ON THE A.C. SPECTRUM OF THE 1D DISCRETE DIRAC OPERATOR

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Abstract. In this paper, under some integrability condition, we prove that an electrical perturbation of the discrete Dirac operator has purely absolutely continuous spectrum for the one dimensional case. We reduce the problem to a non-self-adjoint Laplacian-like operator by using a spin up/down decomposition and rely on a transfer matrices technique.

Contents

1. Introduction

We study properties of relativistic (massive or not) charged particles with spin-1/2. We follow the Dirac formalism, see [Di]. We shall focus on the 1-dimensional discrete version of the problem. In the introduction we stick to the case of $\mathbb{Z}$ and shall discuss the case of $\mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$ in the core of the paper, see Section 5.1. The mass of the particle is given by $m \geq 0$. For simplicity, we re-normalize the speed of light and the reduced Planck constant by 1. The Dirac discrete operator, acting on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, is defined by

$$D_m := \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix},$$

where $d := \mathrm{Id} - \tau$ and τ is the right shift, defined by $\tau f(n) = f(n+1)$, for all $f \in \ell^2(\mathbb{Z}, \mathbb{C})$. The operator D_m is self-adjoint. Moreover, notice that

$$D_m^2 = \begin{pmatrix} \Delta + m^2 & 0 \\ 0 & \Delta + m^2 \end{pmatrix},$$

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where $\Delta f(n) := 2f(n) - f(n+1) - f(n-1)$. This yields that $\sigma(D_m^2) = [m^2, 4+m^2]$. To remove the square above D_m, we define the symmetry S on $\ell^2(\mathbb{Z}, \mathbb{C})$ by $Sf(n) := f(-n)$ and the unitary operator on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$

$$U := \begin{pmatrix} 0 & \mathrm{i}S \\ -\mathrm{i}S & 0 \end{pmatrix}. \tag{1.1}$$

Clearly $U = U^* = U^{-1}$. We have that $UD_mU = -D_m$. We infer that the spectrum of D_m is purely absolutely continuous (ac) and that

$$\sigma(D_m) = \sigma_{\mathrm{ac}}(D_m) = [-\sqrt{m^2+4}, -m] \cup [m, \sqrt{m^2+4}].$$

We shall now perturb the operator by an electrical potential $V = (V_1, V_2)^t \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$. We set

$$H := D_m + \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}. \tag{1.2}$$

Here, V_i denotes also the operator of multiplication by the function V_i. Clearly, the essential spectrum of H is the same as that of D_m if V tends to 0 at infinity. We turn to more refined questions. The singular continuous spectrum, quantum transport, and localization have been studied before [CaOl, PrOl, COP, PrOl2, OlPr, OlPr2]. The question of the purely ac spectrum seems not to have been answered before. This is the purpose of our article.

We recall the following standard result for the Laplacian (the non-relativistic setting). For completeness we sketch the proof in Section 6.

Theorem 1.1. *Take $V \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ and $\nu \in \mathbb{Z}_+ \setminus \{0\}$ such that:*

$$\begin{aligned} &\lim_{n\to\pm\infty} V(n) = 0, \\ &V_{|\mathbb{Z}_+} - \tau^\nu V_{|\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}), \end{aligned} \tag{1.3}$$

then the spectrum of $\Delta + V$ is purely absolutely continuous on $(0, 4)$.

In the case of $\mathbb{Z}_+$ and for $\nu = 1$, the result has been essentially proved in [Wei] (in fact in the quoted reference, one focuses only on the continuous setting). The proof for the discrete setting can be found in [DoNe, Sim]. For $\nu > 1$, it seems that it was first done in [Sto]. Note that for instance, (1.3) is satisfied by potentials like $V(n) = (-1)^n W(n)$, where W is decay to 0. We refer to [GoNe] and to [KaLa] for recent results in this direction.

An amusing and easy remark is the difference between $\mathbb{Z}_+$ and $\mathbb{Z}$. In the latter, it is enough to assume the decay hypothesis on the right part of the potential. This reflects the fact that the particle can always escape to the right even if the left part of the potential would have given some singular continuous spectrum in a half-line setting.

We now turn to the main result of the paper. For simplicity we present the case of $\mathbb{Z}$ with electric perturbations. We refer to Section 3 for the main statements.

Theorem 1.2. *Take $V \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$ and $\nu \in \mathbb{Z}_+ \setminus \{0\}$ with:*

$$\lim_{n\to\pm\infty} V(n) = 0,$$

$$V_{|\mathbb{Z}_+} - \tau^\nu V_{|\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^2),$$

then the spectrum of H is purely absolutely continuous on $\left(-\sqrt{m^2+4}, -m\right) \cup \left(m, \sqrt{m^2+4}\right)$.

To study H we reduce the problem to a non-self-adjoint Laplacian-like operator which depends on the spectral parameter. This is due to a spin-up/down decomposition, see Proposition 4.2. This idea has been efficiently used in the continuous setting, e.g., [DES, BoGo, JeNe] and references therein, and seems to be new in the discrete setting. Then, we adapt the iterative process to the non-self-adjoint Laplacian-like operator and follow the presentation of [FHS]. We refer to [FHS2] for a recent survey about this technique.

Finally we present the organization of the paper. In section 2 we recall general facts about the free discrete Dirac operator and about hyperbolic geometry. Then in Section 3 we present the main results. Next in Section 4, we reduce the problem to a kind of Laplacian and adapt the transfer matrices technique. After that in Section 5, we prove the main results about absolutely continuous spectra. Finally we discuss briefly the case of the Laplacian.

Notation: We denote by $\mathcal{B}(X)$ the space of bounded operators acting on a Banach space X. Let $\mathbb{Z}_k := \mathbb{Z} \cap [k, +\infty[$ for $k \in \mathbb{Z}$ and $\mathbb{Z}_- := \mathbb{Z} \setminus \mathbb{Z}_+$. For $A, B \subset \mathbb{C}$, we set $A \Subset B$ if $\mathrm{cl}A \subset \mathrm{int}B$, where cl and int stand for closure and interior, respectively.

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2. General facts

2.1. **The spectrum of the discrete Dirac operator.** Let $\mathbb{Z}_k := \mathbb{Z} \cap [k, +\infty[$ for $k \in \mathbb{Z}$ and $\mathbb{G} \in \{\mathbb{Z}_k, \mathbb{Z}\}$. We define $d \in \mathcal{B}\left(\ell^2(\mathbb{G}, \mathbb{C})\right)$ by

$$\forall f \in \ell^2(\mathbb{G}, \mathbb{C}) \ df(n) := f(n) - f(n+1).$$

Clearly d is bounded. Its adjoint is given by

$$d^* f(n) = \begin{cases} f(n) & \text{if } \mathbb{G} = \mathbb{Z}_k \text{ and } n = k, \\ f(n) - f(n-1) & \text{otherwise,} \end{cases}$$

for all $f \in \ell^2(\mathbb{G}, \mathbb{C})$. Now for $m \geq 0$ we define the Dirac discrete operator on $\ell^2(\mathbb{G}, \mathbb{C}^2)$ by

$$D_m^{(\mathbb{G})} := \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix}.$$

It is easy to see that $D_m^{(\mathbb{G})}$ is self-adjoint. Let $\Delta^{(\mathbb{G})}$ be the Laplacian on $\ell^2(\mathbb{G}, \mathbb{C})$ defined by

$$\Delta^{(\mathbb{G})} f(n) := \begin{cases} f(n) - f(n+1) & \text{if } \mathbb{G} = \mathbb{Z}_k \text{ and } n = k, \\ 2f(n) - f(n-1) - f(n+1) & \text{otherwise,} \end{cases} \tag{2.1}$$

for all $f \in \ell^2(\mathbb{G}, \mathbb{C})$. We study first the Dirac discrete operator on $\mathbb{Z}$, we have

$$\left(D_m^{(\mathbb{Z})}\right)^2 = \begin{pmatrix} \Delta^{(\mathbb{Z})} + m^2 & 0 \\ 0 & \Delta^{(\mathbb{Z})} + m^2 \end{pmatrix}.$$

By Fourier transformation, we see that $\Delta^{(\mathbb{Z})}$ is non-negative and that its spectrum is $[0, 4]$. Therefore, the spectrum of $\left(D_m^{(\mathbb{Z})}\right)^2$ is $[m^2, 4 + m^2]$. Relying on (1.1), we obtain:

Proposition 2.1. *We have*

$$\sigma\left(D_m^{(\mathbb{Z})}\right) = \sigma_{\mathrm{ess}}\left(D_m^{(\mathbb{Z})}\right) = \sigma\left(D_m^{(\mathbb{Z}_+)}\right) = \sigma_{\mathrm{ess}}\left(D_m^{(\mathbb{Z}_+)}\right) = \left[-\sqrt{m^2+4}, -m\right] \cup \left[m, \sqrt{m^2+4}\right].$$

Proof. We have $\left(-D_m^{(\mathbb{Z})} - \lambda\right)^{-1} = U\left(D_m^{(\mathbb{Z})} - \lambda\right)^{-1} U$ so $\varphi\left(-D_m^{(\mathbb{Z})}\right) = U\varphi\left(D_m^{(\mathbb{Z})}\right)U$, for all φ Borel measurable. Therefore, $\sigma\left(D_m^{(\mathbb{Z})}\right) = \left[-\sqrt{m^2+4}, -m\right] \cup \left[m, \sqrt{m^2+4}\right]$. By writing $\mathbb{Z} = \mathbb{Z}_- \cup \mathbb{Z}_+$, we see easily that $\sigma_{\mathrm{ess}}(D_m^{(\mathbb{Z})}) = \sigma_{\mathrm{ess}}(D_m^{(\mathbb{Z}_+)})$. To conclude, a direct computation shows that $D_m^{(\mathbb{Z}_+)}$ has no eigenvalue. $\square$

2.2. **A few words about the Poincaré half-plane.** We shall use extensively some properties of the *Poincaré half-plane*. It is defined by:

$$\mathbb{H} := \{x + \mathrm{i}y \mid x \in \mathbb{R}, y > 0\}, \text{ endowed with the metric } ds^2 = \frac{dx^2 + dy^2}{y^2},$$

Recall that the geodesic distance is given by:

$$d_{\mathbb{H}}(z_1, z_2) = \cosh^{-1}\left(1 + \frac{1}{2}\frac{|z_1 - z_2|^2}{\Im z_1 \cdot \Im z_2}\right) \leq \frac{|z_1 - z_2|}{\sqrt{\Im(z_1)}\sqrt{\Im(z_2)}}. \tag{2.2}$$

We turn to the study of (hyperbolic-)contractions.

Lemma 2.2. *Given $a, b \in \mathrm{cl}(\mathbb{H})$ and $c > 0$, we set:*

$$\varphi_{a,b,c}(z) := -\left(a - (b + cz)^{-1}\right)^{-1}. \tag{2.3}$$

It is a contraction of $(\mathbb{H}, d_{\mathbb{H}})$.

