilon} \end{array}$$

is commutative. This achieves the proof. □

1.4. Let X_K be a smooth geometrically connected projective curve over a field K, with Jacobian J_K. We define a canonical isomorphism

$$\tau_{X_K} : H^1(X_K, \mathcal{O}_{X_K}) \to H^1(J_K, \mathcal{O}_{J_K}),$$

compatible with base change $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$, as follows. $\mathrm{Pic}_{X_K/K}$ is representable by a K-scheme locally of finite type, whose connected component $\mathrm{Pic}^0_{X_K/K}$ is J_K. Let $\lambda : \mathrm{Pic}^0_{X_K/K} \to \mathrm{Pic}^0_{J_K/K}$ be the canonical isomorphism of J_K with its dual (given by the Θ-divisor, [7], p. 261). Let Z_K denote X_K or J_K. Then $\mathrm{Pic}^0_{Z_K/K}$ is an open subgroup of $\mathrm{Pic}_{Z_K/K}$ and, hence, $\mathrm{Lie}(\mathrm{Pic}^0_{Z_K/K}) = \mathrm{Lie}(\mathrm{Pic}_{Z_K/K})$ (Proposition 1.1). The isomorphism τ_{X_K} is the only isomorphism making the diagram of isomorphisms below commutative:

$$(4)\qquad \begin{array}{ccccc} H^1(X_K, \mathcal{O}_{X_K}) & \xrightarrow{\theta_{X_K}} & \mathrm{Lie}(\mathrm{Pic}_{X_K/K}) & = & \mathrm{Lie}\left(\mathrm{Pic}^0_{X_K/K}\right) \\ {\scriptstyle \tau_{X_K}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Lie}(\lambda)} \\ H^1(J_K, \mathcal{O}_{J_K}) & \xrightarrow{\theta_{J_K}} & \mathrm{Lie}(\mathrm{Pic}_{J_K/K}) & = & \mathrm{Lie}\left(\mathrm{Pic}^0_{J_K/K}\right). \end{array}$$

(In order to lighten the notation, we have denoted $\theta_{X_K}(\mathrm{Spec}\,K)$ simply by θ_{X_K}.) The reader will check that τ_{X_K} is compatible with the base change K'/K.

Corollary 1.5. *Let X_K be a smooth projective geometrically connected curve of genus 1 over a field K such that $X_K(K) \neq \emptyset$. Let E_K be the Jacobian of X_K. Then there exists a K-isomorphism of curves $h : E_K \to X_K$ such that $\tau_{X_K} = H^1(h)$.*

Proof. Let J_{E_K} denote the Jacobian of E_K, and let $\lambda : E_K \to J_{E_K}$ be the canonical isomorphism defined by $x \mapsto [x - 0]$. Let us fix $x_0 \in X_K(K)$. Then there exists a unique isomorphism $f : X_K \to E_K$ such that, over an algebraic closure of K, $f(x) = [x - x_0]$ for all closed points. Let $h = f^{-1}$. Let us check that the morphism $\hat{h} : E_K = J_{X_K} \to J_{E_K}$ is equal to λ. We can suppose K algebraically closed. Let $y \in E_K(K)$. We can write $y = [x - x_0]$ for some $x \in X_K(K)$. Then

$$\hat{h}(y) = \left[h^{-1}(x) - h^{-1}(x_0)\right] = [y - 0] = \lambda(y).$$

Hence, 1.3(c) gives a commutative diagram

$$\begin{CD} H^1(X_K, \mathcal{O}_{X_K}) @>{\theta_{X_K}}>> \mathrm{Lie}\,(\mathrm{Pic}_{X_K/K}) = \mathrm{Lie}\,\left(\mathrm{Pic}^0_{X_K/K}\right) \\ @V{H^1(h)}VV @VV{\mathrm{Lie}\,(\lambda)}V \\ H^1(E_K, \mathcal{O}_{E_K}) @>{\theta_{E_K}}>> \mathrm{Lie}\,(\mathrm{Pic}_{E_K/K}) = \mathrm{Lie}\,\left(\mathrm{Pic}^0_{E_K/K}\right). \end{CD}$$

Therefore, $\tau_{X_K} = H^1(h)$ by diagram (4). □

2. Morphisms of smooth S-group schemes

This section is independent of the rest of the paper, and the reader interested only in its applications to Picard schemes and Néron models of Jacobians may proceed directly to read the next section. Given any artinian $\mathcal{O}_K$-module M, we let $\ell(M)$ denote its length.

Consider an exact sequence of smooth algebraic groups over K

$$0 \longrightarrow G'_K \longrightarrow G_K \xrightarrow{u_K} G''_K \longrightarrow 0.$$

Assume that this sequence is the generic fiber of an exact sequence

$$0 \longrightarrow G' \longrightarrow G \xrightarrow{u} G''$$

of group schemes of finite type over $S := \mathrm{Spec}\,\mathcal{O}_K$, with G and G'' smooth and separated over S. We then have an exact sequence (1.1(a))

$$0 \longrightarrow \mathrm{Lie}\,(G') \longrightarrow \mathrm{Lie}\,(G) \xrightarrow{\mathrm{Lie}\,(u)} \mathrm{Lie}\,(G'').$$

The module Coker (Lie(u)) has finite length because $\mathrm{Lie}(G)_K \to \mathrm{Lie}\,(G'')_K$ is surjective since $G_K \to G''_K$ is smooth (1.1(c)). Let $v : \tilde{G}' \to G'$ be the group smoothening of G' ([7], p. 174). Our goal in this section is to prove the following theorem.

Theorem 2.1. *Assume that $\mathcal{O}_K$ is complete with algebraically closed residue field, and let $u : G \to G''$ be as above. Then*

(a) *If $u(S) : G(S) \to G''(S)$ is surjective, then*

$$\ell(\mathrm{Coker}\,(\mathrm{Lie}\,(u))) = \ell(\mathrm{Lie}\,(G')/\,\mathrm{Lie}\,(\tilde{G}')).$$

(b) *In general, there exists a smooth group scheme D/k of finite type such that $D(k)$ is isomorphic, as abelian group, to* Coker $(u(S))$, *and such that*

$$\ell(\mathrm{Coker}\,(\mathrm{Lie}\,(u))) = \ell(\mathrm{Lie}\,(G')/\,\mathrm{Lie}\,(\tilde{G}')) + \dim(D).$$

Proof. Suppose that $\mathcal{O}_K$ is any strictly henselian discrete valuation ring. The additional assumptions that $\mathcal{O}_K$ is complete and k is perfect will only be used starting in 2.5. Let G'/S be any group scheme of finite type with smooth generic fiber. Its group smoothening $\tilde{G}'/S$ is obtained from G' by a sequence of dilatations ([7], pp. 174–175)

$$\tilde{G}' = G'_n \longrightarrow \ldots \longrightarrow G'_1 \longrightarrow G'_0 = G',$$

where, for each i, the center of the dilatation is a closed smooth subgroup scheme H_i/k of the special fiber $(G'_i)_k$. Consider now the closed immersion $G' \to G$ given in Theorem 2.1. Construct G_1 as the dilatation of the image of H_0 in G_k. Then the natural map $G'_1 \to G_1$ is a closed immersion ([7], 3.2.2(c)). Similarly, construct G_2 as the dilatation of the image of H_1 in G_1. Repeating this process, we find that it is possible to obtain a commutative diagram

$$\begin{tikzcd} \tilde{G}' \arrow[r] \arrow[d, "g'"] & \tilde{G} \arrow[d, "g"] \\ G' \arrow[r] & G, \end{tikzcd}$$

where $\tilde{G}'/S$ is the group smoothening of G', the map $\tilde{G}' \to \tilde{G}$ is a closed immersion, and the map g is a sequence of dilatations with smooth centers. Consider the quotient map $\tilde{u} : \tilde{G} \to \tilde{G}'' := \tilde{G}/\tilde{G}'$. That this quotient exists is proved, for instance, in [3], 4.C. That $\tilde{G}''/S$ is smooth of finite type is proved in [SGA3] VI_B 9.2 (xii), p. 380. The same reference also shows that $\tilde{u}$ is faithfully flat (hence smooth), while (x) implies that $\tilde{G}''/S$ is separated. Note that the map $\tilde{u}(S)$ is surjective: given a section $S \to \tilde{G}''$, then $S \times_{\tilde{G}''} \tilde{G} \to S$ being smooth and S strictly henselian implies that $S \times_{\tilde{G}''} \tilde{G}' \to S$ has a section. We have thus obtained a diagram of group schemes with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{G}' \arrow[r] \arrow[d, "g'"'] & \tilde{G} \arrow[r, "\tilde{u}"] \arrow[d, "g"'] & \tilde{G}'' \arrow[r] \arrow[d, "g''"'] & 0 \\ 0 \arrow[r] & G' \arrow[r] & G \arrow[r, "u"] & G''. \end{tikzcd} \tag{5}$$

Consider the diagram induced by (5) on S-points:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{G}'(S) \arrow[r] \arrow[d, "g'"'] & \tilde{G}(S) \arrow[r, "\tilde{u}(S)"] \arrow[d, "g"'] & \tilde{G}''(S) \arrow[r] \arrow[d, "g''"'] & 0 \\ 0 \arrow[r] & G'(S) \arrow[r] & G(S) \arrow[r, "u(S)"] & G''(S), \end{tikzcd}$$

with $\tilde{G}'(S) \to G'(S)$ surjective by construction of the group smoothening. It follows that

$$0 \to G(S)/\tilde{G}(S) \to G''(S)/\tilde{G}''(S) \to \mathrm{Coker}(u(S)) \longrightarrow 0 \tag{6}$$

is an exact sequence of abelian groups. Consider now the diagram induced by (5) on Lie algebras:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Lie}(\tilde{G}') \arrow[r] \arrow[d, "\mathrm{Lie}(g')"] & \mathrm{Lie}(\tilde{G}) \arrow[r, "\mathrm{Lie}(\tilde{u})"] \arrow[d, "\mathrm{Lie}(g)"] & \mathrm{Lie}(\tilde{G}'') \arrow[r] \arrow[d, "\mathrm{Lie}(g'')"] & 0 \\
0 \arrow[r] & \mathrm{Lie}(G') \arrow[r] & \mathrm{Lie}(G) \arrow[r, "\mathrm{Lie}(u)"] & \mathrm{Lie}(G''),
\end{tikzcd}$$

where $\mathrm{Lie}(\tilde{u})$ is surjective because $\tilde{u}$ is smooth (1.1(c)). We easily extract from this diagram the exact sequence

$$0 \to \mathrm{Coker}(\mathrm{Lie}(g')) \to \mathrm{Coker}(\mathrm{Lie}(g)) \\ \to \mathrm{Coker}(\mathrm{Lie}(g'')) \to \mathrm{Coker}(\mathrm{Lie}(u)) \to 0.$$

It follows that

$$(7) \quad \ell(\mathrm{Coker}(\mathrm{Lie}(u))) - \ell(\mathrm{Lie}(G')/\mathrm{Lie}(\tilde{G}')) \\ = \ell(\mathrm{Coker}(\mathrm{Lie}(g''))) - \ell(\mathrm{Coker}(\mathrm{Lie}(g))).$$

By construction, the morphism g is a sequence of dilatations. Our next proposition and its corollary 2.3 show that the same holds for g''. In the second part of the proof 2.1, starting in 2.5, we use this fact to identify, when k is perfect, the groups $G(S)/\tilde{G}(S)$ and $G''(S)/\tilde{G}''(S)$ with the k-points of two smooth k-group schemes C/k and C''/k, having dimensions $\ell(\mathrm{Coker}(\mathrm{Lie}(g))$ and $\ell(\mathrm{Coker}(\mathrm{Lie}(g''))$, respectively. □

Proposition 2.2. *Let $\mathcal{O}_K$ be a discrete valuation ring. Let $f : F \to G$ be a birational morphism of S-group schemes of finite type. Let $\alpha : \omega_{G/S} \to \omega_{F/S}$ denote the induced map of $\mathcal{O}_S$-modules* (1.1(b)).

(a) *We have a canonical exact sequence of $\mathcal{O}_K$-modules:*

$$(8) \qquad 0 \to \mathrm{Lie}(F) \xrightarrow{\mathrm{Lie}(f)} \mathrm{Lie}(G) \\ \to \mathrm{Ext}^1(\mathrm{Coker}\,\alpha, \mathcal{O}_K) \to \mathrm{Ext}^1(\omega_{F/S}, \mathcal{O}_K).$$

In particular, $\mathrm{Lie}(f)$ *is injective.*

(b) *Suppose that G is smooth over S and that $f : F \to G$ is the dilatation with respect to a closed smooth subgroup scheme H_0 of G_k. Let $d = \mathrm{codim}(H_0, G_k)$. Then*

$$\ell(\mathrm{Lie}(G)/\mathrm{Lie}(F)) = d.$$

(c) *Suppose that S is henselian. Let $S_i := \mathrm{Spec}(\mathcal{O}_K/(\pi^i))$. Let G/S be smooth, and let $f : F \to G$ be a sequence of n dilatations with respect to smooth closed subgroups. Then, for all $i \geq n$, the canonical map*

$$r_i : G(S)/f(F(S)) \to G(S_i)/f(F(S_i))$$

is an isomorphism.

Proof. (a) On the generic fiber, $f_K : F_K \to G_K$ is birational and, hence, is an isomorphism. It follows that both $\operatorname{Ker}\alpha$ and $\operatorname{Coker}\alpha$ are killed by a power of π. Recall that Lie(.) can be identified with $\operatorname{Hom}_{\mathcal{O}_S}(\omega_{./S}, \mathcal{O}_S)$ (1.1(b)). We obtain the exact sequence (8) by taking the dual $\operatorname{Hom}_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ of the exact sequence

$$0 \to \operatorname{Ker}\alpha \to \omega_{G/S} \xrightarrow{\alpha} \omega_{F/S} \to \operatorname{Coker}\alpha \to 0.$$

(b) Since the dilatation F/S of a smooth group G/S is smooth, $\omega_{F/S}$ is locally free, and, hence, the last term in (8) vanishes. Let us compute $\operatorname{Coker}\alpha$. Let $s = \dim H_0$. Then there exists a system of local parameters $\{x_1, ..., x_s, y_1, ..., y_d, \pi\}$ of G at $0 \in G_k$ such that $H_0 = V(y_1, ..., y_d, \pi)$ locally at 0. Since F is an open subset of the blowing-up of G along H_0, we see that $\mathcal{O}_{F,0} = \mathcal{O}_{G,0}[z_1, ..., z_d]/(\pi y_i - z_i, i = 1, \dots, d)$. We have an exact sequence

$$0 \to \Omega_{G/S,0} \otimes_{\mathcal{O}_{G,0}} \mathcal{O}_{F,0} \to \Omega_{F/S,0} \to \Omega_{F/G,0} \simeq \oplus_{1\le i\le d}\mathcal{O}_{G,0}dz_i/\pi dz_i \to 0$$

([31], Exercise 6.3.8). Restricting to the unit section of F/S gives rise to an exact sequence

$$\omega_{G/S} \to \omega_{F/S} \to \oplus_{1\le i\le d}\mathcal{O}_K/(\pi) \simeq k^d \to 0.$$

Therefore, $\operatorname{Coker}\alpha \simeq k^d$ and $\ell(\operatorname{Ext}^1_{\mathcal{O}_K}(\operatorname{Coker}\alpha, \mathcal{O}_K)) = d$.

(c) Since $G(S) \to G(S_i)$ is surjective, it is enough to prove that r_i is injective. Let $\alpha \in G(S)$ be such that $\alpha|_{S_i} \in f(F(S_i))$. Since $F(S) \to F(S_i)$ is surjective, we can change α by a section of $f(F(S))$ and assume that $\alpha|_{S_i} = 0$. Then Part (c) is an immediate consequence of the following property:

(1) $\operatorname{Ker}(G(S) \to G(S_n)) \subseteq f(F(S))$.

We prove now by induction on n that (1) and (2) hold, where

(2) $\operatorname{Ker}(F(S) \to F(S_{n+m}) \to G(S_{n+m})) \subseteq \operatorname{Ker}(F(S) \to F(S_m))$ for all $m \ge 1$.

Let us start with $n = 1$. Then property (1) comes from the universal property of the dilatation. Let ε_F and ε_G denote the zero sections of F and G respectively. Let $\alpha \in F(S)$ be such that $f(\alpha)|_{S_{m+1}} = \varepsilon_G|_{S_{m+1}}$. Then α maps the closed point of S to a point of $F \times_G \operatorname{Spec}\mathcal{O}_{G,0} = \operatorname{Spec}\mathcal{O}_{G,0}[T_1, ..., T_d]/(\pi T_j - y_j)_{1\le j\le d}$ with the above notation. We have $\pi\alpha^\#(T_i) \equiv \varepsilon_G^\#(y_i) = 0 \mod \pi^{m+1}$. Thus $\alpha^\#(T_i) \equiv 0 \mod \pi^m$ and $\alpha^\#(a) \equiv 0 \mod \pi^{m+1}$ for all $a \in \mathcal{O}_{G,0}$. Therefore $\alpha|_{S_m} = \varepsilon_F|_{S_m}$. This proves the property (2) when $n = 1$.

If $n \ge 2$, we decompose $F \to G$ into a single dilatation $F \to F'$ and a sequence of $n-1$ dilatations $F' \to G$. Then (2) is easily deduced from the case $n = 1$ and the induction hypothesis applied to $F' \to G$. Property (1) follows from (2) applied to $F \to F'$ and $F' \to G$. □

The above proposition has the following important consequence for birational morphisms of group schemes over S (when G is affine, see [65], 1.4).

Corollary 2.3. *Let S be the spectrum of a strictly henselian discrete valuation ring $\mathcal{O}_K$. Let $f : F \to G$ be a morphism of smooth S-group schemes of finite type. Suppose that $f_K : F_K \to G_K$ is an isomorphism. If f is separated, then f consists in a finite sequence of dilatations with smooth centers.*

Proof. Let H_0 be the schematic closure of $\mathrm{Im}\,(F(S) \to F(k) \to G(k))$ in G_k. It is a smooth subgroup scheme of G_k, and $F_k \to G_k$ factorizes through $H_0 \to G_k$. If $H_0 = G_k$, then $F_k \to G_k$ has dense image, so it is faithfully flat ([SGA3], VI_A, 5.6). Hence, $F \to G$ is birational and faithfully flat ([EGA], IV.11.3.11). It is an isomorphism by Lemma 2.4 below (where the hypothesis that f is separated is used).

Now suppose that $H_0 \neq G_k$. Let $G_1 \to G$ be the dilatation along H_0. Then by the universal property of the dilatation, $F \to G$ factorizes into $F \to G_1 \to G$. If $F \to G_1$ is not an isomorphism, we start again with a dilatation on G_1. We construct in this way a sequence of dilatations

$$F \to G_n \to G_{n-1} \to \dots \to G_1 \to G.$$

If $\dim(H_0) = \dim(G_k)$, then we find that the map $G_1 \to G$ is an open immersion and $H_1 = G_{1,k}$. It follows that $G_2 = G_1 = F$. If $\dim(H_i) < \dim(G_{i,k})$, we find that $\mathrm{Lie}\,(G_i)/\,\mathrm{Lie}\,(G_{i-1}) \neq 0$ (2.2(b)). Since $\mathrm{Lie}\,(G)/\,\mathrm{Lie}\,(F) \to \mathrm{Lie}\,(G)/\,\mathrm{Lie}\,(G_n)$ is surjective, it follows that there exists $n \leq \ell(\mathrm{Coker}\,(\mathrm{Lie}\,(f)))$ such that $F = G_n$. □

Lemma 2.4. *Let T be an integral locally noetherian scheme with generic point ξ. Let $f : F \to G$ be a morphism of flat group schemes over T. Let $H := \mathrm{Ker}\, f$, with $h : H \to T$ the structural morphism.*

(a) *If f admits a section $\sigma : G \to F$, then f is faithfully flat.*
(b) *Assume that f is separated, of finite type, and birational (i.e., f_ξ is an isomorphism). If f is faithfully flat, then it is an isomorphism.*

Proof. (a) The existence of σ implies that $F \to G$ is an epimorphism. Let $i : H \to F$ be the canonical morphism. Then $(i, \sigma) : H \times_T G \to F$ is an isomorphism of T-schemes. Since $G \to T$ is faithfully flat and $F \to T$ is flat, we find that $H \to T$ is flat. Thus the projection $H \times_T G \to G$ is flat. Since this projection corresponds to the map f, f is flat too.

(b) We claim that f is a monomorphism if h is flat. The morphism h is separated, of finite type, and $H_\xi = \mathrm{Ker}\, f_\xi = \{0\}$. Since h is separated and T is integral, $H(T)$ consists only of the zero section. For any T-scheme T' which is integral and locally noetherian, applying the previous remark to the morphism $H_{T'} \to T'$ and the integral scheme T' gives the equality $H(T') = T(T')$. The scheme H itself is integral since H/T is flat birational and T is integral (see, e.g., [31], 4.3.8). Let $\epsilon_H : T \to H$ be the zero section. The equality $H(H) = T(H)$ shows then that $\mathrm{id}_H = \epsilon_H \circ h$. Thus, h is an isomorphism, and f is a monomorphism.

Let us suppose now that f is faithfully flat. From the above, $H \to T$ is an isomorphism and $F \to G$ is an isomorphism of fppf-sheaves, hence $F \to G$ is an isomorphism. □

2.5. Let us now return to the proof of Theorem 2.1. Assume until the end of the proof that $\mathcal{O}_K$ is complete and that k is algebraically closed. Let us turn to defining the smooth group scheme D/k alluded to in the statement of 2.1.

Let $S_i := \operatorname{Spec}(\mathcal{O}_K/(\pi^i))$. The Greenberg functor Gr_i of level i (see, e.g., [7], p. 276) associates to each S_i-scheme Y_i locally of finite type a k-scheme $\mathrm{Gr}(Y_i)$ locally of finite type in such a way that, functorially in Y_i,

$$Y_i(S_i) = \mathrm{Gr}_i(Y_i)(S_1).$$

When K is of equicharacteristic p, $\mathrm{Gr}_i(Y_i)$ is simply the Weil restriction $\mathrm{Res}_{S_i/S_1} Y_i$, where $S_i \to S_1$ is the natural structure morphism.

Let G/S be any scheme locally of finite type. Let $G_i := G \times_S S_i$. Denote by $\mathrm{Gr}_i(G)$ the k-scheme $\mathrm{Gr}_i(G_i)$, and let $\mathrm{Gr}(G)$ denote the projective system of k-schemes

$$\ldots \longrightarrow \mathrm{Gr}_i(G) \longrightarrow \mathrm{Gr}_{i-1}(G) \longrightarrow \ldots \longrightarrow \mathrm{Gr}_1(G) = G_k.$$

Any morphism $f : G \to G''$ of S-group schemes induces in a natural way morphisms $f_i : \mathrm{Gr}_i(G) \to \mathrm{Gr}_i(G'')$ of k-group schemes, which give a morphism of projective systems $\mathrm{Gr}(f) : \mathrm{Gr}(G) \longrightarrow \mathrm{Gr}(G'')$. We let $\operatorname{Coker}(\mathrm{Gr}(f))$ denote the projective system $\{\operatorname{Coker}(f_i)\}_i$. When G/S is smooth, $\mathrm{Gr}_i(G)/k$ is smooth for all i ([5], 4.1.1).

Lemma 2.6. *Assume that $\mathcal{O}_K$ is complete and that k is algebraically closed. Let $f : G \to G''$ be a morphism of S-group schemes of finite type.*

(a) *If G''/S is smooth and $f(S)$ is surjective, then* $\operatorname{Coker}(\mathrm{Gr}(f)) = \{0\}$.
(b) *If G/S is smooth, then* $\operatorname{Coker}(f(S))$ *is isomorphic to* $\varprojlim_i \operatorname{Coker}(f_i(k))$.
(c) *If G and G'' are smooth and if f is separated and birational, then there exists n such that, for all $i \geq n$,* $\operatorname{Coker}(f_{i+1})(k) \to \operatorname{Coker}(f_i)(k)$ *is an isomorphism and* $\dim \operatorname{Coker}(f_i) = \ell(\operatorname{Coker}(\operatorname{Lie} f))$.

