LIMITING ABSORPTION PRINCIPLE FOR SOME LONG RANGE PERTURBATIONS OF DIRAC SYSTEMS AT THRESHOLD ENERGIES

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Abstract. We establish a limiting absorption principle for some long range perturbations of the Dirac systems at threshold energies. We cover multi-center interactions with small coupling constants. The analysis is reduced to studying a family of non-self-adjoint operators. The technique is based on a positive commutator theory for non self-adjoint operators, which we develop in appendix. We also discuss some applications to the dispersive Helmholtz model in the quantum regime.

Contents

1. Introduction

We study properties of relativistic massive charged particles with spin-1/2 (e.g., electron, positron, (anti-)muon, (anti-)tauon,...). We follow the Dirac formalism, see [17]. Because of the spin, the configuration space of the particle is vector valued. To simplify, we consider finite dimensional and trivial fiber. Let $\nu \geq 2$ be an integer. The movement of the free particle is given by the Dirac equation,

$$\mathrm{i}\hbar\frac{\partial\varphi}{\partial t} = D_m\varphi, \text{ in } L^2(\mathbb{R}^3;\mathbb{C}^{2\nu}),$$

where $m > 0$ is the mass, c the speed of light, $\hbar$ the reduced Planck constant, and

$$D_m := c\hbar\,\alpha\cdot P + mc^2\beta = -\mathrm{i}c\hbar\sum_{k=1}^{3}\alpha_k\partial_k + mc^2\beta. \tag{1.1}$$

Here we set $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\beta := \alpha_4$. The α_i, for $i \in \{1, 2, 3, 4\}$, are linearly independent self-adjoint linear maps, acting in $\mathbb{C}^{2\nu}$, satisfying the anti-commutation relations:

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{i,j}\mathbf{1}_{\mathbb{C}^{2\nu}}, \text{ where } i, j \in \{1, 2, 3, 4\}. \tag{1.2}$$

For instance, when $\nu = 2$, one may choose the Pauli-Dirac representation:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathrm{Id}_{\mathbb{C}^\nu} & 0 \\ 0 & -\mathrm{Id}_{\mathbb{C}^\nu} \end{pmatrix} \tag{1.3}$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

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for $i = 1, 2, 3$. We refer to [66][Appendix 1.A] for various equivalent representations. In this paper we do not choose any specific basis and work intrinsically with (1.2). We refer to [53] for a discussion of the representations of the Clifford algebra generated by (1.2). We also renormalize and consider $\hbar = c = 1$. The operator D_m is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^{2\nu})$ and the domain of its closure is $\mathscr{H}^1(\mathbb{R}^3; \mathbb{C}^{2\nu})$, the Sobolev space of order 1 with values in $\mathbb{C}^{2\nu}$. We denote the closure with the same symbol. Easily, using Fourier transformation and some symmetries, one deduces the spectrum of D_m is purely absolutely continuous and given by $(-\infty, -m] \cup [m, \infty)$.

In this introduction, we focus on the dynamical and spectral properties of the Hamiltonian describing the movement of the particle interacting with n fixed, charged particles. We model them by fixed points $\{a_i\}_{i=1,\dots,n} \in \mathbb{R}^{3n}$ with respective charges $\{z_i\}_{i=1,\dots,n} \in \mathbb{R}^n$. Doing so, we tacitly suppose that the particles $\{a_i\}$ are far enough from one another, so as to neglect their interactions. Note we make no hypothesis on the sign of the charges. The new Hamiltonian is given by

$$H_\gamma := D_m + \gamma V_c(Q), \text{ where } V_c := v_c \otimes \mathrm{Id}_{\mathbb{C}^{2\nu}} \text{ and } v_c(x) := \sum_{k=1,\dots,n} \frac{z_i}{|x - a_i|}, \tag{1.4}$$

acting on $\mathcal{C}_c^\infty(\mathbb{R}^3 \setminus \{a_i\}_{i=1,\dots,n}; \mathbb{C}^{2\nu})$, with $a_i \neq a_j$ for $i \neq j$. The $\gamma \in \mathbb{R}$ is the coupling constant. The index c stands for *coulombic multi-center*. The notation $V(Q)$ indicates the operator of multiplication by V. Here, we identify $L^2(\mathbb{R}^3; \mathbb{C}^{2\nu}) \simeq L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2\nu}$, canonically. Remark the perturbation V_c is not relatively compact with respect to D_m, then one needs to be careful to define a self-adjoint extension for D_m. Assuming

$$Z := |\gamma| \max_{i=1,\dots,n} (|z_i|) < \sqrt{3}/2, \tag{1.5}$$

the theorem of Levitan-Otelbaev ensures that H_γ is essentially self-adjoint and its domain is the Sobolev space $\mathscr{H}^1(\mathbb{R}^3; \mathbb{C}^{2\nu})$, see [2, 45, 49, 51, 52, 50] for various generalizations. This condition corresponds to the nuclear charge $\alpha_{\mathrm{at}}^{-1} Z \leq 118$, where $\alpha_{\mathrm{at}}^{-1} = 137.035999710(96)$. Note that using the Hardy-inequality, the Kato-Rellich theorem will apply till $Z < 1/2$ and is optimal in the matrix-valued case, see [66][Section 4.3] for instance. For $Z < 1$, one shows there exists only one self-adjoint extension so that its domain is included in $\mathscr{H}^{1/2}(\mathbb{R}^3; \mathbb{C}^{2\nu})$, see [58]. This covers the nuclear charges up to $Z = 137$. When $n = 1$ and $Z = 1$, this property still holds true, see [23]. Surprisingly enough, when $n = 1$ and $Z > 1$, there is no self-adjoint extension with domain included in $\mathscr{H}^{1/2}(\mathbb{R}^3; \mathbb{C}^{2\nu})$, see [74][Theorem 6.3]. We mention also the work of [68] for $Z > 1$.

In [58], one shows for $Z < 1$ that the essential spectrum is given by $(-\infty, -m] \cap [m, \infty)$ for all self-adjoint extension. For all Z, one refers to [30][Proposition 4.8.], which relies on [74]. In [29] one gives some criteria of stability of the essential spectrum for some very singular cases. In [4], one proves there is no embedded eigenvalues for a more general model and till the coupling constant $Z < 1$. For all energies being in a compact set included in $(-\infty, -m) \cap (m, \infty)$, [30] obtains some estimates of the resolvent. This implies some propagation estimates and that the spectrum of H_γ is purely absolutely continuous. Similar results have been obtained for magnetic potential of constant direction, see [70] and more recently [64].

In this paper we are interested in uniform estimates of the resolvent at threshold energies. The energy m is called the *electronic threshold* and $-m$ the *positronic threshold*. In Theorem (1.2), we obtain a uniform estimation of the resolvent over $[-m-\delta, -m] \cup [m, m+\delta]$, see (1.8) and deduce some propagation properties, see (1.9). One difficulty is that in the case $n = 1$ and $z_i < 0$, it is well known there are infinitely many eigenvalues in the gap $(-m, m)$ converging to the m as soon as $\gamma \neq 0$ (see for instance [66][Section 7.4] and references therein). This is a difficult problem and, to our knowledge, this result is new for the multi-center case. There is a larger literature for non-relativist models, e.g., $-\Delta + V$ in $L^2(\mathbb{R}^n; \mathbb{C})$. The question is intimately linked with the presence of resonances at threshold energy, [43, 25, 57, 63, 69]. We mention also [14] for applications to Strichartz estimates and [19, 20] for applications to scattering theory. We refer to [8, 9] for perturbations in divergence form and to [36, 37, 67] for some more geometrical setting. We also point out some low energy results in the context of non-relativistic quantum electrodynamics, [26, 27].

Before giving the main result, we shall discuss some commutator methods. The first stone was set by C.R. Putnam, see [61] and for instance [62][Theorem XIII.28]. Let H be a self-adjoint operator acting in a Hilbert space $\mathscr{H}$. One supposes there is a *bounded* operator A so that

$$C := [H, \mathrm{i}A]_\circ > 0, \tag{1.6}$$

where "$>$" means non-negative and injective. The commutator has to be understood in the form sense. When it extends into a bounded operator between some spaces, we denote this extension with the symbol $\circ$

in subscript, see Appendix A. The operator A is said to be *conjugate* to H. One deduces some estimation on the imaginary part of the resolvent, i.e., one finds some *weight* B, a closed injective operator with dense domain, so that

$$\sup_{\Re(z)\in\mathbb{R},\Im(z)>0} \Im\langle f,(H-z)^{-1}f\rangle \leq \|Bf\|^2.$$

This estimation is equivalent to the global propagation estimate, c.f. [46] and [62][Theorem XIII.25]:

$$\int_{\mathbb{R}} \|B^{-1}e^{\mathrm{i}tH}f\|^2 dt \leq 2\|f\|^2$$

One infers that the spectrum of H is purely absolutely continuous with respect to the Lebesgue measure. In particular, H has no eigenvalue. To deal with the presence of eigenvalues, the fact that A is unbounded and with the 3-body-problem, E. Mourre has the idea to localized in energy the estimates and to allow a compact perturbation, see [56]. With further hypotheses, one shows an estimate of the resolvent (and not only on the imaginary part). The applications of this theory are numerous. The theory was immediately adapted to treat the N-body problem, see [60]. The theory was finally improved in many directions and optimized in many ways, see [1] for a more thorough discussion of these matters. We mention also [31, 34, 32] for recent developments. As we are concern about thresholds, Mourre's method does not seem enough, as the estimate of the resolvent is given on an interval which is strictly smaller than the one used in the commutator estimate. In [13] one generalizes the result of Putnam's approach. Under some conditions, one allows A to be unbounded. They obtain a global estimate of the resolvent. Note this implies the absence of eigenvalue. In [25], in the non-relativistic context, by asking some positivity on the Virial of the potential, see below, one is able to conciliate the estimation of the resolvent above the threshold energy and the accumulation of eigenvalues under it. In [63], one presents an abstract version of the method of [25]. To give an idea, we shall compare the theories on a non-optimal example. Take $H := -\Delta + V$ in $L^2(\mathbb{R}^3)$, with V being in the Schwartz space. Consider the generator of dilation $A := (P\cdot Q + Q\cdot P)/2$, where $P := -\mathrm{i}\nabla$. One looks at the quantity

$$[H,\mathrm{i}A]_\circ - cH = -(2-c)\Delta - W_V(Q), \text{ where } W_V(Q) := Q\cdot\nabla V(Q) + cV(Q),$$

with $c\in(0,2)$ and seeks some positivity. The expression W_V is called the *Virial* of V. In [25], one uses extensively that $W_V(x) \leq -c\langle x\rangle^{-\alpha}$ for some $\alpha, c > 0$ and $|x|$ big enough. In [63], one notices that it suffices to suppose that $W_V(x)\leq 0$ and to take advantage of the positivity of the Laplacian. We take the opportunity to mention that it is enough to suppose that $W_V(x)\leq c'|x|^{-2}$, for some small positive constant c', see Theorem C.1. Observe also that these methods give different weights. For instance, [25] obtains better weights in the scale of $\langle Q\rangle^\alpha$ and [63] can obtain singular weights like $|Q|$, see Appendix B. Finally, [25] deals only with low energy estimates and [63] works globally on $[0,\infty)$. We also point out [39] which relies on commutator techniques and deals with smooth homogeneous potentials.

In this article, we revisit the approach of [63] and make several improvements, see Appendix B. Our aim is twofold: to treat dispersive non self-adjoint operator and to obtain estimates of the resolvent uniformly in a parameter. At first sight, these improvements are pointless from the standpoint of the Coulomb-Dirac problem we treat. In reality, they are the key-stone of our approach.

As a direct by-product of the method, we obtain some new results for dispersive Schrödinger operators. The following V_2 term corresponds to the absorption coefficient of the laser energy by material medium absorption term in the Helmholtz model, see [42] for instance.

Theorem 1.1. *Let $n\geq 3$. Suppose that $V_1, V_2 \in L^\infty(\mathbb{R}^n;\mathbb{R})$ satisfy:*

(H1) *∇V_i, $Q\cdot\nabla V_i(Q)$, $\langle Q\rangle(Q\cdot\nabla V_i)^2(Q)$ are bounded, for $i\in\{1,2\}$.*

(H2) *There are $c_1\in[0,2)$ and $c_1'\in\big[0, 4(2-c_1)/(n-2)^2\big)$ such that*

$$W_1(x) := x\cdot(\nabla V_1)(x) + c_1V_1(x) \leq \frac{c_1'}{|x|^2}, \text{ for all } x\in\mathbb{R}^n.$$

and

$$V_2(x)\geq 0 \text{ and } -x\cdot(\nabla V_2)(x)\geq 0, \text{ for all } x\in\mathbb{R}^n.$$

On $\mathcal{C}_c^\infty(\mathbb{R}^n)$, we define $H := -\Delta + V(Q)$, where $V := V_1 + \mathrm{i}V_2$. The closure of H defines a dispersive closed operator with domain $\mathscr{H}^2(\mathbb{R}^n)$. We keep denoting it with H. Its spectrum included in the upper half-plane. The operator H has no eigenvalue in $[0,\infty)$. Moreover,

$$\sup_{\lambda\in[0,\infty),\,\mu>0} \big\| |Q|^{-1}(H-\lambda+\mathrm{i}\mu)^{-1}|Q|^{-1}\big\| < \infty. \tag{1.7}$$

Note we require nor smoothness on the potentials neither that they are relatively compact with respect to the Laplacian. We refer to Appendix C for further comments, the case $c_1 = 0$ and a stronger result.

We come back to the main application, namely the operator H_γ defined by (1.4). As the Dirac operator is vector-valued, coulombic interactions are singular and as we are interested in both thresholds, we were not able to use directly the ideas of [25, 63]. Indeed, it is unclear for us if one can actually deal with thresholds energy and keep the "positivity" of something close to the quantity $[H_\gamma, \mathrm{i}A] - cH_\gamma$, for some self-adjoint operator A. We avoid this fundamental problem. First of all we cut-off the singularities of the potential V_c and consider the operator $H_\gamma^{\mathrm{bd}} = D_m + \gamma V$ in Section 2. We recover the singularities of the operator by perturbation in Proposition 4.1. In Section (2.1), similarly to [21], we explicit the resolvent of $H_\gamma^{\mathrm{bd}} - z$ relatively to a spin-down/up decomposition. This transfers the analysis to the one of an elliptic operator of second order, $\Delta_{m,v,z}$, see Section 2.1. The drawback is that this operator is dispersive and also depends on the spectral parameter z. We bypass the latter difficulty by studying the family $\{\Delta_{m,v,\xi}\}_{\xi\in\mathcal{E}}$ uniformly in $\mathcal{E}$. In Section 2.2, we explain how to deduce the estimation of the resolvent of H_γ^{bd} having the one of $\Delta_{m,v,z}$. In the Section 3, we establish some positive commutator estimates for $\Delta_{m,v,z}$ and derive the sought estimates of the resolvent, see Theorem (3.1). For the last step, we rely on the theory developed in Appendix B. The main result of this introduction is the following one.

Theorem 1.2. *There are $\kappa, \delta, C > 0$ such that*

$$\sup_{|\lambda|\in[m,m+\delta],\, \varepsilon>0, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(H_\gamma - \lambda - \mathrm{i}\varepsilon)^{-1}\langle Q\rangle^{-1}\| \leq C. \tag{1.8}$$

In particular, H_γ has no eigenvalue in $\pm m$. Moreover, there is C' so that

$$\sup_{|\gamma|\leq\kappa} \int_{\mathbb{R}} \|\langle Q\rangle^{-1} e^{-\mathrm{i}tH_\gamma} E_{\mathcal{I}}(H_\gamma) f\|^2 dt \leq C'\|f\|^2, \tag{1.9}$$

where $\mathcal{I} = [-m-\delta, -m] \cup [m, m+\delta]$ and where $E_{\mathcal{I}}(H_\gamma)$ denotes the spectral measure of H_γ.

A more general result is given in Theorem 4.1. In Theorem 4.2 we discuss the weights $\langle P\rangle^{1/2}|Q|$ and in Remark 4.2 the weights $|Q|$. If one is not interested in the uniformity in the coupling constant, using [30], one can consider all $\delta > 0$ and deduce (1.8). The propagation estimate (1.9) refers as Kato smoothness and it is a well-known consequence of (1.8), see [46]. Using some kernel estimates, one can obtain (1.8) directly for the free Dirac operator, i.e., $\gamma = 0$, see for instance [66][Section 1.E] and [48]. One may find an alternative proof of this fact in [41] which relies on some positive commutator techniques.