Moreover, if $a, b \in \mathbb{H}$, then $\varphi_{a,b,c}$ is a strict contraction of $(\mathbb{H}, d_{\mathbb{H}})$. More precisely, we have

$$d_{\mathbb{H}}\left(\varphi_{a,b,c}(z_1), \varphi_{a,b,c}(z_2)\right) \leq \frac{1}{1 + \Im(a) \cdot \Im(b)} d_{\mathbb{H}}\left(z_1, z_2\right), \tag{2.4}$$

for all $z_1, z_2 \in \mathbb{H}$. Moreover we have

$$d_{\mathbb{H}}\left(\varphi_{a,b,c}(z_1), \varphi_{a,b,c}(z_2)\right) \leq \frac{\left(1 + \Im(b)|a|\right)^2}{\left(\Im(b) \cdot \Im(a)\right)^2},$$

for all $z_1, z_2 \in \mathbb{H}$.

Proof. First, since $z \mapsto cz$ is a hyperbolic isometry, it is enough to consider $\varphi_{a,b} := \varphi_{a,b,1}$. Then, using that $z \mapsto z + w$ is a contraction when $w \in \mathrm{cl}(\mathbb{H})$ and that $z \mapsto -1/z$ is an isometry, we obtain that $\varphi_{a,b}$ is a contraction.

We turn to the second statement. A direct computation yields that if $h \in \mathbb{H}$ then

$$-\left(h + \mathbb{H}\right)^{-1} = B_{|\cdot|}\left(\frac{\mathrm{i}}{2\Im(h)}, \frac{1}{2\Im(h)}\right) \subset \mathbb{H}. \tag{2.5}$$

Given $C > 0$, as in the proof of [FHS][Proposition 2.1], if $z_1, z_2 \in \mathbb{H}$ with $\min\left(|z_1|, |z_2|\right) \leq C$ note that

$$d_{\mathbb{H}}(z_1 + a, z_2 + a) \leq \frac{C}{C + \Im(a)} d_{\mathbb{H}}(z_1, z_2).$$

Since $z \mapsto -z^{-1}$ is an isometry of $\mathbb{H}$ and $z \mapsto -(b + z)^{-1}$ is a contraction of $\mathbb{H}$, we use (2.5) for $h = b$ to have that

$$\begin{aligned}
d_{\mathbb{H}}\left(\varphi_{a,b}(z_1), \varphi_{a,b}(z_2)\right) &= d_{\mathbb{H}}\left(a - (b + z_1)^{-1}, a - (b + z_2)^{-1}\right) \\
&\leq \frac{\left(\Im(b)\right)^{-1}}{\left(\Im(b)\right)^{-1} + \Im(a)} d_{\mathbb{H}}\left(-\left(b + z_1\right)^{-1}, -\left(b + z_2\right)^{-1}\right) \\
&\leq \frac{1}{1 + \Im(a) \cdot \Im(b)} d_{\mathbb{H}}\left(z_1, z_2\right),
\end{aligned}$$

for all $z_1, z_2 \in \mathbb{H}$. So we obtain (2.4). By (2.5) for $h = a$ we know that $-\left(a + \mathbb{H}\right)^{-1}$ is an Euclidean ball of diameter $\Im(a)^{-1}$, hence

$$|\varphi_{a,b}(z_1) - \varphi_{a,b}(z_2)| \leq \left(\Im(a)\right)^{-1}$$

for all $z_1, z_2 \in \mathbb{H}$. Given $C > 0$, if $z \in \mathbb{H}$ with $|z| \leq C$ we have

$$\Im\left(-\frac{1}{a+z}\right) \geq \frac{\Im(a)}{(C+|a|)^2}.$$

So if $z \in \mathbb{H}$, by (2.5) for $h = b$, $|-(b+z)^{-1}| \leq (\Im(b))^{-1}$, so

$$\Im\left(\varphi_{a,b}(z)\right) \geq \frac{\Im(a)}{\left((\Im(b))^{-1}+|a|\right)^2}.$$

So we obtain that

$$\begin{aligned} d_{\mathbb{H}}\left(\varphi_{a,b}(z_1), \varphi_{a,b}(z_2)\right) &= \cosh^{-1}\left(1+\frac{1}{2}\frac{|\varphi_{a,b}(z_1)-\varphi_{a,b}(z_2)|^2}{\Im(\varphi_{a,b}(z_1))\Im(\varphi_{a,b}(z_2))}\right) \leq \frac{|\varphi_{a,b}(z_1)-\varphi_{a,b}(z_2)|}{\sqrt{\Im(\varphi_{a,b}(z_1))}\sqrt{\Im(\varphi_{a,b}(z_2))}} \\ &\leq \frac{\left((\Im(b))^{-1}+|a|\right)^2}{(\Im(a))^2} = \frac{(1+\Im(b)|a|)^2}{(\Im(b)\cdot\Im(a))^2}, \end{aligned}$$

for all $z_1, z_2 \in \mathbb{H}$. □

We shall also need the following technical lemmata.

Lemma 2.3. *Suppose that $a, b \in \mathrm{cl}(\mathbb{H})$, $c > 0$, and $z \in \mathbb{H}$. We have:*

$$d_{\mathbb{H}}(\varphi_{a,b,c}(z), \mathrm{i}) \leq \left(\frac{(|b|+c|z|)^2}{c\Im z}+1\right)\frac{(|b|+c|z|)\left(|a|+(c\Im(z))^{-1}\right)}{\sqrt{c\Im(z)}}.$$

Proof. First we have:

$$|\varphi_{a,b,c}(z)| = \left|\frac{1}{a-(b+cz)^{-1}}\right| \leq \frac{1}{\Im\left(a-(b+cz)^{-1}\right)} \leq \frac{1}{\Im\left(-(b+cz)^{-1}\right)} = \frac{|b+cz|^2}{\Im(b+cz)} \leq \frac{(|b|+c|z|)^2}{c\Im(z)}.$$

Then we obtain:

$$\begin{aligned} \Im\left(\varphi_{a,b,c}(z)\right) &= \Im\left(-\frac{1}{a-(b+cz)^{-1}}\right) = \frac{\Im\left(a-(b+cz)^{-1}\right)}{\left|a-(b+cz)^{-1}\right|^2} \geq \frac{\Im\left(-(b+cz)^{-1}\right)}{\left(|a|+|b+cz|^{-1}\right)^2} \\ &\geq \frac{\Im(b+cz)}{|b+cz|^2}\left(|a|+\Im(b+cz)^{-1}\right)^{-2} \geq \frac{c\Im(z)}{(|b|+c|z|)^2\left(|a|+(c\Im(z))^{-1}\right)^2}. \end{aligned}$$

Finally with (2.2) we infer:

$$d_{\mathbb{H}}(\varphi_{a,b,c}(z), \mathrm{i}) \leq \frac{|\varphi_{a,b,c}(z)-\mathrm{i}|}{\sqrt{\Im\left(\varphi_{a,b,c}(z)\right)}} \leq \frac{|\varphi_{a,b,c}(z)|+1}{\sqrt{\Im\left(\varphi_{a,b,c}(z)\right)}},$$

which yields the result. □

Lemma 2.4. *For all $n \in \mathbb{Z}_1$ there exist*

$$A_n, B_n, C_n, D_n \in \mathbb{R}[X_1, Y_1, Z_1, \ldots, X_n, Y_n, Z_n],$$

such that $C_n(\omega) + D_n(\omega)\zeta \neq 0$ and

$$\varphi_{x_1,y_1,z_1} \circ \cdots \circ \varphi_{x_n,y_n,z_n}(\zeta) = -\frac{A_n(\omega)+B_n(\omega)\zeta}{C_n(\omega)+D_n(\omega)\zeta},$$

for all $\omega := (x_1, y_1, z_1, \ldots, x_n, y_n, z_n) \in \left(\mathrm{cl}(\mathbb{H})^2 \times \mathbb{R}_+^*\right)^n$ and $\zeta \in \mathbb{H}$.

We point out that the continuity of A_n, B_n, C_n, and D_n with respect to the coefficients will be crucial in Proposition 4.8. The proof is straightforward. We give it for completeness.

Proof. We prove the result by induction. Let $n = 1$, we have

$$\varphi_{x,y,z}(\zeta) = -\Big(x - (y+z\zeta)^{-1}\Big)^{-1} = -\frac{y+z\zeta}{xy-1+xz\zeta} = -\frac{A_1(x,y,z)+B_1(x,y,z)\zeta}{C_1(x,y,z)+D_1(x,y,z)\zeta}$$

with

$$A_1(x,y,z) := y,\ B_1(x,y,z) := z,\ C_1(x,y,z) := xy-1, \text{ and } D_1(x,y,z) := xz.$$

Moreover

$$C_1(x,y,z) + D_1(x,y,z)\zeta = xy - 1 + xz\zeta = (y+z\zeta)\left(x - \frac{1}{y+z\zeta}\right) \neq 0$$

for all $(x, y, z) \in \mathrm{cl}(\mathbb{H})^2 \times \mathbb{R}_+^*$ and $\zeta \in \mathbb{H}$ because this is the product of two elements of $\mathbb{H}$.

Now suppose that we have proved the existence of A_n, B_n, C_n, D_n and prove the existence of A_{n+1}, B_{n+1}, C_{n+1}, and D_{n+1}. Let $\omega := (x_1, y_1, z_1, \ldots, x_{n+1}, y_{n+1}, z_{n+1}) \in \big(\mathrm{cl}(\mathbb{H})^2 \times \mathbb{R}_+^*\big)^{n+1}$ and $\zeta \in \mathbb{H}$. Let $\tilde{\omega} := (x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$, we have

$$\begin{aligned}
&\varphi_{x_1,y_1,z_1} \circ \cdots \circ \varphi_{x_{n+1},y_{n+1},z_{n+1}}(\zeta) = \varphi_{x_1,y_1,z_1} \circ \cdots \circ \varphi_{x_n,y_n,z_n}(\varphi_{x_{n+1},y_{n+1},z_{n+1}}(\zeta)) \\
&\qquad = -\frac{A_n(\tilde{\omega}) - B_n(\tilde{\omega})\dfrac{A_1(x_{n+1},y_{n+1},z_{n+1}) + B_1(x_{n+1},y_{n+1},z_{n+1})\zeta}{C_1(x_{n+1},y_{n+1},z_{n+1}) + D_1(x_{n+1},y_{n+1},z_{n+1})\zeta}}{C_n(\tilde{\omega}) - D_n(\tilde{\omega})\dfrac{A_1(x_{n+1},y_{n+1},z_{n+1}) + B_1(x_{n+1},y_{n+1},z_{n+1})\zeta}{C_1(x_{n+1},y_{n+1},z_{n+1}) + D_1(x_{n+1},y_{n+1},z_{n+1})\zeta}} = -\frac{A_{n+1}(\omega) + B_{n+1}(\omega)\zeta}{C_{n+1}(\omega) + D_{n+1}(\omega)\zeta},
\end{aligned}$$

with

$$\begin{aligned}
A_{n+1}(\omega) &:= A_n(\tilde{\omega})C_1(x_{n+1},y_{n+1},z_{n+1}) - B_n(\tilde{\omega})A_1(x_{n+1},y_{n+1},z_{n+1}),\\
B_{n+1}(\omega) &:= A_n(\tilde{\omega})D_1(x_{n+1},y_{n+1},z_{n+1}) - B_n(\tilde{\omega})B_1(x_{n+1},y_{n+1},z_{n+1}),\\
C_{n+1}(\omega) &:= C_n(\tilde{\omega})C_1(x_{n+1},y_{n+1},z_{n+1}) - D_n(\tilde{\omega})A_1(x_{n+1},y_{n+1},z_{n+1}),\\
D_{n+1}(\omega) &:= C_n(\tilde{\omega})D_1(x_{n+1},y_{n+1},z_{n+1}) - D_n(\tilde{\omega})B_1(x_{n+1},y_{n+1},z_{n+1}).
\end{aligned}$$