Proof. Since G''/S is smooth, $G''(S_{i+1}) \to G''(S_i)$ is surjective. Since $\mathrm{Gr}_i(G)(k) \to \mathrm{Gr}_i(G'')(k)$ is identified with the map $G(S_i) \to G''(S_i)$ by construction, the hypothesis that $f(S)$ is surjective implies that f_i is surjective for all i, thereby proving (a). To prove (b), note that any exact sequence of projective systems

$$\mathrm{Gr}_i(G)(k) \longrightarrow \mathrm{Gr}_i(G'')(k) \longrightarrow \operatorname{Coker}(f_i(k)) \longrightarrow 0$$

produces an exact sequence of projective limits

$$\varprojlim_i \mathrm{Gr}_i(G)(k) \longrightarrow \varprojlim_i \mathrm{Gr}_i(G'')(k) \longrightarrow \varprojlim_i \operatorname{Coker}(f_i(k)) \longrightarrow 0$$

if all maps $\mathrm{Gr}_i(G) \to \mathrm{Gr}_{i-1}(G)$ of the projective complex on the left are surjective. This is the case when G/S is smooth. Hence, since $\mathcal{O}_K$ is complete, we have an exact sequence

$$G(S) \longrightarrow G''(S) \longrightarrow \varprojlim_i \mathrm{Coker}\,(f_i(k)) \longrightarrow 0$$

which proves (b).

(c) By Corollary 2.3, f consists in a sequence of n dilatations with respect to smooth closed subgroups. By construction, $\mathrm{Coker}\,(f_i)(k) = G''(S_i)/f(G(S_i))$. The isomorphism $\mathrm{Coker}\,(f_{i+1})(k) \to \mathrm{Coker}\,(f_i)(k)$ follows from Proposition 2.2(c). The equality $\dim \mathrm{Coker}(f_i) = \ell(\mathrm{Coker}(\mathrm{Lie}\, f))$ comes from Proposition 2.2(b) when $n = 1$. If $n \geq 2$, we decompose f into $f' : F \to F'$ and $g : F' \to G$ as in the proof of 2.2(c). Then

$$\ell(\mathrm{Coker}\,(\mathrm{Lie}\ f)) = \ell(\mathrm{Coker}\,(\mathrm{Lie}\ f')) + \ell(\mathrm{Coker}\,(\mathrm{Lie}\ g)).$$

On the other hand, we have an exact sequence

$$0 \longrightarrow F'(S)/f'(F(S)) \longrightarrow G(S)/f(F(S)) \longrightarrow G(S)/g(F'(S)) \longrightarrow 0$$

which induces an exact sequence

$$0 \to \mathrm{Coker}\,(f'_i)(k) \to \mathrm{Coker}\,(f_i)(k) \to \mathrm{Coker}\,(g_i)(k) \to 0$$

for all $i \geq n$. Hence, $\dim \mathrm{Coker}\,(f_i) = \dim \mathrm{Coker}\,(f'_i) + \dim \mathrm{Coker}\,(g_i)$, which allows us to achieve the proof by induction on n. □

Remark 2.7. Some hypothesis on G''/S is needed in Lemma 2.6(a). Indeed, let G''/S be the group μ_p/S, kernel of the multiplication by p on $\mathbb{G}_m/S$. Let G/S denote the trivial group scheme, and let $f : G \to G''$ denote the natural closed immersion. Assume that $\mathcal{O}_K$ does not contains the p-th roots of unity. Then $f(S)$ is surjective as $\mu_p(S)$ is the trivial group. The reader will check that the projective system $\mathrm{Coker}\,(\mathrm{Gr}(f))$ is not trivial.

When G/S is a smooth group scheme of finite type, $\dim(\mathrm{Gr}_i(G)) = i \dim(G_K)$ if K is of equal characteristic and $\dim(\mathrm{Gr}_i(G)) = ie \dim(G_K)$ if K is of mixed characteristics and e is the absolute ramification index (see, e.g., [5], 4.1.1). Thus, when G/S and G''/S are smooth with $\dim(G_K) < \dim(G''_K)$, then $\lim_i \dim(\mathrm{Coker}\,(f_i)) = \infty$. Under our assumptions on the morphism $u : G \to G''$ in 2.1, on the other hand, we are going to show that the projective system $\mathrm{Coker}\,(\mathrm{Gr}(u))$ is constant for i large enough.

Let us return to the proof of 2.1. Part (a) of 2.1 follows from Part (b). Indeed, when $u(S)$ is surjective, we find that $D(k) = \{0\}$ in (b). Since k is algebraically closed, we conclude that $\dim(D) = 0$, and the formula in

(b) implies (a). Let us now conclude the proof of 2.1(b). Consider the right hand side of the diagram (5):

$$\begin{array}{ccccc} \tilde{G} & \xrightarrow{\tilde{u}} & \tilde{G}'' & \longrightarrow & 0 \\ {\scriptstyle g}\downarrow & & {\scriptstyle g''}\downarrow & & \\ G & \xrightarrow{u} & G''. & & \end{array}$$

The morphism g'' is separated and an isomorphism on the generic fiber. It follows then from 2.3 that g'' is a finite sequence of dilatations with smooth centers. By construction, the same holds for g.

We apply Lemma 2.6(c) to the maps g and g'' and find that there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$, the natural maps $\operatorname{Coker}(g_i) \to \operatorname{Coker}(g_{i-1})$ and the natural maps $\operatorname{Coker}(g_i'') \to \operatorname{Coker}(g_{i-1}'')$ are all isomorphisms of smooth group schemes. Denote by C and C'' the smooth groups $\operatorname{Coker}(g_{i_0})$ and $\operatorname{Coker}(g_{i_0}'')$, respectively, and let $u_0 : C \to C''$ denote the map between them induced by u. Denote by D/k the (smooth) group $\operatorname{Coker} u_0$. It follows from Lemma 2.6(b) and the exact sequence (6) that we have a commuting diagram with vertical isomorphisms:

$$\begin{array}{ccccccccc} & & C(k) & \xrightarrow{u_0(k)} & C''(k) & \longrightarrow & D(k) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & G(S)/\tilde{G}(S) & \xrightarrow{u} & G''(S)/\tilde{G}''(S) & \longrightarrow & G''(S)/G(S) & \longrightarrow & 0. \end{array}$$

We find that the map $u_0(k)$ is injective and, thus, we can conclude with Lemma 2.6(c) that

$$\dim(D) = \dim(C'') - \dim(C) = \ell(\operatorname{Coker}(\operatorname{Lie}(g''))) - \ell(\operatorname{Coker}(\operatorname{Lie}(g))).$$

To conclude the proof of 2.1, it suffices to apply Equality (7). □

Remarks 2.8. a) Keep the hypotheses of 2.1, and assume in addition that the morphism $u : G \to G''$ is faithfully flat, so that the sequence $0 \to G' \to G \to G'' \to 0$ is exact in the fppf-topology. In this case, the three invariants appearing in the statement of 2.1 depend on G'/S only, and not on the expression of G' as the kernel of a morphism of smooth group schemes. Indeed, this is clearly the case for $\ell(\operatorname{Lie}(G')/\operatorname{Lie}(\tilde{G}'))$. The cokernel $G''(S)/G(S)$ can be identified with the cohomology group $H^1_{fppf}(G')$, as $H^1_{fppf}(G) = \{0\}$ since G/S is smooth. Finally, as G'/S is flat, $\operatorname{Lie}(G'')/\operatorname{Lie}(G)$ can be computed in terms of the cotangent complex of G' (see [21] (4.3.3)).

b) It is natural to wonder whether Part (a) of Theorem 2.1 still holds when k is only assumed to be separably closed. In other words, if $G(S) \to G''(S)$ is surjective, is it true that $\ell(\operatorname{Lie}(G')/\operatorname{Lie}(\tilde{G}')) = \ell(\operatorname{Coker}(\operatorname{Lie}(u)))$? This is the case when G'/S is smooth.

c) Suppose that $\dim(G'_K) = 0$. Then G'_K/K is étale in our context, and we have $\mathrm{Lie}\,(\tilde{G}') = \mathrm{Lie}\,(G')$. When G'_K/K is only assumed to be finite and $G \to G''$ is faithfully flat, the computation of $\dim(D)$ in 2.1(b) is found in [5], 4.2.2.

Examples 2.9. All terms in the formula of 2.1(b) can be non-zero.

a) For instance, $\dim(D) \neq 0$ for any non-trivial dilatation $u : G \to G''$ of any smooth group scheme G''/S. A very interesting sequence of dilatations is obtained as follows. Let L/K be a finite extension, let $S' = \mathrm{Spec}\,\mathcal{O}_L$ and let G_K be an algebraic group. Suppose that the Néron model G/S of G_K and the Néron model H/S' of G_L exist. We obtain in this case a natural map $G \times_S S' \to H$. The length of the cokernel $\mathrm{Lie}\,(H)/\,\mathrm{Lie}\,(G \times_S S')$ is studied in [9] and [10].
b) To find an example where $\ell(\mathrm{Lie}\,(G')/\,\mathrm{Lie}\,(\tilde{G}')) \neq 0$, consider a finite separable totally ramified extension L/K, and let $G \to G''$ be the norm map $N : R_{S'/S}\mathbb{G}_{m,S'} \to \mathbb{G}_{m,S}$ (where $R_{S'/S}$ is the Weil restriction to S, see [7], 7.6). The kernel G' of this map is usually denoted by $R^1_{S'/S}\mathbb{G}_{m,S'}$. The reader will verify that there is a natural commutative diagram with vertical isomorphisms

$$\begin{array}{ccc} \mathrm{Lie}\,(G) & \xrightarrow{\mathrm{Lie}\,(N)} & \mathrm{Lie}\,(G'') \\ \uparrow & & \uparrow \\ \mathcal{O}_L & \xrightarrow{\mathrm{Tr}_{L/K}} & \mathcal{O}_K \end{array}$$

We find that $\ell(\mathrm{Lie}\,(G'')/\,\mathrm{Lie}\,(G)) = \ell(\mathcal{O}_K/\mathrm{Tr}_{L/K}(\mathcal{O}_L))$. Let $\mathcal{D}_{L/K}$ denote the different of L/K. As in [53], V.3, Lemma 4, when $[L : K]$ is prime, we find that $\ell(\mathcal{O}_K/\mathrm{Tr}_{L/K}(\mathcal{O}_L))$ is equal to the integer part of $v_L(\mathcal{D}_{L/K})/[L : K]$.
Assume now that K is complete with algebraically closed residue field. Then it is known that $\mathrm{Norm}(L^*) = K^*$ for any finite separable extension L/K (see, e.g., [53], V.5, prop. 7). Since $v_K(\mathrm{Norm}(\alpha)) = v_L(\alpha)$, we find that $\mathrm{Norm}(\mathcal{O}_L^*) = \mathcal{O}_K^*$, so that the map $N(S)$ is surjective. It follows from 2.1 that $\ell(\mathrm{Lie}\,(G')/\,\mathrm{Lie}\,(\tilde{G}')) = \ell(\mathrm{Lie}\,(G'')/\,\mathrm{Lie}\,(G))$. The latter integer is not zero when L/K is wildly ramified. The smoothening map $\tilde{G}' \to G'$ is described in [33], 5.6, when L/K is cyclic of degree p.
c) Let us return to the general formula of 2.1(b). Suppose that G and G'' are the Néron models of two abelian varieties G_K and G''_K, with G semistable. When $v(p) < p-1$, then $\ell(\mathrm{Lie}\,(G'')/\,\mathrm{Lie}\,(G)) = 0$ ([7], 7.5/4), and when $v(p) = p-1$, then $\ell(\mathrm{Lie}\,(G')/\,\mathrm{Lie}\,(\tilde{G}')) = 0$ and $p-1 \mid \dim(D)$ ([1], Theorem A.1).

It is possible to slightly generalize Theorem 2.1 as follows. Let

$$G^\bullet : \quad \ldots \longrightarrow G^i \xrightarrow{u_i} G^{i+1} \longrightarrow \ldots$$

be a bounded complex of separated smooth S-group schemes of finite type. Assume that $G^\bullet$ is exact on the generic fiber. When k is algebraically closed, we associate to this complex two invariants as follows. Consider the associated complex of Lie algebras

$$\mathrm{Lie}\,(G^\bullet): \quad \ldots \longrightarrow \mathrm{Lie}\,(G^i) \xrightarrow{\mathrm{Lie}\,(u_i)} \mathrm{Lie}\,(G^{i+1}) \longrightarrow \ldots.$$

By hypothesis, this complex of $\mathcal{O}_K$-modules becomes exact when tensored with K. Thus, we can define

$$h^i_{\mathrm{Lie}}\,(G^\bullet) := \ell(\mathrm{Ker}\,(\mathrm{Lie}\,(u_i))/\mathrm{Im}(\mathrm{Lie}\,(u_{i-1})),$$

$$\text{and} \ \ \chi_{\mathrm{Lie}}\,(G^\bullet) := \sum_i (-1)^i h^i_{\mathrm{Lie}}\,(G^\bullet).$$

Note that one can define a Cartier divisor $D := \mathrm{Div}(\mathrm{Lie}\,(G^\bullet))$ on S ([26], pp. 47–48, see also 5.3) such that $\det(\mathrm{Lie}\,(G^\bullet))$ is canonically isomorphic to $\mathcal{O}_S(D)$, and $\chi_{\mathrm{Lie}}\,(G^\bullet)$ is then equal to the degree of D.

Consider a morphism $u : F \to G$ of separated S-group schemes of finite type, with F_K and G_K smooth and $F_K \to G_K$ smooth and surjective. Then the group $\mathrm{Coker}\,(u(S))$ can be endowed with the structure of a smooth algebraic group over k. Indeed, consider the group smoothenings $\tilde{F}$ and $\tilde{G}$ of F and G, and let $\tilde{u} : \tilde{F} \to \tilde{G}$ be the associated morphism. Recall that $\tilde{F}(S) = F(S)$ and $\tilde{G}(S) = G(S)$. Then, as we showed in 2.1, there exists a smooth group scheme of finite type D/k such that $D(k) = \mathrm{Coker}\,(\tilde{u}(S))$.

For each map $u_i : G^i \to G^{i+1}$, consider the associated morphism $G^{i-1} \to \mathrm{Ker}\,(u_i)$, again denoted by u_{i-1}. Let D_i/k denote the group scheme such that $D_i(k) = \mathrm{Ker}\,(u_i(S))/\mathrm{Im}(u_{i-1}(S))$. Define

$$h^i_{\mathrm{points}}(G^\bullet) := \dim(D_i) \qquad \text{and} \qquad \chi_{\mathrm{points}}(G^\bullet) := \sum_i (-1)^i h^i_{\mathrm{points}}(G^\bullet).$$

Theorem 2.10. *Let $G^\bullet$ be a complex of S-group schemes as above. Then*

$$\chi_{\mathrm{Lie}}\,(G^\bullet) = \chi_{\mathrm{points}}(G^\bullet).$$

Proof. Left to the reader. Proceed by induction on the length of the complex.

3. Comparison of Lie algebras

Let $S = \mathrm{Spec}\,\mathcal{O}_K$, and let $f : X \to S$ be a proper and flat curve, with X regular and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let J/S denote the Néron model of $\mathrm{Jac}\,(X_K)$. We want to compare $\mathrm{Lie}\,(\mathrm{Pic}_{X/S})$ with $\mathrm{Lie}\,(J)$. As we shall see, these $\mathcal{O}_K$-modules are naturally isomorphic when k is perfect and X is cohomologically flat in dimension zero ([7], p. 206) or, equivalently, when k is perfect and $H^1(X, \mathcal{O}_X)$ is torsion-free. In the case where X is not cohomologically flat in dimension zero, the functor $\mathrm{Pic}_{X/S}$ is, unfortunately, not representable by an algebraic space over S ([48], 2.4.4, or [7], 8.3/2). To alleviate this technical difficulty, the rigidified Picard functors introduced in [48], 2.1.2 and 2.4.1 (or [7], 8.1/11), allow for a resolution of $\mathrm{Pic}_{X/S}$ by a sequence of group spaces. We recall below this theory, following the notation of [7]. We will show that when S is strictly henselian, $\mathrm{Pic}_{X/S}$ has in fact a resolution by group *schemes* (3.2). Consideration of this resolution and its associated sequence of Lie algebras allows us then to prove our main theorem 3.1.

A *rigidificator* Y/S of the relative Picard functor $\mathrm{Pic}_{X/S}$ is a subscheme $Y \subset X$ which is finite and faithfully flat over S, and such that the map

$$H^0(X_T, \mathcal{O}_{X_T}) \longmapsto H^0(Y_T, \mathcal{O}_{Y_T}),$$

induced from the inclusion $Y_T \to X_T$, is injective for all S-schemes T. It is shown in [48], 2.2.3, that $\mathrm{Pic}_{X/S}$ has a rigidificator. An invertible sheaf on X *rigidified along* Y is defined to be a pair $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible sheaf on X and $\alpha : \mathcal{O}_Y \to \mathcal{L}|_Y$ is an isomorphism. Two pairs $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ are said to be isomorphic if there is an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ such that $\varphi|_Y \circ \alpha = \alpha'$. We let

$$(\mathrm{Pic}_{X/S}, Y) : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$$

denote the group functor that associates to any S-scheme T the set of isomorphism classes of line bundles on X_T rigidified on Y_T. We let

$$h : (\mathrm{Pic}_{X/S}, Y) \longrightarrow \mathrm{Pic}_{X/S}$$

denote the natural transformation of functors obtained by forgetting the rigidification. Both functors are formally smooth ([7], 8.4/2), and h is an epimorphism of étale sheaves. The functor $(\mathrm{Pic}_{X/S}, Y)$ is always representable by an algebraic space over S ([48], 2.3.1). The functor $\mathrm{Pic}_{X/S}$ is representable by an algebraic space over S if and only if X/S is cohomologically flat in dimension 0 ([48], 2.4.4).

Let P and (P, Y) denote respectively the subfunctors of $\mathrm{Pic}_{X/S}$ and $(\mathrm{Pic}_{X/S}, Y)$ consisting of the line bundles of total degree 0 (see [7], p. 265). We will denote again by h the map $h|_{(P,Y)}$.

Consider the subfunctor $\mathrm{Ker}(h)$ of (P, Y). Since the generic fiber of P is representable, so is the generic fiber $\mathrm{Ker}(h_K)$ of $\mathrm{Ker}(h)$. Let then H denote the schematic closure of $\mathrm{Ker}(h_K)$ in (P, Y) (see [48], 3.2 c)). Proposition 9.5.3 in [7] states that the $fppf$-quotient $Q := (P, Y)/H$ is representable by a smooth and separated group scheme over S. The map $(P, Y) \to Q$ factors through $(P, Y) \to P$, to give a map

$$q : P \to Q$$

which is an isomorphism on the generic fiber. Thus, Q_K coincides with the Jacobian J_K of X_K. Since X is regular, the group scheme Q/S is of finite type ([7], 9.5.11). Our main theorem in this section is:

Theorem 3.1. *Let $\mathcal{O}_K$ be a discrete valuation ring. Assume that $f : X \to S$ is a proper and flat curve, with X regular and $f_*\mathcal{O}_X = \mathcal{O}_S$. Consider the natural map of Lie algebras*

$$\mathrm{Lie}\,(q) : \mathrm{Lie}\,(P) \longrightarrow \mathrm{Lie}\,(Q)$$

induced by the map $q : P \to Q$. Then

(a) *The kernel of* $\mathrm{Lie}\,(q)$ *is isomorphic to the torsion subgroup of $H^1(X, \mathcal{O}_X)$.*
(b) *Assume that the residue field k is perfect (in which case Q is the Néron model of J_K, see 3.7). Then the kernel and cokernel of* $\mathrm{Lie}\,(q)$ *have the same length.*

Proof. We saw in Proposition 1.3 that $\mathrm{Lie}\,(P)$ can be canonically identified with the $\mathcal{O}_K$-module $H^1(X, \mathcal{O}_X)$. To prove (a), let us note that the map $q : P \to Q$ is an isomorphism on the generic fiber. Hence, the kernel of the map $\mathrm{Lie}\,(q)$ is equal to the torsion submodule of $\mathrm{Lie}\,(P)$ since the $\mathcal{O}_K$-module $\mathrm{Lie}\,(Q)$ is torsion-free, Q/S being a smooth group scheme.

The proof of Part (b) will occupy the rest of this section. Let us start by noting that it is sufficient to prove the case where $\mathcal{O}_K$ is strictly henselian and even complete. It is only when using the main result of Sect. 2, Theorem 2.1, that we will need to distinguish between separably closed and algebraically closed residue fields. Thus, until the end of this section, we only assume that k is separably closed.

When P/S is an algebraic space, the proof of (b) is substantially simplified: the map $P \to Q$ is an étale map of algebraic spaces and, thus, the map $\mathrm{Lie}\,(P) \to \mathrm{Lie}\,(Q)$ is an isomorphism (1.1(c)).

Let us recall now the description of the kernel $\mathrm{Ker}\,(h)$ of the map $h : (\mathrm{Pic}_{X/S}, Y) \to \mathrm{Pic}_{X/S}$ (see for instance pp. 206–209 of [7]). Clearly, given T/S, we can map a global invertible section a on $Y \times_S T$ to the pair $(\mathcal{O}_{X_T}, \alpha)$ in $\mathrm{Ker}\,(h)(T)$, where the isomorphism $\alpha : (\mathcal{O}_{X_T})_{|Y_T} \to (\mathcal{O}_{X_T})_{|Y_T}$ is the multiplication by a. One shows that given Z/S proper and flat, the functor

$$(\mathrm{Sch})^0 \to (\mathrm{Sets}), \quad T \mapsto H^0(Z_T, \mathcal{O}_{Z_T})$$

is representable by a scheme V_Z/S, and that the subfunctor of units

$$(\mathrm{Sch})^0 \to (\mathrm{Sets}), \quad T \mapsto H^0\big(Z_T, \mathcal{O}^*_{Z_T}\big)$$

is represented by a smooth open subscheme V^*_Z of V_Z. When $Y \subset X$ is a rigidificator, the induced morphism $V_X \to V_Y$ is a closed immersion which induces an immersion $V^*_X \to V^*_Y$ of group schemes. The following sequence is an exact sequence of sheaves with respect to the étale topology ([48], 2.1.2, 2.4.1):

$$(9) \qquad 0 \longrightarrow V^*_X \longrightarrow V^*_Y \longrightarrow (\mathrm{Pic}_{X/S}, Y) \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow 0.$$

This is the 'resolution of $\mathrm{Pic}_{X/S}$ by group spaces' alluded to at the beginning of this section. In order to obtain objects that are of finite type, we may pass to the analogue commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V_X^* \arrow[r] & V_Y^* \arrow[r] \arrow[d] & (P, Y) \arrow[r, "h"] \arrow[d, equal] & P \arrow[r] \arrow[d, "q"'] & 0 \\
 & 0 \arrow[r] & H \arrow[r] & (P, Y) \arrow[r, "\bar{h}"] & Q \arrow[r] & 0.
\end{tikzcd}$$

Since V_Y^*/S is flat, we find that $\mathrm{Ker}(h) \subseteq H$. □

Proposition 3.2. *The functor* $(\mathrm{Pic}_{X/S}, Y)$ *is representable by a smooth group space. Its connected component of zero* $(\mathrm{Pic}_{X/S}, Y)^0$ *is represented by a separated smooth group scheme. If S is strictly henselian, then* $(\mathrm{Pic}_{X/S}, Y)$ *is representable by a smooth group scheme (not separated in general).*

Proof. Recall that since (P, Y) and $(\mathrm{Pic}_{X/S}, Y)$ are represented by algebraic spaces smooth along the unit section, $(P, Y)^0$ and $(\mathrm{Pic}_{X/S}, Y)^0$ are represented by open subspaces (see [SGA3] VI_B, 3.10, p. 344, for group schemes).

As we mentioned above already, $(\mathrm{Pic}_{X/S}, Y)^0$ is represented by a smooth group space. To show that $(\mathrm{Pic}_{X/S}, Y)^0$ is represented by a separated smooth group scheme, it is sufficient to show that $(\mathrm{Pic}_{X/S}, Y)^0$ is a separated group space (definition in [48], 3.2 a)). Indeed, any separated group space G/S is a group scheme ([3], 4.B, or [48], 3.3.1, when G/S is smooth). To show that a group space G/S is separated, it is sufficient to show that the unit section of G is closed. Indeed, let E/S denote the schematic closure in G of the unit section of G_K. Then the quotient G/E is a separated group scheme ([48], 3.3.5).