In this study, we are mainly interested by long range perturbations of Dirac operators. Concerning limiting absorption principle for short range perturbations of Dirac operators there are some interesting works such as [15] for small perturbations without discrete spectrum or [10] for potentials producing discrete spectrum. These authors were mainly interested by time decay estimates similar to (1.9). In the short range case, the limiting absorption principle is a key ingredient to establish Strichartz estimates for perturbed Dirac type equations see [11, 16]. For free Dirac equations there are some direct proofs, see [22, 55, 54]. Time decay estimates such as (1.9) or Strichartz are crucial tool to establish well posedness results [22, 55, 54] and stability results [10, 11] for nonlinear Dirac equations.

The paper is organized as follows. In the second section we reduced the analysis of the resolvent of the Dirac operator perturbed with a bounded potential to the one of family of non self-adjoint operators. In the third part, we analyze these operators and obtain some estimates of the resolvent. In the fourth part, we state the main results of the paper. For the convenience of the reader, we expose some commutator expansions in the Appendix A. In the Appendix B, we develop the abstract positive commutator theory. At last in Appendix C, we give a direct application to the theory in the context of the Helmholtz equation.

Notation: In the following $\Re$ and $\Im$ denote the real and imaginary part, respectively. The smooth function with compact support are denoted by $\mathcal{C}_c^\infty$. Given a complex-valued function F, we denote by $F(Q)$ the operator of multiplication by F. We mention also the notation $P = -\mathrm{i}\nabla$. We use the standard $\langle\cdot\rangle := (1 + |\cdot|^2)^{1/2}$.

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2. Reduction of the problem

In this section, we study the resolvent of the perturbed Dirac operator

$$H_\gamma^{\mathrm{bd}} := D_m + \gamma V, \text{ where } V := v \otimes \mathrm{Id}_{\mathbb{C}^{2\nu}} \text{ and } v \text{ \underline{bounded}}. \tag{2.1}$$

In Section 4, we explain how to cover some singularities. Due to the method, we will consider only small coupling constants. We will show the limiting absorption principle

$$\sup_{|\lambda|\in[m,m+\delta],\, \varepsilon>0,\, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(H_\gamma^{\mathrm{bd}} - \lambda - \mathrm{i}\varepsilon)^{-1}\langle Q\rangle^{-1}\| \leq C, \tag{2.2}$$

for some $\kappa > 0$. We notice this is equivalent to

$$\sup_{\lambda\in[m,m+\delta],\, \varepsilon>0,\, |\gamma|\leq\kappa} \|\langle Q\rangle^{-1}(H_\gamma^{\mathrm{bd}} - \lambda - \mathrm{i}\varepsilon)^{-1}\langle Q\rangle^{-1}\| \leq C, \tag{2.3}$$

Indeed, by setting $\alpha_5 := \alpha_1\alpha_2\alpha_3\alpha_4$ and using the anti-commutation relation (1.2), we infer

$$\alpha_5\,(D_m + \gamma V)\,\alpha_5^{-1} = -D_m + \gamma V.$$

Note that α_5 is unitary in the Sobolev spaces $\mathscr{H}^s(\mathbb{R}^3;\mathbb{C}^{2n})$, for $s \in \mathbb{R}$. This gives

$$\alpha_5\varphi(D_m + \gamma V)\alpha_5^{-1} = \varphi\big(-(D_m - \gamma V)\big), \text{ for all } \varphi \in \mathcal{C}(\mathbb{R};\mathbb{C}). \tag{2.4}$$

Finally notice that $\langle Q\rangle$ commutes to α_5.

2.1. **The non self-adjoint operator.** Here, we relate the resolvent of (2.1) in a point $z \in \mathbb{C}\setminus\mathbb{R}$ with the one of some non self-adjoint Laplacian type operator $\Delta_{m,v,z}$, chosen in (2.8). We fix a *compact* set $\mathcal{I}$ being the area of energy we are concentrating on. In the next section, we explain how to recover a limiting absorption principle for H_γ^{bd} over $\mathcal{I}$ given the one of $\Delta_{m,\gamma v,z}$.

We consider a potential $v \in L^\infty(\mathbb{R}^3;\mathbb{R})$, not necessarily smooth, satisfying

$$\|v\|_\infty \leq m/2 \text{ and } \nabla v \in L^\infty(\mathbb{R}^3;\mathbb{R}^3). \tag{2.5}$$

It particular, $(v(Q) - m - z)^{-1}$ stabilizes $\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^{2\nu})$ for all z in $\mathbb{C}\setminus\mathbb{R}$.

Since $\beta = \alpha_4$ satisfies (1.2), we deduct that β has the eigenvalues ±1 and the eigenspaces have the same dimension. Let P^+ be the orthogonal projection on the spin-up part of the space, i.e., on $\ker(\beta-1)$. Let $P^- := 1 - P^+$. Since α_j satisfies (1.2), for $j \in \{1,2,3\}$, we get $P^\pm\alpha_jP^\pm = 0$. We set:

$$\alpha_j^+ := P^+\alpha_jP^- \text{ and } \alpha_j^- := P^-\alpha_jP^+, \text{ for } j \in \{1,2,3\}.$$

They are partial isometries:

$$\big(\alpha_j^+\big)^* = \alpha_j^-, \quad \alpha_j^+\alpha_j^- = P^+ \text{ and } \alpha_j^-\alpha_j^+ = P^-, \text{ for } j \in \{1,2,3\}.$$

The relation of anti-commutation (1.2) gives:

$$\alpha_i^-\alpha_j^+ + \alpha_j^-\alpha_i^+ = 2\delta_{i,j}P^- \text{ and } \alpha_i^+\alpha_j^- + \alpha_j^+\alpha_i^- = 2\delta_{i,j}P^+, \text{ for } i,j \in \{1,2,3\}. \tag{2.6}$$

We set $\mathbb{C}^\nu_\pm := P^\pm\mathbb{C}^{2\nu}$. In the direct sum $\mathbb{C}^\nu_- \oplus \mathbb{C}^\nu_+$, with a slight abuse of notation, one can write

$$\beta = \begin{pmatrix} \mathrm{Id}_{\mathbb{C}^\nu} & 0 \\ 0 & -\mathrm{Id}_{\mathbb{C}^\nu} \end{pmatrix} \text{ and } \alpha_j = \begin{pmatrix} 0 & \alpha_j^+ \\ \alpha_j^- & 0 \end{pmatrix}, \text{ for } j \in \{1,2,3\}.$$

We now split the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3;\mathbb{C}^{2\nu})$ into the spin-up and down part:

$$\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-, \text{ where } \mathscr{H}^\pm := L^2(\mathbb{R}^3;\mathbb{C}^\nu_\pm) \simeq L^2(\mathbb{R}^3;\mathbb{C}^\nu). \tag{2.7}$$

We define the operator:

$$\Delta_{m,v,z} := \alpha^+\cdot P\frac{1}{m - v(Q) + z}\alpha^-\cdot P + v(Q) \tag{2.8}$$

on $\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^\nu_+)$. It is well defined by (2.5). It is closable as its adjoint has a dense domain. We consider the minimal extension, its closure. We denote its domain by $\mathcal{D}_{\min}(\Delta_{m,v,z})$ and keep the same symbol for the operator. It is well known that even for symmetric operators one needs to be careful with domains as the domain of the adjoint could be much bigger than the one of the closure. In the next Proposition, we care about this problem in our non-symmetric setting.

Proposition 2.1. *Let $z \in \mathbb{C}\setminus\mathbb{R}$ such that $\Re(z) \geq 0$. Under the hypotheses* (2.5)*, we have that*

$$\mathcal{D}_{\min}(\Delta_{m,v,z}) = \mathcal{D}(\Delta^*_{m,v,\overline{z}}) = \mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+) \text{ and } \Delta_{m,v,z} = \Delta^*_{m,v,\overline{z}}.$$

Proof. We mimic the Kato-Rellich approach and compare $\Delta_{m,v,z}$ with the more convenient operator $\tilde{\Delta}_z := \big(1/(m+z)\big)\Delta_{1,0,0}$. Its domain is $\mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$ and its spectrum is $\{(m+\overline{z})t \mid t\in[0,\infty)\}$. We now show there is $a\in[0,1)$ and $b\geq 0$ such that

$$\|Bf\|^2 \leq a\left\|\tilde{\Delta}_z f\right\|^2 + b\|f\|^2, \tag{2.9}$$

holds true for all $f\in\mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{C}^\nu_+)$, where

$$B := \frac{v}{(m-v+z)}\tilde{\Delta}_z - \mathrm{i}\frac{(\alpha^+\cdot\nabla v)(Q)}{(m-v(Q)+z)^2}\,\alpha^-\cdot P + v(Q).$$

Since $\|v\|_\infty \leq m/2$, $\Re(z)\geq 0$ and $\Im(z)>0$, we infer $a_0 := \|v/(m-v+z)\|_\infty < 1$. Set $M := \|(\alpha^+\cdot\nabla v)(\cdot)/(m-v(\cdot)+z)^2\|_\infty$. Take $\varepsilon,\varepsilon'\in(0,1)$

$$\begin{aligned}\|Bf\|^2 &\leq (1+\varepsilon)a_0^2\left\|\tilde{\Delta}_z f\right\|^2 + \left(1+\frac{1}{\varepsilon}\right)\left\|\frac{(\alpha^+\cdot\nabla v)(Q)}{(m-v(Q)+z)^2}\,\alpha^-\cdot Pf + v(Q)f\right\|^2,\\ &\leq (1+\varepsilon)a_0^2\left\|\tilde{\Delta}_z f\right\|^2 + \frac{4M^2}{\varepsilon}\left\|\alpha^-\cdot Pf\right\|^2 + \frac{4\|v\|_\infty}{\varepsilon}\|f\|^2,\\ &\leq \big((1+\varepsilon)a_0^2+\varepsilon'\big)\left\|\tilde{\Delta}_z f\right\|^2 + \left(\frac{4\|v\|_\infty}{\varepsilon} + \frac{2|m+z|^2M^2}{\varepsilon\varepsilon'}\right)\|f\|^2.\end{aligned}$$

By choosing ε and ε' so that the first constant is smaller than 1, (2.9) is fulfilled.

Now, observe that since $\Re z>0$, $\|B(\tilde{\Delta}_z+\mu)^{-1}\|^2 \leq a+b\mu^{-2}$ for $\mu>0$. Fix $\mu_0>0$ such that $\|B(\tilde{\Delta}_z+\mu_0)^{-1}\|<1$. Then $(1+B(\tilde{\Delta}_z+\mu_0)^{-1})$ is bijective. Noticing that

$$\big(\mathrm{Id}+B(\tilde{\Delta}_z+\mu_0)^{-1}\big)(\tilde{\Delta}_z+\mu_0) = \Delta_{m,v,z}+\mu_0,$$

we infer that $\Delta_{m,v,z}+\mu_0$ is bijective from $\mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$ onto $L^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$. In particular $\mathcal{D}_{\min}(\Delta_{m,v,z}) = \mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$. Directly, one has $\mathcal{D}_{\min}(\Delta_{m,v,z})\subset\mathcal{D}(\Delta^*_{m,v,\overline{z}})$ and $\Delta_{m,v,z}\subset\Delta^*_{m,v,\overline{z}}$ (inclusion of graphs). Take now $f\in\mathcal{D}(\Delta^*_{m,v,\overline{z}})$. Since $\Delta_{m,v,z}+\mu_0$ is surjective, there is $g\in\mathcal{D}_{\min}(\Delta_{m,v,z})$ so that

$$(\Delta_{m,v,z}+\mu_0)g = (\Delta^*_{m,v,\overline{z}}+\mu_0)f.$$

In particular, $(\Delta^*_{m,v,\overline{z}}+\mu_0)(f-g)=0$. As $\Delta_{m,v,z}+\mu_0$ is surjective, we derive that $\ker(\Delta^*_{m,v,\overline{z}}+\mu_0)=\{0\}$. In particular $f=g$, $\mathcal{D}_{\min}(\Delta_{m,v,z}) = \mathcal{D}(\Delta^*_{m,v,\overline{z}})$ and $\Delta_{m,v,z} = \Delta^*_{m,v,\overline{z}}$. □

As a corollary, we derive:

Lemma 2.1. *The spectrum of $\Delta_{m,v,z}$ is contained in the lower/upper half-plane which does not contain z. In particular, $c+z$ is always in the resolvent set of $\Delta_{m,v,z}$ for any $c\in\mathbb{R}$.*

Proof. Take now $f\in\mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$. Since

$$\Im\langle f, \Delta_{m,v,z}\, f\rangle = \langle \alpha^-\cdot Pf, \frac{-\Im(z)}{\big(m-v(Q)+\Re(z)\big)^2+\Im(z)^2}\,\alpha^-\cdot Pf\rangle, \tag{2.10}$$

is of the sign of $-\Im(z)$. Since $\Delta_{m,v,z}$ is a closed operator having the same domain of its adjoint, the spectrum of $\Delta_{m,v,z}$ is contained in the closure of its numerical range, see Lemma B.1. □

We give a kind of Schur's Lemma, so as to compute the inverse of the Dirac operator, see also [21, 43].

Lemma 2.2. *Suppose (2.5). Take $z\in\mathbb{C}\setminus\mathbb{R}$ such that $\Re(z)\geq 0$. In the spin-up/down decomposition $\mathscr{H}=\mathscr{H}^+\oplus\mathscr{H}^-$, we have* $(H_1^{\mathrm{bd}}-z)^{-1} =$

$$\begin{pmatrix} (\Delta_{m,v,z}+m-z)^{-1} & (\Delta_{m,v,z}+m-z)^{-1}\alpha^+\cdot P\dfrac{1}{m-v(Q)+z} \\ \dfrac{1}{m-v(Q)+z}\alpha^-\cdot P(\Delta_{m,v,z}+m-z)^{-1} & \dfrac{1}{m-v(Q)+z}\alpha^-\cdot P(\Delta_{m,v,z}+m-z)^{-1}\alpha^+\cdot P\dfrac{1}{m-v(Q)+z} - \dfrac{1}{m-v(Q)+z}\end{pmatrix}$$

Remark 2.1. *The operator $(H_1^{\rm bd}-z)^{-1}$ is bounded from $L^2(\mathbb{R}^3;\mathbb{C}^{2\nu})$ into $\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^{2\nu})$. However, this improvement in the Sobolev scale does not hold if one looks at the matricial terms separately. There is a real compensation coming from the off-diagonal terms. First note that $\alpha^-\cdot P(\Delta_{m,v,z}+m-z)^{-1}\alpha^+\cdot P$ is a bounded operator in $L^2(\mathbb{R}^3;\mathbb{C}^\nu_-)$ and a priori not into $\mathscr{H}^s(\mathbb{R}^3;\mathbb{C}^\nu_-)$, with $s>0$. Indeed, $\alpha^+\cdot P$ sends $L^2(\mathbb{R}^3;\mathbb{C}^\nu_-)$ into $\mathscr{H}^{-1}(\mathbb{R}^3;\mathbb{C}^\nu_+)$, then $(\Delta_{m,v,z}+m-z)^{-1}$ to $\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^\nu_+)$ and the left $\alpha^-\cdot P$ sends again into $L^2(\mathbb{R}^3;\mathbb{C}^\nu_-)$. On the other hand, the term $(\Delta_{m,v,z}+m-z)^{-1}$ is bounded from $L^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$ into $\mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$, which is much better than expected.*

Proof. Let $f\in L^2(\mathbb{R}^3;\mathbb{C}^{2\nu})$. By self-adjointness of $H_1^{\rm bd}$, there is a unique $\psi\in\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^{2\nu})$ such that $(H_1^{\rm bd}-z)\psi=f$. We separate the upper and lower spin components and denote $f=(f_+,f_-)$ and $\psi=(\psi_+,\psi_-)$ in $\mathscr{H}=\mathscr{H}^+\oplus\mathscr{H}^-$. We rewrite the equation $(D_m+V(Q)-z)\psi=f$ to get:

$$\begin{cases}\alpha^+\cdot P\psi_-+m\psi_++v(Q)\psi_+-z\psi_+=f_+,\\ \alpha^-\cdot P\psi_+-m\psi_-+v(Q)\psi_--z\psi_-=f_-.\end{cases}\tag{2.11}$$