Finally since $\varphi_{x_{n+1},y_{n+1},z_{n+1}}$ is a contraction by Lemma 2.2, we have that $\varphi_{x_{n+1},y_{n+1},z_{n+1}}(\zeta) \in \mathbb{H}$ and

$$\begin{aligned}
C_{n+1}(\omega) + D_{n+1}(\omega)\zeta = \big(&C_n(\tilde{\omega}) + D_n(\tilde{\omega})\varphi_{x_{n+1},y_{n+1},z_{n+1}}(\zeta)\big)\\
&\times (C_1(x_{n+1},y_{n+1},z_{n+1}) + D_1(x_{n+1},y_{n+1},z_{n+1})\zeta) \neq 0,
\end{aligned}$$

by induction. This finishes the proof. □

3. Main results

In this paper, we study a Jacobi-like version of $D_m^{(\mathbb{G})} + V$, given $V = (V_1, V_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{R}^2)$ and $W = (W_1, W_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{C}^2)$, let

$$H^{(\mathbb{G})}_{m,V,W} := \begin{pmatrix} m + V_1 & \tilde{d} \\ \tilde{d}^* & -m + V_2 \end{pmatrix}, \tag{3.1}$$

where $\tilde{d} := d + W_1 + W_2\tau$ with $\tau f(n) := f(n+1)$. We stress that we allow W_1 to have non-purely imaginary values. This is not merely a magnetic perturbation.

We start with the main result on $\mathbb{Z}_+$. It will be proved in Section 5.1.

Theorem 3.1. *Take $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^2)$ and $W \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$ with:*

$$\lim_{n\to\infty} V(n) = \lim_{n\to\infty} W(n) = 0, \tag{3.2}$$

then $\sigma_{\mathrm{ess}}(H^{(\mathbb{Z}_+)}_{m,V,W}) = \left[-\sqrt{m^2+4}, -m\right] \cup \left[m, \sqrt{m^2+4}\right]$. *Assuming also that there exist* $\nu_1, \nu_2 \in \mathbb{Z}_+ \setminus \{0\}$ *such that*

$$\begin{aligned} &W_1(n) \neq -1 \text{ and } W_2(n) \neq 1, \text{ for all } n \in \mathbb{Z}_+, \\ &V - \tau^{\nu_1} V \in \ell^1(\mathbb{Z}_+, \mathbb{R}^2) \text{ and } W - \tau^{\nu_2} W \in \ell^1(\mathbb{Z}_+, \mathbb{C}^2), \end{aligned} \tag{3.3}$$

then the spectrum of $H^{(\mathbb{Z}_+)}_{m,V,W}$ *is purely absolutely continuous on* $\left(-\sqrt{m^2+4}, -m\right) \cup \left(m, \sqrt{m^2+4}\right)$.

We discuss briefly the necessity of the first line of (3.3).

Remark 3.2. *Assume that there is* n_0 *such that* $W_2(n_0) = 1$, *then the operator is a direct sum of* $H^{(\mathbb{Z}_{n_0})}_{m,V,W}$ *and of a finite matrix. Therefore, it is easy to construct embedded eigenvalues for this operator and this is an obstruction for the result of the theorem.*

We turn to the case of $\mathbb{Z}$, we have this important symmetry of charge:

$$U H^{(\mathbb{Z})}_{m,V,W} U = -H^{(\mathbb{Z})}_{m,(-SV_2,-SV_1)^t,\left(S\overline{W_1},\tau S\overline{W_2}\right)^t}, \tag{3.4}$$

where U is given by (1.1).

Theorem 3.3. *Take* $V \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$ *and* $W \in \ell^\infty(\mathbb{Z}, \mathbb{C}^2)$ *with:*

$$\lim_{n\to\pm\infty} V(n) = \lim_{n\to\pm\infty} W(n) = 0,$$

then $\sigma_{\mathrm{ess}}(H^{(\mathbb{Z})}_{m,V,W}) = \left[-\sqrt{m^2+4}, -m\right] \cup \left[m, \sqrt{m^2+4}\right]$. *Assuming also that there exists* $\nu_1, \nu_2 \in \mathbb{Z}_+ \setminus \{0\}$ *such that*

$$\begin{aligned} &W_1(n) \neq -1 \text{ and } W_2(n) \neq 1, \text{ for all } n \in \mathbb{Z}, \\ &V_{|\mathbb{Z}_+} - \tau^{\nu_1} V_{|\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{R}^2) \text{ and } W_{|\mathbb{Z}_+} - \tau^{\nu_2} W_{|\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{C}^2), \end{aligned} \tag{3.5}$$

then the spectrum of $H^{(\mathbb{Z})}_{m,V,W}$ *is purely absolutely continuous on* $\left(-\sqrt{m^2+4}, -m\right) \cup \left(m, \sqrt{m^2+4}\right)$.

Remark 3.4. *Note that in Theorem 3.3, supposing alternatively*

$$V_{|\mathbb{Z}_-} - \tau^{\nu_1} V_{|\mathbb{Z}_-} \in \ell^1(\mathbb{Z}_-, \mathbb{R}^2) \text{ and } W_{|\mathbb{Z}_-} - \tau^{\nu_2} W_{|\mathbb{Z}_-} \in \ell^1(\mathbb{Z}_-, \mathbb{C}^2), \tag{3.6}$$

gives the same result by using the transformation U.

4. A Laplacian-like approach

4.1. **Another form for the resolvent.** The objective is to reduce the analysis of the operator $H^{(\mathbb{G})}_{m,V,W}$ on $\ell^2(\mathbb{G}, \mathbb{C}^2)$ to that of two operators which are similar to a Laplacian. Take $\lambda \in \mathbb{H}$, $V = (V_1, V_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{R}^2)$, and $W = (W_1, W_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{C}^2)$, we define

$$\Delta^{(\mathbb{G})}_{1,m,\lambda,V,W} := \tilde{d} \frac{1}{\lambda + m - V_2} \tilde{d}^* - (\lambda - m - V_1)$$

and

$$\Delta^{(\mathbb{G})}_{2,m,\lambda,V,W} := \tilde{d}^* \frac{1}{\lambda - m - V_1} \tilde{d} - (\lambda + m - V_2). \tag{4.1}$$

We first check their invertibility.

Proposition 4.1. *Let* $\lambda \in \mathbb{H}$, $V = (V_1, V_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{R}^2)$, *and* $W = (W_1, W_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{C}^2)$, *then* $\Delta^{(\mathbb{G})}_{1,m,\lambda,V,W}$ *and* $\Delta^{(\mathbb{G})}_{2,m,\lambda,V,W}$ *are invertible.*

Proof. For $b \in \mathcal{B}\left(\ell^2(\mathbb{G}, \mathbb{C})\right)$, $X, Y \in \ell^\infty(\mathbb{G}, \mathbb{R})$ and $\mu \in \mathbb{H}$ let $A_{\mu,b,X,Y} := b^*(\mu - X)^{-1} b + Y$, then

$$\Im\langle f, A_{\mu,b,X,Y} f\rangle = -\Im(\mu) \left\| \frac{1}{|\mu - X|} bf \right\|^2 \leq 0, \tag{4.2}$$

for $f \in \ell^2(\mathbb{G}, \mathbb{C})$. With the Numerical Range Theorem (e.g., [BoGo][Lemma B.1]) we derive that we have $\mathbb{H} \subset \rho(A_{\mu,b,X,Y})$, the resolvent set of $A_{\mu,b,X,Y}$. Since

$$\Delta^{(\mathbb{G})}_{1,m,\lambda,V,W} = A_{\lambda,\tilde{d}^*,V_2-m,V_1+m} - \lambda \quad \text{and} \quad \Delta^{(\mathbb{G})}_{2,m,\lambda,V,W} = A_{\lambda,\tilde{d},V_1+m,V_2-m} - \lambda,$$

we get $\Delta^{(\mathbb{G})}_{1,m,\lambda,V}$ and $\Delta^{(\mathbb{G})}_{2,m,\lambda,V}$ are invertible. □

We give a kind of Schur's Lemma, so as to compute the inverse of the Dirac operator, see also [DES], [BoGo], and [JeNe] for some applications in the continuous setting.

Proposition 4.2. *Let $\lambda \in \mathbb{H}$, $V = (V_1, V_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{R}^2)$, and $W = (W_1, W_2)^t \in \ell^\infty(\mathbb{G}, \mathbb{C}^2)$. Then :*

$$\left(H^{(\mathbb{G})}_{m,V,W} - \lambda\right)^{-1} = \begin{pmatrix} (\Delta^{(\mathbb{G})}_{1,m,\lambda,V,W})^{-1} & 0 \\ 0 & (\Delta^{(\mathbb{G})}_{2,m,\lambda,V,W})^{-1} \end{pmatrix} \begin{pmatrix} 1 & \tilde{d}\frac{1}{\lambda+m-V_2} \\ \tilde{d}^*\frac{1}{\lambda-m-V_1} & 1 \end{pmatrix}.$$

Proof. We set $(H^{(\mathbb{G})}_{m,V,W} - \lambda)f = g$. This gives:

$$\begin{cases} (V_1 - \lambda + m)f_1 + \tilde{d}f_2 &= g_1 \\ \tilde{d}^* f_1 + (V_2 - \lambda - m)f_2 &= g_2 \end{cases}, \quad \begin{cases} f_1 &= \dfrac{1}{V_1 - \lambda + m}(g_1 - \tilde{d}f_2) \\ f_2 &= \dfrac{1}{V_2 - \lambda - m}(g_2 - \tilde{d}^* f_1) \end{cases}$$

$$\begin{cases} f_1 &= \dfrac{1}{V_1 - \lambda + m}\left(g_1 - \tilde{d}\dfrac{1}{V_2 - \lambda - m}(g_2 - \tilde{d}^* f_1)\right) \\ f_2 &= \dfrac{1}{V_2 - \lambda - m}\left(g_2 - \tilde{d}^*\dfrac{1}{V_1 - \lambda + m}(g_1 - \tilde{d}f_2)\right) \end{cases}$$

$$\begin{cases} \left(\tilde{d}\dfrac{1}{\lambda + m - V_2}\tilde{d}^* - (\lambda - m - V_1)\right) f_1 &= g_1 + \tilde{d}\dfrac{1}{\lambda + m - V_2} g_2 \\ \left(\tilde{d}^*\dfrac{1}{\lambda - m - V_1}\tilde{d} - (\lambda + m - V_2)\right) f_2 &= \tilde{d}^*\dfrac{1}{\lambda - m - V_1} g_1 + g_2 \end{cases}$$

Since $\Delta^{(\mathbb{G})}_{1,m,\lambda,V,W}$ and $\Delta^{(\mathbb{G})}_{2,m,\lambda,V,W}$ are invertible, we obtain the result. □

4.2. **Study of the truncated operator.** Note that in (4.1), if we forget about the terms in λ, V, and W, we obtain a Laplacian on $\mathbb{Z}_+$. Moreover, motivated by the results of Sections 5.1 and 5.2, it is enough to focus the analysis on the study of $\Delta^{(\mathbb{Z}_+)}_{2,m,\lambda,V,W}$. Therefore, we stress that we will not study $\Delta^{(\mathbb{Z}_+)}_{1,m,\lambda,V,W}$ at all. In fact, the latter leads to some technical complications and is less natural, i.e., it is not a direct analogue of the Laplacian on $\mathbb{Z}_+$.