To show that the zero section of $G := (\mathrm{Pic}_{X/S}, Y)^0$ is closed, we proceed as follows. The morphism $\epsilon : S \to G$ is an immersion, closed on the generic fiber, and it is closed if, for each étale covering $U \to G$ (with U a scheme), the map $S \times_G U \to U$ is a closed immersion[1]. The latter property can be checked after a base change $T \to S$ which is faithfully flat and quasi-compact ([EGA] IV, 2.6.2), so we can assume that S is strictly henselian. We claim that to prove $\epsilon : S \to G$ is a closed immersion, it suffices to check that the set $G_0(S)$ of sections $\mu \in G(S)$ such that $\mu_K = \epsilon_K$ contains only the section ϵ. Indeed, for any section $\mu : S \to U$ such that $\mu_K = \rho_K$ for some section $\rho : S \to S \times_G U$, we have $\mu \in (S \times_G U)(S)$ (composing μ with $U \to G$ gives rise to a section $S \to G$ in $G_0(S) = \{\epsilon\}$). The reader will check that this property implies that $S \times_G U \to U$ is a closed immersion,

[1] Note that the criterion given in [7], p. 225, states that ϵ is a closed immersion if, for any scheme Y and morphism $Y \to G$, $Y \times_G S$ is a scheme and $Y \times_G S \to Y$ is a closed immersion. Since ϵ is an immersion, we find that $Y \times_G S$ is a scheme ([25], p. 109). The proof of the equivalence between the two criteria is left to the reader.

having in mind that $S \times_G U$ is a union of copies of S, and $S \times_G U \to U$ is an immersion, closed above the generic point of S.

Let us show that the set of sections of $(\mathrm{Pic}_{X/S}, Y)^0(S)$ reducing to ϵ_K consists only in the unit section. Consider the natural exact sequence

(10) $0 \to \mathrm{Pic}\,(T) \to \mathrm{Pic}\,(X \times_S T) \to \mathrm{Pic}_{X/S}(T) \to \mathrm{Br}\,(T) \to \mathrm{Br}\,(X \times_S T).$

Since S is strictly henselian, $\mathrm{Br}\,(S) = (0)$ and we have $\mathrm{Pic}\,(X) = \mathrm{Pic}_{X/S}(S)$. Note that by setting $T = \mathrm{Spec}\,(K)$ in (10) we obtain the exact sequence:

(11) $0 \to \mathrm{Pic}\,(X_K) \to \mathrm{Pic}_{X/S}(K) \to \mathrm{Br}\,(K) \to \mathrm{Br}\,(X_K),$

which will be used in the next sections. Thus, every element of $(\mathrm{Pic}_{X/S}, Y)^0(S)$ is represented by a pair of the form $(\mathcal{L}, \alpha)$, where $\mathcal{L} \in \mathrm{Pic}\,(X)$. (Note that $(\mathrm{Pic}_{X/S}, Y)$ is an fppf-sheaf, while $\mathrm{Pic}_{X/S}$ is only the fppf-sheaf associated with the presheaf $T \to \mathrm{Pic}\,(X \times_S T)$ ([48], 2.1.2). If $\mathcal{L}$ is trivial on the generic fiber, we can identify $(\mathcal{L}, \alpha)$ with $(\mathcal{O}_X(D), \alpha)$, where D is a divisor supported on the special fiber. Now the condition that the element is in $(\mathrm{Pic}_{X/S}, Y)^0(S)$ implies that we can assume that $D = qr^{-1}X_k$ with $0 \le q \le r-1$, where r is the gcd of the multiplicities of the irreducible components of X_k. We have to show that $(\mathcal{O}_X(D), \alpha) \simeq (\mathcal{O}_X, \mathrm{Id})$.

By hypothesis, $(\mathcal{O}_{X_K}, \alpha_K) \simeq (\mathcal{O}_{X_K}, \mathrm{Id})$. So there exists $a \in H^0(X_K, \mathcal{O}^*_{X_K}) = K^*$ such that $\alpha_K : \mathcal{O}_{Y_K} \to \mathcal{O}_{Y_K}$ maps 1 to a. Let $y \in Y$. Consider the isomorphism

$$\alpha_y : \mathcal{O}_{Y,y} \to \mathcal{O}_X(D) \otimes \mathcal{O}_{Y,y}.$$

There exists a basis e of $\mathcal{O}_X(D)_y$ such that $\bar{e} := \alpha_y(1) = a$ in $\mathcal{O}_X(D) \otimes \mathcal{O}_{Y,y} \otimes K$. This implies that $e = a$ in $\mathcal{O}_X(D) \otimes \mathcal{O}_{Y,y}$ and, thus $a = e + eb$ in $\mathcal{O}_X(D)_y$ for some $b \in \mathcal{O}_X(-Y)_y \subset \mathfrak{m}_y \mathcal{O}_{X,y}$. Let Γ be an irreducible component of X_k passing through y, of multiplicity s in X_k. The element a, being, up to a unit, a power of π, is a function having on Γ a pole of order a multiple of s. Since $b+1$ is a unit at y, we see that $e(b+1)$ has a pole on Γ of order exactly qs/r. It follows that $r \mid q$, which implies that $q = 0$ and $a \in \mathcal{O}^*_K$. Thus $\mathcal{O}_X(D) = \mathcal{O}_X$ and $(\mathcal{O}_X(D), \alpha) \simeq (\mathcal{O}_X, \mathrm{Id})$.

Assume now that $(\mathrm{Pic}_{X/S}, Y)^0$ is a scheme. Since $(\mathrm{Pic}_{X/S}, Y)$ is a smooth algebraic space, it is equal to the schematic closure of its generic fiber. Proposition 3.3.6 2) of [48] implies that when k is separably closed and $(\mathrm{Pic}_{X/S}, Y)^0$ is separated, then $(\mathrm{Pic}_{X/S}, Y)$ is a scheme. □

Let us now consider the Lie algebras associated with the resolution of $\mathrm{Pic}_{X/S}$. The functorial definitions of V^*_X and V^*_Y recalled above immediately allows the computations of their Lie algebras, and one find that the following natural diagram with vertical isomorphisms of $\mathcal{O}_K$-modules is commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_Y(Y) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Lie}\,(V^*_X) & \longrightarrow & \mathrm{Lie}\,(V^*_Y). \end{array}$$

Proposition 3.3. *Keep the above notation. The exact sequence of Lie algebras associated with the exact sequence of functors* (9) *is exact:*

$$0 \to \mathrm{Lie}(V_X^*) \longrightarrow \mathrm{Lie}(V_Y^*) \longrightarrow \mathrm{Lie}(\mathrm{Pic}_{X/S}, Y) \longrightarrow \mathrm{Lie}(\mathrm{Pic}_{X/S}) \to 0.$$

Proof. In the proof that the functor $(\mathrm{Pic}_{X/S}, Y)$ is representable ([48], 2.3.1), the existence of a natural exact sequence of the form

$$0 \to \mathcal{O}_X(X) \longrightarrow \mathcal{O}_Y(Y) \longrightarrow \mathrm{Lie}(\mathrm{Pic}_{X/S}, Y) \longrightarrow H^1(X, \mathcal{O}_X) \to 0,$$

is established in $(*)$ on p. 36, taking, in the notation of the bottom of p. 35, $\mathcal{A}_0 = \mathcal{O}_S$ and $\mathcal{A}' = \overline{\mathcal{A}} = \mathcal{O}_{S_\varepsilon}$. We established in 1.3(b) that $H^1(X, \mathcal{O}_X)$ is canonically isomorphic with $\mathrm{Lie}(\mathrm{Pic}_{X/S})$. We leave it to the reader to check that the four-term sequence above can be naturally identified with the four-term sequence in the statement of the proposition. □

Remark 3.4. Let $I \subset \mathcal{O}_X$ denote the sheaf of ideals defining the subscheme Y. The exact sequence of coherent sheaves $0 \to I \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ induces the long exact sequence

$$(12) \qquad 0 \longrightarrow f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y \longrightarrow R^1 f_* I \longrightarrow R^1 f_*\mathcal{O}_X \longrightarrow 0$$

of coherent sheaves on S. The exact sequence of Lie algebras associated with the exact sequence of functors (9) can be canonically identified with the exact sequence of global sections over S associated with the exact sequence (12).

Let S be strictly henselian. Since $(\mathrm{Pic}_{X/S}, Y)$ is a scheme, (P, Y) is an open and closed subscheme and $(P, Y)^0$ is an open subscheme of (P, Y). Consider the closed subscheme $H_1 := H \cap (P, Y)^0$ and its open subscheme H_1^0/S. Let $\tilde{H}_1/S$ be the group smoothening of H_1/S ([7], pp. 174–175).

Lemma 3.5. (a) *The map $V_Y^* \to (P, Y)$ factors through $V_Y^* \to H_1 \subset (P, Y)^0$*

(b) *Let r be the gcd of the multiplicities of the irreducible components of X_k. Then we have an exact sequence*

$$V_Y^*(S) \to H_1(S) \to \mathbb{Z}/r\mathbb{Z} \to 0.$$

(c) *There are natural immersions of group schemes*

$$\mathbb{G}_{m,S} \longrightarrow V_X^* \longrightarrow V_Y^*,$$

with $\mathbb{G}_{m,S} \to V_Y^*$ a closed immersion.

(d) *The map $V_Y^* \to H_1$ factors through $\tilde{H}_1$ and the sequence*

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow V_Y^* \longrightarrow \tilde{H}_1^0 \longrightarrow 0$$

is exact.

(e) *We have a natural diagram with exact lines:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Lie}(\tilde{H}_1) & \longrightarrow & \mathrm{Lie}((P,Y)^0) & \xrightarrow{Lie(h)} & \mathrm{Lie}(P^0) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle Lie(q)} & & \\
0 & \longrightarrow & \mathrm{Lie}(H_1) & \longrightarrow & \mathrm{Lie}((P,Y)^0) & \xrightarrow{\mathrm{Lie}(\bar{h})} & \mathrm{Lie}(Q^0). & &
\end{array}$$

Proof. Let $x = (\mathcal{L}, \alpha) \in H_1(S)$. Then on the generic fiber, $\mathcal{L}_K \simeq \mathcal{O}_{X_K}$. So up to isomorphism of $(\mathcal{L}, \alpha)$, we can write $\mathcal{L} = \mathcal{O}_X(D)$ for some vertical divisor D. On the special fiber, we have $\mathcal{L}|_{X_k} \in P_k^0$, which means that $\deg \mathcal{L}|_\Gamma = 0$ for all irreducible components Γ of X_k. Hence, $D = qr^{-1}X_k$ for some $q \in \mathbb{Z}$.

(a) and (b). Since V_Y^* is connected, it maps to $(P, Y)^0$ with image in H. This gives a canonical morphism $V_Y^* \to H_1$. The map

$$H_1(S) \to \mathbb{Z}/r\mathbb{Z}, \quad \big(\mathcal{O}_X(qr^{-1}X_k), \alpha\big) \mapsto \bar{q} \in \mathbb{Z}/r\mathbb{Z}$$

is surjective. An element $(\mathcal{O}_X(D), \alpha)$ is in the kernel if and only if $D \in \mathbb{Z}X_k$, which is equivalent to $\mathcal{O}_X(D) \simeq \mathcal{O}_X$ and, hence, equivalent to $(\mathcal{O}_X(D), \alpha) \in \mathrm{Ker}(h)(S) = \mathrm{Im}(V_Y^*(S))$.

(c) Recall that the subscheme $Y \subset X$ is finite and flat over S. It follows that the group scheme V_Y^*/S is nothing but the Weil restriction $R_{Y/S}\mathbb{G}_{m,Y}$ ([7], 7.6). Indeed, by definition, the group scheme $R_{Y/S}\mathbb{G}_{m,Y}$ represents the functor $T \to \Gamma(Y_T, \mathcal{O}_{Y_T}^*)$, and is thus equal to V_Y^*. Note now the inclusions

$$\Gamma\big(T, \mathcal{O}_T^*\big) \subseteq \Gamma\big(X_T, \mathcal{O}_{X_T}^*\big) \subseteq \Gamma\big(Y_T, \mathcal{O}_{Y_T}^*\big).$$

We thus find natural morphisms of representable functors

$$\mathbb{G}_{m,S} \to V_X^* \to V_Y^*.$$

The map $Y \to S$ corresponds to a morphism of rings $\mathcal{O}_K \to A$. Using a basis for A over $\mathcal{O}_K$, it is easy to see that the composition $\mathbb{G}_{m,S} \to V_Y^*$ is given by a surjective morphism of rings and is thus a closed immersion.

(d) The quotient $V_Y^*/\mathbb{G}_{m,S}$ is a smooth group scheme (see [3], 4.C, for the existence of the quotient. Since $\mathbb{G}_{m,S}$ is a torus and V_Y^* is affine, see also [SGA3] VIII, 5.7, p. 19. For smoothness, see [SGA3] VI_B 9.2 xii, p. 380) and the map $V_Y^* \to H_1$ factors through $V_Y^*/\mathbb{G}_{m,S}$. Since $V_Y^*/\mathbb{G}_{m,S}$ is smooth, the map $V_Y^*/\mathbb{G}_{m,S} \to H_1$ factors through a map $g : V_Y^*/\mathbb{G}_{m,S} \to \tilde{H}_1$. Let us show that g is an open immersion. Clearly, g_K is an isomorphism. Recall that since $\tilde{H}_1 \to H_1$ is a group smoothening, $\tilde{H}_1(S) \to H_1(S)$ is a bijection. Part (b) shows then that the cokernel of the map g_s on the special fiber is finite. Since both $V_Y^*/\mathbb{G}_{m,S}$ and $\tilde{H}_1$ are flat and the dimension of their generic fibers are equal, the dimension of their special fibers are also equal. It follows that the morphism g_s is quasi-finite and, thus, so is g. Since $V_Y^*/\mathbb{G}_{m,S} \to S$ is affine, g is separated. We may thus apply Zariski's Main

Theorem ([EGA], IV_3, 8.12.10), to show that g is an open immersion. Since the fibers of $V_Y^*/\mathbb{G}_{m,S}$ are connected, g is an isomorphism between $V_Y^*/\mathbb{G}_{m,S}$ and $\tilde{H}_1^0$.

(e) The morphisms $\mathbb{G}_{m,S} \to V_X^* \to V_Y^*$ induce the maps of Lie algebras $\mathcal{O}_S \to f_*\mathcal{O}_X \to \mathrm{Lie}(V_Y^*)$ (see the commutative square before 3.3). Since, by hypothesis, $\mathcal{O}_S = f_*\mathcal{O}_X$, we find that the exact sequence of smooth group schemes in Part (b) of our lemma gives $\mathrm{Lie}(\tilde{H}_1) = \mathrm{Lie}(V_Y^*)/\mathrm{Lie}(V_X^*)$. The four-term sequence of Lie algebras in 3.3 can thus be replaced by the exact sequence:

$$0 \longrightarrow \mathrm{Lie}(\tilde{H}_1) \longrightarrow \mathrm{Lie}((P, Y)^0) \longrightarrow \mathrm{Lie}(P^0) \longrightarrow 0. \qquad \square$$

To conclude the proof of Theorem 3.1, note first that it follows immediately from Diagram 3.5 (e) that

$$\mathrm{Ker}(\mathrm{Lie}(q)) \simeq \mathrm{Lie}(H_1)/\mathrm{Lie}(\tilde{H}_1), \quad \mathrm{Coker}(\mathrm{Lie}(q)) \simeq \mathrm{Coker}(\mathrm{Lie}(\overline{h})).$$

Assume now that k is algebraically closed. Consider the morphism of smooth separated group schemes

$$(P, Y)^0 \xrightarrow{\overline{h}} Q^0$$

and apply to it 2.1. Since the induced map on S-points, $(P, Y)^0(S) \to Q^0(S)$, is surjective ([7], p. 275), Theorem 2.1 implies that $\mathrm{Lie}(H_1)/\mathrm{Lie}(\tilde{H}_1)$ has the same length as $\mathrm{Coker}(\mathrm{Lie}(\overline{h}))$, and Part (b) follows. $\square$

Remark 3.6. When k is imperfect, we have been unable to find examples where the lengths of $\mathrm{Ker}(\mathrm{Lie}(q))$ and $\mathrm{Coker}(\mathrm{Lie}(q))$ are not equal. In a very particular case (8.13), we are able to show that these lengths are in fact equal.

In order to state our applications of Theorem 3.1, let us recall the relationship between the group scheme Q and the Néron model of the Jacobian J_K of X_K.

Facts 3.7. Let $f : X \to S$ be a proper and flat curve, with X regular and $f_*\mathcal{O}_X = \mathcal{O}_S$.

(a) If k is perfect or if X/S has a section on $\mathcal{O}_K^{\mathrm{sh}}$, then the group scheme Q/S is the Néron model of J_K/K.
(b) Suppose S strictly henselian. Then Q is the Néron model of J_K if $\mathrm{Pic}^0(X_K) \to J_K(K)$ is an isomorphism. The latter happens for instance if the residue field k of $\mathcal{O}_K$ is algebraically closed (see also [48], 8.1.4 b)).

Proof. (a) When X is regular and $f_*\mathcal{O}_X = \mathcal{O}_S$, the generic fiber X_K/K is geometrically irreducible. Indeed, $X_{K^{sep}}/K^{sep}$ is regular and connected, so it is irreducible. Since $X_{K^{alg}} \to X_{K^{sep}}$ is a homeomorphism, $X_{K^{alg}}$ is irreducible. Part (a) follows then from [7], 9.5.4. (b) Most of this statement is contained in the proofs of [7], 9.5.4 and 9.5.2. The key is to note that $\mathrm{Pic}^0(X_K) \to P(K) = J_K(K)$ is an isomorphism if and only if the canonical map $P(S) \to P(K)$ is surjective. Indeed, since X is regular, each line bundle on X_K extends to a line bundle on X, and using [7], 9.1/2, we find that this extension is in $P(S)$. So the image of $P(S) \subseteq \mathrm{Pic}(X)$ in $P(K)$ is equal to $\mathrm{Pic}^0(X_K)$. Since Q/S is a smooth separated group scheme of finite type, it follows from [7], 7.1/1, that Q is the Néron model of its generic fiber if and only if $Q(S) \to Q(K)$ is surjective. Since the map $P(K) \to Q(K)$ is the identity, we find that if $P(S) \to P(K)$ is surjective, then $Q(S) \to Q(K)$ is surjective. This achieves the proof. □

Let J/S denote the Néron model of J_K/K. Examples where the natural maps $Q \to J$ and $\mathrm{Lie}(Q) \to \mathrm{Lie}(J)$ are not isomorphisms are given in 9.3. We do not have an example where $P(S) \to Q(S)$ is not surjective.

The following corollary to 3.1 is an essential ingredient in the proof of our theorem 6.6 on the reduction of curves of genus 1.

Theorem 3.8. *Let X_K/K be a smooth projective geometrically connected curve of genus* 1, *and denote by E_K/K its Jacobian. Let X/S and E/S be regular models of X_K and E_K. Then there exists a homomorphism of $\mathcal{O}_K$-modules*

$$\tau_X : H^1(X, \mathcal{O}_X) \longrightarrow H^1(E, \mathcal{O}_E)$$

such that the following diagram is commutative:

$$\begin{tikzcd} H^1(X, \mathcal{O}_X) \arrow[r, "\tau_X"] \arrow[d] & H^1(E, \mathcal{O}_E) \arrow[d] \\ H^1(X_K, \mathcal{O}_{X_K}) \arrow[r, "\tau_{X_K}"] & H^1(E_K, \mathcal{O}_{E_K}), \end{tikzcd}$$

where the vertical maps are the natural ones induced by the open immersions $X_K \subset X$ and $E_K \subset E$, and τ_{X_K} is described in 1.4. *When k is perfect, the kernel and cokernel of τ_X have the same length.*

Proof. We have $\mathrm{Pic}^0_{X_K/K} = E_K$. Since E/S has a section, the Néron model of $\mathrm{Pic}^0_{E_K/K}$ can be obtained using E: it is the group scheme Q_E, quotient of P_E. Consider the group functors P_X and Q_X associated to X. The canonical identification $\lambda : E_K = \mathrm{Pic}^0_{X_K/K} \to \mathrm{Pic}^0_{E_K/K}$ (recalled in 1.4) extends to a morphism $Q_X \longrightarrow Q_E$, since Q_X/S is smooth. The maps

$$P_X \longrightarrow Q_X \longrightarrow Q_E \longleftarrow P_E$$

induce maps of the corresponding Lie algebras

$$\begin{tikzcd} \mathrm{Lie}(P_X) \arrow[r] & \mathrm{Lie}(Q_X) \arrow[r] & \mathrm{Lie}(Q_E) & \mathrm{Lie}(P_E) \arrow[l, "\sim"'] \\ H^1(X, \mathcal{O}_X) \arrow[u, "\theta_X(S)"] & & & H^1(E, \mathcal{O}_E), \arrow[u, "\theta_E(S)"'] \end{tikzcd} \tag{13}$$

where $\theta_X(S)$ is the map on S-sections of the canonical isomorphism of fppf-sheaves of $\mathcal{O}_S$-modules $\theta_X : \underline{R}^1 f_*\mathcal{O}_X \longrightarrow \mathcal{L}\mathrm{ie}(\mathrm{Pic}_{X/S})$ introduced in 1.3(b). The map $\mathrm{Lie}(Q_E) \leftarrow \mathrm{Lie}(P_E)$ is an isomorphism since the map $P_E^0 \to Q_E^0$ is an isomorphism (Fact 3.7(a)). Thus, we can define a map of $\mathcal{O}_K$-modules $\tau_X : H^1(X, \mathcal{O}_X) \to H^1(E, \mathcal{O}_E)$ which renders the above diagram commutative. The compatibility with the map τ_{X_K} follows from the definition of τ_{X_K} in 1.4 and the sheaf properties of θ_X.

When k is perfect, Q_X is the Néron model of E_K (3.7(a)). Thus the morphism $Q_X \longrightarrow Q_E$ is an isomorphism. It follows immediately from 3.1(b) that the kernel and cokernel of τ_X are of the same length. □

4. Application to conjecture d) of Artin and Tate

4.1. Our second application of Theorem 3.1 pertains to the conjectures of Artin-Tate and Birch-Swinnerton-Dyer. Let k be a finite field of characteristic p. Let V/k be a smooth projective geometrically connected curve with function field K. Let X/k be a proper smooth and geometrically connected surface endowed with a proper flat map $f : X \to V$ such that the generic fiber X_K/K is smooth and geometrically connected of genus g. The conjecture of Birch and Swinnerton-Dyer for the Jacobian A_K/K of X_K/K and the conjecture of Artin-Tate for X/k are conjectured by Artin and Tate to be equivalent ([58], conj. d)). This equivalence has been shown when, for each place v of V, the curve X_{K_v} has index 1 (where K_v is the completion of K at the place v). See [58], [16], 6.1 and [38], 1.5. Recall that the *index* $\delta(X_K)$ of a curve over a field K is the least positive degree of a divisor on X_K. The hypothesis $\delta(X_{K_v}) = 1$ for all v implies that f is cohomologically flat in dimension zero.

The condition that f is cohomologically flat in dimension zero is explicitly used in [58] and [16] to express the Euler-Poincaré characteristic $\chi(X, \mathcal{O}_X)$ in terms of the degree on V of the line bundle $\bigwedge^g \omega_{A/V}$ (see (14) below), where A/V denote the Néron model of A_K/K over V. Theorem 3.1 can be used to show that (14) still holds even when f is not cohomologically flat. To see this, let us state first the analogue of Theorem 3.1 in the current context.

