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Méthodes algébriques dans l'analyse spectrale d'opérateurs sur les graphes et les variétés

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“L’imagination, c’est l’art de donner vie à ce qui n’existe pas, de persuader les autres d’accepter un monde qui n’est pas vraiment là.”

Paul Auster
Moon palace

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Sylvain Golénia

Abstract :

In this thesis, we use C^*-algebraical techniques aiming for applications in spectral theory.

In the first two articles, in the context of trees, we adapt the C^*-algebra methods to the study of the spectral and scattering theories of Hamiltonians of the system. We first consider a natural formulation and generalization of the problem in a Fock space context. We then get a Mourre estimate for the free Hamiltonian and its perturbations. Finally, we compute the quotient of a C^*-algebra of energy observables with respect to its ideal of compact operators. As an application, the essential spectrum of highly anisotropic Schrödinger operators is computed.

In the third article, we give powerful critera of stability of the essential spectrum of unbounded operators. We develop an abstract approch in the context of Banach modules. Our applications cover Dirac operators, perturbations of riemannian metrics, differential operators in divergence form. The main point of our approach is that no regularity conditions are imposed on the coefficients.

Table des matières :

Dans cette thèse, produit de techniques issues de la théorie des C^*-algèbres et de la théorie spectrale, nous établissons de nouveaux résultats concernant les propriétés spectrales d'opérateurs agissant sur les arbres et divers critères concernant la stabilité du spectre essentiel d'opérateurs non-bornés. Elle se compose de trois articles.

Les deux premiers [Gol, GG1] traitent de la théorie spectrale et de la diffusion des opérateurs de Schrödinger sur un arbre et de sa généralisation naturelle aux espaces de Fock. Les problèmes abordés sont : la validité de l'estimation de Mourre et la caractérisation du spectre essentiel d'opérateurs anisotropes par des méthodes C^*-algébriques. L'article [Gol] est par ailleurs accepté pour publication dans *Annals of Henri Poincaré*.

Dans le troisième article [GG2], nous nous proposons une recherche de critères de stabilité du spectre essentiel pour des opérateurs agissant sur des modules de Banach. Les applications couvrent les opérateurs de Dirac, les perturbations de métriques riemanniennes, les opérateurs sous forme divergence et bien d'autres. Outre son formalisme algébrique, ce travail est caractérisé par l'absence de conditions de régularité dans les hypothèses.

L'introduction se découpera donc en trois parties : la première regroupera les liens entre arbres et espaces de Fock et nous permettra d'exposer nos résultats relatifs à l'estimation de Mourre, la seconde abordera le problème de l'anisotropie de potentiels et enfin, dans la dernière, on dépeindra le fruit de notre formalisme dans la recherche de critères de stabilité du spectre essentiel.

Notations : Pour $\mathscr{H}, \mathscr{K}$ espaces de Banach, on notera par $\mathcal{B}(\mathscr{H}, \mathscr{K})$ l'espace des applications linéraires continues de $\mathscr{H}$ dans $\mathscr{K}$. Si $\mathscr{H}$ est égal à $\mathscr{K}$, on notera tout simplement cet ensemble par $\mathcal{B}(\mathscr{H})$. L'ensemble des applications linéraires compactes sur $\mathscr{H}$ sera, quant à lui, noté $\mathcal{K}(\mathcal{H})$. Tous les espaces de Hilbert seront par la suite supposés complexes. Pour A opérateur auto-adjoint sur l'espace de Hilbert $\mathscr{H}$, on notera par $\mathcal{D}(A)$ son domaine, par $\sigma(A)$ son spectre, $\sigma_{\text{ess}}(A)$ son spectre essentiel et par $\rho(A)$ son ensemble résolvant.

1 Arbres, espaces de Fock et estimations de Mourre

1.1 Graphes et estimations de Mourre, généralités

Un *graphe* est un couple $\Gamma = (\mathcal{V}, \mathcal{E})$, où $\mathcal{V}$ est un ensemble au plus dénombrable et $\mathcal{E} \subset \mathscr{P}_2(V)$, l'ensemble des parties de $\mathcal{V}$ à deux éléments. On dit que les sommets $x, y \in \mathcal{V}$ sont voisins s'ils sont reliés par une arête, i.e. $\{x, y\} \in \mathcal{E}$. On le note alors par $x \leftrightarrow y$. On supposera par la suite que le nombre maximal de voisins possibles est fini et on identifiera Γ avec l'ensemble de ses sommets $\mathcal{V}$.

Pour deux sommets $x, y \in \Gamma$, on appelle *chemin de longueur* n reliant x à y une suite de sommets x_i tels que $x_0 = x$, $x_n = y$ et $x_i \leftrightarrow x_{i+1}$ pour $i = 0, \ldots, n-1$. La longueur minimale de ces chemins est appellé la distance de x à y, on la note $d(x, y)$. Un graphe Γ est dit *connexe* si pour tous couples de sommets $x, y \in \Gamma$, un chemin relie x à y. Sous cette hypothèse, (Γ, d) est un espace métrique. Un chemin de longueur $n \geq 3$ est appelé *cycle* s'il relie x à lui-même et si les x_i qui le composent sont deux à deux différents pour $i = 1, \ldots, n$.

On appelle *arbre* un graphe connexe sans cycle. Soit $\nu \in \mathbb{N}^*$. Si le nombre de voisins est ν pour tous les sommets, l'arbre est dit *homogène*. Si le nombre de voisins est ν pour tous les sommets sauf un, qui n'a que $\nu - 1$ sommets voisins, l'arbre est dit *enraciné* (en ce dernier point). Ce point est l'origine de l'arbre et sera noté e.

Soit l'espace de Hilbert complexe

$$\ell^2(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \sum_{x \in \Gamma} |f(x)|^2 < \infty\},$$

munit du produit scalaire $\langle f, g \rangle = \sum_{x \in \Gamma} \overline{f(x)} g(x)$. Le Laplacien Δ est l'opérateur borné auto-adjoint agissant sur les $f \in \ell^2(\Gamma)$ donné par :

$$(\Delta f)(x) = \sum_{y \leftrightarrow x} (f(y) - f(x)), \tag{1.1}$$

pour tous $x, y \in \Gamma$. On note par $V(Q)$ l'opérateur de multiplication sur $\ell^2(\Gamma)$ par la fonction $V : \Gamma \to \mathbb{R}$ et on appelle $H := \Delta + V(Q)$ l'opérateur de Schrödinger, Δ l'Hamiltonien libre et V le potentiel.

Nous sommes intéressés par l'étude des propriétés spectrales des opérateurs de Schrödinger sur les arbres et plus particulièrement, par la continuité absolue de la mesure spectrale et l'absence de spectre singulier continu. Nous utiliserons donc, à ces fins, la méthode des opérateurs conjugués initiée par E. Mourre dans [Mou]. On lui connaît depuis 20 ans de nombreuses applications en mécanique quantique (problème à N-corps) et en théorie des champs. Une exposition complète et bien référencée de la théorie se trouve dans [ABG]. Dans ce qui suit, nous essayerons d'en donner un bref aperçu.

Soit H un opérateur auto-adjoint sur un Hilbert $\mathscr{H}$ et J un ouvert dont l'adhérence est incluse dans le spectre $\sigma(H)$ de H. Le but est de trouver un opérateur auto-adjoint A tel qu'il existe $\alpha > 0$ et un opérateur compact K pour lequel l'inégalité suivante (prise au sens des formes) soit satisfaite :

$$E_H(J)[H, iA]E_H(J) \geq \alpha E_H(J) + K. \tag{1.2}$$

Ici $E_H(J)$ est la mesure spectrale de H prise en J et le commutateur a un sens sous des hypothèses convenables sur le couple (H, A). Un tel A est dit *conjugué à H* sur J. L'inégalité est dite *stricte* si $K = 0$. Une conséquence immédiate de (1.2) est l'existence d'un nombre au plus fini de valeurs propres de H dans J. Si $K = 0$, il n'y a aucune valeur propre dans J. Si de plus, H appartient à une certaine classe de régularité par rapport à A, notée $\mathscr{C}^{1,1}(A)$, on montre que le spectre singulier continu de H dans l'intervalle J est vide.

Dans la pratique, on montre généralement l'inégalité stricte pour un opérateur libre, ici Δ, puis on procède de façon perturbative. On peut ainsi mettre en évidence une classe de potentiels V, Δ-compacts (ici, compacts), tel que (1.2) soit satisfaite pour $H = \Delta + V$ et pour le même A que Δ. Pour trouver un opérateur A conjugué à Δ, on procède généralement par “tranformée de Fourier”. Dans cette nouvelle représentation, Δ devient alors un opérateur de multiplication $\varphi(Q)$ par une fonction φ que l'on prend dans $\mathscr{C}^1(\mathbb{R})$ pour fixer les idées. On choisit alors

$$A_\varphi := -1/2(\varphi'(Q)P + P\varphi'(Q)), \tag{1.3}$$

où $P = -id/dx$ et on obtient

$$[\varphi(Q), iA_\varphi] = \varphi'(Q)^2 \tag{1.4}$$

qui est strictement positif en dehors des points critiques de φ'. On choisit ensuite J en dehors de ces points critiques et en revenant dans la représentation initiale, par “transformation de Fourier inverse”, on obtient bien l'inégalité de Mourre stricte pour Δ sur J.

L'exemple le plus simple est celui du laplacien usuel sur $L^2(\mathbb{R})$. Il est défini comme l'unique extension auto-adjointe de l'opérateur défini par $\Delta f = -f''$ pour $f \in C_c^\infty(\mathbb{R})$. L'opérateur conjugué obtenu est alors $A = (PQ + QP)/2$ et on obtient $[\Delta, iA] = 2\Delta$, donc on a un estimation de Mourre stricte sur J si $\overline{J} \subset]0, \infty[$.

L'analyse de Mourre appliquée aux graphes est encore naissante. Nous trouvons principalement deux références : [AlF] pour le cas des arbres et

[BoS] pour celui de $\mathbb{Z}^d$. La deuxième référence est une application directe de la technique décrite plus haut.

Dans le cas d'un arbre homogène, dans [Sun] par exemple, on utilise une transformé de Fourier sphérique pour transformer le laplacien en un opérateur de multiplication. Cela permet de montrer la continuité absolue de la mesure spectrale de Δ et aussi de donner le spectre du laplacien. Cette diagonalisation a cependant un défaut majeur : elle ne permet pas de donner une écriture explicite de l'opérateur conjugué. Ce même problème se pose également dans le cas du demi-plan de Poincaré dans le cas continu.

Cette difficulté peut être levée en se réduisant à l'étude d'arbres enracinés. Ainsi, dans [AlF], en étudiant des sous espaces invariants et en se ramenant ainsi à une étude unidimensionnelle, les auteurs sont capables de donner, explicitement cette fois, un opérateur conjugué à Δ. Ils traitent aussi le cas où l'on perturbe avec un potentiel de classe $C^2(A)$ (au sens de [ABG]) ce qui est un peu plus général que $O(1/n^2)$ (n étant la distance sur le graphe jusqu'à l'origine). Dans [All], on traite aussi la question de la théorie de la diffusion.

Dans [GG1], nous adoptons un point de vue différent qui nous permet de simplifier et de généraliser considérablement les résultats de [AlF, All]. Notre technique ne repose plus sur une étude de sous espaces invariants mais sur le fait que le laplacien s'écrive, à facteur près, sous la forme $\Delta = U + U^*$ où U est une isométrie totalement non unitaire. La perturbation V considérée est de classe $\mathscr{C}^{1,1}(A)$ ce qui peut être vu, en première approximation, comme un potentiel en $O(1/n^{1+\varepsilon})$ pour $\varepsilon > 0$. Concernant l'optimalité du résultat, notons qu'une décroissance du type $O(1/n^{1-\varepsilon})$ permet l'existence d'un spectre ponctuel dense dans le spectre continu, voir [NaY].

1.2 Arbres et espaces de Fock

Si Γ est arbre enraciné, alors $\ell^2(\Gamma)$ peut être naturellement vu comme un espace de Fock à la Boltzmann (i.e. sans statistique).

Nous devons, pour commencer, remarquer la structure de monoïde sous-jacente à un arbre enraciné. En effet, soit A un ensemble constitué de ν éléments et soit

$$\Gamma = \bigcup_{n\geq 0} A^n, \tag{1.5}$$

où A^n est le n-ième produit cartésien de A avec lui même. Pour $n = 0$, A^0

est constitué d'un seul élément e. On notera un élément $x = (a_1, a_2, \dots, a_n) \in A^n$ par $x = a_1 a_2 \dots a_n$. Si de plus, $y = b_1 b_2 \dots b_m \in A^m$ alors $xy = a_1 a_2 \dots a_n b_1 b_2 \dots b_n \in A^{n+m}$ avec la convention que $xe = ex = x$. Cela donne à Γ sa structure naturelle de monoïde. On remarquera que la structure de graphe est liée à celle de monoïde par : $x \leftrightarrow y$ si et seulement s'il existe $a \in A$ tel que $y = xa$ ou $x = ya$. Ainsi, Γ est donc bien un arbre enraciné en e tel que le nombre de voisins d'un élément de $\Gamma \setminus \{e\}$ est $\nu + 1$. Quand $\nu = 2$, il s'agit de l'arbre binaire.

Nous injectons Γ dans $\ell^2(\Gamma)$ en identifiant $x \in \Gamma$ avec la fonction caractéristique de l'ensemble $\{x\}$. Ainsi Γ devient la base orthonormale canonique de $\ell^2(\Gamma)$. En particulier, les combinaisons linéaires d'éléments de Γ sont bien définies dans $\ell^2(\Gamma)$. Par exemple, $\sum_{a \in A} a$ appartient bien à $\ell^2(\Gamma)$ et à pour norme $\sqrt{\nu}$.

Grâce à la structure de monoïde de Γ, chaque élément v du sous espace linéaire engendré par Γ dans $\ell^2(\Gamma)$ définit deux opérateurs bornés λ_v et ρ_v sur $\ell^2(\Gamma)$, les opérateurs de multiplication à droite et à gauche par v. On voit alors facilement que si $v = \sum_{a \in A} a$ alors l'operateur adjoint ρ_v^* agit comme suit : si $x \in \Gamma$ alors $\rho_v^* x = x'$, où x' est l'unique élément de Γ tel que $x = x'a$ pour un certain $a \in A$, si $x \in \Gamma \setminus \{e\}$, et $x' = 0$ si $x = e$. Ainsi, le laplacien défini par (1.1) peut se réécrire par:

$$\Delta = \rho_v + \rho_v^* + e - (\nu + 1).$$

Pour la suite, nous n'inclurons plus les termes $e - (\nu + 1)$ parce que e est une fonction Γ à support égal à $\{e\}$, et sera considérée dès lors comme faisant partie d'un potentiel, et parce que $\nu + 1$ est un réel, qui a donc une contribution triviale sur le spectre. Il est aussi agréable de renormaliser Δ en remplaçant v par un vecteur de norme $1/2$, ce qui revient à considérer $v/(2\sqrt{\nu})$ au lieu de $v = \sum_{a \in A} a$.

Nous expliquons maintenant comment passer des arbres aux espaces de Fock. Si A, B sont deux ensembles, nous avons l'égalité (ou l'isomorphisme canonique) $\ell^2(A \times B) = \ell^2(A) \otimes \ell^2(B)$. Ainsi,

$$\ell^2(A^n) = \ell^2(A)^{\otimes n} \text{ si } n \geq 1 \text{ et } \ell^2(A^0) = \mathbb{C}.$$

Alors, puisque l'union dans (1.5) est disjointe, nous avons :

$$\ell^2(\Gamma) = \bigoplus_{n=0}^{\infty} \ell^2(A^n) = \bigoplus_{n=0}^{\infty} \ell^2(A)^{\otimes n}$$

qui est l'espace de Fock construit sur l'espace de Hilbert à "une particule" $\mathcal{H} = \ell^2(A)$. Par conséquent, on est naturellement amenés à considérer le cadre abstrait d'un espace de Hilbert complexe $\mathcal{H}$ et de l'espace de Fock associé $\mathscr{H}$:

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}. \tag{1.6}$$

Le fait que $\mathcal{H}$ puisse être de dimension infinie n'a pas importance ici. Nous la supposerons finie dans nos applications.

Il n'y a pas de vraie différence entre le modèle de l'arbre et celui de l'espace de Fock si ce n'est le fait que le premier est plus géométrique et le deuxième plus algébrique. En fait, si $\mathcal{H}$ est un espace de Hilbert muni d'une base orthonormée $A \subset \mathcal{H}$ alors l'arbre Γ associé à A peut s'identifier de façon canonique à la base Hilbertienne de $\mathscr{H}$ donnée par les vecteurs de la forme: $a_1 \otimes a_2 \cdots \otimes a_n$ avec $a_k \in A$. En d'autres termes, *la donnée d'un arbre est équivalente à celle d'un espace de Fock sur un espace de Hilbert de dimension finie muni d'une certaine base orthonormale*. Toutefois, ce choix d'une base donne plus de structure à l'espace de Fock : les notions de positivités, de localité intrinsèque à l'espace $\ell^2(\Gamma)$ sont manquantes dans le modèle de Fock, il n'y a pas non plus d'analogue aux espaces $\ell^p(\Gamma)$, etc. Nos résultats montrent néanmoins que cette structure spécifique à l'arbre n'est pas indispensable pour les propriétés spectrale et de diffusion.

Soit $u \in \mathcal{H}$ un vecteur de norme 1 et soit $U \equiv \rho_u : \mathscr{H} \to \mathscr{H}$ defini par $Uf = f \otimes u$ si $f \in \mathcal{H}^{\otimes n}$. Il est clair que U est une isométrie sur $\mathscr{H}$, elle est en fait une isométrie totalement non-unitaire, i.e. $\text{s-}\lim_{k\to\infty} U^k = 0$. Ceci sera d'ailleurs le point clef de notre approche. Nous nous intéresserons donc à l'opérateur auto-adjoint :

$$\Delta = \text{Re}\, U = \frac{1}{2}(U + U^*). \tag{1.7}$$

Notre but sera donc d'étudier les perturbations $H = \Delta + V$ où les conditions sur V seront suggérées par la structure d'espace de Fock de $\mathscr{H}$.

1.3 Opérateur de nombre et estimation de Mourre

Traduire le problème en termes d'espace de Fock ne le résout pas pour autant. Nous nous proposons même de résoudre un problème plus général en fait : étant donné un isométrie U sur un espace de Hilbert, peut-on trouver un opérateur conjugué explicite et simple à sa partie réele Δ ? Peut-on aussi décrire simplement les perturbations autorisées ?

Si U est unitaire, il n'y a pas d'espoir d'obtenir une solution générale à ce problème. En fait, pour la plus part des U, le spectre de Δ est purement singulier continue. D'un autre côté, si U est complètement non-unitaire, une construction simple et en un certain sens canonique pour un opérateur conjugué à Δ peut être menée.

Soit U une isométrie sur un Hilbert $\mathscr{H}$. On appelle *opérateur de nombre associé à une isométrie* U un opérateur auto-adjoint N sur $\mathscr{H}$ tel que $UNU^* = N - 1$. Voici les exemples les plus simples d'opérateurs de nombre.

Exemple 1.1 Soit $\mathscr{H} = \ell^2(\mathbb{Z})$ et $(Uf)(x) = f(x-1)$. Si $\{e_n\}$ est la base orthonormale canonique de $\mathscr{H}$ alors $Ue_n = e_{n+1}$. Il suffit de définir N par la condition $Ne_n = ne_n$. En fait, n'importe quel autre opérateur de nombre est de la forme $N + \lambda$ pour un certain $\lambda \in \mathbb{R}$. On a alors $[N, U] = U$ au sense des formes sur le domaine $\mathcal{D}(N)$ de N.

Exemple 1.2 Soit $\mathscr{H} = \ell^2(\mathbb{N})$ et U définie précédement. Alors $U^*e_n = e_{n-1}$ avec $e_{-1} = 0$, et soit $P_0 = |e_0\rangle\langle e_0|$, le projecteur orthogonal sur e_0. On obtient alors un opérateur de nombre en posant $Ne_n = (n+1)e_n$. Il est facile de voir que c'est la seule possibilité. On remarquera aussi que $\mathscr{H}$ un espace de Fock construit sur un espace vectoriel de dimension 1.

Il est ensuite facile de vérifier que, si S est la partie imaginaire de U, l'opérateur $A := (SN + NS)/2$ vérifie

$$[\Delta, iA] = 1 - \Delta^2. \tag{1.8}$$

Nous obtenons ainsi une estimation de Mourre (stricte) sur $[-a, a]$ pour chaque $a \in]0, 1[$.

L'intuition derrière cette construction est immédiate. Dans les exemples 1.1 et 1.2 les opérateurs Δ sont respectivement les laplaciens sur $\mathbb{Z}$ et $\mathbb{N}$, les discrétisés du laplacien usuel, et S est un opérateur de dérivation, l'analogue de $P = -i\frac{d}{dx}$ sur $\mathbb{R}$, alors il est naturel de chercher l'analogue de l'opérateur position Q et de $A = (PQ+QP)/2$. Il faut remarquer que nous n'avons pas recours à une transformée de Fourrier comme dans [AlF] ou dans [BoS].

Dans le cas unitaire l'existence de N est très restrictive. On montre alors que l'étude de U se ramène à celle de l'exemple 1.1. D'autre part, un fait remarquable est que pour une isométrie complètement non unitaire U, i.e.

telle que $\text{s-}\lim_{k\to\infty} U^k = 0$, N existe et est unique. En fait, formellement, comme $N = 1 + UNU^*$, on obtient par itération que $N = 1 + UU^* + U^2U^{*2} + \ldots$ et cette série converge si et seulement si $\text{s-}\lim_{k\to\infty} U^k = 0$. Les opérateurs ρ_u sur l'espace de Fock sont complètement non-unitaires. Nous pouvons donc appliquer notre construction et montrer que Δ définie en (1.7) satisfait à (1.8) et donc que nous avons une estimation de Mourre comme plus haut.

Notre notion d'opérateur de nombre N ne doit pas être confondue avec l'opérateur de nombre classique sur les espaces de Fock bosonique ou fermionique ; N ne compte pas le nombre de particule mais plutôt le nombre de particules en l'état u, heuristiquement parlant. Si l'on revient sur l'arbre, on remarque qu'il n'est même pas un opérateur local. Pour palier à ce problème, nous allons exprimer nos résultats en termes du, plus géométrique, nombre de particule.

Soit $\boldsymbol{N}$ *l'opérateur nombre de particules* défini sur $\mathscr{H}$ par $\boldsymbol{N}f = nf$ si f appartient à $\mathcal{H}^{\otimes n}$. Sur l'arbre, c'est l'opérateur de multiplication par la distance à l'origine, i.e. $(\boldsymbol{N}f)(x) = d(e,x)f(x)$.

Nous donnons maitenant notre résultat principal pour la théorie spectrale et celle de la diffusion pour l'opérateur H. Tout d'abord fixons quelques notations. Soit $\mathbf{1}_{\mathcal{H}}$ la projection orthogonale de $\mathscr{H}$ sur $\mathcal{H}$ et soient $\mathbf{1}_n$ et $\mathbf{1}_{\geq n}$ celles sur les sous-espaces $\mathcal{H}^{\otimes n}$ et $\bigoplus_{k\geq n}\mathcal{H}^{\otimes k}$. Pour s réel, soit $\mathscr{H}_{(s)}$ l'espace de Hilbert défini par la norme

$$\|f\|^2 = \|\mathbf{1}_0 f\|^2 + \sum_{n\geq 1} n^{2s}\|\mathbf{1}_n f\|^2.$$

Si T est un opérateur sur un espace de dimension finie E alors $\langle T\rangle$ est sa *trace normalisée* : $\langle T\rangle = \mathrm{Tr}(T)/\dim E$. On note par $\sigma_{\mathrm{ess}}(H)$ et par $\sigma_{\mathrm{p}}(H)$ les spectres essentiels et l'ensemble des valeurs propres de H. Alors :

Théorème 1.3 *Soient $\mathcal{H}$ de dimension finie et $u \in \mathcal{H}$ de norme 1 et soit $\Delta = (\rho_u + \rho_u^*)/2$. Soit V un opérateur auto-adjoint de la forme $V = \sum_{n\geq 0} V_n\mathbf{1}_n$, avec $V_n \in \mathcal{B}(\mathcal{H}^{\otimes n})$, $\lim_{n\to\infty}\|V_n\| = 0$, et tel que*

$$\|V_n - \langle V_n\rangle\| + \|V_{n+1} - V_n\otimes \mathbf{1}_{\mathcal{H}}\| \leq \delta(n)$$

où δ est une fonction décroissante telle que $\sum_n \delta(n) < \infty$. Soit W un opérateur borné auto-adjoint satisfaisant $\sum_n \|W\mathbf{1}_{\geq n}\| < \infty$. Si $H_0 = \Delta + V$ et $H = H_0 + W$, alors :
(1) $\sigma_{\mathrm{ess}}(H) = [-1, +1]$;

(2) *les valeurs propres de* H *différentes de* ± 1 *sont de multiplicité finie et ne peuvent s'accumuler qu'en* ± 1 *;*
(3) *si* $s > 1/2$ *et* $\lambda \notin \kappa(H) := \sigma_{\mathrm{p}}(H) \cup \{\pm 1\}$*, alors* $\lim_{\mu \to 0}(H - \lambda - i\mu)^{-1}$ *existe en norme dans* $\mathcal{B}(\mathscr{H}_{(s)}, \mathscr{H}_{(-s)})$ *localement unifomément dans* $\lambda \in \mathbb{R} \setminus \kappa(H)$ *;*
(4) *les opérateurs d'ondes de la paire* (H, H_0) *existent et sont complets.*

Ces résultats sont en complète analogie avec ceux du problème à deux corps pour l'espace Euclidien, l'opérateur nombre de particule $\boldsymbol{N}$ jouant le rôle de l'opérateur position. Les opérateurs V, W sont les analogues directs des composantes à longue portée et à courte portée du potentiel. Dans [GG1], nous obtenons des résultats plus généraux sur les perturbations permises. En particulier, nous pouvons remplacer l'espace $\mathscr{H}_{(s)}$ qui est en fait le domaine de $\boldsymbol{N}^s$ par le domaine de N^s qui est nettement plus gros.

2 Arbres, espaces de Fock, anisotropie et spectre essentiel

Nous allons nous intéresser maintenant à un problème d'une nature complètement différente. Notre but est de calculer le spectre essentiel d'une classe générale d'opérateurs sur un espace de Fock grace à leurs "localisations à l'infini", comme cela a été fait dans [GeI] quand Γ était un groupe abélien localement compact.

L'idée de base dans [GeI] est trés générale : la première étape est d'isoler la classe d'opérateurs que l'on veut étudier en considérant la C^*-algèbre $\mathscr{C}$ engendrée par certains Hamiltonians "élémentaires", la seconde est de calculer le quotient de $\mathscr{C}$ par l'idéal $\mathscr{C}_0 = \mathscr{C} \cap \mathcal{K}(\mathscr{H})$ des opérateurs compacts appartenant à $\mathscr{C}$. Alors, si $H \in \mathscr{C}$, sa projection $\widehat{H}$ dans le quotient $\mathscr{C}/\mathscr{C}_0$ est la localisation de H à l'infini (ou l'ensemble des localisations à l'infini, suivant la façon dont est représentée le quotient). L'intérêt de $\widehat{H}$ vient du fait que $\sigma_{\mathrm{ess}}(H) = \sigma(\widehat{H})$. Dans toutes les situations étudiées dans [GeI], ces localisations à l'infini correspondent à ce que l'intuition donnait.

Nous insistons sur le fait que les deux étapes de cette approche sont non triviales en général. L'algèbre $\mathscr{C}$ doit être choisie avec soin, si elle est trop petite ou trop grande alors le quotient nous donnera que de piètres informations sur notre classe d'opérateurs ou aucune car trop compliqué pour pouvoir être exploité. En outre, il n'existe que peu de techniques efficaces pour calculer de tels quotients. Une des principales observations de [GeI] est que dans beaucoup de situations intéressantes en mécanique quantique, l'espace des configurations est un groupe abélien localement compact et alors les algèbres intéressantes peuvent être construites grâce à des produits croisés ; dans ce contexte, la procédure est alors systématique.

Les techniques de [GeI] ne peuvent pas être utilisées ici car la structure de monoïde de l'arbre est trop pauvre et que celle de l'espace de Fock est pire encore. Toutefois, dans [Gol], on calcule un tel quotient pour une classe naturelle d'opérateurs anisotropes associés à la compactification hyperbolique de l'arbre. Cette algèbre contient les opérateurs compacts sur $\ell^2(\Gamma)$ et l'algèbre quotient s'injecte dans un produit tensoriel ce qui nous permet de calculer de spectres essentiels. Cette approche est par ailleurs constructive et respecte bien l'intuition donné par les localisations à l'infini.

Cette classe de potentiels est en fait facile à construire et la structure de l'arbre nous permet de bien la comprendre. L'espace métrique (Γ, d) est en réalité hyperbolique au sens de Gromov, où d est la métrique naturelle. Nous considérons alors sa compactification hyperbolique $\widehat{\Gamma} = \Gamma \cup \partial\Gamma$, où $\partial\Gamma$ est son bord à l'infini. Un élément $x \in \partial\Gamma$ est une suite de $x = (x_n)_{n\in\mathbb{N}}$ à valeurs dans Γ telle que $d(x_n, e) = n$ et telle que $x_{n+1} \leftrightarrow x_n$ pour tous $n \in \mathbb{N}$. On munit alors $\widehat{\Gamma}$ d'une structure ultramétrique naturelle, pour $x \in \partial\Gamma$ et $(y_n)_{n\in\mathbb{N}}$ une suite à valeurs dans Γ, on a $\lim_{n\to\infty} y_n = x$ si pour chaque $m \in \mathbb{N}$ il existe $N \in \mathbb{N}$ tel que pour chaque $n \geq N$ on ait $y_n \in x_m\Gamma$, où $x_m\Gamma$ est le sous-arbre enraciné d'origine x_m. On note alors par $C(\widehat{\Gamma})$ l'ensemble des fonctions continues à valeurs complexes sur $\widehat{\Gamma}$. Puisque Γ est dense dans $\widehat{\Gamma}$, on peut voir $C(\widehat{\Gamma})$ comme une C^*-sous-algèbre de $C_b(\Gamma)$, l'algèbre des fonctions bornées à valeurs complexes sur Γ. Enfin, pour $\widetilde{V} \in C(\widehat{\Gamma})$, l'opérateur de multiplication par $V = \widetilde{V}|_\Gamma$ agit dans $\ell^2(\Gamma)$.

Dans [GG1], on généralise cette approche et on considère des types d'anisotropie plus généraux. De plus, nous établissons une nouvelle technique pour le calcul effectif de l'algèbre quotient. Dans un souci de clareté, nous allons donner un exemple.

On se place dans un espace de Fock construit sur $\mathcal{H}$ de dimension finie. On fixe un vecteur $u \in \mathcal{H}$ et on note U l'isométrie associée. On s'intéresse aux opérateurs auto-adjoints de la forme $H = D+V$ où D est une "fonction continue" de U et U^*, i.e. elle appartient à la C^*-algèbre engéndrée par U, et V est de la forme $\sum V_n \mathbf{1}_n$ où V_n sont des opérateurs bornés sur $\mathcal{H}^{\otimes n}$ et qui sont asymptotiquement constants dans un certain sense (quand $n \to \infty$). Pour avoir des résultats précis, donnons des hypothèses spécifiques sur V_n.

Soit $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ une C^*-algèbre avec $\mathbf{1}_\mathcal{H} \in \mathcal{A}$. Soit $\mathscr{A}_{\rm vo}$ l'ensembre des opérateurs V comme précédement tels que $V_n \in \mathcal{A}^{\otimes n}$, $\sup \|V_n\| < \infty$

et $\|V_n - V_{n-1} \otimes \mathbf{1}_{\mathcal{H}}\| \to 0$ quand $n \to \infty$. Si $\nu = 1$, i.e. si on se place dans le cadre de l'Exemple 1.2, $\mathscr{A}_{\mathrm{vo}}$ est l'algèbre des suites bornées à "oscillation évanescente" (vanishing oscillation) à l'infini introduite par Cordes dans [Cor].

Les algèbres $\mathcal{A}^{\otimes n}$ s'injectent dans le produit tensoriel infini de C^*-algèbre $\mathcal{A}^{\otimes \infty}$. Ainsi, nous pouvons introduire la C^*-sous-algèbre $\mathscr{A}_\infty$ de $\mathscr{A}_{\mathrm{vo}}$ qui est constituée des opérateurs V tels que $V_\infty := \lim_{n\to\infty} V_n$ existent en norme dans $\mathcal{A}^{\otimes\infty}$. On remarquera que le sous-ensemble $\mathscr{A}_0$ des opérateurs V tels que $\lim_{n\to\infty} V_n = 0$ est un idéal de $\mathscr{A}_{\mathrm{vo}}$.

Les algèbres d'Hamiltoniens qui nous intéressent sont alors définies comme étant les C^*-algèbres $\mathscr{C}_{\mathrm{vo}}$ et $\mathscr{C}_\infty$ engendrées par les opérateurs de la forme $H = D + V$ où D est un polynôme en U, U^* et V est respectivement dans $\mathscr{A}_{\mathrm{vo}}$ et dans $\mathscr{A}_\infty$. On pose $\mathscr{C}_0 = \mathscr{C}_{\mathrm{vo}} \cap \mathcal{K}(\mathscr{H}) = \mathscr{C}_\infty \cap \mathcal{K}(\mathscr{H})$.

Théorème 2.1 *Si $\mathcal{H}$ est de dimension finie plus grande ou égale à 2, alors on a des isomorphismes canoniques :*

$$\mathscr{C}_{\mathrm{vo}}/\mathscr{C}_0 \simeq (\mathscr{A}_{\mathrm{vo}}/\mathscr{A}_0) \otimes \mathscr{D}, \quad \mathscr{C}_\infty/\mathscr{C}_0 \simeq \mathcal{A}^{\otimes\infty} \otimes \mathscr{D}. \tag{2.9}$$

Le cas où $\mathcal{H}$ est de dimension 1, $\ell^2(\mathbb{N})$, est aussi compris dans le formalisme de [GG1], mais son écriture fait en fait intervenir $\ell^2(\mathbb{Z})$.

Comme applications au calcul de spectres essentiels, si par exemple $D \in \mathscr{D}$ et $V \in \mathscr{A}_\infty$ sont des opérateurs auto-adjoints et que $H = D + V$, alors

$$\sigma_{\mathrm{ess}}(H) = \sigma(D) + \sigma(V_\infty). \tag{2.10}$$

La localisation de H à l'infini dans ce cas est $\widehat{H} = 1 \otimes D + V_\infty \otimes 1$.

Pour se replacer dans le cas d'un arbre et revenir au cas concret du laplacien avec un potentiel continu sur la compactification hyperbolique de l'arbre, soit $\mathcal{A}$ une algèbre abélienne sur $\mathcal{H}$. Puisque $\mathcal{H}$ est de dimension finie, le spectre de $\mathcal{A}$ est un ensemble fini A. On a alors $\mathcal{A} \simeq C(A)$ ainsi $\mathcal{A}^{\otimes n} \simeq C(A^n)$ canoniquement. Si $A^\infty \equiv A^{\mathbb{N}^*} (= \partial\Gamma)$ est équipé de la topologie produit, alors nous obtenons un identification naturelle $\mathcal{A}^{\otimes\infty} \simeq C(A^\infty)$. Soit alors $\Gamma := \bigcup_{n\geq 0} A^n$, $\mathscr{A}$ la C^*-algèbre des fonctions bornés $V : \Gamma \to \mathbb{C}$ et $\mathscr{A}_0$ est le sous-ensemble des fonctions continues qui tendent vers 0 en l'infini. Alors $V \in \mathscr{A}_{\mathrm{vo}}$ si et seulement si

$$\lim_{n\to\infty} \sup_{a \in A^n,\, b\in A} |V(a,b) - V(a)| = 0.$$

Soit $\pi_n : A^\infty \to A^n$ la projection sur les n premiers facteurs. Alors $V \in \mathscr{A}_\infty$ si et seulement s'il existe $V_\infty \in C(A^\infty)$ tel que

$$\lim_{n\to\infty} \sup_{a\in A^\infty} |V \circ \pi_n(a) - V_\infty(a)| = 0.$$

Cela revient à dire que la fonction $\widetilde{V}$ définit sur l'espace $\widetilde{\Gamma} = \Gamma \cup A^\infty$ muni de la topologie naturelle d'espace hyperbolique par la condition $\widetilde{V}|_\Gamma = V$ et $\widetilde{V}|_{A^\infty} = V_\infty$ est continue. Réciproquement, chaque fonction continue $\widetilde{V} : \widetilde{\Gamma} \to \mathbb{C}$ définit un élément $\mathscr{A}_\infty$ par $\widetilde{V}|_\Gamma = V$.

Nous pouvons ainsi parler de l'ensemble des localisations à l'infini de H. En utilisant

$$\mathcal{A}^{\otimes\infty} \otimes \mathscr{D} \simeq C(A^\infty; \mathscr{D}),$$

on voit que $\widehat{H}$ est une fonction continue $\widehat{H} : A^\infty \to \mathscr{D}$ et nous pouvons dire que $\widehat{H}(x)$ est la localisation de H au point $x \in A^\infty$ du bord à l'infini de l'arbre. Plus précisément, si $H = D + V$ est comme au dessus, alors on obtient $\widehat{H}(x) = D + V_\infty(x)$ et $\sigma_{\mathrm{ess}}(H) = \sigma(D) + V_\infty(A^\infty)$.

3 Stabilité du spectre essentiel, opérateurs agissant sur des modules de Banach

3.1 Présentation générale

Le but de cette partie est de donner des critères assurant la compacité de la différence des résolvantes de deux opérateurs.

Définition 3.1 *Soient A et B deux opérateurs agissant sur un espace de Banach $\mathscr{H}$. On dira que B* est une perturbation compacte de A *s'il existe $z \in \rho(A) \cap \rho(B)$ tel que $(A - z)^{-1} - (B - z)^{-1}$ soit compact.*

Sous les conditions de cette définition, la différence $(A - z)^{-1} - (B - z)^{-1}$ est compacte pour tout $z \in \rho(A) \cap \rho(B)$. En particulier, *si B est une perturbation compacte de A, alors A et B ont le même spectre essentiel*, et ceci pour toute définition raisonnable du spectre essentiel. Pour être précis, nous prendrons comme définition du spectre essentiel de A l'ensemble des points $\lambda \in \mathbb{C}$ tels que $A - \lambda$ ne soit pas un opérateur de Fredholm.

Nous allons maintenant décrire une méthode standard, simple et assez puissante pour prouver que B est une perturbation compacte de A.

Nous sommes intéressés par des situations où A et B sont des opérateurs différentiels (ou pseudo-différentiels) à coefficients complexes mesurables qui diffèrent peu au voisinage de l'infini. Une remarque importante est que, dans cette situation, on ne connaît pas le domaine des opérateurs en général. On possède toutefois plus d'informations sur le "domaine de forme" de l'opérateur. Nous voulons par ailleurs pouvoir considérer des opérateurs de tout ordre et en particulier des opérateurs de type Dirac. Nous travaillerons alors dans le contexte suivant, plus général que celui des formes acrétives.

Soient $\mathscr{G}, \mathscr{H}, \mathscr{K}$ des espaces de Banach réflexifs tels que $\mathscr{G} \subset \mathscr{H} \subset \mathscr{K}$ continûment et densément. Nous sommes intéressés par les opérateurs agissant dans $\mathscr{H}$ construits comme suit : soient A_0, B_0 des bijections continues $\mathscr{G} \to \mathscr{K}$ et soient A, B leur restriction à $A_0^{-1}\mathscr{H}$ et $B_0^{-1}\mathscr{H}$. Ce sont des opérateurs fermés densément définis sur $\mathscr{H}$. Alors dans $\mathcal{B}(\mathscr{K}, \mathscr{G})$ on a :

$$A_0^{-1} - B_0^{-1} = A_0^{-1}(B_0 - A_0)B_0^{-1}. \tag{3.11}$$

En particulier, on aura $z = 0 \in \rho(A) \cap \rho(B)$ et nous obtenons dans $\mathcal{B}(\mathscr{H})$

$$A^{-1} - B^{-1} = A_0^{-1}(B_0 - A_0)B^{-1}. \tag{3.12}$$

Nous avons alors un critère simple de compacité : si $A_0 - B_0 : \mathscr{G} \to \mathscr{K}$ est compact, alors B est une perturbation compacte de A. Le problème est que dans ce cas nous avons un peu trop : l'opérateur $A_0^{-1} - B_0^{-1} : \mathscr{K} \to \mathscr{G}$ est aussi compact, et cela ne peut se produire pour des opérateurs différentiels A_0, B_0 à parties principales distinctes. Ceci exclut aussi d'emblée des perturbations singulières d'ordre inférieur comme, par exemple, le potentiel Coulombien pour Dirac. Toutefois, l'avantage certain de ce critère est qu'il ne nécessite aucune connaissance particulière sur les domaines de A et de B.

Pour éviter les problèmes évoqués, on pourrait supposer alors que A est plus régulier que B ; au sens où les fonctions de son domaine seront, au moins localement, légèrement meilleures que celle de $\mathscr{G}$. Quand on équipe $\mathcal{D}(A)$ de sa norme graphe, il s'injecte continûment et de façon dense dans $\mathscr{G}$. Nous obtenons alors un deuxième critère de compacité en demandant que $A_0 - B_0 : \mathcal{D}(A) \to \mathscr{K}$ soit compact. Cette fois encore, nous avons plus que désiré : B n'est pas seulement une perturbation compacte de A, l'opérateur $A_0^{-1} - B_0^{-1} : \mathscr{H} \to \mathscr{G}$ est aussi compact. Cependant, les perturbations de la partie principale d'un opérateur différentiel et certaines perturbations singulières de l'opérateur de Dirac sont autorisées.

Dans cette thèse, nous nous sommes intéressés à des situations où nous n'avions aucune information sur les domaines de A et de B (si ce n'est le fait qu'ils ne soient sous-espaces de $\mathscr{G}$). Le cas où A, B sont des opérateurs elliptiques à coefficients mesurables complexes agissant sur $\mathscr{H} = L^2(\mathbb{R}^n)$ a été étudié par Ouhabaz et Stollmann dans [OS] et, à notre connaissance, c'est le seul article où les coefficents de l'opérateur non-perturbé ne soient pas supposés lisses. Leur approche consiste à prouver que la différence $A^{-k} - B^{-k}$ est compacte pour un certain $k \geq 2$ (ce qui implique la compacité de $A^{-1} - B^{-1}$). Ils utilisent ensuite le fait que $\mathcal{D}(A^k)$ est un sous-ensemble d'un espace de Sobolev $W^{1,p}$ pour un certain $p > 2$, ce qui donne un léger gain de régularité. Des techniques propres aux espaces L^p issues de la théorie des équations aux dérivées partielles sont alors nécessaires pour conclure.

Nous allons maintenant expliquer, dans la situation la plus élémentaire, les idées principales de notre approche à cette question. Soient $\mathscr{H} = L^2(\mathbb{R})$ et $P = -i\frac{d}{dx}$. On considère des opérateurs de la forme $A_0 = PaP + V$ et $B_0 = PbP + W$ où a, b sont des opérateurs bornés sur $\mathscr{H}$ tels que Re a et Re b sont bornées inférieurement par un nombre strictement positif. V et W sont supposés être des opérateurs continus $\mathscr{H}^1 \to \mathscr{H}^{-1}$, où $\mathscr{H}^s$ sont les espaces de Sobolev associés à $\mathscr{H}$. Alors $A_0, B_0 \in \mathcal{B}(\mathscr{H}^1, \mathscr{H}^{-1})$ et nous rajoutons une condition sur V et W qui assure l'inversibilité de A_0, B_0. Ainsi, nous sommes dans le contexte abstrait précédent avec $\mathscr{G} = \mathscr{H}^1$ et $\mathscr{K} = \mathscr{H}^{-1} \equiv \mathscr{G}^*$. Alors, grâce à (3.12) nous obtenons

$$A^{-1} - B^{-1} = A_0^{-1}P(b-a)PB^{-1} + A_0^{-1}(W-V)B^{-1}. \tag{3.13}$$

Soit R le premier terme du membre de droite de l'égalité. La compacité du deuxième membre de droite est en général plus facile à traiter car V et W sont d'ordre inférieur en pratique. Regardons maintenant comment prouver celle du terme R.