From the second line, we get $(v(Q)-m-z)\psi_-=f_--\alpha^-\cdot P\psi_+$. Since z is not real, we can take the inverse and infer $\psi_-=(v(Q)-m-z)^{-1}(f_--\alpha^-\cdot P\psi_+)$. Since $\psi_-\in\mathscr{H}^1$, we can apply it $\alpha^+\cdot P$ and obtain a vector of $L^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$. Now, since f_- is in $L^2(\mathbb{R}^3;\mathbb{C}^\nu_-)$ and since $(v(Q)-m-z)^{-1}$ is bounded, we have $\alpha^+\cdot P(v(Q)-m-z)^{-1}f_-\in\mathscr{H}^{-1}(\mathbb{R}^3;\mathbb{C}^\nu_+)$ and since $(v(Q)-m-z)^{-1}\alpha^-\cdot P\psi_+$ is in $L^2(\mathbb{R}^3;\mathbb{C}^\nu_-)$, we rewrite the system:

$$\begin{cases}\left(\alpha^+\cdot P\dfrac{1}{m-v(Q)+z}\,\alpha^-\cdot P+v(Q)+m-z\right)\psi_+= & f_++\alpha^+\cdot P\dfrac{1}{m-v(Q)+z}f_-,\\ \qquad\qquad\qquad\qquad\psi_-= & \dfrac{1}{m-v(Q)+z}\left(\alpha^-\cdot P\psi_+-f_-.\right)\end{cases}$$

To conclude it remains to show that $\Delta_{m,v,z}+m-z$ is invertible in $\mathcal{B}(\mathscr{H}^1,\mathscr{H}^{-1})$, so as to invert it in the system. Using (2.10), we have $|\Im\langle u,(\Delta_{m,v,z}-z)u\rangle|\geq c\|u\|^2_{\mathscr{H}^1}$. Then $\|(\Delta_{m,v,z}+m-z)u\|_{\mathscr{H}^{-1}}\geq c\|u\|_{\mathscr{H}^1}$ and $\|(\Delta_{m,v,z}+m-z)^*u\|_{\mathscr{H}^{-1}}\geq c\|u\|_{\mathscr{H}^1}$ hold. Thus, $\Delta_{m,v,z}-z$ is bijective from $\mathscr{H}^1$ onto $\mathscr{H}^{-1}$. □

2.2. **From one limiting absorption principle to another.** The main motivation for the operator $\Delta_{m,v,z}$ is to deduce a limiting absorption principle for $H_\gamma^{\rm bd}$ starting with one for $\Delta_{m,\gamma v,z}$. Consider the upper right term in Lemma 2.2, the basic idea would be to put by force the weight $\langle Q\rangle^{-1}$ and to say that every terms are bounded. However, we have that

$$\underbrace{\langle Q\rangle^{-1}(\Delta_{m,v,z}+m-z)^{-1}\langle Q\rangle^{-1}}_{\text{bounded from LAP for }\Delta_{m,v,z}}\underbrace{\langle Q\rangle\,\alpha^+\cdot P\langle Q\rangle^{-1}}_{\text{unbounded}}\,\frac{1}{m-v(Q)+z}.$$

One needs to take advantage that one seeks an estimate on a bounded interval of the spectrum. Therefore, we start with a lemma of localization in the momentum space and elicit a solution in Lemma 2.4. Note also that one may consider $\Im z<0$ by taking the adjoints in the two next lemmata. We shall also use estimates which are uniform in the coupling constant, due to Proposition 4.1.

Lemma 2.3. *Set $\mathcal{I}\subset\mathbb{R}$ a compact interval. Let V be a bounded potential and $\kappa>0$. There is an* even *function $\varphi\in\mathcal{C}_c^\infty(\mathbb{R};\mathbb{R})$ such that the following estimations of the resolvent are equivalent:*

$$\sup_{\Re z\in\mathcal{I},\Im z>0,|\gamma|\leq\kappa}\left\|\langle Q\rangle^{-1}\varphi(\alpha\cdot P)(D_m+\gamma V(Q)-z)^{-1}\varphi(\alpha\cdot P)\langle Q\rangle^{-1}\right\|<\infty,\tag{2.12}$$

$$\sup_{\Re z\in\mathcal{I},\Im z>0,|\gamma|\leq\kappa}\left\|\langle Q\rangle^{-1}(D_m+\gamma V(Q)-z)^{-1}\varphi(\alpha\cdot P)\langle Q\rangle^{-1}\right\|<\infty,\tag{2.13}$$

$$\sup_{\Re z\in\mathcal{I},\Im z>0,|\gamma|\leq\kappa}\left\|\langle Q\rangle^{-1}(D_m+\gamma V(Q)-z)^{-1}\langle Q\rangle^{-1}\right\|<\infty.\tag{2.14}$$

Proof. It is enough to consider $\Im z\in(0,1]$. Set $\mathcal{J}:=\mathcal{I}\times(0,1]\times[-\kappa,\kappa]$, $H_\circ:=\alpha\cdot P$ and $H_\gamma:=D_m+\gamma V$. We choose $\varphi_1\in\mathcal{C}_c^\infty(\mathbb{R})$ with value in $[0,1]$, being even and equal to 1 in a neighborhood of 0. We define $\varphi_R(\cdot):=\varphi_1(\cdot/R)$ and $\widetilde{\varphi}_R:=1-\varphi_R$.

We first notice that $\langle Q\rangle\in\mathcal{C}^1(H_\circ)$, see Appendix A. There is a constant $C>0$ so that

$$|\langle\langle Q\rangle f,\alpha\cdot Pf\rangle-\langle\alpha\cdot Pf,\langle Q\rangle f\rangle|=|\langle f,(\alpha\cdot\nabla\langle\cdot\rangle)(Q)f\rangle|\leq C\|f\|^2\tag{2.15}$$

holds true for all $f\in\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^{2\nu})$. This is usually not enough to deduce the $\mathcal{C}^1$ property, see [28]. We use [35][Lemma A.2] with the notations $A:=H_\circ$, $H:=\langle Q\rangle$, $\chi_n(x):=\varphi(x/n)$ and with $\mathscr{D}:=\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^{2\nu})$. The hypotheses are fulfilled and we deduce that $\langle Q\rangle\in\mathcal{C}^1(H_\circ)$.

By the resolvent equality, we have:

$$(H_\gamma - z)^{-1}\tilde\varphi_R(H_\circ)\big(\mathrm{Id} + W(H_\circ - z)^{-1}\tilde\varphi_R(H_\circ)\big) = (H_\circ - z)^{-1}\tilde\varphi_R(H_\circ) - (H_\gamma - z)^{-1}\varphi_R(H_\circ)W(H_\circ - z)^{-1}\tilde\varphi_R(H_\circ), \tag{2.16}$$

where $W := \gamma V + m\beta$. Note that the support of $\tilde\varphi_R$ vanishes as R goes to infinity. We have

$$\|\langle Q\rangle(H_\circ - z)^{-1}\tilde\varphi_R(H_\circ)\langle Q\rangle^{-1}\| \leq \mathcal{O}(1/R), \text{ uniformly in } (z,\gamma) \in \mathcal{J}.$$

Indeed, if we commute with $\langle Q\rangle$, the part in $(H_\circ - z)^{-1}\tilde\varphi_R(H_\circ)$ is a $\mathcal{O}(1/R)$ by functional calculus. For the other part, Lemma A.2 gives

$$\|[\langle Q\rangle, (H_\circ - z)^{-1}\tilde\varphi_R(H_\circ)]\langle Q\rangle^{-1}\| \leq \mathcal{O}(1/R^2), \text{ uniformly in } (z,\gamma) \in \mathcal{J}.$$

Remembering V is bounded and choosing R big enough, we infer there is a constant $c \in (0,1)$, so that

$$\|W\langle Q\rangle(H_\circ - z)^{-1}\tilde\varphi_R(H_\circ)\langle Q\rangle^{-1}\| \leq c, \text{ uniformly in } (z,\gamma) \in \mathcal{J}. \tag{2.17}$$

We fix R and choose $\varphi := \varphi_R$. We now prove the equivalence. Observe that $\langle Q\rangle^{-1}\varphi(H_\circ)\langle Q\rangle$ is bounded, since $\langle Q\rangle \in \mathcal{C}^1(H_\circ)$. One infers directly that (2.14) $\Rightarrow$ (2.13) $\Rightarrow$ (2.12). It remains to prove (2.12) $\Rightarrow$ (2.14). Thanks to (2.17), we deduce from (2.16) that:

$$\begin{aligned}\langle Q\rangle^{-1}(H_\gamma - z)^{-1}\tilde\varphi(H_\circ)\langle Q\rangle^{-1} = \Big(&\langle Q\rangle^{-1}(H_\circ - z)^{-1}\tilde\varphi(H_\circ)\langle Q\rangle^{-1} \\ &- \langle Q\rangle^{-1}(H_\gamma - z)^{-1}\varphi(H_\circ)\langle Q\rangle^{-1} \quad W\langle Q\rangle(H_\circ - z)^{-1}\tilde\varphi(H_\circ)\langle Q\rangle^{-1}\Big) \\ &\times \big(\mathrm{Id} + W\langle Q\rangle(H_\circ - z)^{-1}\tilde\varphi(H_\circ)\langle Q\rangle^{-1}\big)^{-1}.\end{aligned} \tag{2.18}$$

Note that the last line and the right part of the second line of the r.h.s. are uniformly bounded in $(z,\gamma) \in \mathcal{J}$ by (2.17). We multiply on the left by the bounded operator $\langle Q\rangle^{-1}\varphi(H_\circ)\langle Q\rangle$. The first term of the r.h.s. is bounded uniformly by functional calculus. For the second one, we use (2.12). We infer:

$$\sup_{(z,\gamma)\in\mathcal{J}} \big\|\langle Q\rangle^{-1}\varphi(H_\circ)(H_\gamma - z)^{-1}\tilde\varphi(H_\circ)\langle Q\rangle^{-1}\big\| < \infty.$$

Doing like in (2.18), on the left hand side, we get

$$\sup_{(z,\gamma)\in\mathcal{J}} \big\|\langle Q\rangle^{-1}\tilde\varphi(H_\circ)(H_\gamma - z)^{-1}\varphi(H_\circ)\langle Q\rangle^{-1}\big\| < \infty. \tag{2.19}$$

Finally, to control $\langle Q\rangle^{-1}\tilde\varphi(H_\circ)(H_\gamma - z)^{-1}\tilde\varphi(H_\circ)\langle Q\rangle^{-1}$, we multiply (2.18) on the left by the bounded operator $\langle Q\rangle^{-1}\tilde\varphi(H_\circ)\langle Q\rangle$ and deduce the boundedness using (2.19). □

Lemma 2.4. *Take $\kappa \in (0,1]$ and a compact interval $\mathcal{I} \subset [0,\infty)$. Suppose (2.5) and that*

$$\sup_{\Re z\in\mathcal{I}, \Im z\in(0,1], |\gamma|\leq\kappa} \big\|\langle Q\rangle^{-1}(\Delta_{m,\gamma v,z} + m - z)^{-1}\langle Q\rangle^{-1}\big\| < \infty \tag{2.20}$$

hold true. Then, we have

$$\sup_{\Re z\in\mathcal{I}, \Im z>0, |\gamma|\leq\kappa} \big\|\langle Q\rangle^{-1}(D_m + \gamma v(Q)\otimes \mathrm{Id}_{\mathbb{C}^{2\nu}} - z)^{-1}\langle Q\rangle^{-1}\big\| < \infty. \tag{2.21}$$

Proof. Set $H_\circ := \alpha\cdot P$ and $\mathcal{J} := \mathcal{I}\times(0,1]\times[-\kappa,\kappa]$. By Lemma 2.3, it is enough to show (2.12) for a chosen φ. Since φ is even and constant in a neighborhood of 0, by setting $\psi(\cdot) := \varphi(\sqrt{|\cdot|})$, we have $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ and that $\varphi(H_\circ) = \psi\big((\alpha\cdot P)^2\big)$. In particular, we obtain $\varphi(H_\circ)$ stabilizes $\mathscr{H}^\pm$ and have the right to let it appear in spin decomposition of the resolvent of H of Lemma 2.2. We treat only the upper right corner of the expression as the others are managed in the same way. We need to bound the term:

$$\begin{aligned}&\langle Q\rangle^{-1}\varphi(H_\circ)(\Delta_{m,\gamma v,z} + m - z)^{-1}\alpha^+\cdot P\frac{1}{m - \gamma v(Q) + z}\varphi(H_\circ)\langle Q\rangle^{-1} = \\ &\qquad \langle Q\rangle^{-1}\varphi(H_\circ)\langle Q\rangle \quad \langle Q\rangle^{-1}(\Delta_{m,\gamma v,z} + m - z)^{-1}\langle Q\rangle^{-1} \quad \langle Q\rangle\alpha^+\cdot P\frac{1}{m - \gamma v(Q) + z}\varphi(H_\circ)\langle Q\rangle^{-1}.\end{aligned}$$

The middle term is controlled by the hypothesis. Thanks to (2.15), one has that $[\varphi(H_\circ), \langle Q\rangle]$ is bounded; hence the first term is bounded. For the last one, we commute:

$$\begin{aligned}\langle Q\rangle\alpha^+\cdot P\frac{1}{m - \gamma v(Q) + z}\varphi(H_\circ)\langle Q\rangle^{-1} =& \langle Q\rangle\left[\alpha^+\cdot P, \frac{1}{m - \gamma v(Q) + z}\right]\langle Q\rangle^{-1} \quad \langle Q\rangle\varphi(H_\circ)\langle Q\rangle^{-1} \\ &+ \langle Q\rangle\frac{1}{m - \gamma v(Q) + z}\langle Q\rangle^{-1} \quad \langle Q\rangle\alpha^+\cdot P\varphi(H_\circ)\langle Q\rangle^{-1}.\end{aligned}$$

We estimate uniformly in $(z,\gamma)\in\mathcal{J}$. By (2.5) , we get $\|\langle Q\rangle(m-\gamma v(Q)+z)^{-1}\langle Q\rangle^{-1}\|$ is bounded as $\langle Q\rangle$ commute with v. By (2.5), we also obtain that $\|\langle Q\rangle[\alpha^+\cdot P,(m-\gamma v+z)^{-1}]\langle Q\rangle^{-1}\|$ is also controlled. At last, it is enough to consider $\langle Q\rangle\partial_j\varphi(H_\circ)\langle Q\rangle^{-1}$, which is easily bounded by Lemma A.2 for instance. □

We come to other types of weights. Motivated by the non-relativistic case, see Theorem C.1, we are interested in singular weights like $|Q|$. But, as noticed in Remark 4.2, the operator $|Q|^{-1}(H^{\rm bd}-z)^{-1}|Q|^{-1}$ is even not bounded. Therefore, we enlarge the space in momentum and try the first reasonable weight, namely $\langle P\rangle^{1/2}|Q|$. Given $z\in\mathbb{C}\setminus\mathbb{R}$ and using the Hardy inequality, one reaches

$$\begin{aligned}\|\langle P\rangle^{-1}|Q|^{-1}(H^{\rm bd}_\gamma-v-z)^{-1}|Q|^{-1}\| &\leq \|\langle P\rangle^{-1}|P|\,\|\cdot\|\,|P|^{-1}|Q|^{-1}\|^2\cdot\|(H^{\rm bd}_\gamma-v-z)^{-1}|P|\,\|\\ &\leq\ C(\kappa)\langle z\rangle/|\Im(z)|.\end{aligned}$$

By interpolation, one infers

$$\|\langle P\rangle^{-1/2}|Q|^{-1}(H^{\rm bd}_\gamma-v-z)^{-1}|Q|^{-1}\langle P\rangle^{-1/2}\|\leq C(\kappa)\langle z\rangle/|\Im(z)|<\infty.$$

The upper bound seems relatively sharp in z. However, under the same hypotheses as before, we obtain:

Lemma 2.5. *Take $\kappa\in(0,1]$ and a compact interval $\mathcal{I}\subset[0,\infty)$. Suppose (2.5) and that (2.20) hold true. Then, there is $C>0$ so that*

$$\sup_{\Re z\in\mathcal{I},\Im z>0,|\gamma|\leq\kappa}\left\|\langle P\rangle^{-1/2}|Q|^{-1}(D_m+\gamma v(Q)\otimes \mathrm{Id}_{\mathbb{C}^{2\nu}}-z)^{-1}|Q|^{-1}\langle P\rangle^{-1/2}\right\|\leq C. \tag{2.22}$$