Theorem 4.2. *Let k be a perfect field. Let V/k be a smooth proper curve with function field K. Let X/k be a projective smooth geometrically connected surface. Let $f : X \to V$ be a proper and flat curve, with $f_*\mathcal{O}_X = \mathcal{O}_V$. Consider the subfunctor P of* $\mathrm{Pic}_{X/V}$ *consisting in the line bundles of total degree* 0*, and let Q/V denote its largest separated quotient.*

(a) *Then Q/V is represented by a smooth group scheme of finite type, and Q/V is the Néron model of the Jacobian A_K/K of X_K/K.*
(b) *The kernel and cokernel of the canonical map* $\mathrm{Lie}(q) : R^1 f_*\mathcal{O}_X \to \mathcal{L}\mathrm{ie}(Q)$ *are torsion sheaves on V of same length.*

Proof. (a) The results recalled in 3.7 show that the statement of (a) is correct when V is replaced by $\mathrm{Spec}\,(\mathcal{O}_{V,v})$ for any closed point $v \in V$. Corollary 1.2 in [SGA6], Exp. XII, implies that there is a dense open set $U \subseteq V$ such that $\mathrm{Pic}_{f^{-1}(U)/U}$ is representable by a separated smooth scheme locally of finite type. Note that under our hypotheses, X_K/K is not assumed to be smooth. In particular, A_K/K is not necessarily an abelian variety. The curve X_K/K is, however, geometrically irreducible (proof of 3.7 (a)). It follows that by restricting U if necessary, we can assume that $f^{-1}(U) \to U$ has geometrically irreducible fibers ([EGA], IV, 9.7.8). Since the geometric fibers are irreducible, the groups of components of $Q_v/\mathrm{Spec}\,(\mathcal{O}_{V,v})$ are trivial for all $v \in U$ ([7], 9.5/9). It follows that Q_U/U is equal to the connected component of zero of $\mathrm{Pic}_{f^{-1}(U)/U}$. Using [7], 1.2/4, we find that the scheme Q_U/U is then the Néron model of its generic fiber. As in the proof of [7], 1.4/1, we conclude that Q/V is represented by a scheme of finite type, and is a global Néron model for A_K/K.

(b) The kernel and cokernel of $\mathrm{Lie}\,(q)$ are torsion sheaves on V since both of these sheaves are trivial when restricted to the dense open set U. Then (b) follows immediately from the local statement 3.1. □

We return for the remainder of this section to the hypotheses of 4.1, where X_K/K is smooth. Part (b) of 4.2 implies that $\chi(V, R^1 f_*\mathcal{O}_X)) = \chi(V, \mathcal{L}\mathrm{ie}\,(Q)) = \chi(V, \omega^\vee_{Q/V})$. Using this equality, we find, as in [16], 2.4 and 6.5, that

$$\chi(X, \mathcal{O}_X) = \deg\Big(\bigwedge^g \omega_{Q/V}\Big) + (1-g)\chi(V, \mathcal{O}_V). \tag{14}$$

This equality is exactly what is needed to prove the following generalization of the main result of [16], Theorem 6.3. Recall that the *period* $\delta'(X_K)$ of a curve X_K is the order of the cokernel of the degree map $\mathrm{Pic}_{X_K/K}(K) \to \mathbb{Z}$. Clearly, $\delta'(X_K)$ divides $\delta(X_K)$. When K is a local field (i.e., complete with finite residue field), we have $\delta(X_K) = \delta'(X_K)$ or $\delta(X_K) = 2\delta'(X_K)$, and $\delta(X_K) = 2\delta'(X_K)$ if and only if $\delta(X_K)$ does not divide $g-1$ ([30], Theorem 7). In particular, $\delta'(X_K)$ always divide $g-1$. In fact, the results of [30] are stated only for finite extensions of $\mathbb{Q}_p$, but once Tate's duality holds in the equicharacteristic case (see, e.g., [39], III 7.8), the same proofs work over any local field. Let $\text{Ш}(A_K)$ denote the Shafarevich-Tate group of the abelian variety A_K/K. It is well-known that if either $\text{Ш}(A_K)$ or $\mathrm{Br}\,(X)$ is finite, then so is the other. We let $\delta := \delta(X_K)$, $\delta' := \delta'(X_K)$, $\delta_v := \delta(X_{K_v})$, and $\delta'_v := \delta'(X_{K_v})$.

Theorem 4.3. *Let X/k and $f : X \to V$ be as in* 4.1. *Assume that* $\text{Ш}(A_K)$ *and* $\mathrm{Br}\,(X)$ *are finite. Then the equivalence of the Artin-Tate and Birch-Swinnerton-Dyer conjectures holds exactly when*

$$|\text{Ш}(A_K)| \prod_v \delta_v \delta'_v = \delta^2 |\mathrm{Br}\,(X)|. \tag{15}$$

Proof. We claim that the only places in [16] where the hypothesis that f is cohomologically flat in dimension zero is used[2] is in 2.4 and 6.5, to obtain Formula (14), and in the course of the proof of Theorem 6.3, in (6.2.1). As we discussed above, Formula (14) also holds without the hypothesis of cohomological flatness. To complete the proof of (6.2.1) when f is not cohomologically flat, we need the following variation on Proposition 3.3 in [16]. Let $v \in V$ be a closed point with residue field $k(v)$, and let $N_v := |k(v)|$. Let $X_{k(v)} = \sum_{i=a}^{h_v} r_a\, X_a$, and let $X_{a,\overline{k(v)}} = \sum_{j=1}^{q_a} X_{a,j}$. Then the zeta function of $X_{k(v)}/k(v)$ is:

$$Z(X_{k(v)}, T) = \frac{P_1(T)}{(1-T)\prod_{a=1}^{h_v}(1-N_vT)^{q_a}}$$

(see [16], 3.3).

Lemma 4.4. *Let $A_v^0/k(v)$ denote the connected component of zero of the special fiber of the Néron model of* $\mathrm{Jac}\,(X_K)/K$ *over $\mathcal{O}_{K_v}$. Then*

$$P_1\big(N_v^{-1}\big) = \big|A_v^0(k(v))\big| N_v^{-g}.$$

Proof. Let $d := \dim(\mathrm{Pic}_{X_{k(v)}/k(v)})$. It is shown in [16], 3.3, that

$$P_1\big(N_v^{-1}\big) = \big|\mathrm{Pic}^0_{X_{k(v)}/k(v)}(k(v))\big| N_v^{-d}.$$

When k is perfect and X/V is not cohomologically flat, the kernel of the natural faithfully flat ([48], 4.1.2) morphism

$$\mathrm{Pic}^0_{X_{k(v)}} \longrightarrow A_v^0$$

is a non-smooth group scheme $E_v/k(v)$ ([48], 6.4.2). The connected component of zero E_v^0 of E_v is unipotent ([48], 6.3.8 (ii)). Consider the quotient $B_v := \mathrm{Pic}^0_{X_{k(v)}}/E_v^0$. Then B_v is an abelian variety isogenous to A_v^0. Since $k(v)$ is finite,

$$|B_v(k(v))| = \big|A_v^0(k(v))\big|$$

(see [43], Appendix 1, Theorem 2 (c)). When k is perfect, $H^1(\mathrm{Gal}\,(\bar{k}/k), G) = \{0\}$ for any connected (possibly not smooth) unipotent group G/k, since such a group has a filtration with quotients isomorphic to $\mathbb{G}_a/k$ or α_p/k. Therefore,

$$\big|\mathrm{Pic}^0_{X_{k(v)}/k(v)}(k(v))\big| = \big|E_v^0(k(v))\big||B_v(k(v))|.$$

Since E_v^0 is unipotent, $|E_v^0(k(v))| = |E_v^{0,red}(k(v))| = N_v^{\dim(E_v)}$. The lemma follows. □

[2] This hypothesis on f is not necessary in Theorem 5.2(3) of [16], see [7], 9.6/1. Note also a slight correction in the statement and proof of Proposition 4.4 in [16]: the possibly non-reduced proper scheme Pic^0_X should be replaced by the abelian variety $(\mathrm{Pic}^0_X)^{\mathrm{red}}$.

In view of the above remarks, we can use the proof of Theorem 6.3 in [16] in the case f is not cohomologically flat, and obtain from [16], middle of p. 196, that the equivalence of the Artin-Tate and Birch-Swinnerton-Dyer conjectures holds exactly when

$$|\text{Ш}(A_K)| \prod_v d_v^2 \epsilon_v = \alpha^2 |\mathrm{Br}(X)|, \tag{16}$$

where the notation is as follows. In [16], p. 169, α is the index δ. The integer d_v (defined on p. 173) divides the index δ_v, and one defines $\Delta_v := \delta_v/d_v$ (to verify that this definition is consistent with the definition of Δ_v on p. 174 of [16], use Remark 1 after Lemma 16 of [46]). The integer ϵ_v is introduced in Proposition 5.5 of [16]. We compute it now using Theorem 1.17 of [8] (which generalizes 5.3 in [16]). Let $\Phi/k(v)$ denote the group of components of the Néron model of A_{K_v}. Theorem 1.17 expresses $|\Phi(k(v))|$ as the product of a term $|\operatorname{Ker}(\beta)/\operatorname{Im}(\alpha)|$ by $\delta_v/(d_v q)$, where $q = 1$ if δ_v divides $g-1$, and $q = 2$ otherwise. Now, according to the results of [30] recalled above, we have $\delta_v/q = \delta'_v$, thus $\delta_v/d_v q = \delta'_v/d_v$. The definition of ϵ_v in 5.5 and Lemma 5.4 immediately give that $\epsilon_v = \delta_v\delta'_v/d_v^2$. Thus, our theorem follows from Gordon's formula (16). □

Remark 4.5. Let us assume that both $\text{Ш}(A_K)$ and $\mathrm{Br}(X)$ are finite. On the left hand side of (15), $\delta_v\delta'_v$ is a square if and only if δ_v divides $g-1$. Otherwise, it is twice a square ([30], Theorem 7). In [46], a place v is called deficient if δ_v does not divide $g-1$ (see just before Corollary 12, and Remark 1 after Lemma 16). Let d denote the number of deficient places (also equal to the number of places v where $\delta_v = 2\delta'_v$). Corollaries 9 and 12 of [46] show that the order of $\text{Ш}(A_K)$ is a square if d is even, and is twice a square if d is odd. Thus, it follows that the order of the left hand side of (15) is a square.

It is shown in [58] that the prime-to-p part of $\mathrm{Br}(X)$ is endowed with a skew-symmetric non-degenerate pairing. This statement is extended in [36], 2.4, to the p-part of $\mathrm{Br}(X)$. Thus, the prime-to-2 part of $\mathrm{Br}(X)$ has order a square. When $p \neq 2$, it is proved in [60], 0.1- 0.3, that the 2-part of $\mathrm{Br}(X)$ has order a square. Thus, in this case, the 2-part of the right hand side of (15) is also a square. If Formula (15) holds, then the 2-part of $\mathrm{Br}(X)$ has order a square, even when $p = 2$.

We now use the results of [15] to prove new instances where the Artin-Tate and Birch-Swinnerton-Dyer conjectures are equivalent.

Corollary 4.6. *Let X and $f : X \to V$ be as in* 4.1. *Assume that* $\text{Ш}(A_K)$ *and* $\mathrm{Br}(X)$ *are finite. Then the equivalence of the Artin-Tate and Birch-Swinnerton-Dyer conjectures holds if the periods δ'_v are pairwise coprime.*

Proof. Under the assumption that the periods δ'_v are pairwise coprime, we find in [15], Main Theorem, that:

$$|\text{Ш}(A_K)| \prod_v (\delta'_v)^2\, 2^{e+f} = \delta\delta' |\mathrm{Br}(X)|, \tag{17}$$

where $e = \max(0, d-1)$, and $f = 1$ if $d \geq 1$ and $\delta'/\prod_v \delta'_v$ is even (d is as in 4.5). Otherwise, $f = 0$. The proof of the corollary consists in showing that Formula (15) is equivalent to (17).

We claim that if $|A_K(K)/\mathrm{Pic}^0(X_K)| = (\prod_v \delta'_v)/\operatorname{lcm}(\delta'_v)$, then $\delta = \delta'$ or $\delta = 2\delta'$. Indeed, using the global and local versions of the exact sequence (11), as well as the exact sequence $0 \to \mathrm{Br}(K) \to \oplus_v \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0$ (see the proofs of 2.3 and 2.5 in [15]), we obtain a diagram with exact rows and injective vertical maps:

$$\begin{array}{ccccc} 0 & \longrightarrow & A_K(K)/\mathrm{Pic}^0(X_K) & \longrightarrow & \mathrm{Pic}_{X_K/K}(K)/\mathrm{Pic}(X_K) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ker}(\bar{\eta}') & \longrightarrow & \mathrm{Ker}(\bar{\eta}). \end{array}$$

The map $\bar{\eta}$ is the summation map $\oplus_v \mathrm{Br}(K_v) = \oplus_v \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ restricted to the subgroup $\oplus_v \delta_v^{-1}\mathbb{Z}/\mathbb{Z}$. Similarly, the map $\bar{\eta}'$ is the summation map $\oplus_v \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ restricted to the subgroup $\oplus_v (\delta'_v)^{-1}\mathbb{Z}/\mathbb{Z}$. The map $\mathrm{Ker}(\bar{\eta}') \to \mathrm{Ker}(\bar{\eta})$ is the natural inclusion. By definition of the period and index, the cokernel of the first row is cyclic of order δ/δ'. By hypothesis, the left column is an isomorphism. Since $\delta_v = \delta'_v$ or $2\delta'_v$, it is clear that the cokernel of $A_K(K)/\mathrm{Pic}^0(X_K) \to \mathrm{Pic}_{X_K/K}(K)/\mathrm{Pic}(X_K)$ is killed by 2, and our claim is proved.

Let us return to the proof of the corollary. By [15], Theorem 2.5, we have $\left|A_K(K)/\mathrm{Pic}^0(X_K)\right| = 1 = (\prod_v \delta'_v)/\operatorname{lcm}(\delta'_v)$, thus $\delta = \delta'$ or $\delta = 2\delta'$. If $d = 0$, then $\delta_v = \delta'_v$ for all v. Thus, at most one of the groups $\delta_v^{-1}\mathbb{Z}/\mathbb{Z}$ (in the proof of the above claim) has even order and, hence, the kernel of the summation map cannot contain an element of order 2. Thus, δ is odd. Therefore, $\delta = \delta'$, and (15) is equivalent to (17).

Assume that $d \geq 1$. Then for some v, δ_v is even and, thus, δ is even. If δ' is odd, we find that $2\delta' \mid \delta$. Hence $2\delta' = \delta$ by the above claim. Again, (15) is equivalent to (17).

Assume now that $d \geq 1$ and δ' is even. Suppose that there exists an odd δ'_v such that $2\delta'_v = \delta_v$. According to [30], Theorem 7, $(g-1)/\delta'_v$ is odd and, thus, g is even. In particular, $\operatorname{ord}_2(2g-2) = 1$. Since $\delta \mid 2g-2$, we find that $\operatorname{ord}_2(\delta) = 1$. Since $(\delta/\delta') \mid 2$, we find that $\delta' = \delta$. Since g is even, all δ'_v are odd since they divide $g-1$. It follows that $f = 1$, and (15) is equivalent to (17).

Assume now that $d \geq 1$, δ' is even, and there exists no odd δ'_v with $2\delta'_v = \delta_v$. Thus there exists w with δ'_w even and $\delta_w = 2\delta'_w$. If $f = 0$, that is, if $\operatorname{ord}_2(\delta'_w) = \operatorname{ord}_2(\delta')$, then $\operatorname{ord}_2(\delta') + 1 \leq \operatorname{ord}_2(\delta)$. Thus, $2\delta' = \delta$ and (15) is equivalent to (17). If $f = 1$, that is, if $\operatorname{ord}_2(\delta'_w) + 1 \leq \operatorname{ord}_2(\delta')$, note that $(g-1)/\delta'_w$ is odd. Therefore, $\operatorname{ord}_2(\delta') = \operatorname{ord}_2(\delta)$ and $\delta' = \delta$. Again, this shows that (15) is equivalent to (17). □

The results of [15] allow us to state below a variation on Theorem 4.3. Let us return to the general situation of 4.1. Let $\Delta := \operatorname{lcm}(\delta_v)$ and

$\Delta' := \mathrm{lcm}(\delta'_v)$. Let $\mathscr{P} = \mathrm{Pic}_{X_K/K}$. The proof of (17) in [15] exhibits two exact sequences

$$0 \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \mathrm{Br}(X) \longrightarrow Ш(\mathscr{P}) \xrightarrow{\phi} \mathbb{Q}/\Delta^{-1}\mathbb{Z},$$

([15], 2.4) and

$$0 \longrightarrow T_2 \longrightarrow Ш(A_K) \longrightarrow Ш(\mathscr{P}) \xrightarrow{\gamma} \mathrm{Coker}\ D$$

([15], exact sequence (7)) with $|T_0| = (\delta/\delta')|A_K(K)/\mathrm{Pic}^0(X_K)|$, $|T_1| = (\prod_v \delta_v)/\Delta$, $|T_2| = \delta'/\Delta'$, and $|\,\mathrm{Coker}\ D| = (\prod_v \delta'_v)/\Delta'$. The image T_3 of the composition $Ш(A_K) \to Ш(\mathscr{P}) \to \mathbb{Q}/\Delta^{-1}\mathbb{Z}$ has order $|T_3| = \delta'/\Delta' \gcd(\Delta/\Delta', \delta'/\Delta')$ if $Ш(A_K)$ has no nonzero infinitely divisible elements ([15], 2.10).

Let $a := |A_K(K)/\mathrm{Pic}^0(X_K)|$, let $b := |\,\mathrm{Coker}\ D/\mathrm{Im}(\gamma)|$, and let $c := |\mathrm{Im}(\phi)/T_3|$. With this notation, we find that $|T_1||Ш(A_K)||\mathrm{Im}(\gamma)| = |T_0||T_2||\mathrm{Br}(X)||\mathrm{Im}(\phi)|$. In other words,

$$|Ш(A_K)| \prod_v \delta_v \delta'_v = \delta\delta' abc\epsilon |\mathrm{Br}(X)|, \tag{18}$$

where, to simplify the notation, we let $\epsilon := (\Delta/\Delta') \gcd(\Delta/\Delta', \delta'/\Delta')^{-1}$. Note that $\epsilon = 1$ or 2. Comparing (18) with (15), we obtain:

Corollary 4.7. *Assume that $Ш(A_K)$ and $\mathrm{Br}(X)$ are finite. Then the conjectures of Artin-Tate and Birch-Swinnerton-Dyer are equivalent if and only if $\delta = \delta' abc\epsilon$.*

Note that the equality $\delta = \delta' abc\epsilon$ has the following interesting consequences. Since $\delta \mid 2(\delta')^2$ ([30], Thm. 8), we find that $abc\epsilon \mid 2\delta'$. Since $|\,\mathrm{Ker}(\bar{\eta})| = (\prod_v \delta_v)/\Delta$, we find that $a\delta/\delta' \mid (\prod_v \delta_v)/\Delta$. From $\delta/\delta' = abc\epsilon$, we conclude that $a^2 \mid (\prod_v \delta_v)/\Delta$.

5. Comparison between the discriminants of X/S and E/S

Let $g : Y \to T$ be a smooth projective curve over a scheme T with geometrically connected fibers of genus ≥ 1. Let $\omega_{Y/T}$ denote the relative canonical sheaf, equal to $\Omega^1_{Y/T}$ when Y/T is smooth. There exists a unique (up to sign) isomorphism of invertible sheaves on Y:

$$\Delta_{Y/T} : \det Rg_*\big(\omega_{Y/T}^{\otimes 2}\big) \longrightarrow (\det Rg_*(\omega_{Y/T}))^{\otimes 13}, \tag{19}$$

functorial with respect to isomorphisms and compatible with base changes $T' \to T$ (the functor $\det Rg_*$ is briefly reviewed in 5.2). This isomorphism was first noted by Mumford [42], Theorem 5.10. Deligne [12] reduced the sign ambiguity in [42] (one sign for each genus) to a unique sign ambiguity in genus 1. Note that when $Y \to T$ is smooth of genus 1, then $\omega_{Y/T} = g^* g_* \omega_{Y/T}$ and the projection formula 5.2 (b) implies a canonical isomorphism $\det Rg_*(\omega_{Y/T}^{\otimes 2}) \simeq \det Rg_*(\omega_{Y/T})$. Hence, $\Delta_{Y/T} : \det Rg_*(\omega_{Y/T}^{\otimes 2}) \to (\det Rg_*(\omega_{Y/T}))^{\otimes 13}$ can be identified with a canonical map, again denoted by $\Delta_{Y/T}$:

$$\Delta_{Y/T} : \mathcal{O}_T \to (g_*\omega_{Y/T})^{\otimes 12}. \tag{20}$$

Let $X_K \to \operatorname{Spec}(K)$ be a proper smooth and geometrically connected curve over a discrete valuation field K, with canonical isomorphism $\Delta_{X_K/K}$ as in (19). Let $f : X \to S$ be a proper regular model of X_K. Consider the invertible sheaves $\det Rf_*(\omega_{X/S}^{\otimes 2})$ and $(\det Rf_*(\omega_{X/S}))^{\otimes 13}$ on S, identified with their respective images in $\det Rf_{K*}(\omega_{X_K/K}^{\otimes 2})$ and $(\det Rf_{K*}(\omega_{X_K/K}))^{\otimes 13}$ (see Lemma 5.3). Then there exists $\lambda \in K^*$ such that

$$\Delta_{X_K/K}\big(\det Rf_*\big(\omega_{X/S}^{\otimes 2}\big)\big) = \lambda(\det Rf_*(\omega_{X/S}))^{\otimes 13}. \tag{21}$$

The integer $\operatorname{disc}(X) := v(\lambda)$ is called the (valuation of the) *discriminant of* X. When X/S is the minimal model of X_K/K, we may call $\operatorname{disc}(X)$ *the discriminant of* X_K.

We prove in this section that the discriminant of a curve X_K of genus 1 and the discriminant of its Jacobian E_K are equal when k is algebraically closed. This statement is not true anymore if k is not assumed to be perfect (9.2). For the convenience of the reader, we start by collecting below some results on determinants, as found in [26], and similarly quoted in [40], 1.1. Note that we cannot assume in our context the simplifying assumption that all sheaves involved (for instance, $R^1 f_*\mathcal{O}_X$) are flat.

Let X be a scheme. A complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules is *perfect* ([SGA6] I.0, p. 80) if locally (for the Zariski topology) on X, there is a quasi-isomorphism of complexes $\mathcal{G}^\bullet \to \mathcal{F}^\bullet$, where $\mathcal{G}^\bullet$ is bounded and its terms are free and finitely generated $\mathcal{O}_X$-modules. If X is noetherian and regular, then any bounded complex of coherent $\mathcal{O}_X$-modules is perfect.

Facts 5.1 ([26], Theorem 2). Let X be a scheme. There is a unique way (up to unique isomorphism) to associate to any perfect complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules an invertible sheaf $\det \mathcal{F}^\bullet$ on X such that the following properties are true.

(a) The map det is a functor from the category of perfect complexes on X with quasi-isomorphisms to the category of invertible sheaves on X with isomorphisms.
(b) If $\mathcal{F}^\bullet$ consists of a single locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank, then $\det \mathcal{F}^\bullet$ is just the classical determinant $\det \mathcal{F}$.
(c) Let $0 \to \mathcal{F}'^\bullet \xrightarrow{\alpha} \mathcal{F}^\bullet \xrightarrow{\beta} \mathcal{F}''^\bullet \to 0$ be a true triangle of perfect complexes (see [26], def. 2 on p. 37). Then there exists a canonical isomorphism

$$i(\alpha, \beta) : \det \mathcal{F}'^\bullet \otimes \det \mathcal{F}''^\bullet \simeq \det \mathcal{F}^\bullet.$$

(d) (Base change) If $\rho : X' \to X$ is a morphism of schemes, then there exists a canonical isomorphism of $\mathcal{O}_{X'}$-modules $\det(\rho^*\mathcal{F}^\bullet) \simeq \rho^* \det \mathcal{F}^\bullet$.

Recall that by convention, the determinant of the zero sheaf on X is $\mathcal{O}_X$. Note that to be precise, $\det \mathcal{F}^\bullet$ is an invertible sheaf with a sign, but it does not matter in our work.

Facts 5.2. Let $f : X \to Y$ be a proper flat morphism of schemes with Y noetherian. Then there exists a functor $\mathcal{F}^\bullet \mapsto \det Rf_*\mathcal{F}^\bullet$ from the category of perfect complexes $\mathcal{F}^\bullet$ on X with quasi-isomorphisms to the category of invertible sheaves on Y with isomorphisms such that the following properties are true.