Tout d'abord, on remarque que $R\mathscr{H} \subset \mathscr{H}^1$. On peut alors écrire $R = \psi(P)R_1$ pour un certain $\psi \in B_0(\mathbb{R})$ (l'ensemble des fonctions boréliennes qui tendent vers zéro à l'infini) et $R_1 \in \mathcal{B}(\mathscr{H})$. Ceci ne représente néanmoins que la moitié des conditions nécessaires pour la compacité. En fait, R est compact si et seulement s'il existe $\varphi \in B_0(\mathbb{R})$ et $R_2 \in \mathcal{B}(\mathscr{H})$ tels que $R = \varphi(Q)R_2$. Evidemmement, le seul facteur pouvant apporter une telle décroissance est $b - a$. On suppose alors que $b - a$ s'écrive $\xi(Q)U$ avec ξ dans $B_0(\mathbb{R})$ et U dans $\mathscr{B}(\mathscr{H})$. Puisque $P : \mathscr{H} \to \mathscr{H}^{-1}$ et $A_0^{-1} : \mathscr{H}^{-1} \to \mathscr{H}^1$ sont bornés, l'opérateur $S = A_0^{-1}P$ est borné sur $\mathscr{H}$. Par suite, $R = S\xi(Q)UPB^{-1}$ et $UPB^{-1} \in \mathcal{B}(\mathscr{H})$. Ainsi, R sera compact si l'opérateur $S \in \mathcal{B}(\mathscr{H})$ a la propriété que, pour chaque $\xi \in B_0(\mathbb{R})$, il existe $\varphi \in B_0(\mathbb{R})$ et $T \in \mathcal{B}(\mathscr{H})$ tels que $S\xi(Q) = \varphi(Q)T$. Un opérateur S possédant cette propriété sera appellée *quasilocal*.

3.2 Exemples

Dans cette section, nous donnerons des résultats obtenus grâce à un formalisme abstrait issu de la situation exposée plus haut.

Pour commencer, prenons l'exemple des opérateurs de Dirac. Soient $X = \mathbb{R}^n$ et E un espace de Hilbert complexe de dimension finie. On considère l'espace de Hilbert $\mathscr{H} = L^2(X; E)$ des fonctions de carré intégrables à valeurs dans E. Soient $m \in \mathbb{R}$ et $\alpha_0 \equiv \beta, \alpha_1, \ldots, \alpha_n$ des opérateurs symétriques sur E tels que $\alpha_j\alpha_k + \alpha_k\alpha_j = \delta_{jk}$. On étudie l'Hamiltonien libre de Dirac $D = \sum_{k=1}^n \alpha_k P_k + m\beta$. L'opérateur D est auto-adjoint sur $\mathscr{H}$ et son domaine d'injecte continûment et densément dans $\mathscr{G} = \mathscr{H}^{1/2}$, $\mathscr{H}^s$ étant cette fois les espaces de Sobolev usuels à valeurs dans E.

En identifiant $\mathscr{H}$ avec son dual grâce à l'isomorphisme de Riesz, nous obtenons alors le triplet de Gelfand $\mathscr{G} \subset \mathscr{H} = \mathscr{H}^* \subset \mathscr{G}^*$, où les injections sont denses et continues. Pour tout $s \geq 0$ réel, on note $\mathscr{M}_s$ la fermeture dans $\mathscr{B}(\mathscr{H}^s)$ de l'ensemble des opérateurs de multiplication par des fonctions de classe C^∞ à support compact. Soit $\mathcal{B}_0(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$ l'ensemble des $S \in \mathscr{B}(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$ tel qu'il existe $T, T' \in \mathscr{B}(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$ et $M, M' \in \mathscr{M}_{1/2}$ vérifiant $S = MT = T'M'$. Cet ensemble représente, dans un certain sens, une classe *d'opérateurs petits à l'infini*. On montre alors :

Théorème 3.2 *Soient V etW des fonctions mesurables sur X à valeurs dans les opérateurs symétriques sur E. Supposons que les opérateurs de multiplication par V et W définissent des fonctions continues $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ et telles que $V - W \in \mathcal{B}_0(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$. Supposons que $D+V+i$ et que $D+W+i$ sont des fonctions bijectives $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$. Alors $D+V$ et $D+W$ induisent des opérateurs auto-adjoints A et B sur $\mathscr{H}$, où B est une perturbation compacte de A et donc $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.*

J'insiste sur le fait que la principale nouveauté est que l'opérateur “non perturbé” A est localement aussi singulier que B. Notons aussi que les hypothèses sur V et W sont très générales.

Nous donnons maintenant un résultat concernant la perturbation d'une métrique d'une variété. Soient X une variété différentielle C^1, *non-compacte* et T^*X son fibré cotangent. On suppose que X est équipée d'une structure riemannienne mesurable et localement bornée. La fibre au dessus de $x \in X$ est notée par T_x^*X. Chaque T_x^*X est alors équipé d'un produit scalaire $\langle\cdot|\cdot\rangle_x$ et l'on note par $\|\cdot\|_x$ la norme associée.

Soit μ l'élément de volume riemannien. On prendra $\mathscr{H} = L^2(X,\mu)$ et $\mathscr{K}$ le complété de l'espace de sections continues sur T^*X à support compact pour la norme :

$$\|v\|_{\mathscr{K}}^2 = \int_X \|v(x)\|_x^2 d\mu(x).$$

En fait, $\mathscr{K}$ est l'espaces des sections de carré intégrable sur T^*X.

Soit $\mathrm{d} : C_{\mathrm{c}}^1(X) \subset \mathscr{H} \to \mathscr{K}$ l'opérateur (fermable) de différentiation exterieure. On note encore d sa fermeture et son domaine $\mathscr{G}$ est l'espace de Sobolev $\mathscr{H}^1$ qui est le complété de $C_{\mathrm{c}}^1(X)$ sous la norme :

$$\|u\|_{\mathscr{H}^1}^2 = \int_X \Big(|u(x)|^2 + \|\mathrm{d}u(x)\|_x^2\Big) d\mu(x).$$

L'opérateur de Laplace-Beltrami est donné par $\Delta = \mathrm{d}^*\mathrm{d}$.

Théorème 3.3 *Soit X une variété de classe C^1 munie d'une structure riemannienne localement mesurable. On suppose donné une nouvelle structure riemannienne mesurable sur X telle que les normes associées $\|\cdot\|_x'$ vérifient*

$$\alpha(x)\|\cdot\|_x \leq \|\cdot\|_x' \leq \beta(x)\|\cdot\|_x$$

pour α, β tels que $\lim_{x\to\infty}\alpha(x) = \lim_{x\to\infty}\beta(x) = 1$. On suppose que l'espace métrique X est complet pour la métrique associée à une de ces deux structures (et donc à l'autre aussi). Soient Δ et Δ' les opérateurs de Laplace-Beltrami respectifs. On a alors $\sigma_{\text{ess}}(\Delta) = \sigma_{\text{ess}}(\Delta')$.

3.3 Perturbations faiblement petites à l'infini

Dans les exemples précédents, la notion de convergence à l'infini était donnée par celle de convergence suivant le filtre de Fréchet. Dans un espace localement compact X, ce filtre est donné par les complémentaires des ensembles compacts. Il s'associe à la compactification à un point de l'espace, dite d'Alexandrov. Nous allons maintenant considérer des familles de filtres plus fins. On pourra alors remplacer $B_0(X)$ par ces nouvelles classes de fonctions.

On présentera ici seulement un cas particulier de notre résultat principal dans cette direction. Soit $B_{\mathrm{w}}(\mathbb{R}^n)$ la classe des fonctions $\varphi \in L^\infty(\mathbb{R}^n)$ telles que

$$\lim_{a\to\infty} \int_{|x-a|} |\varphi(x)|dx = 0.$$

Cette classe est une C^*-algèbre dont on peut trouver une description en terme de filtres dans [GG2].

Soit $\Delta_a = \sum_{|\alpha|,|\beta|\leq m} P^\alpha a_{\alpha\beta} P^\beta$ avec $a_{\alpha\beta} \in L^\infty(\mathbb{R}^n)$ et supposons que l'opérateur $\Delta_a : \mathscr{H}^m \to \mathscr{H}^{-m}$ ait la propriété qu'il existe des nombres $\mu, \nu > 0$ tels que pour tout $u \in \mathscr{H}^m$:

$$\mathrm{Re}\ \langle u, \Delta_a u\rangle \geq \mu\|u\|^2_{\mathscr{H}^m} - \nu\|u\|^2_{\mathscr{H}}.$$

Soit $\Delta_b = \sum_{|\alpha|,|\beta|\leq m} P^\alpha b_{\alpha\beta} P^\beta$ un autre opérateur de ce type.

Théorème 3.4 *Si $b_{\alpha\beta} - a_{\alpha\beta} \in B_{\mathrm{w}}(\mathbb{R}^n)$ pour tous les α, β, alors l'opérateur Δ_b est une perturbation compacte de Δ_a, en particulier Δ_a et Δ_b ont les mêmes spectres essentiels.*

On pourra remarquer que les résultats de ce type n'étaient connus que dans le cas $m = 1$, voir [OS].

References

[All] Allard, C. : *Asymptotic completeness via Mourre theory for a Schrödinger operator on a binary tree*, Master's thesis, UBC (1997).

[ABG] Amrein, W., Boutet de Monvel, A., Georgescu, V. : *C_0-Groups, commutator methods and spectral theory of N-body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.

[AlF] Allard, C., Froese, R. : *A Mourre estimate for a Schrödinger operator on a binary tree* , Reviews in Mathematical Physics, Vol. 12, No. 12 (2000), 1655–1667.

[BoS] Boutet de Monvel, A., Sahbani, J. : *On the spectral properties of discrete Schrödinger operators* C.R.Acad.Sci Paris Sér I Math. 326, no. 9 (1998), 1145–1150.

[Cor] Cordes, H.O. : *On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators.* J. Funct. Anal. **18** (1975), 115–131

[GG1] Georgescu, V., Golénia, S. : *Isometries, Fock Spaces, and Spectral Analysis of Schrödinger Operators on Trees*, prépublication.

[GG2] Georgescu, V., Golénia, S. : *Quasilocal Operators and Stability of the Essential Spectrum*, prépublication.

[GeI] Georgescu, V., Iftimovici, A. : *Crossed products of C^*-algebras and spectral analysis of quantum Hamiltonians*, Comm. Math. Phys. **228** (2002), no. 3, 519–560.

[Gol] Golénia, S. : *C^*-algebra of anisotropic Schrödinger operators on trees*, (à parraître dans Annales Henri Poincaré).

[Mou] Mourre, E. : Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. **91** (1981), 391–408.

[NaY] Naboko, S.N., Yakovlev, S.I. : *The discrete Schrödinger operator. The point spectrum lying on the continuous spectrum*, ST. Petersburg Math J. Vol.4 , **3** (1993), 559–568.

[OS] E.M. Ouhabaz, P. Stollmann : *Stability of the ensential spectrum of second-order complex elliptic operators*, J. Reine Angew. Math. **500** (1998), 113–126.

[Sun] Sunada, T. : *Fundamental groups and Laplacians*, Amer. Math. Soc. Transl. Ser. 2, **160** (1994) 19–32.

[We] J. Weidmann : *Linear operators in Hilbert space*, Springer, 1980.

C^*-algebras of anisotropic Schrödinger operators on trees

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Abstract

We study a C^*-algebra generated by differential operators on a tree. We give a complete description of its quotient with respect to the compact operators. This allows us to compute the essential spectrum of self-adjoint operators affiliated to this algebra. The results cover Schrödinger operators with highly anisotropic, possibly unbounded potentials.

1 Introduction

Given a ν-fold tree Γ of origin e with its canonical metric d, we write $x \sim y$ when x and y are connected by an edge and we set $|x| = d(x, e)$. For each $x \in \Gamma \setminus \{e\}$, we denote by $x' \equiv x^{(1)}$ the unique element $y \sim x$ such that $|y| = |x| - 1$ and we set $x^{(p)} = (x^{(p-1)})'$ for $1 \leq p \leq |x|$. Let $x\Gamma = \{y \in \Gamma \mid |y| \geq |x| \text{ and } y^{(|y|-|x|)} = x\}$, where the convention $x^{(0)} = x$ has been used.

On $\ell^2(\Gamma)$ we define the bounded operator ∂ given by $(\partial f)(x) = \sum_{y'=x} f(y)$. Its adjoint is given by $(\partial^* f)(e) = 0$ and $(\partial^* f)(x) = f(x')$ for $|x| \geq 1$. Let $\mathscr{D}$ be the C^*-algebra generated by ∂.

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In order to obtain our algebra of potentials, we consider the "hyperbolic" compactification $\widehat{\Gamma} = \Gamma \cup \partial\Gamma$ of Γ constructed as follows. An element x of the boundary at infinity $\partial\Gamma$ is a Γ-valued sequence $x = (x_n)_{n\in\mathbb{N}}$ such that $|x_n| = n$ and $x_{n+1} \sim x_n$ for all $n \in \mathbb{N}$. We set $|x| = \infty$ for $x \in \partial\Gamma$. The space $\widehat{\Gamma}$ is equipped with a natural ultrametric space structure. For $x \in \partial\Gamma$ and $(y_n)_{n\in\mathbb{N}}$ a sequence in Γ we have $\lim_{n\to\infty} y_n = x$ if for each $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that for each $n \geq N$ we have $y_n \in x_m\Gamma$. We denote by $C(\widehat{\Gamma})$ the set of complex-valued continuous functions defined on $\widehat{\Gamma}$. Since Γ is dense in $\widehat{\Gamma}$, we can view $C(\widehat{\Gamma})$ as a C^*-subalgebra of $C_b(\Gamma)$, the algebra of bounded complex-valued functions defined on Γ. For $V \in C(\widehat{\Gamma})$, we denote by $V(Q)$ the operator of multiplication by V in $\ell^2(\Gamma)$.

Let us now denote by $\mathscr{C}(\widehat{\Gamma})$ the C^*-algebra generated by $\mathscr{D}$ and $C(\widehat{\Gamma})$. It contains the set $\mathbb{K}(\Gamma)$ of compact operators on $\ell^2(\Gamma)$. Following the strategy exposed in [6], we shall first compute its quotient with respect to the ideal of compact operators. We stress that the crossed product technique introduced in [6] in order to compute quotients cannot be used in our case. Instead, we shall use the Theorem 4.5 in order to calculate the essential spectrum of self-adjoint operators related to $\mathscr{C}(\widehat{\Gamma})$. In this introduction we consider only the most important case, when $\nu > 1$.

Theorem 1.1 *Let $\nu > 1$. There is a unique morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D} \otimes C(\partial\Gamma)$ such that $\Phi(D) = D \otimes 1$ for all $D \in \mathscr{D}$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$. This morphism is surjective and its kernel is $\mathbb{K}(\Gamma)$.*

The rest of this introduction is devoted to some applications of this theorem to spectral analysis. Let $\nu > 1$ and $H = \sum_{\alpha,\beta} a_{\alpha,\beta}(Q)\partial^{*\alpha}\partial^{\beta} + K$, where K is a compact operator, $a_{\alpha,\beta} \in C(\widehat{\Gamma})$ and $a_{\alpha,\beta} = 0$ for all $(\alpha,\beta) \in \mathbb{N}^2$ but a finite number of pairs. Clearly $H \in \mathscr{C}(\widehat{\Gamma})$. As a consequence of the Theorem 1.1, there is Φ such that $\Phi(H) = \sum_{\alpha,\beta} \partial^{*\alpha}\partial^{\beta} \otimes (a_{\alpha,\beta})|_{\partial\Gamma}$, and, if H self-adjoint, its essential spectrum is:

$$\sigma_{\text{ess}}(H) = \bigcup_{\gamma\in\partial\Gamma} \sigma\Big(\sum_{\alpha,\beta} a_{\alpha,\beta}(\gamma)\partial^{*\alpha}\partial^{\beta}\Big).$$

This result can be made quite explicit in the particular case of a Schrödinger operator

$H = \Delta + V(Q)$ with potential V in $C(\widehat{\Gamma})$. Since Δ is a bounded operator on $\ell^2(\Gamma)$ defined by $(\Delta f)(x) = \sum_{y\sim x}(f(y) - f(x))$, it belongs to $\mathscr{C}(\widehat{\Gamma})$. We then set $\Delta_0 = \partial + \partial^* - \nu \mathrm{Id}$ (which belongs to $\mathscr{D}$) and notice that $\Delta - \Delta_0$ is compact. One then gets (see [1] for instance):

$$\sigma_{ess}(\partial + \partial^*) = \sigma_{ac}(\partial + \partial^*) = \sigma(\partial + \partial^*) = [-2\sqrt{\nu}, 2\sqrt{\nu}\,],$$

where $\sigma_{ac}(T)$ denotes the absolute continuous part of the spectrum of a given self-adjoint operator T. On the other hand, Theorem 1.1 gives us directly $\sigma_{ess}(\partial^* + \partial) = \sigma(\partial^* + \partial)$. We thus get

$$\sigma_{ess}(\Delta + V(Q)) = \sigma(\Delta_0) + V(\partial\Gamma) = [-\nu - 2\sqrt{\nu}, -\nu + 2\sqrt{\nu}\,] + V(\partial\Gamma).$$

In fact this result holds (and is trivial) in the case of $\nu = 1$, i.e. when $\Gamma = \mathbb{N}$.

Given a continuous function on $\partial\Gamma$, the Tietze theorem allows us to extend it to a continuous function on $\widehat{\Gamma}$, so one may construct a large class of Hamiltonians with given essential spectra. Nevertheless, we are able to point out a concrete class of non-trivial potentials $V \in C(\widehat{\Gamma})$ with uniform behaviour at infinity which form a dense family of $C(\widehat{\Gamma})$. Namely, for each bounded function $f : \Gamma \to \mathbb{R}$ and each real $\alpha > 1$ let

$$V(x) = \sum_{k=1}^{|x|} \frac{f(x_k)}{k^\alpha}, \tag{1.1}$$

where $x_k = x^{|x|-k}$ for $x \in \Gamma$ (V belongs to $C(\widehat{\Gamma})$ because of Proposition 2.3).

Concerning finer spectral features, based mainly on the Mourre estimate, we mention that in the case $H = \Delta + V(Q)$, with V as in (1.1) where $\alpha \geq 3$ and such that $V(\partial\Gamma) = 0$, the results of [1] can be applied (the hypotheses of the Lemmas 6 and 7 from [1] are verified since $V(x) = O(|x|^{-\alpha+1})$ when $|x| \to \infty$). The aim of our work in preparation [8] is to prove that the Mourre estimate holds for more general classes of Hamiltonians affiliated to $\mathscr{C}(\widehat{\Gamma})$ and to develop a scattering theory for them. Theorem 1.1 remains the key technical point for these purposes.

The preceding results on trees allow us to treat more general graphs. We recall that a graph is said to be *connected* if two of its elements can be joined by a sequence of neighbours. Let $G = \bigcup_{i=1}^n \Gamma_i \cup G_0$ be a finite disjoint union of Γ_i, each Γ_i being a ν_i-fold branching tree with $\nu_i \geq 1$ and of G_0, a compact connected graph. We endow G with a connected graph structure that respects the graph structure of each Γ_i and the one of G_0, such that Γ_i is connected to Γ_j ($i \neq j$) only through G_0 and such that Γ_i is connected to G_0 only through e_i, the origin of Γ_i. The graph G is hyperbolic and its boundary at infinity ∂G is the disjoint union $\cup_{i=1}^n \partial\Gamma_i$. We now choose $V \in C(G \cup \partial G)$. One has $V|_{\widehat{\Gamma}_i} \in C(\widehat{\Gamma}_i)$ for all $i = 1, \ldots, n$ and we easily obtain:

$$\sigma_{ess}(\Delta + V(Q)) = \bigcup_{i=1}^n \big([-\nu_i - 2\sqrt{\nu_i}, -\nu_i + 2\sqrt{\nu_i}\,] + V(\partial\Gamma_i) \big).$$

This covers in particular the case of the Cayley graph of a free group with finite system of generators. We recall that the Cayley graph of a group G with a system of generators S is the graph defined on the set G with the relation $x \sim y$ if $xy^{-1} \in S$ or $yx^{-1} \in S$. Let G be a free group with a system of generators S such that $S = S^{-1}$. We denote by e its neutral element and we set $|S| = \nu + 1$. One may associate the restriction of the Cayley graph to the set of words starting with a given generator with a ν-fold branching tree having as origin the generator. Hence, the Cayley graph of G will be $\cup_{i=1}^{\nu} \Gamma_i \cup \{e\}$ where Γ_i is a ν-fold branching tree with the above graph structure.

We now go further by taking $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$ such that $V(\Gamma) \subset \mathbb{R}$ (here $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the Alexandrov compactification of $\mathbb{R}$). More precisely, $V \in C(\widehat{\Gamma}, \overline{\mathbb{R}})$ if and only if for each $\gamma \in \partial\Gamma$ we have either $\lim_{x\to\gamma} V(x) = l$ where $l \in \mathbb{R}$ or for each $M \geq 0$ there is $N \in \mathbb{N}$ such that $|V(x)| \geq M$ for all $n \geq N$ and $x \in \gamma_n\Gamma$ (see Proposition 2.3). We set

$$D(V) = \{f \in \ell^2(\Gamma) \mid \|V(Q)f\|^2 < \infty\}.$$

Let $T \in \mathscr{D}$ and $T_0 = \Phi(T)$. Since T is bounded, the operator $H = T + V(Q)$ with domain $D(V)$ is self-adjoint and it is affiliated to $\mathscr{C}(\widehat{\Gamma})$ (i.e. its resolvent belongs to $\mathscr{C}(\widehat{\Gamma})$). Indeed, we have $(V(Q) + z)^{-1} \in C(\widehat{\Gamma})$ for each $z \in \mathbb{C} \setminus \mathbb{R}$, and for large such z,

$$(H + z)^{-1} = (V(Q) + z)^{-1} \sum_{n\geq 0} (T(V(Q) + z)^{-1})^n,$$

where the series is norm convergent. Now, with the same z, we use the Theorem 1.1 and the fact that $\mathscr{D} \otimes C(\partial\Gamma) \simeq C(\partial\Gamma, \mathscr{D})$ to obtain

$$\Phi_\gamma\big((H+z)^{-1}\big) \equiv \Phi\big((H+z)^{-1}\big)(\gamma) = (V(\gamma)+z)^{-1} \sum_{n\geq 0} (T_0(V(\gamma)+z)^{-1})^n.$$

Note that $(V(\gamma)+z)^{-1} = 0$ if $V(\gamma) = \infty$. By analytic continuation we get $\Phi_\gamma((T + V(Q) + z)^{-1}) = (T_0 + V(\gamma) + z)^{-1}$, for all $z \in \mathbb{C} \setminus \mathbb{R}$. We used the convention $(T_0 + V(\gamma) + z)^{-1} = 0$ if $V(\gamma) = \infty$.

We now compute the essential spectrum of H. If $V(\gamma) = \infty$ then $\sigma(\Phi_\gamma(H)) = \emptyset$. Otherwise, one has $\sigma(\Phi_\gamma(H)) = \sigma(T_0+V(\gamma)) = \sigma(T_0)+V(\gamma)$. Hence we obtain:

$$\sigma_{\text{ess}}(T + V(Q)) = \sigma(T_0) + V(\partial\Gamma_0),$$

where $\partial\Gamma_0$ is the set of $\gamma \in \partial\Gamma$ such that $V(\gamma) \in \mathbb{R}$.

Remark: We mention an interesting question which has not been studied in this paper. In fact, one could replace the algebra $\mathscr{D}$ by the (much bigger) C^*-algebra generated by all the right translations ρ_a (see Subsection 3.4 for notations) and consider the corresponding algebra $\mathscr{C}(\widehat{\Gamma})$. This is a natural object, since it contains all the “right-differential” operators acting on the tree (not only polynomials in ∂ and ∂^*). A combination of the techniques that we use and that of [9, 10] could allow one to compute the quotient in this case too. We also note that in [9, 10] a certain connection with the notion of crossed-product is pointed out, and this could be useful in further investigations. I would like to thank the referee for bringing to my attention the two papers of A. Nica quoted above.

2 Trees and related objects

2.1 The free monoïd Γ

Let $\mathscr{A}$ be a finite set consisting of ν objects. Let Γ be the free monoïd over $\mathscr{A}$; its elements are *words* and those of $\mathscr{A}$ *letters*. We refer to [3, Chapter I, §7] for a detailed discussion of these notions, but we recall that a word x is an $\mathscr{A}$-valued map defined on a set of the form[1] $[\![1, n]\!]$ with $n \in \mathbb{N}$, $x(i)$

[1] We use the notation $[\![1, n]\!] = [1, n] \cap \mathbb{N}$ where $\mathbb{N}$ is the set of integers ≥ 0 and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

being the i-th letter of the word x. The integer n (the number of letters of x) is the length of the word and will be denoted $|x|$. There is a unique word e of length 0, its domain being the empty set. This is the neutral element of Γ. We will also identify $\mathscr{A}$ with the set of words of length 1.

The monoïd Γ will be endowed with the discrete topology. If $x \in \Gamma$, we denote $x\Gamma$ and Γx the right and left ideals generated by x. We have on Γ a canonical order relation which is by definition:

$$x \leq y \Leftrightarrow y \in x\Gamma.$$

We recall some terminology from the theory of ordered sets. If Γ is an arbitrary ordered set and $x, y \in \Gamma$, then one says that y *covers* x if $x < y$ and if $x \leq z \leq y \Rightarrow z = x$ or $z = y$. If $x \in \Gamma$, we denote $\widetilde{x} = \{y \in \Gamma \mid y$ covers $x\}$

In our case, y covers x if $x \leq y$ and $|y| = |x| + 1$. Notice that each element $x \in \Gamma \backslash \{e\}$ covers a unique element x', its *father*, and each element $x \in \Gamma$ is covered by ν elements, its *sons*. The set of sons of x clearly is $\widetilde{x} = \{x\varepsilon \mid \varepsilon \in \mathscr{A}\}$. Hence:

$$y \text{ covers } x \Leftrightarrow y' = x \Leftrightarrow y \in \widetilde{x}.$$

For $|x| \geq n$, we define $x^{(n)}$ inductively by setting $x^{(0)} = x$ and $x^{(m+1)} = (x^{(m)})'$ for $m \leq n-1$. One may also notice that: $|x^{(\alpha)}| = |x| - \alpha$, if $\alpha \leq |x|$, and for $\alpha \leq |ab|$:

$$(ab)^{(\alpha)} = \begin{cases} ab^{(\alpha)}, & \text{if } \alpha \leq |b| \\ a^{(\alpha-|b|)}, & \text{if } \alpha \geq |b|. \end{cases}$$

We remark that if $\nu = 1$ then $\Gamma = \mathbb{N}$ and if $\nu > 1$ then Γ is the set of monoms of ν non-commutative variables.

2.2 The tree Γ and the extended tree associated to $\mathscr{A}$

Recall that a graph is a couple $G = (V, E)$, where V is a set (of *vertices*) and E is a set of pairs of elements of V (the *edges*). If x and y are joined by an edge, one says that they are *neighbours* and one abbreviates $x \sim y$. The graph structure allows one to endow V with a canonical metric d, where $d(x, y)$ is the length of the shortest path in G joining x to y.

The graph G_Γ associated to the free monoïd Γ is defined as follows: $V = \Gamma$ and $x \sim y$ if x covers y or y covers x. It is usual to identify Γ and G_Γ, the so-called ν-fold branching tree. For all $x \in \Gamma$, we have $|x| = d(e,x)$. We set $B(x,r) = \{y \in \Gamma \mid d(x,y) < r\}$ and $S^n = \{x \in \Gamma \mid |x| = n\}$.

We shall now define an extended tree by mimicking the definition of a free monoïd over $\mathscr{A}$. We choose $o \in \mathscr{A}$; this element will be fixed from now on. For each integer r, we set $\mathbb{Z}_r = \{i \in \mathbb{Z} \mid i \leq r\}$. The *extended tree* $\widetilde{\Gamma}$ *associated to* $\mathscr{A}$ is the set of $\mathscr{A}$-valued maps x defined on sets of the form $\mathbb{Z}_r$ such that $\{i \mid x(i) \neq o\}$ is finite. For $x \in \widetilde{\Gamma}$, the unique $r \in \mathbb{Z}$ such that x is a map $\mathbb{Z}_r \to \mathscr{A}$ will be denoted $|x|$ and will be called *length* of x.

We shall identify Γ with the set $\{x \mid |x| \geq 0 \text{ and } x(i) = o \text{ if } i \leq 0\}$ as follows: if $x \in \Gamma$ then we associate to it the element of $\widetilde{\Gamma}$ defined on $\mathbb{Z}_{|x|}$ by extending x with $x(i) = o$ if $i \leq 0$. The element e will be identified with the map $e \in \widetilde{\Gamma}$ such that $|e| = 0$ and $e(i) = o, \forall i \leq 0$. Notice that the two notions of length are consistent on Γ.

There is a natural right action of Γ on $\widetilde{\Gamma}$ by concatenation, i.e. for $x \in \widetilde{\Gamma}$ and $y \in \Gamma$, xy will be the function z defined on $\mathbb{Z}_{|x|+|y|}$ such that $z(i) = x(i)$, for $i \in \mathbb{Z}_{|x|}$ and $z(|x|+i) = y(i)$ for $i \in [\![1, |y|\,]\!]$. Then we equip $\widetilde{\Gamma}$ with an order relation by setting:

$$x \leq y \Leftrightarrow y \in x\Gamma.$$

As before, y covers x if and only if $x \leq y$ and $|y| = |x| + 1$. Now, each $x \in \widetilde{\Gamma}$ covers a unique $x' \in \widetilde{\Gamma}$ and each $x \in \widetilde{\Gamma}$ is covered by ν elements, namely those of $\widetilde{x} = \{x\varepsilon \mid \varepsilon \in \mathscr{A}\}$. We still have: $y \text{ covers } x \Leftrightarrow y' = x \Leftrightarrow y \in \widetilde{x}$. Observe that $x' = x|_{\mathbb{Z}_{|x|-1}}$. We will set $x^{(\alpha)} = x|_{\mathbb{Z}_{|x|-\alpha}}$ for all $\alpha \in \mathbb{Z}$. As we did it for Γ, we shall indentify the graph $G_{\widetilde{\Gamma}}$ with $\widetilde{\Gamma}$. This justifies the notion of extended *tree* used for $\widetilde{\Gamma}$.

2.3 The boundary at infinity of Γ

We shall see in the ending remark of this subsection that the boundary at infinity of Γ can be thought as the boundary of a 0-hyperbolic space in the sense of Gromov. We prefer, however, to give a simpler presentation that is closer to the theory of p-adic numbers (see [11] for instance). In fact, if ν is prime the boundary will be the set of ν-adic integers.

Definition 2.1 *The boundary at infinity of* Γ *is the set* $\partial\Gamma = \{x : \mathbb{N}^* \to \mathscr{A}\}$. *For* $x \in \partial\Gamma$, *we set* $|x| = \infty$.

Let $\widehat{\Gamma}$ be $\Gamma \cup \partial\Gamma$. For $x \in \widehat{\Gamma}$, we define the sequence $(x_n)_{n\in[\![0,|x|]\!]}$ with values in Γ by setting $x_0 = e$ and $x_n = x|_{[\![1,n]\!]}$ for $n \geq 1$. Observe that the map $x \mapsto (x_n)_{n\in[\![0,|x|]\!]}$ is injective. There is a natural left action of Γ on $\widehat{\Gamma}$. For $x \in \Gamma$ and $y \in \widehat{\Gamma}$, xy will be defined on the set[2] $[\![1, |x| + |y|]\!]$ by $x(i)$ for $i \leq |x|$ and by $y(i - |x|)$ for $i > |x|$.

We will now equip $\widehat{\Gamma}$ with a structure of ultrametric space. We define a kind of valuation v on $\widehat{\Gamma} \times \widehat{\Gamma}$ by

$$v(x, y) = \begin{cases} \max\{n \mid x_n = y_n\} & \text{if} \quad x \neq y \\ \infty & \text{if} \quad x = y. \end{cases} \tag{2.1}$$

If $x, y, z \in \widehat{\Gamma}$ it is easy to see that:

$$v(x, y) \geq \min(v(x, z), v(z, y)). \tag{2.2}$$

Let us set on $\widehat{\Gamma}$:

$$\widehat{d}(x, y) = \exp(-v(x, y)).$$

The relation (2.2) clearly implies that $(\widehat{\Gamma}, \widehat{d})$ is an ultrametric space, i.e. a metric space such that $\widehat{d}(x, y) \leq \max(\widehat{d}(x, z), \widehat{d}(z, y))$, for $x, y, z \in \widehat{\Gamma}$. We will denote, for $r > 0$, $\widehat{B}(x, r) = \{y \in \widehat{\Gamma} \mid \widehat{d}(x, y) < r\}$. Notice that ultrametricity implies that $\widehat{B}(x, r)$ is closed for all $x \in \widehat{\Gamma}$ and $r > 0$.

The topology induced by $\widehat{\Gamma}$ on Γ coincides with the initial topology of Γ, the discrete one. For $x \in \partial\Gamma$ and $n \in \mathbb{N}$,

$$x_n\widehat{\Gamma} \;=\; \{y \in \widehat{\Gamma} \mid v(x, y) \geq n\} = \widehat{B}(x, \exp(-n + 1))$$

which is the closure of $x_n\Gamma$ in $\widehat{\Gamma}$. Hence for each $x \in \partial\Gamma$, $\{x_n\widehat{\Gamma}\}_{n\in\mathbb{N}}$ is a basis of neighbourhoods of x in $\widehat{\Gamma}$. Observe that if $x \in \Gamma$ then $x\partial\Gamma = x\widehat{\Gamma} \cap \partial\Gamma$.

[2] We use the convention $[\![1, \infty]\!] = \mathbb{N}^* \cup \{\infty\}$.

Proposition 2.2 *$\widehat{\Gamma}$ and $\partial\Gamma$ are compact spaces. $\widehat{\Gamma}$ is a compactification of Γ.*

Proof: $\partial\Gamma = \mathscr{A}^{\mathbb{N}^*}$, thus the set $\partial\Gamma$ endowed with the product topology is compact. This topology coincides with the one induced by the restriction of $\widehat{d}$ on $\partial\Gamma$ (for $x \in \partial\Gamma$, the product topology gives us the same basis of neighbourhoods $\{x_n\partial\Gamma\}_{n\in\mathbb{N}}$ as $\widehat{d}|_{\partial\Gamma}$).

Since $\partial\Gamma$ is compact, in order to show that $\widehat{\Gamma}$ is compact, it suffices to remark that $\cup_{x\in\partial\Gamma}\widehat{B}(x, \exp(-k)) = \{y\widehat{\Gamma} \mid |y| = k+1\}$ has a finite complementary in $\widehat{\Gamma}$, for all $k \in \mathbb{N}$. Since Γ is dense in $\widehat{\Gamma}$, $\widehat{\Gamma}$ is a compactification of Γ. $\square$

Notice also that if $\nu > 1$, the topological space $\partial\Gamma$ is perfect.

The C^*-algebra $C(\widehat{\Gamma})$ of continuous complex-valued functions on $\widehat{\Gamma}$ plays an important rôle. The dense embedding $\Gamma \subset \widehat{\Gamma}$ gives a canonical inclusion $C(\widehat{\Gamma}) \subset C_b(\Gamma)$ ($C_b(\Gamma)$ is the space of bounded complex-valued functions on Γ). Moreover, we have

$$C_0(\Gamma) = \{f \in C(\widehat{\Gamma}) \mid f|_{\partial\Gamma} = 0\}, \tag{2.3}$$

where $C_0(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \forall\varepsilon > 0, \exists M > 0 \mid |x| > M \Rightarrow |f(x)| < \varepsilon\}$. We shall often abbreviate $C_0(\Gamma)$ by C_0.

The following proposition gives us a better understanding of the functions in $C(\widehat{\Gamma})$.

Proposition 2.3 *Let E be a metrisable topological space. A function $V : \Gamma \to E$ extends to a continuous function $\widehat{V} : \widehat{\Gamma} \to E$ if and only if for each $x \in \partial\Gamma$ the limit of $V(y)$, when $y \in \Gamma$ converges to x, exists.*

Proof: Let $x \in \partial\Gamma$ and $\widehat{V}(x)$ be the above limit. Let F be a closed neighbourhood of $\widehat{V}(x)$ in E; there is k such that $V(x_k\Gamma) \subset F$. Then $x_k\widehat{\Gamma}$ is a neighbourhood of x in $\widehat{\Gamma}$ and, since F is closed, we have $\widehat{V}(x_k\widehat{\Gamma}) \subset F$. $\square$

Later on, we will need the next ultrametricity result. We will say that $\mathscr{U} = \{x_i\Gamma\}$ is a *covering of* $\partial\Gamma$ if $\widehat{\mathscr{U}} = \{x_i\widehat{\Gamma}\}$ is a covering of $\partial\Gamma$.

Proposition 2.4 *For each open covering $\{\mathscr{O}_i\}_{i\in I}$ of $\partial\Gamma$, there is a disjoint and finite covering $\{x_j\Gamma\}_{j\in J}$ of $\partial\Gamma$ such that for each $j \in J$ there is $i \in I$ such that $x_j\widehat{\Gamma} \subset \mathscr{O}_i$.*

Proof: For each $x \in \partial\Gamma$ there is i such that x belongs to the open set $\mathscr{O}_i$ and there is $n = n(x,i)$ such that $x_n\widehat{\Gamma} \subset \mathscr{O}_i$. Since $\partial\Gamma$ is compact, there is a finite sub-covering of $\partial\Gamma$ made by sets $\{y_j\widehat{\Gamma}\}_{j\in[\![1,m]\!]}$ such that each of its elements is a subset of some $\mathscr{O}_i$. But in ultrametric spaces two balls are either disjoint or one of them is included in the other one. Since $\{y_j\widehat{\Gamma}\}$ are balls, we get the result. One may also choose $\{y\widehat{\Gamma} \mid |y| = \max_{j\in[\![1,m]\!]} |y_j|\}$ as the required covering. □

Remark: As we said previously, this section could be presented from the perspective of hyperbolicity in the sense of Gromov, see [2, Chapter V] (a deeper investigation can be found in [4] and [7]). Let (M,d) be a metric space. For $x, y \in M$ and a given $O \in M$, we define the *Gromov product* as:

$$(x,y)_O = \frac{1}{2}(d(O,x) + d(O,y) - d(x,y)). \tag{2.4}$$

The space (M,d) is called δ-*hyperbolic* if there is δ such that for all x, y, z, $O \in M$,

$$(x,y)_O \geq \min((x,z)_O, (z,y)_O) - \delta. \tag{2.5}$$

A metric space is *hyperbolic* if it is δ-hyperbolic for a certain δ. In fact, if there is δ such that (2.5) holds for all $x, y, z \in M$ and a given O then (M,d) is 2δ-hyperbolic. Classical examples of 0-hyperbolic spaces are trees (connected graphs with no cycle) and real trees (see [7] for this notion). Cartan-Hadamard manifolds, the Poincaré half-plane and, more generally, complete simply connected manifolds with sectional curvature bounded by $\kappa < 0$ are δ-hyperbolic spaces with $\delta > 0$.

We equip the set of sequences with values in M with an equivalence relation between (u_n) and (v_n) defined by the condition $\lim_{(n,m)\to\infty}(u_n, v_m)_O = \infty$. The boundary at infinity ∂M is the set of equivalence classes. A basis of open sets of ∂M is given by

$$\widetilde{\mathscr{O}} = \{\gamma \in \partial M \mid \gamma \text{ is not associated to any sequence of } M \setminus \mathscr{O}\},$$

where $\mathscr{O}$ is an open set of M. The boundary of a 0-hyperbolic space is ultrametric.

In our context, if we drop the convention $v(x,x) = \infty$, our valuation (2.1) is exactly (2.4). Hence (2.2) implies that Γ is 0-hyperbolic. We define a *geodesic ray* as being $\gamma : \mathbb{N} \to \Gamma$ such that $|\gamma(n)| = n$ and $\gamma(n+1) \sim$

$\gamma(n)$. Geodesic rays are representative elements of the above equivalence classes. The two notions of boundary at infinity are identified by setting $x_n = \gamma(n)$.

3 Operators in $\ell^2(\Gamma)$

3.1 Bounded and compact operators

We are interested in operators acting on the Hilbert space $\ell^2(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \sum_{x\in\Gamma} |f(x)|^2 < \infty\}$ endowed with the inner product: $\langle f, g\rangle = \sum_{x\in\Gamma} \overline{f(x)}g(x)$. We embed $\Gamma \subset \ell^2(\Gamma)$ by identifying x with $\chi_{\{x\}}$, where χ_A is the characteristic function of the set A. Observe that Γ is the canonical orthonormal basis in $\ell^2(\Gamma)$ and each $f \in \ell^2(\Gamma)$ writes as $f = \sum_{x\in\Gamma} f(x)x$.

We denote by $\mathbb{B}(\Gamma)$, $\mathbb{K}(\Gamma)$ the sets of bounded, respectively compact operators in $\ell^2(\Gamma)$. For $T \in \mathbb{B}(\Gamma)$, we will denote by T^* its adjoint. Given $A \subset \Gamma$ we denote by $\mathbf{1}_A$ the operator of multiplication by χ_A in $\ell^2(\Gamma)$. The orthogonal projection associated to $\{x \in \Gamma \mid |x| \geq r\}$ is denoted by $\mathbf{1}_{\geq r}$. For $T \in \Gamma$, we have the following compacity criterion for bounded operators T in $\ell^2(\Gamma)$:

Proposition 3.1 $T \in \mathbb{K}(\Gamma) \iff \|\mathbf{1}_{\geq r}T\| \underset{r\to\infty}{\longrightarrow} 0 \iff \|T\mathbf{1}_{\geq r}\| \underset{r\to\infty}{\longrightarrow} 0.$

Proof: If one has for example $\|\mathbf{1}_{\geq r}T\| \to 0$, then T is the norm limit of the sequence of finite rank operators $\mathbf{1}_{B(e,r)}T$, hence is compact. $\square$

3.2 The operator ∂

We now extend $x \mapsto x'$ to a map $\ell^2(\Gamma) \to \ell^2(\Gamma)$. We set $e' = 0$ and define the derivative of any $f \in \ell^2(\Gamma)$ as:

$$(\partial f)(x) \equiv f'(x) = \sum_{y\in\Gamma} f(y)y'(x) = \sum_{y'=x} f(y) = \sum_{y\in\widetilde{x}} f(y).$$

Thus $\partial \in \mathbb{B}(\Gamma)$. Indeed, $\|f'\|^2 = \sum_{x\in\Gamma} |f'(x)|^2 \leq \nu \sum_{x\in\Gamma} \sum_{y\in\widetilde{x}} |f(y)|^2 \leq \nu\|f\|^2$. The adjoint ∂^* acts on each $f \in \ell^2(\Gamma)$ as follows:

$$\partial^* f(x) = \chi_{\Gamma\setminus\{e\}}(x)f(x').$$

Indeed, $\langle \partial f, f\rangle = \sum_{x\in\Gamma}\sum_{y\in\widetilde{x}} \overline{f(y)}f(x) = \sum_{x\in\Gamma}\overline{f(x)}\chi_{\Gamma\setminus\{e\}}(x)f(x') = \langle f, \partial^* f\rangle$. Moreover, $\|\partial^* f\|^2 = \sum_{x\in\Gamma\setminus\{e\}}|f(x')|^2 = \nu\sum_{x\in\Gamma}|f(x)|^2 = \nu\|f\|^2$ shows that

$$\partial\partial^* = \nu\mathrm{Id}. \tag{3.1}$$

Thus $\partial^*/\sqrt{\nu}$ is isometric on $\ell^2(\Gamma)$ and $\|\partial\| = \|\partial^*\| = \sqrt{\nu}$.

For $\alpha \in \mathbb{N}$ we set $f^{(\alpha)} = \partial^\alpha f$. Thus for each $x \in \Gamma$, $x^{(\alpha)}$ is well defined in $\ell^2(\Gamma)$ and $x^{(\alpha)} = 0 \Leftrightarrow \alpha > |x|$. For $|x| \geq \alpha$ the notation is consistent with our old definition.

3.3 C^*-algebras of energy observables related to Γ

We first summarize the method used in [6] to study the essential spectrum of large families of operators. Let $\mathscr{H}$ be a Hilbert space and H a bounded self-adjoint operator on $\mathscr{H}$. If $C(\mathscr{H}) = B(\mathscr{H})/K(\mathscr{H})$ is the Calkin C^*-algebra, we denote by $S \mapsto \widehat{S}$ the canonical surjection of $B(\mathscr{H})$ onto $C(\mathscr{H})$ and we recall that $\sigma_{ess}(H) = \sigma(\widehat{H})$ (this is a version of Weyl's Theorem). If $\mathfrak{C}$ is a C^*-subalgebra of $B(\mathscr{H})$ which contains the compact operators, then one has a canonical embedding $\mathfrak{C}/K(\mathscr{H}) \subset C(\mathscr{H})$. Thus, in order to determine the essential spectrum of an operator $H \in \mathfrak{C}$ it suffices to give a good description of the quotient $\mathfrak{C}/K(\mathscr{H})$ and to compute $\widehat{H}$ as element of it. As explained in [6], we can actually go further by taking H as an unbounded operator over $\mathscr{H}$ such that $(H+i)^{-1} \in \mathfrak{C}$. We shall apply this strategy in our context.