Proof. It is enough to consider $\Im z\in(0,1]$. Set $H_\gamma:=D_m+\gamma v(Q)\otimes\mathrm{Id}_{\mathbb{C}^{2\nu}}$ and $\mathcal{J}:=\mathcal{I}\times(0,1]\times[-\kappa,\kappa]$. Let $f=(f_+,f_-)$, with $f_\pm\in\mathcal{C}^\infty_c(\mathbb{R}^3\setminus\{0\};\mathbb{C}^\nu_\pm)$. Lemma 2.2 and (2.5) give a constant $C>0$, uniform in $(z,\gamma)\in\mathcal{J}$, so that:

$$\begin{aligned}\left|\langle f,(H_\gamma-z)^{-1}f\rangle\right| &\leq 4/m^2\|f_-\|^2+2\left\||Q|^{-1}(\Delta_{m,\gamma v,z}+m-z)^{-1}|Q|^{-1}\right\|\times\\ &\quad\left(\left\||Q|f_+\right\|^2+\left\||Q|\alpha^+\cdot P(m-v(Q)+z)^{-1}f_-\right\|^2+\left\||Q|\alpha^+\cdot P(m-v(Q)+\overline{z})^{-1}f_-\right\|^2\right)\\ &\leq C\left(\left\||Q|f_+\right\|^2+\left\||Q|\alpha^+\cdot Pf_-\right\|^2+\|f_-\|^2\right).\end{aligned}$$

Note that the Hardy inequality gives that $\|f_-\|\leq 2\left\||Q|\alpha^+\cdot Pf_-\right\|$. Then, by commuting $|Q|$ with $\alpha^+\cdot P$ over $\mathcal{C}^\infty_c(\mathbb{R}^3\setminus\{0\};\mathbb{C}^\nu)$, we find $C'>0$, so that:

$$\sup_{(z,\gamma)\in\mathcal{J}}\left|\langle f,|Q|^{-1}(H_\gamma-z)^{-1}|Q|^{-1}f\rangle\right|\leq C'\left(\left\|\mathrm{Id}\otimes P_+\,f\right\|^2+\left\|\langle P\rangle\otimes P_-\,f\right\|^2\right),$$

for all $f\in\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^{2\nu})$, since $\mathcal{C}^\infty_c(\mathbb{R}^3\setminus\{0\};\mathbb{C}^{2\nu})$ is dense in $\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^{2\nu})$. Here we identify, $L^2(\mathbb{R}^3;\mathbb{C}^{2\nu})\simeq L^2(\mathbb{R}^3)\otimes\mathbb{C}^{2\nu}$. We now exchange the role of P_+ and P_-. Considering the operator

$$\alpha^-\cdot P\frac{1}{m+v(Q)+z}\alpha^+\cdot P-v(Q)\text{ in }L^2(\mathbb{R}^3;\mathbb{C}^\nu_-),$$

which leads to the same arguments as for $\Delta_{m,-v,z}$ if one identifies $\mathbb{C}^\nu_-\simeq\mathbb{C}^\nu_+$, one obtains also that

$$\left|\langle f,|Q|^{-1}(H_\gamma-z)^{-1}|Q|^{-1}f\rangle\right|\leq C'\left(\left\|\mathrm{Id}\otimes P_-\,f\right\|^2+\left\|\langle P\rangle\otimes P_+\,f\right\|^2\right),$$

for all $f\in\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^{2\nu})$. By interpolation, e.g., [7][Theorem 4.4.1 and Theorem 6.4.5.(7)], we infer:

$$\left|\langle f,|Q|^{-1}(H_\gamma-z)^{-1}|Q|^{-1}f\rangle\right|\leq C''\left\|\langle P\rangle^{1/2}f\right\|^2,$$

for all $f\in\mathscr{H}^{1/2}(\mathbb{R}^3;\mathbb{C}^{2\nu})$. □

3. Positive commutator estimates.

In the previous section, we saw how to deduce some estimate of the resolvent for $D_m+V(Q)$ starting with some of $\Delta_{m,v,z}$, namely (2.20). First, one technical problem is that these operators depend on the spectral parameter; hence we will study a family of operators uniformly in the spectral parameter. Secondly, we are concerned about the interval $[m,m+\delta]$ and we know that there is no such estimate above $(m-\varepsilon,m)$ as eigenvalues usually accumulates to m from below. Since the theory developed in Appendix B gives some estimates for $\Re z\in[0,\infty)$, we will perform a shift. Therefore, we study the operator

$$\Delta_{2m,\gamma v,\xi},\ \text{uniformly in }(\gamma,\xi)\in\mathcal{E}=\mathcal{E}(\kappa,\delta):=[-\kappa,\kappa]\times[0,\delta]\times(0,1]. \tag{3.1}$$

Here we use a slight abuse of notation identifying $\mathbb{C}\simeq\mathbb{R}^2$. One should read $\Re\xi\in[0,\delta],\Im\xi\in(0,1]$ and $|\gamma|\leq\kappa$. Note the uniformity in the coupling constant is used in Proposition 4.1.

To show (2.20), and therefore (2.14) with the help of Lemma 2.3, it is enough to prove the following fact. Note we strengthen the hypothesis (2.5).

Theorem 3.1. *Suppose that $v \in L^\infty(\mathbb{R}^3;\mathbb{R})$ satisfies the hypotheses (H1) and (H2) from Theorem 4.1. Then there are $\delta, \kappa, C_{\text{LAP}} > 0$ such that*

$$\sup_{\Re z\geq 0, \Im z>0, (\gamma,\xi)\in\mathcal{E}} \left\| |Q|^{-1}(\Delta_{2m,\gamma v,\xi} - z)^{-1}|Q|^{-1}\right\| \leq C_{\text{LAP}}. \tag{3.2}$$

We will show the theorem in the end of the section. We proceed by checking the hypothesis of Appendix B. We recall (2.6) and fix some notation:

$$S := \Delta_{1,0,0} = \alpha^+ \cdot P\, \alpha^- \cdot P = -\Delta_{\mathbb{R}^3} \otimes \mathrm{Id}_{\mathbb{C}^\nu_+} \text{ in } \mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+) \simeq \mathscr{H}^2(\mathbb{R}^3) \otimes \mathbb{C}^\nu_+$$

and set $\mathscr{S} := \dot{\mathscr{H}}^1(\mathbb{R}^3;\mathbb{C}^\nu_+)$, the homogeneous Sobolev space of order 1, i.e., the completion of $\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^\nu_+)$ under the norm $\|f\|_{\mathscr{S}} := \|S^{1/2}f\|^2$. Consider the strongly continuous one-parameter unitary group $\{W_t\}_{t\in\mathbb{R}}$ acting by:

$$(W_t f)(x) = e^{3t/2} f(e^t x), \text{ for all } f \in L^2(\mathbb{R}^3;\mathbb{C}^\nu_+).$$

This is the C_0-group of dilatation. Easily, by interpolation and duality, one gets

$$W_t\mathscr{S} \subset \mathscr{S} \text{ and } W_t\mathscr{H}^s(\mathbb{R}^3;\mathbb{C}^\nu_+) \subset \mathscr{H}^s(\mathbb{R}^3;\mathbb{C}^\nu_+), \text{ for all } s \in \mathbb{R}. \tag{3.3}$$

Consider now its generator A in $L^2(\mathbb{R}^3;\mathbb{C}^\nu_+)$. It acts as follows:

$$A = \frac{1}{2}(P\cdot Q + Q\cdot P) \otimes \mathrm{Id}_{\mathbb{C}^\nu_+} \text{ on } \mathcal{C}^\infty_c(\mathbb{R}^3;\mathbb{C}^\nu_+) \simeq \mathcal{C}^\infty_c(\mathbb{R}^3) \otimes \mathbb{C}^\nu_+.$$

By the Nelson lemma, it is essentially self-adjoint on $\mathcal{C}^\infty_c(\mathbb{R}^3;\mathbb{C}^\nu_+)$.

In the next Proposition, we will choose the upper bound κ of the coupling constant and state the commutator estimates.

Proposition 3.1. *Let $\delta \in (0, 2m)$. Suppose that the hypotheses (H1) and (H2) are fulfilled. Then there are $c_1, \kappa > 0$ such that*

$$\mathcal{D}(\Delta_{2m,\gamma v,\xi}) = \mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^\nu_+), \quad (\Delta_{2m,\gamma v,\xi})^* = \Delta_{2m,\gamma v,\bar{\xi}}, \tag{3.4}$$

$$[\Re(\Delta_{2m,\gamma v,\xi}), \mathrm{i}A]_\circ - c_v\Re(\Delta_{2m,\gamma v,\xi}) \geq c_1 S > 0, \tag{3.5}$$

$$\mp\Im(\Delta_{2m,\gamma v,\Re(\xi)\pm\mathrm{i}\Im(\xi)}) \geq 0, \quad \mp[\Im(\Delta_{2m,\gamma v,\Re(\xi)\pm\mathrm{i}\Im(\xi)}), \mathrm{i}A]_\circ \geq 0, \tag{3.6}$$

hold true in the sense of forms on $\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^\nu_+)$, for all $(\gamma,\xi) \in \mathcal{E}$.

Proof. The first part of (3.6) follows from (2.10). We start with a first restriction on κ. We impose $\kappa \leq (2m-\delta)/\|v\|_\infty$. Hence,

$$\delta \leq 2m - \gamma v(\cdot) + \Re(\xi) \leq 4m, \text{ for all } (\gamma,\xi) \in \mathcal{E}. \tag{3.7}$$

In particular, 0 is not in the essential image of $2m - \gamma v + \Re(\xi)$; Proposition 2.1 gives (3.4).

We turn to the commutator estimates. It is enough to compute the commutators in the sense of form on $\mathcal{C}^\infty_c(\mathbb{R}^3;\mathbb{C}^\nu)$, since it is a core for $\Delta_{2m,v,\xi}$ and A.

$$\begin{aligned}\left[\Delta_{2m,\gamma v,\xi}, \mathrm{i}A\right] &= \left[\alpha^+\cdot P\,\frac{1}{2m-\gamma v+\xi}\alpha^-\cdot P, \mathrm{i}A\right] + \gamma[v,\mathrm{i}A]\\ &= 2\,\alpha^+\cdot P\,\frac{1}{2m-\gamma v+\xi}\alpha^-\cdot P - \gamma\,\alpha^+\cdot P\,\frac{Q\cdot\nabla v(Q)}{(2m-\gamma v+\xi)^2}\alpha^-\cdot P - \gamma\,Q\cdot\nabla v(Q).\end{aligned} \tag{3.8}$$

Then, we have $[\Re(\Delta_{2m,\gamma v,\xi}), \mathrm{i}A] - c_v\Re(\Delta_{2m,\gamma v,\xi}) =$

$$\begin{aligned}&= (2-c_v)\,\alpha^+\cdot P\frac{2m-\gamma v+\Re(\xi)}{\big(2m-\gamma v+\Re(\xi)\big)^2+\Im(\xi)^2}\alpha^-\cdot P\\ &\quad - \gamma\,\alpha^+\cdot P\left(\frac{Q\cdot\nabla v(Q)\big(\big(2m-\gamma v+\Re(\xi)\big)^2-\Im(\xi)^2\big)}{\big(\big(2m-\gamma v+\Re(\xi)\big)^2+\Im(\xi)^2\big)^2}\right)\alpha^-\cdot P - \gamma\,Q\cdot\nabla v(Q) - c_v\gamma v(Q).\\ &\geq (2-c_v)\frac{\delta}{16m^2+1}S - \kappa\,\|Q\cdot\nabla v(Q)\|\,\frac{16m^2+1}{\delta^4}S - \kappa\frac{c'_v}{|Q|^2} \geq c_1 S,\end{aligned}$$

where $c_1 := \dfrac{\delta(2-c_v)}{32m^2+2}$ and by assuming that $\kappa \leq \dfrac{c_1}{(4c_v' + \|Q\cdot\nabla v(Q)\|(16m^2+1)/\delta^4)}$. Note the "4" comes from the Hardy inequality. This gives (3.5).

At last, we have:

$$[\Im\Delta_{2m,\gamma v,\xi}, \mathrm{i}A] = -2\Im(\xi)\,\alpha^+\cdot P\frac{\big(2m-\gamma v+\Re(\xi)\big)^2+\Im(\xi)^2-\gamma Q\cdot\nabla v(Q)\big(2m-\gamma v+\Re(\xi)\big)}{\Big(\big(2m-\gamma v+\Re(\xi)\big)^2+\Im(\xi)^2\Big)^2}\alpha^-\cdot P.$$

This is of the sign of $-\Im(\xi)$, when we further impose $\kappa \leq \delta^2/(8m\|Q\cdot\nabla v(Q)\|)$. □

We now bound some commutators.

Proposition 3.2. *Let $\delta\in(0,2m)$. Suppose that the hypotheses (H1) and (H2) are fulfilled. Consider the $c_1,\kappa>0$ from Proposition 3.1. There is c and C depending on c_v,δ,κ and v, such that*

$$|\langle\Delta_{2m,\gamma v,\overline{\xi}}\, f, Ag\rangle - \langle Af, \Delta_{2m,\gamma v,\xi}\, g\rangle| \leq c\|f\|\cdot\|(\Delta_{2m,\gamma v,\xi}\pm \mathrm{i})g\|, \tag{3.9}$$

holds true, for all $f,g\in\mathscr{H}^2(\mathbb{R}^3;\mathbb{C}^+_\nu)\cap\mathcal{D}(A)$ and

$$|\langle f, [[\Delta_{2m,\gamma v,\xi},\mathrm{i}A]_\circ,\mathrm{i}A]_\circ f\rangle| \leq C\langle f, Sf\rangle. \tag{3.10}$$

holds true for all $f\in\mathscr{H}^1(\mathbb{R}^3;\mathbb{C}^\nu_+)$.

Proof. As the domain of A and S are explicit, one easily sees that $\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^\nu_+)$ is dense in $\mathcal{D}(A)\cap\mathcal{D}(S)$ endowed with the norm $\|\cdot\|+\|A\cdot\|+\|S\cdot\|$. More generally, this also follows from the fact that $S\in\mathcal{C}^1(A)$, see Theorem 6.2.10 of [1]. Therefore, it is enough to prove (3.9) on this core. We take κ as in the proof of Proposition 3.1. We first find $c>0$, uniform in $(\gamma,\xi)\in\mathcal{E}$, so that

$$|\langle f, [\Delta_{2m,\gamma v,\xi},\mathrm{i}A]_\circ g\rangle| \leq c\big(\|f\|\cdot\|g\|+\|f\|\cdot\|Sg\|\big),\ \text{for all } f,g\in\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^\nu_+). \tag{3.11}$$

Taking in account (3.8), observe that

$$\left|\frac{2}{2m-\gamma v+\xi} - \gamma\frac{Q\cdot\nabla v(Q)}{(2m-\gamma v+\xi)^2}\right| \leq \frac{2}{\delta}+\kappa\frac{\|Q\cdot\nabla v(Q)\|}{\delta^2}.$$

It remains to find $a,b>0$, which are uniform in $(\gamma,\xi)\in\mathcal{E}$, such that the following estimation holds:

$$\|\Delta_{2m,\gamma v,\xi}f\| \geq a\|Sf\| - b\|f\|,\ \text{for all } f\in\mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{C}^\nu_+).$$

This follows from $\|\Delta_{2m,\gamma v,\xi}f\|^2 \geq a^2\|Sf\|^2 - b^2\|f\|^2$. Take $\varepsilon,\varepsilon'\in(0,1)$.

$$\begin{aligned}
\|\Delta_{2m,v,z}f\|^2 &\geq (1-\varepsilon)\left\|\frac{1}{2m-\gamma v(Q)+\xi}Sf\right\|^2 + \left(1-\frac{1}{\varepsilon}\right)\left\|\frac{\gamma(\alpha^+\cdot\nabla v)(Q)}{(2m-\gamma v(Q)+\xi)^2}\alpha^-\cdot Pf\right\|^2.\\
&\geq (1-\varepsilon)\frac{1}{1+16m^2}\|Sf\|^2 + \left(1-\frac{1}{\varepsilon}\right)\frac{\kappa\|\alpha^+\cdot\nabla v(Q)\|}{\delta^4}\left\|\alpha^-\cdot Pf\right\|^2,\\
&\geq \left((1-\varepsilon)\frac{1}{1+16m^2}+\varepsilon'(\varepsilon-1)\frac{\kappa\|\alpha^+\cdot\nabla v(Q)\|}{2\varepsilon\delta^4}\right)\|Sf\|^2 + (\varepsilon-1)\frac{\kappa\|\alpha^+\cdot\nabla v(Q)\|}{2\varepsilon\varepsilon'\delta^4}\|f\|^2.
\end{aligned}$$

Choosing ε' small enough, we infer (3.9).