(a) Suppose that Y is regular. Let $\mathcal{F}$ be a perfect coherent sheaf on X. Then we have a functorial isomorphism

$$\det Rf_*\mathcal{F} \simeq \otimes_{i\geq 0}(\det(R^i f_*\mathcal{F}))^{\otimes(-1)^i}.$$

(b) (Projection formula). Let $\mathcal{F}$ be a perfect coherent sheaf on X, flat on Y. Let $\chi_f(\mathcal{F})$ be the locally constant map

$$y \mapsto \chi_{k(y)}(\mathcal{F}_y) := \sum_{i\geq 0}(-1)^i \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

on Y. Then for any invertible sheaf $\mathcal{L}$ on Y, we have a canonical isomorphism

$$\det Rf_*(\mathcal{F} \otimes f^*\mathcal{L}) \simeq (\det Rf_*\mathcal{F}) \otimes \mathcal{L}^{\otimes \chi_f(\mathcal{F})}.$$

Proof. (a) See [26], Proposition 8. (b) is stated in [40], 1.1.7, and is an immediate consequence of op. cit., 1.1.6. □

Let M be a finitely generated module over a discrete valuation ring $\mathcal{O}_K$. Let $r := \dim_K(M \otimes K)$. When M is free of rank r, then $\det M = \wedge^r M$ by definition. When M is not free, there is no unique choice of $\det M$, since $\det M$ is only determined up to canonical isomorphisms. However, having chosen a det functor, we can consider the image of $\det M$ under the following sequence of canonical maps:

$$\det M \to (\det M) \otimes K \simeq \det(M \otimes K) = \wedge^r(M \otimes K). \tag{22}$$

Clearly, $\det M$ and the isomorphism $\simeq$ above depend on the choice of the functor det, but $\wedge^r(M \otimes K)$ does not. The following lemma shows that the image of $\det M$ in $\wedge^r(M \otimes K)$ is independent of the choice of a det functor. Thus, so is the discriminant $\mathrm{disc}\,(X)$ defined in (21).

Lemma 5.3. *Let $\varphi : M \to N$ be a map of finitely generated $\mathcal{O}_K$-modules such that $\varphi_K : M \otimes K \to N \otimes K$ is an isomorphism. Then the following properties are true.*

(a) *Let* $c := -\ell(\operatorname{Ker}\varphi) + \ell(\operatorname{Coker}\varphi)$. *Then the image of the canonical injective map*

$$\det M \to (\det M) \otimes K \simeq \det(M \otimes K) \stackrel{\det(\varphi_K)}{\longrightarrow} \det(N \otimes K)$$

is $\pi^c \det N$, *where we abuse notation and denote again by* $\det(N)$ *the image of* $\det(N)$ *under the canonical map* (22).

(b) *The canonical surjection* $M \to L := M/M_{\mathrm{tors}}$ *induces a canonical isomorphism* $\det(M \otimes K) \simeq \det(L \otimes K)$. *The image of* $\det M$ *in* $\det(M \otimes K)$ *is equal to the image of* $\pi^{-\ell(M_{\mathrm{tors}})} \det L$ *in* $\det(L \otimes K)$. *As* L *is free,* $\det L$ *is independent of the choice of* $\det$ *and, thus, so is the image of* $\det(M)$.

Proof. (a) The map of coherent sheaves $M^{\sim} \to N^{\sim}$ on S is a good map in the sense of [26], p. 47, where a Cartier divisor $\operatorname{Div}(M^{\sim} \to N^{\sim})$ (that we denote by $\operatorname{Div}(M \to N)$ for simplicity) on S is defined. By construction, the image of $\det M$ in $(\det N)\otimes K$ is nothing but $\mathcal{O}_S(-\operatorname{Div}(M \to N)) \det N$. By decomposing φ into $M \to \operatorname{Im}\varphi$ and $\operatorname{Im}\varphi \to N$, we are reduced to the case where φ is either injective or surjective ([26], Theorem 3(i)). Suppose for example that φ is surjective. Let $A = \operatorname{Ker}\varphi$. Then there exist resolutions by bounded complexes of finite free $\mathcal{O}_K$-modules $\mathcal{A} \to A$, $\mathcal{M} \to M$, and $\mathcal{N} \to N$, forming a true triangle $0 \to \mathcal{A} \to \mathcal{M} \to \mathcal{N} \to 0$ ([26], Prop. 4). The latter induces an isomorphism

$$\det \mathcal{M} \simeq \det \mathcal{N} \otimes \det \mathcal{A},$$

and we have $\operatorname{Div}(M \to N) = \operatorname{Div}(\mathcal{M} \to \mathcal{N}) = -\operatorname{Div}(0 \to A)$. Since $\operatorname{Div}(0 \to C) = \pi^{-\ell(C)}\mathcal{O}_K$ for any $\mathcal{O}_K$-torsion module C ([26], Thm. 3(vi)), the lemma is proved when φ is surjective. The proof is similar when φ is injective.

(b) is an immediate consequence of (a). □

Until the end of the section, $\mathcal{O}_K$ is a discrete valuation ring, $S = \operatorname{Spec}\mathcal{O}_K$, and $\det M$ is canonically identified with its image in $\det(M \otimes K)$.

Corollary 5.4. *Let* $f : X \to S$ *be a flat projective curve with* X *regular. Let* $\mathcal{F}$ *be a coherent sheaf on* X. *Then we have a canonical isomorphism*

$$\det Rf_*\mathcal{F} \simeq \det f_*\mathcal{F} \otimes (\det R^1 f_*\mathcal{F})^\vee.$$

If, moreover, $\mathcal{F}$ *is killed by some power of* π, *then* $\det Rf_*\mathcal{F} = \pi^{-\chi_{\mathrm{len}}(\mathcal{F})}\mathcal{O}_K$, *where* $\chi_{\mathrm{len}}(\mathcal{F}) := \ell(H^0(X, \mathcal{F})) - \ell(H^1(X, \mathcal{F}))$.

Proof. Note that X/S being a curve, $R^i f_*\mathcal{F} = 0$ if $i > 1$. Since on a regular noetherian scheme, any coherent sheaf is perfect, we can apply 5.2(a) and 5.3(b). □

Proposition 5.5. *Let* $f : X \to S$ *be the minimal regular model of a smooth projective geometrically connected curve* $X_K \to \operatorname{Spec}(K)$ *of genus* 1. *Then,*

(a) *For each m, we have a canonical isomorphism*

$$\det Rf_*\big(\omega_{X/S}^{\otimes m}\big) \simeq \big(\det R^1 f_*\mathcal{O}_X\big)^\vee.$$

(b) *Let* $a := \ell(H^1(X, \mathcal{O}_X)_{\mathrm{tors}})$. *Then we have a canonical isomorphism*

$$\big(\det R^1 f_*\mathcal{O}_X\big)^\vee \simeq \pi^a(f_*\omega_{X/S}).$$

The proposition will follow from the next two lemmas.

Lemma 5.6. *Let X_K/K be as in* 5.5, *and let X be a regular model of X_K over S. Let $X_k = \sum_{1\le i\le n} r_i\Gamma_i$, with $r := \gcd(r_1, \dots, r_n)$. Let $V := r^{-1}X_k$, considered as a scheme over k. Let $\mathcal{F}$ be an invertible sheaf on X with* $\deg \mathcal{F}|_{X_k} = 0$. *For any $n \in \mathbb{Z}$, let $\mathcal{F}(nV) := \mathcal{F} \otimes \mathcal{O}_X(nV)$. Then we have a canonical isomorphism*

$$\det Rf_*(\mathcal{F}(nV)) \simeq \det Rf_*\mathcal{F}.$$

Proof. Let $j : V \to X$ denote the closed embedding. Consider the exact sequence

$$0 \to \mathcal{F}((n-1)V) \to \mathcal{F}(nV) \to j_*(\mathcal{F}(nV)|_V) \to 0.$$

It induces a canonical isomorphism:

$$\det Rf_*(\mathcal{F}(nV)) \simeq \det Rf_*(\mathcal{F}((n-1)V)) \otimes \det Rf_*(j_*\mathcal{F}(nV)|_V).$$

Since $j_*(\mathcal{F}(nV)|_V)$ is killed by π, Corollary 5.4 implies that $\det Rf_*(j_*\mathcal{F}(nV)|_V) = \pi^{-\chi_k(j_*\mathcal{F}(nV)|_V)}\mathcal{O}_K$. To prove the lemma, it remains to show that $\chi_k(j_*\mathcal{F}(nV)|_V) = 0$. We claim that $\chi_k(\mathcal{L}) = 0$ for any invertible sheaf $\mathcal{L}$ on V of degree 0. Indeed, the Riemann-Roch for the l.c.i. curve V/k states ([31], Theorem 7.3.17 and Corollary 7.3.31):

$$\chi_k(\mathcal{L}) = \deg \mathcal{L} + \chi_k(\mathcal{O}_V) = \chi_k(\mathcal{O}_V) = -\frac{1}{2}\deg \omega_{V/k}.$$

The adjunction formula $\omega_{V/k} \simeq (\mathcal{O}_X(V) \otimes \omega_{X/S})|_V$ ([31], Theorem 9.1.37) implies:

$$\deg \omega_{V/k} = V^2 + V \cdot \omega_{X/S} = r^{-1}X_k \cdot \omega_{X/S} = -2r^{-1}\chi_K(\mathcal{O}_{X_K}) = 0.$$

Hence, $\chi_k(\mathcal{L}) = 0$. Since $\mathcal{F}(nV)|_V$ has degree 0, the lemma is proved. □

Lemma 5.7. *Keep the hypotheses of* 5.5. *Then there exists an integer* $0 \le q \le r-1$ *such that $\omega_{X/S}$ is canonically isomorphic to $f^*f_*\omega_{X/S} \otimes \mathcal{O}_X(qV)$.*

Proof. Since X_K has genus 1, $f_*\omega_{X/S}$ is free of rank 1 on S. Let ω_0 be a basis of $f_*\omega_{X/S}$. Then $f^*f_*\omega_{X/S} \to \omega_{X/S}$ is nothing but the inclusion $\omega_0\mathcal{O}_X \to \omega_{X/S}$. The invertible sheaf $f^*f_*\omega_{X/S} \otimes \omega_{X/S}^\vee$ is canonically a subsheaf of $\mathcal{O}_X$, whose restriction to the generic fiber is equal to $\mathcal{O}_X$. So it is equal to $\mathcal{O}_X(-D)$ for some vertical divisor $D \geq 0$ on X.

Note now that for any irreducible component Γ of X_k, we have $D \cdot \Gamma = \deg \omega_{X/S}|_\Gamma = 0$. To see the second equality, recall the adjunction formula: $0 = -2\chi_K(\mathcal{O}_X) = \sum_i r_i \deg(\omega_{X/S}|_{\Gamma_i})$. Since $\deg(\omega_{X/S}|_{\Gamma_i}) \geq 0$ because X is minimal ([31], Prop. 9.3.10(b)), we have $\deg(\omega_{X/S}|_{\Gamma_i}) = 0$ for all i. Thus $D = qV \in \mathbb{Z}V$ ([31], Theorem 9.1.23). We have $0 \leq q \leq r-1$ since $f^*f_*\omega_{X/S}$ and $\omega_{X/S}$ have the same global sections (otherwise, if $q \geq r$, ω_0/π is a global section of $\omega_{X/S} = \omega_0\mathcal{O}_X(qV)$). □

Proof of 5.5. (a) Let $\underline{\omega} = f_*\omega_{X/S}$ and let $m \in \mathbb{Z}$. Lemma 5.7 gives us a canonical isomorphism

$$\omega_{X/S}^{\otimes m} \simeq f^*\underline{\omega}^{\otimes m} \otimes \mathcal{O}_X(mqV).$$

Lemma 5.6 then produces a canonical isomorphism

$$\det Rf_*\big(\omega_{X/S}^{\otimes m}\big) \simeq \det Rf_*(f^*\underline{\omega}^{\otimes m}).$$

The projection formula 5.2(b) applied with $\mathcal{L} = \underline{\omega}^{\otimes m}$ produces a canonical isomorphism

$$\det Rf_*(f^*\underline{\omega}^{\otimes m}) \simeq (\underline{\omega}^{\otimes m})^{\otimes \chi_f(\mathcal{O}_X)} \otimes \det Rf_*\mathcal{O}_X = \underline{\omega}^{\otimes m\chi_f(\mathcal{O}_X)} \otimes \det Rf_*\mathcal{O}_X.$$

Since $\chi_f(\mathcal{O}_X) = 0$ and $f_*\mathcal{O}_X = \mathcal{O}_S$, Corollary 5.4 produces canonical isomorphisms

$$\det Rf_*\big(\omega_{X/S}^{\otimes m}\big) \simeq \det Rf_*\mathcal{O}_X \simeq \big(\det R^1f_*\mathcal{O}_X\big)^\vee.$$

This proves (a).

(b) By Grothendieck's duality, we have a canonical isomorphism $(R^1f_*\mathcal{O}_X)^\vee \simeq \underline{\omega}$. By Lemma 5.3, $\det R^1f_*\mathcal{O}_X \simeq \pi^{-a}\mathcal{M}$ where $\mathcal{M} = R^1f_*\mathcal{O}_X/(R^1f_*\mathcal{O}_X)_{\mathrm{tors}}$ is free of rank 1 on S. Hence $\underline{\omega} \simeq \mathcal{M}^\vee \simeq \pi^{-a}(\det R^1f_*\mathcal{O}_X)^\vee$. □

Our main theorem in this section, Theorem 5.9 below, is proved using Theorem 3.8, which we now recall in the following form.

Corollary 5.8. *Keep the hypotheses of* Theorem 3.8. *Then $a = \ell(\mathrm{Ker}\,\tau_X)$, and we let $b := \ell(\mathrm{Coker}(\tau_X))$. Identify as in* (22) $\det H^1(X, \mathcal{O}_X)$ *and* $\det H^1(E, \mathcal{O}_E)$ *with their images in* $\det H^1(X_K, \mathcal{O}_{X_K})$ *and* $\det H^1(E_K, \mathcal{O}_{E_K})$, *respectively. Then the isomorphism*

$$\det \tau_{X_K} : \det H^1(X_K, \mathcal{O}_{X_K}) \to \det H^1(E_K, \mathcal{O}_{E_K})$$

(which is equal to τ_{X_K}) maps $\det H^1(X, \mathcal{O}_X)$ *onto* $\pi^{b-a} \det H^1(E, \mathcal{O}_E)$.

Proof. This follows immediately from Corollary 3.8, Lemma 5.3, and the fact that $H^1(E, \mathcal{O}_E)$ is torsion free. □

Theorem 5.9. *Let S be the spectrum of a discrete valuation ring $\mathcal{O}_K$ with residue field k. Let X_K be a smooth projective geometrically connected curve of genus 1 over K, with Jacobian E_K. Let $f : X \to S$ and $g : E \to S$ be their respective minimal regular models. Then*

$$\operatorname{disc}(X) = \operatorname{disc}(E) + 12(b-a). \tag{23}$$

Moreover, when k is perfect, then $\operatorname{disc}(X) = \operatorname{disc}(E)$.

Proof. Recall (see (20)) that $\Delta_{X_K/K}$ can be identified with a map

$$\Delta_{X_K/K} : K \to \left(\det H^1\left(X_K, \mathcal{O}_{X_K}\right)^{\vee}\right)^{\otimes 12}.$$

The equality (23) will follow from the commutativity of the diagram:

$$\begin{tikzcd}
K \arrow[r, "\Delta_{X_K/K}"] \arrow[d, equal] & \left(\det H^1\left(X_K, \mathcal{O}_{X_K}\right)^{\vee}\right)^{\otimes 12} \\
K \arrow[r, "\Delta_{E_K/K}"] & \left(\det H^1\left(E_K, \mathcal{O}_{E_K}\right)^{\vee}\right)^{\otimes 12}. \arrow[u, "\left(\tau_{X_K}^{\vee}\right)^{\otimes 12}"']
\end{tikzcd}$$

To prove this commutativity, we first note that this statement is only related to the curve X_K over K. Using the compatibility with base change, we can enlarge K and assume that $X_K(K) \neq \emptyset$. By Corollary 1.5, $\tau_{X_K} = H^1(h)$ for some isomorphism $h : E_K \to X_K$. Then the commutativity of the above diagram comes from the functoriality of $\Delta_{X_K/K}$ with respect to isomorphisms of curves. □

Remark 5.10. As in the proof of 3.8, we obtain the map τ_X as the composition of the canonical maps $\theta_X : H^1(X, \mathcal{O}_X) = \operatorname{Lie}(P_X) \to \operatorname{Lie}(Q_X)$ and $\operatorname{Lie}(Q_X) \to \operatorname{Lie}(Q_E)$. The latter map is injective because $\operatorname{Lie}(Q_X)$ is torsion free (Q_X is smooth). Hence

$$b - a = \ell(\operatorname{Lie}(Q_E)/\operatorname{Lie}(Q_X)) + \ell(\operatorname{Coker}\theta_X) - \ell(\operatorname{Ker}\theta_X).$$

For an example where $b - a \neq 0$ or, more precisely, where $\operatorname{Lie}(Q_X) \to \operatorname{Lie}(Q_E)$ is not surjective, see 9.3. We do not know of examples of curves X_K where $\ell(\operatorname{Ker}\theta_X) \neq \ell(\operatorname{Coker}\theta_X)$. It would be interesting to determine whether $\operatorname{disc}(X) \geq \operatorname{disc}(E)$ holds in general. If the reduction of E is multiplicative, $\operatorname{disc}(X) = \operatorname{disc}(E)$, even when k is not perfect (8.11).

6. Néron models and curves of genus 1 when k is perfect

We prove in this section our main theorem 6.6 describing the relationship between the reduction of a curve of genus 1 and the reduction of its Jacobian, when k is algebraically closed. We start by reviewing various general results needed for the proof.

Let $S = \mathrm{Spec}\,(\mathcal{O}_K)$. Let $f : X \to S$ be a proper and flat curve, with X regular and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $\Gamma_1, \ldots, \Gamma_n$ be the irreducible components of X_k, of respective multiplicities $r_1, \ldots, r_n$, and respective geometric multiplicities $e_i := \mathrm{length}(k(\Gamma_i) \otimes_k \bar{k})$. Consider the (modified) intersection matrix $M := (e_i^{-1}\Gamma_i \cdot \Gamma_j)_{i,j}$ of X_k and let

$$\Phi_X := \text{torsion subgroup of } \mathbb{Z}^n/M\mathbb{Z}^n.$$

Let J/S denote the Néron model of the Jacobian J_K/K of X_K/K. Recall that the group scheme Q/S introduced in Sect. 3 is smooth and has generic fiber J_K. Thus, the universal property of the Néron model implies the existence of a morphism $Q \to J$ which restricts to the identity map on the generic fiber. This morphism induces a canonical map of component groups $\Phi_Q \to \Phi_J$.

Facts 6.1. Let $\mathcal{O}_K$ be a strictly henselian discrete valuation ring.

(a) There exists a canonical surjective map $\Phi_X \to \Phi_Q$. This map is injective if X/S is cohomologically flat in dimension zero.
(b) Assume that k is algebraically closed. Then Q/S is the Néron model of J_K, and the map $\Phi_X \to \Phi_Q$ is an isomorphism.

Proof. See [7], 9.6.1 and 9.6.3 (these references apply since X_K is geometrically irreducible (proof of 3.7)). It is proved in [48], 8.1.2 (iii), that if X is cohomologically flat in dimension zero, then the natural map $\Phi_X \to \Phi_Q$ is injective. That the map is always surjective is proved in [7], 9.5.9. □

6.2. Let us recall next the following definitions. Let C/k be a projective and geometrically connected curve over a field k. Over an algebraic closure $\bar{k}$ of k, the group scheme $\mathrm{Pic}^0_{C_{\bar{k}/\bar{k}}}$ can be decomposed as

$$0 \to T \times U \to \mathrm{Pic}^0_{C_{\bar{k}/\bar{k}}} \to A \to 0 \qquad (24)$$

where A, T, and U are, respectively, abelian, toric, and unipotent, smooth group varieties. We will call their respective dimensions the *abelian, toric*, and *unipotent*, ranks of C, denoted by a_C, t_C, and u_C. We have

$$a_C + t_C + u_C = \dim_{\bar{k}} H^1(C_{\bar{k}}, \mathcal{O}_{C_{\bar{k}}}) = \dim_k H^1(C, \mathcal{O}_C).$$

For the existence of the above exact sequence, see for instance [7], p. 244. The integers a_C and t_C depend only on $(C_{\bar{k}})_{\mathrm{red}}$. We will say that $\mathrm{Pic}^0_{C/k}$ is *semi-abelian* if $u_C = 0$.

Lemma 6.3. *Let ℓ be a prime number different from the characteristic of k. Let*

$$T_\ell\big(\mathrm{Pic}^0_{C/k}\big) := \varprojlim_n \mathrm{Pic}^0(C_{\bar{k}})[\ell^n]$$

be the Tate module of $\mathrm{Pic}^0_{C/k}$.

(a) $T_\ell(\mathrm{Pic}^0_{C/k})$ *is a free* $\mathbb{Z}_\ell$*-module of rank* $2a_C + t_C$.
(b) *Let n denote the number of irreducible components of $C_{\bar{k}}$. Then*

$$\chi_{\text{ét}}(C_{\bar{k}}) := \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\text{ét}}(C_{\bar{k}}, \mathbb{Q}_\ell) = n + 1 - (2a_C + t_C).$$

Proof. (a) is well-known. (b) We have (see, e.g., [59], 10.3.5) $\dim_{\mathbb{Q}_\ell} H^0_{\text{ét}}(C_{\bar{k}}, \mathbb{Q}_\ell) = 1$, $\dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(C_{\bar{k}}, \mathbb{Q}_\ell) = n$, and

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(C_{\bar{k}}, \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell} \Big(T_\ell\big(\mathrm{Pic}^0_{C/k}\big)^\vee \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\Big) = 2a_C + t_C.$$

□

6.4. Let X_K/K be a projective, smooth, geometrically connected curve of genus $g \geq 1$. Let X/S be a regular model of X_K. Let $a_X := a_{X_k}$, $t_X := t_{X_k}$ and $u_X := u_{X_k}$. One can show that these integers do not depend on the choice of a regular model for X_K/K. The connected component of zero J^0_k of the special fiber J_k/k of the Néron model J of the Jacobian of X_K is a smooth group scheme, and we similarly let a_J, t_J, and u_J be the abelian, toric, and unipotent ranks of $J^0_{\bar{k}}$, respectively. When $\mathrm{Pic}^0_{X/S}$ is isomorphic to J^0, we find that $a_X = a_J$ and $t_X = t_J$. We prove in 7.1 that these equalities are always true. Denote by n_X the number of irreducible components of $X_{\bar{k}}$.

When k is perfect, the Swan conductor δ_{X_K} is the Swan conductor ([54], 2.1) associated with the ℓ-adic representation

$$\mathrm{Gal}(K^s/K) \to \mathrm{Aut}(H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)).$$

The Artin conductor of X is defined by

$$\begin{aligned} \mathrm{Art}(X/S) &:= \chi_{\text{ét}}(X_{\bar{k}}) - \chi_{\text{ét}}(X_{\bar{K}}) + \delta_{X_K} \\ &= (2g(X_K) - 2a_X - t_X + \delta_{X_K}) + n_X - 1 \\ &= (t_J + 2u_J + \delta_{J_K}) + n_X - 1, \end{aligned}$$

where δ_{J_K} is the Swan conductor associated to the Tate module $T_\ell(J_K)$ of J_K. The second equality above comes from 6.3(b). Since

$$H^1_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell) \simeq T_\ell(J_K)^\vee \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

we have $\delta_{X_K} = \delta_{J_K}$. The third equality is then a consequence of Proposition 7.1. Hence, we see that $\mathrm{Art}(X/S) - n_X$ depends only on J_K.

Corollary 6.5. *Assume that k is perfect. Let X_K be a smooth, projective, and geometrically connected curve of genus 1 over K, with Jacobian E_K. Let X and E be the minimal regular models over $\mathcal{O}_K$ of X_K and E_K, respectively. Let n_X and n_E denote the number of irreducible components of $X_{\bar{k}}$ and $E_{\bar{k}}$, respectively. Then $n_X = n_E$.*

Proof. We have shown in Theorem 5.9 that $\operatorname{disc}(X) = \operatorname{disc}(E)$. The relationship between the Artin conductor and the discriminant is expressed in a theorem of T. Saito [51]:

$$\operatorname{disc}(X) = \operatorname{Art}(X/S).$$

It follows that $\operatorname{Art}(X/S) = \operatorname{Art}(E/S)$. As discussed above, $\operatorname{Art}(X/S) - n_X = \operatorname{Art}(E/S) - n_E$, and the corollary follows. □

We are now ready to prove our main theorem on the reduction of curves of genus 1 when k is algebraically closed.