Let $\mathscr{D}_{\mathrm{alg}}$ be the $*$-algebra of operators in $\ell^2(\Gamma)$ generated by ∂ and $\mathscr{D}$ the C^*-algebra of operators in $\ell^2(\Gamma)$ generated by ∂. Because of (3.1), $\mathscr{D}_{\mathrm{alg}}$ is unital. We denote by $\varphi(Q)$ the operator of multiplication by φ on $\ell^2(\Gamma)$. If C is a C^*-subalgebra of $\ell^\infty(\Gamma)$ then we embed C in $\mathbb{B}(\Gamma)$ by $\varphi \mapsto \varphi(Q)$. Let $\langle\mathscr{D}, C\rangle$ be the C^*-algebra generated by $\mathscr{D}\cup C$. In this paper we shall take $\mathfrak{C} = \langle\mathscr{D}, C\rangle$. This algebra contains many Hamiltonians of physical interest, for instance Schrödinger operators with potentials in C. We recall that given a graph G the Laplace operator acts on $\ell^2(G)$ as follows:

$$(\Delta f)(x) = \sum_{y\sim x}(f(y) - f(x)).$$

With our definitions $\Delta = \partial + \partial^* - \nu \mathrm{Id} + \chi_{\{e\}}$. Notice that if $\nu > 1$ then $\mathscr{D}$ does not contain compact operators (see below), so $\Delta \notin \mathscr{D}$. On the other hand, if $C \supset C_0$ and $V \in C$ then the Schrödinger operator $\Delta + V(Q)$ clearly belongs to $\langle \mathscr{D}, C \rangle$.

We now give a new description of $\mathbb{K}(\Gamma)$.

Proposition 3.2 *If $\mathscr{C}_0$ be the C^*-algebra generated by $\mathscr{D} \cdot C_0$ then $\mathscr{C}_0 = \mathbb{K}(\Gamma)$.*

Proof: For each $\varphi \in C_0$, Proposition 3.1 shows $\varphi(Q) \in \mathbb{K}(\Gamma)$. Hence $\mathscr{C}_0 \subset \mathbb{K}(\Gamma)$. For the opposite inclusion, let $T \in \mathbb{K}(\Gamma)$ and fix $\varepsilon > 0$. Proposition 3.1, shows that there is an operator T' with compactly supported kernel such that $\|T - T'\| \leq \varepsilon$. Define $\delta_{x,y} \in \mathbb{K}(\Gamma)$ by $(\delta_{x,y} f)(z) = f(y)$ if $z = x$ and 0 elsewhere. We have $\delta_{x,x} = \chi_{\{x\}}(Q) \in C_0$. As T' is a linear combination of $\delta_{x,y}$, it suffices to show that $\delta_{x,y}$ is in $\mathscr{C}_0$. But this follows from $\delta_{x,y} = \delta_{x,x} (\partial^*)^{|x|} \partial^{|y|} \delta_{y,y}$. $\square$

If C is a C^*-subalgebra of $\ell^\infty(\Gamma)$ that contains C_0, then $\mathbb{K}(\Gamma) \subset \langle \mathscr{D}, C \rangle$. Hence, in order to apply the technique described above, we have to give a sufficiently explicit description of the quotient $\langle \mathscr{D}, C \rangle / \mathbb{K}(\Gamma)$. In this paper we concentrate on the case $C \equiv C(\widehat{\Gamma})$ which is, geometrically speaking, the most interesting one (see the last Remark in §2.3). *The C^*-algebra generated by ∂ and $C(\widehat{\Gamma})$ will be denoted by $\mathscr{C}(\widehat{\Gamma})$ and the $*$-subalgebra generated by ∂ and $C(\widehat{\Gamma})$ will be denoted by $\mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$.* We will need the next fundamental property.

Proposition 3.3 $[\partial, C(\widehat{\Gamma})] \subset \mathbb{K}(\Gamma)$.

Proof: For each $\varphi \in C(\widehat{\Gamma})$ one has $([\partial, \varphi(Q)] f)(x) = \sum_{y'=x} (\varphi(y) - \varphi(x)) f(y) = (\partial \circ \psi(Q) f)(x)$, where ψ belongs to $C(\widehat{\Gamma})$ and is defined by $\psi(y) = \varphi(y) - \varphi(y')$ when $|y| \geq 1$ and $\psi(e) = 0$. Observe that for $\gamma \in \partial\Gamma$ we have $\psi(\gamma) = \varphi(\gamma) - \varphi(\gamma) = 0$. Hence by (2.3), $\psi \in C_0$. Proposition 3.2 implies $\psi(Q) \in \mathbb{K}(\Gamma)$. $\square$

Remark: The algebra $\mathscr{D}$ is the tree analogous of the algebra generated by the momentum operator on the real line. However, these algebras are rather different: $\mathscr{D}$ is not commutative and the spectrum and the essential spectrum of the operators from $\mathscr{D}$ are not connected sets in general. For instance, one has $\sigma(\partial^* \partial) = \sigma_{\mathrm{ess}}(\partial^* \partial) = \{0, \nu\}$ if $\nu > 1$. Indeed, we remind that if A, B are elements of a Banach algebra we have $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$ and, as noticed below, $\dim \operatorname{Ker} \partial$ is infinite for $\nu > 1$.

3.4 Translations in $\ell^2(\Gamma)$

Γ acts on itself to the left and to the right: for each $a \in \Gamma$ we may define $\lambda_a, \rho_a : \Gamma \to \Gamma$ by $\lambda_a(x) = ax$ and $\rho_a(x) = xa$ respectively. Clearly, for $a, b \in \Gamma$, $\lambda_a \rho_b = \rho_b \lambda_a$ and for any $x \in a\Gamma$ we define $a^{-1}x$ as being the y for which $x = ay$. For each $x \in \Gamma a = \{y \in \Gamma \mid \exists z \in \Gamma \text{ s.t. } y = za\}$, we define $y = xa^{-1}$ by $x = ya$. We extend now these translations to $\ell^2(\Gamma)$. The translation λ_a acts on each $f \in \ell^2(\Gamma)$ as $\sum_{x\in\Gamma} f(x)ax$, i.e. $(\lambda_a f)(x) = \chi_{a\Gamma}(x) f(a^{-1}x)$. In the same manner, we define $(\rho_a f)(x) = \chi_{\Gamma a}(x) f(xa^{-1})$. The operators λ_a and ρ_a are isometries:

$$\lambda_a^* \lambda_a = \text{Id and } \rho_a^* \rho_a = \text{Id}. \tag{3.2}$$

It is easy to check that the adjoints act on any $f \in \ell^2(\Gamma)$ as $(\lambda_a^* f)(x) = f(ax)$ and $(\rho_a^* f)(x) = f(xa)$. Moreover,

$$\lambda_a \lambda_a^* = \mathbf{1}_{a\Gamma} \text{ and } \rho_a \rho_a^* = \mathbf{1}_{\Gamma a}. \tag{3.3}$$

Note also that $\partial^* = \sum_{|a|=1} \rho_a$ and $\partial = \sum_{|a|=1} \rho_a^*$.

3.5 Localizations at infinity

In order to study $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$ we have to define the localizations at infinity of $T \in \mathscr{C}(\widehat{\Gamma})$ by looking at the behavior of the translated operator $\lambda_a^* T \lambda_a$ as a converges to γ in $\widehat{\Gamma}$ (abbreviated $a \to \gamma$), for each $\gamma \in \partial\Gamma$.

If $T \in \mathbb{K}(\Gamma)$ then $\text{u-}\lim_{a\to\gamma} \lambda_a^* T \lambda_a = 0$, where $\text{u-}\lim$ means convergence in norm. Indeed, by (3.2), (3.3) and Proposition 3.1 we get $\|\lambda_a^* T \lambda_a\| = \|\mathbf{1}_{a\Gamma} T \mathbf{1}_{a\Gamma}\| \to 0$, as $a \to \gamma$. Now, we compute the uniform limit of $\lambda_a^* T \lambda_a$ when $T \in \mathscr{C}(\widehat{\Gamma})_{\text{alg}}$. There is P, a non-commutative complex polynomial in $m+2$ variables, and functions $\varphi_i \in C(\widehat{\Gamma})$ for $i = [\![1, m]\!]$, such that $T = P(\varphi_1, \varphi_2, \ldots, \varphi_m, \partial, \partial^*)$. We set $T(\gamma) = P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \partial, \partial^*)$.

Lemma 3.4 *There is* $a_0 \in \Gamma$ *such that* $\text{u-lim}_{a\to\gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma)\lambda_{a_0}$.

Proof: The Proposition 3.3 and (3.1) give some $\phi_k \in C(\widehat{\Gamma})$, $K \in \mathbb{K}(\Gamma)$ and $\alpha_k, \beta_k \in \mathbb{N}$ such that $T = \sum_{k=1}^n \phi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k} + K$ and $T(\gamma) = \sum_{k=1}^n \phi_k(\gamma)\partial^{*\alpha_k}\partial^{\beta_k}$. Thus, it suffices to compute a limit of the form $\text{u-lim}_{a\to\gamma} \lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a$ with $\varphi \in C(\widehat{\Gamma})$. We suppose $|a| \geq \alpha$ and take $f \in \ell^2(\Gamma)$. We first show the result for $\varphi = 1$. Since

$$(\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a f)(x) = \sum_{\{y|y^{(\beta)}=(ax)^{(\alpha)}\}} (\lambda_a f)(y) = \sum_{\{y|(ay)^{(\beta)}=(ax)^{(\alpha)}\}} f(y), \tag{3.4}$$

it suffices to show that the set $\{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\}$ is independent of a if $|a| \geq \alpha$. But this is precisely what asserts the Lemma 3.5 below.

We now treat the general case $\varphi \in C(\widehat{\Gamma})$. The identity $(\lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a f)(x) = \varphi(ax)(\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a f)(x)$ gives us that $\|\lambda_a^*\varphi(Q)\partial^{*\alpha}\partial^\beta\lambda_a - \varphi(\gamma)\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a\| \leq \|\varphi(aQ)-\varphi(\gamma)\|\cdot\|\partial^{*\alpha}\partial^\beta\| \to 0$ as $a \to \gamma$. On the other hand, by the Lemma 3.5, $\varphi(\gamma)\lambda_a^*\partial^{*\alpha}\partial^\beta\lambda_a$ is constant for $|a| \geq \alpha$. Thus, it suffices to choose $|a_0| \geq \max\{\alpha_k \mid k = 1, \ldots, n\}$ in the statement of the lemma to end the proof. □

Lemma 3.5 *For* $|a| \geq \alpha$ *we have (see Subsection 2.2 for notations):*

$$\{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\} = \begin{cases} \emptyset & \text{for } |x| + \beta - \alpha < 0, \\ S^{|x|+\beta-\alpha} & \text{for } |x| < \alpha \text{ and } |x| + \beta - \alpha \geq 0, \\ x^{(\alpha)}S^\beta & \text{for } |x| \geq \alpha \text{ and } |x| + \beta - \alpha \geq 0. \end{cases} \tag{3.5}$$

Proof: Let $J_x = \{y \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\}$. Then

$$\begin{aligned} aJ_x &= \{ay \mid (ay)^{(\beta)} = (ax)^{(\alpha)}\} = \{y \mid y^{(\beta)} = (ax)^{(\alpha)}\} \cap a\Gamma \\ &= ((ax)^{(\alpha)}S^\beta(\Gamma)) \cap a\Gamma. \end{aligned}$$

We first notice that $(ax)^{(\alpha)}S^\beta \subset S^{|a|+|x|-\alpha+\beta}$. If $|x| - \alpha + \beta < 0$ then $((ax)^{(\alpha)}S^\beta) \cap a\Gamma = \emptyset$, so $aJ_x = \emptyset$. This implies $J_x = \emptyset$. If $|x| - \alpha + \beta \geq 0$ then $((ax)^{(\alpha)}S^\beta) \cap a\Gamma \neq \emptyset$. If we suppose that $|x| < \alpha$, i.e. $|(ax)^{(\alpha)}| < |a|$, we have $a \in (ax)^{(\alpha)}\Gamma$. Let b such that $a = (ax)^{(\alpha)}b$. Thus

$$\begin{aligned} ((ax)^{(\alpha)}S^\beta) \cap a\Gamma &= ((ax)^{(\alpha)}S^\beta) \cap (ax)^{(\alpha)}b\Gamma = (ax)^{(\alpha)}(S^\beta \cap b\Gamma) \\ &= (ax)^{(\alpha)}bS^{\beta-|b|} = aS^{\beta-|b|} = aS^{\beta+|x|-\alpha}, \end{aligned}$$

so we have $aJ_x = aS^{\beta+|x|-\alpha}$, hence $J_x = S^{\beta+|x|-\alpha}$.

Finally, if $|x| \geq \alpha$, i.e. $|(ax)^{(\alpha)}| \geq |a|$, one has $(ax)^{(\alpha)} \in a\Gamma$. Thus we obtain $aJ_x = (ax)^{(\alpha)}S^\beta = ax^{(\alpha)}S^\beta$, hence $J_x = x^{(\alpha)}S^\beta$. $\square$

Remark: As seen in the proof of lemma 3.4, one may choose any a_0 such that $|a_0| \geq \deg(P)$. On the other hand, we stress that the limit is not a multiplicative function of T. Indeed,

$$\text{u-}\lim_{a\to\gamma} \lambda_a^* \partial^* \partial \lambda_a \neq (\text{u-}\lim_{a\to\gamma} \lambda_a^* \partial^* \lambda_a) \cdot (\text{u-}\lim_{a\to\gamma} \lambda_a^* \partial \lambda_a).$$

Therefore, in order to describe the morphism of the algebra $\mathscr{C}(\widehat{\Gamma})$ onto its quotient $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$ we have to improve our definition of the localizations at infinity.

3.6 Extensions to $\widetilde{\Gamma}$

The space $\ell^2(\widetilde{\Gamma})$ is defined similarly to $\ell^2(\Gamma)$. Since $\Gamma \subset \widetilde{\Gamma}$, we have $\ell^2(\Gamma) \hookrightarrow \ell^2(\widetilde{\Gamma})$. As before, we embed $\widetilde{\Gamma}$ in $\ell^2(\widetilde{\Gamma})$ by sending x on $\chi_{\{x\}}$ and we notice that $\widetilde{\Gamma}$ is an orthonormal basis of $\ell^2(\widetilde{\Gamma})$. We define $\widetilde{\partial} : \ell^2(\widetilde{\Gamma}) \to \ell^2(\widetilde{\Gamma})$ by

$$(\widetilde{\partial} f)(x) = f'(x) = \sum_{y'=x} f(y).$$

For $\alpha \in \mathbb{N}$, we set $f^{(\alpha)} = \widetilde{\partial}^\alpha f$, notation which is consistent with our old definition of $x^{(\alpha)}$ as the restriction of x to $\mathbb{Z}_{|x|-\alpha}$. Obviously $\widetilde{\partial} \in \mathbb{B}(\Gamma)$, its adjoint $\widetilde{\partial}^*$ acts as $(\widetilde{\partial}^* f)(x) = f(x')$, $\widetilde{\partial}^*/\sqrt{\nu}$ is an isometry on $\ell^2(\widetilde{\Gamma})$:

$$\widetilde{\partial}\widetilde{\partial}^* = \nu \mathrm{Id}, \tag{3.6}$$

thus $||\widetilde{\partial}|| = ||\widetilde{\partial}^*|| = \nu$. *We denote by $\widetilde{\mathscr{D}}$ the C^*-algebra generated by $\widetilde{\partial}$ and by $\widetilde{\mathscr{D}}_{\mathrm{alg}}$ the $*$-algebra generated by $\widetilde{\partial}$.* Both of them are unital.

We now make the connection between $\mathscr{D}_{alg}$ and $\widetilde{\mathscr{D}}_{alg}$.

Lemma 3.6 *For $|a| \geq \alpha$, one has:* $\lambda_a^* \partial^{*\alpha} \partial^\beta \lambda_a = \mathbf{1}_\Gamma \widetilde{\partial}^{*\alpha} \widetilde{\partial}^\beta \mathbf{1}_\Gamma$.

Proof: For any $f \in \ell^2(\widetilde{\Gamma})$, one has $(\mathbf{1}_\Gamma \widetilde{\partial}^{*\alpha} \widetilde{\partial}^\beta \mathbf{1}_\Gamma f)(x) = \mathbf{1}_\Gamma(x) \sum_{\{y|y^{(\beta)}=x^{(\alpha)}\}} \mathbf{1}_\Gamma(y) f(y)$. Using the same arguments as in the proof of the Lemma 3.5, one shows that for each $x \in \Gamma$ the set $\{y \in \Gamma \mid y^{(\beta)} = x^{(\alpha)}\}$ equals the r.h.s. of (3.5). Thus the above sum is the same as that of the r.h.s. of (3.4).$\square$

We will also need a result concerning the localization of the norm on $\widetilde{\mathscr{D}}_{\mathrm{alg}}$.

Lemma 3.7 *If $\widetilde{T} \in \widetilde{\mathscr{D}}_{alg}$, then $\|\widetilde{T}\| = \|\mathbf{1}_\Gamma \widetilde{T} \mathbf{1}_\Gamma\|$.*

Proof: Because of (3.6), we can suppose that $\widetilde{T} = \sum_{k=1}^{n} c_k \widetilde{\partial}^{*\alpha_k} \widetilde{\partial}^{\beta_k}$. We denote by β the integer $\max\{\beta_k \mid k \in [\![1, n]\!]\}$. For each $\varepsilon > 0$, there is some $g \in \ell^2(\widetilde{\Gamma})$ with compact support such that $\|g\| = 1$ and $\|\widetilde{T}g\| \geq \|\widetilde{T}\| - \varepsilon$. Note that if $y_1, y_2, \ldots, y_m$ are distinct points of Γ, $a_1, a_2, \ldots, a_m$ are complex numbers and $x_1, x_2 \in \widetilde{\Gamma}$, we have

$$\|\sum_{i=1}^{m} a_i x_1 y_i\|^2 = \sum_{i=1}^{m} |a_i|^2 = \|\sum_{i=1}^{m} a_i x_2 y_i\|^2. \tag{3.7}$$

Thus, since g has compact support, there are $x \in \widetilde{\Gamma}$, $m \in \mathbb{N}^*$ and $y_i \in \Gamma$, $|y_i| \geq \beta$, $a_i \in \mathbb{C}$, for all $i \in [\![1, m]\!]$ such that $g = \sum_{k=1}^{m} a_i x y_i$. We set $f = \sum_{k=1}^{m} a_i e y_i$. Then (3.7) gives us $\|f\| = \|g\| = 1$. Using $|y_i| \geq \beta$, we get $f \in \ell^2(\Gamma)$ and $\widetilde{T}f \in \ell^2(\Gamma)$. Also with (3.7) we obtain for $z \in \Gamma$,

$$\begin{aligned}
\|\widetilde{T}g\| &= \|\sum_{k=1}^{n} \sum_{i=1}^{m} c_k a_i \widetilde{\partial}^{*\alpha_k} \widetilde{\partial}^{\beta_k} x y_i\| = \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i (x y_i)^{(\beta_k)} z\| \\
&= \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i x (y_i)^{(\beta_k)} z\| = \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i e (y_i)^{(\beta_k)} z\| \\
&= \|\sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{|z|=\alpha_k} c_k a_i (e y_i)^{(\beta_k)} z\| = \|\sum_{k=1}^{n} \sum_{i=1}^{m} c_k a_i \widetilde{\partial}^{*\alpha_k} \widetilde{\partial}^{\beta_k} e y_i\| = \|\widetilde{T}f\|.
\end{aligned}$$

Hence, there is $f \in \ell^2(\widetilde{\Gamma})$ such that $\|\mathbf{1}_\Gamma \widetilde{T} \mathbf{1}_\Gamma f\| = \|\widetilde{T}f\| = \|\widetilde{T}g\| \geq \|\widetilde{T}\| - \varepsilon$. $\square$

4 The main results

4.1 The morphism

In the sequel, a morphism will be understood as a morphism of C^*-algebras. To describe the quotient $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$, we need to find an adapted morphism.

Theorem 4.1 *For each $\gamma \in \partial\Gamma$ there is a unique morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}$ such that $\Phi_\gamma(\partial) = \widetilde{\partial}$ and $\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)$, for all $\varphi \in C(\widehat{\Gamma})$. One has $\mathbb{K}(\Gamma) \subset \operatorname{Ker}\Phi_\gamma$.*

Proof: We use the notations from §3.5. If $T \in \mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$ then by Lemma 3.4 we have $\text{u-}\lim_{a\to\gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma)\lambda_{a_0}$. Let $\widetilde{T}(\gamma)$ be $P(\varphi_1(\gamma), \varphi_2(\gamma), \dots, \varphi_m(\gamma), \widetilde{\partial}, \widetilde{\partial}^*)$. By Lemma 3.6 and (3.6) one can choose a_0 such that $\lambda_{a_0}^* T(\gamma)\lambda_{a_0} = \mathbf{1}_\Gamma \widetilde{T}(\gamma)\mathbf{1}_\Gamma$. Lemma 3.7 implies

$$\|\widetilde{T}(\gamma)\| = \|\mathbf{1}_\Gamma \widetilde{T}(\gamma)\mathbf{1}_\Gamma\| = \|\lambda_{a_0}^* T(\gamma)\lambda_{a_0}\| = \|\text{u-}\lim_{a\to\gamma} \lambda_a^* T \lambda_a\| \le \|T\|.$$

Thus there is a linear multiplicative contraction $\Phi_\gamma^0 : \mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}} \to \widetilde{\mathscr{D}}$, $\Phi_\gamma^0(T) = T(\gamma)$. The density of $\mathscr{C}(\widehat{\Gamma})_{\mathrm{alg}}$ in $\mathscr{C}(\widehat{\Gamma})$ allows us to extend Φ_γ^0 to a morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}$ which clearly satisfies the conditions of the theorem. The uniqueness of Φ_γ is obvious and the last assertion of the theorem follows from the Proposition 3.2. □

4.2 The case $\nu > 1$

In this case, we are able to improve the Theorem 4.1. We recall first that an isometry is said to be *proper* if it is not unitary. The operators ∂^* and $\widetilde{\partial}^*$ are proper isometries and the dimensions of the kernels of ∂ and $\widetilde{\partial}$ are infinite: in the case of ∂, if one lets a, b be two different letters of $\mathscr{A}$, and one chooses $g \in \ell^2(\Gamma a)$ and $h \in \ell^2(\Gamma b)$ such that $h(xb) = g(xa)$ for all $x \in \Gamma$, then $g - h$ is in $\operatorname{Ker}\partial$.

Let $\mathbb{T}$ be the unit circle of $\mathbb{R}^2$ and H^2 the closure of the subspace spanned by $\{e^{inQ}, n \in \mathbb{N}\}$ in $\ell^2(\mathbb{T})$. For $g \in L^\infty(\mathbb{T})$, we define the *Toeplitz operator* T_g on H^2 by $T_g h = P_{H^2} g h$, where P_{H^2} is the projection on H^2.

We denote by $\mathscr{T}$ the C^*-algebra generated by T_u, where we u is the map $u(z) = z$. The next theorem is due to Coburn (see [5] for a proof).

Theorem 4.2 *If S is a proper isometry, then there is a unique isomorphism $\mathscr{J}$ of $\mathscr{T}$ onto $\mathscr{S}$, the C^*-algebra generated by S, such that $\mathscr{J}(T_u) = S$.*

If $\nu > 1$ then ∂^* and $\widetilde{\partial}^*$ are proper isometries, hence there is a unique isomorphism $\mathscr{J}$ of $\mathscr{D}$ onto $\widetilde{\mathscr{D}}$ such that $\mathscr{J}(\partial) = \mathscr{J}(\widetilde{\partial})$. So we can rewrite our Theorem 4.1 as follows:

Theorem 4.3 *Assume that $\nu > 1$ and let $\gamma \in \partial\Gamma$. Then there is a unique morphism $\Phi_\gamma : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D}$ such that $\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)$ for all $\varphi \in C(\widehat{\Gamma})$ and $\Phi_\gamma(D) = D$ for all $D \in \mathscr{D}$.*

Remark: When $\nu = 1$, there is no isomorphism $\mathscr{J} : \mathscr{D} \to \widetilde{\mathscr{D}}$ such that $\mathscr{J}(\partial) = \widetilde{\partial}$ because $\widetilde{\mathscr{D}}$ is commutative. Thus, in this case, one cannot hope in a result as above. There is an other way of proving Theorem 4.3 which uses the next proposition.

Proposition 4.4 *If $\nu \geq 1$ then $\{\partial^{*\alpha}\partial^\beta\}_{\{\alpha,\beta\in\mathbb{N}\}}$ is a basis of the vector space $\mathscr{D}_{\text{alg}}$. One has $\nu > 1$ if and only if $\{\widetilde{\partial}^{*\alpha}\widetilde{\partial}^\beta\}_{\{\alpha,\beta\in\mathbb{N}\}}$ is a basis of space $\widetilde{\mathscr{D}}_{\text{alg}}$.*

Proof: Let $\lambda_i \neq 0$ for all $i \in [\![1, n]\!]$. Assume that $\sum_{i=1}^n \lambda_i \partial^{*\alpha_i}\partial^{\beta_i} = 0$, where (α_i, β_i) are distinct couples. We set $\underline{\alpha} = \min\{\alpha_i \mid i \in [\![1, n]\!]\}$ and $I = \{ i \mid \alpha_i = \underline{\alpha}\}$. We take $x \in \Gamma$ such that $|x| = \underline{\alpha}$ and we obtain $\sum_{i\in I} \lambda_i(\partial^{\beta_i} f)(e) = 0$. Notice that $\{\beta_i\}_{i\in I}$ are pairwise distinct by hypothesis. Now, by taking $i_0 \in I$ and f the characteristic function of $S_{\beta_{i_0}}$, we get that $\lambda_{i_0} = 0$ which is a contradiction. Hence $\sum_{i=1}^n \lambda_i \partial^{*\alpha_i}\partial^{\beta_i} \neq 0$, i.e. the familly is free. Let now $\nu > 1$ and $\lambda_i \neq 0$ for all $i \in [\![1, n]\!]$. We suppose $\sum_{i=1}^n \lambda_i \widetilde{\partial}^{*\alpha_i}\widetilde{\partial}^{\beta_i} = 0$, with (α_i, β_i) pairwise distinct. We fix $x \in \widetilde{\Gamma}$ and set $\bar{\alpha} = \max\{\alpha_i, i \in [\![1, n]\!]\}$. One has $(\sum_{i=1}^n \lambda_i \widetilde{\partial}^{*\alpha_i}\widetilde{\partial}^{\beta_i} f)(x) = \sum_{i=1}^n \lambda_i \sum_{y\in x^{(\alpha_i)}S^{\beta_i}} f(y) = 0$. Notice that $x^{(\alpha)}S^\beta \cap x^{(\alpha')}S^{\beta'} = \emptyset$ if and only if $\alpha' - \alpha \neq \beta' - \beta$. Taking $f \in \ell^2(S^{|x|-\alpha_1+\beta_1})$, we see that one can reduce oneself o the case when there is some k such that $\alpha_i - \beta_i = k$ for all $i \in [\![1, n]\!]$. Since $x^{(\bar{\alpha}-l)}S^{\bar{\alpha}-k-l} \subset x^{(\bar{\alpha}-1)}S^{\bar{\alpha}-k-1} \subsetneq x^{(\bar{\alpha})}S^{\bar{\alpha}-k}$ for all $l \in [\![1, (\bar{\alpha} - k)]\!]$, there is some $y_0 \in x^{(\bar{\alpha})}S^{\bar{\alpha}-k} \setminus \cup_{\alpha_i\neq\bar{\alpha}} x^{(\alpha_i)}S^{\beta_i}$. Then, taking $f = \chi_{\{y_0\}}$ we get some i_0 such that $\lambda_{i_0} = 0$, which is a contradiction. Hence $\sum_{i=1}^{n} \lambda_i \widetilde{\partial}^{*\alpha_i} \widetilde{\partial}^{*\beta_i} \neq 0$. Finally, since when $\nu = 1$ one has $\widetilde{\partial}\widetilde{\partial}^* = \widetilde{\partial}^* \widetilde{\partial} = \mathrm{Id}$, $\{\widetilde{\partial}^{*\alpha} \widetilde{\partial}^{\beta}\}_{\alpha,\beta \in \mathbb{N}}$ is obviously not a basis. □

4.3 Description of $\mathscr{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma)$

Theorem 4.5 i) *For any $\nu \geq 1$, there is a unique morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$ such that $\Phi(\partial) = \widetilde{\partial} \otimes 1$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$. This morphism is surjective and its kernel is $\mathbb{K}(\Gamma)$.*
ii) *For $\nu > 1$, there is a unique surjective morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \mathscr{D} \otimes C(\partial\Gamma)$ such that $\Phi(\partial) = \partial \otimes 1$, $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$ and $\operatorname{Ker}\Phi = \mathbb{K}(\Gamma)$.*

Once again, as in Remark 4.2, the statement (ii) of the theorem is false if $\nu = 1$. As a corollary of Theorem 4.5 we obtain the following result.

Proposition 4.6 *If $\nu > 1$ then $\mathscr{D} \cap \mathbb{K}(\Gamma) = \{0\}$ and if $\nu = 1$ one has $\mathbb{K}(\Gamma) \subset \mathscr{D}$.*

Proof: Let $\nu > 1$ and $T \in \mathscr{D} \cap \mathbb{K}(\Gamma)$. Theorem 4.5 gives us both $\Phi(T) = T \otimes 1$ and $\Phi(T) = 0$ (since T is compact). For $\nu = 1$, as in the proof of Proposition 3.2, it suffices to prove that $\delta_{x,x}$ is in $\mathscr{D}$. But this is clear since $\delta_{x,x} = \partial^{*|x+1|}\partial^{|x+1|} - \partial^{*|x|}\partial^{|x|}$. □

We devote the rest of the section to the proof of the Theorem 4.5.

Proof: By Theorem 4.1 there is a morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}}^{\partial\Gamma}$ such that $(\Phi(\partial))(\gamma) = \widetilde{\partial}$ and $(\Phi(\varphi(Q)))(\gamma) = \varphi(\gamma)$, for all $\gamma \in \partial\Gamma$, $\varphi \in C(\widehat{\Gamma})$. Since the images of ∂ and $\varphi(Q)$ through Φ belong to the C^*-subalgebra $C(\partial\Gamma, \widetilde{\mathscr{D}})$, and since $\mathscr{C}(\widehat{\Gamma})$ is generated by ∂ and such $\varphi(Q)$, it follows that the range of Φ is included in $C(\partial\Gamma, \widetilde{\mathscr{D}})$. We have $C(\partial\Gamma, \widetilde{\mathscr{D}}) \cong \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$, so we get the required morphism $\Phi : \mathscr{C}(\widehat{\Gamma}) \to \widetilde{\mathscr{D}} \otimes C(\partial\Gamma)$. Now since $\Phi(\partial) = \widetilde{\partial} \otimes 1$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$, and since any function in $C(\partial\Gamma)$ is the restriction of some function from $C(\widehat{\Gamma})$, it follows that Φ is surjective. Its uniqueness is clear. It remains to compute the kernel.

As seen in the Theorem 4.1, $\mathbb{K}(\Gamma) \subset \operatorname{Ker}\Phi$. In the remainder of this section we shall prove the reverse inclusion. For this we need some preliminary lemmas.

Lemma 4.7 *Let $R = \varphi(Q)\partial^{*\alpha}\partial^\beta$ and $\mathscr{U} = \{a_i\Gamma\}_{i\in[\![1,n]\!]}$ be a disjoint covering of $\partial\Gamma$. For each $\varepsilon > 0$ there are $c_1, c_2, \dots, c_m \in \mathrm{Ran}(\varphi)$ and there is a disjoint covering $\mathscr{U}' = \{b_j\Gamma\}_{j\in[\![1,m]\!]}$ of $\partial\Gamma$ finer than $\mathscr{U}$ such that $\|\mathbf{1}_{U'}R - R'\| \le \varepsilon$, where $R' = \sum_{j=1}^m \mathbf{1}_{b_j\Gamma} c_j \partial^{*\alpha}\partial^\beta$ and $U' = \cup_{j=1}^m b_j\Gamma$.*

Proof: Let $\varepsilon > 0$ and denote $\varepsilon/\|\partial^{*\alpha}\partial^\beta\|$ by ε'. Since $\varphi(\partial\Gamma)$ is compact, there are $\gamma_1, \gamma_2, \dots, \gamma_N \subset \partial\Gamma$ such that $\varphi(\partial\Gamma) \subset \cup_{k=1}^N D(\varphi(\gamma_k), \varepsilon')$, where $D(z,r)$ is the complex open disk of center z and ray r. The open sets $\mathscr{O}_{i,k} = a_i\widehat{\Gamma} \cap \varphi^{-1}(D(\varphi(\gamma_k), \varepsilon'))$ cover $\partial\Gamma$. The Proposition 2.4 gives us a disjoint covering $\{b_j\Gamma\}_{j\in[\![1,m]\!]}$ of $\partial\Gamma$ such that for each $j \in [\![1,m]\!]$ there are i and k such that $b_j\widehat{\Gamma} \subset \mathscr{O}_{i,k}$. To simplify the notations, we will denote by γ_j those γ_k associated to $b_j\Gamma$. We set $\mathscr{U}' = \{b_j\Gamma\}_{j\in[\![1,m]\!]}$ and $R' = \sum_{j=1}^n \mathbf{1}_{b_j\Gamma}\varphi(\gamma_j)\partial^{*\alpha}\partial^\beta$. Recall that $\sup_{x\in b_j\Gamma} |\varphi(\gamma_j) - \varphi(x)| \le \varepsilon'$, so

$$\begin{aligned}
\|(R' - \mathbf{1}_{U'}R)f\|^2 &= \sum_{x\in\Gamma} |\sum_{j=1}^m \mathbf{1}_{b_j\Gamma}(x)(\varphi(\gamma_j) - \varphi(x))(\partial^{*\alpha}\partial^\beta f)(x)|^2 \\
&= \sum_{j=1}^m \sum_{x\in b_j\Gamma} |(\varphi(\gamma_j) - \varphi(x))(\partial^{*\alpha}\partial^\beta f)(x)|^2 \\
&\le \sum_{j=1}^m \sup_{x\in b_j\Gamma} |\varphi(\gamma_j) - \varphi(x)|^2 \sum_{x\in b_j\Gamma} |(\partial^{*\alpha}\partial^\beta f)(x)|^2 \\
&\le \varepsilon'^2 \sum_{j=1}^m \sum_{x\in b_j\Gamma} |(\partial^{*\alpha}\partial^\beta f)(x)|^2 \\
&\le \varepsilon^2 \|\partial^{*\alpha}\partial^\beta\|^{-2} \cdot \|\partial^{*\alpha}\partial^\beta\|^2 \cdot \|f\|^2 = \varepsilon^2\|f\|^2.
\end{aligned}$$

Denoting $\varphi(\gamma_j)$ by c_j we obtain the result. □

Lemma 4.8 *Let $T = \sum_{k=1}^n \varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k}$ with $\varphi_k \in C(\widehat{\Gamma})$ and let $\varepsilon > 0$. There are a compact operator K, a disjoint covering $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$ of $\partial\Gamma$ and $S = \sum_{k=1}^n \sum_{j=1}^m \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}$, with $\min_{j\in[\![1,m]\!]} |a_j| \ge \max_{k\in[\![1,n]\!]} \alpha_k$ and $\gamma_{j,k} \in \partial\Gamma$ such that $\|T - S - K\| \le \varepsilon$.*

Proof: We denote by $\alpha = \max\{\alpha_k \mid k \in [\![1,n]\!]\}$. Let T_k be $\varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k}$. Setting $\mathscr{U}_0 = \cup_{\{a||a|=\alpha\}}\{a\Gamma\}$, we apply the Lemma 4.7 inductively for $k \in$

$[\![1, n]\!]$ with ε/n instead of ε, $\mathscr{U} = \mathscr{U}_{k-1}$ and $R = T_k$, denoting $\mathscr{U}'$ by $\mathscr{U}_k$ and R' by S_k. Then, for $k \in [\![1, n]\!]$ we get $\|\mathbf{1}_{U_k} T_k - S_k\| \leq \varepsilon/k$. Since $\mathscr{U}_{k+1}$ is finer than $\mathscr{U}_k$ for $k \in [\![1, n-1]\!]$, we obtain $\|\mathbf{1}_{U_n} \sum_{k=1}^n (T_k - S_k)\| \leq \varepsilon$, hence $\|T - \mathbf{1}_{U_n^c} T - \mathbf{1}_{U_n} \sum_{k=1}^n S_k\| \leq \varepsilon$. To finish the proof, we denote the compact operator $\mathbf{1}_{U_n^c} T$ by K, $\mathbf{1}_{U_n} \sum_{k=1}^n S_k$ by S and $\mathscr{U}_n$ by $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$. $\square$

We now go back to the proof of Theorem 4.5. Let $T \in \operatorname{Ker}\Phi$. For each $\varepsilon > 0$ there is $T' \in \mathscr{C}(\widehat{\Gamma})_{\text{alg}}$ such that $\|T - T'\| \leq \varepsilon/4$. By relation (3.1) and Proposition 3.3, we can write $T' = \sum_{k=1}^n \varphi_k(Q)\partial^{*\alpha_k}\partial^{\beta_k} + K$, where $K \in \mathbb{K}(\Gamma)$ and $\varphi_k \in C(\widehat{\Gamma})$. Thus $\|\Phi(T')\| \leq \varepsilon/4$. Using Lemma 4.8, we get an operator S and a compact operator K_1 such that $\|T' - S - K_1\| \leq \varepsilon/4$. This implies that $\|\Phi(S)\| \leq \varepsilon/2$.

Lemma 4.9 *There is $K_2 \in \mathbb{K}(\Gamma)$ such that $\|S - K_2\| \leq \|\Phi(S)\|$.*

Before proving the lemma, let us remark that it implies

$$\|T - K_1 - K_2\| \;\leq\; \|T - T'\| + \|T' - S - K_1\| + \|S - K_2\| \leq \varepsilon.$$

Hence $T \in \mathbb{K}(\Gamma)$. Thus Theorem 4.5 is proved.$\square$

Proof of Lemma 4.9. First, we remark that for each $a \in \Gamma$ and α, $\beta \geq 0$, the Proposition 3.3 gives us that $\mathbf{1}_{a\Gamma}\partial^{*\alpha}\partial^{\beta} - \mathbf{1}_{a\Gamma}\partial^{*\alpha}\partial^{\beta}\mathbf{1}_{a\Gamma}$ is a compact operator. We define $S' = \sum_{k=1}^n \sum_{j=1}^m \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}$ and we set $K_2 = S - S'$, which is a compact operator. Since $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$ is a disjoint covering of $\partial\Gamma$, for any $f \in \ell^2(\Gamma)$:

$$\begin{aligned}
\|S'f\|^2 &= \sum_{x\in\Gamma} |\sum_{k=1}^n \sum_{j=1}^m (\mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma} f)(x)|^2 \\
&= \sum_{j=1}^m \sum_{x\in\Gamma} |\sum_{k=1}^n (\mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma} f)(x)|^2 \\
&\leq \sum_{j=1}^m \|\sum_{k=1}^n \mathbf{1}_{a_j\Gamma}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\mathbf{1}_{a_j\Gamma}\|^2 \cdot \|\mathbf{1}_{a_j\Gamma} f\|^2.
\end{aligned}$$

Now we use (3.2) and (3.3) and get:

$$\|\mathbf{1}_{a_j\Gamma}\big(\sum_{k=1}^n \varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\big)\mathbf{1}_{a_j\Gamma}\| = \|\lambda_{a_j}^*\big(\sum_{k=1}^n \varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\big)\lambda_{a_j}\|.$$

Since $|a_j| \geq \max\{\alpha_k \mid k \in [\![1,n]\!]\}$, the Lemmas 3.6 and 3.7 give us:

$$\begin{aligned} \|\lambda^*_{a_j}\big(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\partial^{*\alpha_k}\partial^{\beta_k}\big)\lambda_{a_j}\| &= \|\mathbf{1}_\Gamma\big(\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial}^{*\alpha_k}\widetilde{\partial}^{\beta_k}\big)\mathbf{1}_\Gamma\| \\ &= \|\sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial}^{*\alpha_k}\widetilde{\partial}^{\beta_k}\|. \end{aligned}$$

For each j we choose $\gamma_j \in a_j\partial\Gamma$. The family $\{a_j\Gamma\}_{j\in[\![1,m]\!]}$ is a disjoint covering of $\partial\Gamma$, so we have $\lim_{x\to\gamma_j}\chi_{a_j\Gamma}(x) = 1$ and $\lim_{x\to\gamma_j}\chi_{a_i\Gamma}(x) = 0$ for $i \neq j$. Hence $\Phi_{\gamma_j}(S') = \sum_{k=1}^{n}\varphi_k(\gamma_{j,k})\widetilde{\partial}^{*\alpha_k}\widetilde{\partial}^{\beta_k}$. We obtain

$$\|S'f\|^2 \leq \sum_{j=1}^{m}\|\Phi_{\gamma_j}(S')\|^2 \cdot \|\mathbf{1}_{a_j\Gamma}f\|^2 \leq \sup_{\gamma\in\partial\Gamma}\|\Phi_\gamma(S')\|^2 \cdot \|f\|^2.$$

Finally, since $\mathbb{K}(\Gamma) \subset \operatorname{Ker}\Phi$, $\|\Phi(S)\| = \|\Phi(S')\| = \sup_{\gamma\in\partial\Gamma}\|\Phi_\gamma(S')\|$. $\square$

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References

[1] Allard, C., Froese, R.: A Mourre estimate for a Schrödinger operator on a binary tree. Rev. Math. Phys **12**, No.12, 1655–1667 (2000)

[2] Ancona, A.: Théorie du potentiel sur les graphes et les variétés. *Ecole d'été de Probabilités de Saint-Flour XVIII – 1988*, Lecture Notes in Mathematics **1427**, Springer-Verlag, Berlin, 1990, pp. 1–112

[3] Bourbaki, N.: *Eléments de mathématiques*, *Algèbre, chapitres 1 à 3*. Diffusion C.C.L.S. Paris, 1970.

[4] Coornaert, M., Delzant, T., Papadopoulos, A.: *Géométrie et théorie des groupes: les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics **1441**, Springer-Verlag, Berlin, 1990.

[5] Davidson, K.R.: *C^*-algebra by examples*. American Mathematical Society (1996).

[6] Georgescu, V., Iftimovici, A.: Crossed products of C^*-algebras and spectral analysis of quantum Hamiltonians. Commun. Math. Phys. **228**, No. 3, 519–560 (2002) .

[7] Ghys, E., De la Harpe, P.: *Sur les groupes hyperboliques d'après Mikhael Gromov*. Birkhäuser, 1990.

[8] Golénia, S.: Mourre estimates for anisotropic operators on trees (in preparation).

[9] Nica, A.: C^*-algebras generated by isometries and Wiener-Hopf operators. J. Operator Theory **27**, No. 1, 17–52 (1992).

[10] Nica, A.: On a groupoid construction for actions of certain inverse semigroups. Internat. J. Math. **5**, No. 3, 349–372 (1994).

[11] Robert, A.M.: *A Course in p-adic Analysis*. Graduate Texts in Mathematics **198**, Springer-Verlag, New York, 2000.

Isometries, Fock Spaces, and Spectral Analysis of Schrödinger Operators on Trees

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Abstract

We construct conjugate operators for the real part of a completely non unitary isometry and we give applications to the spectral and scattering theory of a class of operators on (complete) Fock spaces, natural generalizations of the Schrödinger operators on trees. We consider C^*-algebras generated by such Hamiltonians with certain types of anisotropy at infinity, we compute their quotient with respect to the ideal of compact operators, and give formulas for the essential spectrum of these Hamiltonians.

1 Introduction

The Laplace operator on a graph Γ acts on functions $f : \Gamma \to \mathbb{C}$ according to the relation

$$(\Delta f)(x) = \sum_{y \leftrightarrow x} (f(y) - f(x)), \tag{1.1}$$

where $y \leftrightarrow x$ means that x and y are connected by an edge. The spectral analysis and the scattering theory of the operators on $\ell^2(\Gamma)$ associated to expressions of the form $L = \Delta + V$, where V is a real function on Γ, is an interesting question which does not seem to have been much studied (we have in mind here only situations involving non trivial essential spectrum).

Our interest on these questions has been aroused by the work of C. Allard and R. Froese [All, AlF] devoted to the case when Γ is a binary tree: their main results are the construction of a conjugate operator for L under suitable conditions on the potential V and the proof of the Mourre estimate. As it is well known, this allows one to deduce various non trivial spectral properties of L, for example the absence of the singularly continuous spectrum.

The starting point of this paper is the observation that if Γ is a tree then $\ell^2(\Gamma)$ can be naturally viewed as a Fock space[1] over a finite dimensional Hilbert space and that the operator L has a very simple interpretation in this framework. This suggests the consideration of a general class of operators, abstractly defined only in terms of the Fock space structure. Our purpose then is twofold: first, to construct conjugate operators for this class of operators, hence to point out some of their basic spectral properties, and second to reconsider the kind of anisotropy studied in [Gol] in the present framework.