We turn to (3.10). Again, it is enough to compute in the form sense on $\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{C}^\nu)$.

$$\begin{aligned}
[[\Delta_{2m,v,z},\mathrm{i}A],\mathrm{i}A] = {}& 4\,\alpha^+\cdot P\frac{1}{2m-v+z}\alpha^-\cdot P + 4\,\alpha^+\cdot P\frac{Q\cdot\nabla v(Q)}{(2m-v+z)^2}\alpha^-\cdot P\\
&-\alpha^+\cdot P\frac{Q\cdot\nabla\big(Q\cdot\nabla v(Q)\big)}{(2m-v+z)^2}\alpha^-\cdot P + 2\,\alpha^+\cdot P\frac{(Q\cdot\nabla v(Q))^2}{(2m-v+z)^3}\alpha^-\cdot P + (Q\cdot\nabla)^2 v(Q).
\end{aligned}$$

Note that (H1) ensures that $\|(Q\cdot\nabla)^2 v(Q)f\|^2 \leq 4\|\,|Q|(Q\cdot\nabla)^2 v(Q)\|^2\|Sf\|^2$ is controlled by S. Relying again on (3.7), the bound (3.10) follows. □

We finally turn to the proof of the main result of this section.

Proof of Theorem 3.1. We check the hypotheses of Theorem B.1 for $H^-(p) := \Delta_{2m,\gamma v,\xi}$ and $p := (\gamma,\xi)$ and where $\mathcal{E}$ is defined in (3.1). Clearly, $S\in\mathcal{C}^1(A)$ and (B.4) is given by (3.3). Now, observe that

$$\langle f, (W_tH^-(p)-H^-(p)W_t)g\rangle = \int_0^t \frac{d}{ds}\langle W^*_{t-s}f, H^-(p)W_sg\rangle ds = \int_0^t\langle W^*_{t-s}f, [H^-(p),\mathrm{i}A]W_sg\rangle ds,$$

for all $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{C}^\nu_+)$. Using (3.11) and by density, we derive

$$(3.12) \qquad \big(W_t H^-(p) - H^-(p) W_t\big) g = \int_0^t W_{t-s}[H^-(p), \mathrm{i}A]_\circ W_s g\, ds, \text{ for all } g \in \mathscr{H}^2$$

and where the integral exists in the strong sense. By dividing by t, letting t go to 0 and using the fact that $\{W_t\}$ is a C_0-group in $\mathscr{H}$ and in $\mathscr{H}^2$, we derive that $\Delta_{2m,\gamma v,\xi} \in \mathcal{C}^1(A, \mathscr{H}^2, \mathscr{H})$, for all $(\gamma, \xi) \in \mathcal{E}$. Note that (3.12) ensures that the strong limit of $(\mathrm{i}t)^{-1}[H^-(p), W_t]$ is $[H^-(p), A]_\circ$. By interpolation, we deduce that $\Delta_{2m,\gamma v,\xi} \in \mathcal{C}^1(A, \mathscr{H}^1, \mathscr{H}^{-1})$. Now taking in account (3.10), we infer in the same way that $\Delta_{2m,\gamma v,\xi} \in \mathcal{C}^2(A, \mathscr{H}^1, \mathscr{H}^{-1})$, for all $(\gamma, \xi) \in \mathcal{E}$.

Using Propositions 3.1 and 3.2, we can apply Theorem B.1. We derive there is a finite C' so that

$$\sup_{\Re z \geq 0, \Im z > 0, (\gamma,\xi) \in \mathcal{E}} |\langle f, (\Delta_{2m,\gamma v,\xi} - z)^{-1} f\rangle| \leq C' \left(\|S^{-1/2} f\|^2 + \|S^{-1/2} A f\|^2 \right).$$

The Hardy inequality concludes. □

4. Main result

In this section, we will prove the main result of this paper and deduce Theorem 1.2.

Theorem 4.1. *Let $\gamma \in \mathbb{R}$. Suppose that $v \in L^\infty(\mathbb{R}^3; \mathbb{R})$ satisfies the hypothesis:*

(H1) *$\|v\|_\infty \leq m/2$ and ∇v, $Q \cdot \nabla v(Q)$, $\langle Q\rangle (Q \cdot \nabla v)^2(Q)$ are bounded.*

(H2) *There are $c_v \in [0, 2)$ and $c'_v \geq 0$ such that*

$$x \cdot (\nabla v)(x) + c_v v(x) \leq \frac{c'_v}{|x|^2}, \text{ for all } x \in \mathbb{R}^3 \setminus \{0\}.$$

Set $V_1(Q) := v(Q) \otimes \mathrm{Id}_{\mathbb{C}^{2n}}$, where $L^2(\mathbb{R}^3; \mathbb{C}^{2\nu}) \simeq L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2\nu}$.

(H3) *Consider $V_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^{2\nu})$ satisfying:*

$$\langle Q\rangle^2 V_2(Q) \in \mathcal{B}\big(\mathscr{H}^1(\mathbb{R}^3; \mathbb{C}^{2\nu}), L^2(\mathbb{R}^3; \mathbb{C}^{2\nu})\big).$$

Then, there are $\kappa, \delta, C > 0$, such that $H_\gamma := D_m + \gamma V(Q)$, where $V := V_1 + V_2$, is self-adjoint with domain $\mathscr{H}^1(\mathbb{R}^3; \mathbb{C}^{2\nu})$. Moreover,

$$(4.1) \qquad \sup_{|\lambda| \in [m, m+\delta],\, \varepsilon > 0, |\gamma| \leq \kappa} \|\langle Q\rangle^{-1} (H_\gamma - \lambda - \mathrm{i}\varepsilon)^{-1} \langle Q\rangle^{-1}\| \leq C.$$

In particular, H_γ has no eigenvalue in $\pm m$. Moreover, there is C' so that

$$(4.2) \qquad \sup_{|\gamma| \leq \kappa} \int_{\mathbb{R}} \|\langle Q\rangle^{-1} e^{-\mathrm{i}tH_\gamma} E_{\mathcal{I}}(H_\gamma) f\|^2 dt \leq C' \|f\|^2, \text{ for all } f \in L^2(\mathbb{R}^3, \mathbb{C}^4),$$

where $\mathcal{I} = [-m - \delta, -m] \cup [m, m + \delta]$ and where $E_{\mathcal{I}}(H_\gamma)$ denotes the spectral measure of H_γ.

Remark 4.1. *In [25] and in [63], one takes advantage that the Virial of the potential is negative, in order to prove the limiting absorption principle for some self-adjoint Schrödinger operators, see Remark C.1. Here, we cannot allow this hypothesis as we are also interested in positronic threshold, i.e., we seek a result for v and $-v$, see (2.4). We recover the positivity using some Hardy inequality and small coupling constants.*

Proof of Theorem 4.1. First note that (4.2) is a consequence of (4.1), see [46]. Consider the case $V_2 = 0$. Note that, in Section 2, the operator H_γ is denoted by H_γ^{bd}.

The self-adjointness is clear. We first apply Theorem 3.1 and obtain (3.2). We now choose $\xi = z$. As $\||Q|f\| \leq \|\langle Q\rangle f\|$, we infer (2.20). In turn, it implies (2.14). Finally, using the unitary transformation α_5, (4.1) follows from (2.4). For a general V_2, we use Proposition 4.1. □

It remains to explains how to add the singular part V_2 of the potential by perturbing the limiting absorption principal. This is somehow standard. Note that unlike [43], for instance, we do not distinguish the nature of the singularity at the threshold energy, as we work with small coupling constants.

Proposition 4.1. *Assume that Theorem 4.1 holds true for $V_2 = 0$. Take now V_2 satisfying (H3). Then there is $\kappa' \in (0, \kappa]$, so that*

$$H_\gamma := D_m + \gamma(V + V_2)(Q)$$

is self-adjoint with domain $\mathscr{H}^1(\mathbb{R}^3; \mathbb{C}^{2\nu})$, for all $|\gamma| \leq \kappa$. Moreover,

$$\sup_{\Re z \in [m, m+\delta], \Im z > 0, |\gamma| \leq \kappa'} \left\| \langle Q\rangle^{-1} (H_\gamma - z)^{-1} \langle Q\rangle^{-1} \right\| < \infty.$$

Proof. Up to a smaller κ, Kato-Rellich ensures the self-adjointness. We turn to the estimate of the resolvent. Easily, one reduces to the case $|\Im(z)| \leq 1$. From the resolvent identity, we have:

$$\langle Q\rangle^{-1}(H_\gamma - z)^{-1}\langle Q\rangle^{-1} \quad \langle Q\rangle \left\{\mathrm{Id} + \gamma V_2(H_\gamma^{\mathrm{bd}} - z)^{-1}\right\}\langle Q\rangle^{-1} = \langle Q\rangle^{-1}(H_\gamma^{\mathrm{bd}} - z)^{-1}\langle Q\rangle^{-1}$$

Considering Lemma 2.4 and Theorem 3.1, the result follows if we can invert the second term of the l.h.s. uniformly in the parameters. Therefore, we show there is $\kappa' \in (0, \kappa]$ so that

$$\sup_{\Re(z)\in[m,m+\delta],\Im(z)\in(0,1],|\gamma|\leq\kappa'} \|\langle Q\rangle\gamma V_2(H_\gamma^{\mathrm{bd}} - z)^{-1}\langle Q\rangle^{-1}\| < 1.$$

Using the identity of the resolvent, we get

$$\begin{aligned}\langle Q\rangle V_2(H_\gamma^{\mathrm{bd}} - z)^{-1}\langle Q\rangle^{-1} =& \langle Q\rangle V_2(H_0^{\mathrm{bd}} - \mathrm{i})^{-1} \quad \langle Q\rangle^{-1} \\ &- \langle Q\rangle V_2(H_0^{\mathrm{bd}} - \mathrm{i})^{-1}\langle Q\rangle \quad (\gamma V - z + \mathrm{i}) \quad \langle Q\rangle^{-1}(H_\gamma^{\mathrm{bd}} - z)^{-1}\langle Q\rangle^{-1}.\end{aligned}$$

The first term of the r.h.s. is bounded by using (H3). To control the last term, remember that z is bounded and use again Lemma 2.4 and Theorem 3.1. It remains to notice that

$$\langle Q\rangle V_2(H_0^{\mathrm{bd}} - \mathrm{i})^{-1}\langle Q\rangle = \langle Q\rangle^2 V_2(H_0^{\mathrm{bd}} - \mathrm{i})^{-1} - \langle Q\rangle V_2(H_0^{\mathrm{bd}} - \mathrm{i})^{-1} \quad [H_0^{\mathrm{bd}}, \langle Q\rangle]_\circ (H_0^{\mathrm{bd}} - \mathrm{i})^{-1}$$

is bounded. Indeed, the assumption (H3) controls the terms in V_2 and $\langle Q\rangle \in \mathcal{C}^1(H_0)$ and $[H_0^{\mathrm{bd}}, \langle Q\rangle]_\circ$ is bounded, see proof of Lemma 2.3. □

At last, Theorem 1.2 is an immediate corollary of Theorem 4.1. Indeed, one has:

Example 4.1 (Multi-center)**.** *For $i = 1, \ldots, n$, we choose $a_i \in \mathbb{R}^3$ the site of the poles and $Z_i \in \mathbb{R}$ its charge. We set:*

$$v_c := \sum_{i=1}^{n} \frac{z_i}{|\cdot - a_i|}$$

Note that

$$Q \cdot \nabla v_c(Q) + v_c := \sum_{i=1}^{n} a_i \cdot \nabla v(Q).$$

Choose now $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ radial with values in $[0, 1]$. Moreover, we ask that φ restricted to the ball $B(0, 2\max(|a_i|))$ is 1. Consider the support large enough, so that $\|\tilde\varphi v\|_\infty \leq m/2$, where $\tilde\varphi :\ 1 - \varphi$. Set $v := \tilde\varphi v_c$. Straightforwardly, the hypothesis (H1) and (H2) are satisfied. Note that (H3) follows from the Hardy inequality.

Example 4.2 (Smooth homogeneous potentials)**.** *In* [39], *one considers smooth potential independent of $|x|$ of the form $v(x) := \tilde v(|x|/x)$, with $v \in \mathcal{C}^\infty(S^2)$, see also Remark C.2. Here, by taking $c_v = 0$ in Theorems 4.1 and 4.2, one obtains a relativistic equivalent of this result. We point out that this perturbation is not relatively compact with respect to the Dirac operator.*

We now discuss singular weights in $|Q|$.

Remark 4.2. *It is important to note that unlike in the non-relativistic case, see Theorem C.1, one cannot replace the weights $\langle Q\rangle$ in (2.2) by $|Q|$. Indeed, with the notation of Theorem 4.1, $V_2 = 0$ and $z \in \mathbb{C}$, consider a function f in $\mathcal{C}_c^\infty(\mathbb{R}^3 \setminus \{0\}; \mathbb{C}^{2\nu})$ and notice the expression $R^{-3/2}\big\| |Q|(H_\gamma - z)|Q| f(\cdot/R)\big\|_2$ tends to 0, as R goes to 0. Therefore, there is no $z \in \mathbb{C}$ such that the operator $|Q|(H_\gamma - z)|Q|$ has a bounded inverse.*

We finally give a second result with a weight allowing some singularity in $|Q|$. Using Lemma 2.5 instead of Lemma 2.4 in the proof of Theorem 4.1, we infer straightforwardly:

Theorem 4.2. *Let $\gamma \in \mathbb{R}$ and take $v \in L^\infty(\mathbb{R}^3; \mathbb{R})$ satisfying (H1) and (H2). Then, there are $\kappa, \delta, C > 0$, such that $H_\gamma := D_m + \gamma v(Q) \otimes \mathrm{Id}_{\mathbb{C}^{2\nu}}$ satisfies*

$$\sup_{|\lambda|\in[m,m+\delta],\,\varepsilon>0,|\gamma|\leq\kappa} \|\langle P\rangle^{-1/2}|Q|^{-1}(H_\gamma - \lambda - \mathrm{i}\varepsilon)^{-1}|Q|^{-1}\langle P\rangle^{-1/2}\| \leq C. \tag{4.3}$$

Moreover, there is C' so that

$$\sup_{|\gamma|\leq\kappa} \int_{\mathbb{R}} \|\langle P\rangle^{-1/2}|Q|^{-1} e^{-\mathrm{i}tH_\gamma} E_{\mathcal{I}}(H_\gamma) f\|^2 dt \leq C'\|f\|^2, \tag{4.4}$$

where $\mathcal{I} = [-m - \delta, -m] \cup [m, m + \delta]$ and where $E_{\mathcal{I}}(H_\gamma)$ denotes the spectral measure of H_γ.

Keeping in mind Proposition 4.1, one sees that one can only add trivial potentials in the perturbation theory of the limiting absorption principle. Hence, it is an open question whether one can cover the example 4.1 with the weights $\langle P\rangle^{1/2}|Q|$.

Appendix A. Commutator expansions.

This section is a small improvement of [34][Appendix B], see also [18, 40]. We start with some generalities. Given a bounded operator B and a self-adjoint operator A acting in a Hilbert space $\mathscr{H}$, one says that $B \in \mathcal{C}^k(A)$ if $t \mapsto e^{-\mathrm{i}tA}Be^{\mathrm{i}tA}$ is strongly $\mathcal{C}^k$. Given a closed and densely defined operator B, one says that $B \in \mathcal{C}^k(A)$ if for some (hence any) $z \notin \sigma(B)$, $t \mapsto e^{-\mathrm{i}tA}(B-z)^{-1}e^{\mathrm{i}tA}$ is strongly $\mathcal{C}^k$. The two definitions coincide in the case of a bounded self-adjoint operator. We recall a result following from Lemma 6.2.9 and Theorem 6.2.10 of [1].