As in [FHS], we reduce the problem to $\mathbb{Z}_k$ for $k \in \mathbb{Z}_+$. We define the truncated operator $\tilde{d}^{(n)} \in \mathcal{B}\left(\ell^2(\mathbb{Z}_n, \mathbb{C})\right)$ by

$$\tilde{d}^{(n)} f(k) := (1 + W_1(k))f(k) + (-1 + W_2(k))f(k+1)$$

for all $k \geq n$. Now we define $\Delta^{(n)}_{m,\lambda,V,W} \in \mathcal{B}\left(\ell^2(\mathbb{Z}_n, \mathbb{C})\right)$ by

$$\Delta^{(n)}_{m,\lambda,V,W} := \left(\tilde{d}^{(n)}\right)^* \frac{1}{\lambda - m - V_{1|_{\mathbb{Z}_n}}} \tilde{d}^{(n)} - (\lambda + m - V_{2|_{\mathbb{Z}_n}})$$

We point out that:

$$\Delta^{(0)}_{m,\lambda,V,W} = \Delta^{(\mathbb{Z}_+)}_{2,m,\lambda,V,W}.$$

Proposition 4.3. *Let $\lambda \in \mathbb{H}$, $V = (V_1, V_2)^t \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^2)$, and $W = (W_1, W_2)^t \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$, then $\Delta^{(n)}_{m,\lambda,V,W}$ is invertible for all $n \in \mathbb{Z}_+$.*

Proof. This is essentially the same proof as for Proposition 4.1. □

We study the related Green function:

$$\alpha_n := \left\langle \delta_n, \left(\Delta^{(n)}_{m,\lambda,V,W}\right)^{-1} \delta_n \right\rangle,$$

where $\delta_n(m) := 1$ if and only if $n = m$ and 0 otherwise. The objective is to bound α_0 independently of λ. We give the first property of α_n.

Proposition 4.4. *Take $\lambda \in \mathbb{H}$, $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^2)$, and $W \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$ then*

$$\alpha_n = \left\langle \delta_n, \left(\Delta^{(n)}_{m,\lambda,V,W}\right)^{-1} \delta_n \right\rangle \in \mathbb{H},$$

for all $n \in \mathbb{Z}_+$.

Proof. We have

$$\Im(\alpha_n) = \Im(\lambda) \left(\left\| \frac{1}{|\lambda - m - V_{1|_{\mathbb{Z}_n}}|} \tilde{d}^{(n)} \left(\Delta^{(n)}_{m,\lambda,V,W}\right)^{-1} \delta_n \right\|^2 + \left\| \left(\Delta^{(n)}_{m,\lambda,V,W}\right)^{-1} \delta_n \right\|^2 \right).$$

So $\Im(\alpha_n) > 0$ because $\lambda \in \mathbb{H}$ and $\left(\Delta^{(n)}_{m,\lambda,V,W}\right)^{-1} \delta_n \neq 0$. □

We follow the strategy of [FHS] and express α_n with the help of α_{n+1}. The aim is to use a fixed point argument in order to recover some bounds on α_0.

Proposition 4.5. *Take $\lambda \in \mathbb{H}$, $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^2)$, $W \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$ with $W_1(k) \neq -1$ for all $k \in \mathbb{Z}_+$, and $n \in \mathbb{Z}_+$. By setting*

$$\begin{aligned} \Phi_n(z) &:= \varphi_{a_n,b_n,c_n}(z) = -\left(a_n - (b_n + c_n z)^{-1}\right)^{-1} \\ a_n &:= \lambda + m - V_2(n) \in \mathbb{H} \\ b_n &:= (\lambda - m - V_1(n))|1 + W_1(n)|^{-2} \in \mathbb{H} \\ c_n &:= \left|\frac{1 - W_2(n)}{1 + W_1(n)}\right|^2 \in \mathbb{R}_+, \end{aligned} \tag{4.3}$$

we obtain $\alpha_n = \Phi_n(\alpha_{n+1})$.

Proof. We define in $\ell^2(\mathbb{Z}_n, \mathbb{C})$ and in $\ell^2(\mathbb{Z}_{n+1}, \mathbb{C})$

$$f := \left(\Delta^{(n)}_{m,\lambda,V,W}\right)^{-1} \delta_n \quad \text{and} \quad g := \left(\Delta^{(n+1)}_{m,\lambda,V,W}\right)^{-1} \delta_{n+1},$$

respectively. Clearly $\alpha_n = f(n)$ and $\alpha_{n+1} = g(n+1)$. By definition f is the unique solution in $\ell^2(\mathbb{Z}_n, \mathbb{C})$ of $\Delta^{(n)}_{m,\lambda,V,W} f = \delta_n$, i.e.,

$$\begin{aligned} &\frac{1 + \overline{W_1}(k)}{\lambda - m - V_1(k)} \left((1 + W_1(k))f(k) + (-1 + W_2(k))f(k+1)\right) \\ &+ \frac{1 - \overline{W_2}(k-1)}{\lambda - m - V_1(k-1)} \left((1 - W_2(k-1))f(k) + (-1 - W_1(k-1))f(k-1)\right) - (\lambda + m - V_2(k))f(k) = 0, \end{aligned} \tag{4.4}$$

for all $k \geq n+1$ and

$$\frac{1+\overline{W_1}(n)}{\lambda - m - V_1(n)} \left((1+W_1(n))f(n) + (-1+W_2(n))f(n+1)\right) - (\lambda + m - V_2(n))f(n) = 1. \tag{4.5}$$

We see that $f_{|\mathbb{Z}_{n+1}}$ is solution of (4.4) for all $k \geq n+2$ and

$$\begin{aligned}\frac{1+\overline{W_1}(n+1)}{\lambda - m - V_1(n+1)} &\left((1+W_1(n+1))f(n+1) + (-1+W_2(n+1))f(n+2)\right) \\ &- (\lambda + m - V_2(n+1))f(n+1) \\ &\qquad = \frac{1-\overline{W_2}(n)}{\lambda - m - V_1(n)} \left((1+W_1(n))f(n) + (-1+W_2(n))f(n+1)\right).\end{aligned}$$

So we obtain that

$$\Delta^{(n+1)}_{m,\lambda,V,W} f_{|\mathbb{Z}_{n+1}} = \frac{1-\overline{W_2}(n)}{\lambda - m - V_1(n)} \left((1+W_1(n))f(n) + (-1+W_2(n))f(n+1)\right) \delta_{n+1}.$$

Because $\Delta^{(n+1)}_{m,\lambda,V,W} g = \delta_{n+1}$ we have

$$\Delta^{(n+1)}_{m,\lambda,V,W} f_{|\mathbb{Z}_{n+1}} = \frac{1-\overline{W_2}(n)}{\lambda - m - V_1(n)} \left((1+W_1(n))f(n) + (-1+W_2(n))f(n+1)\right) \Delta^{(n+1)}_{m,\lambda,V,W} g$$

But $\Delta^{(n+1)}_{m,\lambda,V,W}$ is invertible, so

$$f_{|\mathbb{Z}_{n+1}} = \frac{1-\overline{W_2}(n)}{\lambda - m - V_1(n)} \left((1+W_1(n))f(n) + (-1+W_2(n))f(n+1)\right) g.$$

Note that

$$f(n+1) = \frac{1-\overline{W_2}(n)}{\lambda - m - V_1(n)} \left((1+W_1(n))f(n) + (-1+W_2(n))f(n+1)\right) g(n+1).$$

Straightforwardly, using (4.5) we conclude that $\alpha_n = f(n) = \Phi_n(g(n+1)) = \Phi_n(\alpha_{n+1})$. $\square$

4.3. **An iterative process.** The key to the process relies is the fact that Φ_n is a strict contraction.

Proposition 4.6. *Given $\lambda \in \mathbb{H}$, $n \in \mathbb{Z}_+$, $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$, and $W \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$ with $W_1(n) \neq -1$ and $W_2(n) \neq 1$. Then Φ_n is a strict contraction. More precisely, we have*

$$d_{\mathbb{H}}\left(\Phi_n(z_1), \Phi_n(z_2)\right) \leq \frac{1}{1 + (\Im(\lambda))^2 \left(1 + \|W_1\|_\infty\right)^{-1}} d_{\mathbb{H}}\left(z_1, z_2\right), \tag{4.6}$$

for all $z_1, z_2 \in \mathbb{H}$ and $n \in \mathbb{Z}_+$. Moreover we obtain

$$d_{\mathbb{H}}\left(\Phi_n(z_1), \Phi_n(z_2)\right) \leq \frac{\left(1 + \|a_n\|_\infty \|b_n\|_\infty\right)^2 \left(1 + \|W_1\|_\infty\right)^2}{(\Im(\lambda))^4},$$

for all $z_1, z_2 \in \mathbb{H}$ and $n \in \mathbb{Z}_+$.

Proof. Using Lemma 2.2, we obtain that $\Phi_n = \varphi_{a_n,b_n,c_n}$ is a strict contraction. More precisely, we get

$$d_{\mathbb{H}}\left(\Phi_n(z_1), \Phi_n(z_2)\right) \leq \frac{1}{1 + \Im(a_n)\Im(b_n)} d_{\mathbb{H}}\left(z_1, z_2\right) \leq \frac{1}{1 + (\Im(\lambda))^2 \left(1 + \|W_1\|_\infty\right)^{-1}} d_{\mathbb{H}}\left(z_1, z_2\right),$$

for all $z_1, z_2 \in \mathbb{H}$, and

$$d_{\mathbb{H}}\left(\Phi_n(z_1), \Phi_n(z_2)\right) \leq \frac{(1 + \Im(b_n)|a_n|)^2}{(\Im(b_n)\Im(a_n))^2} \leq \frac{\left(1 + \|a_n\|_\infty \|b_n\|_\infty\right)^2 \left(1 + \|W_1\|_\infty\right)^2}{(\Im(\lambda))^4},$$

for all $z_1, z_2 \in \mathbb{H}$. $\square$

Now we have an asymptotic property. That is an analogue of [FHS][Theorem 2.3]. It relies strongly on the fact that Φ_n is a strict contraction.