It seems interesting to emphasize the non technical character of our approach: once the correct objects are isolated (the general framework, the notion of number operator associated to an isometry, the C^*-algebras of anisotropic potentials), the proofs are very easy, of a purely algebraic nature, the arguments needed to justify some formally obvious computations being very simple.

We recall the definition of a ν-fold tree with origin e, where ν is a positive integer and $\nu = 2$ corresponds to a binary tree (see [Gol]). Let A be a set consisting of ν elements and let

$$\Gamma = \bigcup_{n \geq 0} A^n \tag{1.2}$$

where A^n is the n-th Cartesian power of A. If $n = 0$ then A^0 consists of a single element that we denote e. An element $x = (a_1, a_2, \ldots, a_n) \in A^n$ is written $x = a_1 a_2 \ldots a_n$ and if $y = b_1 b_2 \ldots b_m \in A^m$ then $xy = a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n \in A^{n+m}$ with the convention $xe = ex = x$. This provides Γ with a monoïd structure. The graph structure on Γ is defined as follows: $x \leftrightarrow y$ if and only if there is $a \in A$ such that $y = xa$ or $x = ya$.

We embed Γ in $\ell^2(\Gamma)$ by identifying $x \in \Gamma$ with the characteristic function of the set $\{x\}$. Thus Γ becomes the canonical orthonormal basis of $\ell^2(\Gamma)$. In particular, linear combinations of elements of Γ are well defined elements of $\ell^2(\Gamma)$, for example $\sum_{a \in A} a$ belongs to $\ell^2(\Gamma)$ and has norm equal to $\sqrt{\nu}$.

Due to the monoïd structure of Γ, each element v of the linear subspace generated by Γ in $\ell^2(\Gamma)$ defines two bounded operators λ_v and ρ_v on $\ell^2(\Gamma)$, namely the operators of left and right multiplication by v. It is then easy to see that if $v = \sum_{a \in A} a$ then the adjoint operator ρ_v^* acts as follows: if $x \in \Gamma$ then $\rho_v^* x = x'$, where $x' = 0$ if $x = e$ and x' is the unique element in Γ such that $x = x'a$ for some

[1] Note that we use the notion of Fock space in a slightly unusual sense, since no symmetrization or anti-symmetrization is involved in its definition. Maybe we should say “total Fock space”.

$a \in A$ otherwise. Thus the Laplace operator defined by (1.1) can be expressed as follows:

$$\Delta = \rho_v + \rho_v^* + e - (\nu + 1)$$

In the rest of this paper we shall not include in Δ the terms $e - (\nu + 1)$ because e is a function on Γ with support equal to $\{e\}$, hence can be considered as part of the potential, and $\nu + 1$ is a number, so has a trivial contribution to the spectrum. It will also be convenient to renormalize Δ by replacing v by a vector of norm $1/2$, hence by $v/(2\sqrt{\nu})$ if $v = \sum_{a \in A} a$.

We shall explain now how to pass from trees to Fock spaces. We use the following equality (or, rather, canonical isomorphism): if A, B are sets, then

$$\ell^2(A \times B) = \ell^2(A) \otimes \ell^2(B).$$

Thus $\ell^2(A^n) = \ell^2(A)^{\otimes n}$ if $n \geq 1$ and clearly $\ell^2(A^0) = \mathbb{C}$. Then, since the union in (1.2) is disjoint, we have

$$\ell^2(\Gamma) = \bigoplus_{n=0}^{\infty} \ell^2(A^n) = \bigoplus_{n=0}^{\infty} \ell^2(A)^{\otimes n}$$

which is the Fock space constructed over the "one particle" Hilbert space $H = \ell^2(A)$. Thus we are naturally led to the following abstract framework. Let H be a complex Hilbert space and let $\mathscr{H}$ be the Fock space associated to it:

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} H^{\otimes n}. \tag{1.3}$$

Note that H could be infinite dimensional, but this is not an important point here and in the main applications we assume it finite dimensional. We choose an arbitrary vector $u \in H$ with $\|u\| = 1$ and consider the operator $U \equiv \rho_u : \mathscr{H} \to \mathscr{H}$ defined by $Uf = f \otimes u$ if $f \in H^{\otimes n}$. It is clear that U is an isometry on $\mathscr{H}$ and the self-adjoint operator of interest for us is

$$\Delta = \mathrm{Re}\, U = \frac{1}{2}(U + U^*), \tag{1.4}$$

our purpose being to study perturbations $L = \Delta + V$ where the conditions on V are suggested by the Fock space structure of $\mathscr{H}$. In the second part of the paper we shall replace Δ by an arbitrary self-adjoint operator in the C^*-algebra generated by U.

Translating the problem into a Fock space language does not solve it. The main point of the first part of our paper is that we treat a more general problem.

The question is: given an arbitrary isometry on a Hilbert space $\mathscr{H}$ and defining Δ by (1.4), can one construct a conjugate operator for it? We also would like that this conjugate operator be relatively explicit and simple, because we should be able to use it also for perturbations L of Δ.

If U is unitary, there is no much hope to have an elegant solution to this problem. Indeed, for most unitary U the spectrum of Δ will be purely singular. On the other hand, we show that in the opposite case of completely non unitary U, there is a very simple prescription for the construction of a "canonical" conjugate operator. Sections 2 and 3 are devoted to this question in all generality and in Section 4 we give applications in the Fock space framework.

The construction is easy and elementary. Let U be an isometry on a Hilbert space $\mathscr{H}$. We call *number operator associated to* U a self-adjoint operator N on $\mathscr{H}$ such that $UNU^* = N - 1$. The simplest examples of such operators are described in Examples 2.5 and 2.6. It is trivial then to check that, if S is the imaginary part of U, the operator $A := (SN + NS)/2$, satisfies $[\Delta, iA] = 1 - \Delta^2$, hence we have a (strict) Mourre estimate on $[-a, a]$ for each $a \in]0, 1[$.

The intuition behind this construction should be immediate for people using the positive commutator method: in Examples 2.5 and 2.6 the operator Δ is the Laplacian on $\mathbb{Z}$ or $\mathbb{N}$ respectively and S is the operator of derivation, the analog of $P = -i\frac{d}{dx}$ on $\mathbb{R}$, so it is natural to look after something similar to the position operator Q and then to consider the analog of $(PQ + QP)/2$. Note that we got such a simple prescription because we did *not* make a Fourier transform in order to realize Δ as a multiplication operator, as it is usually done when studying discrete Laplacians (e.g. in [AlF]). Note also that the relation $UNU^* = N - 1$ is a discrete version of the canonical commutation relations, cf. (2) of Lemma 2.4.

In the unitary case the existence of N is a very restrictive condition, see Example 2.5. The nice thing is that in the completely non unitary case N exists and is uniquely defined. This is an obvious fact: the formal solution of the equation $N = 1 + UNU^*$ obtained by iteration $N = 1 + UU^* + U^2U^{*2} + \dots$ exists as a densely defined self-adjoint operator if and only if $U^{*n} \to 0$ strongly on $\mathscr{H}$, which means that U is completely non unitary. Finally, observe that the operators ρ_u on the Fock space are completely non unitary, so we can apply them this construction.

Our notation N should not be confused with that used in [AlF]: our N is proportional to their $R - N + 1$, in our notations R being the particle number operator $\boldsymbol{N}$ (see below). We could have used the notation Q for our N, in view of the intuition mentioned above. We have preferred not to do so, because the number operator associated to U in the tree case has no geometric interpretation, as we explain below.

There is no essential difference between the tree model and the Fock space model, besides the fact that we tend to emphasize the geometric aspects in the first representation and the algebraic aspects in the second one. In fact, if H is a finite dimensional Hilbert space equipped with an orthonormal basis $A \subset H$ then the tree Γ associated to A can be identified with the orthonormal basis of $\mathscr{H}$ canonically associated to A, namely the set of vectors of the form $a_1 \otimes a_2 \cdots \otimes a_n$ with $a_k \in A$. In other terms, giving a tree is equivalent with giving a Fock space over a finite dimensional Hilbert space equipped with a certain orthonormal basis. However, this gives more structure than usual on a Fock space: the notions of positivity and locality inherent to the space $\ell^2(\Gamma)$ are missing in the pure Fock space situation, there is no analog of the spaces $\ell^p(\Gamma)$, etc. But our results show that this structure specific to the tree is irrelevant for the spectral and scattering properties of L.

We stress, however, that an important operator in the Fock space setting has a simple geometric interpretation in any tree version. More precisely, let $\boldsymbol{N}$ be the *particle number operator* defined on $\mathscr{H}$ by the condition $\boldsymbol{N} f = nf$ if f belongs to $H^{\otimes n}$. Clearly, if $\mathscr{H}$ is represented as $\ell^2(\Gamma)$, then $\boldsymbol{N}$ becomes the operator of multiplication by the function d, where $d(x) \equiv d(x, e)$ is the distance from the point x to the origin e (see [Gol]).

On the other hand, the number operator N associated to an isometry of the form $U = \rho_u$ is quite different from $\boldsymbol{N}$, it has not a simple geometrical meaning and is not a local operator in the tree case, unless we are in rather trivial situations like the case $\nu = 1$ (see Example 2.6). For this reason we make an effort in Section 4 to eliminate the conditions from Section 3 involving the operator N and to replace them by conditions involving $\boldsymbol{N}$. This gives us statements like that of the Theorem 1.1 below, a particular case of our main result concerning the spectral and scattering theory of the operators L.

We first have to introduce some notations. Let $\mathbf{1}_n$ and $\mathbf{1}_{\geq n}$ be the orthogonal projections of $\mathscr{H}$ onto the subspaces $H^{\otimes n}$ and $\bigoplus_{k \geq n} H^{\otimes k}$ respectively. For real s let $\mathscr{H}_{(s)}$ be the Hilbert space defined by the norm

$$\|f\|^2 = \|\mathbf{1}_0 f\|^2 + \sum_{n \geq 1} n^{2s} \|\mathbf{1}_n f\|^2.$$

If T is an operator on a finite dimensional space E then $\langle T \rangle$ is its *normalized trace*: $\langle T \rangle = \operatorname{Tr}(T)/\dim E$. We denote by $\sigma_{\text{ess}}(L)$ and $\sigma_{\text{p}}(L)$ the essential spectrum and the set of eigenvalues of L. As a consequence of Theorem 4.6, we have:

Theorem 1.1 *Assume that H is finite dimensional, choose $u \in H$ with $\|u\| = 1$, and let us set $\Delta = (\rho_u + \rho_u^*)/2$. Let V be a self-adjoint operator of the form $V = \sum_{n \geq 0} V_n \mathbf{1}_n$, with $V_n \in \mathcal{B}(H^{\otimes n})$, $\lim_{n \to \infty} \|V_n\| = 0$, and such that $\|V_n - \langle V_n \rangle\| + \|V_{n+1} - V_n \otimes \mathbf{1}_H\| \leq \delta(n)$ where δ is a decreasing function such that $\sum_n \delta(n) < \infty$. Let W be a bounded self-adjoint operator satisfying $\sum_n \|W \mathbf{1}_{\geq n}\| < \infty$. We set $L_0 = \Delta + V$ and $L = L_0 + W$. Then:*

(1) $\sigma_{\text{ess}}(L) = [-1, +1]$*;*
(2) *the eigenvalues of L distinct from ± 1 are of finite multiplicity and can accumulate only toward ± 1;*
(3) *if $s > 1/2$ and $\lambda \notin \kappa(L) := \sigma_{\text{p}}(L) \cup \{\pm 1\}$, then $\lim_{\mu \to 0}(L - \lambda - i\mu)^{-1}$ exists in norm in $\mathcal{B}(\mathscr{H}_{(s)}, \mathscr{H}_{(-s)})$, locally uniformly in $\lambda \in \mathbb{R} \setminus \kappa(L)$;*
(4) *the wave operators for the pair (L, L_0) exist and are complete.*

These results show a complete analogy with the standard two body problem on an Euclidean space, the particle number operator $\boldsymbol{N}$ playing the rôle of the position operator. Note that V, W are the analogs of the long range and short range components of the potential. See Proposition 4.4 for a result of a slightly different nature, covering those from [AlF]. Our most general results in the Fock space setting are contained in Theorem 4.6.

The second part of the paper (Section 5) is devoted to a problem of a completely different nature[2]. Our purpose is to compute the essential spectrum of a general class of operators on a Fock space in terms of their "localizations at infinity", as it was done in [GeI] for the case when Γ is an abelian locally compact group.

The basic idea of [GeI] is very general and we shall use it here too: the first step is to isolate the class of operators we want to study by considering the C^*-algebra $\mathscr{C}$ generated by some elementary Hamiltonians and the second one is to compute the quotient of $\mathscr{C}$ with respect to the ideal $\mathscr{C}_0 = \mathscr{C} \cap \mathcal{K}(\mathscr{H})$ of compact operators belonging to $\mathscr{C}$. Then, if $L \in \mathscr{C}$ the projection $\widehat{L}$ of L in the quotient $\mathscr{C}/\mathscr{C}_0$ is the localization of L at infinity we need (or the set of such localizations, depending on the way the quotient is represented). The interest of $\widehat{L}$ comes from the relation $\sigma_{\text{ess}}(L) = \sigma(\widehat{L})$. In all the situations studied in [GeI] these localizations at infinity correspond effectively with what we would intuitively expect.

We stress that both steps of this approach are non trivial in general. The algebra $\mathscr{C}$ must be chosen with care, if it is too small or too large then the quotient will either be too complicated to provide interesting information, or the information we get will be less precise than expected. Moreover, there does not seem to be many techniques for the effective computation of the quotient. One of the main observations in [GeI] is that in many situations of interest in quantum mechanics the configuration space of the system is an abelian locally compact group and then the algebras of interest can be constructed as crossed products; in such a case there is a systematic procedure for computing the quotient.

The techniques from [GeI] cannot be used in the situations of interest here, because the monoïd structure of the tree is not rich enough and in the Fock space

[2] Our initial purpose was also to prove a Mourre estimate for the class of anisotropic operators $\mathscr{C}_\infty$, but we have not succeeded yet.

version the situation is even worse. However, a natural C^*-algebra of anisotropic operators associated to the hyperbolic compactification of a tree has been pointed out in [Gol]. This algebra contains the compact operators on $\ell^2(\Gamma)$ and an embedding of the quotient algebra into a tensor product, which allows the computation of the essential spectrum, has also been described in [Gol]. In Section 5 and in the Appendix we shall improve these results in two directions: we consider more general types of anisotropy and we develop new abstract techniques for the computation of the quotient algebra. To clarify this, we give an example below.

We place ourselves in the Fock space setting with H finite dimensional and we fix a vector $u \in H$ and the isometry U associated to it. We are interested in self-adjoint operators of the form $L = D + V$ where D is a "continuous function" of U and U^*, i.e. it belongs to the C^*-algebra $\mathscr{D}$ generated by U, and V is of the form $\sum V_n \mathbf{1}_n$ where V_n are bounded operators on $H^{\otimes n}$ and are asymptotically constant in some sense (when $n \to \infty$). In order to get more precise results, we make more specific assumptions on the operators V_n.

Let $\mathcal{A} \subset \mathcal{B}(H)$ be a C^*-algebra with $\mathbf{1}_H \in \mathcal{A}$. Let $\mathscr{A}_{\rm vo}$ be the set of operators V as above such that $V_n \in \mathcal{A}^{\otimes n}$, $\sup \|V_n\| < \infty$ and $\|V_n - V_{n-1} \otimes \mathbf{1}_H\| \to 0$ as $n \to \infty$. If $\nu = 1$, i.e. in the setting of Example 2.6, $\mathscr{A}_{\rm vo}$ is the algebra of bounded sequences of vanishing oscillation at infinity. We mention that the C^*-algebra of bounded continuous functions with vanishing oscillation at infinity on a group has first been considered in the context of [GeI] in [Man] (cf. also references therein).

Observe that the algebras $\mathcal{A}^{\otimes n}$ are embedded in the infinite tensor product C^*-algebra $\mathcal{A}^{\otimes\infty}$. Thus we may also introduce the C^*-subalgebra $\mathscr{A}_\infty$ of $\mathscr{A}_{\rm vo}$ consisting of the operators V such that $V_\infty := \lim V_n$ exists in norm in $\mathcal{A}^{\otimes\infty}$. Note that the subset $\mathscr{A}_0$ of operators V such that $\lim V_n = 0$ is an ideal of $\mathscr{A}_{\rm vo}$.

The algebras of Hamiltonians of interest for us can now be defined as the C^*-algebras $\mathscr{C}_{\rm vo}$ and $\mathscr{C}_\infty$ generated by the operators of the form $L = D + V$ where D is a polynomial in U, U^* and $V \in \mathscr{A}_{\rm vo}$ or $V \in \mathscr{A}_\infty$ respectively. Let us denote $\mathscr{C}_0 = \mathscr{C}_{\rm vo} \cap \mathcal{K}(\mathscr{H})$.

Theorem 1.2 *There are canonical isomorphisms*

$$\mathscr{C}_{\rm vo}/\mathscr{C}_0 \simeq (\mathscr{A}_{\rm vo}/\mathscr{A}_0) \otimes \mathscr{D}, \quad \mathscr{C}_\infty/\mathscr{C}_0 \simeq \mathcal{A}^{\otimes\infty} \otimes \mathscr{D}. \tag{1.5}$$

For applications in the computation of the essential spectrum, see Propositions 5.15 and 5.16. For example, if $D \in \mathscr{D}$ and $V \in \mathscr{A}_\infty$ are self-adjoint operators and $L = D + V$, then

$$\sigma_{\rm ess}(L) = \sigma(D) + \sigma(V_\infty). \tag{1.6}$$

The localization of L at infinity in this case is $\widehat{L} = 1 \otimes D + V_\infty \otimes 1$.

To cover perturbations of the Laplacian on a tree by functions V, it suffices to consider an abelian algebra $\mathcal{A}$, see Example 5.13. In this case, if A is the spectrum of $\mathcal{A}$, then $\mathcal{A}^{\otimes\infty} = C(A^\infty)$ where $A^\infty = A^{\mathbb{N}}$ is a compact topological space with the product topology, and then we can speak of the set of localizations at infinity of L. Indeed, we have then

$$\mathcal{A}^{\otimes\infty} \otimes \mathscr{D} \simeq C(A^\infty, \mathscr{D}),$$

hence $\widehat{L}$ is a continuous map $\widehat{L} : A^\infty \to \mathscr{D}$ and we can say that $\widehat{L}(x)$ is the localization of L at the point $x \in A^\infty$ on the boundary at infinity of the tree (or in the direction x). More explicitly, if $L = D + V$ as above, then $\widehat{L}(x) = D + V_\infty(x)$.

Plan of the paper: The notion of number operator associated to an isometry is introduced and studied in Section 2. The spectral theory of the operators L is studied via the Mourre estimate in Section 3: after some technicalities in the first two subsections, our main abstract results concerning these matters can be found in Subsection 3.3 and the applications in the Fock space setting in Subsection 4.2. Section 5 is devoted to the study of several C^*-algebras generated by more general classes of anisotropic Hamiltonians on a Fock space. Subsections 5.1 and 5.2 contain some preparatory material which is used in Subsection 5.3 in order to prove our main result in this direction, Theorem 5.10. The Appendix, concerned with the representability of some C^*-algebras as tensor products, is devoted to an important ingredient of this proof. The case $\nu = 1$, which is simpler but not covered by the techniques of Section 5, is treated at the end of the Appendix.

Notations: $\mathcal{B}(\mathscr{H})$, $\mathcal{K}(\mathscr{H})$ are the spaces of bounded or compact operators on a Hilbert space $\mathscr{H}$. If S, T are operators such that $S - T \in \mathcal{K}(\mathscr{H})$, we write $S \approx T$. If S, T are quadratic forms with the same domain and $S - T$ is continuous for the topology of $\mathscr{H}$, we write $S \sim T$. $\mathcal{D}(T)$ is the domain of the operator T. We denote by 1 the identity of a unital algebra, but for the clarity of the argument we sometimes adopt a special notation, e.g. the identity operator on $\mathscr{H}$ could be denoted $\mathbf{1}_{\mathscr{H}}$. A *morphism* between two C^*-algebras is a $*$-homomorphism and an *ideal* of a C^*-algebra is a closed bilateral ideal.

Acknowledgments: We are grateful to George Skandalis for a very helpful conversation related to the questions we treat in the Appendix (see the comments before Proposition A.2 and in its proof).

2 Number operator associated to an isometry

2.1 Definition and first examples

Let U be an isometry on a Hilbert space $\mathscr{H}$. Thus $U^*U = 1$ and UU^* is the (orthogonal) projection onto the closed subspace $\operatorname{ran} U = U\mathscr{H}$, hence $P_0 := [U^*, U] = 1 - UU^*$ is the projection onto $(\operatorname{ran} U)^\perp = \ker U^*$.

Definition 2.1 *A* number operator associated to U *is a self-adjoint operator* N *satisfying* $UNU^* = N - 1$.

In fact, N *is a number operator for* U *if and only if* $U^*\mathcal{D}(N) \subset \mathcal{D}(N)$ *and* $UNU^* = N - 1$ *holds on* $\mathcal{D}(N)$. Indeed, this means $N - 1 \subset UNU^*$ and $N - 1$ is a self-adjoint operator, so it cannot have a strict symmetric extension.

In this section we discuss several aspects of this definition. If the operator U is unitary (situation of no interest in this paper), then $U^k NU^{-k}$ is a well defined self-adjoint operator for each $k \in \mathbb{Z}$ and the equality $UNU^* = N - 1$ is equivalent to $U^k NU^{-k} = N - k$ for all $k \in \mathbb{Z}$. In particular, a number operator associated to a unitary operator cannot be semibounded. Example 2.5 allows one to easily understand the structure of a unitary operator which has an associated number operator.

Note that if U is unitary, than N does not exist in general and if it exists, then it is not unique, since $N + \lambda$ is also a number operator for each real λ. On the other hand, we will see in the Subsection 2.2 that N exists, is positive and is uniquely defined if U is a completely non unitary isometry.

In order to express Definition 2.1 in other, sometimes more convenient, forms, we recall some elementary facts. If A, B are linear operators on $\mathscr{H}$ then the domain of AB is the set of $f \in \mathcal{D}(B)$ such that $Bf \in \mathcal{D}(A)$. It is then clear that if A is closed and B is bounded, then AB is closed, but in general BA is not. However, if B is isometric, then BA is closed. Thus, if N is self-adjoint and U is isometric, then UNU^* is a closed symmetric operator.

Lemma 2.2 *Let* N *be a number operator associated to* U. *Then* $\mathcal{D}(N)$ *is stable under* U *and* U^* *and we have* $NU = U(N+1)$ *and* $NU^* = U^*(N-1)$. *Moreover,* $\operatorname{ran} P_0 \subset \ker(N - 1)$ *and* $NP_0 = P_0 N = P_0$.

Proof: From $UNU^* = N - 1$ and $U^*U = 1$ we get $U^*\mathcal{D}(N) \subset \mathcal{D}(N)$ and $NU^* = U^*(N - 1)$ on the domain on N. Moreover, since $U^*P_0 = 0$, we have $P_0\mathscr{H} \subset \mathcal{D}(UNU^*) = \mathcal{D}(N)$ and $(N - 1)P_0 = 0$, so $NP_0 = P_0$, which clearly implies $P_0N = P_0$. If $f, g \in \mathcal{D}(N)$ then

$$\langle (N-1)f, Ug\rangle = \langle U^*(N-1)f, g\rangle = \langle NU^*f, g\rangle = \langle f, UNg\rangle$$

hence $Ug \in \mathcal{D}(N^*) = \mathcal{D}(N)$ and $UNg = (N-1)Ug$. Thus $U\mathcal{D}(N) \subset \mathcal{D}(N)$ and $NU = U(N+1)$ on the domain on $\mathcal{D}(N)$. If $f \in \mathscr{H}$ and $Uf \in \mathcal{D}(N)$ then $f = U^*Uf \in \mathcal{D}(N)$, so we have $NU = U(N+1)$ as operators. If $f \in \mathscr{H}$ and $U^*f \in \mathcal{D}(N)$ then $UU^*f \in \mathcal{D}(N)$ and $P_0 f \in \mathcal{D}(N)$, so $f = UU^*f + P_0 f$ belongs to $\mathcal{D}(N)$, hence $NU^* = U^*(N-1)$ as operators. ■

Note that the relation $NU = U(N+1)$ can also be written $[N,U] = U$. Reciprocally, we have:

Lemma 2.3 *If a self-adjoint operator N satisfies $[N,U] = U$ in the sense of forms on $\mathcal{D}(N)$ and $P_0 N = P_0$ on $\mathcal{D}(N)$, then N is a number operator associated to U.*

Proof: The first hypothesis means $\langle Nf, Ug\rangle - \langle U^*f, Ng\rangle = \langle f, Ug\rangle$ for all f, g in $\mathcal{D}(N)$. But this clearly implies $U^*f \in \mathcal{D}(N)$ and $NU^*f = U^*(N-1)f$ for all $f \in \mathcal{D}(N)$. Then we get

$$UNU^*f = UU^*(N-1)f = (N-1)f - P_0(N-1)f = (N-1)f$$

for all such f, so N is a number operator by the comment after Definition 2.1. ■

Observe that by induction we get $[N, U^n] = nU^n$, hence $\|[N, U^n]\| = n$ if $U \neq 0$. In particular, N is not a bounded operator.

Lemma 2.4 *If N is a self-adjoint operator, then the condition $[N,U] = U$ in the sense of forms on $\mathcal{D}(N)$ is equivalent to each of the following ones:*

(1) $U\mathcal{D}(N) \subset \mathcal{D}(N)$*and* $[N,U] = U$ *as operators on* $\mathcal{D}(N)$*;*

(2) $e^{itN} U e^{-itN} = e^{it} U$ *for all* $t \in \mathbb{R}$*;*

(3) $\varphi(N) U = U \varphi(N+1)$ *for all* $\varphi : \mathbb{R} \to \mathbb{C}$ *bounded and Borel.*

Proof: The implications (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (0) are immediate, condition (0) being that $[N,U] = U$ in the sense of forms on $\mathcal{D}(N)$. If (0) holds, then for all $f, g \in \mathcal{D}(N)$ one has $\langle Nf, Ug\rangle - \langle f, UNg\rangle = \langle f, Ug\rangle$. This gives us $Ug \in \mathcal{D}(N^*) = \mathcal{D}(N)$, hence we get (1). If (2) is satisfied then $\langle e^{-itN} f, Ue^{-itN} g\rangle = e^{it}\langle f, Ug\rangle$ for all $f, g \in \mathcal{D}(N)$, so by taking the derivatives at $t = 0$, we get (0). If (1) holds then by using $NU = U(N+1)$ we get $(N+z)^{-1}U = U(1+N-z)^{-1}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, hence by standard approximation procedures we obtain (3). ■

It is easy to check that the map $\mathscr{U}$ defined by $S \mapsto USU^*$ is a morphism of $\mathcal{B}(\mathscr{H})$ onto $\mathcal{B}(U\mathscr{H})$. We identify $\mathcal{B}(U\mathscr{H})$ with the C^*-subalgebra of $\mathcal{B}(\mathscr{H})$ consisting of the operators T such that $TP_0 = P_0 T = 0$; note that $P_0^{\perp}$ is the identity of the algebra $\mathcal{B}(U\mathscr{H})$ and that the linear positive map $T \mapsto U^*TU$ is a right-inverse for $\mathscr{U}$. Clearly

$$U\varphi(N)U^* = \varphi(N-1)P_0^\perp \quad \text{for all bounded Borel functions} \quad \varphi : \mathbb{R} \to \mathbb{C}. \tag{2.1}$$

By standard approximation procedures we now see that each of the following conditions is necessary and sufficient in order that N be a number operator associated to U: (i) $Ue^{itN}U^* = e^{-it}e^{itN}P_0^\perp$ for all $t \in \mathbb{R}$; (ii) $U(N-z)^{-1}U^* = (N-1-z)^{-1}P_0^\perp$ for some $z \in \mathbb{C} \setminus \mathbb{R}$.

We now give the simplest examples of number operators.

Example 2.5 Let $\mathscr{H} = \ell^2(\mathbb{Z})$ and $(Uf)(x) = f(x-1)$. If $\{e_n\}$ is the canonical orthonormal basis of $\mathscr{H}$ then $Ue_n = e_{n+1}$. It suffices to define N by the condition $Ne_n = ne_n$. Any other number operator is of the form $N + \lambda$ for some real λ. It is an easy exercise to show that if (U, N) is an abstract irreducible couple consisting of a unitary operator U and a self-adjoint operator N such that $[N, U] = U$ in the sense of forms on $\mathcal{D}(N)$, then there is a unique real λ such that this couple is unitarily equivalent to the couple $(U, N+\lambda)$ constructed above.

Example 2.6 Let $\mathscr{H} = \ell^2(\mathbb{N})$ and U as above. Then $U^*e_n = e_{n-1}$ with $e_{-1} = 0$, so $P_0 = |e_0\rangle\langle e_0|$. We obtain a number operator by defining $Ne_n = (n+1)e_n$ and it is easy to see that this is the only possibility. We shall prove this in a more general context below.

2.2 Completely non unitary isometries

An isometry U is called *completely non unitary* if $\text{s-}\lim_{k\to\infty} U^{*k} = 0$. This is equivalent to the fact that the only closed subspace $\mathscr{K}$ such that $U\mathscr{K} = \mathscr{K}$ is $\mathscr{K} = \{0\}$. We introduce below several objects naturally associated to such an isometry, see [Bea].

Consider the decreasing sequence $\mathscr{H} = U^0\mathscr{H} \supset U^1\mathscr{H} \supset U^2\mathscr{H} \supset \dots$ of closed subspaces of $\mathscr{H}$. Since U^k is an isometric operator with range $U^k\mathscr{H}$, the operator $P^k := U^kU^{*k}$ is the orthogonal projection of $\mathscr{H}$ onto $U^k\mathscr{H}$ and we have $1 = P^0 \geq P^1 \geq P^2 \dots$ and $\text{s-}\lim_{k\to\infty} P^k = 0$, because $\|P^kf\| = \|U^{*k}f\| \to 0$.

Recall that $P_0 = 1 - UU^* = 1 - P^1$ is the projection onto $\ker U^*$. More generally, let $\mathscr{H}_k$ be the closed subspace

$$\mathscr{H}_k = \ker U^{*k+1} \ominus \ker U^{*k} = \operatorname{ran} U^k \ominus \operatorname{ran} U^{k+1} = U^k(\ker U^*)$$

and let P_k be the projection onto it, so

$$P_k = P^k - P^{k+1} = U^kU^{*k} - U^{k+1}U^{*k+1} = U^kP_0U^{*k}$$

Notice that $P_{k+1} = UP_kU^*$, hence $UP_k = P_{k+1}U$, and

$$P_kP_m = 0 \text{ if } k \neq m \text{ and } \sum_{k=0}^{\infty} P_k = 1. \tag{2.2}$$

We have $\dim \mathscr{H}_k = \dim \mathscr{H}_0 \neq 0$ for all $k \in \mathbb{N}$. Indeed, it suffices to show that $U_k := U|_{\mathscr{H}_k} : \mathscr{H}_k \to \mathscr{H}_{k+1}$ is a bijective isometry with inverse equal to $U^*|_{\mathscr{H}_{k+1}}$.

In fact, from $UP_k = P_{k+1}U$ we get $U\mathscr{H}_k \subset \mathscr{H}_{k+1}$ so U_k is isometric from $\mathscr{H}_k$ to $\mathscr{H}_{k+1}$. To prove surjectivity, note that $U^*P_{k+1} = P_kU^*$, hence $U^*\mathscr{H}_{k+1} \subset \mathscr{H}_k$ and $UU^*P_{k+1} = UP_kU^* = P_{k+1}$. Thus $U_k : \mathscr{H}_k \to \mathscr{H}_{k+1}$ is bijective and its inverse is $U^*|_{\mathscr{H}_{k+1}}$.

Proposition 2.7 *If U is a completely non unitary isometry then there is a unique number operator associated to it, and we have*

$$N \equiv N_U = \sum_{k=0}^{\infty} P^k = \sum_{k=0}^{\infty} (k+1)P_k, \tag{2.3}$$

the sums being interpreted in form sense. Thus each $k+1$, with $k \in \mathbb{N}$, is an eigenvalue of N_U of multiplicity equal to $\dim\ker U^*$ and $\mathscr{H}_k$ is the corresponding eigenspace.

Proof: Since $P_k = P^k - P^{k+1}$, the two sums from (2.3) are equal and define a self-adjoint operator N_U with $\mathbb{N}+1$ as spectrum and $\mathscr{H}_k$ as eigenspace of the eigenvalue $k+1$. Since $UP_k = P_{k+1}U$, condition (3) of Lemma 2.4 is clearly verified, hence N_U is a number operator for U by Lemma 2.3. Of course, one can also check directly that the conditions of the Definition 2.1 are satisfied. It remains to show uniqueness.

It is clear that an operator N is a number operator if and only if it is of the form $N = M + 1$ where M is a self-adjoint operator such that $M = UU^* + UMU^*$. With a notation introduced above, this can be written $M = UU^* + \mathscr{U}(M)$ hence we get a unique formal solution by iteration: $M = \sum_{k\geq 0} \mathscr{U}^k(UU^*) = \sum_{k\geq 1} P^k$ which gives (2.3). In order to make this rigorous, we argue as follows.

Recall that, by Lemma 2.2, U and U^* leave invariant the domain of M. Hence by iteration we have on $\mathcal{D}(M)$:

$$M = P^1 + UMU^* = P^1 + UP^1U^* + U^2MU^{*2} = P^1 + P^2 + \ldots + P^n + U^nMU^{*n}$$

for all $n \in \mathbb{N}$. It is clear that $P^m\mathcal{D}(M) \subset \mathcal{D}(M)$ for all m and $(1-P^n)U^n = U^{*n}(1-P^n) = 0$, hence

$$M(1-P^n) = (1-P^n)M = \sum_{1\leq k\leq n-1} P^k(1-P^n) = \sum_{1\leq k\leq n-1} kP^k$$

Then $MP_k = P_kM = kP_k$ for all $k \in \mathbb{N}$, hence $M = \sum_k kP_k$. ∎

3 The Mourre estimate

3.1 The free case

Our purpose in this section is to construct a conjugate operator A and to establish a Mourre estimate for the "free" operator

$$\Delta := \text{Re } (U) = \frac{1}{2}(U + U^*) \tag{3.1}$$

where U is an isometry which admits a number operator N on a Hilbert space $\mathscr{H}$. The operator A will be constructed in terms of N and of the imaginary part of U:

$$S := \text{Im } (U) = \frac{1}{2}(U - U^*). \tag{3.2}$$

More precisely, we define A as the closure of the operator

$$A_0 = \frac{1}{2}(SN + NS), \quad \mathcal{D}(A_0) = \mathcal{D}(N). \tag{3.3}$$

We shall prove below that A_0 is essentially self-adjoint and we shall determine the domain of A. That A_0 is not self-adjoint is clear in the situations considered in Examples 2.5 and 2.6. Note that in these examples S is an analog of the derivation operator. Before, we make some comments concerning the operators introduced above.

We have $U = \Delta + iS$ and $\|\Delta\| = \|S\| = 1$. In fact, by using [Mur, Theorem 3.5.17] in case U is not unitary and (2) of Lemma 2.4 if U is unitary, we see that $\sigma(\Delta) = \sigma(S) = [-1, 1]$. By Lemma 2.2 the polynomials in U, U^* (hence in Δ, S) leave invariant the domain of N. If not otherwise mentioned, the computations which follow are done on $\mathcal{D}(N)$ and the equalities are understood to hold on $\mathcal{D}(N)$. The main relations

$$NU = U(N + 1) \text{ and } NU^* = U^*(N - 1) \tag{3.4}$$

will be frequently used without comment. In particular, this gives us

$$[N, S] = -i\Delta \text{ and } [N, \Delta] = iS \tag{3.5}$$

These relations imply that Δ and S are of class $C^\infty(N)$ (we use the terminology of [ABG]). We also have

$$[U, \Delta] = -P_0/2, \quad [U^*, \Delta] = P_0/2, \quad [S, \Delta] = iP_0/2. \tag{3.6}$$

A simple computation gives then:

$$\Delta^2 + S^2 = 1 - P_0/2. \tag{3.7}$$

It follows that we have on the domain of N:

$$A_0 = NS + \frac{i}{2}\Delta = SN - \frac{i}{2}\Delta = \frac{1}{2i}\big((N - \frac{1}{2})U - U^*(N - \frac{1}{2})\big). \tag{3.8}$$

Remark: If we denote $a = iU^*(N - 1/2)$ then on the domain of N we have $A = (a + a^*)/2$. Note that a looks like a bosonic annihilation operator (the normalization with respect to N being, however, different) and that

$$aa^* = (N + 1/2)^2,\ a^*a = (N - 1/2)^2 P_0^\perp,\ [a, a^*] = 2N + P_0/4,\ [N, a] = a.$$

Lemma 3.1 *A is self-adjoint with $\mathcal{D}(A) = \mathcal{D}(NS) = \{f \in \mathscr{H} \mid Sf \in \mathcal{D}(N)\}$.*

Proof: Note that NS is closed on the specified domain and that $\mathcal{D}(N) \subset \mathcal{D}(NS)$, because $S\mathcal{D}(N) \subset \mathcal{D}(N)$. Let us show that $\mathcal{D}(N)$ is dense in $\mathcal{D}(NS)$ (i.e. NS is the closure of $NS|\mathcal{D}(N)$). Let $f \in \mathcal{D}(NS)$, then $f_\varepsilon = (1 + i\varepsilon N)^{-1} f \in \mathcal{D}(N)$ and $\|f_\varepsilon - f\| \to 0$ when $\varepsilon \to 0$. Then, since $S \in C^1(N)$:

$$\begin{aligned} NSf_\varepsilon &= NS(1 + i\varepsilon N)^{-1} f \\ &= N(1 + i\varepsilon N)^{-1}[i\varepsilon N, S](1 + i\varepsilon N)^{-1} f + N(1 + i\varepsilon N)^{-1} Sf \\ &= \varepsilon N(1 + i\varepsilon N)^{-1}\Delta(1 + i\varepsilon N)^{-1} f + (1 + i\varepsilon N)^{-1} NSf. \end{aligned}$$

The last term converges to NSf as ε tends to 0. So it suffices to observe that $\varepsilon N(1 + i\varepsilon N)^{-1} \to 0$ strongly as $\varepsilon \to 0$.

Let $A_0 = SN - i\Delta/2$, $\mathcal{D}(A_0) = \mathcal{D}(N)$. It is trivial to prove that $A_0^* = NS + i\Delta/2$, $\mathcal{D}(A_0^*) = \mathcal{D}(NS)$. By what we proved and the fact that $A_0^*|_{\mathcal{D}(N)} = A_0$, we see that A_0^* is the closure of A_0. So A_0 is essentially self-adjoint. ∎

The next proposition clearly implies the Mourre estimate for Δ outside ± 1.

Proposition 3.2 *$\Delta \in C^\infty(A)$ and $[\Delta, iA] = 1 - \Delta^2 = S^2 + P_0/2$.*

Proof: On $\mathcal{D}(N)$ we have

$$\begin{aligned} [\Delta, iA] &= [\Delta, iNS] = [\Delta, iN]S + N[\Delta, iS] \\ &= S^2 + NP_0/2 = S^2 + P_0/2 = 1 - \Delta^2, \end{aligned}$$

which implies $\Delta \in C^\infty(A)$ by an obvious induction argument. ∎

We mention two other useful commutation relations:

$$[iA, S] = \text{Re}\,(S\Delta) \quad \text{and} \quad [iA, N] = -\text{Re}\,(N\Delta). \tag{3.9}$$

Indeed:

$$[iA, S] = [iSN + \frac{1}{2}\Delta, S] = iS[N, S] + \frac{1}{2}[\Delta, S] = S\Delta + \frac{1}{2}[\Delta, S]$$

and

$$[iA, N] = [iSN + \frac{1}{2}\Delta, N] = [iS, N]N + \frac{1}{2}[\Delta, N] = -\Delta N + \frac{1}{2}[\Delta, N].$$

3.2 Commutator bounds

The following abbreviations will be convenient. For $T \in \mathcal{B}(\mathscr{H})$ we set $\dot{T} \equiv T^{\cdot} = [iN, T]$, interpreted as a form on $\mathcal{D}(N)$, and $T' = [S, T]$, $T_\Delta = [\Delta, T]$, which are bounded operators on $\mathscr{H}$. Iterated operations like $\ddot{T} \equiv T^{\cdot\cdot}$, $T'^{\cdot}$ or $\dot{T}' \equiv T^{\cdot\prime}$ are obviously defined. Note that

$$\dot{T}' - T'^{\cdot} = [S, [iN, T]] - [iN, [S, T]] = [T, [iN, S]]] = -T_\Delta \tag{3.10}$$

because of the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ and (3.5).

If T is a bounded operator then both NT and TN are well defined quadratic forms with domain $\mathcal{D}(N)$. We write $\|NT\| = \infty$, for example, if NT is not continuous for the topology of $\mathscr{H}$. If NT is continuous, then $T\mathcal{D}(N) \subset \mathcal{D}(N)$ and the operator NT with domain $\mathcal{D}(N)$ extends to a unique bounded operator on $\mathscr{H}$ which will also be denoted NT and whose adjoint is the continuous extension of T^*N to $\mathscr{H}$. If $T^* = \pm T$ then the continuity of NT is equivalent to that of TN. Such arguments will be used without comment below.

Proposition 3.3 *For each $V \in \mathcal{B}(\mathscr{H})$ we have, in the sense of forms on $\mathcal{D}(N)$,*

$$[iA, V] = \dot{V}S + iNV' - \frac{1}{2}V_\Delta. \tag{3.11}$$

In particular

$$\|[iA, V]\| \leq \|\dot{V}\| + \|NV'\| + \frac{1}{2}\|V\|. \tag{3.12}$$

Moreover, for the form $[iA, [iA, V]]$ with domain $\mathcal{D}(N^2)$, we have

$$\begin{aligned}\frac{1}{4}\|\,[iA, [iA, V]]\,\| \quad &\leq \quad \|V\| + \|\dot{V}\| + \|\ddot{V}\| + \|V'\| \\ &+ \quad \|NV'\| + \|NV_\Delta\| + \|N\dot{V}'\| + \|N^2V''\|.\end{aligned} \tag{3.13}$$

Proof: The relation (3.11) follows immediately from $A = iNS - \frac{1}{2}\Delta$. For the second commutator, note that $A\mathcal{D}(N^2) \subset \mathcal{D}(N)$, hence in the sense of forms on $\mathcal{D}(N^2)$ we have:

$$\begin{aligned}[iA, [iA, V]] \quad &= \quad [iA, \dot{V}S] + [iA, iNV'] - \frac{1}{2}[iA, V_\Delta] \\ &= \quad [iA, \dot{V}]S + \dot{V}[iA, S] + [iA, iN]V' + iN[iA, V'] - \frac{1}{2}[iA, V_\Delta].\end{aligned}$$

By (3.9) we have $\|\dot{V}[iA, S]\| \leq \|\dot{V}\|$ and then (3.5) gives

$$\begin{aligned}[iA, iN]V' \quad &= \quad -i\mathrm{Re}\,(N\Delta)V' = -\frac{i}{2}(N\Delta V' + \Delta NV') \\ &= \quad -\frac{i}{2}[N, \Delta]V' - i\Delta NV' = \frac{1}{2}SV' - i\Delta NV'\end{aligned}$$

Thus, we have

$$\|[iA,[iA,V]]-[iA,\dot V]S-iN[iA,V']+\frac{1}{2}[iA,V_\Delta]\| \le \|\dot V\|+\|V'\|/2+\|NV'\|.$$

We now apply (3.11) three times with V replaced successively by $\dot V$, V' and V_Δ. First, we get

$$\|[iA,\dot V]S\| = \|\ddot V S^2 + iN\dot V' S - \dot V_\Delta S/2\| \le \|\ddot V\| + \|N\dot V'\| + \|\dot V\|.$$

Then, by using also (3.10) and the notation $V'_\Delta = (V')_\Delta$, we get

$$N[iA,V'] = NV'^{\cdot}S + iN^2V'' - NV'_\Delta/2 = N(\dot V' + V_\Delta)S + iN^2V'' - NV'_\Delta/2.$$

Now (3.5) gives

$$NV'_\Delta = N\Delta V' - NV'\Delta = [N,\Delta]V' + [\Delta, NV'] = iSV' + [\Delta, NV']$$

hence

$$\|N[iA,V']\| \le \|N\dot V'\| + \|NV_\Delta\| + \|N^2V''\| + \|V'\|/2 + \|NV'\|.$$

Then

$$[iA,V_\Delta] = (V_\Delta)^{\cdot} + iN(V_\Delta)' - (1/2)V_{\Delta\Delta}.$$

The first two terms on the right hand side are estimated as follows:

$$(V_\Delta)^{\cdot} = [iN,[\Delta,V]] = -[\Delta,[V,iN]] - [V,[iN,\Delta]] = [\Delta,\dot V] + [V,S]$$

and

$$\begin{aligned} N(V_\Delta)' &= N[S,[\Delta,V]] = -N[\Delta,[V,S]] - N[V,[S,\Delta]] = N[\Delta,V'] \\ &- \frac{i}{2}N[V,P_0] = [N,\Delta]V' + \Delta NV' - NV'\Delta - \frac{i}{2}N[V,P_0] \\ &= iSV' + [\Delta,NV'] - \frac{i}{2}N[V,P_0]. \end{aligned}$$

Since $NP_0 = P_0$ we have

$$N[V,P_0] = NVP_0 - NP_0V = [N,V]P_0 + VNP_0 - NP_0V = -i\dot V + [V,P_0].$$

hence we get

$$\|[iA,V_\Delta]\| \le 5\|V\| + (5/2)\|\dot V\| + \|V'\| + \|NV'\|.$$

Adding all these estimates we get a more precise form of the inequality (3.13). ■

The following result simplifies later computations. The notation $X \sim Y$ means that X, Y are quadratic forms on the domain of N or N^2 and $X - Y$ extends to a bounded operator. *From now on we suppose* $0 \notin \sigma(N)$. In fact, in the case of interest for us we have $N \ge 1$.