Theorem A.1. *Let A and B be two self-adjoint operators in the Hilbert space $\mathscr{H}$. For $z \notin \sigma(A)$, set $R(z) := (B-z)^{-1}$. The following points are equivalent to $B \in \mathcal{C}^1(A)$:*

(1) *For one (then for all) $z \notin \sigma(B)$, there is a finite c such that*

$$|\langle Af, R(z)f\rangle - \langle R(\overline{z})f, Af\rangle| \leq c\|f\|^2, \text{ for all } f \in \mathcal{D}(A). \tag{A.1}$$

(2) a. *There is a finite c such that for all $f \in \mathcal{D}(A)\cap\mathcal{D}(B)$:*

$$|\langle Af, Bf\rangle - \langle Bf, Af\rangle| \leq c\big(\|Bf\|^2 + \|f\|^2\big). \tag{A.2}$$

b. *For some (then for all) $z \notin \sigma(B)$, the set $\{f \in \mathcal{D}(A) \mid R(z)f \in \mathcal{D}(A) \text{ and } R(\overline{z})f \in \mathcal{D}(A)\}$ is a core for A.*

Note that the condition (2.b) could be uneasy to check, see [28]. We mention [35][Lemma A.2] and [33][Lemma 3.2.2] to overcome this subtlety. As $(B+\mathrm{i})^{-1}$ is a homeomorphism between $\mathscr{H}$ onto $\mathcal{D}(B)$, $(B+\mathrm{i})^{-1}\mathcal{D}(A)$ is dense in $\mathcal{D}(B)$, endowed with the graph norm. Moreover, (A.1) gives $(B+\mathrm{i})^{-1}\mathcal{D}(A) \subset \mathcal{D}(A)$. Therefore $(B+\mathrm{i})^{-1}\mathcal{D}(A) \subset \mathcal{D}(B)\cap\mathcal{D}(A)$ are dense in $\mathcal{D}(B)$ for the graph norm. Remark that $\mathcal{D}(B)\cap\mathcal{D}(A)$ is usually not dense in $\mathcal{D}(A)$, see [31].

Note that (A.1) yields the commutator $[A, R(z)]$ extends to a bounded operator, in the form sense. We shall denote the extension by $[A, R(z)]_\circ$. In the same way, since $\mathcal{D}(B)\cap\mathcal{D}(A)$ is dense in $\mathcal{D}(B)$, (A.2) ensures that the commutator $[B, A]$ extends to a unique element of $\mathcal{B}\big(\mathcal{D}(B), \mathcal{D}(B)^*\big)$ denoted by $[B, A]_\circ$. Moreover, when $B \in \mathcal{C}^1(A)$, one has:

$$\big[A, (B-z)^{-1}\big]_\circ = \underbrace{(B-z)^{-1}}_{\mathscr{H}\leftarrow\mathcal{D}(B)^*} \quad \underbrace{[B, A]_\circ}_{\mathcal{D}(B)^*\leftarrow\mathcal{D}(B)} \quad \underbrace{(B-z)^{-1}}_{\mathcal{D}(B)\leftarrow\mathscr{H}}.$$

Here we use the Riesz lemma to identify $\mathscr{H}$ with its anti-dual $\mathscr{H}^*$.

We now recall some well known facts on symbolic calculus and almost analytic extensions. For $\rho \in \mathbb{R}$, let $\mathcal{S}^\rho$ be the class of function $\varphi \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ such that

$$\forall k \in \mathbb{N}, \quad C_k(\varphi) := \sup_{t\in\mathbb{R}} \langle t\rangle^{-\rho+k}|\varphi^{(k)}(t)| < \infty. \tag{A.3}$$

Equipped with the semi-norms defined by (A.3), $\mathcal{S}^\rho$ is a Fréchet space. Leibniz' formula implies the continuous embedding: $\mathcal{S}^\rho \cdot \mathcal{S}^{\rho'} \subset \mathcal{S}^{\rho+\rho'}$. We shall use the following result, e.g., [18].

Lemma A.1. *Let $\varphi \in \mathcal{S}^\rho$ with $\rho \in \mathbb{R}$. For all $l \in \mathbb{N}$, there is a smooth function $\varphi^{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, such that:*

$$\varphi^{\mathbb{C}}|_{\mathbb{R}} = \varphi, \qquad \left|\frac{\partial\varphi^{\mathbb{C}}}{\partial\overline{z}}(z)\right| \leq c_1\langle\Re(z)\rangle^{\rho-1-l}|\Im(z)|^l \tag{A.4}$$

$$\operatorname{supp}\varphi^{\mathbb{C}} \subset \{x+\mathrm{i}y \mid |y| \leq c_2\langle x\rangle\}, \tag{A.5}$$

$$\varphi^{\mathbb{C}}(x+\mathrm{i}y) = 0, \text{ if } x \notin \operatorname{supp}\varphi. \tag{A.6}$$

for some constants c_1, c_2 depending on the semi-norms (A.3) *of φ in $\mathcal{S}^\rho$ and not on φ.*

One calls $\varphi^{\mathbb{C}}$ an *almost analytic extension* of φ. Let A be a self-adjoint operator, $\rho < 0$ and $\varphi \in \mathcal{S}^\rho$. By functional calculus, one has $\varphi(A)$ bounded. The Helffer-Sjöstrand's formula, see [38] and [18] for instance, gives that for all almost analytic extension of φ, one has:

$$\varphi(A) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\varphi^{\mathbb{C}}}{\partial\overline{z}}(z-A)^{-1}dz\wedge d\overline{z}. \tag{A.7}$$

Note the integral exists in the norm topology, by (A.4) with $l = 1$. Next we come to a commutator expansion. Here B is not necessarily bounded while in [34], one considers the case B bounded. We denote by $\mathrm{ad}^j_A(B)$ the extension of the j-th commutator of A with B defined inductively by $\mathrm{ad}^p_A(B) := [\mathrm{ad}^{p-1}_A(B), A]_\circ$, when it exists.

Proposition A.1. *Let $k \in \mathbb{N}^*$ and $B \in \mathcal{C}^k(A)$ be self-adjoint. Suppose $\mathrm{ad}^j_A(B)$ are bounded operators, for $j = 1, \dots, k$. Let $\rho < k$ and $\varphi \in \mathcal{S}^\rho$. Suppose that $\mathcal{D}(B) \cap \mathcal{D}(\langle A\rangle^\rho)$ is dense in $\mathcal{D}(\langle A\rangle^\rho)$ for the graph norm. Then, the commutator $[\varphi(A), B]_\circ$ belongs to $\mathcal{B}\big(\mathcal{D}(\langle A\rangle^{\rho-1})), \mathscr{H}\big)$ and satisfies*

$$[\varphi(A), B]_\circ = \sum_{j=1}^{k-1} \frac{1}{j!}\varphi^{(j)}(A)\mathrm{ad}^j_A(B) + \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial \varphi^{\mathbb{C}}}{\partial \overline{z}}(z-A)^{-k}\mathrm{ad}^k_A(B)(z-A)^{-1} dz \wedge d\overline{z}, \tag{A.8}$$

where the integral exists for the topology of $\mathcal{B}(\mathscr{H})$.

Proof. We cannot use the (A.7) directly with φ as the integral does not seem to exist. We proceed as in [34]. Take $\chi_1 \in \mathcal{C}^\infty_c(\mathbb{R};\mathbb{R})$ with values in $[0,1]$ and being 1 on $[-1,1]$. Set $\chi_R := \chi(\cdot/R)$. As R goes to infinity, χ_R converges pointwise to 1. Moreover, $\{\chi_R\}_{R\in[1,\infty]}$ is bounded in $\mathcal{S}^0$. We infer $\varphi_R := \varphi\chi_R$ tends pointwise to φ and that $\{\varphi_R\}_{R\in[1,\infty]}$ is bounded in $\mathcal{S}^\rho$. Now, note that

$$\begin{aligned}[\varphi_R(A), B] = &\sum_{j=1}^{k-1} \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial \varphi_R^{\mathbb{C}}}{\partial \overline{z}}(z-A)^{-j-1}\mathrm{ad}^j_A(B) dz \wedge d\overline{z} \\ &+ \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial \varphi_R^{\mathbb{C}}}{\partial \overline{z}}(z-A)^{-k}\mathrm{ad}^k_A(B)(z-A)^{-1} dz \wedge d\overline{z}.\end{aligned} \tag{A.9}$$

in the form sense on $\mathcal{D}(B)$. Using (A.4), the integral converges in norm. We write $[\varphi_R(A), B]_\circ$ on the l.h.s. The first term of the r.h.s. is $\sum_{j=1}^{k-1} \varphi_R^{(j)}(A)\mathrm{ad}^j_A(B)/j!$. Now we let R goes to infinity. For the l.h.s. and the first term of the r.h.s., we expand the commutator in (A.9) in the form sense on $\mathcal{D}(\langle A\rangle^\rho) \cap \mathcal{D}(B)$, take the limit by functional calculus and finish by density in $\mathcal{D}(\langle A\rangle^\rho)$. For the remainder term of the r.h.s., we use the Lebesgue convergence theorem. It remains to note that the operator of the r.h.s. is in $\mathcal{B}(\langle A\rangle^{\rho-1}, \mathscr{H})$ since $\varphi \in \mathcal{S}^\rho$. □

The hypothesis on the density of $\mathcal{D}(B) \cap \mathcal{D}(\langle A\rangle^\rho)$ in $\mathcal{D}(\langle A\rangle^\rho)$ could be delicate to check. It follows by the Nelson Lemma from the fact that the C_0-group $\{e^{\mathrm{i}tA^k}\}_{t\in\mathbb{R}}$ stabilizes $\mathcal{D}(B)$. We mention that for $k = 1$, since $[B, \mathrm{i}A]_\circ$ is bounded, [28][Lemma 2] ensures this invariance of the domain.

The rest of the previous expansion is estimated as in [34]. We rely on the following important bound. Let $c > 0$ and $s \in [0,1]$, there exists some $C > 0$ so that, for all $z = x + \mathrm{i}y \in \{a + \mathrm{i}b \mid 0 < |b| \le c\langle a\rangle\}$:

$$\big\|\langle A\rangle^s(A-z)^{-1}\big\| \le C\langle x\rangle^s \cdot |y|^{-1}. \tag{A.10}$$

Lemma A.2. *Let $B \in \mathcal{C}^k(A)$ self-adjoint. Suppose $\mathrm{ad}^j_A(B)$ are bounded operators, for $j = 1, \dots, k$. Let $\varphi \in \mathcal{S}^\rho$, with $\rho < k$. Let $I_k(\varphi)$ the rest of the development of order k of $[\varphi(A), B]$ in (A.8). Let $s, s' \in [0,1]$ such that $\rho + s + s' < k$. Then $\langle A\rangle^s I_k(\varphi)\langle A\rangle^{s'}$ is bounded and it is uniformly bounded when φ stays in a bounded subset of $\mathcal{S}^\rho$. Let $R > 0$. If φ stays in a bounded subset of $\{\psi \in \mathcal{S}^\rho \mid [-R;R] \cap \mathrm{supp}(\varphi) = \emptyset\}$ then $\langle R\rangle^{k-\rho-s-s'}\|\langle A\rangle^s I_k(\varphi)\langle A\rangle^{s'}\|$ is uniformly bounded.*

Proof. We will follow ideas from [18][Lemma C.3.1]. In this proof, all the constants are denoted by C, independently of their value. Given a complex number z, x and y will denote its real and imaginary part, respectively. Since $B \in \mathcal{C}^k(A)$, $\mathrm{ad}^k_A(B)$ is bounded. We start with the second assertion. Let $\varphi \in \mathcal{S}^\rho$, $R > 0$ such that $[-R;R] \cap \mathrm{supp}(\varphi) = \emptyset$. Notice that, by (A.6), $\varphi^{\mathbb{C}}(x + \mathrm{i}y) = 0$ for $|x| \le R$. By (A.10),

$$\begin{aligned}\|\langle A\rangle^s I_k(\varphi)\langle A\rangle^{s'}\| &\le \frac{1}{\pi}\int \big|\frac{\partial \varphi^{\mathbb{C}}}{\partial \overline{z}}\big| \cdot \frac{\langle x\rangle^s}{|y|^k} \cdot \|\mathrm{ad}^k_A(B)\| \cdot \frac{\langle x\rangle^{s'}}{|y|} dx \wedge dy \\ &\le C(\varphi)\int_{|x|\ge R}\int_{|y|\le c_2\langle x\rangle} \langle x\rangle^{\rho+s+s'-1-l}|y|^l|y|^{-k-1} dx \wedge dy,\end{aligned}$$

for any l, by (A.4). Recall that $dz \wedge d\overline{z} = -2idx \wedge dy$. We choose $l = k+1$. We have,

$$\|\langle A\rangle^s I_k(\varphi)\langle A\rangle^{s'}\| \le C(\varphi)\int_{|x|\ge R} \langle x\rangle^{\rho+s+s'-k-1} dx \le C(\varphi)\langle R\rangle^{\rho+s+s'-k}.$$

Since $C(\varphi)$ is bounded when φ stays in a bounded subset of $\mathcal{S}^\rho$, this yields the second assertion. For the first one, we can follow the same lines, replacing R by 0 in the integrals, and arrive at the result. □

Appendix B. A non-selfadjoint weak Mourre theory

In this section, we adapt ideas coming from [25] and [63] in order to obtain a limiting absorption principle for a family of closed operators $\{H^{\pm}(p)\}_{p\in\mathcal{E}}$. We ask that they have a common domain

$$\mathscr{D} := \mathcal{D}\big(H^+(p)\big) = \mathcal{D}\big(H^-(p)\big), \text{ for all } p \in \mathcal{E}. \tag{B.1}$$

We choose $p_0 \in \mathcal{E}$ and endow $\mathscr{D}$ with the graph norm of $H^+(p_0)$. We also ask that

$$\big(H^+(p)\big)^* = H^-(p), \text{ for all } p \in \mathcal{E}. \tag{B.2}$$

In particular, we have that $\mathcal{D}\big((H^{\pm}(p))^*\big) = \mathscr{D}$. In the sequel, we forgo p, when no confusion can arises.

Since $H^{\pm}$ are densely defined, share the same domain and are adjoint of the other, we have that $\Re(H^{\pm})$ and $\Im(H^{\pm})$ are closable operators on $\mathscr{D}$, indeed their adjoints are densely defined. We denote by $\Re(H^{\pm})$ and by $\Im(H^{\pm})$ the closure of these operators. It is possible that they are not self-adjoint, albeit there are symmetric. However, $\mathscr{D}$ is a core for them. Their domain is possibly bigger than $\mathscr{D}$. We suppose that H^+ is *dissipative*, i.e.,

$$\langle f, \Im(H^+)f\rangle \geq 0, \text{ for all } f \in \mathscr{D}.$$

This gives also that $\Im(H^-) \leq 0$. By the numerical range theorem (see Lemma B.1), we infer that $\sigma(H^{\pm})$ is included in the half-plan containing $\pm\mathrm{i}$. Take now a non-negative self-adjoint operator S, *independent* of $p \in \mathcal{E}$, with form domain $\mathscr{G} := \mathcal{D}(S^{1/2}) \supset \mathscr{D}$. We assume that S is injective. We have $\langle f, Sf\rangle > 0$ for all $f \in \mathscr{G}\setminus\{0\}$ and simply write $S > 0$. One defines $\mathscr{S}$ as the completion of $\mathscr{G}$ under the norm $\|f\|^2_{\mathscr{S}} := \langle f, Sf\rangle$. We obtain $\mathscr{G} \subset \mathscr{S}$ with dense and continuous embedding. Moreover, since $\mathscr{G} = \langle S^{1/2}\rangle^{-1}\mathscr{H}$, $\mathscr{S}$ is also the completion of $\mathscr{H}$ under the norm given by $\|S^{1/2}\langle S^{1/2}\rangle^{-1}\cdot\|$. We use the Riesz Lemma to identify $\mathscr{H}$ with $\mathscr{H}^*$, its anti-dual. The adjoint space $\mathscr{S}^*$ of $\mathscr{S}$ is exactly the domain of $\langle S^{1/2}\rangle S^{-1/2}$ in $\mathscr{H} \simeq \mathscr{H}^*$. Note that S^{-1} is an isomorphism between $\mathscr{S}$ and $\mathscr{S}^*$. We get the following scale with continuous and dense embeddings:

$$\begin{array}{ccccccccccc} & & & & & & \mathscr{S}^* & & & & \\ & & & & & & \downarrow & \searrow & & & \\ \mathscr{D} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{H} & \simeq & \mathscr{H}^* & \longrightarrow & \mathscr{G}^* & \longrightarrow & \mathscr{D}^*. \\ & & & \searrow & \downarrow & & & & & & \\ & & & & \mathscr{S} & & & & & & \end{array} \tag{B.3}$$

To perform this analysis, we consider an external operator, the conjugate operator. Let A be a self-adjoint operator in $\mathscr{H}$. We assume $S \in \mathcal{C}^1(A)$. Let $W_t := e^{\mathrm{i}tA}$ be the C_0-group associated to A in $\mathscr{H}$. We ask:

$$W_t\mathscr{G} \subset \mathscr{G} \text{ and } W_t\mathscr{S} \subset \mathscr{S}, \text{ for all } t \in \mathbb{R}. \tag{B.4}$$

By duality, we have W_t stabilizes $\mathscr{G}^*$ and also $\mathscr{S}^*$ (but may be not $\mathscr{D}$ or $\mathscr{D}^*$). The restricted group to these spaces is also a C_0-group. We denote the generator by A with the subspace in subscript. Given $\mathscr{H}_i \subset \mathscr{H}_j$ be two of those spaces. One easily shows that $A|_{\mathscr{H}_i} \subset A|_{\mathscr{H}_j}$ and that $A|_{\mathscr{H}_j}$ is the closure of $A|_{\mathscr{H}_i}$ in $\mathscr{H}_j$. Moreover, one has

$$\mathcal{D}(A|_{\mathscr{H}_i}) = \big\{f \in \mathcal{D}\big(A|_{\mathscr{H}_j}\big) \cap \mathscr{H}_i \text{ such that } A|_{\mathscr{H}_j}f \in \mathscr{H}_i\big\}. \tag{B.5}$$

We now explain how to check the second hypothesis of (B.4), see also [63]. We mention this result is due to [24] when $\mathcal{D}(S) \subset \mathcal{D}(A)$.

