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On the Brauer group of a surface

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Our goal in this note is to complete the proof of the following theorem.

Theorem 1. *Let k be a finite field, of characteristic p. Let X/k be a smooth proper geometrically connected surface. Assume that for some prime ℓ, the ℓ-part of the group* $\mathrm{Br}(X)$ *is finite. Then* $|\mathrm{Br}(X)|$ *is a square.*

Artin and Tate [15], 5.1, have shown in 1966 the existence of a canonical skew-symmetric pairing on the non-p part of $\mathrm{Br}(X)$, whose kernel is exactly the set of divisible elements. It follows from this fact that if the non-p part of $\mathrm{Br}(X)$ is finite, then its order is a square or twice a square. A few years later, Manin published examples of rational surfaces (that is, surfaces birational over $\bar{k}$ to the projective plane) with Brauer groups equal to $\mathbb{Z}/2\mathbb{Z}$. It is only in 1996 that the examples of Manin were revisited by Urabe, who found a mistake in them. For rational surfaces, the Brauer group is relatively easy to understand, and Urabe [16], remark after 1.17, showed, improving on a result of Milne [9], that the Brauer group of a rational surface has order a square. In [17], 0.3, Urabe then proves in full generality that when $p \neq 2$, the 2-part of $\mathrm{Br}(X)$ modulo its divisible subgroup has order a square. Thus, to complete the proof of Theorem 1, it remains to treat the case where $p = 2$ and X/k is not rational.

As we remarked above, it was wrongly assumed for about 30 years that $|\mathrm{Br}(X)|$ was not always a square. Our method of proof provides for any p that the 2-part of $\mathrm{Br}(X)$ has order a square via the knowledge of the 2-part

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of a Shafarevich-Tate group Ш(A). It is amusing to remark that the 2-part of the order of the Shafarevich-Tate group of a Jacobian was wrongly assumed for some 30 years to be always a square, until the subject was revisited by Poonen and Stoll [14] in 1999.

Let V/k be a proper smooth geometrically connected curve over a finite field. Let $K := k(V)$ denote the function field of V. Let X/k be a smooth proper and geometrically connected surface endowed with a proper flat map $f: X \to V$ such that the generic fiber X_K/K is a proper smooth geometrically connected curve of genus g. Let A_K/K denote the Jacobian of X_K/K. Artin and Tate conjectured (Conj. d) in [15]) that the full Birch and Swinnerton-Dyer conjecture for A_K/K ([15], Conj. B) is equivalent to the Artin-Tate conjecture for the surface X/k ([15], Conj. C). As Leslie Saper pointed out, the recent result of Kato-Trihan implies that Conjecture d) holds. Indeed, Artin and Tate [15], 5.1, and Milne [10], 4.1 and 6.1, proved[1] that if, for some prime ℓ, the ℓ-part of the Brauer group $\mathrm{Br}(X)$ is finite, then Conjecture C) of Artin-Tate holds for X/k. Kato and Trihan established in 2003 in [6], main theorem, that if the ℓ-part of the Shafarevich-Tate group Ш(A) is finite for some prime ℓ, then the Birch and Swinnerton-Dyer conjecture holds for A_K/K. As Ш(A) is finite if and only if $\mathrm{Br}(X)$ is finite ([2], 4.7), we find:

Theorem 2. *Conjecture d) of Artin-Tate is true.*

Let K_v denote the completion of K at a place $v \in V$. Let δ_v and δ'_v denote respectively the index and the period of X_{K_v}. Let δ denote the index of X_K/K.

Corollary 3. *Let $f: X \to V$ be as above. Assume that for some prime ℓ, the ℓ-part of the group* $\mathrm{Br}(X)$ *or of the group* Ш(A) *is finite. Then* $|Ш(A)| \prod_v \delta_v \delta'_v = \delta^2 |\mathrm{Br}(X)|$*, and* $|\mathrm{Br}(X)|$ *is a square.*

Proof. Apply Theorem 2, with 4.3 and 4.5 in [8]. □

The formula $|Ш(A)| \prod_v \delta_v \delta'_v = \delta^2 |\mathrm{Br}(X)|$ is known to hold independently of the Kato-Trihan result [6] only when the periods δ'_v are pairwise coprime ([8], 4.7).

Proof of Theorem 1. Since a smooth proper surface over a field is projective (see, e.g., [7], 9.3.5), we may consider an embedding (over k) of X into a projective space $\mathbb{P}^n_k$. Gabber ([1], 1.6) proved that some hypersurface in $\mathbb{P}^n_k$ intersects X in a smooth section. Replacing the embedding by a d-uple embedding if necessary, we can assume, using [13], 1.1, and Remark 3), that a geometrically integral hyperplane section of X is smooth. Consider a second section, and the associated rational map $f: X \dashrightarrow \mathbb{P}^1$ over k. Let

[1] In [10], 4.1, Milne assumes that $p \neq 2$. He notes on his web page [12] that this hypothesis can be removed if one replaces his reference to a preprint of Bloch in his paper [11] (used in 2.1 of [10]) by the reference [5].