Lemma 3.4 *Let V be a bounded self-adjoint operator. If $[U,V]N$ is bounded, then $[U^*,V]N$ is bounded, so $\|NV'\| + \|NV_\Delta\| < \infty$. If $[U,V]N$ is compact, then $[U^*,V]N$ is compact, so NV' is compact. If $\dot{V}$ and $[U,\dot{V}]N$ are bounded, then $\|N\dot{V}'\| < \infty$. If $[U,[U,V]]N^2$ is bounded, then $\|N^2V''\| < \infty$.*

Proof: We have

$$N = UU^*N + P_0N = U(N+1)U^* + P_0 \tag{3.14}$$

hence

$$[U^*,V]N = U^*[V,U](N+1)U^* + [U^*,V]P_0, \tag{3.15}$$

which proves the first two assertions. The assertion involving $\dot{V}$ is a particular case, because $\dot{V}$ is self-adjoint if it is bounded.

For the rest of the proof we need the following relation:

$$N = P_0 + 2P_1 + U^2(N+2)U^{*2}. \tag{3.16}$$

This follows easily directly from the definition of N:

$$\begin{aligned} N &= 1 + UNU^* = 1 + U(1+UNU^*)U^* = 1 + UU^* + U^2NU^{*2} \\ &= (1-UU^*) + 2(UU^* - U^2U^{*2}) + U^2(N+2)U^{*2}. \end{aligned}$$

Since $P_kU^2 = U^{*2}P_k = 0$ for $k = 0,1$, we get from (3.17):

$$N^2 = P_0 + 4P_1 + U^2(N+2)^2U^{*2}. \tag{3.17}$$

We clearly have:

$$-4N^2V'' = N^2[U^*,[U^*,V]] + N^2[U,[U,V]] - N^2([U^*,[U,V]] + [U,[U^*,V]]$$

We shall prove that the three terms from the right hand side are bounded. Since $N^2[U^*,[U^*,V]] = ([U,[U,V]]N^2)^*$, this is trivial for the first one. The second term is the adjoint of $[U^*,[U^*,V]]N^2$ and due to (3.17) we have

$$\begin{aligned} [U^*,[U^*,V]]N^2 &= (U^{*2}V - 2U^*VU^* + VU^{*2})N^2 \\ &\sim (U^{*2}V - 2U^*VU^* + VU^{*2})U^2(N+2)^2U^{*2} \\ &= U^{*2}[U,[U,V]](N+2)^2U^{*2}, \end{aligned}$$

hence we have the required boundedness. Finally, the third term is the adjoint of $([U,[U^*,V]] + [U^*,[U,V]])N^2$ and by a simple computation this is equal to

$$2(V - UVU^* - U^*VU + VUU^*)N^2 \sim -2U^*[U,[U,V]](N+1)^2U^*$$

where we used $N^2 = UU^*N^2 + P_0N^2 = U(N+1)^2U^* + P_0$. ■

If the right hand side of the relation (3.12) or (3.13) is finite, then the operator V is of class $C^1(A)$ or $C^2(A)$ respectively. We shall now point out criteria which are less general than (3.12), (3.13) but are easier to check.

Proposition 3.5 *Let $\Lambda \in \mathcal{B}(\mathscr{H})$ be a self-adjoint operator such that $[\Lambda, N] = 0$ and $[U, \Lambda]N \in \mathcal{B}(\mathscr{H})$. Let V be a bounded self-adjoint operator.*
(1) *If $(V - \Lambda)N$ is bounded, then $V \in C^1(A)$.*
(2) *If $[U, [U, \Lambda]]N^2$ and $(V - \Lambda)N^2$ are bounded, then $V \in C^2(A)$.*
(3) *If $[U, \Lambda]N$, $[\Delta, V]$ and $(V - \Lambda)N$ are compact, then $[iA, V]$ is compact.*

Proof: We have $-i\dot{V} = [N, V] = [N, V - \Lambda] = N(V - \Lambda) - (V - \Lambda)N$ so this is a bounded (or even compact) operator under the conditions of the proposition. Then by using (3.5) we get

$$\begin{aligned} NV' &= N[S, \Lambda] + N[S, V - \Lambda] = N[S, \Lambda] + NS(V - \Lambda) - N(V - \Lambda)S \\ &= N[S, \Lambda] - i\Delta(V - \Lambda) + [S, N(V - \Lambda)] \end{aligned}$$

hence NV' is bounded (or compact). Now in order to get (1) and (3) it suffices to use (3.11) and (3.12) and Lemma 3.4 with V replaced by Λ.

Now we prove (2). We have $V \in C^1(A)$ by what we have shown above. The assumption $\|(V - \Lambda)N^2\| < \infty$ implies $\|N^2(V - \Lambda)\| < \infty$ and then by interpolation $\|N(V - \Lambda)N\| < \infty$. Thus

$$\begin{aligned} -\ddot{V} &= [N, [N, V]] = [N, [N, V - \Lambda]] \\ &= N^2(V - \Lambda) - 2N(V - \Lambda)N + (V - \Lambda)N^2 \end{aligned}$$

is bounded. Moreover,

$$\begin{aligned} -iN\dot{V}' &= N[S, [N, V]] = N[S, [N, V - \Lambda]] = NSN(V - \Lambda) \\ &- NS(V - \Lambda)N - N^2(V - \Lambda)S + N(V - \Lambda)NS, \end{aligned}$$

is bounded by (3.5). Lemma 3.4 shows that $[U^*, \Lambda]N$ is a bounded operator. Hence, by using again (3.5),

$$\begin{aligned} NV_\Delta &= N[\Delta, V - \Lambda] + N[\Delta, \Lambda] \sim N[\Delta, V - \Lambda] \\ &= N\Delta(V - \Lambda) - N(V - \Lambda) \sim \Delta N(V - \Lambda) + iS(V - \Lambda). \end{aligned}$$

So NV_Δ is bounded. At last $N^2V'' = N^2[S, [S, V]] \sim N^2[S, [S, V - \Lambda]]$ by Lemma 3.4 applied to Λ, and this is a bounded operator. ■

3.3 Spectral and scattering theory

We shall now study the spectral theory of abstract self-adjoint operators of the form $L = \Delta + V$ with the help of the theory of conjugate operators initiated in [Mou] and the estimates. We first give conditions which ensure that a Mourre estimate holds. Recall that U is an arbitrary isometry on a Hilbert space $\mathscr{H}$ which admits a number operator N such that $0 \notin \sigma(N)$ and $\Delta = \operatorname{Re} U$. In this subsection the operator V is assumed to be at least self-adjoint and compact. We recall the notation: $S \approx 0$ if $S \in \mathcal{K}(\mathscr{H})$.

Definition 3.6 *We say that the self-adjoint operator L has* normal spectrum *if $\sigma_{\text{ess}}(L) = [-1, +1]$ and the eigenvalues of L different from ± 1 are of finite multiplicity and can accumulate only toward ± 1. Let $\sigma_{\text{p}}(L)$ be the set of eigenvalues of L; then $\kappa(L) = \{-1, +1\} \cup \sigma_{\text{p}}(L)$ is the set of* critical values *of L.*

Theorem 3.7 *Let V be a compact self-adjoint operator on $\mathscr{H}$ such that $[N, V]$ and $[U, V]N$ are compact operators. Then L has normal spectrum and if J is a compact subset of $]-1, +1[$, then there are a real number $a > 0$ and a compact operator K such that $E(J)[L, iA]E(J) \geq aE(J) + K$, where E is the spectral measure of L.*

Proof: We have $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(\Delta) = [-1, +1]$ because V is compact. This also implies that $\varphi(L) - \varphi(\Delta)$ is compact if φ is a continuous function. From (3.11) and Lemma 3.4 it follows that $[V, iA]$ is a compact operator, so V is of class $C^1(A)$ in the sense of [ABG]. Then, if $\operatorname{supp} \varphi$ is a compact subset of $]-1, +1[$ we have

$$\varphi(L)^*[L, iA]\varphi(L) \approx \varphi(\Delta)^*[\Delta, iA]\varphi(\Delta) \geq a|\varphi(\Delta)|^2 \approx a|\varphi(L)|^2$$

because $[\Delta, iA] = 1 - \Delta^2 \geq a$ on $\varphi(\Delta)\mathscr{H}$. This clearly implies the Mourre estimate, which in turn implies the the assertions concerning the eigenvalues, see [Mou] or [ABG, Corollary 7.2.11]. ■

The next result summarizes the consequences of the Mourre theorem [Mou], with an improvement concerning the regularity of the boundary values of the resolvent, cf. [GGM] and references there. If s is a positive real number we denote by $\mathcal{N}_s$ the domain of $|N|^s$ equipped with the graph topology and we set $\mathcal{N}_{-s} := (\mathcal{N}_s)^*$, where the adjoint spaces are defined such as to have $\mathcal{N}_s \subset \mathscr{H} \subset \mathcal{N}_{-s}$. If J is a real set then $J_{\pm}$ is the set of complex numbers of the form $\lambda \pm i\mu$ with $\lambda \in J$ and $\mu > 0$.

Theorem 3.8 *Let V be a compact self-adjoint operator on $\mathscr{H}$ such that $[N, V]$ and $[U, V]N$ are compact operators. Assume also that $[N, [N, V]]$, $[U, [N, V]]N$*

and $[U,[U,V]]N^2$ are bounded operators. Then L has no singularly continuous spectrum. Moreover, if J is a compact real set such that $J \cap \kappa(L) = \emptyset$, then for each real $s \in]1/2, 3/2[$ there is a constant C such that for all $z_1, z_2 \in J_\pm$

$$\|(L - z_1)^{-1} - (L - z_2)^{-1}\|_{B(\mathcal{N}_s, \mathcal{N}_{-s})} \leq C|z_1 - z_2|^{s-1/2}. \tag{3.18}$$

We have used the obvious fact that $\mathcal{N}_s \subset \mathcal{D}(|A|^s)$ for all real $s > 0$ (for our purposes, it suffices to check this for $s = 2$). The theorem can be improved by using [ABG, Theorem 7.4.1], in the sense that one can eliminate the conditions on the second order commutators, replacing them with the optimal Besov type condition $V \in \mathscr{C}^{1,1}(A)$, but we shall consider this question only in particular cases below.

With the terminology of [ABG], the rôle of the conditions on the second order commutators imposed in Theorem 3.8 is to ensure that V (hence L) is of class $C^2(A)$. We shall now consider more general operators, which admit short and long range type components which are less regular. We also make a statement concerning scattering theory under short range perturbations.

Definition 3.9 *Let W be a bounded self-adjoint operator. We say that W is* short range with respect to N*, or* N-short range*, if*

$$\int_1^\infty \|W\chi_0(|N|/r)\| dr < \infty, \tag{3.19}$$

where χ_0 is the characteristic function of the interval $[1,2]$ in $\mathbb{R}$. We say that W is long range with respect to N*, or* N-long range*, if $[N,W]$ and $[U,W]N$ are bounded operators and*

$$\int_1^\infty \Big(\|[N,W]\chi(|N|/r)\| + \|[U,W]N\chi(|N|/r)\|\Big)\frac{dr}{r} < \infty, \tag{3.20}$$

where χ is the characteristic function of the interval $[1,\infty[$ in $\mathbb{R}$.

The condition (3.19) is obviously satisfied if there is $\varepsilon > 0$ such that

$$\|W|N|^{1+\varepsilon}\| < \infty. \tag{3.21}$$

Similarly, (3.20) is a consequence of

$$\|[N,W]\,|N|^\varepsilon\| + \|[U,W]\,|N|^{1+\varepsilon}\| < \infty. \tag{3.22}$$

Lemma 3.10 *If W is compact and N-short range, then WN is a compact operator. If W is N-long range, then $\int_1^\infty \|[U^*,W]N\chi(|N|/r)\|dr/r < \infty$.*

Proof: Let φ be a smooth function on $\mathbb{R}$ such that $\varphi(x) = 0$ if $x < 1$ and $\varphi(x) = 1$ if $x > 2$ and let $\theta(x) = x\varphi(x)$. Then $\int_0^\infty \theta(x)dx/x = 1$ hence $\int_0^\infty \theta(|N|/r)dr/r = 1$ in the strong topology. If $\theta_1(x) = x\theta(x)$ then we get $\int_0^\infty W\theta_0(|N|/r)dr = W|N|$ on the domain of N, which clearly proves the first part of the lemma. The second part follows from (3.15) and (3) of Lemma 2.4. ■

Theorem 3.11 *Let V be a compact self-adjoint operator such that $[N, V]$ and $[U, V]N$ are compact. Assume that we can decompose $V = V_s + V_\ell + V_m$ where V_s is compact and N-short range, V_ℓ is N-long range, and V_m is such that*

$$[N, [N, V_m]], \quad [U, [N, V_m]]N \quad \textit{and} \quad [U, [U, V_m]]N^2$$

are bounded operators. Then $L = \Delta + V$ has normal spectrum and no singularly continuous spectrum. Moreover, $\lim_{\mu\to 0}(L - \lambda - i\mu)^{-1}$ exists in norm in $\mathcal{B}(\mathcal{N}_s, \mathcal{N}_{-s})$ if $s > 1/2$ and $\lambda \notin \kappa(L)$, and the convergence is locally uniform in λ outside $\kappa(L)$. Let $L_0 = \Delta + V_\ell + V_m$ and let Π_0, Π be the projections onto the subspaces orthogonal to the set of eigenvectors of L_0, L respectively. Then the wave operators

$$\Omega_\pm := \text{s-}\lim_{t\to\pm\infty} e^{itL}e^{-itL_0}\Pi_0$$

exist and are complete, i.e. $\Omega_\pm\mathscr{H} = \Pi\mathscr{H}$.

Proof: From the Lemma 3.10 it follows easily that $[N, V_s]$ and $[U, V_s]N$ are compact operators, hence the potentials V and $V_\ell + V_m$ satisfy the hypotheses of Theorem 3.7, so the Mourre estimate holds for L and L_0 on each compact subset of $]-1, +1[$. From [ABG, Theorem 7.5.8] it follows that the operator V_s is of class $\mathscr{C}^{1,1}(A)$. By using (3.11), the second part of Lemma 3.10 and [ABG, Proposition 7.5.7] we see that $[iA, V_\ell]$ is of class $\mathscr{C}^{0,1}(A)$, hence V_ℓ is of class $\mathscr{C}^{1,1}(A)$. Finally, V_m is of class $C^2(A)$ by Proposition 3.3 and Lemma 3.4. Thus, L_0 and L are of class $\mathscr{C}^{1,1}(A)$. Then an application of [ABG, Theorem 7.4.1] gives the spectral properties of L and the existence of the boundary values of the resolvent. Finally, the existence and completeness of the wave operators is a consequence of [ABG, Proposition 7.5.6] and [GeM, Theorem 2.14]. ■

4 A Fock space model

4.1 The Fock space

Let H be a complex Hilbert space and let $\mathscr{H} = \bigoplus_{n=0}^\infty H^{\otimes n}$ be the (complete) Fock space associated to it. We make the conventions $H^{\otimes 0} = \mathbb{C}$ and $H^{\otimes n} = \{0\}$

if $n < 0$. We fix $u \in H$ with $\|u\| = 1$. Let $U = \rho_u$ be the right multiplication by u. More precisely:

$$\begin{aligned}
\rho_u h_1 \otimes \ldots \otimes h_n &= h_1 \otimes \ldots \otimes h_n \otimes u \\
\rho_u^* h_1 \otimes \ldots \otimes h_n &= \begin{cases} h_1 \otimes \ldots \otimes h_{n-1} \langle u, h_n \rangle & \text{if} \quad n \geq 1 \\ 0 \quad \text{if} \quad n = 0. \end{cases}
\end{aligned}$$

Clearly $\rho_u^* \rho_u = 1$, so U is an isometric operator. Then $\Delta = (U + U^*)/2$ acts as follows:

$$\Delta h_1 \otimes \ldots \otimes h_n = h_1 \otimes \ldots \otimes h_{n-1} \otimes (h_n \otimes u + \langle u, h_n \rangle)$$

if $n \geq 1$ and $\Delta h = hu$ if $h \in \mathbb{C} = H^{\otimes 0}$. We have

$$UH^{\otimes n} \subset H^{\otimes n+1}, \quad U^* H^{\otimes n} \subset H^{\otimes n-1}. \tag{4.1}$$

In particular $U^{*n} H^{\otimes m} = 0$ if $n > m$, hence we have s–$\lim_{n\to\infty} U^{*n} = 0$.

Thus U is a completely non unitary isometry, hence there is a unique number operator $N_U \equiv N$ associated to it. We shall keep the notations $P^k = \rho_u^k \rho_u^{*k}$ and $P_k = \rho_u^k [\rho_u^*, \rho_u] \rho_u^{*k}$ introduced in the general setting of Subsection 2.2.

Let us denote by $p_u = |u\rangle\langle u|$ the orthogonal projection in H onto the subspace $\mathbb{C}u$. Then it is easy to check that

$$P^k | H^{\otimes n} = \begin{cases} 0 \quad \text{if} \quad 0 \leq n < k \\ \mathbf{1}_{n-k} \otimes p_u^{\otimes k} \quad \text{if} \quad n \geq k. \end{cases} \tag{4.2}$$

Here $\mathbf{1}_n$ is the identity operator in $H^{\otimes n}$ and the tensor product refers to the natural factorization $H^{\otimes n} = H^{\otimes n-k} \otimes H^{\otimes k}$. In particular, we get $P^k H^{\otimes n} \subset H^{\otimes n}$ or $[P^k, \mathbf{1}_n] = 0$ for all $k, n \in \mathbb{N}$ and similarly for the P_k.

Lemma 4.1 *N leaves stable each $H^{\otimes n}$. We have*

$$N_n := N | H^{\otimes n} = \sum_{k=0}^{n} (k+1) P_k | H^{\otimes n} \tag{4.3}$$

and $\sigma(N_n) = \{1, 2, \ldots n+1\}$, hence $1 \leq N_n \leq n+1$ and $\|N_n\| = n+1$.

Proof: The first assertion is clear because each spectral projection P_k of N leaves $H^{\otimes n}$ invariant. We obtain (4.3) from $P_k = P^k - P^{k+1}$ and the relations (2.3) and (4.2). To see that each $k+1$ is effectively an eigenvalue, one may check that

$$N_n w \otimes v \otimes u^{\otimes k} = (k+1) w \otimes v \otimes u^{\otimes k}$$

if $k < n$, $w \in H^{n-k-1}$ and $v \in H$ with $v \perp u$, and $N_n u^{\otimes n} = (n+1)u^{\otimes n}$. ■

The following more explicit representations of N_n can be proved without difficulty. Let $p_u^\perp$ be the projection in H onto the subspace K orthogonal to u. Then:

$$\begin{aligned}
N_n &= \mathbf{1}_n + \mathbf{1}_{n-1} \otimes p_u + \mathbf{1}_{n-2} \otimes p_u^{\otimes 2} + \cdots + p_u^{\otimes n} \\
&= \mathbf{1}_{n-1} \otimes p_u^\perp + 2\mathbf{1}_{n-2} \otimes p_u^\perp \otimes p_u + 3\mathbf{1}_{n-3} \otimes p_u^\perp \otimes p_u^{\otimes 2} + \ldots \\
&+ \ (n+1)p_u^{\otimes n}.
\end{aligned}$$

The last representation corresponds to the following orthogonal decomposition:

$$H^{\otimes n} = \oplus_{k=0}^{n}(H^{\otimes n-k-1} \otimes K \otimes u^{\otimes k})$$

where the term corresponding to $k = n$ must be interpreted as $\mathbb{C}u^{\otimes n}$.

The number operator N associated to U should not be confused with the *particle number operator* $\boldsymbol{N}$ acting on the Fock space according to the rule $\boldsymbol{N}f = nf$ if $f \in H^{\otimes n}$. In fact, while $\boldsymbol{N}$ counts the total number of particles, $N - 1$ counts (in some sense, i.e. after a symmetrization) the number of particles in the state u. From (4.3) we get a simple estimate of N in terms of $\boldsymbol{N}$:

$$N \leq \boldsymbol{N} + 1. \tag{4.4}$$

It is clear that an operator $V \in \mathcal{B}(\mathscr{H})$ commutes with $\boldsymbol{N}$ if and only if it is of the form

$$V = \sum_{n \geq 0} V_n \mathbf{1}_n, \quad \text{with} \quad V_n \in \mathcal{B}(H^{\otimes n}) \quad \text{and} \quad \sup_n \|V_n\| < \infty. \tag{4.5}$$

Note that we use the same notation $\mathbf{1}_n$ for the identity operator in $H^{\otimes n}$ and for the orthogonal projection of $\mathscr{H}$ onto $H^{\otimes n}$. For each operator V of this form we set $V_{-1} = 0$ and then we define

$$\delta(V) = \sum_{n \geq 0}(V_{n-1} \otimes \mathbf{1}_H - V_n)\mathbf{1}_n, \tag{4.6}$$

which is again a bounded operator which commutes with $\boldsymbol{N}$. We have:

$$[U, V] = \delta(V)U. \tag{4.7}$$

Indeed, if $f \in H^{\otimes n}$ then

$$UVf = UV_n f = (V_n f) \otimes u = (V_n \otimes \mathbf{1}_H)(f \otimes u) = (V_n \otimes \mathbf{1}_H)Uf.$$

On the other hand, since $Uf \in H^{\otimes n+1}$, we have $VUf = V_{n+1}Uf$ and $\delta(V)Uf = (V_n \otimes \mathbf{1}_H - V_{n+1})Uf$, which proves the relation (4.7).

Lemma 4.2 *If V is a bounded self-adjoint operator which commutes with $\boldsymbol{N}$ then the quadratic forms $\dot{V}$ and $\ddot{V}$ are essentially self-adjoint operators. With the notations from (4.5), the closures of these operators are given by the direct sums*

$$\begin{aligned} \dot{V} &= \sum_{n\geq 0}[iN_n, V_n]\mathbf{1}_n \equiv \sum_{n\geq 0}\dot{V}_n\mathbf{1}_n, && (4.8)\\ \ddot{V} &= \sum_{n\geq 0}[iN_n[iN_n, V_n]]\mathbf{1}_n \equiv \sum_{n\geq 0}\ddot{V}_n\mathbf{1}_n. && (4.9)\end{aligned}$$

The proof is easy and will not be given. In particular: $\dot{V}$ *is bounded if and only if* $\sup_n \|[N_n, V_n]\| < \infty$ *and* $\ddot{V}$ *is bounded if and only if* $\sup_n \|[N_n[N_n, V_n]]\| < \infty$.

4.2 The Hamiltonian

In this subsection *we assume that H is finite dimensional* and we apply the general theory of Section 3 to the Hamiltonian of the form $L = \Delta + V$ where V is a compact self-adjoint operator on $\mathscr{H}$ such that $[V, \boldsymbol{N}] = 0$, so V preserves the number of particles (but V *does not commute with* N in the cases of interest for us). Equivalently, this means that V has the form

$$V = \sum_{n\geq 0} V_n\mathbf{1}_n, \quad \text{with} \quad V_n \in \mathcal{B}(H^{\otimes n}) \quad \text{and} \quad \lim_{n\to\infty}\|V_n\| = 0. \qquad (4.10)$$

We shall also consider perturbations of such an L by potentials which do not commute with $\boldsymbol{N}$ but satisfy stronger decay conditions.

The following results are straightforward consequences of the theorems proved in Subsection 3.3, of the remarks at the end of Subsection 4.1, and of the relation (4.7). For example, in order to check the compactness of $[U, V]N$, we argue as follows: we have $[U, V]N = \delta(V)UN = \delta(V)(N-1)U$ and $(\boldsymbol{N}+1)^{-1}N$ is bounded, hence the compactness of $\delta(V)\boldsymbol{N}$ suffices. Note also the relations

$$\begin{aligned} [U,[U,V]] &= [U,\delta(V)U] = [U,\delta(V)]U = \delta^2(V)U^2 && (4.11)\\ \delta^2(V) &= \sum_{n\geq 0}(V_{n-2}\otimes \mathbf{1}_{H^{\otimes 2}} - 2V_{n-1}\otimes\mathbf{1}_H + V_n)\mathbf{1}_n. && (4.12)\end{aligned}$$

Proposition 4.3 *Assume that H is finite dimensional and let V be a self-adjoint operator of the form (4.10) and such that $\|\dot{V}_n\| + n\|V_{n-1}\otimes\mathbf{1}_H - V_n\| \to 0$ when $n \to \infty$. Then the spectrum of L is normal and the Mourre estimate holds on each compact subset of $]-1, +1[$.*

Proposition 4.4 *Assume that H is finite dimensional and let V be a self-adjoint operator of the form (4.10) and such that*
(1) $\|\dot{V}_n\| + n\|V_{n-1} \otimes \mathbf{1}_H - V_n\| \to 0$ *when* $n \to \infty$
(2) $\|\ddot{V}_n\| + n\|\dot{V}_{n-1} \otimes \mathbf{1}_H - \dot{V}_n\| + \|(V_{n-2} \otimes \mathbf{1}_{H^{\otimes 2}} - 2V_{n-1} \otimes \mathbf{1}_H + V_n\| \leq C < \infty$
Then L has normal spectrum and no singularly continuous spectrum.

This result is of the same nature as those of C. Allard and R. Froese. To see this, we state a corollary with simpler and explicit conditions on the potential. If T is a linear operator on a finite dimensional Hilbert space E, we denote by $\langle T \rangle$ its normalized trace:

$$\langle T \rangle = \frac{1}{\dim E} \operatorname{Tr} T \tag{4.13}$$

Observe that $|\langle T \rangle| \leq \|T\|$.

Corollary 4.5 *Let H be finite dimensional and let V be as in (4.10) and such that:*
(1) $\|V_n - \langle V_n \rangle\| = O(1/n^2)$,
(2) $\langle V_{n+1} \rangle - \langle V_n \rangle = o(1/n)$,
(3) $\langle V_{n+1} \rangle - 2\langle V_n \rangle + \langle V_{n-1} \rangle = O(1/n^2)$.
Then L has normal spectrum and no singularly continuous spectrum, the Mourre estimate holds on each compact subset of $]-1, +1[$, and estimates of the form (3.18) are valid.

This follows easily from Proposition 3.5 with $\Lambda = \sum_{n \geq 0} \langle V_n \rangle \mathbf{1}_n$. In the case when V is a function on a tree, the conditions (1)-(3) of the corollary are equivalent to those of Lemma 7 and Theorem 8 in [AlF]. Note, however, that even in the tree case we do not assume that the V_n are functions. Now we improve these results.

Let $\mathbf{1}_{\geq n} = \sum_{k \geq n} \mathbf{1}_k$ be the orthogonal projection of $\mathscr{H}$ onto $\bigoplus_{k \geq n} H^{\otimes k}$.

Theorem 4.6 *Let H be finite dimensional and let V be a self-adjoint operator of the form (4.10) and such that*

$$\sum_{k \geq 0} \sup_{n \geq k} \|V_n - \langle V_n \rangle\| < \infty \quad \textit{and} \quad \langle V_{n+1} \rangle - \langle V_n \rangle = o(1/n). \tag{4.14}$$

Furthermore, assume that $\langle V_n \rangle = \lambda_n + \mu_n$ where $\{\lambda_n\}, \{\mu_n\}$ are sequences of real numbers which converge to zero and such that:
(1) $\lambda_{n+1} - \lambda_n = o(1/n)$ *and* $\lambda_{n+1} - 2\lambda_n + \lambda_{n-1} = O(1/n^2)$,
(2) $\sum_{n \geq 0} \sup_{m \geq n} |\mu_{m+1} - \mu_m| < \infty$.
Finally, let W be a bounded self-adjoint operator satisfying $\sum_n \|W \mathbf{1}_{\geq n}\| < \infty$. Then the operators $L_0 = \Delta + V$ and $L = L_0 + W$ have normal spectrum and no singularly continuous spectrum, and the wave operators for the pair (L, L_0) exist and are complete.

Proof: Let $\Lambda = \sum \lambda_n \mathbf{1}_n$ and $M = \sum \mu_n \mathbf{1}_n$. We shall apply Theorem 3.11 to L with the following identifications: $V_s = V + W - (\Lambda + M)$, $V_\ell = M$ and $V_m = \Lambda$. Note that the condition imposed on W implies that W is a compact N-short range operator (in fact, the condition says that W is $\boldsymbol{N}$-short range). Moreover, the first condition in (4.14) is of the same nature, so it implies that $V - (\Lambda + M)$ is N-short range. Hence V_s is compact and N-short range. The fact that M is N-long range is an easy consequence of $[M, N] = 0$ and of the condition (2) (which says, in fact, that M is $\boldsymbol{N}$-long range). Finally, the fact that V_m satisfies the conditions required in Theorem 3.11 is obvious, by (1) and by what we have seen before. The compactness of $[N, V]$ and $[U, V]N$ is proved as follows. Since $V - (\Lambda + M)$ is N-short range and due to Lemma 3.10, it suffices to show the compactness of the operators $[N, \Lambda + M]$ and $[U, \Lambda + M]N$. But the first one is zero and for the second one we use the first part of condition (1) and condition (2). In the case of $V + W$ one must use again Lemma 3.10 ■

Under the conditions of the preceding theorem, we also have the following version of the "limiting absorption principle", cf. Theorem 3.11. For real s let $\mathscr{H}_{(s)}$ be the Hilbert space defined by the norm

$$\|f\|^2 = \|\mathbf{1}_0 f\|^2 + \sum_{n \geq 1} n^{2s} \|\mathbf{1}_n f\|^2.$$

Then, if $s > 1/2$ and $\lambda \notin \kappa(L)$, the limit $\lim_{\mu \to 0} (L - \lambda - i\mu)^{-1}$ exists in norm in the space $\mathcal{B}(\mathscr{H}_{(s)}, \mathscr{H}_{(-s)})$, the convergence being locally uniform on $\mathbb{R} \setminus \kappa(L)$.

5 The anisotropic tree algebra

5.1 The free algebra

Our purpose now is to study more general operators of the form $L = D + V$, where D is a function of U and U^* (in the sense that it belongs to the C^*-algebra generated by U) and V has the same structure as in Subsection 4.2, i.e. is a direct sum of operators V_n acting in $H^{\otimes n}$, but V_n does not vanish as $n \to \infty$, so V is anisotropic in a sense which will be specified later on.

In this section we keep the assumptions and notations of Subsection 4.1 but *assume that H is of dimension $\nu \geq 2$* (possibly infinite). Then both the range of U and the kernel of U^* are infinite dimensional. It follows easily that each P_k is a projection of infinite rank.

The free algebra $\mathscr{D}$ is the C^*-algebra of operators on $\mathscr{H}$ generated by the isometry U. Since $U^*U = 1$ on $\mathscr{H}$, the set $\mathscr{D}_0$ of operator of the form

$$D = \sum_{n,m\geq 0} \alpha_{nm} U^n U^{*m} \tag{5.1}$$

with $\alpha_{nm} \in \mathbb{C}$ and $\alpha_{nm} \neq 0$ only for a finite number of n, m, is a $*$-subalgebra of $\mathscr{D}$, dense in $\mathscr{D}$. Observe that the projections $P^k = U^k U^{*k}$ and $P_k = P^k - P^{k+1}$ belong to $\mathscr{D}_0$. In the tree case the elements of $\mathscr{D}$ are interpreted as "differential" operators on the tree, which justifies our notation.

We introduce now a formalism needed for the proof of Lemma 5.4, a result important for what follows. For each operator $S \in \mathcal{B}(\mathscr{H})$ we define

$$S^\circ = \sum_{n=0}^{\infty} \mathbf{1}_n S \mathbf{1}_n. \tag{5.2}$$

It is clear that the series is strongly convergent and that $\|S^\circ\| \leq \|S\|$. Thus $S \mapsto S^\circ$ is a linear contraction of $\mathcal{B}(\mathscr{H})$ into itself such that $1^\circ = 1$. This map is also positive and faithful in the following sense:

$$S \geq 0 \text{ and } S \neq 0 \Rightarrow S^\circ \geq 0 \text{ and } S^\circ \neq 0 \tag{5.3}$$

Indeed, $S^\circ \geq 0$ is obvious and if $S^\circ = 0$ then $(\sqrt{S}\mathbf{1}_n)^*(\sqrt{S}\mathbf{1}_n) = \mathbf{1}_n S \mathbf{1}_n = 0$ hence $\sqrt{S}\mathbf{1}_n = 0$ for all n, so $\sqrt{S} = 0$ and then $S = 0$.

We need one more property of the map $S \mapsto S^\circ$:

$$S \in \mathcal{K}(\mathscr{H}) \Rightarrow S^\circ \in \mathcal{K}(\mathscr{H}). \tag{5.4}$$

In fact, this follows from

$$\|S^\circ - \sum_{0\leq m\leq n} \mathbf{1}_m S \mathbf{1}_m\| \leq \sup_{m>n} \|\mathbf{1}_m S \mathbf{1}_m\|$$

because $\|\mathbf{1}_n S \mathbf{1}_n\| \to 0$ as $n \to 0$ if S is compact.

Lemma 5.1 *The restriction to $\mathscr{D}$ of the map $S \mapsto S^\circ$ is a map $\theta : \mathscr{D} \to \mathscr{D}$ whose range is equal to the (abelian, unital) C^*-algebra $\mathscr{P}$ generated by the projections P^k, $k \geq 0$. Moreover, θ is a norm one projection of $\mathscr{D}$ onto its linear subspace $\mathscr{P}$, i.e. $\theta(D) = D$ if and only if $D \in \mathscr{P}$.*

Proof: Since $U^n U^{*m} H^{\otimes k} \subset H^{\otimes(k-m+n)}$, we have $\mathbf{1}_k U^n U^{*m} \mathbf{1}_k \neq 0$ only if $n = m$. Thus, if $D \in \mathscr{D}_0$ is as in (5.1), then

$$\mathbf{1}_k D \mathbf{1}_k = \sum_n \alpha_{n,n} \mathbf{1}_k U^n U^{*n} \mathbf{1}_k = \sum_n \alpha_{n,n} P^n \mathbf{1}_k,$$

because $[P^n, \mathbf{1}_k] = 0$. Thus we get $D^\circ = \sum_n \alpha_{n,n} P^n \in \mathscr{P}$. Since $D \mapsto D^\circ$ is a linear contraction and $\mathscr{D}_0$ is dense in $\mathscr{D}$, we get that $D^\circ \in \mathscr{P}$ for all $D \in \mathscr{D}$. To finish the proof, note that $(P^n)^\circ = P^n$ for all n and $\mathscr{P}$ is the closed linear subspace of $\mathscr{D}$ generated by the operators P^n, hence $D^\circ = D$ for all $D \in \mathscr{P}$. ■

The pairwise orthogonal projections P_n belong to $\mathscr{P}$ but the C^*-algebra (equal to the norm closed subspace) generated by them is strictly smaller than $\mathscr{P}$. On the other hand, the Von Neumann algebra $\mathscr{P}_w$ generated by $\mathscr{P}$ (i.e. the strong closure of $\mathscr{P}$) coincides with that generated by $\{P_n\}_{n\geq 0}$. Indeed, for each $n \geq 0$ we have $P^n = \sum_{m\geq n} P_m$ the series being strongly convergent.

Lemma 5.2 *For each $D \in \mathscr{D}$ there is a unique bounded sequence $\{\alpha_n\}_{n\geq 0}$ of complex numbers such that $D^\circ = \sum_{n\geq 0} \alpha_n P_n$. If $D \geq 0$ then $\alpha_n \geq 0$ for all n. If $D \in \mathscr{D}$, $D \geq 0$ and $D \neq 0$, one has $D^\circ \geq \alpha P_n$ for some real $\alpha > 0$ and some $n \in \mathbb{N}$.*

Proof: Since $P_n P_m = 0$ if $n \neq m$ and $\sum_{k\geq 0} P_k = 1$, each element of the Von Neumann algebra generated by $\{P_n\}_{n\geq 0}$ can be written as $\sum_{n\geq 0} \alpha_n P_n$ for some unique bounded sequence of comples numbers α_n. If $D \geq 0$, then $D^\circ \geq 0$ and this is equivalent to $\alpha_n \geq 0$ for all n. If $D \geq 0$ and $D \neq 0$, then $D^\circ \neq 0$ by (5.3) hence $\alpha_n > 0$ for some n. ■

Corollary 5.3 $\mathscr{D} \cap \mathcal{K}(\mathscr{H}) = \{0\}$.

Proof: $\mathscr{D} \cap \mathcal{K}(\mathscr{H})$ is a C^*-algebra, so that if the intersection is not zero, then it contains some D with $D \geq 0$ and $D \neq 0$. But then D° is a compact operator by (5.4) and we have $D^\circ \geq \alpha P_n$ for some $\alpha > 0$ and $n \in \mathbb{N}$.

We note that if $0 \leq S \leq K$ and $K \approx 0$ then $S \approx 0$. Indeed, for each $\varepsilon > 0$ there is a finite range projection F such that $\|F'KF'\| \leq \varepsilon$, where $F' = 1 - F$. Thus $0 \leq F'SF' \leq \varepsilon$ and so $S = FS + F'SF + F'SF'$ is the sum of a finite range operator and of an operator of norm $\leq \varepsilon$. Hence $S \approx 0$.

Thus P_n is compact, or P_n is an infinite dimension projection. ■

Finally, we are able to prove the result we need.

Lemma 5.4 *Let $V \in \mathcal{B}(\mathscr{H})$ such that $V = V^\circ$ and $[V, U] \in \mathcal{K}(\mathscr{H})$. If there is $D \in \mathscr{D}$, $D \neq 0$, such that $VD \in \mathcal{K}(\mathscr{H})$, then $VP_0 \in \mathcal{K}(\mathscr{H})$.*

Proof: From $VD \approx 0$ it follows that $VDD^*V^* \approx 0$. Then (5.4) gives

$$V(DD^*)^\circ V^* = (VDD^*V^*)^\circ \approx 0.$$

By Lemma 5.2, since $DD^* \in \mathscr{D}$ is positive and not zero, we have $DD^* \geq \alpha P_n$ for some $n \geq 0$, with $\alpha > 0$. Thus $0 \leq VP_nV^* \leq \alpha^{-1}VDD^*V^*$. Or $VDD^*V^* \approx 0$ so $VP_nV^* \approx 0$ and since $VP_n = \sqrt{VP_nV^*}J$ for some partial isometry J we see that $VP_n \approx 0$. But $P_n = U^nP_0U^{*n}$ and $U^*U = 1$ so $VU^nP_0 \approx 0$. If $n \geq 1$ then $UVU^{n-1}P_0 = [U,V]U^{n-1}P_0 + VU^nP_0 \approx 0$ and since $U^*U = 1$ we get $VU^{n-1}P_0 \approx 0$. Repeating, if necessary, the argument, we obtain that $VP_0 \approx 0$. ■

5.2 The interaction algebra

The classes of interaction operators $V \in \mathcal{B}(\mathscr{H})$ we isolate below must be such that $V = V^\circ$ and $VP_0 \approx 0 \Rightarrow V \approx 0$. We shall use the embedding ($n \geq 0$)

$$\mathcal{B}(H^{\otimes n}) \hookrightarrow \mathcal{B}(H^{\otimes n+1}) \text{ defined by } S \mapsto S \otimes \mathbf{1}_H. \tag{5.5}$$

Let us set $\mathcal{A}_0 = \mathbb{C}$ and for each $n \geq 1$ let $\mathcal{A}_n$ be a C^*-algebra of operators on $H^{\otimes n}$ such that

$$\mathcal{A}_n \otimes \mathbf{1}_H \subset \mathcal{A}_{n+1}. \tag{5.6}$$

Note that this implies $\mathbf{1}_n \in \mathcal{A}_n$. The convention (5.5) gives us natural embeddings

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A}_n \subset \ldots \tag{5.7}$$

and we can define $\mathcal{A}_\infty$ as the completion of the $*$-algebra $\cup_{n=0}^{\infty}\mathcal{A}_n$ under the unique C^*-norm we have on it (note that $\mathcal{A}_{n+1}$ induces on $\mathcal{A}_n$ the initial norm of $\mathcal{A}_n$). Thus $\mathcal{A}_\infty$ is a unital C^*-algebra, each $\mathcal{A}_n$ is a unital subalgebra of $\mathcal{A}_\infty$ and we can write:

$$\mathcal{A}_\infty = \overline{\bigcup_{n\geq 0} \mathcal{A}_n} \quad \text{(norm closure).} \tag{5.8}$$

We emphasize that $\mathcal{A}_\infty$ *has not* a natural realization as algebra of operators on $\mathscr{H}$. On the other hand, the following is a unital C^*-algebra of operators on $\mathscr{H}$:

$$\mathscr{A} = \prod_{n\geq 0} \mathcal{A}_n = \big\{V = (V_n)_{n\geq 0} \mid V_n \in \mathcal{A}_n \text{ and } \|V\| := \sup_{n\geq 0} \|V_n\| < \infty\big\}. \tag{5.9}$$

Indeed, if $f = (f_n)_{n\geq 0} \in \mathscr{H}$ and V is as above, we set $Vf = (V_nf_n)_{n\geq 0}$. In other terms, we identify

$$V = \sum_{n=0}^{\infty} V_n \mathbf{1}_n \tag{5.10}$$

the right hand side being strongly convergent on $\mathscr{H}$. Observe that

$$\mathscr{A}_0 = \bigoplus_{n\geq 0} \mathcal{A}_n = \big\{V \in \mathscr{A} \mid \lim_{n\to\infty} \|V_n\| = 0\big\}. \tag{5.11}$$

is an ideal in $\mathscr{A}$.

Lemma 5.5 *We have $\mathscr{A} \cap \mathcal{K}(\mathscr{H}) \subset \mathscr{A}_0$ and the inclusion becomes an equality if H is finite dimensional.*

Proof: We have $\mathbf{1}_n \to 0$ strongly on $\mathscr{H}$ if $n \to \infty$, hence if V is compact then $\|V\mathbf{1}_n\| \to 0$. In the finite dimensional case, note that $\sum_{m=0}^n V_m \mathbf{1}_m$ is compact for all n and converges in norm to V if $V \in \mathscr{A}_0$. ■

Let $\tau : \mathscr{A} \to \mathscr{A}$ be the morphism defined by:

$$\tau(V_0, V_1, V_2, \ldots) = (0, V_0 \mathbf{1}_H, V_1 \otimes \mathbf{1}_H, V_2 \otimes \mathbf{1}_H, \ldots),$$

or $\tau(V)_n = V_{n-1} \otimes \mathbf{1}_H$, where $V_{-1} = 0$. Clearly $\tau^n(V) \to 0$ as $n \to \infty$ strongly on $\mathscr{H}$, for each $V \in \mathscr{A}$. Observe that the map $\delta = \tau - \mathrm{Id}$ coincides with that defined in (4.6), because

$$\delta(V)_n = V_{n-1} \otimes \mathbf{1}_H - V_n.$$

Since $\delta(V'V'') = \delta(V')\tau(V'') + V'\delta(V'')$ and since $\mathscr{A}_0$ is an ideal of $\mathscr{A}$, the space

$$\mathscr{A}_{\mathrm{vo}} = \{V \in \mathscr{A} \mid \delta(V) \in \mathscr{A}_0\} \tag{5.12}$$

is a C^*-subalgebra of $\mathscr{A}$ which contains $\mathscr{A}_0$. This algebra is an analog of the algebra of bounded continuous functions with vanishing oscillation at infinity on $\mathbb{R}$, or that of bounded functions with vanishing at infinity derivative on $\mathbb{Z}$ or $\mathbb{N}$.