Remark B.1. *The second invariance of the domains of* (B.4) *follows from the first one and from*

$$|\langle Sf, Af\rangle - \langle Af, Sf\rangle| \leq c\|S^{1/2}f\|^2, \text{ for all } f \in \mathcal{D}(S)\cap\mathcal{D}(A). \tag{B.6}$$

As $(S+\mathrm{i})^{-1}$ is a homeomorphism between $\mathscr{H}$ onto $\mathcal{D}(S)$, $(S+\mathrm{i})^{-1}\mathcal{D}(A)$ is dense in $\mathcal{D}(S)$, endowed with the graph norm. Moreover, since $S \in \mathcal{C}^1(A)$, one has $(S+\mathrm{i})^{-1}\mathcal{D}(A) \subset \mathcal{D}(A)$. Therefore $(S+\mathrm{i})^{-1}\mathcal{D}(A) \subset \mathcal{D}(S)\cap\mathcal{D}(A)$ are dense in $\mathcal{D}(S)$, hence in $\mathscr{G}$ and in $\mathscr{S}$. The commutator $[S, A]$ has a unique extension to an element of $\mathcal{B}(\mathscr{S}, \mathscr{S}^*)$, in the form sense. We denote it by $[S, A]_\circ$. Take now $f \in \mathscr{G}\cap\mathcal{D}(A)$, which is a dense set in $\mathscr{G}$. On one hand we have $\tau \mapsto \|W_\tau f\|^2_{\mathscr{S}}$ is bounded when τ is in a compact set (since $\mathscr{G} \hookrightarrow \mathscr{S}$. On the other hand, the Gronwall lemma concludes by noticing:

$$\|W_t f\|^2_{\mathscr{S}} = \langle f, Sf\rangle + \int_0^t \langle W_\tau f, [S, \mathrm{i}A]_\circ W_\tau f\rangle\, d\tau \leq \|S^{1/2}f\|^2 + c\int_0^{|t|} \|W_\tau f\|^2_{\mathscr{S}}\, d\tau.$$

Let $\mathscr{K} \subset \mathscr{H}$ be a space which is stabilized by W_t. Consider $L \in \mathcal{B}(\mathscr{K}, \mathscr{K}^*)$. We say that $L \in \mathcal{C}^k(A; \mathscr{K}, \mathscr{K}^*)$, when $t \to W_{-t}LW_t$ is strongly $\mathcal{C}^k$ from $\mathscr{K}$ into $\mathscr{K}^*$. When $\mathscr{K} = \mathscr{H}$, using the resolvent equality, one observes that this class is the same as $\mathcal{C}^k(A)$, see for instance [1][Theorem 6.3.4 a.].

Theorem B.1. *Let $H^\pm = H^\pm(p)$, with $p\in\mathcal{E}$ as above. Let A be self-adjoint such that* (B.4) *holds true. Suppose that $H^\pm\in\mathcal{C}^2(A;\mathscr{G},\mathscr{G}^*)$ and that there is a constant c, independent of p, such that*

$$|\langle H^\mp f, Ag\rangle - \langle Af, H^\pm g\rangle| \le c\|f\|\cdot\|(H^\pm\pm \mathrm{i})g\|, \text{ for all } f,g\in\mathscr{D}\cap\mathcal{D}(A). \tag{B.7}$$

Take $c_1\ge 0$ independent of p and assume that

$$[\Re(H^\pm),\mathrm{i}A]_\circ - c_1\Re(H^\pm)\ge S>0, \tag{B.8}$$

$$\pm c_1[\Im(H^\pm),\mathrm{i}A]_\circ\ge 0,\quad \pm\Im(H^\pm)\ge 0, \tag{B.9}$$

in the sense of forms on $\mathscr{G}$. Suppose also there exists $C>0$ independent of $p\in\mathcal{E}$ such that

$$\big|\langle f, \big[\big[H^\pm,A\big]_\circ, A\big]_\circ f\rangle\big| \le C\|S^{1/2}f\|^2, \text{ for all } f\in\mathscr{G}. \tag{B.10}$$

Then, there are c and $\mu_0>0$, both independent of p, such that

$$|\langle f, (H^\pm-\lambda\pm\mathrm{i}\mu)^{-1}f\rangle|\le c\left(\|S^{-1/2}f\|^2+\|S^{-1/2}Af\|^2\right)\le c\|f\|_{\mathcal{D}(A|_{\mathscr{G}^*})}, \tag{B.11}$$

for all $p\in\mathcal{E}$, $\mu\in(0,\mu_0)$ and $\lambda\ge 0$, in the case $c_1>0$ and $\lambda\in\mathbb{R}$ if $c_1=0$.

In the self-adjoint setting, the case $c_1=0$ is treated in [12, 13]. Comparing with [63], who deal with the case of one self-adjoint operator and for $c_1>0$. We give some few improvements. First, we do not ask $\mathscr{D}$ to be the domain of S. Moreover, we drop the hypothesis that the first commutator $[H,\mathrm{i}A]_\circ$ is bounded from below. For the latter, we use more carefully the numerical range theorem in our proof. Finally, unlike [63], we shall not go into interpolation theory so as to improve the norm in the limiting absorption principle. Indeed, in the context of the model we are considering here, we reach the weights we are interested in without it. We stick to an intermediate and explicit result, which is closer to [41]. Therefore, for the sake of clarity, we present then the easiest proof possible and pay an important care about domains.

We also mention that there exists other Mourre-like theory for non-self-adjoint operators, [3, 65].

Proof. We focus on the case $c_1>0$, as for the case $c_1=0$, one replaces "$\lambda\ge 0$" by "$\lambda\in\mathbb{R}$". Since $H^\pm\in\mathcal{C}^1(A,\mathscr{G},\mathscr{G}^*)$, by the resolvent equality, we obtain

$$\big[(H^\pm\pm\mathrm{i})^{-1},W_t\big] = -\underbrace{(H^\pm\pm\mathrm{i})^{-1}}_{\mathscr{H}\longleftarrow\mathscr{G}^*}\ \underbrace{\big[H^\pm,W_t\big]}_{\mathscr{G}^*\longleftarrow\mathscr{G}}\ \underbrace{(H^\pm\pm\mathrm{i})^{-1}}_{\mathscr{G}\longleftarrow\mathscr{H}}.$$

We take the derivative with respect to t. It exists strongly in $\mathscr{H}$, then $H^\pm\in\mathcal{C}^1(A)$. In particular, as in Remark B.1, one has $\mathscr{D}\cap\mathcal{D}(A)$ dense in $\mathscr{D}$ for the graph norm. Thus, (B.7) gives $[H^\pm,\mathrm{i}A]_\circ\in\mathcal{B}(\mathscr{D},\mathscr{H})$. We define $H^\pm_\varepsilon := H^\pm\pm\mathrm{i}\varepsilon[H^\pm,\mathrm{i}A]_\circ$ with the common domain $\mathscr{D}$ for $\varepsilon\ge 0$. Since $H^\pm\pm\mathrm{i}$ is bijective, by writing $H^\pm_\varepsilon\pm\mathrm{i}=\big(1\pm\mathrm{i}\varepsilon[H^\pm,\mathrm{i}A]_\circ(H^\pm\pm\mathrm{i})^{-1}\big)(H^\pm\pm\mathrm{i})$ and using (B.7), we get there is ε_0 such that $H^\pm_\varepsilon(p)\pm\mathrm{i}$ is bijective and closed for all $|\varepsilon|\le\varepsilon_0$ and all $p\in\mathcal{E}$. Therefore $(H^\pm_\varepsilon\pm\mathrm{i})^*$ is also bijective from $\mathcal{D}\big((H^\pm_\varepsilon)^*\big)$ onto $\mathscr{H}$. Now since $(H^\pm_\varepsilon\pm\mathrm{i})^*$ is an extension of $H^\mp_\varepsilon\mp\mathrm{i}$ which is also bijective, we infer the equality of the domains and that $(H^\pm_\varepsilon)^*=H^\mp_\varepsilon$ for $\varepsilon\le\varepsilon_0$.

Since $H^\pm\in\mathcal{C}^1(A;\mathscr{G},\mathscr{G}^*)$, we obtain that $\Re(H^\pm)$ and $\Im(H^\pm)$ are in $\mathcal{C}^1(A;\mathscr{G},\mathscr{G}^*)$. In this space we have $[H^\pm,A]_\circ=[\Re(H^\pm),A]_\circ+\mathrm{i}[\Im(H^\pm),A]_\circ$. Now, take $f\in\mathscr{G}$. Take $\varepsilon,\lambda,\mu\ge 0$. We get:

$$\begin{aligned}
&-c_1\varepsilon\left\langle f,\Re(H^\pm_\varepsilon-\lambda\pm\mathrm{i}\mu)f\right\rangle \pm\left\langle f,\Im(H^\pm_\varepsilon-\lambda\pm\mathrm{i}\mu)f\right\rangle =\\
&\quad = -c_1\varepsilon\left\langle f,\big(\Re(H^\pm)\mp\varepsilon[\Im(H^\pm),\mathrm{i}A]_\circ-\lambda\big)f\right\rangle \pm\left\langle f,\big(\Im(H^\pm)\pm\mu\pm\varepsilon[\Re(H^\pm),\mathrm{i}A]_\circ\big)f\right\rangle\\
&\quad = \varepsilon\left\langle f,\big([\Re(H^\pm),\mathrm{i}A]_\circ-c_1\Re(H^\pm)\big)f\right\rangle + (c_1\lambda\varepsilon+\mu)\|f\|^2 \pm\left\langle f,\big(c_1\varepsilon^2[\Im(H^\pm),\mathrm{i}A]_\circ+\Im(H^\pm)\big)f\right\rangle
\end{aligned}$$

$$\ge (c_1\lambda\varepsilon+\mu)\|f\|^2+\varepsilon\|S^{1/2}f\|^2. \tag{B.12}$$

We start with a crude bound. For $\varepsilon,\mu>0$, we get:

$$(c_1\varepsilon+1)\|(H^\pm_\varepsilon-\lambda\pm\mathrm{i}\mu)f\|_{\mathscr{G}^*}\ge\min(c_1\lambda\varepsilon+\mu,\varepsilon)\|f\|_{\mathscr{G}}.$$

Since $H^\pm_\varepsilon-\lambda\pm\mathrm{i}\mu\in\mathcal{B}(\mathscr{G},\mathscr{G}^*)$ and since they are adjoint of the other, we infer the injectivity and that the ranges are closed. They are bijective and the inverse is bounded by the open mapping theorem.

$$G^\pm_\varepsilon := G^\pm_\varepsilon(\lambda,\mu)=(H^\pm_\varepsilon-\lambda\pm\mathrm{i}\mu)^{-1} \text{ exists in } \mathcal{B}(\mathscr{G}^*,\mathscr{G}), \text{ for } \lambda\ge 0 \text{ and } \varepsilon,\mu>0.$$

Here we lighten the notation but keep in mind the dependency in λ and μ. Moreover,

$$\|G^\pm_\varepsilon\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})}\le(c_1\varepsilon+1)/\min(c_1\lambda\varepsilon+\mu,\varepsilon), \text{ for } \lambda\ge 0 \text{ and } \varepsilon,\mu>0. \tag{B.13}$$

This bound seems not enough to lead the whole analysis. Then, we first restrict the domain of $G^\pm_\varepsilon$ to $\mathscr{H}$ and improve it. Since this inequality (B.12) holds also true on the common domain of $H^\pm_\varepsilon$ (and of its adjoint), we can apply the numerical range theorem, Lemma B.1. Since $S \geq 0$, we get the spectrum of $H_\varepsilon^+ - \lambda + i\mu$ is contained in the lower half-plane delimited by the equation $y \leq -c_1\varepsilon x - \mu$. Hence, for $\varepsilon \in (0, \varepsilon_0]$ and $\mu > 0$, $H_\varepsilon^\pm - \lambda \pm i\mu$ is bijective and by taking ε_0 smaller, one has the distance from 0 to the boundary of the cone bigger than $\mu/2$. Then,

$$\|G_\varepsilon^\pm\|_{\mathcal{B}(\mathscr{H})} \leq 2/\mu, \text{ for } \mu > 0 \text{ and } \varepsilon \in [0, \varepsilon_0]. \tag{B.14}$$

Note also that $(G_\varepsilon^\pm)^* = G_\varepsilon^\mp$. Take $\varepsilon, \mu > 0$. We fix $f \in \mathscr{H}$ and set:

$$F_\varepsilon^\pm := \langle f, G_\varepsilon^\pm f \rangle .$$

Since $G_\varepsilon^\pm \mathscr{H} \subset \mathscr{D} \subset \mathscr{S}$ and using (B.12), we infer

$$\begin{aligned}\left\|S^{1/2}G_\varepsilon^\pm f\right\|^2 &\leq c_1 \left|\Re \langle G_\varepsilon^\pm f, (H_\varepsilon^\pm - \lambda \pm i\mu)G_\varepsilon^\pm f\rangle\right| + \frac{1}{\varepsilon}\left|\Im \langle G_\varepsilon^\pm f, (H_\varepsilon^\pm - \lambda \pm i\mu)G_\varepsilon^\pm f\rangle\right| \\ &\leq \max\left(c_1, \frac{1}{\varepsilon}\right) \left|F_\varepsilon^\pm\right| . \end{aligned} \tag{B.15}$$

Hence up to a smaller $\varepsilon_0 > 0$, we obtain $\left\|S^{1/2}G_\varepsilon^\pm f\right\|^2 \leq \left|F_\varepsilon^\pm\right|/\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$. Moreover, if $f \in \mathcal{D}(S^{-1/2})$, we obtain

$$\left|F_\varepsilon^\pm\right| \leq \left\|S^{-1/2}f\right\| \left\|S^{1/2}G_\varepsilon^\pm f\right\| \leq \left\|S^{-1/2}f\right\| \frac{\sqrt{\left|F_\varepsilon^\pm\right|}}{\sqrt{\varepsilon}}$$

and deduce

$$\left|F_\varepsilon^\pm\right| \leq \frac{1}{\varepsilon}\left\|S^{-1/2}f\right\|^2, \text{ for all } \varepsilon \in (0, \varepsilon_0]. \tag{B.16}$$