Corollary 4.7. *Take $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^2)$, and $W \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$ with $W_1(n) \neq -1$ and $W_2(n) \neq 1$ for all $n \in \mathbb{Z}_+$. Then for all $\lambda \in \mathbb{H}$ and $(\zeta_n)_n \in \mathbb{H}^{\mathbb{Z}_+}$ we have*

$$d_{\mathbb{H}}\text{-}\lim_{n\to\infty} \Phi_0 \circ \cdots \circ \Phi_n(\zeta_n) = \alpha_0.$$

Proof. With Proposition 4.4, for all $n \in \mathbb{Z}_+$ we have $\alpha_n \in \mathbb{H}$. With Proposition 4.6 there exist $\delta \in (0,1)$ and $\eta > 0$ such that

$$d_{\mathbb{H}}\left(\Phi_n(z_1), \Phi_n(z_2)\right) \leq \min\left(\delta d_{\mathbb{H}}\left(z_1, z_2\right), \eta\right),$$

for all $n \in \mathbb{Z}_+$ and $z_1, z_2 \in \mathbb{H}$. So, using that $\alpha_n = \Phi_n(\alpha_{n+1})$ for all $n \in \mathbb{Z}_+$, we obtain that for $n \in \mathbb{Z}_+$

$$\begin{aligned} d_{\mathbb{H}}\left(\Phi_0 \circ \cdots \circ \Phi_n(\zeta_n), \alpha_0\right) &= d_{\mathbb{H}}\left(\Phi_0 \circ \cdots \circ \Phi_n(\zeta_n), \Phi_0 \circ \cdots \circ \Phi_n(\alpha_{n+1})\right) \\ &\leq \delta^n d_{\mathbb{H}}\left(\Phi_n(\zeta_n), \Phi_n(\alpha_{n+1})\right) \leq \eta\delta^n. \end{aligned}$$

Therefore, $d_{\mathbb{H}}\text{-}\lim_{n\to\infty} \Phi_0 \circ \cdots \circ \Phi_n(\zeta_n) = \alpha_0$. □

From now on, set

$$\nu := \nu_1 \cdot \nu_2.$$

Now unlike in [FHS][Lemma 4.5] or in [FHS2][Proposition 3.4] we shall not rely directly on a fixed point of Φ_n but on one of $\Phi_n \circ \cdots \circ \Phi_{n+\nu-1}$. The proof is unfortunately more complicated but the improvement is real as we can treat potentials satisfying $V - \tau^{\nu_1} V \in \ell^1$. Recall that with the approach of [FHS], one covers only the case $\nu = 1$. We localize in energy and introduce:

$$K_{x_1,x_2,\varepsilon} := (x_1, x_2) + \mathrm{i}(0, \varepsilon). \tag{4.7}$$

Proposition 4.8. *Take $x \in \left(-\sqrt{m^2+4}, -m\right) \cup \left(m, \sqrt{m^2+4}\right)$, $\nu \in \mathbb{Z}_+ \setminus \{0\}$, and assume that* (3.2) *and* (3.3) *hold true, then there exist $x_1, x_2 \in \mathbb{R}$ such that $x \in (x_1, x_2)$ and $M_1, \varepsilon > 0$ so that*

$$d_{\mathbb{H}}\left(\alpha_0, \mathrm{i}\right) = d_{\mathbb{H}}\left(\left\langle \delta_0, \left(\Delta^{(0)}_{m,\lambda,V,W}\right)^{-1} \delta_0 \right\rangle, \mathrm{i}\right) \leq M_1$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$. In particular there exists $M_2 > 0$ such that

$$\left| \left\langle \delta_0, \left(\Delta^{(\mathbb{Z}_+)}_{2,m,\lambda,V,W}\right)^{-1} \delta_0 \right\rangle \right| \leq M_2$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$.

Proof. Using Lemma 2.4, there exist some polynomials $A, B_1, B_2, C \in \mathbb{R}[X_1, \ldots, X_{3\nu}]$ such that

$$\begin{aligned} \Phi_n \circ \cdots \circ \Phi_{n+\nu-1}(z) &= \varphi_{a_n,b_n,c_n} \circ \cdots \circ \varphi_{a_{n+\nu-1},b_{n+\nu-1},c_{n+\nu-1}}(z) \\ &= -\frac{C(\omega_{n,\lambda}) + B_2(\omega_{n,\lambda})z}{B_1(\omega_{n,\lambda}) + A(\omega_{n,\lambda})z}, \end{aligned}$$

for all $\lambda \in \mathbb{H}$ and $n \in \mathbb{Z}_+$, where

$$\omega_{n,\lambda} := (a_n, b_n, c_n, \ldots, a_{n+\nu-1}, b_{n+\nu-1}, c_{n+\nu-1}),$$

and where $B_1(\omega) + A(\omega)z \neq 0$ for all $\omega \in (\mathrm{cl}(\mathbb{H})^2 \times \mathbb{R}_+^*)^\nu$ and $z \in \mathbb{H}$.

We now work in a neighbourhood of x. First notice that the fixed points of $\varphi_{x+m,x-m,1}$ are given by:

$$-\frac{x-m}{2} \pm \frac{1}{2}\mathrm{i}\sqrt{\frac{x-m}{x+m}(4+m^2-x^2)}. \tag{4.8}$$

Then

$$\underbrace{\varphi_{x+m,x-m,1} \circ \cdots \circ \varphi_{x+m,x-m,1}}_{\nu \text{ times}} = -\frac{C(\omega_{\infty,x}) + B_2(\omega_{\infty,x})\cdot}{B_1(\omega_{\infty,x}) + A(\omega_{\infty,x})\cdot},$$

where $\omega_{\infty,x} := (x+m, x-m, 1, \dots, x+m, x-m, 1)$, has at least (4.8) as fixed points. As it is a homography it has exactly at most two fixed points. Note also that $A(\omega_{\infty,x}) \neq 0$, because there are two different fixed points.

Now we would like to study the fixed points of

$$R(\omega, z) := -\frac{C(\omega) + B_2(\omega)z}{B_1(\omega) + A(\omega)z}$$

with respect to z, for ω being in a neighbourhood of $\omega_{\infty,x}$. As the Inverse Function Theorem does not seem to apply we rely on a direct approach. Since A is continuous, there exists a neighbourhood Ω_1 of $\omega_{\infty,x}$ such that $A(\omega) \neq 0$ for all $\omega \in \Omega_1$. We define on Ω_1

$$Z(\omega) := -\frac{B_1(\omega) + B_2(\omega)}{2A(\omega)} + \frac{1}{2}\mathrm{i}\sqrt{4\frac{C(\omega)}{A(\omega)} - \left(\frac{B_1(\omega) + B_2(\omega)}{A(\omega)}\right)^2}, \tag{4.9}$$

where we have chosen the square root in order to guarantee that:

$$\Re\left(\sqrt{4\frac{C(\omega)}{A(\omega)} - \left(\frac{B_1(\omega) + B_2(\omega)}{A(\omega)}\right)^2}\right) \geq 0,$$

for all $\omega \in \Omega_1$. A direct computation gives that $Z(\omega)$ is a fixed point of $R(\omega, \cdot)$ on Ω_1. Since A, B_1, and B_2 are polynomials with real coefficients, we infer that $\Im(Z(\omega_{\infty,x})) \geq 0$, by the choice of the square root. On the other hand $Z(\omega_{\infty,x})$ belongs to (4.8). Therefore we infer that

$$Z(\omega_{\infty,x}) = -\frac{x-m}{2} + \frac{1}{2}\mathrm{i}\sqrt{\frac{x-m}{x+m}(4+m^2-x^2)} \in \mathbb{H}. \tag{4.10}$$

In particular, since A, B_1, B_2, and C are polynomials with real coefficients,

$$4\frac{C(\omega_{\infty,x})}{A(\omega_{\infty,x})} - \left(\frac{B_1(\omega_{\infty,x}) + B_2(\omega_{\infty,x})}{A(\omega_{\infty,x})}\right)^2 = \frac{x-m}{x+m}(4+m^2-x^2) > 0.$$

Therefore there exists a neighbourhood $\Omega_2 \subset \Omega_1$ of $\omega_{\infty,x}$ such that

$$4\frac{C(\omega)}{A(\omega)} - \left(\frac{B_1(\omega) + B_2(\omega)}{A(\omega)}\right)^2 \notin \mathbb{R}_-, \text{ for all } \omega \in \Omega_2.$$

We infer that we can take the principal value of the square root in the definition of (4.9) when $\omega \in \Omega_2$. In particular, $Z \in \mathcal{C}^\infty(\Omega_2, \mathbb{C})$. Hence, recalling (4.10), there exists a compact neighbourhood $\Omega_3 \subset \Omega_2$ of $\omega_{\infty,x}$ and $\eta_1, M_1 > 0$ such that

$$\Im(Z(\omega)) > \eta_1 \text{ and } |Z(\omega)| \leq M_1,$$

for all $\omega \in \Omega_3$. Now there exist $x_1, x_2 \in \mathbb{R}$, $\varepsilon > 0$, and $n_0 \in \mathbb{Z}_+$ such that $x \in (x_1, x_2)$ and $\omega_{n,\lambda} \in \Omega_3$, for all $\lambda \in K_{x_1,x_2,\varepsilon}$ and $n \geq n_0$. We define now

$$z_n(\lambda) := Z(\omega_{n,\lambda})$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$ and $n \geq n_0$. Notice that

$$\Im(z_n(\lambda)) > \eta_1 \text{ and } |z_n(\lambda)| \leq M_1, \tag{4.11}$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$ and $n \geq n_0$. Moreover, by definition of Z we have

$$\Phi_n \circ \cdots \circ \Phi_{n+\nu-1}(z_n(\lambda)) = z_n(\lambda) \tag{4.12}$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$ and $n \geq n_0$. Next, there is $M_2 > 0$ such that

$$(|a_k - a_{k+\nu}| + |b_k - b_{k+\nu}| + |c_k - c_{k+\nu}|) \leq M_2 \left(\|V(k) - V(k+\nu)\| + \|W(k) - W(k+\nu)\|\right)$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$ and $k \in \mathbb{Z}_+$. Now since Z is $\mathcal{C}^\infty(\Omega_3, \mathbb{C})$ and Ω_3 is compact, there exists a Lipschitz constant $M_3 > 0$ such that