Proposition 5.6 *Assume that H is finite dimensional and let $V \in \mathscr{A}_{\mathrm{vo}}$. If $D \in \mathscr{D}$, $D \neq 0$, and $VD \in \mathcal{K}(\mathscr{H})$, then $V \in \mathcal{K}(\mathscr{H})$.*

Proof: We have $\delta(V) \approx 0$ and $[U, V] \approx 0$ by (4.7) and Lemma 5.5. Now according to Lemma 5.4, it remains to prove that $V \approx 0$ follows from $VP_0 \approx 0$. Since $\mathbf{1}_n \to 0$ strongly as $n \to \infty$ and since $[\mathbf{1}_n, P_0] = 0$ and $V\mathbf{1}_n = V_n \mathbf{1}_n$, we get $\|V_n P_0 \mathbf{1}_n\| \to 0$ as $n \to \infty$. By using $P_0 = 1 - P^1$ we get

$$P_0 \mathbf{1}_n = \mathbf{1}_n - \mathbf{1}_{n-1} \otimes p_u = \mathbf{1}_{n-1} \otimes p'_u,$$

where $p'_u = \mathbf{1}_H - p_u$ is the projection of H onto the subspace orthogonal to u, hence $\|p'_u\| = 1$ (recall that $\dim H = \nu \geq 2$). Thus we have $\|V_n \cdot \mathbf{1}_{n-1} \otimes p'_u\| \to 0$. But $\delta(V) \in \mathscr{A}_0$ means $\|V_n - V_{n-1} \otimes \mathbf{1}_H\| \to 0$. So

$$\|V_{n-1}\| = \|V_{n-1} \otimes p'_u\| \leq \|(V_n - V_{n-1} \otimes \mathbf{1}_H) \cdot \mathbf{1}_{n-1} \otimes p'_u\| + \|V_n \cdot \mathbf{1}_{n-1} \otimes p'_u\|$$

converges to 0 as $n \to \infty$. ■

We are mainly interested in the particular class of algebras $\mathcal{A}_n$ constructed as follows. Let $\mathcal{A}$ be a C^*-algebra of operators on H such that $\mathbf{1}_H \in \mathcal{A}$ and let us set:

$$\mathcal{A}_0 = \mathcal{A}^{\otimes 0} = \mathbb{C} \quad \text{and} \quad \mathcal{A}_n = \mathcal{A}^{\otimes n} \quad \text{if} \quad n \geq 1. \tag{5.13}$$

Then $\mathcal{A}_\infty$ *is just the infinite tensor product* $\mathcal{A}^{\otimes\infty}$. Note that the embedding $\mathcal{A}^{\otimes n} \subset \mathcal{A}^{\otimes\infty}$ amounts now to identify $V_n \in \mathcal{A}^{\otimes n}$ with $V_n \otimes \mathbf{1}_H \otimes \mathbf{1}_H \otimes \ldots \in \mathcal{A}^{\otimes\infty}$. We summarize the preceeding notations and introduce new ones specific to this situation:

$$\begin{aligned}
\mathscr{A} &= \prod_{n\geq 0} \mathcal{A}^{\otimes n} = \{V = (V_n)_{n\geq 0} \mid V_n \in \mathcal{A}^{\otimes n}, \|V\| = \sup_{n\geq 0} \|V_n\| < \infty\} \\
\mathscr{A}_0 &= \bigoplus_{n\geq 0} \mathcal{A}^{\otimes n} = \{V \in \mathscr{A} \mid \lim_{n\to\infty} \|V_n\| = 0\} \\
\mathscr{A}_{\mathrm{vo}} &= \{V \in \mathscr{A} \mid \delta(V) \in \mathscr{A}_0\} \\
\mathscr{A}_\infty &= \{V \in \mathscr{A} \mid V_\infty := \lim_{n\to\infty} V_n \text{ exists in } \mathcal{A}^{\otimes\infty}\} \\
\mathscr{A}_{\mathrm{f}} &= \{V \in \mathscr{A} \mid \exists N \text{ such that } V_n = V_N \text{ if } n \geq N\}.
\end{aligned}$$

Note that $V_n = V_N$ means $V_n = V_N \otimes \mathbf{1}_{n-N}$ if $n > N$. The space of main interest for us is the C^*-algebra $\mathscr{A}_\infty$. Clearly, $\mathscr{A}_0$ is a closed self-adjoint ideal in $\mathscr{A}_\infty$ and

$$V \in \mathscr{A}_\infty \Rightarrow \delta(V) \in \mathscr{A}_0, \tag{5.14}$$

in other terms $\mathscr{A}_\infty \subset \mathscr{A}_{\mathrm{vo}}$.

Proposition 5.7 *The map $V \mapsto V_\infty$ is a surjective morphism of the C^*-algebra $\mathscr{A}_\infty$ onto $\mathcal{A}^{\otimes\infty}$ whose kernel is $\mathscr{A}_0$. Thus, we have a canonical isomorphism*

$$\mathscr{A}_\infty / \mathscr{A}_0 \simeq \mathcal{A}^{\otimes\infty}. \tag{5.15}$$

Moreover, $\mathscr{A}_{\mathrm{f}}$ is a dense $*$-subalgebra of $\mathscr{A}_\infty$ and we have

$$\mathscr{A}_{\mathrm{f}} = \Big\{V \in \mathscr{A}_\infty \mid V_\infty \in \bigcup_{n\geq 0} \mathcal{A}^{\otimes n}\Big\}. \tag{5.16}$$

Proof: That $V \mapsto V_\infty$ is a morphism and is obvious. $\mathscr{A}_{\mathrm{f}}$ is clearly a $*$-subalgebra. If $V \in \mathscr{A}_\infty$ and if we set $V_n^N = V_n$ for $n \leq N$, $V_n^N = V_N$ for $n > N$, then $V^N \in \mathscr{A}_{\mathrm{f}}$ and $\|V - V^N\| = \sup_{n>N} \|V_n - V_N\| \to 0$ as $N \to \infty$. Thus $\mathscr{A}_{\mathrm{f}}$ is dense in $\mathcal{A}_\infty$.

If $W \in \mathcal{A}^{\otimes N}$ and if we define $V \in \mathscr{A}$ by $V_n = 0$ for $n < N$, $V_n = W$ if $n \geq N$, then $V \in \mathscr{A}_{\mathrm{f}}$ and $V_\infty = W$. Thus the range of the morphism $V \mapsto V_\infty$

contains the dense subset $\cup_{n\geq 0}\mathcal{A}^{\otimes n}$ of $\mathcal{A}^{\otimes\infty}$. Since the range of a morphism is closed, the morphism is surjective. ■

The following remarks concerning the linear map $\mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ defined by $S \mapsto U^*SU$ will be needed below (see also the comments after Lemma 2.4). If we use the natural embedding $\mathcal{B}(H^{\otimes n}) \hookrightarrow \mathcal{B}(\mathscr{H})$ then we clearly have

$$U^*\mathcal{B}(H^{\otimes n+1})U \subset \mathcal{B}(H^{\otimes n})$$

and if $S' \in \mathcal{B}(H^{\otimes n})$ and $S'' \in \mathcal{B}(H)$ then

$$U^*(S' \otimes S'')U = S'\langle u, S''u\rangle.$$

Of course, $U^*SU = 0$ if $S \in \mathcal{B}(H^{\otimes 0})$. It is clear then that $\omega(V) := U^*VU$ defines a linear positive contraction $\omega : \mathscr{A} \to \mathscr{A}$ which leaves invariant the subalgebras $\mathscr{A}_0$ and $\mathscr{A}_{\mathrm{f}}$, hence $\mathscr{A}_\infty$ too. From (4.7) we then get for all $V \in \mathscr{A}$:

$$UV = [V + \delta(V)]U \quad \text{and} \quad U^*V = [V - \omega \circ \delta(V)]U^*. \tag{5.17}$$

We make two final remarks which are not needed in what follows. First, note that the map ω could be defined with the help of [Tak, Corollary 4.4.25]. Then, observe that for $S \in \mathcal{B}(H^{\otimes n})$ we have $USU^* = S \otimes p_u$. Thus in general the morphism $S \mapsto USU^*$ does not leave invariant the algebras we are interested in.

5.3 The anisotropic tree algebra

In this subsection we study C^*-algebras of operators on the Fock space $\mathscr{H}$ generated by self-adjoint Hamiltonians of the form $L = D+V$, where D is a polynomial in U and U^* and V belongs to a C^*-subalgebra of $\mathscr{A}$. We are interested in computing the quotient of such an algebra with respect to the ideal of compact operators. The largest algebra for which this quotient has a rather simple form is obtained starting with $\mathscr{A}_{\mathrm{vo}}$ and the quotient becomes quite explicit if we start with $\mathscr{A}_\infty$.

More precisely, we fix a vector $u \in H$ with $\|u\| = 1$ and a C^*-algebra $\mathcal{A}$ of operators on H containing $\mathbf{1}_H$. Recall that H is a Hilbert space of dimension $\nu \geq 2$. Throughout this subsection *we assume that H is finite dimensional*, although part of the results hold in general. Then we define $U = \rho_u$ as in Section 4 and we consider the C^*-algebras on $\mathscr{H}$

$$\mathscr{A}_0 \subset \mathscr{A}_\infty \subset \mathscr{A}_{\mathrm{vo}} \subset \mathscr{A}$$

associated to $\mathcal{A}$ as in Subsection 5.2. Then we define

$$\begin{aligned}
\mathscr{C}_{\mathrm{vo}} &= \text{norm closure of } \mathscr{A}_{\mathrm{vo}} \cdot \mathscr{D}, \\
\mathscr{C}_\infty &= \text{norm closure of } \mathscr{A}_\infty \cdot \mathscr{D}, \\
\mathscr{C}_0 &= \text{norm closure of } \mathscr{A}_0 \cdot \mathscr{D}.
\end{aligned}$$

We recall the notation: if A, B are subspaces of an algebra C, then $A \cdot B$ is the linear subspace of C generated by the products ab with $a \in A$ and $b \in B$. Observe that, $\mathscr{D}$ and $\mathscr{A}_{\rm vo}$ being unital algebras, we have and $\mathscr{D} \cup \mathscr{A}_{\rm vo} \subset \mathscr{C}_{\rm vo}$ and, similarly, $\mathscr{D} \cup \mathscr{A}_\infty \subset \mathscr{C}_\infty$. Clearly $\mathscr{C}_0 \subset \mathscr{C}_\infty \subset \mathscr{C}_{\rm vo}$.

Lemma 5.8 *$\mathscr{C}_{\rm vo}$ and $\mathscr{C}_\infty$ are C^*-algebras and $\mathscr{C}_0$ is an ideal in each of them.*

Proof: Indeed, from (5.17) it follows easily that for each $V \in \mathscr{A}_\infty$ there are $V', V'' \in \mathscr{A}_\infty$ such that $UV = V'U$ and $U^*V = V''U^*$ and similarly in the case of $\mathscr{A}_{\rm vo}$. This proves the first part of the lemma. Then note that $V', V'' \in \mathscr{A}_0$ if $V \in \mathscr{A}_0$ and use (5.14). ■

It is not difficult to prove that $\mathscr{C}_{\rm vo}$ is the C^*-algebra generated by the operators $L = D + V$, where D and V are self-adjoint elements of $\mathscr{D}$ and $\mathscr{A}_{\rm vo}$ respectively, and similarly for $\mathscr{C}_\infty$ (see the proof of Proposition 4.1 from [GeI]). Since only the obvious fact that such operators belong to the indicated algebras matters here, we do not give the details.

Lemma 5.9 *If H finite dimensional, then $\mathscr{C}_0 = \mathcal{K}(\mathscr{H}) \cap \mathscr{C}_\infty = \mathcal{K}(\mathscr{H}) \cap \mathscr{C}_{\rm vo}$. If, moreover, u is a cyclic vector for $\mathcal{A}$ in H, then we have $\mathscr{C}_0 = \mathcal{K}(\mathscr{H})$.*

Proof: Since H is finite dimensional, we have $\mathscr{A}_0 \subset \mathcal{K}(\mathscr{H})$, hence $\mathscr{C}_0 \subset \mathcal{K}(\mathscr{H})$. Reciprocally, let $S \in \mathscr{C}_{\rm vo}$ be a compact operator. Let π_n be the projection of $\mathscr{H}$ onto $\bigoplus_{0 \le m \le n} H^{\otimes m}$. Then $\pi_n = \sum_{0 \le m \le n} \mathbf{1}_m \in \mathscr{A}_0$ and $\pi_n \to \mathbf{1}_{\mathscr{H}}$ strongly when $n \to \infty$. Since S is compact, we get $\pi_n S \to S$ in norm, so it suffices to show that $\pi_n S \in \mathscr{C}_0$ for each n. We prove that this holds for any $S \in \mathscr{C} =$ norm closure of $\mathscr{A} \cdot \mathscr{D}$: it suffices to consider the case $S = VD$ with $V \in \mathscr{A}$ and $D \in \mathscr{D}$, and then the assertion is obvious.

Since H is finite dimensional, u is cyclic for $\mathcal{A}$ if and only if $\mathcal{A}u = H$. If this is the case, then $u^{\otimes n}$ is cyclic for $\mathcal{A}^{\otimes n}$ on $H^{\otimes n}$ for each n. Let $n, m \in \mathbb{N}$ and $f \in H^{\otimes n}$, $g \in H^{\otimes m}$. Then there are $V \in \mathcal{A}^{\otimes n}$ and $W \in \mathcal{A}^{\otimes m}$ such that $f = Vu^{\otimes n} = VU^n e$ and $g = Wu^{\otimes m} = WU^m e$, where $e = 1 \in \mathbb{C} = H^{\otimes 0}$. So we have $|f\rangle\langle g| = VU^n|e\rangle\langle e|U^*W^*$. Clearly V, W and $|e\rangle\langle e|$ belong to $\mathscr{A}_0$, so $|f\rangle\langle g| \in \mathscr{C}_0$. An easy approximation argument gives then $\mathcal{K}(\mathscr{H}) \subset \mathscr{C}_0$. ■

We can now describe the quotient $\mathscr{C}_{\rm vo}/\mathscr{C}_0$ of the algebra $\mathscr{C}_{\rm vo}$ with respect to the ideal of compact operators which belong to it.

Theorem 5.10 *Assume that H is finite dimensional. Then there is a unique morphism $\Phi : \mathscr{C}_{\rm vo} \to (\mathscr{A}_{\rm vo}/\mathscr{A}_0) \otimes \mathscr{D}$ such that $\Phi(VD) = \widehat{V} \otimes D$ for all $V \in \mathscr{A}_{\rm vo}$ and $D \in \mathscr{D}$, where $V \mapsto \widehat{V}$ is the canonical map $\mathscr{A}_{\rm vo} \to \mathscr{A}_{\rm vo}/\mathscr{A}_0$. This morphism is surjective and* $\ker \Phi = \mathscr{C}_0$*, hence we get a canonical isomorphism*

$$\mathscr{C}_{\rm vo}/\mathscr{C}_0 \simeq (\mathscr{A}_{\rm vo}/\mathscr{A}_0) \otimes \mathscr{D}. \tag{5.18}$$

Proof: We shall check the hypotheses of Corollary A.4 with the choices:

$$u \equiv U,\ B = \mathscr{A}_{\mathrm{vo}},\ C = \mathscr{C}_{\mathrm{vo}},\ C_0 = \mathscr{C}_0 = \mathscr{C}_{\mathrm{vo}} \cap \mathcal{K}(\mathscr{H}).$$

Thus $A = \mathscr{D}$. From Corollary 5.3 we get $A_0 = \{0\}$ and then

$$B_0 = \mathscr{A}_{\mathrm{vo}} \cap \mathscr{C}_0 = \mathscr{A}_{\mathrm{vo}} \cap \mathscr{C}_{\mathrm{vo}} \cap \mathcal{K}(\mathscr{H}) = \mathscr{A}_{\mathrm{vo}} \cap \mathcal{K}(\mathscr{H}) = \mathscr{A}_0$$

by Lemma 5.5. Then we use Proposition 5.6 and the fact that $[V, U] \in \mathcal{K}(\mathscr{H})$ if $V \in \mathscr{A}_{\mathrm{vo}}$ (see (4.7) and note that $\delta(V) \in \mathscr{A}_0 \in \mathcal{K}(\mathscr{H})$). ∎

The quotient $\mathscr{C}_\infty/\mathscr{C}_0$ has a more explicit form. This follows immediately from Theorem 5.10 and Proposition 5.7.

Corollary 5.11 *If H is finite dimensional, then there is a unique morphism $\Phi : \mathscr{C}_\infty \to \mathcal{A}^{\otimes\infty} \otimes \mathscr{D}$ such that $\Phi(VD) = V_\infty \otimes D$ for all $V \in \mathscr{A}_\infty$ and $D \in \mathscr{D}$. This morphism is surjective and* $\ker \Phi = \mathscr{C}_0$, *hence we have a canonical isomorphism*

$$\mathscr{C}_\infty/\mathscr{C}_0 \simeq \mathcal{A}^{\otimes\infty} \otimes \mathscr{D}. \tag{5.19}$$

Example 5.12 The simplest choice is $\mathcal{A} = \mathbb{C}\mathbf{1}_H$. Then $\mathcal{A}^{\otimes n} = \mathbb{C}\mathbf{1}_n$ and $\mathscr{A}_\infty$ is the set of operators $V \in \mathcal{B}(\mathscr{H})$ of the form $V = \sum_{n\geq 0} V_n \mathbf{1}_n$, where $\{V_n\}$ is a convergent sequence of complex numbers, and $V_\infty = \lim_{n\to\infty} V_n$. In this case, Theorem 5.10 gives us a canonical isomorphism $\mathscr{C}_\infty/\mathscr{C}_0 \simeq \mathscr{D}$. On the other hand, $\mathscr{A}_{\mathrm{vo}}$ corresponds to the bounded sequences $\{V_n\}$ such that $\lim |V_{n+1} - V_n| = 0$, and the quotient $\mathscr{A}_{\mathrm{vo}}/\mathscr{A}_0$ is quite complicated (it can be described in terms of the Stone-Cech compactification of $\mathbb{N}$).

Example 5.13 In order to cover the tree case considered in [Gol] (see the Introduction) it suffices to choose $\mathcal{A}$ an abelian algebra. Since H is finite dimensional, the spectrum of $\mathcal{A}$ is a finite set A and we have $\mathcal{A} \simeq C(A)$ hence $\mathcal{A}^{\otimes n} \simeq C(A^n)$ canonically. If $A^\infty \equiv A^{\mathbb{N}^*}$ equipped with the product topology, then we get a natural identification $\mathcal{A}^{\otimes\infty} \simeq C(A^\infty)$. Let $\Gamma := \bigcup_{n\geq 0} A^n$, then $\mathscr{A}$ can be identified with the set of bounded functions $V : \Gamma \to \mathbb{C}$ and $\mathscr{A}_0$ is the subset of functions which tend to zero at infinity. The embedding (5.6) is obtained by extending a function $\varphi : A^n \to \mathbb{C}$ to a function on A^{n+1} by setting $\varphi(a_1, \ldots, a_n, a_{n+1}) = \varphi(a_1, \ldots, a_n)$. Thus $V \in \mathscr{A}_{\mathrm{vo}}$ if and only if

$$\lim_{n\to\infty} \sup_{a \in A^n,\, b \in A} |V(a, b) - V(a)| = 0.$$

Let $\pi_n : A^\infty \to A^n$ be the projection onto the n first factors. Then $V \in \mathscr{A}_\infty$ if and only if there is $V_\infty \in C(A^\infty)$ such that

$$\lim_{n\to\infty} \sup_{a \in A^\infty} |V \circ \pi_n(a) - V_\infty(a)| = 0.$$

This means that the function $\widetilde{V}$ defined on the space $\widetilde{\Gamma} = \Gamma \cup A^\infty$ equipped with the natural hyperbolic topology (see [Gol]) by the conditions $\widetilde{V}|\Gamma = V$ and $\widetilde{V}|A^\infty = V_\infty$ is continuous. And reciprocally, each continuous function $\widetilde{V} : \widetilde{\Gamma} \to \mathbb{C}$ defines by $\widetilde{V}|\Gamma = V$ an element of $\mathscr{A}_\infty$. This shows that our results cover those of [Gol].

We mention that in order to have a complete equivalence with the tree model as considered in [Gol] the vector u must be a cyclic vector of $\mathcal{A}$, in particular $\mathcal{A}$ must be maximal abelian. Indeed, in this case A can be identified with an orthonormal basis of H diagonalizing $\mathcal{A}$ (the vectors a are uniquely determined modulo a factor of modulus 1 and the associated character of $\mathcal{A}$ is $V \mapsto \langle a, Va\rangle$). Then $u = \sum_{a\in A} c_a a$ is cyclic for $\mathcal{A}$ if and only if $c_a \neq 0$ for all a. If $c_a = |A|^{-1/2}$ with $|A|$ the number of elements of A, we get the standard tree case.

Example 5.14 Another natural choice is $\mathcal{A} = \mathcal{B}(H)$. Then u is a cyclic vector for $\mathcal{A}$ because $u \neq 0$, so $\mathscr{C}_0 = \mathcal{K}(\mathscr{H})$. In this case we have

$$\mathscr{C}_\infty / \mathcal{K}(\mathscr{H}) \simeq \mathcal{B}(H)^{\otimes\infty} \otimes \mathscr{D}$$

and $\mathcal{B}(H)^{\otimes\infty}$ is a simple C^*-algebra.

We give an application to the computation of the essential spectrum. Note that if $L = \sum_{k=1}^n V^k D_k$, with $V^k \in \mathscr{A}_{\text{vo}}$ and $D_k \in \mathscr{D}$, then $\Phi(L) = \sum_{k=1}^n \widehat{V^k} \otimes D_k$. In particular, we get

Proposition 5.15 *Let $L = D + V$ with $D \in \mathscr{D}$ and $V \in \mathscr{A}_{\text{vo}}$ self-adjoint. Then*

$$\sigma_{\text{ess}}(L) = \sigma(D) + \sigma(\widehat{V}). \tag{5.20}$$

If $V \in \mathscr{A}_\infty$, then

$$\sigma_{\text{ess}}(L) = \sigma(D) + \sigma(V_\infty). \tag{5.21}$$

Proof: It suffices to note that $\Phi(L) = 1 \otimes D + \widehat{V} \otimes 1$ and to use the general relation: if A, B are self-adjoint then $\sigma(A \otimes 1 + 1 \otimes B) = \sigma(A) + \sigma(B)$. ■

In the abelian case the result is more general and more explicit.

Proposition 5.16 *Assume that we are in the framework of Example 5.13 and let $L = \sum_{k=1}^n V^k D_k$ be a self-adjoint operator with $V^k \in \mathscr{A}_\infty$ and $D_k \in \mathscr{D}$. Then*

$$\sigma_{\text{ess}}(L) = \bigcup_{a \in A^\infty} \sigma\Big(\sum_k V_\infty^k(a) D_k \Big). \tag{5.22}$$

For the proof, observe that $a \mapsto \sum_k V_\infty^k(a) D_k$ is a norm continuous map on the compact space A^∞, which explains why the right hand side above is a closed set. A formula similar to (5.22) holds if $\mathscr{A}_\infty$ is replaced by $\mathscr{A}_{\mathrm{vo}}$, the only difference being that A^∞ must be replaced with the spectrum of the abelian algebra $\mathscr{A}_{\mathrm{vo}}/\mathscr{A}_0$.

Remarks: We shall make some final comments concerning various natural generalizations of the algebras considered above. Assume that $\mathcal{A}_n$ are C^*-algebras as at the beginning of Subsection 5.2 and let $\mathscr{A}$ be given by (5.9). Then

$$\mathscr{A}_{\mathrm{rc}} = \big\{ V = (V_n)_{n \geq 0} \mid V_n \in \mathcal{A}_n \text{ and } \{V_n \mid n \geq 0\} \text{ is relatively compact in } \mathcal{A}_\infty \big\}$$

is a C^*-subalgebra of $\mathscr{A}$ which contains $\mathscr{A}_{\mathrm{vo}}$. Interesting subalgebras of $\mathscr{A}_{\mathrm{rc}}$ can be defined as follows (this is the analog of a construction from [GeI]): let α be a filter on $\mathbb{N}$ finner than the Fréchet filter and let $\mathscr{A}_\alpha$ be the set of $V = (V_n) \in \mathscr{A}$ such that $\lim_\alpha V_n$ exists in $\mathcal{A}_\infty$, where $\lim_\alpha$ means norm limit along the filter α. Note that $\mathscr{A}_\alpha = \mathscr{A}_{\mathrm{rc}}$ if α is an ultrafilter. Now it is natural to consider the C^*-algebra $\mathscr{C}_{\mathrm{rc}}$ generated by the Hamiltonians with potentials $V \in \mathscr{A}_{\mathrm{rc}}$, so the C^*-algebra generated by $\mathscr{A}_{\mathrm{rc}} \cup \mathscr{D}$, and the similarly defined algebras $\mathscr{C}_\alpha$. It would be interesting to describe the quotient $\mathscr{C}_\alpha/\mathscr{C}_0$, but neither the techniques of the Appendix nor those from [GeI] do not seem to be of any use for this. Indeed, the main ingredients of our proof where Proposition 5.6 and the fact that the commutator of a potential with U is compact, or these properties will not hold in general. Moreover, the examples treated in [GeI], more precisely the Klaus (or bumps) algebra, which has an obvious analog here, show that we cannot expect a simple embedding of the quotient into a tensor product. Note that "localizations at infinity" in the sense of [GeI] can be defined for the elements of $\mathscr{C}_{\mathrm{rc}}$ by using iterations of the operators λ_v of *left* multiplication by elements $v \in H$ in the Fock space $\mathscr{H}$, a technique already used in [GeI, GoI], and this could be used in order to define the canonical morphism which describes the quotient.

A Appendix

Let us consider two C^*-subalgebras A and B of a C^*-algebra C satisfying the following conditions:

- A or B is nuclear,
- $ab = ba$ if $a \in A$ and $b \in B$.

We denote by $A \otimes B$ the minimal C^*-algebra tensor product of the two algebras A and B. Since, by the nuclearity assumption, $A \otimes B$ is also the maximal tensor product of A and B, there is a unique morphism $\phi : A \otimes B \to C$ such that $\phi(a \otimes b) = ab$, see [Mur, Theorem 6.3.7].

Our purpose is to find conditions which ensure that ϕ is injective. Then ϕ is isometric and so it gives a canonical identification of the tensor product $A \otimes B$ with the C^*-subalgebra of C generated by A and B. The following simple observation is useful.

Lemma A.1 *The morphism ϕ is injective if and only if the following condition is satisfied: if $b_1, \ldots, b_n$ is a linearly independent family of elements of B, then*

$$a_1, \ldots, a_n \in A \text{ and } a_1 b_1 + \cdots + a_n b_n = 0 \Rightarrow a_1 = \cdots = a_n = 0. \qquad \text{(A.1)}$$

Proof: This condition is clearly necessary. Reciprocally, let $A \odot B$ be the algebraic tensor product of A and B, identified with a dense subspace of $A \otimes B$. Then each $x \in A \odot B$ can be written $x = \sum a_i \otimes b_i$ for some linearly independent family $b_1, \ldots, b_n$ of elements of B and then $\phi(x) = \sum a_i b_i$. It follows immediately that $x \mapsto \|\phi(x)\|$ is a C^*-norm on $A \odot B$. But the nuclearity of A or B ensures that there is only one such norm, hence $\|\phi(x)\| = \|x\|$, so that ϕ extends to an isometry on $A \otimes B$. ■

The condition (A.1) is not easy to check in general, so it would be convenient to replace it with the simpler:

$$a \in A, b \in B, b \neq 0 \text{ and } ab = 0 \Rightarrow a = 0. \qquad \text{(A.2)}$$

Exercise 2 in [Tak, Sec. 4.4] treats the case when A is abelian. The following result, which was suggested to us by a discussion with Georges Scandalis, is more suited to our purposes.

Let us say that a self-adjoint projection p in a C^*-algebra K is *minimal* if $p \neq 0$ and if the only projections $q \in K$ such that $q \leq p$ are 0 and p. We say that the algebra is *generated by minimal projections* if for each positive non zero element $a \in K$ there is a minimal projection p and a real $\alpha > 0$ such that $a \geq \alpha p$.

We also recall that an ideal K of A is called *essential* if for $a \in A$ the relation $aK = 0$ implies $a = 0$.

Proposition A.2 *If (A.2) is fulfilled and if A contains an essential ideal K which is generated by its minimal projections, then ϕ is injective.*

Proof: The following proof of the proposition in the case $A = \mathscr{D}$, which is the only case of interest in this paper, is due to Georges Scandalis: since $\mathscr{D}$ is isomorphic to the Toeplitz algebra, $\mathscr{D}$ contains a copy K of the algebra of compact operators on $\ell^2(\mathbb{N})$ as an essential ideal. Then it is clear that it suffices to assume that $A = K$

and in this case the assertion is essentially obvious, because $\ker(\varphi \otimes \psi)$ is an ideal of $K \otimes B$. These ideas are certainly sufficient to convince an expert in C^*-algebras, but since we have in mind a rather different audience, we shall develop and give the details of the preceding argument. We also follow a different idea in the last part of the proof.

(i) We first explain why it suffices to consider the case $A = K$. Note that one can identify $K \otimes B$ with the closed subspace of $A \otimes B$ generated by the elements of the form $a \otimes b$ with $a \in K, b \in B$ (see [Mur, Theorem 6.5.1]) and so $K \otimes B$ is an ideal in $A \otimes B$. Let us show that this is an essential ideal.

We can assume that K and B are faithfully and non-degenerately represented on Hilbert spaces $\mathscr{E}, \mathscr{F}$. Since K is essential in A, the representation of K extends to a faithful and non-degenerate representation of A on $\mathscr{E}$ (this is an easy exercise). Thus we are in the situation $K \subset A \subset \mathcal{B}(\mathscr{E})$, $B \subset \mathcal{B}(\mathscr{F})$, the action of K on $\mathscr{E}$ being non-degenerate. Let $\{k_\alpha\}$ be an approximate unit of K. Then $\text{s-}\lim k_\alpha = 1$ on $\mathscr{E}$, because $\|k_\alpha\| \leq 1$ and the linear subspace generated by the vectors ke, with $k \in K$ and $e \in \mathscr{E}$, is dense in $\mathscr{E}$ (in fact $K\mathscr{E} = \mathscr{E}$). Similarly, if $\{b_\beta\}$ is an approximate unit for B then $\text{s-}\lim b_\beta = 1$ on $\mathscr{F}$ and then clearly $\text{s-}\lim_{\alpha,\beta} k_\alpha \otimes b_\beta = 1$ on $\mathscr{E} \otimes \mathscr{F}$. From our assumptions (the tensor products are equal to the minimal ones) we get $K \otimes B \subset A \otimes B \subset \mathcal{B}(\mathscr{E} \otimes \mathscr{F})$. Let $x \in A \otimes B$ such that $x \cdot K \otimes B = 0$. Then $x \cdot k_\alpha \otimes b_\beta = 0$ for all α, β, hence $x = \text{s-}\lim_{\alpha,\beta} x \cdot k_\alpha \otimes b_\beta = 0$. Thus $K \otimes B$ is an essential ideal in $A \otimes B$.

Now it is obvious that a morphism $A \otimes B \to C$ whose restriction to $K \otimes B$ is injective, is injective. Thus it suffices to show that the restriction of ϕ to $K \otimes B$ is injective, so from now on we may, and we shall, assume that $A = K$.

(ii) We make a preliminary remark: let P be the set of minimal projections in A; then for each $p \in P$ we have $pAp = \mathbb{C}p$. Note that this is equivalent to the fact that for each $p \in P$ there is a state τ_p of A such that $pap = \tau_p(a)p$ for all $a \in A$.

Since pAp is the C^*-subalgebra of A consisting of the elements a such that $ap = pa = a$, it suffices to show that each $a \in pAp$ with $a \geq 0$, $a \neq 0$, is of the form λp for some real λ. Let $q \in P$ such that $a \geq \varepsilon q$ for some real $\varepsilon > 0$. Then $\varepsilon q \leq a = pap \leq \|a\|p$ from which it is easy to deduce that $q \leq p$, hence $q = p$ (p and q being minimal). Let λ be the largest positive number such that $a \geq \lambda p$. If $a - \lambda p \neq 0$, then there is $r \in P$ and a real $\nu > 0$ such that $a - \lambda p \geq \nu r$. In particular $a \geq \nu r$ and so $r = p$ by the preceding argument. Hence $a \geq (\lambda + \nu)p$, which contradicts the maximality of λ. Thus $a = \lambda p$.

(iii) Finally, we check (A.1). Let $b_1, \dots, b_n$ be a linearly independent family of elements of B and $a_1, \dots, a_n \in A$ such that $\sum a_i b_i = 0$. Then for all $a \in A$ and $p \in P$ we have

$$p\left(\sum \tau_p(aa_i)b_i\right) = \sum paa_ipb_i = pa\left(\sum a_ib_i\right)p = 0.$$

Since $p \in A$, $p \neq 0$, and $\sum \tau_p(aa_i)b_i \in B$, we must have $\sum \tau_p(aa_i)b_i = 0$. But $\tau_p(aa_i)$ are complex numbers, so $\tau_p(aa_i) = 0$ for each i and all $a \in A$. In particular, we have $\tau_p(a_i^* a_i) = 0$, which is equivalent to $pa_i^* a_i p = 0$ for all $p \in P$. If $a_i^* a_i \neq 0$, then there are $\alpha > 0$ and $q \in P$ such that $a_i^* a_i \geq \alpha q$. By taking $p = q$, we get $0 = qa_i^* a_i q \geq \alpha q$, which is absurd. Thus $a_i^* a_i = 0$, i.e. $a_i = 0$. ■

The next proposition is a simple extension of the preceding one. We recall that a C^*-algebra is called *elementary* if it is isomorphic with the C^*-algebra of all compact operators on some Hilbert space.

Proposition A.3 *Let A, B be C^*-subalgebras of a C^*-algebra C, let C_0 be an ideal of C, and let $A_0 = A \cap C_0$ and $B_0 = B \cap C_0$ be the corresponding ideals of A and B respectively. Denote by $\widehat{A} = A/A_0$, $\widehat{B} = B/B_0$ and $\widehat{C} = C/C_0$ the associated quotient algebras and assume that:*

- *$\widehat{A}$ contains an essential ideal K which is an elementary algebra and such that $\widehat{A}/K$ is nuclear (e.g. abelian)*
- *if $a \in A, b \in B$ then $[a, b] \in C_0$*
- *if $a \in A, b \in B$ and $ab \in C_0$ then either $a \in C_0$ or $b \in C_0$.*
- *C is the C^*-algebra generated by $A \cup B$*

Then there is a unique morphism $\Phi : C \to \widehat{A} \otimes \widehat{B}$ such that $\Phi(ab) = \widehat{a} \otimes \widehat{b}$ for all $a \in A, b \in B$. This morphism is surjective and has C_0 as kernel. In other terms, we have a canonical isomorphism

$$C/C_0 \simeq (A/A_0) \otimes (B/B_0). \tag{A.3}$$

Proof: It is clear that an elementary algebra is generated by minimal projections and is nuclear hence, by [Mur, Theorem 6.5.3], the conditions we impose on A imply the nuclearity of $\widehat{A}$. Note that $\widehat{A}$ and $\widehat{B}$ are C^*-subalgebras of $\widehat{C}$ and that they generate $\widehat{C}$. Moreover, we have $\widehat{a}\widehat{b} = \widehat{b}\widehat{a}$ for all $a \in A, b \in B$ and if $\widehat{a}\widehat{b} = 0$ then $\widehat{a} = 0$ or $\widehat{b} = 0$. By Proposition A.2 the natural morphism $\widehat{A} \otimes \widehat{B} \to \widehat{C}$ is an isomorphism. Denote ψ its inverse, let $\pi : C \to \widehat{C}$ be the canonical map, and let $\Phi = \psi \circ \pi$. This proves the existence of a morphism with the required properties. Its uniqueness is obvious. ■

Now we summarize the facts needed in this paper.

Corollary A.4 *Let C be a C^*-algebra, C_0 an ideal of C, B a C^*-subalgebra of C, $B_0 = B \cap C_0$, and $u \in C$ a non unitary isometry such that $B \cup \{u\}$ generates C.*

Let A be the C^*-subalgebra generated by u and let us assume that $A \cap C_0 = \{0\}$ and that $[u, b] \in C_0$ for all $b \in B$. Finally, assume that:

$$a \in A,\ b \in B \ \text{ and } \ ab \in C_0 \Rightarrow a \in C_0 \ \text{ or } \ b \in C_0.$$

Then there is a unique morphism $\Phi : C \to A \otimes (B/B_0)$ such that $\Phi(ab) = a \otimes \widehat{b}$ for all $a \in A, b \in B$ (where $\widehat{b}$ is the image of b in B/B_0). This morphism is surjective and has C_0 as kernel. In other terms, we have a canonical isomorphism

$$C/C_0 \simeq A \otimes (B/B_0). \tag{A.4}$$

Proof: The assumption $[u, b] \in C_0$ for all $b \in B$ clearly implies $[a, b] \in C_0$ for all $a \in A, b \in B$. Moreover, the algebra $A = \widehat{A}$ is isomorphic with the Toeplitz algebra, see [Mur, Theorem 3.5.18], and so all the conditions imposed on it in Proposition A.3 are satisfied, see [Mur, Example 6.5.1]. ■

We shall now study a more elementary situation which is relevant in the context of Section 5. Our purpose is to treat the case when the Hilbert space H is of dimension 1 (this situation, although much simpler, is not covered by the arguments from Section 5).

This is in fact the case considered in Example 2.6, namely we take $\mathscr{H} = \ell^2(\mathbb{N})$ and define the isometry U by $Ue_n = e_{n+1}$. Then the C^*-algebra $\mathscr{D}(\mathbb{N})$ generated by U is just the Toeplitz algebra [Mur, Section 3.5]. We also consider the situation of Example 2.5, where $\mathscr{H} = \ell^2(\mathbb{Z})$ and U acts in the same way, but now it is a unitary operator and the C^*-algebra $\mathscr{D}(\mathbb{Z})$ generated by it is isomorphic to the algebra $C(T)$ of continuous functions on the unit circle T (make a Fourier transformation). Let $\mathscr{K}(\mathbb{N}) := \mathcal{K}(\ell^2(\mathbb{N}))$ and $\mathscr{K}(\mathbb{Z}) := \mathcal{K}(\ell^2(\mathbb{Z}))$ be the ideals of compact operators on $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ respectively.

It is clear that $\mathscr{D}(\mathbb{Z}) \cap \mathscr{K}(\mathbb{Z}) = \{0\}$ and it is easily shown that $\mathscr{K}(\mathbb{N}) \subset \mathscr{D}(\mathbb{N})$. From [Mur, Theorem 3.5.11] it follows that we have a canonical isomorphism $\mathscr{D}(\mathbb{N})/\mathscr{K}(\mathbb{N}) \simeq \mathscr{D}(\mathbb{Z})$. This isomorphism is uniquely defined by the fact that it sends the shift operator U on $\mathbb{N}$ into the the shift operator U on $\mathbb{Z}$, cf. the Coburn theorem [Mur, Theorem 3.5.18]).

We identify $\ell^\infty(\mathbb{N})$ with the set of bounded multiplication operators on $\ell^2(\mathbb{N})$.

Proposition A.5 *Let $\mathscr{A}$ be a unital C^*-subalgebra of $\ell^\infty(\mathbb{N})$ such that for each $V \in \mathscr{A}$ the operator $[U, V]$ is compact. Let $\mathscr{C}$ be the C^*-algebra generated by $\mathscr{A} \cup \{U\}$ and let us denote $\mathscr{A}_0 = \mathscr{A} \cap \mathscr{K}(\mathbb{N})$ and $\mathscr{C}_0 = \mathscr{C} \cap \mathscr{K}(\mathbb{N})$. Then*

$$\mathscr{C}/\mathscr{C}_0 \simeq (\mathscr{A}/\mathscr{A}_0) \otimes \mathscr{D}(\mathbb{Z}). \tag{A.5}$$

This relation holds also if $\mathbb{N}$ is replaced with $\mathbb{Z}$.

Proof: Clearly $[D, V] \in \mathscr{K}(\mathbb{N})$ for all $D \in \mathscr{D}(\mathbb{N})$ and $V \in \mathscr{A}$, hence we have a natural surjective morphism $(\mathscr{A}/\mathscr{A}_0) \otimes \mathscr{D}(\mathbb{Z}) \to \mathscr{C}/\mathscr{C}_0$. It remains to show that this is an injective map. According to [Tak, Sec. 4.4, Exercice 2], it suffices to prove the following: if $D \in \mathscr{D}(\mathbb{N})$ is not compact and if $V \in \ell^\infty(\mathbb{N})$ has the property $VD \in \mathscr{K}(\mathbb{N})$, then V is compact. We may assume that $D \geq 0$, otherwise we replace it by DD^*.

To each $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ we associate a unitary operator S_α on $\ell^2(\mathbb{N})$ by the rule $S_\alpha e_n = \alpha^n e_n$. We clearly have $S_\alpha U S_\alpha^* = \alpha U$, thus $A \mapsto A_\alpha := S_\alpha A S_\alpha^*$ is an automorphism of $\mathcal{B}(\ell^2(\mathbb{N}))$ which leaves invariant the algebra $\mathscr{D}(\mathbb{N})$ and the ideal $\mathscr{K}(\mathbb{N})$ and reduces to the identity on $\ell^\infty(\mathbb{N})$. Thus $VD_\alpha \in \mathscr{K}(\mathbb{N})$ for each such α. We shall prove the following: there are $\alpha_1, \dots, \alpha_n$ such that $\sum D_{\alpha_i} = A + K$, where A is an invertible operator and K is compact. Then VA is compact and $V = VAA^{-1}$ too, which finishes the proof of the proposition.

We shall denote by $\widehat{S}$ the image of an operator $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ in the Calkin algebra $\mathcal{B}(\ell^2(\mathbb{N}))/\mathcal{K}(\ell^2(\mathbb{N}))$. Thus we have $\widehat{D} \geq 0$, $\widehat{D} \neq 0$. As explained before the proof, we have $\mathscr{D}(\mathbb{N})/\mathscr{K}(\mathbb{N}) \simeq \mathscr{D}(\mathbb{Z}) \simeq C(T)$. Let θ_α be the automorphism of $C(T)$ defined by $\theta_\alpha(\varphi)(z) = \varphi(z\alpha)$. Then we have $\widehat{D_\alpha} = \theta_\alpha(\widehat{D})$ (because this holds for U, hence for all the elements of the C^*-algebra generated by U). But $\widehat{D}$ is a positive continuous function on T which is strictly positive at some point, hence the sum of a finite number of translates of the function is strictly positive, thus invertible in $C(T)$. So there are $\alpha_1, \dots, \alpha_n$ such that the image of $\sum D_{\alpha_i}$ be invertible in the Calkin algebra and this is exactly what we need. ■

References

[All] Allard, C. : *Asymptotic completeness via Mourre theory for a Schrödinger operator on a binary tree*, Master's thesis, UBC (1997).

[ABG] Amrein, W., Boutet de Monvel, A., Georgescu, V. : *C_0-Groups, commutator methods and spectral theory of N-body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.

[AlF] Allard, C., Froese, R. : *A Mourre estimate for a Schrödinger operator on a binary tree* , Reviews in Mathematical Physics, Vol. 12, No. 12 (2000), 1655–1667.

[Bea] Beauzamy, B. : *Introduction to operator theory and invariant subspaces* , Elsevier Science, 1988.

[GGM] Georgescu, V., Gérard, C., Møller, J. : *Commutators, C_0-semigroups and resolvent estimates*, to appear in J. Func. Analysis, preprint at http://www.ma.utexas.edu/mp_arc/.

[GeI] Georgescu, V., Iftimovici, A. : *Crossed products of C^*-algebras and spectral analysis of quantum Hamiltonians*, Comm. Math. Phys. 228 (2002), no. 3, 519–560.

[GeM] Georgescu, V., Măntoiu, M. : *On the spectral theory of Dirac type Hamiltonians*, J. Operator Theory 46 (2001), 289–321.

[Gol] Golénia, S. : *C^*-algebra of anisotropic Schrödinger operators on trees*, (to appear) see preprint at http://www.ma.utexas.edu/mp_arc/.

[Man] Măntoiu, M. : *C^*-algebras, dynamical systems at infinity and the essential spectrum of generalized Schrdinger operators*, J. Reine Angew. Math. 550 (2002), 211–229.

[Mou] Mourre, E. : Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. **91** (1981) 391–408.

[Mur] Murphy, G. : *C^*-algebras and operator theory*, Academic Press (1990).

[Tak] Takesaki, M. : *Theory of operator algebras*, volume I, Springer (2002).

Quasilocal Operators and Stability of the Essential Spectrum

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Abstract

We establish criteria for the stability of the essential spectrum for unbounded operators acting in Banach modules. The Banach module structure allows one to give a meaning to notions like vanishing at infinity or quasilocal operators which covers many situations of practical interest. Our abstract results can be applied to large classes of differential operators of any order with complex measurable coefficients, singular Dirac operators, Laplace-Beltrami operators on Riemannian manifolds with measurable metrics, operators acting on sections of vector fiber bundles over non-smooth manifolds or locally compact abelian groups.