Theorem 6.6. *Assume that k is algebraically closed. Let X_K/K be a smooth, geometrically connected projective curve of genus 1 and let E_K/K be its Jacobian. Let X/S and E/S be the minimal regular models of X_K and E_K, respectively. Let m denote the order of the element of $H^1(K, E_K)$ corresponding to the torsor X_K. If T denotes the type of E_k, then X_k is of type mT.*

Proof. We can assume that $\mathcal{O}_K$ is strictly henselian. Let us first show that the type of X_k is of the form rT for some r. Since $a_X = a_E$ and $t_X = t_E$ (7.1), we are reduced to consider three cases:

1) Assume that $a_X = a_E = 1$. In this case, E_k is of Kodaira type I_0. The classification of the possible types of reduction shows that the type of X_k is rI_0 for some $r \geq 1$.

2) Assume that $t_X = t_E = 1$. In this case, E_k is of Kodaira type I_ν, $\nu > 0$. The classification of the possible types of reduction shows that the type of X_k is rI_μ for some $\mu \geq 1$. In particular, $|\Phi_E| = \nu$ and $|\Phi_X| = \mu$. Since the group of components Φ_X and Φ_E are isomorphic (6.1(b)), we find that $\nu = \mu$.

3) Assume that $u_X = u_E = 1$. Then E_k is of Kodaira type T and X_k is of type rT', with $T, T' \in \{II, II^*, III, III^*, IV, IV^*, I_n^*\}$. It follows from 6.5 that X_k and E_k have the same number of irreducible components. Moreover, $|\Phi_E| = |\Phi_X|$. Upon inspection of the types in the above list (see, e.g., [55], p. 365), we find that these two conditions are satisfied only when $T = T'$.

To conclude the proof of 6.6, we recall the following facts. The element of $H^1(K, E_K)$ corresponding to the torsor X_K is equal to the image of 1 under the natural coboundary map $\mathbb{Z} \to H^1(K, E_K)$ associated with the exact sequence $0 \to E_K \to \operatorname{Pic}_{X_K/K} \to \mathbb{Z} \to 0$. Indeed, the image of 1 under the above map is the class of the torsor $\operatorname{Pic}^1_{X_K/K}$ ([17], Remarque 4.2(e)), which is isomorphic to X_K/K. Thus, the element of $H^1(K, E_K)$ corresponding to the torsor X_K has order equal to the period δ' of X_K.

On the other hand, it is well-known that the index δ of X_K is equal to the gcd of the multiplicities of the irreducible components of X_k ([7], 9.1.5 and 9.1.10). Since $\mathrm{Br}(K)$ is trivial, we find that $\mathrm{Pic}(X_K) = \mathrm{Pic}_{X_K/K}(K)$. Hence $\delta = \delta'$. This shows that if X_K is a torsor under E_K whose minimal model has a special fiber of type rT, then the element of $H^1(K, E_K)$ associated with X_K has order r. □

Since the torsors under E_K are classified by the group $H^1(K, E_K)$, it is natural to wonder whether the group $H^1(K, E_K)$ can be trivial and, if it is not, what is its structure. When k is algebraically closed, the field K is C_1, and the multiplication-by-p map on $H^1(K, E_K)$ is then surjective. Thus, in this case, the group $H^1(K, E_K)$ is divisible by p. No such result seems to be known when k is imperfect.

When k is algebraically closed and K is complete, the first results concerning the non-triviality and structure of the p-part of $H^1(K, E_K)$ are due to Vvedenskii, [62] and [63]. In [64], he corrects an earlier assertion of Néron that $H^1(K, E_K) = (0)$ when E_K has additive reduction ([44], Thm. 3). The general structure of $H^1(K, E_K)$ can be understood through a non-degenerate pairing between $H^1(K, E_K)$ and a huge (in particular, non-trivial) profinite group associated with the Greenberg realization of the Néron model of E_K/K. The existence of such a pairing was conjectured by Shafarevich, and proved by Bégueri [5] in the mixed characteristic case, and by Bertapelle [6] in the equicharacteristic case. Little is known about the non-triviality and structure of $H^1(K, E_K)$ when k is imperfect. Note that $H^1(K, E_K)$ is a p-group if the type of E_k is additive (7.4). The following corollary answers positively a question of Néron ([45], last sentence).

Corollary 6.7. *Let $\mathcal{O}_K$ be complete with k algebraically closed. Let E_K be an elliptic curve, E/S its minimal regular model, and let T denote the type of E_k. Let $m = p^n \geq 1$ if the type is additive, and let $m > 0$ be an arbitrary integer otherwise. Then there exists a proper flat map $X \to S$, with X regular, such that the generic fiber X_K is a torsor under E_K, and the special fiber X_k is of type mT.*

Proof. The results of [5] and [6] recalled above show that under our hypotheses on $\mathcal{O}_K$, the group $H^1(K, E_K)$ has an element x of order m as in the statement of the corollary. Theorem 6.6 shows that the torsor X_K corresponding to x has a minimal regular model X/S whose special fiber has type mT. □

Remark 6.8. Theorem 6.6 does not hold as stated when the hypothesis that k is algebraically closed is dropped, and is replaced either by k imperfect and separably closed, or by k perfect but not algebraically closed (8.8).

The analogue of Corollary 6.7 for curves of genus $g > 1$ is not known in general. In other words, given a type T of genus g, it is not known whether there exists a regular model X/S with X_K/K a smooth proper geometrically connected curve of genus g and X_k of type T (if $T = mT'$, we have $m \mid 2g - 2$). This problem is not completely solved even when the gcd m of the multiplicities of T is equal to 1 (see [61] and [66]), and is mostly open when $\mathcal{O}_K$ is of mixed characteristic.

The analogue of Corollary 6.7 when k is separably closed and imperfect is also open. Remarks 8.7 and 9.4 seem to indicate that the existence of a given reduction type over $\mathcal{O}_K$ may depend on the existence of certain cyclic extensions of K.

Examples of multiple fibers can be found in [18], Theorem 0.1, in [24], Sect. 8, in [23], Sect. 3, and in [48], 9.4.1.

7. Remarks on Néron models

We keep the notation introduced in the previous section, and prove the following proposition, which slightly extends 6.1 when k is imperfect.

Proposition 7.1. *Let $\mathcal{O}_K$ be a discrete valuation ring, with separably closed residue field k of characteristic $p \geq 0$. Let $f : X \to S$ be a proper and flat curve, with X regular and $f_*\mathcal{O}_X = \mathcal{O}_S$.*

(a) *Let a_Q and t_Q denote the abelian and toric ranks of $Q^0_{\bar{k}}$. Then $a_X = a_Q = a_J$ and $t_X = t_Q = t_J$.*
(b) *The maps $\Phi_X \to \Phi_Q \to \Phi_J$ induce isomorphisms $\Phi_X^{(p)} \to \Phi_Q^{(p)} \to \Phi_J^{(p)}$ on their prime-to-p parts.*

Examples where the p-part of Φ_X is not equal to the p-part of Φ_J are given in 8.8 and 9.3. We do not know of examples where the map $\Phi_X \to \Phi_J$ is not injective. We start the proof of 7.1 with several lemmas. Let G be any abelian group. For any integer d, let us denote by d_G the multiplication-by-d map on G, and by $G[d]$ the kernel of d_G.

Lemma 7.2. *Let $\mathcal{O}_K$ be strictly henselian. Let d be an integer prime to p.*

(a) *Let G be a smooth group scheme of finite type over S with connected fibers. Then $G(S) = dG(S)$ and the canonical homomorphisms $G(S)[d] \to G(k)[d] \to G(\bar{k})[d]$ are isomorphisms.*
(b) *Assume $\mathcal{O}_K$ complete. Then* (a) *also holds for the fppf-sheaf $G = P^0$, with P as in* 3.1.

Proof. (a) The map d_G is étale ([7], 7.3/2) and surjective on the fibers over S. This implies the surjectivity of $d_{G(S)}$ since S is strictly henselian. The second part of the statement follows immediately from the fact that $G[d]$ is étale over S.

(b) Applying (a) to $P^0_k \to \mathrm{Spec}\,(k)$, we get $P^0(k) = dP^0(k)$ and $P^0(k)[d] \simeq P^0(\bar{k})[d]$. The kernel of $P^0(S) \to P^0(k)$ is uniquely d-divisible (proved as in [4], 2.1). This implies that $P^0(S) = dP^0(S)$ and $P^0(S)[d] \simeq P^0(k)[d]$. □

Lemma 7.3. *Let d be an integer prime to p.*

(a) *The canonical homomorphism $Q(k)[d] \to J(k)[d]$ is an isomorphism;*
(b) *There exists an integer m_0 such that, for all d prime to m_0, the natural map $P^0(k)[d] \to Q^0(k)[d]$ is an isomorphism.*

Proof. (a) Since $P(S) = \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_K)$ is surjective by the regularity of X, we have $\mathrm{Pic}^0(X_K) \subseteq Q(S) \subseteq Q(K) = J(K) = J(S)$. Since $\mathrm{Br}(K)$ has no d-torsion ([17], 1.4), $\mathrm{Pic}^0(X_K)[d] = J(K)[d]$. Consider the commutative diagram induced by the map $Q \to J$:

$$\begin{tikzcd} Q(S)[d] \arrow[r, equal] \arrow[d] & J(S)[d] \arrow[d] \\ Q(k)[d] \arrow[r] & J(k)[d]\,. \end{tikzcd}$$

The vertical arrows are isomorphisms (same proof as for Lemma 7.2 (a)). Then, so is $Q(k)[d] \to J(k)[d]$.

(b) Let E be the schematic closure of the zero section in P (see [48], 3.2 c)). The group $E(S)$ is the subgroup of $P(S) \subset \mathrm{Pic}(X)$ generated by the vertical divisors. Let r be the gcd of the multiplicities of the irreducible components of X_k. Then it is easy to see that $E(S)[d]$ is generated by the class of $(r/(r,d))(r^{-1}X_k) \in P^0(S)$. Using Lemma 7.2(b) (we can suppose $\mathcal{O}_K$ complete), we have $\mathrm{Ker}(P^0(S)[d] \to Q^0(S)[d]) \simeq \mathbb{Z}/(d,r)\mathbb{Z}$ for all d prime to p.

On the other hand, since $P_k \to Q_k$ is surjective ([48], 4.1.2) and the group P_k/P_k^0 is a finitely generated abelian group, we find that $P_k^0 \to Q_k^0$ is surjective. Let $E' = E \cap P^0$, then

$$P^0(k)[d] \to Q^0(k)[d] \to E'(\bar{k})/dE'(\bar{k}) = \Phi_{E'_{\bar{k}}}/d\Phi_{E'_{\bar{k}}}$$

is exact (the last equality comes from the fact that the multiplication by d on $(E'^0_{\bar{k}})_{\mathrm{red}}$ is surjective). Let m_0 be equal to $r|\Phi_{E'_{\bar{k}}}|$, multiplied by p if $p > 0$. Then $P^0(k)[d] \simeq Q^0(k)[d]$ for all d prime to m_0. □

Proof of 7.1. Lemma 7.3(a) implies that $\mathrm{Ker}(Q_k \to J_k)$ contains no semi-abelian subvarieties because, otherwise, $\mathrm{Ker}(Q(k)[d] \to J(k)[d])$ would contain a non-trivial subgroup for any order d prime to p. Hence, $a_Q \le a_J$ and $t_Q \le t_J$. Since $dQ^0(\bar{k}) = Q^0(\bar{k})$, we have an exact sequence

$$0 \longrightarrow Q^0(k)[d] \longrightarrow Q(k)[d] \longrightarrow \Phi_Q[d] \longrightarrow 0.$$

Therefore, $|Q(k)[d]| = d^{2a_Q+t_Q}|\Phi_Q[d]|$. Similarly, $|J(k)[d]| = d^{2a_J+t_J}|\Phi_J[d]|$. Lemma 7.3(a) implies that there are infinitely many values of d such that $d^{2a_Q+t_Q}\left|\Phi_Q^{(p)}\right| = d^{2a_J+t_J}\left|\Phi_J^{(p)}\right|$. It follows that $2a_Q + t_Q = 2a_J + t_J$. Hence, $a_Q = a_J$ and $t_Q = t_J$, and $\left|\Phi_Q^{(p)}\right| = \left|\Phi_J^{(p)}\right|$. The map $\Phi_Q[d] \to \Phi_J[d]$ is then an isomorphism since it is surjective (because $Q_k[d] \to J_k[d]$ is).

Since Ker $(P_k[d] \to Q_k[d])$ is trivial for most integers d by Lemma 7.3(b), we can argue as above to find that $a_X \le a_Q$ and $t_X \le t_Q$. Since $P_k \to Q_k$ is surjective as noted in the proof of 7.3(b), we can conclude that $a_X = a_Q$ and $t_X = t_Q$, and Part (a) is proved.

It remains to compare Φ_X and Φ_Q. We can assume $\mathcal{O}_K$ complete. Let $M \subseteq Q^0(S)$ be the image of $P^0(S)$. Since Φ_X is canonically $P(S)/(P^0(S) + E(S))$ ([7], proof of 9.5.9), the map $P(S) \to Q(S)$ induces a canonical injection $\Phi_X \to \Phi := Q(S)/M$, whose composition with $Q(S)/M \to Q(S)/Q^0(S)$ is the canonical map $\Phi_X \to \Phi_Q$. Let d be prime to p. Then $P^0(S) = dP^0(S)$, so $M = dM$. Hence, $\Phi[d] \simeq Q(S)[d]/M[d]$. Let us compute $|M[d]|$. Since $E(S) \cap P^0(S)$ is a finite group (generated by $r^{-1}X_k$), the exact sequence

$$0 \to E(S) \cap P^0(S) \to P^0(S) \to M \to 0$$

implies that $|M[d]| = |P^0(S)[d]| = d^{2a_X + t_X} = d^{2a_Q + t_Q}$ by (a). So $|M[d]| = |Q^0(S)[d]|$. Hence $|\Phi[d]| = |\Phi_Q[d]|$ and $\left|\Phi_X^{(p)}\right| \le |\Phi^{(p)}| = \left|\Phi_Q^{(p)}\right|$. Since $\Phi_X^{(p)} \to \Phi_Q^{(p)}$ is surjective by 6.1(a), it is an isomorphism. □

Corollary 7.4. *Let X be as in 7.1. Assume that $a_X = t_X = 0$, which happens if and only if the Jacobian of X_K has purely additive reduction. If $r := \gcd(r_1, \dots, r_n) > 1$, then $\mathrm{char}(k) = p > 0$ and $r = p^s$ for some $s \ge 1$.*

Proof. Let $V = \frac{1}{r}X_k$. Then V defines a line bundle of order exactly r in $P^0(S)$. Thus, if r is not a power of p, then $\mathrm{Pic}^0_{X_k/k}(k) = P^0(k)$ contains a torsion element of order prime to p, which implies that $a_X + t_X > 0$. The assertion on the Jacobian of X_K follows from 7.1(a). □

Corollary 7.5. *Let X be as in 7.1. If J/S has semi-stable reduction, then $Q \to J$ is an open immersion. Moreover, if J/S has good reduction, then $Q \to J$ is an isomorphism.*

Proof. Since Q/S is separated, so is $Q \to J$, and we can apply 2.3 to find that $Q \to J$ factorizes as a sequence of dilatations. As shown in 7.1(a), $\dim Q_k = \dim J_k = a_J + t_J = a_Q + t_Q$. Thus, Q_k does not contain any unipotent subgroup. Hence, the sequence of dilatations $Q \to J$ can only be the dilatation of a union of connected components of J_k. When J_k is an abelian variety, we find that Φ_J is trivial and, thus, $Q \to J$ is an isomorphism. □

An example where the map $Q \to J$ is not an isomorphism and J has purely toric reduction is given in 8.8.

Remark 7.6. Let us note that when Q/S is not the Néron model, $p \mid 2g-2$. Indeed, recall that $\mathrm{Br}(K)$ is a p-torsion group ([17], 1.4). When $Q \ne J$, $\mathrm{Pic}^0(X_K) \ne J_K(K)$ (3.7). Let η be a non-trivial element in the image of $J_K(K) \to \mathrm{Br}(K)$. Clearly, when X_M/M has a section, the natural map

$J(M) \to \mathrm{Br}(M)$ is the trivial map. Thus, η is in the kernel of the natural map $\mathrm{Br}(K) \to \mathrm{Br}(M)$, which is killed by $[M : K]$. It follows that the p-part of the index δ_{X_K} kills the cokernel of $J_K(K) \to \mathrm{Br}(K)$. Thus, if this cokernel is not trivial, then $p \mid \delta_{X_K}$. Since the canonical divisor has degree $2g-2$, we find that $\delta_{X_K} \mid 2g-2$.

8. The case of semi-stable reduction

Let K be an arbitrary discrete valuation field. Let X_K be a proper smooth and geometrically connected curve of genus 1 over K, and let E_K be its Jacobian. Let X/S and E/S denote the minimal regular models of X_K and E_K, respectively. In this section, we investigate the possible relationships between the special fibers X_k and E_k when E_k is assumed to be semi-stable (generalizing [44], Thm 1').

The invariants of X_k can be explicitly computed when the reduction of E is multiplicative, allowing us for instance to show in 8.8 that the analogue of 6.6 does not hold when k is imperfect, and to prove in 8.11 that $\mathrm{disc}(X) = \mathrm{disc}(E)$. We begin with the case of good reduction in arbitrary dimension.

Proposition 8.1. *Let K be a discrete valuation field. Let X_K be a torsor under an abelian variety A_K having a proper smooth model A/S. Then X_K admits a proper regular model X/S endowed with an action $A \times_S X \to X$ extending the structure of torsor of X_K under A_K, and such that the map $A \times_S X \to X \times_S X$, $(a, x) \mapsto (ax, x)$ is surjective.*

Let $V := X_k^{red}$ and $k_0 := H^0(V, \mathcal{O}_V)$. Then V/k_0 is smooth and k_0/k is purely inseparable. Moreover, $V \times_{k_0} \bar{k}$ is a torsor under an abelian variety isogenous to $A_{\bar{k}}$.

Proof. The first part of 8.1 is proved in [49], c) at the bottom of p. 82 (see also [29]). Since $X_{\bar{k}}$ is a homogeneous space under $A_{\bar{k}}$, $X_{\bar{k}}$ is irreducible. We can write $X_k = rV$ with V reduced and geometrically irreducible over k. Hence, k_0/k is finite and purely inseparable. Since A_k is geometrically reduced, the structure of X under A makes V a homogeneous space under A_k.

If k is perfect, then $k_0 = k$. It follows that V is geometrically integral. Hence, $V_{\bar{k}}$ is a principal homogeneous space under $A_{\bar{k}}/\mathrm{Stab}_x$, where Stab_x is the stabilizer of any point $x \in V(\bar{k})$. This implies then that V/k is smooth.

When k is not perfect, we can prove that $V \to \mathrm{Spec}\, k_0$ is smooth as follows. As $\dim A_k = \dim V$, the morphism $A_k \times_k V \to V \times_k V$, defined by $(a, v) \mapsto (av, v)$, is quasi-finite. Consider the groupoid constructed as in [SGA3], Exemple 2 a) with A_k acting on V. Applying [SGA3], exposé V, Thm 8.1 to this groupoid, we obtain that the fppf-quotient $T := V/A_k$ is representable by a scheme. It follows then that $V \to T$ is smooth because A_k is smooth. The transitivity of the action implies that T is a single point. Since $V \to T$ is faithfully flat, $k \subseteq \mathcal{O}(T) \subseteq k_0$. As k_0 is purely inseparable over k and $V \to T$ is geometrically reduced, we obtain that $\mathcal{O}(T) = k_0$. □

Remark 8.2. In case $\dim(A) = 1$, the proper regular model X/S constructed in the above proposition is clearly the regular minimal model of X_K/K, and is thus endowed with an action of the Néron model J/S of A_K. When the reduction of A_K is not good, one can still show (by a method similar to [31], 10.2.12(c)) that the regular minimal model X of a torsor X_K under A_K is also endowed with an action of J extending the action of A_K on X_K. When X is semi-stable, one can prove that J_k^0 acts non trivially on every irreducible component of X_k.

In the remainder of the section, we consider the case of multiplicative reduction. We say that E_K has *split multiplicative reduction*[3] *of type* I_n, $n \geq 1$, if the unit component of the special fiber of the Néron model of E_K is isomorphic to $\mathbb{G}_{m,k}$ and if $|\Phi_{E_K}(\bar{k})| = n$. Note that $n = -v_K(j(E)) \geq 1$. If k is separably closed, a multiplicative reduction is always split. In general, a multiplicative reduction becomes split after a quadratic étale extension.

To study the reduction of torsors under E_K, we may and will assume that K is complete, since the minimal regular model commutes with the completion of $\mathcal{O}_K$.

Proposition 8.3. *Let K be a complete discrete valuation field. Let X_K be a smooth and geometrically connected projective curve of genus 1 over K, such that its Jacobian E_K has split multiplicative reduction of type I_n, $n \geq 1$.*

(a) *There exists a unique cyclic extension K'/K such that $X_K(K') \neq \emptyset$ and which is contained in any extension of K trivializing X_K (i.e., any extension M/K such that $X_K(M) \neq \emptyset$).*
(b) *Let X/S be the minimal regular model of X_K. Let k' be the residue field of K'. Let $f := f_{K'/K} = [k' : k]$ and $e := e_{K'/K}$. Then $f \mid n$. Moreover, if $f \neq n$, then X_k is a cycle of n/f projective lines over k', each of multiplicity e in X_k. The intersection points between reduced components of X_k are rational over k' with associated intersection number over k' equal to 1. If $f = n$, then $(X_k)^{red}$ is irreducible with a k'-rational double point, and its normalization is a projective line over k'.*

Note that when k is algebraically closed, it follows from Part (b) that X_k is of type $[K' : K]I_n$. Thus, Proposition 8.3 provides a different proof of 6.6 when the reduction is multiplicative. The proof of 8.3 uses Tate's uniformization of E_K (see e.g., [55], V.3, or [14], 5.1). Let $\Gamma_{K^s} := \mathrm{Gal}(K^s/K)$ and let $v_K : K^s \to \mathbb{Q} \cup \{\infty\}$ be the valuation on K^s extending the normalized valuation on K. Denote by $|\ |$ an associated absolute value. There exist

[3] Note that when E_K has split multiplicative reduction of type I_n, the minimal regular model E/S has reduction of type I_n. Indeed, the reduction type of E_k can only be I_m, $I_{2m-2,2}$ or $I_{2m-1,2}$ for some m (see [31], pp. 486–487 for the reduction types of elliptic curve having multiplicative reduction). It is shown in [8], 4.3, that $\Phi_{E_K}(k) = \Phi_{E_K}(\bar{k})$. This forces E_k to have reduction type I_n (see [31], 10.2.24).

$q \in K^*$ with $v_K(q) = n$, and an exact sequence of Γ_{K^s}-modules:

$$1 \longrightarrow q^{\mathbb{Z}} \longrightarrow (K^s)^* \longrightarrow E(K^s) \longrightarrow 0. \tag{25}$$

For any $z \in (K^s)^*$, we denote by $\tilde{z}$ the image of z in $E(K^s)$.

Proof of Part (a). This is well-known: the exact sequence (25) gives rise to the exact sequence

$$0 = H^1(\Gamma_{K^s}, (K^s)^*) \to H^1(\Gamma_{K^s}, E(K^s)) \to H^2(\Gamma_{K^s}, \mathbb{Z}) \simeq \mathrm{Hom}(\Gamma_{K^s}, \mathbb{Q}/\mathbb{Z}).$$

The torsor X_K corresponds to an element x of $H^1(\Gamma_{K^s}, E(K^s))$, and the torsor is trivial if and only if x is trivial in $H^1(\Gamma_{K^s}, E(K^s))$. One easily shows that the extension K'/K corresponds to the kernel of the map $\Gamma_{K^s} \to \mathbb{Q}/\mathbb{Z}$, image of x under the functorial injection $H^1(\Gamma_{K^s}, E(K^s)) \to \mathrm{Hom}(\Gamma_{K^s}, \mathbb{Q}/\mathbb{Z})$. The extension K'/K is cyclic because any finite subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic. □

8.4. Before proving Part (b) in 8.6, we need some preliminary results. The Néron model J of E_K is the smooth part of the minimal regular model E (see e.g., [31] 10.2.14, or [7], 1.5/1). Let Φ_{E_K} be the group of components of J. Using rigid analytic geometry, one can describe the reduction map $r_K : E(K^s) \to E(k^{alg})$ as follows (see [14], §5.3). Let $z \in (K^s)^*$. Then there exists an integer i such that $|\pi|^{i+1} < |z| \leq |\pi|^i$. Any isomorphism $\mathbb{Z}/n\mathbb{Z} \to \Phi_{E_K}$ allows us to number the connected components of E_k and J_k as $E_k^{(i)} \supset J_k^{(i)}$, with $\bar{i} \in \mathbb{Z}/n\mathbb{Z}$. There exists such a numbering such that, when $|z| = |\pi|^i$, $r_K(\tilde{z})$ belongs to the component $J_k^{(i)}$, and when $|\pi|^{i+1} < |z| < |\pi|^i$, then $r_K(\tilde{z})$ is an intersection point $p_{i,i+1}$ of the irreducible components $E_k^{(i)}$ and $E_k^{(i+1)}$. This intersection point is unique if $n \geq 3$.