$$\begin{aligned}
|z_{n+\nu}(\lambda) - z_n(\lambda)| &\leq M_3 \|\omega_{n+\nu,\lambda} - \omega_{n,\lambda}\| \\
&\leq M_3 \left(|a_{n+\nu}(\lambda) - a_n(\lambda)| + |b_{n+\nu}(\lambda) - b_n(\lambda)| + |c_{n+\nu}(\lambda) - c_n(\lambda)| + \cdots \right.\\
&\left. + |a_{n+2\nu-1}(\lambda) - a_{n+\nu-1}(\lambda)| + |b_{n+2\nu-1}(\lambda) - b_{n+\nu-1}(\lambda)| + |c_{n+2\nu-1}(\lambda) - c_{n+\nu-1}(\lambda)|\right) \\
&\leq M_2 M_3 \sum_{k=0}^{\nu-1} \left(\|V(n+k+\nu) - V(n+k)\| + \|W(n+k+\nu) - W(n+k)\|\right) \qquad (4.13)
\end{aligned}$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$ and $n \geq n_0$. By Corollary 4.7, for all $\lambda \in K_{x_1,x_2,\varepsilon}$, we have:

$$\begin{aligned}
d_{\mathbb{H}}(\mathrm{i}, \alpha_0) &= \lim_{n\to\infty} d_{\mathbb{H}}(\mathrm{i}, \Phi_0 \circ \cdots \circ \Phi_{n_0+\nu(n+1)-1}(z_{n_0+\nu n}(\lambda))) \\
&\leq \lim_{n\to\infty} \Bigg(d_{\mathbb{H}}(\mathrm{i}, \Phi_0(\mathrm{i})) + \sum_{k=0}^{n_0+\nu-3} d_{\mathbb{H}}(\Phi_0 \circ \cdots \circ \Phi_k(\mathrm{i}), \Phi_0 \circ \cdots \circ \Phi_{k+1}(\mathrm{i})) \\
&\quad + d_{\mathbb{H}}(\Phi_0 \circ \cdots \circ \Phi_{n_0+\nu-2}(\mathrm{i}), \Phi_0 \circ \cdots \circ \Phi_{n_0+\nu-1}(z_{n_0})) \\
&\quad + \sum_{k=0}^{n-1} d_{\mathbb{H}}(\Phi_0 \circ \cdots \circ \Phi_{n_0+\nu(k+1)-1}(z_{n_0+\nu k}), \Phi_0 \circ \cdots \circ \Phi_{n_0+\nu(k+2)-1}(z_{n_0+\nu(k+1)})) \Bigg) \\
&\leq \sum_{k=0}^{n_0+\nu-2} d_{\mathbb{H}}(\mathrm{i}, \Phi_k(\mathrm{i})) + d_{\mathbb{H}}(\mathrm{i}, \Phi_{n_0+\nu-1}(z_{n_0})) \qquad (4.14) \\
&\quad + \sum_{k\geq 0} d_{\mathbb{H}}(z_{n_0+\nu k}, \Phi_{n_0+\nu(k+1)} \circ \cdots \circ \Phi_{n_0+\nu(k+2)-1}(z_{n_0+\nu(k+1)})) \\
&= \sum_{k=0}^{n_0+\nu-2} d_{\mathbb{H}}(\mathrm{i}, \Phi_k(\mathrm{i})) + d_{\mathbb{H}}(\mathrm{i}, \Phi_{n_0+\nu-1}(z_{n_0})) + \sum_{k\geq 0} d_{\mathbb{H}}(z_{n_0+\nu k}, z_{n_0+\nu(k+1)}) \qquad (4.15) \\
&\leq \sum_{k=0}^{n_0+\nu-2} d_{\mathbb{H}}(\mathrm{i}, \Phi_k(\mathrm{i})) + d_{\mathbb{H}}(\mathrm{i}, \Phi_{n_0+\nu-1}(z_{n_0})) + \sum_{k\geq 0} \frac{\left|z_{n_0+\nu k} - z_{n_0+\nu(k+1)}\right|}{\left(\Im(z_{n_0+\nu k})\right)^{1/2} \left(\Im(z_{n_0+\nu(k+1)})\right)^{1/2}}. \qquad (4.16)
\end{aligned}$$

Here in (4.14) we have used the fact that Φ_n is a contraction, in (4.15) we exploited (4.12), and in (4.16) we relied on (2.2).

Coming back to (4.3), one finds easily $M_4, \eta_4 > 0$ such that

$$\max\left(|a_n|, |b_n|\right) \leq M_4 \quad \text{and} \quad \eta_4 < c_n \leq M_4,$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$ and for all $n \in \mathbb{Z}_+$. Then, Lemma 2.3 ensures that

$$d_{\mathbb{H}}(\Phi_n(z), \mathrm{i}) \leq \left(\frac{(M_4 + M_4|z|)^2}{\eta_4 \Im(z)} + 1\right) \frac{(M_4|z| + M_4)\left(M_4 + (\eta_4 \Im(z))^{-1}\right)}{\sqrt{\eta_4 \Im(z)}}. \qquad (4.17)$$

for all $z \in \mathbb{H}$, $\lambda \in K_{x_1,x_2,\varepsilon}$, and $n \in \mathbb{Z}_+$. Finally combining (4.16) and estimates (4.11), (4.13), and (4.17), we infer:

$$\begin{aligned} d_{\mathbb{H}}(\alpha_0, \mathrm{i}) \leq & (n_0 + \nu - 1) \left(\frac{(2M_4)^2}{\eta_4} + 1 \right) \frac{2M_4 \left(M_4 + \eta_4^{-1} \right)}{\sqrt{\eta_4}} \\ & + \left(\frac{(M_4 + M_4 M_1)^2}{\eta_4 \eta_1} + 1 \right) \frac{(M_4 M_1 + M_4) \left(M_4 + (\eta_4 \eta_1)^{-1} \right)}{\sqrt{\eta_4 \eta_1}} \\ & + \frac{M_2 M_3}{\eta_1} \left(\|V - \tau^\nu V\|_1 + \|W - \tau^\nu W\|_1 \right), \end{aligned}$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$. The second point comes by recalling that $\Delta^{(0)}_{m,\lambda,V,W} = \Delta^{(\mathbb{Z}_+)}_{2,m,\lambda,V,W}$. $\square$

5. The absolutely continuous spectrum

We recall the following standard result, e.g., [ReSi][Theorem XIII.19].

Theorem 5.1. *Let H be a self-adjoint operator of $\mathcal{H}$, let (x_1, x_2) be an interval and $f \in \mathcal{H}$. Suppose*

$$\limsup_{\varepsilon \downarrow 0^+} \sup_{x \in (x_1,x_2)} \left| \langle f, (H - (x + \mathrm{i}\varepsilon))^{-1} f \rangle \right| < +\infty, \tag{5.1}$$

then the measure $\langle f, \mathbf{1}_{(\cdot)}(H) f \rangle$ is purely absolutely continuous w.r.t. the Lebesgue measure on (x_1, x_2).

5.1. The case of $\mathbb{Z}_+$.

In the previous section we have estimated the resolvent of $\Delta^{(\mathbb{Z}_+)}_{2,m,\lambda,V,W}$. Keeping in mind Proposition 4.2, we explain how to transfer the result to $H^{(\mathbb{Z}_+)}_{m,V,W}$. We start by reducing the study to a unique vector.

Lemma 5.2. *Given $A \subset \mathbb{C} \backslash \mathbb{R}$ bounded, $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^2)$, and $W \in \ell^\infty(\mathbb{Z}_+, \mathbb{C}^2)$ with $W_1(n) \neq -1$ and $W_2(n) \neq 1$ for all $n \in \mathbb{Z}_+$. Suppose that there exists $C_1 > 0$ such that*

$$\left| \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0 \right\rangle \right| \leq C_1,$$

for all $\lambda \in A$. Then for all $x_1, y_1, x_2, y_2 \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{Z}_+$ there exists $C_2 > 0$ such that

$$\left| \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \delta_{n_1}, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda \right)^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \delta_{n_2} \right\rangle \right| \leq C_2,$$

for all $\lambda \in A$.

Proof. Let

$$f := \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \overline{\lambda} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0.$$

We have clearly

$$|f_2(0)| = \left| \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0 \right\rangle \right| \leq C_1,$$

for all $\lambda \in A$. By definition, f is the unique solution in $\ell^2(\mathbb{Z}_+, \mathbb{C}^2)$ of

$$\begin{cases} (m + V_1(n) - \overline{\lambda}) f_1(n) + (1 + W_1(n)) f_2(n) + (-1 + W_2(n)) f_2(n+1) = 0 \\ (1 + \overline{W_1}(n)) f_1(n) + (-1 + \overline{W_2}(n-1)) f_1(n-1) + (-m + V_2(n) - \overline{\lambda}) f_2(n) = 0 \end{cases}$$

for all $n \geq 1$ and of

$$\begin{cases} (m + V_1(0) - \overline{\lambda}) f_1(0) + (1 + W_1(0)) f_2(0) + (-1 + W_2(0)) f_2(1) = 0 \\ (1 + \overline{W_1}(0)) f_1(0) + (-m + V_2(0) - \overline{\lambda}) f_2(0) = 1. \end{cases}$$

So by induction on $n \in \mathbb{Z}_+$ there exists $D_n > 0$ such that

$$|f_i(n)| \leq D_n,$$

for all $\lambda \in A$ and $i \in \{1,2\}$. Therefore we obtain that

$$\left|\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda\right)^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \delta_{n_2} \right\rangle\right| = \left|\left\langle \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \delta_{n_2}, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \overline{\lambda}\right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0 \right\rangle\right|$$
$$= |x_2 f_1(n_2) + y_2 f_2(n_2)| \leq (|x_2| + |y_2|) D_{n_2} =: C_2.$$

for all $\lambda \in A$. Now let

$$g := \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda\right)^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \delta_{n_2},$$

we have

$$|g_2(0)| = \left|\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda\right)^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \delta_{n_2} \right\rangle\right| \leq C_2.$$

for all $\lambda \in A$. By definition, g is the unique solution in $\ell^2(\mathbb{Z}_+, \mathbb{C}^2)$ of

$$\begin{cases} (m + V_1(n) - \lambda) g_1(n) + (1 + W_1(n)) g_2(n) + (-1 + W_2(n)) g_2(n+1) = x_2 \delta_{n_2}(n) \\ (1 + \overline{W_1}(n)) g_1(n) + (-1 + \overline{W_2}(n-1)) g_1(n-1) + (-m + V_2(n) - \lambda) g_2(n) = y_2 \delta_{n_2}(n) \end{cases}$$

for all $n \geq 1$ and of

$$\begin{cases} (m + V_1(0) - \lambda) g_1(0) + (1 + W_1(0)) g_2(0) + (-1 + W_2(0)) g_2(1) = x_2 \delta_{n_2}(0) \\ (1 + \overline{W_1}(0)) g_1(0) + (-m + V_2(0) - \lambda) g_2(0) = y_2 \delta_{n_2}(0). \end{cases}$$

So by induction for all $n \in \mathbb{Z}_+$ there exists $D'_n > 0$ such that

$$|g_i(n)| \leq D'_n,$$

for all $\lambda \in A$ and $i \in \{1,2\}$. Therefore we obtain that

$$\left|\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \delta_{n_1}, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda\right)^{-1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \delta_{n_2} \right\rangle\right| = |x_1 g_1(n_1) + y_1 g_2(n_1)| \leq (|x_1| + |y_1|) D'_{n_1} =: C_3,$$

for all $\lambda \in A$. This concludes the proof. □

We are now in position to conclude with our main result.