$X' \to X$ denote a finite sequence of blowups such that the map f extends to a morphism $f' : X' \to \mathbb{P}^1$ over k.

Let us check that $f' \colon X' \to \mathbb{P}^1$ satisfies all the hypotheses of Corollary 3. It is trivial to note that the morphism $f' : X' \to \mathbb{P}^1$ is flat and proper. We claim that the generic fiber of f' is smooth. This can be checked after extension to the algebraic closure $\bar{k}$ of k. By construction, one hyperplane section of the pencil $f \colon X \dashrightarrow \mathbb{P}^1$ over $\bar{k}$ is smooth. The classical Bertini theorem applied to the surface $X_{\bar{k}}$ in some $\mathbb{P}^m_{\bar{k}}$ shows that the set of hyperplanes H such that the hyperplane section $H \cap X_{\bar{k}}$ is smooth is an open set in the projective space of all hyperplanes ([4], II, 8.18). Since the map $X' \to X$ is a finite sequence of blowups, we find that all but finitely many fibers of the morphism $f' \colon X' \to \mathbb{P}^1$ over $\bar{k}$ are isomorphic to smooth hyperplane sections $H \cap X_{\bar{k}}$. Since the smooth locus of a morphism is open ([3], 12.2.4, (iii)), we find that the generic fiber of f' is smooth.

Since X/k is a smooth and geometrically connected surface, we can use the proof of III.7.9 in [4] to find that any hyperplane section is geometrically connected. Thus, the smooth closed fibers of the morphism $f' \colon X' \to \mathbb{P}^1$ are all geometrically connected. Since the locus of the points $y \in \mathbb{P}^1$ such that the fiber over y is geometrically connected and geometrically reduced is open ([3], 12.2.4, (vi)), we find that the generic fiber of f' is geometrically connected.

Since the Brauer group is a birational invariant ([2], 7.2), $\mathrm{Br}(X)$ and $\mathrm{Br}(X')$ are isomorphic. We apply Corollary 3 to obtain that the order of $\mathrm{Br}(X')$ is a square. □

The question of whether $\mathrm{Br}(X)$, when finite, carries a non-degenerate alternating bilinear form with values in $\mathbb{Q}/\mathbb{Z}$ is addressed in [17], 0.4, and [14], Sect. 11. The existence of such a form would imply that $|\mathrm{Br}(X)|$ is a square.

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References

1. Gabber, O.: On space filling curves and Albanese varieties. Geom. Funct. Anal. **11**, 1192–1200 (2001)
2. Grothendieck, A.: Le groupe de Brauer III, in: Dix exposés sur la cohomologie des schémas. North Holland 1968
3. Grothendieck, A., Dieudonné, J.: Eléments de géométrie algébrique IV. Publ. Math., Inst. Hautes Étud. Sci. **28**, 5–251 (1966)
4. Hartshorne, R.: Algebraic Geometry. Springer 1977
5. Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. Éc. Norm. Supér. **12**, 501–661 (1979)
6. Kato, K., Trihan, F.: On the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$. Invent. Math. **153**, 537–592 (2003)
7. Liu, Q.: Algebraic geometry and arithmetic curves. Oxford Graduate Texts in Mathematics **6**. Oxford University Press 2002

8. Liu, Q., Lorenzini, D., Raynaud, M.: Néron models, Lie algebras, and reduction of curves of genus one. Invent. Math. **157**, 455–518 (2004)
9. Milne, J.: The Brauer group of a rational surface. Invent. Math. **11**, 304–307 (1970)
10. Milne, J.: On a conjecture of Artin and Tate. Ann. Math. **102**, 517–533 (1975)
11. Milne, J.: Duality in the flat cohomology of a surface. Ann. Sci. Éc. Norm. Supér. **9**, 171–201 (1976)
12. Milne, J.: http://www.jmilne.org/math/index.html, link: Addenda/Errata
13. Poonen, B.: Bertini theorems over finite fields. To appear in Ann. Math.
14. Poonen, B., Stoll, M.: The Cassels-Tate pairing on polarized abelian varieties. Ann. Math. **150**, 1109–1149 (1999)
15. Tate, J.: On the conjectures of Birch and Swinnerton-Dyer and a geometric analogue. Séminaire Bourbaki 1965/66, Exposé **306**. New York: Benjamin
16. Urabe, T.: Calculation of Manin's invariant for Del Pezzo surfaces. Math. Comp. **65**, 247–258 (1996)
17. Urabe, T.: The bilinear form of the Brauer group of a surface. Invent. Math. **125**, 557–585 (1996)