Lemma 8.5. *Let $w \in K^*$ with $v_K(w) = m$ and $m \not\equiv 0 \mod n$. The translation by $\tilde{w} \in E(K)$ is an automorphism of E_K which extends to an automorphism $t_{\tilde{w}} : E \to E$. This automorphism acts freely on $E(k^{alg})$ and Φ_{E_K}.*

Proof. By the uniqueness of E, any automorphism of E_K (considered as a curve) extends to an automorphism of E. By the description we reviewed above, $t_{\tilde{w}}$ acts as the addition-by-m on $\mathbb{Z}/n\mathbb{Z} \simeq \Phi_{E_K}$. (Note that $n \neq 1$ by the hypothesis on m.) Thus, $t_{\tilde{w}}$ has no fixed point on Φ_{E_K} and, hence, no fixed point on J_k (which is identified with the smooth part of E_k).

Let us suppose now that $n \geq 3$. Then $t_{\tilde{w}}$ maps the intersection point $p_{i,i+1} \in E_k^{(i)} \cap E_k^{(i+1)}$ to $p_{i+m,i+m+1}$. Since $(i, i+1)$ is neither congruent to $(i+m, i+m+1)$ nor to $(i+m+1, i+m)$ modulo n, we see that $p_{i,i+1}$ is not a fixed point.

When $n = 2$, pick z with $|\pi|^i < |z| < |\pi|^{i+1}$, which reduces to an intersection point in E_k. Then $t_{\tilde{w}}(\tilde{z}) = \tilde{w}\tilde{z}$ reduces to the other intersection point. Hence $t_{\tilde{w}}$ has no fixed point in $E(k^{alg})$. □

Let F/K be a finite Galois extension, with group Γ_F and ramification index $e_{F/K}$. Then (25) induces an exact sequence of Γ_F-modules

$$0 \to q^{\mathbb{Z}} \to F^* \to E(F) \to 0.$$

The canonical map $c_F : E(F) \to \Phi_{E_F}$ induced by the Néron model can be understood using the following commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathcal{O}_F^* \arrow[r, equal] \arrow[d] & \mathcal{O}_F^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q^{\mathbb{Z}} \arrow[r] \arrow[d, "v_K"'] & F^* \arrow[r] \arrow[d, "v_K"'] & E(F) \arrow[r] \arrow[d, "c_F"'] & 0 \\
0 \arrow[r] & n\mathbb{Z} \arrow[r] & e_{F/K}^{-1}\mathbb{Z} \arrow[r] & \Phi_{E_F} \arrow[r] & 0.
\end{tikzcd} \tag{26}$$

The last two columns are exact, and the long exact sequence of Galois cohomology applied to these two short exact sequences gives (remembering that Γ_F acts trivially on Φ_{E_F})

$$\begin{tikzcd}
e_{F/K}^{-1}\mathbb{Z} \arrow[r] \arrow[d] & H^1(\Gamma_F, \mathcal{O}_F^*) \arrow[r] \arrow[d, equal] & H^1(\Gamma_F, F^*) = 1 & \\
\Phi_{E_F} \arrow[r] & H^1(\Gamma_F, \mathcal{O}_F^*) \arrow[r] & H^1(\Gamma_F, E(F)) \arrow[r] & H^1(\Gamma_F, \Phi_{E_F}).
\end{tikzcd}$$

Hence, $\Phi_{E_F} \to H^1(\Gamma_F, \mathcal{O}_F^*)$ is surjective and the bottom sequence above induces the exact sequence

$$0 \to H^1(\Gamma_F, E(F)) \to H^1(\Gamma_F, \Phi_{E_F}) = \mathrm{Hom}(\Gamma_F, \Phi_{E_F}) \xrightarrow{\alpha_F} H^2(\Gamma_F, \mathcal{O}_F^*). \tag{27}$$

8.6. *Proof of 8.3 b).* Since X_K and E_K become isomorphic over K', our knowledge of the minimal model $E/\mathcal{O}_K$ will provide us with a description of the minimal regular model X' of $X_{K'}$ over $\mathcal{O}_{K'}$. We can recover the minimal regular model $X/\mathcal{O}_K$ as the quotient of X' by a (twisted) action of $\Gamma := \Gamma_{K'}$, as follows. Let $(\xi_\sigma)_\sigma$ be a 1-cocyle whose class in $H^1(\Gamma, E(K'))$ defines the torsor X_K. Then its image $\rho \in \mathrm{Hom}(\Gamma, \Phi_{E_{K'}})$ in the exact sequence (27) is the 1-cocycle $\sigma \mapsto c_{K'}(\xi_\sigma)$. Thus $\rho(\sigma) = c_{K'}(\xi_\sigma)$. Note that ρ is injective by the minimality of K'.

We identify $X_{K'}$ to $E_{K'}$ endowed with a twisted action of Γ_{K^s}: let $\tau \in \Gamma_{K^s}$ with image σ in Γ, then

$$\tau * x := \tau(x) + \xi_\sigma, \qquad \text{for all } x \in E(K^s),$$

where $\tau(x)$ is the usual Galois action of Γ_{K^s} on $E(K^s)$. Let $E' = X'$ be the minimal regular model of $E_{K'}$ over $\mathcal{O}_{K'}$. Then $\Phi_{E_{K'}}$ is cyclic of order ne. Applying the description of the reduction as in (8.4) to E', we see easily that the usual Galois action of Γ on $E'_{k'}$ fixes the intersection points and the generic points of the irreducible components of $E'_{k'}$. Let $w_\sigma \in K'^*$ with image $\xi_\sigma \in E(K')$. Since $\rho(\sigma) \neq 0$, the commutative diagram (26) with $F = K'$ implies that $v_{K'}(w_\sigma) \neq 0 \mod ne$. The action of σ on X' is induced by the usual Galois action of σ followed by the multiplication by w_σ on $(K^s)^*$. Applying Lemma 8.5 to $E_{K'}/K'$, we see that σ acts freely on $X'_{k'}$ whenever $\sigma \neq 1$. Hence, Γ acts freely on $X'_{k'}$ and on the set of the irreducible components of $X'_{k'}$.

Let $Y := X'/\Gamma$. Since the action of Γ is free, Y is a regular model of X_K over $\mathcal{O}_K$. Looking at the action of Γ on $X'_{k'}$, we see that Y_k is a union of $ne/[K' : K] = n/f$ irreducible components as in the statement of the proposition, each of them having multiplicity e in Y_k. Moreover, two consecutive components meet each other transversally at a point y (if $x \in X'_{k'}$ is a point lying over y, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X',x'}$ is étale) and y is rational over k'. In particular, Y_k does not contain any exceptional divisor, and, hence, $Y = X$. Note that it is important to take K'/K minimal. Otherwise X'/Γ is not necessarily regular. □

Let us now turn to proving the existence of curves of genus 1 having Jacobians with multiplicative reduction.

Proposition 8.7. a) *Let E_K/K be an elliptic curve parametrized as $\mathbb{G}_{m,K}/q^{\mathbb{Z}}$. Let K'/K be a cyclic extension of degree d. Then $H^1(\Gamma, E_K(K'))$ contains an element of order d if and only if $q \in N_{K'/K}(K'^*)$ (where $N_{K'/K}$ denotes the norm map).*

b) *Given any positive integer n and any cyclic extension K'/K with $f_{K'/K} | n$, there exists a smooth projective curve X_K/K of genus* 1 *minimally trivialized by K' and whose Jacobian has split multiplicative reduction of type I_n.*

Proof. a) In view of the exact sequence (27), we need to understand when there exists an injective homomorphism $\rho : \Gamma \to \Phi_{E_{K'}}$ in $\mathrm{Ker}(\alpha_{K'})$. After fixing a generator of Γ, we can identify $\mathrm{Hom}(\Gamma, \Phi_{E_{K'}})$ with $\Phi_{E_{K'}}[d]$, and the set of injective ρ's with set of elements of $\Phi_{E_{K'}}[d]$ of order d. The group $H^2(\Gamma, \mathcal{O}^*_{K'})$ can be identified with $\mathcal{O}^*_K/N_{K'/K}(\mathcal{O}^*_{K'})$ (see [53], VIII, §4, for a review of the Galois cohomology of cyclic groups). The map $\alpha_{K'}$ is then described as follows. Let $x \in \Phi_{E_{K'}}[d]$, and let $y \in E(K')$ be a preimage of x under $c_{K'}$. The trace of y is in $E(K)$ and belongs to $\mathrm{Ker}(c_K)$. It is thus the image of an element in $\mathcal{O}^*_K$. Then $\alpha_{K'}(x)$ is equal to the image of this element of $\mathcal{O}^*_K$ in $\mathcal{O}^*_K/N_{K'/K}(\mathcal{O}^*_{K'})$.

We need to show that $q \in N_{K'/K}(K'^*)$ if and only if $\Phi_{E_{K'}}[d] \subseteq \mathrm{Ker}\,\alpha_{K'}$ or, equivalently, if an element of $\Phi_{E_{K'}}[d]$ of order d is in $\mathrm{Ker}\,\alpha_{K'}$.

Let $n := v_K(q)$. If $\Phi_{E_{K'}}[d]$ has an element of order d or if $q \in N_{K'/K}(K'^*)$, then $f \mid n$. Let $z := (\pi')^{n/f} \in K'$, where π' is a uniformizing element of K'. Then $|z| = |q|^{1/d}$ and $x := c_{K'}(\tilde{z})$ is a generator of $\Phi_{E_{K'}}[d]$. We have $N_{K'/K}(z) = qu$ for some $u \in \mathcal{O}^*_K$, and $\mathrm{Tr}_{K'/K}(\tilde{z}) = \tilde{u}$. Hence, $\alpha_{K'}(x) = u$ in $\mathcal{O}^*_K/N_{K'/K}(\mathcal{O}^*_{K'})$. Then we see immediately that $\alpha_{K'}(x)$ is trivial if and only if $q \in N_{K'/K}(K'^*)$.

b) Given K'/K cyclic with $f \mid n$, let $z := (\pi')^{n/f} \in K'$, and use $q := N_{K'/K}(z)$ to define $E_K := \mathbb{G}_{m,K}/q^{\mathbb{Z}}$. We let X_K be the torsor under E_K defined by the element of order d in $H^1(K, E_K)$ found in Part a). Then $v_K(q) = n$, with E_K and X_K as in Proposition 8.3. □

Example 8.8. Let $n \geq 2$ and let K'/K be a cyclic extension of degree $d \neq 1$ dividing n and such that $e_{K'/K} = 1$. According to Proposition 8.7, there exists a smooth projective curve X_K/K of genus 1 whose Jacobian E_K has split multiplicative reduction of type I_n. Then 8.3 implies that X_k has n/d geometric irreducible components. Thus, Theorem 6.6 does not hold as stated when k is imperfect, or when k is perfect but not algebraically closed.

Assume that k is imperfect. To produce an explicit example, consider a field K of mixed characteristic p which contains the f-th roots of unity (with f a power of p). Then the extension $K' := K[x]/(x^f - a)$, with $a \in \mathcal{O}_K$ not a p-th power modulo π, is cyclic over K with residual index f.

Note that in this example, Φ_X and Φ_E have orders n/d and n, respectively. In particular, their p-parts are not isomorphic. It follows that the natural map $Q \to J$ is an open immersion (7.5), but not an isomorphism.

Remark 8.9. Propositions 8.3 and 8.7 show the existence of torsors X_K of some shape once a cyclic extension K'/K exists with residue extension k'/k of degree f. Let us remark here that a cyclic extension K'/K of degree $d := p^r$, totally ramified, and with $f > 1$, may not exist for a given K. For instance, a theorem of Miki (see, e.g., [56] 9.1) states that if K has characteristic zero and $v_K(p) < p - 1$, then any cyclic extension K'/K of degree p^r has separable residue field extension.

Remark 8.10. Let us return to 8.3 b). Let $Z := (X_k)^{red}$. Since Z/k is geometrically connected (see, e.g., [31], 5.3.17), we know that $H^0(Z, \mathcal{O}_Z)/k$ is purely inseparable. It is in fact equal to $(k')^{\mathrm{Aut}(k'/k)}$, the largest purely inseparable extension of k in k'. Indeed, we saw in the proof of 8.3 b) that Γ acts freely on $X_{k'}$. We find then that $Z = X_{k'}/\Gamma$ and $H^0(Z, \mathcal{O}_Z) = (k')^{\Gamma}$. Since Γ acts on $\mathcal{O}_{K'}$ as $\mathrm{Gal}(K'/K)$, it acts on k' as $\mathrm{Aut}(k'/k)$.

Keep the notation and hypotheses of Proposition 8.3. We turn now to computing the integers $\mathrm{disc}(X)$ and $\ell(H^1(X, \mathcal{O}_X)_{\mathrm{tors}})$ explicitly. It follows from 8.3 that the greatest common divisor of the multiplicities of the components of X_k is e. Let h be the integer such that $0 \leq h \leq e - 1$ and $\omega_{X/S} = \omega_0 \mathcal{O}_X(he^{-1}X_k)$ for some basis ω_0 of $H^0(X, \omega_{X/S})$ (Lemma 5.7).

Let $S' := \mathrm{Spec}(\mathcal{O}_{K'})$ and let π' be a uniformizing element of K'. Recall that $\omega_{S'/S}$ is the sheaf associated with the module $\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_{K'}, \mathcal{O}_K)$. This latter module is isomorphic to $W_{K'/K} := \{\beta \in K' \mid \mathrm{Tr}_{K'/K}(\beta\mathcal{O}_{K'}) \subseteq \mathcal{O}_K\}$ (see, e.g., [31], 6.4.25 and Exer. 6.4.8). Write $W_{K'/K} = (\pi')^{-\delta_{K'/K}}\mathcal{O}_{K'}$.

Proposition 8.11. *Keep the above notation. Let* $a := \ell(H^1(X, \mathcal{O}_X)_{\mathrm{tors}})$. *Then*

(a) $\mathrm{disc}\,(X/S) = \mathrm{disc}\,(E/S) = n$.
(b) *Write* $\delta_{K'/K} = se + r$ *with* $0 \le r < e = e_{K'/K}$. *Then* $a = s$, *and* $h = r$.
(c) X/S *is cohomologically flat if and only if* K'/K *is tamely ramified. When* X/S *is cohomologically flat, then* $h = e - 1$.

Proof. Using the results recalled in 9.1, we obtain that $\mathrm{disc}\,(E/S) = n$. Let $X' \simeq E_{S'}$ be the minimal regular model of $X_{K'}$ over S'. We conclude then that $\mathrm{disc}\,(X'/S') = ne$. Let us now compute $\mathrm{disc}\,(X'/S')$ in a different way.

Recall as in 9.1 that $\Delta_{X_K/K}$ can be regarded as a map $K \to (H^0(X_K, \omega_{X_K/K}))^{\otimes 12}$. Using the identification from 5.5, we find that $\Delta_{X_K/K}(1)$ can be identified with the element $\lambda(\pi^a\omega_0)^{\otimes 12}$ for some $\lambda \in K^*$ with $v_K(\lambda) = \mathrm{disc}\,(X/S)$.

Let $\mu : X' \to X$ be the canonical (quotient) morphism. We saw in the proof of Proposition 8.3 that μ is étale. Thus,

$$\omega_{X'/S} = \mu^*\omega_{X/S} = \omega_0\mathcal{O}_{X'}(hX'_{k'}) = (\pi')^{-h}\omega_0\mathcal{O}_{X'}.$$

Let $\rho : X' \to S'$ be the natural map. Using the adjunction formula (see, e.g., [31] 6.4.9), we find that

$$\omega_{X'/S'} = \omega_{X'/S} \otimes_{\mathcal{O}_{X'}} (\rho^*\omega_{S'/S})^{\vee} = (\pi')^{\delta_{K'/K}}\omega_{X'/S} = (\pi')^{\delta_{K'/K}-h}\omega_0\mathcal{O}_{X'}.$$

We may thus compute $\Delta_{X_{K'}/K'}(1)$ in $H^0(X_{K'}, \omega_{X_{K'}/K'})^{\otimes 12}$ as the image of $\Delta_{X_K/K}(1)$, namely: let $\omega_0' := (\pi')^{\delta_{K'/K}-h}\omega_0$. Then $\Delta_{X_{K'}/K'}(1) = \lambda\pi^{12a}(\pi')^{12(-\delta_{K'/K}+h)}(\omega_0')^{\otimes 12}$. It follows then from the definitions that

$$\mathrm{disc}\,(X'/S') = e\mathrm{disc}\,(X/S) + 12(ea - \delta_{K'/K} + h).$$

Putting both expressions for $\mathrm{disc}\,(X'/S')$ together, we obtain:

$$\mathrm{disc}\,(X/S) = n + 12e^{-1}(\delta_{K'/K} - h - ea). \tag{28}$$

Theorem 5.9 shows that $\mathrm{disc}\,(X) - \mathrm{disc}\,(E)$ is divisible by 12. Since $\mathrm{disc}\,(E) = n$, we obtain from (28) that $e \mid (\delta_{K'/K} - h)$. Thus, $h = r$. We turn now to proving that $a = s$. Once this fact is known, Part a) is immediate from b) and (28).

For any integer $N \ge 0$ and any scheme Y over S, we denote by $Y_N = Y \times_S \mathrm{Spec}\,(\mathcal{O}_K/\pi^{N+1}\mathcal{O}_K)$. Let $\Gamma = \mathrm{Gal}\,(K'/K)$. Since $X' \to X$ is étale, so is $X'_N \to X_N$ and we have $X_N = X'_N/\Gamma$. Therefore $H^0(X_N, \mathcal{O}_{X_N}) = H^0(X'_N, \mathcal{O}_{X'_N})^{\Gamma}$. Since $X' \to \mathrm{Spec}\,\mathcal{O}_{K'}$ is cohomologically flat, we have

$$H^0(X_N, \mathcal{O}_{X_N}) = \left(H^0(X', \mathcal{O}_{X'})\otimes_{\mathcal{O}_{K'}}\mathcal{O}_{K'}/\pi^{N+1}\mathcal{O}_{K'}\right)^{\Gamma} = \left(\mathcal{O}_{K'}/\pi^{N+1}\mathcal{O}_{K'}\right)^{\Gamma}.$$

The exact sequence $0 \to \mathcal{O}_X \xrightarrow{\cdot\pi^N} \mathcal{O}_X \to \mathcal{O}_{X_{N-1}} \to 0$ induces an exact sequence

$$0 \to \mathcal{O}_K/\pi^N\mathcal{O}_K \to \left(\mathcal{O}_{K'}/\pi^N\mathcal{O}_{K'}\right)^{\Gamma} \to H^1(X, \mathcal{O}_X)[\pi^N] \to 0$$

(where $[\pi^N]$ means π^N-torsion). Similarly, the exact sequence $0 \to \mathcal{O}_{K'} \xrightarrow{\cdot\pi^N} \mathcal{O}_{K'} \to \mathcal{O}_{K'}/\pi^N\mathcal{O}_{K'} \to 0$ induces an exact sequence

$$0 \to \mathcal{O}_K/\pi^N\mathcal{O}_K \to \left(\mathcal{O}_{K'}/\pi^N\mathcal{O}_{K'}\right)^\Gamma \to H^1(\Gamma, \mathcal{O}_{K'})[\pi^N] \to 0.$$

Taking $N \gg 0$, we find $\tau = \ell(H^1(\Gamma, \mathcal{O}_{K'})[\pi^\infty])$. Since $H^1(\Gamma, \mathcal{O}_{K'})$ is finitely generated on $\mathcal{O}_K$ and $H^1(\Gamma, K') = 0$, we find that $H^1(\Gamma, \mathcal{O}_{K'})[\pi^\infty] = H^1(\Gamma, \mathcal{O}_{K'})$. As noted in [52], Remark after Theorem 2, one can use Herbrand's quotient to show that $\ell(H^1(\Gamma, \mathcal{O}_{K'})) = \ell(\hat{H}^0(\Gamma, \mathcal{O}_{K'}))$, and this latter module is isomorphic to $\mathcal{O}_K/\mathrm{Tr}(\mathcal{O}_{K'})$. That $\mathrm{Tr}(\mathcal{O}_{K'}) = \pi^s\mathcal{O}_K$ is well-known ([53], V.3, Lemma 4 when $|\Gamma|$ is prime). Hence, $a = s$.

The cohomological flatness of X/S in Part c) is equivalent to $a = 0$, and, thus, equivalent to $\delta_{K'/K} \leq e - 1$. This last condition is equivalent to K'/K tamely ramified and to $\delta_{K'/K} = e - 1$. The equality $h = e - 1$ then follows from (b), and Proposition 8.11 is proved. □

Remark 8.12. We are now in a position to give an example of a wild fiber of strange type (see [24], added in proof, where the authors indicate that such an example is not known.) We need to produce a model X/S which is not cohomologically flat, and such that its fiber has multiplicity e and $h = e-1$. It suffices to find a totally and wildly ramified extension K'/K of degree e, such that $\delta_{K'/K} = se+e-1$ for some $s > 0$. Choose K with an algebraically closed residue field, and consider an extension $K' := K(\alpha)$, where α is the root of an Eisenstein polynomial $f(T) \in \mathcal{O}_K[T]$. The formula for $\delta_{K'/K}$ in terms of the valuation of $f'(\alpha)$ shows that it is possible to find examples of wild extensions with $\delta_{K'/K}$ congruent to any integer $0 < h < e$, at least when the valuation of e is large enough if K has characteristic zero.

Remark 8.13. Keep the notation of this section. Consider the natural maps introduced in 3.8 (13),

$$H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Lie}(Q_X) \longrightarrow H^1(E, \mathcal{O}_E).$$

Under our current hypotheses 8.3, we can show that the cokernel and kernel of $H^1(X, \mathcal{O}_X) \to \mathrm{Lie}(Q_X)$ have same length, even when k is imperfect. Indeed, Theorem 5.9 shows that $\mathrm{disc}(X) = \mathrm{disc}(E) + 12(b - a)$. Since $\mathrm{disc}(X) = \mathrm{disc}(E)$ (8.11), we find that $b = a$. Since $\mathrm{Lie}(Q_X)$ is isomorphic to $H^1(E, \mathcal{O}_E)$ (7.5), our claim follows.

9. Some examples

Let X_K be a smooth, projective, and geometrically connected curve of genus 1, and let E_K be its Jacobian. Let X and E denote the minimal regular models over $S = \mathrm{Spec}\,\mathcal{O}_K$ of X_K and E_K, respectively. When k is separably closed but not algebraically closed, we saw in 8.8 examples where the number of irreducible components n_X and n_E of X_k and E_k are not equal, and where Φ_X and Φ_E are not isomorphic. We exhibit below a different example where the reduction is additive and where, in addition, E and X do not have the same discriminant.

9.1. Let us recall here the relationship between the discriminant disc(X) and the usual discriminant of a curve of genus 1 with a rational point. Let T be any scheme. Let $Y \to T$ be a smooth projective curve with geometrically connected fibers of genus 1. Then $\Delta_{Y/T}$ is defined as an isomorphism $\Delta_{Y/T} : \mathcal{O}_T \to (g_*\omega_{Y/T})^{\otimes 12}$ (see the begining of Sect. 5). Suppose that T is affine and that Y can be defined by a Weierstrass equation

$$y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathcal{O}_T(T).$$

Let $\Delta \in \mathcal{O}_T(T)$ be the discriminant of the above equation, and set

$$\omega_0 := \frac{dx}{2y + a_1x + a_3}. \tag{29}$$

Then $\omega_{Y/T} = \omega_0\mathcal{O}_Y$. It is well-known that $\Delta_{Y/T}(1) = \Delta\omega_0^{\otimes 12}$ up to sign.