Proof of Theorem 3.1. Since $D^{(\mathbb{Z}_+)}_m - H^{(\mathbb{Z}_+)}_{m,V,W}$ is compact, then Weyl Theorem gives the first point. We prove the second one. Take $x \in \left(-\sqrt{m^2+4}, -m\right) \cup \left(m, \sqrt{m^2+4}\right)$. By Propositions 4.2 and 4.8 we have that there exists $\varepsilon, C > 0$ and $x_1, x_2 \in \mathbb{R}$ such that $x \in (x_1, x_2)$ and

$$\left|\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0, \left(H^{(\mathbb{Z}_+)}_{m,V,W} - \lambda\right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_0 \right\rangle\right| = \left|\left\langle \delta_0, (\Delta^{(\mathbb{Z}_+)}_{2,m,\lambda,V,W})^{-1} \delta_0 \right\rangle\right| \leq C$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$. Then Theorem 5.1 and Lemma 5.2 conclude by density. □

5.2. **The case of $\mathbb{Z}$.** Now we express $\Delta^{(\mathbb{Z})}_{i,m,\lambda,V,W}$ with the help of $\Delta^{(0)}_{m,\lambda,\cdot,\cdot}$.

Lemma 5.3. *Take $\lambda \in \mathbb{H}$, $V \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$, $W \in \ell^\infty(\mathbb{Z}, \mathbb{C}^2)$ with $W_1(0) \neq -1$ and $W_2(-1) \neq 1$. Set:*

$$\Phi(z_1, z_2) := -\left(a - (b + c z_1)^{-1} - (b' + c' z_2)^{-1}\right)^{-1}$$
$$a := \lambda + m - V_2(0) \in \mathbb{H},$$
$$b := (\lambda - m - V_1(0))|1 + W_1(0)|^{-2} \in \mathbb{H}, \quad b' := (\lambda - m - V_1(-1))|1 - W_2(-1)|^{-2} \in \mathbb{H},$$
$$c := \left|\frac{1 - W_2(0)}{1 + W_1(0)}\right|^2 \in \mathbb{R}^*_+, \qquad c' := \left|\frac{1 + W_1(-1)}{1 - W_2(-1)}\right|^2 \in \mathbb{R}^*_+,$$

we have

$$\left\langle \delta_0, \left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0\right\rangle = \Phi\left(\left\langle \delta_0, \Delta^{(0)}_{m,\lambda,(\tau V)_{|\mathbb{Z}_+},(\tau W)_{|\mathbb{Z}_+}}\delta_0\right\rangle, \left\langle \delta_0, \Delta^{(0)}_{m,\lambda,(\tau^2 SV_1,\tau SV_2)^t_{|\mathbb{Z}_+},(\tau-\tau^2 SW_2,-\tau^2 SW_1)^t_{|\mathbb{Z}_+}}\delta_0\right\rangle\right).$$

Proof. We define

$$f := \left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0,$$

$$g := \left(\Delta^{(0)}_{m,\lambda,(\tau V)_{|\mathbb{Z}_+},(\tau W)_{|\mathbb{Z}_+}}\right)^{-1}\delta_0 \text{ and } h := \left(\Delta^{(0)}_{m,\lambda,(\tau^2 SV_1,\tau SV_2)^t_{|\mathbb{Z}_+},(\tau-\tau^2 SW_2,-\tau^2 SW_1)^t_{|\mathbb{Z}_+}}\right)^{-1}\delta_0.$$

Clearly

$$\left\langle \delta_0, \left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0\right\rangle = f(0), \quad \left\langle \delta_0, \Delta^{(0)}_{m,\lambda,(\tau V)_{|\mathbb{Z}_+},(\tau W)_{|\mathbb{Z}_+}}\delta_0\right\rangle = g(0)$$
$$\left\langle \delta_0, \Delta^{(0)}_{m,\lambda,(\tau^2 SV_1,\tau SV_2)^t_{|\mathbb{Z}_+},(\tau-\tau^2 SW_2,-\tau^2 SW_1)^t_{|\mathbb{Z}_+}}\delta_0\right\rangle = h(0).$$

By definition f is the unique solution in $\ell^2(\mathbb{Z},\mathbb{C})$ of $\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}f = \delta_0$, i.e.,

$$\begin{aligned}&\frac{1+\overline{W_1}(n)}{\lambda-m-V_1(n)}\left((1+W_1(n))f(n)+(-1+W_2(n))f(n+1)\right)\\ &\quad+\frac{1-\overline{W_2}(n-1)}{\lambda-m-V_1(n-1)}\left((1-W_2(n-1))f(n)+(-1-W_1(n-1))f(n-1)\right)\\ &\quad-(\lambda+m-V_2(n))f(n) = \delta_0(n),\end{aligned} \tag{5.2}$$

for all $n\in\mathbb{Z}$. Let $f' := (\tau f)_{|\mathbb{Z}_+}$, we see that f' is solution of

$$\begin{aligned}&\frac{1+\overline{W_1}(n+1)}{\lambda-m-V_1(n+1)}\left((1+W_1(n+1))f'(n)+(-1+W_2(n+1))f'(n+1)\right)\\ &\quad+\frac{1-\overline{W_2}(n)}{\lambda-m-V_1(n)}\left((1-W_2(n))f'(n)+(-1-W_1(n))f'(n-1)\right)\\ &\quad-(\lambda+m-V_2(n+1))f'(n) = 0\end{aligned}$$

for all $n\geq 1$ and

$$\begin{aligned}&\frac{1+\overline{W_1}(1)}{\lambda-m-V_1(1)}\left((1+W_1(1))f'(0)+(-1+W_2(1))f'(1)\right)-(\lambda+m-V_2(1))f'(0)\\ &\quad=\frac{1-\overline{W_2}(0)}{\lambda-m-V_1(0)}\left((1+W_1(0))f(0)+(-1+W_2(0))f(1)\right).\end{aligned}$$

So we obtain that

$$\begin{aligned}\Delta^{(0)}_{m,\lambda,(\tau V)_{|\mathbb{Z}_+},(\tau W)_{|\mathbb{Z}_+}}f' &= \frac{1-\overline{W_2}(0)}{\lambda-m-V_1(0)}\left((1+W_1(0))f(0)+(-1+W_2(0))f(1)\right)\delta_0\\ &= \frac{1-\overline{W_2}(0)}{\lambda-m-V_1(0)}\left((1+W_1(0))f(0)+(-1+W_2(0))f(1)\right)\Delta^{(0)}_{m,\lambda,(\tau V)_{|\mathbb{Z}_+},(\tau W)_{|\mathbb{Z}_+}}g.\end{aligned}$$

Since $\Delta^{(0)}_{m,\lambda,(\tau V)_{|\mathbb{Z}_+},(\tau W)_{|\mathbb{Z}_+}}$ is invertible, we get

$$\frac{1-\overline{W_2}(0)}{\lambda-m-V_1(0)}\left((1+W_1(0))f(0)+(-1+W_2(0))f(1)\right)g(0)=f'(0)=f(1). \tag{5.3}$$

Now let $f'':=(\tau Sf)_{|\mathbb{Z}_+}$ We see that f'' is solution of

$$\begin{aligned}&\frac{1+\overline{W_1}(-n-1)}{\lambda-m-V_1(-n-1)}\left((1+W_1(-n-1))f''(n)+(-1+W_2(-n-1))f''(n-1)\right)\\&\qquad+\frac{1-\overline{W_2}(-n-2)}{\lambda-m-V_1(-n-2)}\left((1-W_2(-n-2))f''(n)+(-1-W_1(-n-2))f''(n+1)\right)\\&\qquad\qquad-(\lambda+m-V_2(-n-1))f''(n)=0\end{aligned}$$

for all $n\geq 1$ and

$$\begin{aligned}&\frac{1-\overline{W_2}(-2)}{\lambda-m-V_1(-2)}\left((1-W_2(-2))f''(0)+(-1-W_1(-2))f''(1)\right)-(\lambda+m-V_2(-1))f''(0)\\&\qquad=\frac{1+\overline{W_1}(-1)}{\lambda-m-V_1(-1)}\left((1-W_2(-1))f(0)+(-1-W_1(-1))f(-1)\right).\end{aligned}$$

By setting $\tilde{\Delta}:=\Delta^{(0)}_{m,\lambda,(\tau^2SV_1,\tau SV_2)^t_{|\mathbb{Z}_+},(\tau-\tau^2SW_2,-\tau^2SW_1)^t_{|\mathbb{Z}_+}}$, we obtain

$$\begin{aligned}\tilde{\Delta}f''&=\frac{1+\overline{W_1}(-1)}{\lambda-m-V_1(-1)}\left((1-W_2(-1))f(0)+(-1-W_1(-1))f(-1)\right)\delta_0\\&=\frac{1+\overline{W_1}(-1)}{\lambda-m-V_1(-1)}\left((1-W_2(-1))f(0)+(-1-W_1(-1))f(-1)\right)\tilde{\Delta}h.\end{aligned}$$

Since $\tilde{\Delta}$ is invertible, we infer

$$\frac{1+\overline{W_1}(-1)}{\lambda-m-V_1(-1)}\left((1-W_2(-1))f(0)+(-1-W_1(-1))f(-1)\right)h(0)=f''(0)=f(-1). \tag{5.4}$$

Straightforwardly, using (5.2) for $n=0$, (5.3) and (5.4) we have that $f(0)=\Phi(g(0),h(0))$. □

Now with this Lemma we can obtain that $\left\langle\delta_0,\left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0\right\rangle$ is bounded independently of λ with the Proposition 4.8.

Corollary 5.4. *Take $x\in\left(-\sqrt{m^2+4},-m\right)\cup\left(m,\sqrt{m^2+4}\right)$, $V\in\ell^\infty(\mathbb{Z},\mathbb{R}^2)$, and $W\in\ell^\infty(\mathbb{Z},\mathbb{C}^2)$ with $W_1(n)\neq-1$ and $W_2(n)\neq 1$, for all $n\in\mathbb{Z}$. Suppose that there exists $\nu\in\mathbb{Z}_+\setminus\{0\}$ such that* (3.5) *or* (3.6) *holds true. Then there exist $C,\varepsilon>0$ and $x_1,x_2\in\mathbb{R}$ such that $x\in(x_1,x_2)$ and*

$$\left|\left\langle\delta_0,\left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0\right\rangle\right|\leq C$$

for all $\lambda\in K_{x_1,x_2,\varepsilon}$.