1 Introduction

The main purpose of this paper is to establish criteria which ensure that the difference of the resolvents of two operators is compact. In order to simplify later statements, we use the following definition (our notations are quite standard; we recall however the most important ones at the end of this section).

Definition 1.1 *Let A and B be two closed operators acting in a Banach space $\mathscr{H}$. We say that B* is a compact perturbation of A *if there is $z \in \rho(A) \cap \rho(B)$ such that $(A - z)^{-1} - (B - z)^{-1}$ is a compact operator.*

Under the conditions of this definition the difference $(A - z)^{-1} - (B - z)^{-1}$ is a compact operator for all $z \in \rho(A) \cap \rho(B)$. In particular, *if B is a compact* *perturbation of A, then A and B have the same essential spectrum*, and this for any reasonable definition of the essential spectrum, see [GW]. To be precise, in this paper we define the essential spectrum of A as the set of points $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not Fredholm.

We shall describe now a standard and simple, although quite powerful, method of proving that B is a compact perturbation of A. Note that we are interested in situations where A and B are differential (or pseudo-differential) operators with complex measurable coefficients which differ little on a neighborhood of infinity. An important point in such situations is that one has not much information about the domains of the operators. However, one often knows explicitly a generalized version of the "quadratic form domain" of the operator. Since we want to consider operators of any order (in particular Dirac operators) we shall work in the following framework, which goes beyond the theory of accretive forms.

Let $\mathscr{G}, \mathscr{H}, \mathscr{K}$ be reflexive Banach spaces such that $\mathscr{G} \subset \mathscr{H} \subset \mathscr{K}$ continuously and densely. We are interested in operators in $\mathscr{H}$ constructed according to the following procedure: let A_0, B_0 be continuous bijective maps $\mathscr{G} \to \mathscr{K}$ and let A, B be their restrictions to $A_0^{-1}\mathscr{H}$ and $B_0^{-1}\mathscr{H}$. These are closed densely defined operators in $\mathscr{H}$ and we take $z = 0 \in \rho(A) \cap \rho(B)$. Then in $\mathcal{B}(\mathscr{K}, \mathscr{G})$ we have

$$A_0^{-1} - B_0^{-1} = A_0^{-1}(B_0 - A_0)B_0^{-1}. \tag{1.1}$$

In particular, we get in $\mathcal{B}(\mathscr{H})$

$$A^{-1} - B^{-1} = A_0^{-1}(B_0 - A_0)B^{-1}. \tag{1.2}$$

We get the simplest compactness criterion: if $A_0 - B_0 : \mathscr{G} \to \mathscr{K}$ is compact, then B is a compact perturbation of A. But in this case we have more: the operator $A_0^{-1} - B_0^{-1} : \mathscr{K} \to \mathscr{G}$ is also compact, and this can not happen if A_0, B_0 are differential operators with distinct principal part (cf. below). This also excludes singular lower order perturbations, e.g. Coulomb potentials in the Dirac case.

The advantage of the preceding criterion is that no knowledge of the domains $\mathcal{D}(A), \mathcal{D}(B)$ is needed. To avoid the mentioned disadvantages, one may assume that one of the operators is more regular than the second one, so that the functions in its domain are, at least locally, slightly better than those from $\mathscr{G}$. Note that $\mathcal{D}(B)$ when equipped with the graph topology is such that $\mathcal{D}(B) \subset \mathscr{G}$ continuously and densely and we get a second compactness criterion by asking that $A_0 - B_0 : \mathcal{D}(B) \to \mathscr{K}$ be compact. This time again we get more than needed, because not only B is a compact perturbation of A, but also $A_0^{-1} - B_0^{-1} : \mathscr{H} \to \mathscr{G}$ is compact. However, perturbations of the principal part of a differential operator are allowed and also much more singular perturbations of the lower order terms, cf. [N1] for the Dirac case.

In this paper we are interested in situations where we have really no information concerning the domains of A and B (besides the fact that they are subspaces of $\mathscr{G}$). The case when A, B are second order elliptic operators with measurable complex coefficients acting in $\mathscr{H} = L^2(\mathbb{R}^n)$ has been studied by Ouhabaz and Stollmann in [OS] and, as far as we know, this is the only paper where the "unperturbed" operator is not smooth. Their approach consists in proving that the difference $A^{-k} - B^{-k}$ is compact for some $k \geq 2$ (which implies the compactness of $A^{-1} - B^{-1}$). In order to prove this, they take advantage of the fact that $\mathcal{D}(A^k)$ is a subset of the Sobolev space $W^{1,p}$ for some $p > 2$, which means that we have a certain gain of local regularity. Of course, L^p techniques from the theory of partial differential equations are required for their methods to work.

We shall explain now in the most elementary situation the main ideas of our approach to these questions. Let $\mathscr{H} = L^2(\mathbb{R})$ and $P = -i\frac{d}{dx}$. We consider operators of the form $A_0 = PaP + V$ and $B_0 = PbP + W$ where a, b are bounded operators on $\mathscr{H}$ such that Re a and Re b are bounded below by strictly positive numbers. V and W are assumed to be continuous operators $\mathscr{H}^1 \to \mathscr{H}^{-1}$, where $\mathscr{H}^s$ are Sobolev spaces associated to $\mathscr{H}$. Then $A_0, B_0 \in \mathcal{B}(\mathscr{H}^1, \mathscr{H}^{-1})$ and we put some conditions on V, W which ensure that A_0, B_0 are invertible (e.g. we could include the constant z in them). Thus we are in the preceding abstract framework with $\mathscr{G} = \mathscr{H}^1$ and $\mathscr{K} = \mathscr{H}^{-1} \equiv \mathscr{G}^*$. Then from (1.2) we get

$$A^{-1} - B^{-1} = A_0^{-1}P(b-a)PB^{-1} + A_0^{-1}(W-V)B^{-1}. \tag{1.3}$$

Let R be the first term on the right hand side and let us see how we could prove that it is a compact operator on $\mathscr{H}$. Note that the second term should be easier to treat since we expect V and W to be operators of order less than 2.

We have $R\mathscr{H} \subset \mathscr{H}^1$, so we can write $R = \psi(P)R_1$ for some $\psi \in B_0(\mathbb{R})$ (bounded Borel function which tends to zero at infinity) and $R_1 \in \mathcal{B}(\mathscr{H})$. This is just half of the conditions needed for compactness, in fact R will be compact if and only if one can also find $\varphi \in B_0(\mathbb{R})$ and $R_2 \in \mathcal{B}(\mathscr{H})$ such that $R = \varphi(Q)R_2$ (the notations are standard, see the paragraph after Proposition 2.23 if needed). Of course, the only factor which can help to get such a decay is $b-a$. So let us suppose that we can write $b-a = \xi(Q)U$ for some $\xi \in B_0(\mathbb{R})$ and a bounded operator U on $\mathscr{H}$. We denote $S = A_0^{-1}P$ and note that this is a bounded operator on $\mathscr{H}$, because $P : \mathscr{H} \to \mathscr{H}^{-1}$ and $A_0^{-1} : \mathscr{H}^{-1} \to \mathscr{H}^1$ are bounded. Then $R = S\xi(Q)UPB^{-1}$ and $UPB^{-1} \in \mathcal{B}(\mathscr{H})$, hence R will be compact if the operator $S \in \mathcal{B}(\mathscr{H})$ has the following property: for each $\xi \in B_0(\mathbb{R})$ there are $\varphi \in B_0(\mathbb{R})$ and $T \in \mathcal{B}(\mathscr{H})$ such that $S\xi(Q) = \varphi(Q)T$.

An operator S with the property specified above will be called *quasilocal*. For reasons that will become clear later on, we should be more precise and say "right quasilocal with respect to the module structure defined by $B_0(\mathbb{R})$". Anyway, we see that the compactness of R follows from the quasilocality of S and our main point is that it is easy to check this property under very general assumptions on A, cf. Corollary 2.14, and Proposition 2.22 for abstract criteria and Lemmas 3.7, 5.2 and 6.11 for more concrete examples. The perturbative technique used in the proof of Lemma 6.11 seems to us most interesting since it shows that for the quasilocality question it suffices in fact to consider operators with smooth coefficients.

The applications that we have in mind are of a much more general nature than the example considered above. In fact, an abstract formulation of the ideas described above, see Proposition 2.6, allows one to treat pseudo-differential operators on finite dimensional vector spaces over a local (e.g. p-adic) field (see [Sa, Ta] for the corresponding calculus), in particular differential operators of arbitrary order on $\mathbb{R}^n$, and also abstractly defined classes of operators acting on sections of vector bundles over locally compact spaces, in particular an abstract version of the Laplace operator on manifolds with locally L^∞ Riemannian metrics. Sections 4-6 are devoted to such applications. We stress once again that, in the applications to differential operators, we are interested only in situations where the coefficients are not smooth and the lower order terms are quite singular.

Plan of the paper: In Section 2 we introduce an algebraic formalism which allows us to treat in a unified and simple way operators which have an algebraically complex structure, e.g. operators acting on sections of vector fiber bundles over a locally compact space. The class of "vanishing at infinity" operators is defined through an a priori given algebra of operators on a Banach space $\mathscr{H}$, that we call multiplier algebra of $\mathscr{H}$, and this allows us to define the notion of quasilocality in a natural and general context, that of Banach modules. Several examples of multiplier algebras are given Subsections 2.4, 2.5 and 6.1. We stress that Section 2 is only an accumulation of definitions and straightforward consequences.

We mention that this algebraic framework allows one to study differential operators in L^p spaces. However, this question will not be considered in the present version of our work.

Section 3 contains several abstract compactness criteria which formalize in the context of Banach modules the ideas involved in the example discussed above.

In Section 4 we give our first concrete examples of the abstract theory. In Subsection 4.1 we discuss operators in divergence form on $\mathbb{R}^n$, hence of order $2m$ with $m \geq 1$ integer, with coefficients of a rather general form (they do not have to be functions, for example). In the next subsection we consider pseudo-differential operators on abelian groups and Dirac operators on $\mathbb{R}^n$.

Perturbations of the Laplace operator on a Riemannian manifold with locally L^∞ metric are considered in Section 5. We introduce and study an abstract model of this situation which fits very naturally in our algebraic framework. We also have results on Laplace operators acting on differential forms of any order, but we shall include them only in the final version of the paper.

In Section 6 we discuss the question of "weakly vanishing at infinity" functions, a notion which is easily expressed in terms of filters finner than the Fréchet filter. The quasilocality result presented in Theorem 6.8 is, technically speaking, the deepest assertion of this paper: the proof requires nontrivial tools from the modern theory of Banach spaces, cf. the second part of the Appendix. Theorem 6.12 is a last application of our formalism: we prove a compactness result for operators of order $2m$ in divergence form assuming that the difference between their coefficients vanishes at infinity in a weak sense. Such results were known before only in the case $m = 1$, see [OS].

In the first part of the Appendix we collect some general facts concerning operators acting in scales of spaces which are often used without comment in the rest of the paper. In the second part we prove a version of the Maurey's factorization theorem that we need in Section 6.

Notations: If $\mathscr{G}$ and $\mathscr{H}$ are Banach spaces then $\mathcal{B}(\mathscr{G},\mathscr{H})$ is the space of bounded linear operators $\mathscr{G} \to \mathscr{H}$, the subspace of compact operators is denoted $\mathcal{K}(\mathscr{G},\mathscr{H})$, and we set $\mathcal{B}(\mathscr{H}) = \mathcal{B}(\mathscr{H},\mathscr{H})$ and $\mathcal{K}(\mathscr{H}) = \mathcal{K}(\mathscr{H},\mathscr{H})$. The domain and the resolvent set of an operator S will be denoted by $\mathcal{D}(S)$ and $\rho(S)$ respectively. The norm of a Banach space $\mathscr{G}$ is denoted by $\|\cdot\|_{\mathscr{G}}$ and we omit the index if the space plays a central rôle. The adjoint space (space of antilinear continuous forms) of a Banach space $\mathscr{G}$ is denoted $\mathscr{G}^*$ and if $u \in \mathscr{G}$ and $v \in \mathscr{G}^*$ then we set $v(u) = \langle u, v\rangle$. The embedding $\mathscr{G} \subset \mathscr{G}^{**}$ is realized by defining $\langle v, u\rangle = \overline{\langle u, v\rangle}$.

If $\mathscr{G},\mathscr{H},\mathscr{K}$ are Banach spaces such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathscr{H} \subset \mathscr{K}$ continuously then we always identify $\mathcal{B}(\mathscr{H})$ with a subset of $\mathcal{B}(\mathscr{G},\mathscr{K})$ with the help of the natural continuous embedding $\mathcal{B}(\mathscr{H}) \hookrightarrow \mathcal{B}(\mathscr{G},\mathscr{K})$

A *Friedrichs couple* $(\mathscr{G},\mathscr{H})$ is a pair of Hilbert spaces $\mathscr{G},\mathscr{H}$ together with a continuous dense embedding $\mathscr{G} \subset \mathscr{H}$. The *Gelfand triplet* associated to it is obtained by identifying $\mathscr{H} = \mathscr{H}^*$ with the help of the Riesz isomorphism and then taking the adjoint of the inclusion map $\mathscr{G} \to \mathscr{H}$. Thus we get $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ with continuous and dense embeddings. Now if $u \in \mathscr{G}$ and $v \in \mathscr{H} \subset \mathscr{G}^*$ then $\langle u, v\rangle$ is the scalar product in $\mathscr{H}$ of u and v and also the action of the functional v on u. As noted above, we have $\mathcal{B}(\mathscr{H}) \subset \mathcal{B}(\mathscr{G},\mathscr{G}^*)$.

If X is a locally compact topological space then $B(X)$ is the C^*-algebra of bounded Borel complex functions on X, with norm $\sup_{x\in X} |\varphi(x)|$, and $B_0(X)$ is the subalgebra consisting of functions which tend to zero at infinity. Then $C(X)$, $C_{\mathrm{b}}(X)$, $C_0(X)$ and $C_{\mathrm{c}}(X)$ are the spaces of complex functions on X which are continuous, continuous and bounded, continuous and convergent to zero at infinity, and continuous with compact support respectively. The characteristic function of a subset $S \subset X$ is denoted χ_S .

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2 Banach modules and quasilocal operators

2.1 Banach modules

We use the terminology of [FD] but with some abbreviations, e.g. a *morphism* is a linear multiplicative map between two algebras, and a $*$-*morphism* is a morphism between two $*$-algebras which commutes with the involutions. We recall that an *approximate unit* in a Banach algebra $\mathcal{M}$ is a net $\{J_\alpha\}$ in $\mathcal{M}$ such that $\|J_\alpha\| \leq C$ for some constant C and all α and $\lim_\alpha \|J_\alpha M - M\| = \lim_\alpha \|M J_\alpha - M\| = 0$ for all $M \in \mathcal{M}$. It is well known that any C^*-algebra has an approximate unit. If $\mathscr{H}$ is a Banach space, we shall say that a Banach subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathscr{H})$ is *non-degenerate* if the linear subspace of $\mathscr{H}$ generated by the elements Mu, with $M \in \mathcal{M}$ and $u \in \mathscr{H}$, is dense in $\mathscr{H}$.

Definition 2.1 *A* Banach module *is a couple* $(\mathscr{H}, \mathcal{M})$ *consisting of a Banach space* $\mathscr{H}$ *and a non-degenerate Banach subalgebra* $\mathcal{M}$ *of* $\mathcal{B}(\mathscr{H})$ *which has an approximate unit. If* $\mathscr{H}$ *is a Hilbert space and* $\mathcal{M}$ *is a* C^**-algebra of operators on* $\mathscr{H}$*, we say that* $\mathscr{H}$ *is a* Hilbert module.

We shall adopt the usual *abus de language* and say that $\mathscr{H}$ is a Banach module. The distinguished subalgebra $\mathcal{M}$ will be called *multiplier algebra of* $\mathscr{H}$ and, when required by the clarity of the presentation, we shall denote it $\mathcal{M}(\mathscr{H})$. We are especially interested in the case when $\mathcal{M}$ does not have a unit: the operators from $\mathcal{M}$ are the prototype of "vanishing at infinity operators", or the identity cannot vanish at infinity. Note that it is implicit in Definition 2.1 that if $\mathscr{H}$ is a Hilbert module then its adjoint space $\mathscr{H}^*$ is identified with $\mathscr{H}$ with the help of the Riesz isomorphism.

If $\{J_\alpha\}$ is an approximate unit of $\mathcal{M}$, then the density in $\mathscr{H}$ of the linear subspace generated by the elements Mu is equivalent to

$$\lim_\alpha \|J_\alpha u - u\| = 0 \quad \text{for all} \quad u \in \mathscr{H}. \tag{2.4}$$

But much more is true:

$$u \in \mathscr{H} \Rightarrow u = Mv \text{ for some } M \in \mathcal{M} \text{ and } v \in \mathscr{H}. \tag{2.5}$$

This follows from the Cohen-Hewitt theorem, see Theorem A.3. By using (2.4) we could avoid any reference to this result in our later arguments; this would make them more elementary but less simple.

If $\mathscr{H}$ is a Banach module and the Banach space $\mathscr{H}$ is reflexive we say that $\mathscr{H}$ is a *reflexive Banach module*. In this case the adjoint Banach space $\mathscr{H}^*$ *is equipped with a canonical Banach module structure*, its multiplier algebra being $\mathcal{M}(\mathscr{H}^*) := \{A^* \mid A \in \mathcal{M}(\mathscr{H})\}$. This is a closed subalgebra of $\mathcal{B}(\mathscr{H}^*)$ which clearly has an approximate unit and the linear subspace generated by the elements of the form A^*v, with $A \in \mathcal{M}(\mathscr{H})$ and $v \in \mathscr{H}^*$, is weak*-dense, hence dense, in $\mathscr{H}^*$. Indeed, if $u \in \mathscr{H}$ and $\langle u, A^*v\rangle = 0$ for all such A, v then $Au = 0$ for all $A \in \mathcal{M}(\mathscr{H})$ hence $u = 0$ because of (2.4).

Definition 2.2 *A couple $(\mathscr{G}, \mathscr{H})$ consisting of two Hilbert modules such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely will be called a* Friedrichs module. *If $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H})$, we say that $(\mathscr{G}, \mathscr{H})$ is a* compact Friedrichs module.

There is no a priori relation between the multiplier algebras of $\mathscr{H}$ and $\mathscr{G}$ and the choice $\mathcal{M}(\mathscr{G}) = \mathcal{B}(\mathscr{G})$ is allowed. We observe that in general it is not possible to take $\mathcal{M}(\mathscr{G})$ equal to the set of operators $M \in \mathcal{M}(\mathscr{H})$ which leave $\mathscr{G}$ invariant: it may happen that this algebra has not an approximate unit.

In the situation of this definition we always identify $\mathscr{H}$ with its adjoint space, which gives us a Gelfand triplet $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$. Note that, $\mathscr{G}$ being reflexive, $\mathscr{G}^*$ is also a Hilbert module.

If $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs module then each operator M from $\mathcal{M}(\mathscr{H})$ extends to a compact operator $M : \mathscr{H} \to \mathscr{G}^*$ (this is the adjoint of the compact operator $M^* : \mathscr{G} \to \mathscr{H}$). Thus we shall have $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H}) \cap \mathcal{K}(\mathscr{H}, \mathscr{G}^*)$.

2.2 Operators vanishing at infinity

Let $\mathscr{H}$ and $\mathscr{K}$ be Banach spaces. If $\mathscr{K}$ is a Banach module then we shall denote by $\mathcal{B}^l_0(\mathscr{H}, \mathscr{K})$ the norm closed linear subspace generated by the operators MT, with $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $M \in \mathcal{M}(\mathscr{K})$. We say that an operator in $\mathcal{B}^l_0(\mathscr{H}, \mathscr{K})$ *left vanishes at infinity* (with respect to $\mathcal{M}(\mathscr{K})$, if this is not obvious from the context). If J_α is an approximate unit for $\mathcal{M}(\mathscr{K})$, then for an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ we have:

$$\begin{aligned} S \in \mathcal{B}^l_0(\mathscr{H}, \mathscr{K}) \quad &\Leftrightarrow \quad \lim_\alpha \|J_\alpha S - S\| = 0 \\ &\Leftrightarrow \quad S = MT \text{ for some } M \in \mathcal{M}(\mathscr{K}) \text{ and } T \in \mathcal{B}(\mathscr{H}, \mathscr{K}). \end{aligned} \tag{2.6}$$

The second equivalence follows from the Cohen-Hewitt theorem.

If $\mathscr{H}$ is a Banach module then one can similarly define $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ as the norm closed linear subspace generated by the operators TM with $T \in \mathcal{B}(\mathscr{H},\mathscr{K})$ and $M \in \mathcal{M}(\mathscr{H})$. We say that the elements of $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ *right vanish at infinity*. As above, if J_α is an approximate unit for $\mathcal{M}(\mathscr{H})$ we have

$$\begin{aligned} S \in \mathcal{B}_0^r(\mathscr{H},\mathscr{K}) \quad &\Leftrightarrow \quad \lim_\alpha \|SJ_\alpha - S\| = 0 \\ &\Leftrightarrow \quad S = TM \text{ for some } M \in \mathcal{M}(\mathscr{H}) \text{ and } T \in \mathcal{B}(\mathscr{H},\mathscr{K}). \end{aligned} \tag{2.7}$$

If both $\mathscr{H}$ and $\mathscr{K}$ are Banach modules we set

$$\mathcal{B}_0(\mathscr{H},\mathscr{K}) = \mathcal{B}_0^l(\mathscr{H},\mathscr{K}) \cap \mathcal{B}_0^r(\mathscr{H},\mathscr{K}). \tag{2.8}$$

The elements of $\mathcal{B}_0(\mathscr{H},\mathscr{K})$ are called *vanishing at infinity*.

If $(\mathscr{G},\mathscr{H})$ is a Friedrichs module then the space $\mathcal{B}_0^l(\mathscr{G},\mathscr{G}^*)$ for example is well defined, but it could be too large for some purposes (it is equal to $\mathcal{B}(\mathscr{G},\mathscr{G}^*)$ if the multiplier algebra of $\mathscr{G}$ is $\mathcal{B}(\mathscr{G})$). For this reason we introduce the next spaces. Recall that we have a natural continuous embedding $\mathcal{B}(\mathscr{H}) \subset \mathcal{B}(\mathscr{G},\mathscr{G}^*)$. Let

$$\mathcal{B}_{00}^l(\mathscr{G},\mathscr{G}^*) = \text{ norm closure of } \mathcal{B}_0^l(\mathscr{H}) \text{ in } \mathcal{B}(\mathscr{G},\mathscr{G}^*). \tag{2.9}$$

The spaces $\mathcal{B}_{00}^r(\mathscr{G},\mathscr{G}^*)$ and $\mathcal{B}_{00}(\mathscr{G},\mathscr{G}^*)$ are similarly defined. We have

$$\mathcal{K}(\mathscr{G},\mathscr{G}^*) \subset \mathcal{B}_{00}(\mathscr{G},\mathscr{G}^*) \tag{2.10}$$

because $\mathcal{K}(\mathscr{H})$ is a dense subset of $\mathcal{K}(\mathscr{G},\mathscr{G}^*)$ and $\mathcal{K}(\mathscr{H}) \subset \mathcal{B}_0(\mathscr{H})$, see below.

Some simple properties of these spaces are described below.

Proposition 2.3 *If $\mathscr{K}$ is a reflexive Banach module and $S \in \mathcal{B}_0^l(\mathscr{H},\mathscr{K})$ then S^* belongs to $\mathcal{B}_0^r(\mathscr{K}^*,\mathscr{H}^*)$.*

Proof: We have $S = MT$ with $M \in \mathcal{M}(\mathscr{K})$ and $T \in \mathcal{B}(\mathscr{H},\mathscr{K})$ by (2.6), which implies $S^* = T^*M^*$ and we have $M^* \in \mathcal{M}(\mathscr{K}^*)$ by definition. ■

Corollary 2.4 *If $\mathscr{H}$ is a Hilbert module then $\mathcal{B}_0(\mathscr{H})$ is a C^*-algebra and S belongs to $\mathcal{B}_0(\mathscr{H})$ if and only if $S = MTN$ with $M, N \in \mathcal{M}(\mathscr{H})$ and $T \in \mathcal{B}(\mathscr{H})$.*

Proof: $\mathcal{B}_0(\mathscr{H})$ is a C^*-algebra, so $S = S_1S_2$ for some operators $S_1, S_2 \in \mathcal{B}_0(\mathscr{H})$. Thus $S_1 = MT_1$ and $S_2 = T_2N$ for some $M, N \in \mathcal{M}(\mathscr{H})$ and $T_1, T_2 \in \mathcal{B}(\mathscr{H})$, hence $S = MT_1T_2N$. ■

Proposition 2.5 *If $\mathscr{K}$ is a Banach module then $\mathcal{K}(\mathscr{H},\mathscr{K}) \subset \mathcal{B}_0^l(\mathscr{H},\mathscr{K})$. If $\mathscr{H}$ is a reflexive Banach module, then $\mathcal{K}(\mathscr{H},\mathscr{K}) \subset \mathcal{B}_0^r(\mathscr{H},\mathscr{K})$.*

Proof: If $\{J_\alpha\}$ is an approximate unit for $\mathcal{M}(\mathscr{K})$ then s-$\lim_\alpha J_\alpha u = u$ uniformly in u if u belongs to a compact subset of $\mathscr{K}$. Hence if $S \in \mathcal{K}(\mathscr{H}, \mathscr{K})$ then $\lim_\alpha \|J_\alpha S - S\| = 0$ and thus $S \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ by (2.6). To prove the second part of the proposition, observe that if $S \in \mathcal{K}(\mathscr{H}, \mathscr{K})$ then $S^* \in \mathcal{K}(\mathscr{K}^*, \mathscr{H}^*)$, hence $S^* \in \mathcal{B}_0^l(\mathscr{K}^*, \mathscr{H}^*)$ by what we just proved, so $S^{**} \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K}^{**})$ by Proposition 2.3. So $\lim_\alpha \|S^{**} J_\alpha - S^{**}\| = 0$ if $\{J_\alpha\}$ is an approximate unit for $\mathcal{M}(\mathscr{H})$. But clearly$S = S^{**}$, hence $S \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$. ■

Proposition 2.6 *Let $\mathscr{H}$ be a Banach module and $\mathscr{G}$ a Banach space continuously embedded in $\mathscr{H}$ and such that $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H})$. If $R \in \mathcal{B}_0^l(\mathscr{H})$ and $R\mathscr{H} \subset \mathscr{G}$, then $R \in \mathcal{K}(\mathscr{H})$.*

Proof: According to (2.6) we have $R = \lim_\alpha J_\alpha R$, the limit being taken in norm. But $R \in \mathcal{B}(\mathscr{H}, \mathscr{G})$ by the closed graph theorem and $J_\alpha \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ by hypothesis, so that $J_\alpha R \in \mathcal{K}(\mathscr{H})$. ■

Corollary 2.7 *If $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs module and $R \in \mathcal{B}_0^l(\mathscr{H})$ is such that $R\mathscr{H} \subset \mathscr{G}$, then $R \in \mathcal{K}(\mathscr{H})$.*

2.3 Quasilocal operators

Definition 2.8 *Let $\mathscr{H}$, $\mathscr{K}$ be Banach modules and let $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$. We say that S is* left quasilocal *if for each $M \in \mathcal{M}(\mathscr{K})$ we have $MS \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$. We say that S is* right quasilocal *if for each $M \in \mathcal{M}(\mathscr{H})$ we have $SM \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$. If S is left and right quasilocal, we say that S is* quasilocal*.*

We denote $\mathcal{B}_q^l(\mathscr{H}, \mathscr{K})$, $\mathcal{B}_q^r(\mathscr{H}, \mathscr{K})$ and $\mathcal{B}_q(\mathscr{H}, \mathscr{K})$ these classes of operators. These are clearly closed subspaces of $\mathcal{B}(\mathscr{H}, \mathscr{G})$. The next result is obvious; a similar assertion holds for right quasilocality.

Proposition 2.9 *Let $\{J_\alpha\}$ be an approximate unit for $\mathcal{M}(\mathscr{K})$ and let S be an operator in $\mathcal{B}(\mathscr{H}, \mathscr{K})$. Then S is left quasilocal if and only if one of the following conditions is satisfied:*
(1) *$J_\alpha S \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$ for all α.*
(2) *for each $M \in \mathcal{M}(\mathscr{K})$ there are $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $N \in \mathcal{M}(\mathscr{H})$ such that $MS = TN$.*

The next proposition, which says that the set of quasilocal operators is stable under the usual algebraic operations, is an immediate consequence of Proposition 2.9. There is, of course, a similar statement with “left” and “right” interchanged. We denote by $\mathscr{G}, \mathscr{H}$ and $\mathscr{K}$ Banach modules.

Proposition 2.10 (1) $S \in \mathcal{B}^l_q(\mathscr{H}, \mathscr{K})$ *and* $T \in \mathcal{B}^l_q(\mathscr{G}, \mathscr{H}) \Rightarrow ST \in \mathcal{B}^l_q(\mathscr{G}, \mathscr{K})$.
(2) *If* $\mathscr{H}, \mathscr{K}$ *are reflexive and* $S \in \mathcal{B}^l_q(\mathscr{H}, \mathscr{K})$, *then* $S^* \in \mathcal{B}^r_q(\mathscr{K}^*, \mathscr{H}^*)$.
(3) *If* $\mathscr{H}$ *is a Hilbert module then* $\mathcal{B}_q(\mathscr{H})$ *is a unital* C^**-subalgebra of* $\mathcal{B}(\mathscr{H})$.

Obviously $\mathcal{B}^l_0(\mathscr{H}, \mathscr{K}) \subset \mathcal{B}^r_q(\mathscr{H}, \mathscr{K})$ and $\mathcal{B}^r_0(\mathscr{H}, \mathscr{K}) \subset \mathcal{B}^l_q(\mathscr{H}, \mathscr{K})$. But our main results depend on finding other, more interesting examples of quasilocal operators.

Remark: A more natural and suggestive name for "quasilocal operators" would be *decay preserving operators*. We did not use it because the french version of this terminology is rather heavy to use.

2.4 X-modules over locally compact spaces

In the next two subsections we give examples of Banach modules important for this paper. We always denote by X a locally compact non-compact topological space; later we equip it with some more structure.

Definition 2.11 *A* Banach X-module *is a Banach space* $\mathscr{H}$ *equipped with a continuous morphism* $Q : C_0(X) \to \mathcal{B}(\mathscr{H})$ *such that the linear subspace generated by the vectors of the form* $Q(\varphi)u$, *with* $\varphi \in C_0(X)$ *and* $u \in \mathscr{H}$, *is dense in* $\mathscr{H}$. *If* $\mathscr{H}$ *is a Hilbert space and* Q *is a* $*$*-morphism, we say that* $\mathscr{H}$ *is a* Hilbert X-module.

A Friedrichs module $(\mathscr{G}, \mathscr{H})$ such that $\mathscr{H}$ is a Hilbert X-module will be called *Friedrichs* X*-module*. Note that here there are no assumptions concerning the module structure of $\mathscr{G}$.

We shall use the notation $\varphi(Q) \equiv Q(\varphi)$. The Banach module structure on $\mathscr{H}$ is defined by the closure $\mathcal{M}$ in $\mathcal{B}(\mathscr{H})$ of the set of operators of the form $\varphi(Q)$ with $\varphi \in C_0(X)$. In the case of a Hilbert X-module the closure is not needed and we get a Hilbert module structure (recall that a $*$-morphism between two C^*-algebras is continuous and its range is a C^*-algebra).

We note that the morphism Q *has an extension, also denoted* Q, *to a unital continuous morphism of* $C_{\mathrm{b}}(X)$ *into* $\mathcal{B}(X)$ which is uniquely determined by the following strong continuity property: if $\{\varphi_n\}$ is a bounded sequence in $C_{\mathrm{b}}(X)$ such that $\varphi_n \to \varphi$ locally uniformly, then $\varphi_n(Q) \to \varphi(Q)$ strongly on $\mathscr{H}$. Indeed, using once again the Cohen-Hewitt theorem we see that for each $u \in \mathscr{H}$ there are $\psi \in C_0(X)$ and $v \in \mathscr{H}$ such that $u = \psi(Q)v$ hence we can define $\varphi(Q)u = (\varphi\psi)(Q)v$ for each $\varphi \in C_{\mathrm{b}}(X)$; then if e_α is an approximate unit for $C_0(X)$ with $\|e_\alpha\| \leq 1$ we get $\varphi(Q)u = \lim(\varphi e_\alpha)(Q)u$ hence $\|\varphi(Q)\| \leq \|Q\| \sup|\varphi|$.

In the case of a Hilbert X-module we can say more.

Lemma 2.12 *If $\mathscr{H}$ is a Hilbert X-module, then the $*$-morphism Q canonically extends to a $*$-morphism $\varphi \mapsto \varphi(Q)$ of $B(X)$ into $\mathcal{B}(\mathscr{H})$ having the property : if $\{\varphi_n\}$ is a bounded sequence in $B(X)$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in X$, then $\varphi_n(Q)$ converges strongly to $\varphi(Q)$ on $\mathscr{H}$.*

Proof: Q extends, by standard integration theory, to a $*$-morphism of $B(X)$ into $\mathcal{B}(\mathscr{H})$ which is uniquely determined by the following property: if $U \subset X$ is open then $\chi_U(Q) = \sup_\varphi \varphi(Q)$, where φ runs over the set of continuous functions with compact support such that $0 \leq \varphi \leq \chi_U$. We note that if X is second countable then this property is equivalent to the convergence condition from the statement of the lemma. ■

Remark: A separable Hilbert X-module is essentially a direct integral of Hilbert spaces over X, see [Di, Ch. II], but we shall not need this fact. On the other hand, Banach X-modules appear naturally in differential geometry as spaces of sections of vector fiber bundles over a manifold X, and this is the point of interest for us.

The *support* supp $u \subset X$ of an element u of a Banach X-module $\mathscr{H}$ is defined as the smallest closed set such that its complement U has the property $\varphi(Q)u = 0$ if $\varphi \in C_{\mathrm{c}}(U)$. Clearly, the set $\mathscr{H}_{\mathrm{c}}$ of elements $u \in \mathscr{H}$ such that supp u is compact is a dense subspace of $\mathscr{H}$.

If $\mathscr{H}$ and $\mathscr{K}$ are Banach X-modules, then a map $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is called *local* if supp $Su \subset$ supp u for each $u \in \mathscr{H}$; clearly locality implies right quasilocality. Now we look for more interesting criteria of quasilocality.

Let $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $\varphi, \psi \in C(X)$, not necessarily bounded. We say that $\varphi(Q)S\psi(Q)$ is a bounded operator if there is a constant C such that

$$\|\xi(Q)\varphi(Q)S\psi(Q)\eta(Q)\| \leq C \sup|\xi| \sup|\eta|$$

for all $\xi, \eta \in C_{\mathrm{c}}(X)$. The lower bound of the admissible constants C in this estimate is denoted $\|\varphi(Q)S\psi(Q)\|$. If $\mathscr{K}$ is a reflexive Banach X-module, then the product $\varphi(Q)S\psi(Q)$ is well defined as sesquilinear form on the dense subspace $\mathscr{K}_{\mathrm{c}}^* \times \mathscr{H}_{\mathrm{c}}$ of $\mathscr{K}^* \times \mathscr{H}$ and the preceding boundedness notion is equivalent to the continuity of this form for the topology induced by $\mathscr{K}^* \times \mathscr{H}$. We similarly define the boundedness of the commutator $[S, \varphi(Q)]$.

Proposition 2.13 *Assume that $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and let $\Theta : X \to [1, \infty[$ be a continuous function such that $\lim_{x\to\infty} \Theta(x) = \infty$. If $\Theta^{-1}(Q)S\Theta(Q)$ is a bounded operator then S is left quasilocal. If $\Theta(Q)S\Theta^{-1}(Q)$ is a bounded operator then S is right quasilocal.*

Proof: Let $K \subset X$ be compact, let $U \subset X$ be a neighbourhood of infinity in X, and let $\varphi, \psi \in C_{\mathrm{b}}(X)$ such that supp $\varphi \subset K$, supp $\psi \subset U$ and $|\varphi| \leq 1, |\psi| \leq 1$.

Then $\Theta\varphi$ is a bounded function and $\psi\Theta^{-1}$ is bounded and can be made as small as we wish by choosing U conveniently. Thus given $\varepsilon > 0$ we have

$$\|\varphi(Q)S\psi(Q)\| \leq \|\varphi\Theta\| \cdot \|\Theta^{-1}(Q)S\Theta(Q)\| \cdot \|\Theta^{-1}\psi\| \leq \varepsilon$$

if U is a suficiently small neighbourhood of infinity. ∎

The boundedness of $\Theta^{-1}(Q)S\Theta(Q)$ is usually checked by estimating the commutator $[S, \Theta(Q)]$; we give an example for the case of metric spaces. Note that on metric spaces one has a natural class of regular functions, namely the Lipschitz functions, for example the functions which give the distance to subsets: $\rho_K(x) = \inf_{y \in K} \rho(x, y)$ for $K \subset X$.

We say that a locally compact metric space (X, ρ) is *proper* if the metric ρ has the property $\lim_{y \to \infty} \rho(x, y) = \infty$ for some (hence for all) points $x \in X$. Equivalently, if X is not compact but the closed balls are compact.

Corollary 2.14 *Let (X, ρ) be a proper locally compact metric space. If S belongs to $\mathcal{B}(\mathscr{H}, \mathscr{K})$ and if $[S, \theta(Q)]$ is bounded for each positive Lipschitz function θ, then S is quasilocal .*

Proof: Indeed, by taking $\theta = 1 + \rho_K$ and by using the notations of the proof of Proposition 2.13, we easily get the following estimate: there is $C < \infty$ depending only on K such that

$$\|\varphi(Q)S\psi(Q)\| \leq C(1 + \rho(K, U))^{-1},$$

where $\rho(K, U)$ is the distance from K to U. Since S^* has the same properties as S, this proves the quasilocality of S. Note that the boundedness of $[S, \rho_x(Q)]$ for some $x \in X$ suffices in this argument. ∎

2.5 X-modules over locally compact groups

If X is a locally compact abelian group one can associate to it more interesting classes of Banach modules. We always assume X non-compact and we denote additively the group operation. For example, X could be $\mathbb{R}^n, \mathbb{Z}^n$, or a finite dimensional vector space over a local field, e.g. over the field of p-adic numbers. Let X^* be the abelian locally compact group dual to X. One can construct interesting Banach subalgebras of $C_0(X)$ by using the Fourier transformation and submultiplicative functions on X^*, but the approach we adopt is more intrinsic.

Definition 2.15 *If X is a locally compact abelian group, then a* Banach X-module *is a Banach space $\mathscr{H}$ equipped with a strongly continuous representation $\{V_k\}$ of the dual group X^* on $\mathscr{H}$.*

Note that we shall use the same notation V_k for the representations of X^* in different spaces $\mathscr{H}$ whenever this does not lead to ambiguities.

Such a Banach X-module has a canonical structure of Banach module that we now define. We choose Haar measures dx and dk on X and X^* normalized by the following condition: if the Fourier transform of a function φ on X is given by $(\mathcal{F}\varphi)(k) \equiv \widehat{\varphi}(k) = \int_X \overline{k(x)}\varphi(x)dx$ then $\varphi(x) = \int_{X^*} k(x)\widehat{\varphi}(k)dk$. Recall that $X^{**} = X$. Let $C^{(a)}(X) := \mathcal{F}L^1_{\mathrm{c}}(X^*)$ be the set of Fourier transforms of integrable functions with compact support on X^*. It is easy to see that $C^{(a)}(X)$ is a $*$-algebra for the usual algebraic operations; more precisely, it is a dense subalgebra of $C_0(X)$ stable under conjugation. For $\varphi \in C^{(a)}(X)$ we set

$$\varphi(Q) = \int_{X^*} V_k \widehat{\varphi}(k)dk. \tag{2.11}$$

This definition is determined by the formal requirement $k(Q) = V_k$. Then

$$\mathcal{M} := \text{ norm closure of } \{\varphi(Q) \mid \varphi \in C^{(a)}(X)\} \text{ in } \mathcal{B}(\mathscr{H}) \tag{2.12}$$

is a Banach subalgebra of $\mathcal{B}(\mathscr{H})$.

Lemma 2.16 *The algebra $\mathcal{M}$ has an approximate unit consisting of elements of the form $e_\alpha(Q)$ with $e_\alpha \in C^{(a)}(X)$.*

Proof: Indeed, let us fix a compact neighborhood K of the identity in X^*. The set of compact neighborhoods of the identity α such that $\alpha \subset K$ is ordered by $\alpha_1 \geq \alpha_2 \Leftrightarrow \alpha_1 \subset \alpha_2$. For each such α define e_α by $\widehat{e}_\alpha = \chi_\alpha/|\alpha|$, where $|\alpha|$ is the Haar measure of α. Then $\|e_\alpha(Q)\| \leq \sup_{k\in K}\|V_k\| < \infty$, from which it is easy to infer that $\lim_\alpha \|e_\alpha(Q)\varphi(Q) - \varphi(Q)\| = 0$ for all $\varphi \in C^{(a)}(X)$. ■

It is easily seen now that the couple $(\mathscr{H}, \mathcal{M})$ satisfies the conditions of Definition 2.1, which gives us the canonical Banach module structure on $\mathscr{H}$.

Remark 2.17 Assume that $\mathcal{A}$ is a Banach algebra with approximate unit and that a morphism $\Phi : \mathcal{A} \to \mathcal{M}(\mathscr{H})$ with dense image is given. Then the Cohen-Hewitt theorem shows that each $u \in \mathscr{H}$ can be written as $u = Av$ where $A \in \Phi(\mathcal{A})$ and $v \in \mathscr{H}$. We give now examples of such algebras in the preceding context. If ω is a sub-multiplicative function on X^*, i.e. a Borel map $X^* \to [1, \infty[$ satisfying $\omega(k'k'') \leq \omega(k')\omega(k'')$ (this implies local boundedness), let $C^{(\omega)}(X)$ be the set of functions φ whose Fourier transform $\widehat{\varphi}$ satisfies

$$\|\varphi\|_{C^{(\omega)}} := \int_{X^*} |\widehat{\varphi}(k)|\omega(k)dk < \infty. \tag{2.13}$$

Then $C^{(\omega)}(X)$ is a subalgebra of $C_0(X)$ and is a Banach algebra for the norm (2.13). Moreover, $C^{(a)}(X) \subset C^{(\omega)}(X)$ densely and the net $\{e_\alpha\}$ defined in the proof of Lemma 2.16 is an approximate unit of $C^{(\omega)}(X)$. If $\|V_k\|_{\mathcal{B}(\mathscr{H})} \leq c\omega(k)$ for some number $c > 0$ then $\varphi(Q)$ is well defined for each $\varphi \in C^{(\omega)}(X)$ by the relation (2.11) and $\Phi(\varphi) = \varphi(Q)$ is a continuous morphism $C^{(\omega)}(X) \to \mathcal{M}(\mathscr{H})$ with dense range. We could take $\omega(k) = \sup(1, \|V_k\|_{\mathcal{B}(\mathscr{H})})$ but if a second Banach X-module $\mathscr{K}$ is given then it is more convenient to take $\omega(k) = \sup\{1, \|V_k\|_{\mathcal{B}(\mathscr{H})}, \|V_k\|_{\mathcal{B}(\mathscr{K})}\}$.

The adjoint of a reflexive Banach X-module has a natural structure of Banach X-module. Indeed, it is known and easy to prove that a weakly continuous representation is strongly continuous. Thus we can equip the adjoint space $\mathscr{H}^*$ with the Banach X-module structure defined by the representation $k \mapsto (V_{\bar{k}})^*$, where $\bar{k}$ is the complex conjugate of k (i.e. its inverse in X^*).

The group X is, in particular, a locally compact topological space, hence the notion of Banach X-module in the sense of Definition 2.11 makes sense. But this is in fact a particular case of that of Banach X-module in the sense of Definition 2.15. Indeed, according to the comments after Definition 2.11, we get a strongly continuous representation of X^* on $\mathscr{H}$ by setting $V_k = k(Q)$. In the case of Hilbert X-modules we have a more precise fact.

Lemma 2.18 *If $\mathscr{H}$ is a Hilbert space then giving a Hilbert X-module structure on $\mathscr{H}$ is equivalent with giving a Banach X-module structure on $\mathscr{H}$ such that the representation $\{V_k\}_{k\in X^*}$ is unitary. The relation between the two structures is determined by the condition $V_k = k(Q)$.*

Proof: If $\mathscr{H}$ is a Hilbert X-module then we can define $V_k = k(Q) \in \mathcal{B}(\mathscr{H})$ and check that $\{V_k\}_{k\in X^*}$ is a strongly continuous unitary representation of X^* on $\mathscr{H}$ with the help of Lemma 2.12. Reciprocally, it is well known that such a representation allows one to equip $\mathscr{H}$ with a Hilbert X-module structure. The main point is that the estimate $\|\varphi(Q)\| \leq \sup|\varphi|$ holds, see [Lo]. ■

If X is a locally compact abelian group, then Banach X-modules which are not Hilbert X-modules often appear in the following context.