We now show that $G_\varepsilon^\pm \in \mathcal{C}^1(A)$. First note that $G_\varepsilon^\pm$ is a bijection from $\mathscr{H}$ onto $\mathscr{D}$. Then by taking the adjoint, it is also a bijection from $\mathscr{D}^*$ onto $\mathcal{H}$. Remember now that W_t stabilizes $\mathscr{G}$ and $\mathscr{G}^*$. By the resolvent equality in $\mathcal{B}(\mathscr{H})$, we have:

$$[G_\varepsilon^\pm, W_t] = - \underbrace{G_\varepsilon^\pm}_{\mathscr{H} \longleftarrow \mathscr{G}^*} \quad \underbrace{\left[H^\pm \pm i\varepsilon[H^\pm, iA], W_t\right]}_{\mathscr{G}^* \longleftarrow \mathscr{G}} \quad \underbrace{G_\varepsilon^\pm}_{\mathscr{G} \longleftarrow \mathscr{H}}$$

Let now take the derivative in 0. Since $H^\pm$ and $[H^\pm, iA]$ are in $\mathcal{C}^1(A; \mathscr{G}, \mathscr{G}^*)$ (the former being in $\mathcal{C}^2(A; \mathscr{G}, \mathscr{G}^*)$), the right hand side has a strong limit for all element in $\mathscr{H}$. Hence, $H_\varepsilon^\pm \in \mathcal{C}^1(A)$. As in Remark B.1, it follows that $G_\varepsilon^\pm \mathcal{D}(A) \subset \mathcal{D}(A) \cap \mathscr{D}$ and one can safely expand the commutator in the next computation. Take $f \in \mathcal{D}(A)$.

$$\begin{aligned}\frac{d}{d\varepsilon}F_\varepsilon^\pm &= \left\langle f, \frac{d}{d\varepsilon}G_\varepsilon^\pm f\right\rangle = \pm i \left\langle G_\varepsilon^\mp f, [H^\pm, iA]_\circ G_\varepsilon^\pm f\right\rangle \\ &= \pm \left\langle G_\varepsilon^\mp f, Af\right\rangle \mp \left\langle Af, G_\varepsilon^\mp f\right\rangle - \varepsilon \left\langle G_\varepsilon^\mp f, \left[[H, iA]_\circ, iA\right]G_\varepsilon^\pm f\right\rangle .\end{aligned}$$

Here the last commutator in taken in the form sense. Now use three times (B.15) and the bound (B.10), which is uniform in $p \in \mathcal{E}$, then integrate to obtain

$$\left|F_\varepsilon^\pm - F_{\varepsilon'}^\pm\right| \leq \int_\varepsilon^{\varepsilon'} \left\{2\frac{\sqrt{\left|F_s^\pm\right|}}{\sqrt{s}}\left\|S^{-1/2}Af\right\| + C\left|F_s^\pm\right|\right\} ds, \text{ for } 0 < \varepsilon \leq \varepsilon' \leq \varepsilon_0 \tag{B.17}$$

and for all $f \in \mathcal{D}(S^{-1/2}A) \cap \mathcal{D}(A)$.

We give a first estimation. Using (B.16) and the Gronwall lemma, see [1][Lemma 7.A.1] with $\theta = 1/2$ or [59][Lemma 2.6] with $p = 1/2$, we infer there are some constants C, C', C'', C''', independent of $\varepsilon \in (0, \varepsilon_0]$, $\lambda \geq 0$, $\mu > 0$ and of $p \in \mathcal{E}$, so that

$$\begin{aligned}\left|F_\varepsilon^\pm\right| &\leq e^{C(\varepsilon - \varepsilon_0)}\left(\left|F_{\varepsilon_0}^\pm\right|^{1/2} + \int_\varepsilon^{\varepsilon_0}\left\{\frac{1}{\sqrt{\eta}}e^{-\frac{1}{2}C(\eta - \varepsilon_0)}\right\} d\eta \left\|S^{-1/2}Af\right\|\right)^2 \\ &\leq C''\left(\left|F_{\varepsilon_0}^\pm\right| + \left(\sqrt{\varepsilon} - \sqrt{\varepsilon_0}\right)^2\left\|S^{-1/2}Af\right\|^2\right) \\ &\leq C''\left(\frac{1}{\varepsilon_0}\left\|S^{-1/2}f\right\|^2 + \left(\sqrt{\varepsilon} - \sqrt{\varepsilon_0}\right)^2\left\|S^{-1/2}Af\right\|^2\right) \leq C'''\|f\|^2_{\tilde{\mathscr{S}}^*}\end{aligned} \tag{B.18}$$

for $f \in \mathcal{D}(S^{-1/2}) \cap \mathcal{D}(S^{-1/2}A) \cap \mathcal{D}(A)$ and where $\tilde{\mathscr{S}}^*$ is the completion of $\mathcal{D}(A|_{\mathscr{S}^*})$ under the norm $\|f\|^2_{\tilde{\mathscr{S}}^*} := \|S^{-1/2}f\|^2 + \|S^{-1/2}Af\|^2$. Here one notices that the norm is well defined for elements of $\mathcal{D}(A|_{\mathscr{S}^*})$ by taking in account (B.5). We now plug this back in (B.17). Since the inverse of the square root is integrable around 0, we find C'''' with the same independence so that

$$\left|F^\pm_\varepsilon - F^\pm_{\varepsilon'}\right| \le \int_\varepsilon^{\varepsilon'} \left\{ 2\frac{\sqrt{C'''}}{\sqrt{s}} + CC''' \right\} ds\ \|f\|^2_{\tilde{\mathscr{S}}^*} = C''''\big(\sqrt{\varepsilon'} - \sqrt{\varepsilon}\big)\|f\|^2_{\tilde{\mathscr{S}}^*}.$$

Then, $\{F^\pm_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is a Cauchy sequence. We denote by $F^\pm_{0^+}$ the limit, as ε goes to 0. It remains to notice that $F^\pm_{0^+} = F^\pm_0$. Indeed, using (B.14) and (B.7), one has the stronger fact that

$$\|G^\pm_0 - G^\pm_\varepsilon\|_{\mathcal{B}(\mathscr{H})} \le \varepsilon\|G^\pm_\varepsilon\|_{\mathcal{B}(\mathscr{H})} \cdot \|[H^\pm(p), \mathrm{i}A](H^\pm(p) - \lambda \pm \mathrm{i}\mu)^{-1}\|_{\mathcal{B}(\mathscr{H})} \le \frac{c\varepsilon}{\mu^2}.$$

This gives us (B.11). □

For the convenience of the reader, we give a proof of the following well known fact:

Lemma B.1 (Numerical Range Theorem)**.** *Let H be a closed operator. Suppose that $\mathscr{D} := \mathcal{D}(H) = \mathcal{D}(H^*)$. The numerical range of H is defined by $\mathcal{N} := \{\langle f, Hf\rangle$ with $f \in \mathscr{D}$ and $\|f\| = 1\}$. We have that $\sigma(H) \subset \overline{\mathcal{N}}$, the closure of $\mathcal{N}$. Moreover, if $\lambda \notin \sigma(H)$, then $\|(H-\lambda)^{-1}\| \le 1/d(\lambda, \mathcal{N})$.*

Proof. Let $\lambda \notin \overline{\mathcal{N}}$. There is $c := d(\lambda, \mathcal{N}) > 0$, such that $|\langle f, Hf\rangle - \lambda| \ge c$. Then,

$$\|(H-\lambda)f\| \ge c\|f\|, \quad \|(H^* - \overline{\lambda})f\| \ge c\|f\|,$$

for all $f \in \mathscr{D}$ and $\|f\| = 1$. From the second part, we get the range of $(H-\lambda)$ is dense. Then, since H is closed, the first part gives that the range of $(H-\lambda)$ is closed. Hence, using again the first inequality, $H - \lambda$ is bijective. The open mapping theorem concludes. □

Appendix C. Application to non-relativistic dispersive Hamiltonians

In this section, we give an immediate application to the theory exposed in Appendix B. We do not discuss the uniformity with respect to the external parameter. The latter would be used in the heart of our approach, see Section 3. We discuss shortly the Helmholtz equation, see [5, 6, 72, 73]. In [65], one studies the size of the resolvent of

$$H_h := -h^2\Delta + V_1(Q) - \mathrm{i}hV_2(Q), \text{ as } h \to 0.$$

This operator models accurately the propagation of the electromagnetic field of a laser in material medium. The important improvement between [65] and the previous ones, is that he allows V_2 to be a smooth function tending to 0 without any assumption on the size of $\|V_2\|_\infty$. Note he supposes the coefficients are smooth as some pseudo-differential calculus is used to applied the non self-adjoint Mourre theory he develops. Then, he discusses trapping conditions in the spirit of [72]. Here, we will stick to the quantum case and choose $h = -1$. To simplify the presentation and expose some key ideas of Section 3, we focus on $L^2(\mathbb{R}^n;\mathbb{C})$, with $n \ge 3$. For dimensions 1 and 2, one needs to adapt the first part of (H2) and the weights in (C.1).

Theorem C.1. *Suppose that $V_1, V_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R})$ satisfy:*

(H0) *V_i are Δ-operator bounded with a relative bound $a < 1$, for $i \in \{1, 2\}$.*
(H1) *∇V_i, $Q \cdot \nabla V_i(Q)$ are in $\mathcal{B}(\mathscr{H}^2(\mathbb{R}^n); L^2(\mathbb{R}^n))$ and $\langle Q\rangle(Q\cdot\nabla V_i)^2(Q)$ is bounded, for $i \in \{1, 2\}$.*
(H2) *There are $c_1 \in [0, 2)$ and $c_1' \in \big[0, 4(2-c_1)/(n-2)^2\big)$ such that*

$$W_{V_1}(x) := x\cdot(\nabla V_1)(x) + c_1V_1(x) \le \frac{c_1'}{|x|^2}, \text{ for all } x \in \mathbb{R}^n.$$

and

$$V_2(x) \ge 0 \text{ and } -c_1x\cdot(\nabla V_2)(x) \ge 0, \text{ for all } x \in \mathbb{R}^n.$$

On $\mathcal{C}^\infty_c(\mathbb{R}^n)$, we define $H := -\Delta + V(Q)$, where $V := V_1 + \mathrm{i}V_2$. The closure of H defines a dispersive closed operator with domain $\mathscr{H}^2(\mathbb{R}^n)$. We keep denoting it with H. Its spectrum included in the upper half-plane. Moreover, H has no eigenvalue in $[0,\infty)$ and

$$\sup_{\lambda\in[0,\infty),\,\mu>0} \big\|\,|Q|^{-1}(H-\lambda+\mathrm{i}\mu)^{-1}|Q|^{-1}\big\| < \infty. \tag{C.1}$$

If $c_1 = 0$, H has no eigenvalue in $\mathbb{R}$ and (C.1) holds true for $\lambda \in \mathbb{R}$.

The quantity W_{V_1} is called the *virial* of V_1. For h fixed and for a compact $\mathcal{I}$ included in $(0,\infty)$, [65] shows some estimates of the resolvent above $\mathcal{I}$. Here we deal with the threshold 0 and with high energy estimates. On the other hand, as he avoids the threshold, he reaches some very sharp weights. As mentioned above, one can improve the weights $|Q|$ to some extend by the use of Besov spaces, see [63]. In [65] one makes an hypothesis on the sign of V_2 but not on the one of $x \cdot (\nabla V_2)(x)$. Note that if one supposes $c_1 = 0$, we are also in this situation. We take the opportunity to point out [71], where one discusses the presence of possible eigenvalues in 0 for non self-adjoint problems.

Remark C.1. *Taking $V_2 = 0$, we can compare the results with* [25, 63]. *In* [25]*, one uses in a crucial way that $W_{V_1}(x) \leq -c\langle x\rangle^{\alpha}$ in a neighborhood of infinity, for some $\alpha, c > 0$. In* [63]*, one remarks that the condition $W_{V_1}(x) \leq 0$ is enough to obtain the estimate. Here we mention that the condition (H2) is sufficient. Note this example is not explicitly discussed in* [63] *but is covered by his abstract approach. In* [12]*, for the special case $c_1 = 0$, one uses extensively the condition (H2). This implies* (C.1) *for $\lambda \in \mathbb{R}$.*

Remark C.2. *Unlike in* [65]*, we stress that V is* not *supposed to be a relatively compact perturbation of H and that the essential spectrum of H can be different of* $[0,\infty)$*. In* [39]*, see also* [12]*, one studies $V_2 = 0$ and $V_1(x) := v(x/|x|)$, with $v \in \mathcal{C}^{\infty}(S^{n-1})$. We improve the weights of* [39]*[Theorem 3.2] from $\langle Q\rangle$ to $|Q|$. We can also give a non-self-adjoint version. Consider V_1 satisfying (H1) and being relatively compact with respect to Δ and $V_2(x) := v(x/|x|)$, where $v \in \mathcal{C}^0(S^{n-1})$, non-negative. If $v^{-1}(0)$ is non-empty, one shows* $[0,\infty)$ *is included in the essential spectrum of H by using some Weyl sequences.*

Proof of Theorem C.1. Using (H0) and adapting the proof of Kato-Rellich, e.g., [62][Theorem X.12], one obtains easily $\mathcal{D}(H) = \mathcal{D}(H^*) = \mathscr{H}^2(\mathbb{R}^n)$. Let $S := c_s(-\Delta)^{1/2}$, with $c_s := 2 - c_1 - (n-2)^2 c_1'/4 > 0$. Set $\mathscr{S} := \dot{\mathscr{H}}^1(\mathbb{R}^n)$, the homogeneous Sobolev space of order 1, i.e., the completion of $\mathscr{H}^1(\mathbb{R}^n)$ under the norm $\|f\|_{\mathscr{S}} := \|S^{1/2}f\|^2$. Consider the strongly continuous one-parameter unitary group $\{W_t\}_{t\in\mathbb{R}}$ acting by: $(W_t f)(x) = e^{nt/2} f(e^t x)$, for all $f \in L^2(\mathbb{R}^3)$. This is the C_0-group of dilatation. By interpolation and duality, one derives $W_t\mathscr{S} \subset \mathscr{S}$ and $W_t\mathscr{H}^s(\mathbb{R}^3) \subset \mathscr{H}^s(\mathbb{R}^n)$, for all $s \in \mathbb{R}$. Consider now its generator A in $L^2(\mathbb{R}^n)$. By the Nelson lemma, it is essentially self-adjoint on $\mathcal{C}_c^{\infty}(\mathbb{R}^n)$ and acts as follows: $A = (P \cdot Q + Q \cdot P)/2$ on $\mathcal{C}_c^{\infty}(\mathbb{R}^n)$. By computing on $\mathcal{C}_c^{\infty}(\mathbb{R}^n)$ in the form sense, we obtain that

$$[\Re(H), \mathrm{i}A] - c_1\Re(H) = -(2-c_1)\Delta - W_{V_1} \geq S, \tag{C.2}$$

here we used the Hardy inequality for the last step. Furthermore, $\Im(H) = V_2(Q) \geq 0$,

$$[\Im(H), \mathrm{i}A] = -Q \cdot \nabla(V_2)(Q), \tag{C.3}$$

and also

$$[[H, \mathrm{i}A], \mathrm{i}A] = -4\Delta + (Q \cdot \nabla V)^2(Q). \tag{C.4}$$

Since W_t stabilizes $\mathscr{G} := \mathscr{H}^1$ and as (C.2), (C.3) and (C.4) extend to bounded operators from $\mathscr{H}^1$ into $\mathscr{H}^{-1}$, we infer that H and H^* are in $\mathcal{C}^2(A; \mathscr{H}^1, \mathscr{H}^{-1})$ and also (B.8) and (B.9). Now since $\mathcal{C}_c^{\infty}(\mathbb{R}^3)$ is a core for H, H^* and A, (C.2) and (C.3) give (B.7), with notation $H^+ = H$ and $H^- = H^*$. In addition (B.10) follows from the Hardy inequality and (H1), as $\|(Q\cdot\nabla)^2 v(Q) f\|^2 \leq c\| \, |Q|(Q\cdot\nabla)^2 v(Q)\|^2 \|Sf\|^2$. Therefore, we can apply Theorem B.1 and derive the weight $|Q|$ by the Hardy inequality. □

Finally, we recall the Hardy inequality. Take E a finite dimensional vector space. One has:

$$\left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \left|\frac{1}{|x|} f(x)\right|^2 dx \leq \left|\langle f, -\Delta f\rangle\right|, \text{ where } n \geq 3 \text{ and } f \in \mathcal{C}_c^{\infty}(\mathbb{R}^n; E). \tag{C.5}$$

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