Proof. Let

$$\begin{aligned}\alpha_\lambda&:=\left\langle\delta_0,\Delta^{(0)}_{m,\lambda,(\tau V)_{|\mathbb{Z}_+},(\tau W)_{|\mathbb{Z}_+}}\delta_0\right\rangle\in\mathbb{H},\\ \alpha'_\lambda&:=\left\langle\delta_0,\Delta^{(0)}_{m,\lambda,(\tau^2SV_1,\tau SV_2)^t_{|\mathbb{Z}_+},(\tau-\tau^2SW_2,-\tau^2SW_1)^t_{|\mathbb{Z}_+}}\delta_0\right\rangle\in\mathbb{H}.\end{aligned}$$

With Lemma 5.3 we have

$$\left|\left\langle \delta_0, \left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0\right\rangle\right| = \left|a - (b + c\alpha_\lambda)^{-1} - (b' + c'\alpha'_\lambda)^{-1}\right|^{-1}$$
$$\leq \min\left(\left(\Im\left(-(b + c\alpha_\lambda)^{-1}\right)\right)^{-1}, \left(\Im\left(-(b' + c'\alpha'_\lambda)^{-1}\right)\right)^{-1}\right).$$

Now if $V_{|\mathbb{Z}_+} - \tau^{\nu_1}V_{|\mathbb{Z}_+}, W_{|\mathbb{Z}_+} - \tau^{\nu_2}W_{|\mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+, \mathbb{C}^2)$ with Proposition 4.8 there exist $C_1, \varepsilon > 0$ and $x_1, x_2 \in \mathbb{R}$ such that $x \in (x_1, x_2)$ and $d_{\mathbb{H}}(\alpha_\lambda, \mathrm{i}) \leq C_1$ for all $\lambda \in K_{x_1,x_2,\varepsilon}$, so there exists $C_2 > 0$ such that

$$\left|\left\langle \delta_0, \left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0\right\rangle\right| \leq \left(\Im\left(-(b + c\alpha_\lambda)^{-1}\right)\right)^{-1} \leq C_2$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$. Now if $V_{|\mathbb{Z}_-} - \tau^{\nu_1}V_{|\mathbb{Z}_-}, W_{|\mathbb{Z}_-} - \tau^{\nu_2}W_{|\mathbb{Z}_-} \in \ell^1(\mathbb{Z}_-, \mathbb{C}^2)$ with Proposition 4.8 there exist $C_1, \varepsilon > 0$ and $x_1, x_2 \in \mathbb{R}$ such that $x \in (x_1, x_2)$ and $d_{\mathbb{H}}(\alpha'_\lambda, \mathrm{i}) \leq C_1$ for all $\lambda \in K_{x_1,x_2,\varepsilon}$, so there exists $C_2 > 0$ such that

$$\left|\left\langle \delta_0, \left(\Delta^{(\mathbb{Z})}_{2,m,\lambda,V,W}\right)^{-1}\delta_0\right\rangle\right| \leq \left(\Im\left(-(b' + c'\alpha'_\lambda)^{-1}\right)\right)^{-1} \leq C_2$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$. □

Finally we conclude with the help of the symmetry of charge.

Proof of Theorem 3.3. Since $D^{(\mathbb{Z})}_m - H^{(\mathbb{Z})}_{m,V,W}$ is compact, then Weyl Theorem gives the first point. We turn to the second one. Let $n \in \mathbb{Z}$, let $x \in \left(-\sqrt{m^2+4}, -m\right) \cup \left(m, \sqrt{m^2+4}\right)$. Proposition 4.2 and Corollary 5.4 ensure that there exist $\varepsilon_1, C_1 > 0$ and $x_1, x_2 \in \mathbb{R}$ such that $x \in (x_1, x_2)$ and

$$\left|\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta_n, \left(H^{(\mathbb{Z})}_{m,V,W} - \lambda\right)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta_n\right\rangle\right| = \left|\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta_0, \left(H^{(\mathbb{Z})}_{m,\tau^n V,\tau^n W} - \lambda\right)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta_0\right\rangle\right|$$
$$= \left|\left\langle \delta_0, (\Delta^{(\mathbb{Z})}_{2,m,\lambda,\tau^n V,\tau^n W})^{-1}\delta_0\right\rangle\right| \leq C_1,$$

for all $\lambda \in K_{x_1,x_2,\varepsilon_1}$. Then, Theorem 5.1 yields that the measure

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta_n, \mathbf{1}_{(\cdot)}(H^{(\mathbb{Z}_+)}_{m,V,W})\begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta_n\right\rangle$$

is purely absolutely continuous on (x_1, x_2). Now we use U, see (3.4). Let

$$V' := (-S\tau^n V_2, -S\tau^n V_1)^t \quad \text{and} \quad W' := \left(S\tau^n\overline{W_1}, S\tau^{n-1}\overline{W_2}\right)^t,$$

there exist $\varepsilon_2, C_2 > 0$ and $x_3, x_4 \in \mathbb{R}$ such that $x \in (x_3, x_4)$ and

$$\left|\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta_n, \left(H^{(\mathbb{Z})}_{m,V,W} - \lambda\right)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta_n\right\rangle\right| = \left|\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta_0, \left(H^{(\mathbb{Z})}_{m,\tau^n V,\tau^n W} - \overline{\lambda}\right)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta_0\right\rangle\right|$$
$$= \left|\left\langle \begin{pmatrix} 0 \\ \delta_0 \end{pmatrix}, \left(H^{(\mathbb{Z})}_{m,V',W'} - (-\overline{\lambda})\right)^{-1}\begin{pmatrix} 0 \\ \delta_0 \end{pmatrix}\right\rangle\right|$$
$$= \left|\left\langle \delta_0, (\Delta^{(\mathbb{Z})}_{2,m,-\overline{\lambda},V',W'})^{-1}\delta_0\right\rangle\right| \leq C_2$$

for all $\lambda \in K_{x_3,x_4,\varepsilon_2}$. Again Theorem 5.1 gives that the measure

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta_n, \mathbf{1}_{(\cdot)}(H^{(\mathbb{Z}_+)}_{m,V,W})\begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta_n\right\rangle$$

is purely absolutely continuous (x_3, x_4). Finally, remembering that x is arbitrary and by an argument of density, we infer that $H^{(\mathbb{Z}_+)}_{m,V,W}$ has pure ac spectrum $\left(-\sqrt{m^2+4}, -m\right) \cup \left(m, \sqrt{m^2+4}\right)$. □

6. The case of the Laplacian

In this section we explain briefly how to adapt our proofs in order to prove Theorem 1.1. For the sake of clarity, we stick to the case $\Delta + V$ and compare our proof directly to [FHS].

We start by the case of $\mathbb{Z}_+$.

Theorem 6.1. *Take $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R})$ and $\tau \in \mathbb{Z}_+$ such that:*

$$\lim_{n\to+\infty} V(n) = 0, \tag{6.1}$$
$$V - \tau^\nu V \in \ell^1(\mathbb{Z}_+, \mathbb{R}),$$

then the spectrum of $\Delta + V$ is purely absolutely continuous on $(0, 4)$.

Apart from Proposition 6.3, our presentation is very close to that of [FHS]. We start with the truncated case. Set:

$$\alpha_n := \left\langle \delta_n, \left(\Delta^{(n)} + V_{|\mathbb{Z}_n} - \lambda\right)^{-1} \delta_n \right\rangle \in \mathbb{H},$$

where $\Delta^{(n)}$ is the Laplacian on $\mathbb{Z}_n$, see (2.1). As in Proposition 4.5 we have $\alpha_n = \Phi_n(\alpha_{n+1})$ with

$$\Phi_n(z) := \varphi_{\lambda - V(n),1,1}(z) = -\left(\lambda - V(n) - (1+z)^{-1}\right)^{-1}.$$

Note that Φ_n is a contraction of $\mathbb{H}$. However, unlike in Proposition 4.6, this is not a strict contraction. However, $\Phi_n \circ \Phi_{n+1}$ is a strict contraction, see also [FHS][Proposition 2.1]. This infers:

Proposition 6.2. *Take $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R})$, then for all $\lambda \in \mathbb{H}$ and $(\zeta_n)_n \in \mathbb{H}^{\mathbb{Z}_+}$ we have*

$$d_{\mathbb{H}}\text{–}\lim_{n\to\infty} \Phi_0 \circ \cdots \circ \Phi_n(\zeta_n) = \alpha_0.$$

Proof. See the proof of Corollary 4.7 and [FHS][Theorem 2.3]. □

Now unlike in [FHS][Lemma 4.5] or in [FHS2][Proposition 3.4] but as in Proposition 4.8 we use the fixed point of $\Phi_n \circ \cdots \circ \Phi_{n+\nu-1}$. We obtain:

Proposition 6.3. *Take $x \in (0, 4)$, $V \in \ell^\infty(\mathbb{Z}_+, \mathbb{R})$, and $\nu \in \mathbb{Z}_+ \setminus \{0\}$ with $\lim_{n\to+\infty} V(n) = 0$ and $V - \tau^\nu V \in \ell^1(\mathbb{Z}_+, \mathbb{R})$. Then there exist $x_1, x_2 \in \mathbb{R}$ such that $x \in (x_1, x_2)$ and $M_1, \varepsilon > 0$ so that*

$$d_{\mathbb{H}}(\alpha_0, \mathrm{i}) \le M_1$$

for all $\lambda \in K_{x_1,x_2,\varepsilon}$.

Recall that $K_{x_1,x_2,\varepsilon}$ is defined in (4.7).

Proof. This is the same proof as in Proposition 4.8. We study the fixed points of $\Phi_n \circ \cdots \circ \Phi_{n+\nu}$ in a neighbourhood of

$$\omega_{\infty,x} := (x, 1, 1, \ldots, x, 1, 1).$$

The fixed points of $\varphi_{x,1,1}$ are

$$-\frac{1}{2} \pm \frac{1}{2}\mathrm{i}\sqrt{\frac{4}{x} - 1},$$

The rest remains the same. □

Finally Theorem 5.1 concludes the proof of Theorem 6.1.

We turn to the case of the line. As in Lemma 5.3, we reduce the problem to the case of $\mathbb{Z}_+$ because

$$\begin{aligned}\left|\left\langle \delta_0, \left(\Delta^{(\mathbb{Z})} + V - \lambda\right)^{-1} \delta_0 \right\rangle\right| &= \left|\left(\lambda - V(0) - (1+\alpha_\lambda)^{-1} - (1+\alpha'_\lambda)^{-1}\right)^{-1}\right| \\ &\le \frac{1}{\Im(-(1+\alpha_\lambda)^{-1})},\end{aligned}$$

where

$$\alpha_\lambda := \left\langle \delta_1, \left(\Delta^{(\mathbb{Z}_1)} + V_{|_{\mathbb{Z}_1}} - \lambda\right)^{-1} \delta_1 \right\rangle \in \mathbb{H}$$

$$\alpha'_\lambda := \left\langle \delta_{-1}, \left(\Delta^{(-\mathbb{Z}_1)} + V_{|_{-\mathbb{Z}_{-1}}} - \lambda\right)^{-1} \delta_{-1} \right\rangle \in \mathbb{H},$$

with $\Delta^{(\mathbb{Z}_1)}$ and $\Delta^{(-\mathbb{Z}_1)}$ the Laplacian on $\mathbb{Z}_1$ and $-\mathbb{Z}_1$ respectively. This gives Theorem 1.1.

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