Let E_K be an elliptic curve over a discrete valuation field K. Let Δ be the minimal discriminant of E_K, and E be the minimal regular model of E_K over S. We claim that disc$(E) = v(\Delta)$. Indeed, let

$$y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathcal{O}_K$$

be a minimal Weierstrass equation of E_K. Let W/S be the Weierstrass model of E_K associated to the above equation, and let $\eta : E \to W$ be the natural contraction map (see for instance [31], Cor. 9.4.37). Let ω_0 be given by (29). Then, as we mentioned above, $\Delta_{E_K/K}(1) = \pm\Delta\omega_0^{\otimes 12}$. It is well-known that $\eta^*\omega_0$ is a basis of $\omega_{E/S}$ ([31], loc. cit.). Thus disc$(E) = v(\Delta)$.

Proposition 9.2. *Let K be a discrete valuation field of mixed characteristic 3, with imperfect residue field k and uniformizing element π. Let $e_K := v(3)$. Fix an element $c \in \mathcal{O}_K$ whose image in k is not a cube, and consider the cubic curve X_K/K defined by the equation*

$$x^3 + cy^3 + \pi z^3 = 0. \tag{30}$$

Then the regular minimal model X/S of X_K is the plane curve in $\mathbb{P}^2/S$ defined by equation (30)*. Let E_K be the Jacobian of X_K. Then*

$$[\Phi_E,\ n_E,\ \mathrm{disc}\,(E)] = \begin{cases} [\mathbb{Z}/3\mathbb{Z},\ 3,\ 3e_K + 4] & \text{if } e_K \text{ is even,} \\ [\{0\},\ 9,\ 3e_K + 10] & \text{if } e_K \text{ is odd.} \end{cases}$$
$$[\Phi_X,\ n_X,\ \mathrm{disc}\,(X)] = \quad [\{0\},\ 1,\ 9e_K + 4].$$

In particular, $n_X \neq n_E$, disc$(X) \neq$ disc(E) *except when $e_K = 1$, and*

$$\mathrm{disc}\,(X) - (n_X - 1) \neq \mathrm{disc}\,(E) - (n_E - 1). \tag{31}$$

Moreover, if e_K is even, then $\Phi_X \neq \Phi_E$.

Proof. The Jacobian E_K is defined by the Weierstrass equation (see [2], §3.2)

$$y^2 = 4x^3 - 27c^2\pi^2.$$

We easily find that the reduction of E_K is of type IV if e_K is even, and of type II* otherwise (to obtain the minimal model, it suffices in this example to divide the equation by the appropriate power of π^6; see also [57]). This gives Φ_E, n_E, and disc (E) (which by 9.1 is the minimal discriminant of E_K).

Let us compute this data for X_K. Equation (30) defines a plane curve over S which is easily checked to be regular, with integral special fiber. Thus, it is the minimal regular model X/S of X_K, with $n_X = 1$, and $\Phi_X = \{0\}$. Since X is a global complete intersection, it is cohomologically flat over S (see, e.g., [31], Exer. 5.3.14). Moreover, $\omega_{X/S}$ is generated by its global section (5.7). Consider the rational functions $x_1 := x/z$, $y_1 := y/z \in K(X)$ on X. Then the rational differential

$$\omega_1 := \frac{dx_1}{3cy_1^2} \in \Omega^1_{K(X)/K}$$

is a basis of $\omega_{X/S}$ (this can be seen by a direct computation, similar to the proof of [31], 9.4.26 (c)). Thus, we need to compute the valuation of $\lambda \in K$ such that $\Delta_{X_K/K}(1) = \lambda\omega_1^{\otimes 12}$ (see 9.1). Let $K' := K(c^{1/3})$. Let

$$u := \frac{-3\pi z}{x + c^{1/3}y}, \quad v := \frac{9\pi c^{1/3}y}{x + c^{1/3}y} \in K(X_{K'}).$$

Then $v^2 - 9\pi v = u^3 - 27\pi^2$. The discriminant Δ of this Weierstrass equation is $-3^9\pi^4$, and its canonical differential is $\omega_0 = \dfrac{du}{2v - 9\pi}$. One checks that $\omega_0 = c^{1/3}\omega_1$. We may then compute disc (X) as follows. By construction, $\Delta_{X_K/K}(1) = \lambda\omega_1^{\otimes 12}$, and $\Delta_{X_{K'}/K'}(1) = \Delta\omega_0^{\otimes 12}$ (see 9.1). So, as $v(c) = 0$, disc $(X) = v(\lambda) = v(\Delta)$. This achieves the proof. □

Remark 9.3. In the above example, the group scheme Q/S associated with X_K/K is not isomorphic to the Néron model J/S of the Jacobian of X_K, except when $e_K = 1$. This is easily seen as follows when e_K is even: the group Φ_X is trivial, and since the natural map $\Phi_X \to \Phi_Q$ is surjective, Φ_Q is also trivial. On the other hand, the group Φ_E has order 3.

When $e_K > 1$, we can show that the natural map $Q \to J$ does not induce an isomorphism Lie $(Q) \to$ Lie (J) (or, in other words, that $Q \to J$ is not an open immersion). Note first that in the above example, disc $(X) -$ disc $(E) = 6e_K$ or $6(e_K - 1)$, depending on whether e_K is even or odd. Since X/S is cohomologically flat as noted in the proof of 9.2, Theorem 5.9 shows that disc $(X) -$ disc $(E) = 12c$, where c is the length of the cokernel of Lie $(Q) \to H^1(E, \mathcal{O}_E)$. Hence, in this example, $c = e_K/2$ or $(e_K - 1)/2$, depending on whether e_K is even or odd.

Recall that when the natural map of S-group schemes $Q \to J$ is not an isomorphism, then $\mathrm{Pic}^0(X_K) \neq J_K(K)$ (3.7). It is in general very difficult to check directly whether $\mathrm{Pic}^0(X_K) \neq J_K(K)$. The above result shows that for the curve (30), $\mathrm{Pic}^0(X_K) \neq J_K(K)$ if $e_K > 1$.

Remark 9.4. The Brauer group $\mathrm{Br}(K)$, when k is imperfect and separably closed, is a huge torsion group (see, e.g., [22], Theorem 3, [27] 5.8, or [50]). We note below that it is sometimes possible to use torsors under elliptic curves to obtain explicit elements in $\mathrm{Br}(K)$.

In 8.8 and in 9.2, we have been able to construct torsors X_K/K such that the natural map $Q_X \to J$ is not an isomorphism. As we saw above, the cokernel of $\mathrm{Pic}^0(X_K) \to J_K(K)$ is then non-trivial and, thus, we obtain in this way non-trivial elements of $\mathrm{Br}(K)$ (use the exact sequence (11) restricted to Pic^0). When the reduction of X is additive, we find that the cokernel of $\mathrm{Pic}^0(X_K) \to J_K(K)$ maps surjectively to an additive group $(k, +)$.

We consider below curves X_K/K of genus 1 whose Jacobians E_K/K have good reduction. As noted in 7.5, in this case $Q \to J$ is an isomorphism. Thus we cannot use the cokernel of $\mathrm{Pic}^0(X_K) \to J_K(K)$ to produce non-trivial elements of $\mathrm{Br}(K)$. Nevertheless, given a cyclic extension K'/K with residual index $f = [K' : K] = p^n$, we are going to exhibit curves X_K/K of genus 1 whose period δ' strictly divides its index δ. We can then use the exact sequence (11) to produce elements in $\mathrm{Br}(K)$ of order divisible by δ/δ'.

Fix two integers $0 < r \leq n$. Let $\mathcal{O}_K$ be complete with k separably closed. Assume that there exists a cyclic extension K'/K of degree p^n, with associated residue extension k'/k also of degree p^n. Then we construct curves X_K/K of genus 1 with index $\delta = p^{n+r}$ and period $\delta' = p^n$ as follows[4]. Choose an elliptic curve E_K/K with ordinary reduction, as in [48], 9.4.1 (iii), and assume in addition that E/S is a Serre-Tate lifting of its special fiber (as in [48], 9.4.3 b)). In other words, the p-divisible group B associated with E/S splits as the sum of B^{rad} and B^{et}. That many such curves E_K/K exist over K is proved as follows. Let $k_0 \subset k$ be a perfect subfield of k. Let $W(k_0)$ denote the Witt ring associated with k_0, with $S_0 := \mathrm{Spec}\, W(k_0)$. Given any ordinary elliptic curve E_{k_0}/k_0, there exists a unique smooth lifting E_0/S_0 of E_{k_0} to S_0 (the canonical Serre-Tate lifting, see [35], V (3.3)). Using the natural map $W(k_0) \to \mathcal{O}_K$, we can consider the pull back $E = E_0 \times_{S_0} S$. It is easy to check that E/S has the required properties.

For such E/S, the p-part of $H^1(K, E_K)$ is the direct sum of two terms, one isomorphic to $\mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}_p/\mathbb{Z}_p)$, and the other one isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$. Moreover, given any extension F/K of ramification index e, the natural map $H^1(K, E_K) \to H^1(F, E_F)$ restricted to the factor $\mathbb{Q}_p/\mathbb{Z}_p$ is the multiplication-by-e map $\mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p$. We define now a torsor X_K/K as the torsor corresponding to the following class $x = x(K'/K, r)$ in the p-part of $H^1(K, E_K)$. On the factor $\mathbb{Q}_p/\mathbb{Z}_p$, choose the class of $1/p^r$. On the other factor $\mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}_p/\mathbb{Z}_p)$, pick the element corresponding

[4] It is known that $\delta \mid (\delta')^2$ ([30], Theorem 8). Examples where $\delta = (\delta')^2$ are given in [28], end of Sect. 4.

to the map

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(K'/K) \simeq \mathbb{Z}/p^n\mathbb{Z} \subset \mathbb{Q}_p/\mathbb{Z}_p.$$

It is clear that the order of x (that is, the period δ') is equal to p^n. To compute the index of X_K/K, we note that any extension F/K such that x is mapped to 0 in $H^1(F, E_F)$ under the natural map $H^1(K, E_K) \to H^1(F, E_F)$ is such that $F \supseteq K'$ and F/K has ramification index divisible by p^r. Thus, $p^{n+r} \mid \delta$. Since it is always possible to find a totally ramified extension of K' of degree p^r, we find that $\delta = p^{n+r}$.

Remark 9.5. Let X_K be a torsor under an elliptic curve E_K over K. If we naively extend the definition of Artin conductor of X when k is imperfect by setting

$$\mathrm{Art}^{imp}(X) := \chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) + \delta_X^{imp}$$

where δ_X^{imp} is some 'Swan conductor' associated to the Galois module $H^1_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$, then $\mathrm{Art}^{imp}(X) = t_E + 2u_E + \delta_E^{imp} + n_X - 1$ (see 6.4). Proposition 9.2 shows that in general, it is possible that either $\mathrm{disc}\,(X) \neq \mathrm{Art}^{imp}(X)$ or $\mathrm{disc}\,(E) \neq \mathrm{Art}^{imp}(E)$, contrary to the case where k is perfect.

Remark 9.6. Let X_K/K and Y_K/K be two smooth projective geometrically connected curves of genus $g \geq 1$, with minimal regular models X/S and Y/S, respectively. Assume that $\mathrm{Jac}\,(X_K)/K$ and $\mathrm{Jac}\,(Y_K)/K$ are isomorphic as abelian varieties over K (but possibly not as polarized abelian varieties). It is natural to wonder what are the possible relationships between X_k/k and Y_k/k.

Under our assumption, the Néron models J_X/S and J_Y/S of $\mathrm{Jac}\,(X_K)/K$ and $\mathrm{Jac}\,(Y_K)/K$ are isomorphic. Assuming that k is algebraically closed, we can use [7], 9.6.1, and obtain, for instance, that the intersection matrices of X_k and Y_k can be used to compute the group of components $\Phi_{\mathrm{Jac}(X_K)}$ $(= \Phi_{\mathrm{Jac}(Y_K)})$, thus providing a non-trivial relationship between the intersection matrices. One may wonder what further relationships exist between these matrices. Let us note here that, in general, these matrices do not have the same size, contrary to the case of curves of genus 1. Indeed, the two curves of genus 2 over $\mathbb{Q}$:

$$-y^2 = (x^2+4)(x^4+3x^2+1) \quad \text{and} \quad 3y^2 = (x^2-4)(x^4+7x^2+1)$$

have $\mathbb{Q}$-isomorphic Jacobians. This example is due to E. Howe ([20], Example 11). Using [32], one finds that at $p = 3$, one curve has good reduction, while the other degenerates into the union of two elliptic curves.

Appendix A. The reduction types of curves of genus 1

We exhibit below all possible types of reduction of curves of genus 1 that have more than one component. We show that this list of reduction types is complete using the classification of affine Cartan matrices. Let us start by recalling some intersection theory. Let $\mathcal{O}_K$ be strictly henselian with $\mathrm{char}(k) = p > 0$. Let X_K/K be a projective, smooth, geometrically connected curve of genus g. Let X/S be a regular model of X_K. Write the special fiber X_k as $\sum_{i=1}^n r_i\Gamma_i$. Fix a component Γ of X_k. We let $h^i(\Gamma) = \dim_k H^i(\Gamma, \mathcal{O}_\Gamma)$. We also let $e(\Gamma)$ denote the geometric multiplicity of Γ ([7], 9.1.3), and $r(\Gamma)$ denote its multiplicity in X_k. It is shown in [7], 9.1.8, that $e(\Gamma)$ divides $\Gamma \cdot \Delta$ for any divisor Δ on X_k. It is clear that $h^0(\Gamma)$ divides $e(\Gamma)$ (because k is separably closed) and $h^1(\Gamma)$. The integers $h^0(\Gamma)$ and $h^1(\Gamma)$ are related by the adjunction formula

$$\Gamma \cdot \Gamma + \Gamma \cdot K_{X/S} = 2h^1(\Gamma) - 2h^0(\Gamma),$$

where $K_{X/S}$ denotes the relative canonical divisor on X. The same formula applied to X_k instead of Γ reads:

$$2g - 2 = \sum_{i=1}^{n} r(\Gamma_i)\big(- \Gamma_i \cdot \Gamma_i - 2h^0(\Gamma_i) + 2h^1(\Gamma_i)\big). \tag{32}$$

Recall that any component Γ such that $h^0(\Gamma) = -\Gamma\cdot\Gamma$ and $h^1(\Gamma) = 0$ can be blown down to a regular point (see, e.g., [31], 9.3.1). Thus, a regular model is *minimal* if X_k has no such components. When the model is minimal, we find ([31], 9.3.10) that for all $i = 1, \dots, n$,

$$-\Gamma_i \cdot \Gamma_i - 2h^0(\Gamma_i) + 2h^1(\Gamma_i) \geq 0. \tag{33}$$

To be able to study the combinatorics of the special fiber X_k, we introduce the following terminology. We call a *p-type* $T = (G, M, R, E, H^0, H^1)$ the following data. We let $M = ((\Gamma_i \cdot \Gamma_j))$ be a symmetric $(n \times n)$-matrix with integer coefficients. When $n > 1$, we assume that M has negative diagonal entries, and non-negative entries otherwise. When $n = 1$, we set $M = (0)$. The matrix M will be called the *intersection matrix*. We let R denote a vector of positive integers, with $R := {}^t(r_1, \dots, r_n)$, such that $MR = 0$. The vector R is called the *vector of multiplicities*. We let $E := {}^t(e(\Gamma_1), \dots, e(\Gamma_n))$, where $e(\Gamma_i) = p^{f_i}$ for some non-negative integer f_i. We let $H^0 :={}^t (h^0(\Gamma_1), \dots, h^0(\Gamma_n))$, where $h^0(\Gamma_i) = p^{s_i}$ for some non-negative integer s_i. We let $H^1 := {}^t(h^1(\Gamma_1), \dots, h^1(\Gamma_n))$ where $h^1(\Gamma_i)$ is a non-negative integer. We assume that for each i, $e(\Gamma_i)$ divides $(\Gamma_i \cdot \Gamma_j)$ for all j, and divides $2h^1(\Gamma_i) - 2h^0(\Gamma_i)$. We also assume that $h^0(\Gamma_i)$ divides $e(\Gamma_i)$ for all i. We let G denote the graph on n vertices such that the i-th vertex is linked to the j-th vertex by exactly $(\Gamma_i \cdot \Gamma_j)$ edges. Since the graph G is completely determined by M, we may sometimes drop it from the notation, and call *p-type* simply the data $T = (M, R, E, H^0, H^1)$. The genus of the p-type is the number g defined by the right-hand side of the adjunction formula (32). We will say that a type is minimal if the inequality (33) holds for each $i = 1, \dots, n$.

Let us consider the case where $g = 1$. Let $T = (M, R, E, H^0, H^1)$ be a p-type of genus 1 with $n > 1$. For any pair of integers $e = p^s$ and m, we obtain a new p-type $(e, m)T := (eM, mR, eE, eH^0, eH^1)$. The reader will easily check using (32) that $(e, m)T$ has genus 1. When the residue field k is algebraically closed, all geometric multiplicities are equal to 1, and we denote the type $(1, m)T$ simply by mT. We shall call $(e, m)T$ a *multiple* of T.

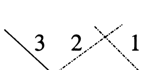
A.1. Note that when $n > 1$, $H^1 = (0, \dots, 0)$. In terms of curves of genus 1: *Let X/S be minimal. Let Γ be any irreducible component of X_k. If $h^1(\Gamma) > 0$, then X_k is irreducible.* Indeed, since $g = 1$, Formula (32) gives $\Gamma \cdot \Gamma = 2h^1(\Gamma) - 2h^0(\Gamma)$. Since $h^0(\Gamma)$ divides $h^1(\Gamma)$, we find that $\Gamma \cdot \Gamma \leq 0$ implies $h^1(\Gamma) = h^0(\Gamma)$ and $\Gamma \cdot \Gamma = 0$. Since the intersection matrix of X_k is semi-definite negative, we find that X_k is a multiple of Γ.

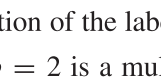
The meaning of the diagrams below is as follows. A plain segment adorned only with an integer r represents a smooth projective line defined over k having multiplicity r. A dotted segment adorned with an integer r represents a smooth projective line defined over purely inseparable extension of k degree p, and having multiplicity r. A plain segment adorned with the symbols $h^0(C) = 1$ or $h^0(C) = 4$ represents a component of multiplicity 1 that is not geometrically reduced, but has $h^0(C) = 1$ or $h^0(C) = 4$; in case $h^0(C) = 1$, we set $e(C) = 2$, and when $h^0(C) = 4$, we set $e(C) = 4$. Each irreducible component appearing in a type given below is such that $h^1(C) = 0$. A dotted segment intersects any other segment with intersection multiplicity p, except in the case of the last such component on the right of the diagram $BC_\ell^{(2)}$, where the intersection multiplicity is 4.

We have adopted a notation compatible with the notation used for affine Cartan matrices, where the size of a matrix is denoted by $\ell+1$. Set $\ell := n-1$. Then, in the notation of each type below, the lower index is always equal to $n - 1$.

Case $p \geq 5$. Any p-type with $p \geq 5$ is a multiple of a classical Kodaira type. The list of the classical Kodaira types can be found, for instance, in [55], p. 365.

Case $p = 3$. Any p-type with $p = 3$ is a multiple of a classical Kodaira type, or a multiple of one of the following two types:

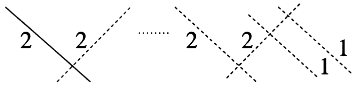
Type $G_2^{(2)}$: 

Type $G_2^{(1)}$: (torsor) 

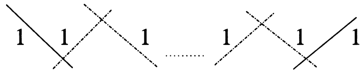
(See Remark A.3 for the explanation of the label "torsor").

Case $p = 2$. Any p-type with $p = 2$ is a multiple of a classical Kodaira type, or a multiple of one of the following 9 families of 2-types. In the diagrams below, $\ell \geq 1$ unless otherwise indicated.

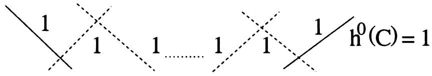
Type $B_\ell^{(1)}$ ($\ell \geq 3$): (torsor)



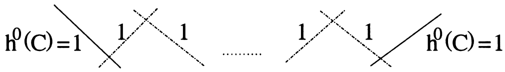
Type $B_\ell^{(21)}$:



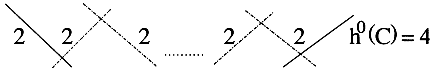
Type $B_\ell^{(22)}$:



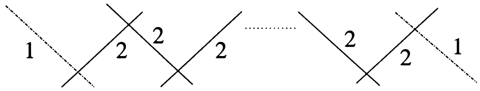
Type $B_\ell^{(23)}$: (torsor)



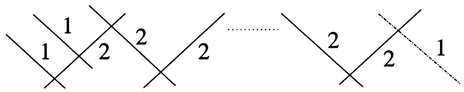
Type $BC_\ell^{(2)}$: (torsor)



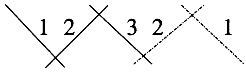
Type $C_\ell^{(1)}$: (torsor)



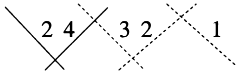
Type $C_\ell^{(2)}$ ($\ell \geq 3$):



Type $F_4^{(2)}$:



Type $F_4^{(1)}$: (torsor)



Let us explain why the above list of p-types $T = (M, R, E, H^0, H^1)$ is complete. Recall that since we list only the types which have more than one component ($n > 1$), then $H^1 = (0, \dots, 0)$ (A.1), and the vector H^0 is completely determined by the matrix M. Indeed, the adjunction formula gives $\Gamma \cdot \Gamma = -2h^0(\Gamma)$. Let H denote the diagonal matrix $\mathrm{diag}(h^0(\Gamma_1), \dots, h^0(\Gamma_n))$. Since $h^0(\Gamma)$ divides $(\Gamma \cdot \Gamma')$ for all Γ', we find that the matrix $A := -MH^{-1}$ has integer coefficients, and each coefficient on the diagonal of A is equal to 2. Moreover, we have $({}^tR)A = 0$. The reader will easily check that A is an affine Cartan matrix in the sense of [41], p. 258. Such matrices have been classified, for instance in Proposition 2 of [41], p. 265. Starting with an affine Cartan matrix A with associated vector R such that $({}^tR)A = 0$, we find from Proposition 2 that this matrix is symmetrizable, that is, that there exists a diagonal integer matrix H such that AH is symmetric. The reader will check that $-AH$ is the intersection matrix associated with a p-type of genus 1, taking as vector H^0 the transpose of the vector $(1, \dots, 1)H$. In all cases but for the affine Cartan matrix $B_\ell^{(2)}$, we take the vector E to be equal to H^0. In the case of $B_\ell^{(2)}$, we obtain three different 2-types, $B_\ell^{(21)}$, $B_\ell^{(22)}$, and $B_\ell^{(23)}$, each having the same intersection matrix and vector $H^0 = (1, \dots, 1)$, but vectors $E^{(21)} = (1, \dots, 1)$, $E^{(22)} = (1, \dots, 1, 2)$ and $E^{(23)} = (2, 1, \dots, 1, 2)$.

A.2. Let us note the following fact: *If* $t_X = 1$*, then* X_k *is either irreducible, or its type is a multiple of* I_n *for some* $n \geq 2$. This statement follows immediately from the classification of affine Cartan matrices.

Remark A.3. All possible types of reduction not labeled with the word 'torsor' appear as the reduction type of an elliptic curve. Specific equations can be found in [57]. All types labeled 'torsor' probably appear as reduction types of curves of genus 1, although we are not able to prove it. An example of reduction of type $BC_1^{(2)}$ is given in [34], 1.5. We prove the existence of reductions of the form $(p^s, m)I_n$ with $n \geq 2$ in 8.8.

An example of reduction $G_2^{(1)}$ is given as follows. Let K be of mixed characteristic 3, with imperfect residue field k and uniformizing element π. Fix an element $c \in \mathcal{O}_K$ whose image in k is not a cube, and consider the cubic curve X/S defined by the equation $x^3 + \pi y^3 + c\pi z^3 = 0$. In the affine chart $x^3 + \pi y^3 + c\pi = 0$, this curve has a singular point $(\pi, x, y^3 + c)$. The special fiber of this chart is an affine line over k of multiplicity 3. The reader will check that the special fiber of the blow-up of this singular point consists of two rational curves defined over $k[y]/(y^3 + c)$, and that this blow-up is regular.

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