Definition 2.19 *If X is a locally compact abelian group then a* stable Friedrichs X-module *is a Friedrichs X-module $(\mathscr{G}, \mathscr{H})$ satisfying $V_k\mathscr{G} \subset \mathscr{G}$ for all $k \in X^*$ and such that if $u \in \mathscr{G}$ and if $K \subset X^*$ is compact then* $\sup_{k\in K} \|V_k u\|_{\mathscr{G}} < \infty$.

Here $V_k = k(Q)$. It is clear that $V_k\mathscr{G} \subset \mathscr{G}$ implies $V_k \in \mathcal{B}(\mathscr{G})$ and that the local boundedness condition implies that the map $k \mapsto V_k \in \mathcal{B}(\mathscr{G})$ is a weakly, hence strongly, continuous representation of X^* on $\mathscr{G}$ (not unitary in general). The local boundedness condition is automatically satisfied if X^* is second countable.

Thus, *if $(\mathscr{G},\mathscr{H})$ is a stable Friedrichs X-module, then $\mathscr{G}$ is equipped with a canonical Banach X-module structure*. Then, by taking adjoints, we get a natural Banach X-module structure on $\mathscr{G}^*$ too. Our definitions are such that after the identifications $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ the restriction to $\mathscr{H}$ of the operator V_k acting in $\mathscr{G}^*$ is just the initial V_k. Indeed, we have $V_k^* = V_k^{-1} = V_{\bar{k}}$ in $\mathscr{H}$. Thus there is no ambiguity in using the same notation V_k for the representation of X^* in the three spaces $\mathscr{G}, \mathscr{H}$ and $\mathscr{G}^*$.

Remark 2.20 We stress that if $(\mathscr{G},\mathscr{H})$ is a stable Friedrichs X-module then $\mathscr{G}$ is always equipped with the Banach module structure associated to its X-module structure defined above (we recall that in the case of a general Friedrichs X-module there was no restriction on the module structure of $\mathscr{G}$). As a consequence, *we have $\mathcal{B}_0^l(\mathscr{K},\mathscr{G}) \subset \mathcal{B}_0^l(\mathscr{K},\mathscr{H})$ for an arbitrary Banach space $\mathscr{K}$, hence also $\mathcal{B}_q^r(\mathscr{K},\mathscr{G}) \subset \mathcal{B}_q^r(\mathscr{K},\mathscr{H})$ if $\mathscr{K}$ is a Banach module*. Indeed, if $S \in \mathcal{B}_0^l(\mathscr{K},\mathscr{G})$ then $S = \varphi(Q)T$ for some $\varphi \in C^{(\omega)}(X)$ with $\omega(k) = \sup(1, \|V_k\|_{\mathcal{B}(\mathscr{G})})$, see Remark 2.17, and some $T \in \mathcal{B}(\mathscr{K},\mathscr{G})$. But clearly such a $\varphi(Q)$ belongs to the multiplier algebra of $\mathscr{H}$ and $T \in \mathcal{B}(\mathscr{K},\mathscr{H})$.

We show now that, in the case of Banach X-modules over locally compact groups, quasilocality is related to regularity in the sense of the next definition.

Definition 2.21 *Let $\mathscr{H}$ and $\mathscr{K}$ be Banach X-modules. We say that a continuous operator $S : \mathscr{H} \to \mathscr{K}$ is of class $C^{\mathrm{u}}(Q)$, and we write $S \in C^{\mathrm{u}}(Q;\mathscr{H},\mathscr{K})$, if the map $k \mapsto V_k^{-1}SV_k \in \mathcal{B}(\mathscr{H},\mathscr{K})$ is norm continuous.*

Note that norm continuity at the origin implies norm continuity everywhere. The class of regular operators is stable under algebraic operations:

Proposition 2.22 *Let $\mathscr{G},\mathscr{H},\mathscr{K}$ be Banach X-modules.*
(i) If $S \in C^{\mathrm{u}}(Q;\mathscr{H},\mathscr{K})$ and $T \in C^{\mathrm{u}}(Q;\mathscr{G},\mathscr{H})$ then $ST \in C^{\mathrm{u}}(Q;\mathscr{G},\mathscr{K})$.
(ii) If $S \in C^{\mathrm{u}}(Q;\mathscr{H},\mathscr{K})$ is bijective, then $S^{-1} \in C^{\mathrm{u}}(Q;\mathscr{K},\mathscr{H})$.
(iii) If $S \in C^{\mathrm{u}}(Q;\mathscr{H},\mathscr{K})$ and $\mathscr{H},\mathscr{G}$ are reflexive, then $S^* \in C^{\mathrm{u}}(Q;\mathscr{K}^*,\mathscr{H}^*)$.

Proof: We prove only (ii). If we set $S_k = V_k^{-1}SV_k$ then $V_k^{-1}S^{-1}V_k = S_k^{-1}$, hence

$$\|V_k^{-1}S^{-1}V_k - S^{-1}\| = \|S_k^{-1} - S^{-1}\| = \|S_k^{-1}(S - S_k)S^{-1}\| \le C\|S - S_k\|$$

which tends to zero as $k \to 0$. ■

Proposition 2.23 *If $T \in C^{\mathrm{u}}(Q;\mathscr{H},\mathscr{K})$ then T is quasilocal.*

Proof: We show that $\varphi(Q)T \in \mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ if $\varphi \in C^{(a)}(X)$. A similar argument gives $T\varphi(Q) \in \mathcal{B}_0^l(\mathscr{H},\mathscr{K})$. Set $T_k = V_k T V_k^{-1}$, then

$$\varphi(Q)T = \int_{X^*} \widehat{\varphi}(k) V_k T dk = \int_{X^*} T_k \widehat{\varphi}(k) V_k dk.$$

Since $k \mapsto T_k$ is norm continuous on the compact support of $\widehat{\varphi}$, for each $\varepsilon > 0$ we can construct, with the help of a partition of unity, functions $\theta_i \in C_{\mathrm{c}}(X^*)$ and operators $S_i \in \mathcal{B}(\mathscr{H},\mathscr{K})$, such that $\|T_k - \sum_{i=1}^n \theta_i(k) S_i\| < \varepsilon$ if $\widehat{\varphi}(k) \neq 0$. Thus

$$\|\varphi(Q)T - \sum_{i=1}^n \int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k dk\| \leq \varepsilon \sum_{i=1}^n \int_{X^*} |\widehat{\varphi}(k)| \|V_k\|_{\mathcal{B}(\mathscr{H})} dk.$$

Now, since $\mathcal{B}_0(\mathscr{H},\mathscr{K})$ is a norm closed subspaces, it suffices to show that the operator $\int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k dk$ belongs to $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ for each i. But if ψ_i is the inverse Fourier transform of $\theta_i\widehat{\varphi}$ then this is $S_i\psi_i(Q)$ and $\psi_i \in C^{(a)}(X)$. ■

We recall that if X is an abelian locally compact group then there is enough structure in order to develop a rich pseudo-differential calculus in $L^2(X)$, but we give only elementary examples. If φ and ψ are bounded Borel functions on X and X^* respectively then, following standard quantum mechanical conventions, we denote by $\varphi(Q)$ the operator of multiplication by φ in $L^2(X)$ and we set $\psi(P) = \mathcal{F}^{-1} M_\psi \mathcal{F}$, where M_ψ is the operator of multiplication by ψ in $L^2(X^*)$. Then one gets more general pseudo-differential operators of order zero by considering C^*-algebras generated by products $\varphi(Q)\psi(P)$. We recall that the C^*-algebra generated by such products with φ and ψ bounded Borel and convergent to zero at infinity is the algebra of compact operators on $L^2(X)$.

Let $C_{\mathrm{b}}^{\mathrm{u}}(X)$ and $C_{\mathrm{b}}^{\mathrm{u}}(X^*)$ be the algebras of bounded uniformly continuous functions on X and X^* respectively. Below the space $L^2(X)$ is equipped with its natural Hilbert X-module structure.

Proposition 2.24 *The C^*-algebra generated by the operators $\varphi(Q)$ and $\psi(P)$ with $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ and $\psi \in C_{\mathrm{b}}^{\mathrm{u}}(X^*)$ consists of quasilocal operators on $L^2(X)$.*

Proof: Since the set of quasilocal operators in $\mathcal{B}(L^2(X))$ is a C^*-algebra, it suffices to show that each $\varphi(Q)$ and $\psi(P)$ is quasilocal. For $\varphi(Q)$ the assertion is trivial while for $\psi(P)$ we apply Proposition 2.23. ■

3 Abstract compactness results

In this section $(\mathscr{G},\mathscr{H})$ will always be a *compact Friedrichs module*, see Definition 2.2. As usual, we associate to it a Gelfand triplet $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ and we set

$\|\cdot\| = \|\cdot\|_{\mathscr{H}}$. We are interested in criteria which ensure that an operator B is a compact perturbation of an operator A, both operators being unbounded operators in $\mathscr{H}$ obtained as restrictions of some bounded operators $\mathscr{G} \to \mathscr{G}^*$. More precisely, the following is a general assumption (suggested by the statement of Theorem 2.1 in [OS]) which will always be fulfilled:

$$
\textbf{(}AB\textbf{)} \begin{cases} A, B \text{ are closed densely defined operators in } \mathscr{H} \text{ with } \rho(A) \cap \rho(B) \neq \emptyset \\ \text{and having the following properties: } \mathcal{D}(A) \subset \mathscr{G} \text{ densely, } \mathcal{D}(A^*) \subset \mathscr{G}, \\ \mathcal{D}(B) \subset \mathscr{G} \text{ and } A, B \text{ extend to continuous operators } \widetilde{A}, \widetilde{B} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*). \end{cases}
$$

The rôle of the assumption (AB) is to allow us to give a rigorous meaning to the formal relation, where $z \in \rho(A) \cap \rho(B)$,

$$
(A-z)^{-1} - (B-z)^{-1} = (A-z)^{-1}(B-A)(B-z)^{-1}. \tag{3.14}
$$

Recall that $z \in \rho(A)$ if and only if $\bar{z} \in \rho(A^*)$ and then $(A^* - \bar{z})^{-1} = (A-z)^{-1*}$. Thus we have $(A-z)^{-1*}\mathscr{H} \subset \mathscr{G}$ by the assumption (AB) and this allows one to deduce that $(A-z)^{-1}$ extends to a unique continuous operator $\mathscr{G}^* \to \mathscr{H}$, that we shall denote for the moment by R_z. From $R_z(A-z)u = u$ for $u \in \mathcal{D}(A)$ we get, by density of $\mathcal{D}(A)$ in $\mathscr{G}$ and continuity, $R_z(\widetilde{A}-z)u = u$ for $u \in \mathscr{G}$, in particular

$$
(B-z)^{-1} = R_z(\widetilde{A}-z)(B-z)^{-1}.
$$

On the other hand, the identity

$$
(A-z)^{-1} = (A-z)^{-1}(B-z)(B-z)^{-1} = R_z(\widetilde{B}-z)(B-z)^{-1}
$$

is trivial. Subtracting the last two relations we get

$$
(A-z)^{-1} - (B-z)^{-1} = R_z(\widetilde{B}-\widetilde{A})(B-z)^{-1}
$$

Since R_z is uniquely determined as extension of $(A-z)^{-1}$ to a continuous map $\mathscr{G}^* \to \mathscr{H}$, we shall keep the notation $(A-z)^{-1}$ for it. With this convention, the rigorous version of (3.14) that we shall use is:

$$
(A-z)^{-1} - (B-z)^{-1} = (A-z)^{-1}(\widetilde{B}-\widetilde{A})(B-z)^{-1}. \tag{3.15}
$$

Theorem 3.1 *Let A, B satisfy assumption (AB) and let us assume that there are a Banach module $\mathscr{K}$ and operators $S \in \mathcal{B}(\mathscr{K}, \mathscr{G}^*)$ and $T \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{K})$ such that $\widetilde{B} - \widetilde{A} = ST$ and $(A-z)^{-1}S \in \mathcal{B}_q^r(\mathscr{K}, \mathscr{H})$ for some $z \in \rho(A) \cap \rho(B)$. Then the operator B is a compact perturbation of the operator A and $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.*

Proof: It suffices to show that $R \equiv (A-z)^{-1} - (B-z)^{-1} \in \mathcal{B}_0^l(\mathscr{H})$, because the domains of A and B are included in $\mathscr{G}$, hence $R\mathscr{H} \subset \mathscr{G}$, which finishes the proof because of Corollary 2.7. Now due to (3.15) and to the factorization assumption, we can write R as a product $R = [(A-z)^{-1}S][T(B-z)^{-1}]$ where the first factor is in $\mathcal{B}_q^r(\mathscr{K},\mathscr{H})$ and the second in $\mathcal{B}_0^l(\mathscr{H},\mathscr{K})$, so the product is in $\mathcal{B}_0^l(\mathscr{H})$. ■

Remarks 3.2 (1) We could have stated the assumptions of Theorem 3.1 in an apparently more general form, namely $B - A = \sum_{k=1}^n S_k T_k$ with operators $S_k \in \mathcal{B}(\mathscr{K}_k, \mathscr{G}^*)$ and $T_k \in \mathcal{B}(\mathscr{G}, \mathscr{K}_k)$. But we are reduced to the stated version of the assumption by considering the Hilbert module $\mathscr{K} = \oplus \mathscr{K}_k$ and $S = \oplus S_k, T = \oplus T_k$.
(2) If $V \in \mathcal{K}(\mathscr{G}, \mathscr{G}^*)$ and if $\mathscr{K}$ is an infinite dimensional module, then there are operators $S \in \mathcal{B}(\mathscr{K}, \mathscr{G}^*)$ and $T \in \mathcal{K}(\mathscr{G}, \mathscr{K})$ such that $V = ST$ (the proof is an easy exercise). This and the preceding remark show that compact contributions to $\widetilde{B} - \widetilde{A}$ are trivially covered by the factorization assumption.

Example 3.3 One can construct interesting classes of operators with the properties required in (AB) as follows. Let $\mathscr{G}_a, \mathscr{G}_b$ be Hilbert spaces such that $\mathscr{G} \subset \mathscr{G}_a \subset \mathscr{H}$ and $\mathscr{G} \subset \mathscr{G}_b \subset \mathscr{H}$ continuously and densely. Thus we have two scales

$$\mathscr{G} \subset \mathscr{G}_a \subset \mathscr{H} \subset \mathscr{G}_a^* \subset \mathscr{G}^*,$$
$$\mathscr{G} \subset \mathscr{G}_b \subset \mathscr{H} \subset \mathscr{G}_b^* \subset \mathscr{G}^*.$$

Then let $A_0 \in \mathcal{B}(\mathscr{G}_a, \mathscr{G}_a^*)$ and $B_0 \in \mathcal{B}(\mathscr{G}_b, \mathscr{G}_b^*)$ such that $A_0 - z : \mathscr{G}_a \to \mathscr{G}_a^*$ and $B_0 - z : \mathscr{G}_b \to \mathscr{G}_b^*$ are bijective for some number z. According to Lemma A.1 we can associate to A_0, B_0 closed densely defined operators $A = \widehat{A_0}, B = \widehat{B_0}$ in $\mathscr{H}$, such that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense subspaces of $\mathscr{G}_a$ and the domains $\mathcal{D}(B)$ and $\mathcal{D}(B^*)$ are dense subspaces of $\mathscr{G}_b$. If we also have $\mathcal{D}(A) \subset \mathscr{G}$ densely, $\mathcal{D}(A^*) \subset \mathscr{G}$ and $\mathcal{D}(B) \subset \mathscr{G}$, then all the conditions of the assumption (AB) are fulfilled with $\widetilde{A} = A_0|\mathscr{G}$ and $\widetilde{B} = B_0|\mathscr{G}$.

The case when one of the operators, for example A, is self-adjoint is worth to be considered separately. As explained in the Appendix, the conditions on A in assumption (AB) are satisfied if $\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ densely. Moreover, if A is semibounded, then this condition is also necessary. In particular, we have:

Corollary 3.4 *Let A, B be self-adjoint operators on $\mathscr{H}$ such that*

$$\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2}) \text{ and } \mathcal{D}(B) \subset \mathscr{G} \subset \mathcal{D}(|B|^{1/2}) \text{ densely.}$$

Let $\widetilde{A}, \widetilde{B}$ be the unique extensions of A, B to operators in $\mathscr{B}(\mathscr{G}, \mathscr{G}^*)$. Assume that there is a Hilbert module $\mathscr{K}$ and that $\widetilde{B} - \widetilde{A} = S^*T$ for some $S \in \mathcal{B}(\mathscr{G}, \mathscr{K})$ and $T \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{K})$ such that $S(A-z)^{-1} \in \mathcal{B}_q^l(\mathscr{H}, \mathscr{K})$ for some $z \in \rho(A) \cap \rho(B)$. Then B is a compact perturbation of A and $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.

The results which follow are either corollaries of Theorem 3.1 or are versions of the theorem based on essentially the same proof. We shall use the results and the terminology of the Appendix. We begin with the simplest corollary which nevertheless covers interesting examples. Note that X is always assumed non-compact.

Theorem 3.5 *Assume that $(\mathscr{G},\mathscr{H})$ is a compact stable Friedrichs X-module over a locally compact abelian group X and that condition (AB) is satisfied. Assume, furthermore, that $\widetilde{A}-z:\mathscr{G}\to\mathscr{G}^*$ is bijective for some $z\in\rho(A)\cap\rho(B)$ and that $\widetilde{A}\in C^{\mathrm{u}}(Q;\mathscr{G},\mathscr{G}^*)$. If $\widetilde{B}-\widetilde{A}\in\mathcal{B}_0^l(\mathscr{G},\mathscr{G}^*)$, then the operator B is a compact perturbation of the operator A.*

Proof: We apply Theorem 3.1 with $\mathscr{K}=\mathscr{G}^*$, S the identity operator and $T=\widetilde{B}-\widetilde{A}$. Then $(\widetilde{A}-z)^{-1}$ is of class $C^{\mathrm{u}}(Q;\mathscr{G}^*,\mathscr{G})$ by (ii) of Proposition 2.22, hence $(\widetilde{A}-z)^{-1}\in\mathcal{B}_q(\mathscr{G}^*,\mathscr{G})$ by Proposition 2.23. But this is stronger than $(\widetilde{A}-z)^{-1}\in\mathcal{B}_q^r(\mathscr{G}^*,\mathscr{H})$, as follows from the Remark 2.20. ■

The next results are convenient for applications to differential operators in divergence form. In these statements we implicitly use Lemma A.1: we note that the operators D^*aD and D^*bD considered below belong to $\mathcal{B}(\mathscr{G},\mathscr{G}^*)$ and we denote by Δ_a and Δ_b the operators on $\mathscr{H}$ associated to them. The notation $\mathcal{B}_{00}^l(\mathscr{E},\mathscr{E}^*)$ is introduced in (2.9).

Theorem 3.6 *Let $(\mathscr{G},\mathscr{H})$ be a compact Friedrichs module, let $(\mathscr{E},\mathscr{K})$ be an arbitrary Friedrichs module, and assume that we are given operators $D\in\mathcal{B}(\mathscr{G},\mathscr{E})$ and $a,b\in\mathcal{B}(\mathscr{E},\mathscr{E}^*)$ and a complex number z such that:*
(1) *The operators D^*aD-z and D^*bD-z are bijective maps $\mathscr{G}\to\mathscr{G}^*$,*
(2) $a-b\in\mathcal{B}_{00}^l(\mathscr{E},\mathscr{E}^*)$,
(3) $D(\Delta_a^*-\bar{z})^{-1}\in\mathcal{B}_q^l(\mathscr{H},\mathscr{K})$.
Then Δ_b is a compact perturbation of Δ_a.

Proof: We give a proof independent of Theorem 3.1, although we could apply this theorem. From Lemma A.1 it follows that the operators Δ_a-z and Δ_b-z extend to bijections $\mathscr{G}\to\mathscr{G}^*$ and the identity

$$R:=(\Delta_a-z)^{-1}-(\Delta_b-z)^{-1}=(\Delta_a-z)^{-1}D^*(b-a)D(\Delta_b-z)^{-1}$$

holds in $\mathcal{B}(\mathscr{G}^*,\mathscr{G})$, hence in $\mathcal{B}(\mathscr{H})$. Since the domains of Δ_a and Δ_b are included in $\mathscr{G}$, we have $R\mathscr{H}\subset\mathscr{G}$. Thus, according to Corollary 2.7, it suffices to show that $R\in\mathcal{B}_0^l(\mathscr{H})$. Since the space $\mathcal{B}_0^l(\mathscr{H})$ is norm closed and since by hypothesis we can approach $b-a$ in norm in $\mathcal{B}(\mathscr{E},\mathscr{E}^*)$ by operators in $\mathcal{B}_0^l(\mathscr{K})$, it suffices to show that

$$(D(\Delta_a^*-\bar{z})^{-1})^*cD(\Delta_b-z)^{-1}\in\mathcal{B}_0^l(\mathscr{H})$$

if $c \in \mathcal{B}_0^l(\mathscr{K})$. But this is clear because $cD(\Delta_b - z)^{-1} \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ and $(D(\Delta_a^* - \bar{z})^{-1})^* \in \mathcal{B}_q^r(\mathscr{K}, \mathscr{H})$ by Proposition 2.10. ■

By (2.10) we have $\mathcal{K}(\mathscr{E}, \mathscr{E}^*) \subset \mathcal{B}_{00}^l(\mathscr{E}, \mathscr{E}^*)$, but the case $a - b \in \mathcal{K}(\mathscr{E}, \mathscr{E}^*)$ is trivial from the point of view of this paper. Although the space $\mathcal{B}_{00}^l(\mathscr{E}, \mathscr{E}^*)$ is much larger than $\mathcal{K}(\mathscr{E}, \mathscr{E}^*)$, we can allow still more general perturbations and obtain more explicit results if we impose more structure on the modules, cf. Remark 4.2. We now describe such an improvement for the case of X-modules, where X is a locally compact abelian group. We shall need the following fact.

Lemma 3.7 *Let X be an abelian locally compact group and let $(\mathscr{G}, \mathscr{H})$ and $(\mathscr{E}, \mathscr{K})$ be stable Friedrichs X-modules. Let $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ be operators of class $C^{\mathrm{u}}(Q)$ such that $D^*aD - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some complex number z and denote Δ_a the operator on $\mathscr{H}$ associated to D^*aD. Then the operator $D(\Delta_a - z)^{-1} \in \mathcal{B}(\mathscr{H}, \mathscr{E})$ is quasilocal.*

Proof: The lemma is an easy consequence of Propositions 2.22 and 2.23. Indeed, due to Proposition 2.23, it suffices to show that the operator $D(\Delta_a - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{E})$. We shall prove more, namely that $D(D^*aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{E})$. Since D is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$, and due to (i) of Proposition 2.22, it suffices to show that $(D^*aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{G})$. But $D^*aD - z$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{G}^*)$ by (i) and (iii) of Proposition 2.22 and is a bijective map $\mathscr{G} \to \mathscr{G}^*$, so the result follows from (ii) of Proposition 2.22. ■

Theorem 3.8 *Let X be an abelian locally compact group and let $(\mathscr{G}, \mathscr{H})$ be a compact stable Friedrichs X-module and $(\mathscr{E}, \mathscr{K})$ a stable Friedrichs X-modules. Assume that $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a, b \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ are operators of class $C^{\mathrm{u}}(Q)$ such that the operators $D^*aD - z$ and $D^*bD - z$ are bijective maps $\mathscr{G} \to \mathscr{G}^*$ for some complex number z. If $a - b \in \mathcal{B}_0^l(\mathscr{E}, \mathscr{E}^*)$ then Δ_b is a compact perturbation of Δ_a.*

Proof: The proof is a repetition of that of Theorem 3.6. The only difference is that we write directly

$$R = (D(\Delta_a^* - \bar{z})^{-1})^*(b - a)D(\Delta_b - z)^{-1}$$

and observe that $(b - a)D(\Delta_b - z)^{-1} \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{E}^*)$ and that $(D(\Delta_a^* - \bar{z})^{-1})^*$ as an operator $\mathscr{E}^* \to \mathscr{H}$ is quasilocal by (2) of Proposition 2.10 and by the fact that the operator $D(\Delta_a^* - \bar{z})^{-1} : \mathscr{H} \to \mathscr{E}$ is quasilocal, cf Lemma 3.7. ■

4 Pseudo-differential operators

4.1 Operators in divergence form on Euclidean spaces

Our first example is in the context of Theorem 3.8. Here $X = \mathbb{R}^n$ equipped with the Lebesgue measure and $\mathscr{H} = L^2(X)$ with the obvious Hilbert X-module structure. If $s \in \mathbb{R}$ we denote by $\mathscr{H}^s$ the usual Sobolev space.

For each $s > 0$ the couple $(\mathscr{H}^s, \mathscr{H})$ is a clearly a compact Friedrichs module. Indeed, for each $\varphi \in C_0(X)$ the operator $\varphi(Q) : \mathscr{H}^s \to \mathscr{H}$ is compact. But we have more: $(\mathscr{H}^s, \mathscr{H})$ is also a stable Friedrichs X-module with respect to the additive group structure on X. In fact, if we identify as usual X^* with X with the help of the exponential function, the representation of X in $\mathscr{H}$ which defines the Hilbert X-module structure of $\mathscr{H}$ is $(V_k u)(x) = \exp(i\langle x, k\rangle)u(x)$, where $\langle x, k\rangle$ is the scalar product. Then we easily get $V_k\mathscr{H}^s \subset \mathscr{H}^s$ and $\|V_k\| \le C(1 + |k|)^s$.

Let us describe the objects which appear in Theorem 3.8 in the present context. We fix an integer $m \ge 1$ and take $\mathscr{G} = \mathscr{H}^m$. Let $\mathscr{K} = \bigoplus_{|\alpha|\le m} \mathscr{H}_\alpha$, where $\mathscr{H}_\alpha \equiv \mathscr{H}$, with the natural direct sum Hilbert X-module structure. Here α are multi-indices $\alpha \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then we define

$$\mathscr{E} = \bigoplus_{|\alpha|\le m} \mathscr{H}^{m-|\alpha|} = \{(u_\alpha)_{|\alpha|\le m} \in \mathscr{K} \mid u_\alpha \in \mathscr{H}^{m-|\alpha|}\}$$

equipped with the Hilbert direct sum structure. It is obvious that $(\mathscr{E}, \mathscr{K})$ is a stable Friedrichs X-module (but not compact).

We set $P_k = -i\partial_k$, where ∂_k is the derivative with respect to the k-th variable, and $P^\alpha = P_1^{\alpha_1} \dots P_n^{\alpha_n}$ if $\alpha \in \mathbb{N}^n$. Then for $u \in \mathscr{G}$ let $Du = (P^\alpha u)_{|\alpha|\le m} \in \mathscr{K}$. Since

$$\|Du\|^2 = \sum_{|\alpha|\le m} \|P^\alpha u\|^2 = \|u\|^2_{\mathscr{H}^m}$$

we see that $D : \mathscr{G} \to \mathscr{K}$ is a linear isometry. Moreover, we have defined $\mathscr{E}$ such as to have $D\mathscr{G} \subset \mathscr{E}$, hence $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$. We shall prove now that $D \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$ (in fact, much more). We have, with natural notations,

$$V_k^{-1} D V_k = (V_k^{-1} P^\alpha V_k)_{|\alpha|\le m} = ((P + k)^\alpha)_{|\alpha|\le m}$$

and this a polynomial in k with coefficients in $\mathcal{B}(\mathscr{G}, \mathscr{E})$, hence the assertion.

We shall identify $\mathscr{H}^* = \mathscr{H}$ and $\mathscr{K}^* = \mathscr{K}$, which implies $\mathscr{G}^* = \mathscr{H}^{-m}$ and

$$\mathscr{E}^* = \oplus_{|\alpha|\le m} \mathscr{H}^{|\alpha|-m}.$$

The operator $D^* \in \mathcal{B}(\mathscr{E}^*, \mathscr{G}^*)$ acts as follows:

$$D^*(u_\alpha)_{|\alpha|\le m} = \sum_{|\alpha|\le m} P^\alpha u_\alpha \in \mathscr{H}^{-m},$$

because $u_\alpha \in \mathscr{H}^{|\alpha|-m}$.

By taking into account the given expressions for $\mathscr{E}$ and $\mathscr{E}^*$ we see that we can identify an operator $a \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ with a matrix of operators $a = (a_{\alpha\beta})_{|\alpha|,|\beta|\leq m}$, where $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$ and

$$a(u_\beta)_{|\beta|\leq m} = \big(\sum_{|\beta|\leq m} a_{\alpha\beta} u_\beta\big)_{|\alpha|\leq m}.$$

Then we clearly have

$$D^* a D = \sum_{|\alpha|,|\beta|\leq m} P^\alpha a_{\alpha\beta} P^\beta. \tag{4.16}$$

which is a general version of a differential operator in divergence form. We must, however, emphasize, that our $a_{\alpha\beta}$ are not necessarily functions, they could be pseudo-differential or more general operators.

In view of the statement of the next theorem, we note that, since the Sobolev spaces are Banach X-modules, the class of regularity $C^{\mathrm{u}}(Q; \mathscr{H}^s, \mathscr{H}^t)$ is well defined for all real s, t. A bounded operator $S : \mathscr{H}^s \to \mathscr{H}^t$ belongs to this class if and only if the map $k \mapsto V_{-k} S V_k \in \mathcal{B}(\mathscr{H}^s, \mathscr{H}^t)$ is norm continuous. In particular, this condition is trivially satisfied if S is the operator of multiplication by a function, because then V_k commutes with S. Since the coefficients $a_{\alpha\beta}$ of the differential expression (4.16) are usually assumed to be functions, this is barely a restriction in the setting of the next theorem. The condition $S \in \mathcal{B}_0^l(\mathscr{H}^s, \mathscr{H}^t)$ is also well defined and it is easily seen that it is equivalent to

$$\lim_{r\to\infty} \|\theta(Q/r)S\|_{\mathscr{H}^s \to \mathscr{H}^t} \to 0 \tag{4.17}$$

where θ is a C^∞ function on X equal to zero on a neighborhood of the origin and equal to one on a neighborhood of infinity. Now we can state the following immediate consequence of Theorem 3.8.

Proposition 4.1 *Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be operators of class $C^{\mathrm{u}}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$ and such that the operators $D^* a D - z$ and $D^* b D - z$ are bijective maps $\mathscr{H}^m \to \mathscr{H}^{-m}$ for some complex z. Let Δ_a and Δ_b be the operators in $\mathscr{H}$ associated to $D^* a D$ and $D^* b D$ respectively. Assume that*

$$\lim_{r\to\infty} \|\theta(Q/r)(a_{\alpha\beta} - b_{\alpha\beta})\|_{\mathscr{H}^{m-|\alpha|} \to \mathscr{H}^{|\alpha|-m}} = 0 \tag{4.18}$$

for each α, β, where θ is a function as above. Then Δ_b is a compact perturbation of Δ_a and the operators Δ_a and Δ_b have the same essential spectrum.

Example: In the simplest case the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the principal parts (i.e. $|\alpha| = |\beta| = m$) are functions. Then the conditions become: $a_{\alpha\beta}$ and $b_{\alpha\beta}$ belong to $L^\infty(X)$ and $|a_{\alpha\beta}(x) - b_{\alpha\beta}(x)| \to 0$ as $|x| \to \infty$. Of course, the assumptions on the lowest order coefficients are much more general.

Example: We show here that "highly oscillating potentials" do not modify the essential spectrum. If $m = 1$ then the terms of order one of D^*aD are of the form $S = \sum_{k=1}^n (P_k v'_k + v''_k P_k)$, where $v'_k \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ and $v''_k \in \mathcal{B}(\mathscr{H}, \mathscr{H}^{-1})$. Choose $v_k \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ symmetric in $\mathscr{H}$ and let $v'_k = iv_k, v''_k = -iv_k$. Then $S = [iP, v] \equiv \operatorname{div} v$, with natural notations, can also be thought as a term of order zero. Now assume that v_k are bounded Borel functions and consider a similar term $T = [iP, w]$ for D^*bD. Then the condition $|v_k(x) - w_k(x)| \to 0$ as $|x| \to \infty$ suffices to ensure the stability of the essential spectrum. However, the difference $S - T$ could be a function which does not tend to zero at infinity in a simple sense, being only "highly oscillating". An explicit example in the case $n = 1$ is the following: a perturbation of the form $\exp(x)(1 + |x|)^{-1} \cos(\exp(x))$ is allowed because it is the derivative of $(1 + |x|)^{-1} \sin(\exp(x))$ plus a function which tends to zero at infinity.

In order to apply Proposition 4.1 we need that $D^*aD - z : \mathscr{H}^m \to \mathscr{H}^{-m}$ be bijective for some $z \in \mathbb{C}$, and similarly for b. A standard way of checking this is to require the following *coercivity condition*:

$$(C) \begin{cases} \text{there are } \mu, \nu > 0 \text{ such that for all } u \in \mathscr{H}^m : \\ \sum_{|\alpha|,|\beta| \le m} \operatorname{Re} \langle P^\alpha u, a_{\alpha\beta} P^\beta u \rangle \ge \mu \|u\|^2_{\mathscr{H}^m} - \nu \|u\|^2_{\mathscr{H}} \end{cases}$$

Example: One often imposes a stronger ellipticity condition that we describe below. Observe that the coefficients of the highest order part of D^*aD defined by $A_0 = \sum_{|\alpha|=|\beta|=m} P^\alpha a_{\alpha\beta} P^\beta$ are operators $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$. Then ellipticity means:

$$(Ell) \begin{cases} \text{there is } \mu > 0 \text{ such that if } u_\alpha \in \mathscr{H} \text{ for } |\alpha| = m \text{ then} \\ \sum_{|\alpha|=|\beta|=m} \operatorname{Re} \langle u_\alpha, a_{\alpha\beta} u_\beta \rangle \ge \mu \sum_{|\alpha|=m} \|u_\alpha\|^2_{\mathscr{H}}. \end{cases}$$

But we emphasize that, our conditions on the lower order terms being very general (e.g. the $a_{\alpha\beta}$ could be differential operators, so the terms of formally lower order could be of order $2m$ in fact), we have to supplement the ellipticity condition **(**Ell**)** with a condition saying that the rest of the terms $A_1 = \sum_{|\alpha|+|\beta|<2m} P^\alpha a_{\alpha\beta} P^\beta$ is small with respect to A_0. For example, we may require the existence of some $\delta < \mu$ and $\gamma > 0$ such that

$$\Big| \sum_{|\alpha|+|\beta|<2m} \operatorname{Re} \langle P^\alpha u, a_{\alpha\beta} P^\beta u \rangle \Big| \le \delta \|u\|^2_{\mathscr{H}^m} + \gamma \|u\|^2_{\mathscr{H}}. \tag{4.19}$$

This is satisfied if $A_1\mathscr{H}^m \subset \mathscr{H}^{-m+\theta}$ for some $\theta > 0$, because for each $\varepsilon > 0$ there is $c(\varepsilon) < \infty$ such that $\|u\|_{\mathscr{H}^{m-\theta}} \le \varepsilon\|u\|_{\mathscr{H}^m} + c(\varepsilon)\|u\|_{\mathscr{H}}$.

Remark 4.2 To understand the relation between $\mathcal{B}^l_{00}(\mathscr{E},\mathscr{E}^*)$ and $\mathcal{B}^l_0(\mathscr{E},\mathscr{E}^*)$ it suffices to consider that between $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-t})$ and $\mathcal{B}^l_0(\mathscr{H}^s,\mathscr{H}^{-t})$ for $s,t \ge 0$, where $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-t})$ is the closure of $\mathcal{B}^l_0(\mathscr{H})$ in $\mathcal{B}(\mathscr{H}^s,\mathscr{H}^{-t})$. If $s = t = 0$ then these spaces are the same, hence we get the same conditions on the coefficients $a_{\alpha\beta} - b_{\alpha\beta}$ of the principal part ($|\alpha| = |\beta| = m$) of the operator $a - b$ if we use Theorem 3.6 or 3.8. But if $s + t > 0$ then $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-t})$ does not contain operators of order $s + t$, while $\mathcal{B}^l_0(\mathscr{H}^s,\mathscr{H}^{-t})$ contains such operators.

4.2 A class of pseudo-differential operators on abelian groups

In this subsection X will be a locally compact non-compact non-discrete abelian group. We also fix a finite dimensional complex Hilbert space E and take $\mathscr{H} = L^2(X;E)$ equipped with its natural Hilbert X-module structure. Note that, according to our conventions, the unitary representation of X^* is given by the multiplication operators $V_k = k(Q)$.

Let $w : X^* \to [1,\infty[$ be a continuous function satisfying $w(k) \to \infty$ as $k \to \infty$ and such that $w(k'k) \le \omega(k')w(k)$ holds for some function ω and all k',k. We shall assume that ω is the smallest function satisfying the preceding estimate. It is clear then that ω is sub-multiplicative in the sense defined in Remark 2.17 (see [Ho, Section 10.1] for this construction).

Then $w(P)$ is a self-adjoint operator on $\mathscr{H}$ with $w(P) \ge 1$. We denote $\mathscr{H}^w = \mathcal{D}(w(P))$ and equip it with the Banach X-module structure given by the norm $\|u\|_w = \|w(P)u\|$ and the representation $V_k|\mathscr{H}^w$. Obviously, this space is a generalization of the usual notion of Sobolev spaces.

Lemma 4.3 *$(\mathscr{H}^w,\mathscr{H})$ is a compact stable Friedrichs X-module.*

Proof: If $\varphi \in C_0(X)$ then $\varphi(Q)w(P)^{-1}$ is a compact operator because w^{-1} belongs to $C_0(X)$, hence $\varphi(Q) \in \mathcal{K}(\mathscr{H}^w,\mathscr{H})$. Then observe that $V_k^{-1}w(P)V_k = w(kP)$ and $w(kP) \le \omega(k)w(P)$. Thus V_k leaves stable $\mathscr{H}^w$ and we have the estimate $\|V_k\|_{\mathcal{B}(\mathscr{H}^w)} \le \omega(k)$. ■

We shall consider now an operator A on $\mathscr{H}$ such that there are w as above and an operator $\widetilde{A} \in \mathcal{B}(\mathscr{H}^w,\mathscr{H}^{w*})$ such that $\widetilde{A} - z : \mathscr{H}^w \to \mathscr{H}^{w*}$ is bijective for some complex z and such that A is the operator induced by $\widetilde{A}$ in $\mathscr{H}$ (see the Appendix). For example, the constant coefficients case with $E = \mathbb{C}$ corresponds to the choice $A = h(P)$ with $h : X^* \to \mathbb{C}$ a Borel function such that $c'w^2 \le 1 + |h| \le c''w^2$ and such that the range of h is not dense in $\mathbb{C}$.

Theorem 3.5 is quite well adapted to show the stability of the essential spectrum of such operators under perturbations which are small at infinity. We stress that the differential operators covered by these results can be of any order and that in the usual case when the coefficients are complex measurable functions a condition of the type $\widetilde{A} \in C^{\mathrm{u}}(Q; \mathscr{H}^w, \mathscr{H}^{w*})$ is very general, if not automatically satisfied (see the remark at the end of this subsection). Hence the only condition really relevant in this context is $\widetilde{B} - \widetilde{A} \in \mathcal{B}_0^l(\mathscr{H}^w, \mathscr{H}^{w*})$ and the main point is that it allows perturbations of the higher order coefficients even in the non-smooth case.

It is clear that these results can be used to establish the stability of the essential spectrum of pseudo-differential operators on finite dimensional vector spaces over local fields, cf. [Sa, Ta], under perturbations of the same order.

We shall give only one explicit example of some physical interest, that of Dirac operators. Let $X = \mathbb{R}^n$ and let $\alpha_0 \equiv \beta, \alpha_1, \dots, \alpha_n$ be symmetric operators on E such that $\alpha_j\alpha_k + \alpha_k\alpha_j = \delta_{jk}$. Then the free Dirac operator is $D = \sum_{k=1}^n \alpha_k P_k + m\beta$ for some real number m. The natural compact stable Friedrichs X-module now is $(\mathscr{H}^{1/2}, \mathscr{H})$ where $\mathscr{H}^s$ are usual Sobolev spaces of E-valued functions.

Proposition 4.4 *Let V, W be measurable functions on X with values symmetric operators on E and such that the operators of multiplication by V and W define continuous maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ and $V - W \in \mathcal{B}_0(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$. Assume that $D + V + i$ and $D + W + i$ are bijective maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$. Then $D + V$ and $D + W$ induce self-adjoint operators A and B in $\mathscr{H}$, B is a compact perturbation of A, and $\sigma_{\mathrm{ess}}(B) = \sigma_{\mathrm{ess}}(A)$.*

This follows immediately from Theorem 3.5. We stress that the main new feature of this result is that the “unperturbed” operator A is locally as singular as the “perturbed” one B. The assumptions imposed on V, W are quite general, compare with [Ar, AY, Kl, N1, N2].

Remark: We shall discuss here the relation between the abstract class of operators A considered in this subsection and the notion of hypoellipticity due to Hörmander. For this we shall consider the case of differential operators on $\mathbb{R}^n$ (which is identified with its dual group in the standard way). Assume first that h is a polynomial on $\mathbb{R}^n$ and that $A = h(P)$. Then the function defined by $w(k)^4 = \sum_\alpha |h^{(\alpha)}(k)|^2$ satisfies $w(k'+k) \le (1+c|k'|)^{m/2} w(k)$, where c is a number and m is the order of h, see [Ho, Example 10.1.3]. Now the “form domain” of the operator $h(P)$ in $L^2(\mathbb{R}^n)$ is the space $\mathscr{G} = \mathcal{D}(|h(P)|^{1/2})$ and this domain is stable under $V_k = \exp i\langle k, Q\rangle$ if and only the function w satisfies $w^2 \le c(1 + |h|)$, see Lemma 7.6.7 in [ABG]. On the other hand, Definition 11.1.2 and Theorem 11.1.3 from [Ho] show that A is hypoelliptic if and only if $h^{(\alpha)}(k)/h(k) \to 0$ when $k \to \infty$, for all $\alpha \neq 0$. So in this case we have $c'w^2 \leq 1 + |h| \leq c''w^2$ and the operator $A = h(P)$ satisfies the conditions of this subsection if $h(\mathbb{R}^n)$ is not dense in $\mathbb{C}$. If $n = 2$ then $h(k) = k_1^4 + k_2^2$ is a simple example of polynomial which has all these properties but is not elliptic. See [GM, Subsections 2.7-2.10] for the case of matrix valued functions h.

5 Abstract Riemannian manifolds

Let $\mathscr{H}, \mathscr{K}$ be two Hilbert spaces identified with their adjoints and d a closed densely defined operator mapping $\mathscr{H}$ into $\mathscr{K}$. Let $\mathscr{G} = \mathcal{D}(\mathrm{d})$ equipped with the graph norm, so $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathrm{d} \in \mathcal{B}(\mathscr{G}, \mathscr{K})$.

Then the quadratic form $\|\mathrm{d}u\|^2_{\mathscr{K}}$ on $\mathscr{H}$ with domain $\mathscr{G}$ is positive densely defined and closed. Let Δ be the positive self-adjoint operator on $\mathscr{H}$ associated to it. In fact $\Delta = \mathrm{d}^*\mathrm{d}$, where the adjoint d^* of d is a closed densely defined operator mapping $\mathscr{K}$ into $\mathscr{H}$.

Now let $\lambda \in \mathcal{B}(\mathscr{H})$ and $\Lambda \in \mathcal{B}(\mathscr{K})$ be self-adjoint and such that $\lambda \geq c$ and $\Lambda \geq c$ for some real $c > 0$. Then we can define new Hilbert spaces $\widetilde{\mathscr{H}}$ and $\widetilde{\mathscr{K}}$ as follows:

$$
(*) \quad \begin{cases} \widetilde{\mathscr{H}} = \mathscr{H} \text{ as vector space and } \langle u \mid v \rangle_{\widetilde{\mathscr{H}}} = \langle u \mid \lambda v \rangle_{\mathscr{H}}, \\ \widetilde{\mathscr{K}} = \mathscr{K} \text{ as vector space and } \langle u \mid v \rangle_{\widetilde{\mathscr{K}}} = \langle u \mid \Lambda v \rangle_{\mathscr{K}}. \end{cases}
$$

Since $\mathscr{H} = \widetilde{\mathscr{H}}$ and $\mathscr{K} = \widetilde{\mathscr{K}}$ as topological vector spaces; the operator $\mathrm{d} : \mathscr{G} \subset \widetilde{\mathscr{H}} \to \widetilde{\mathscr{K}}$ is still a closed densely defined operator, hence the quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\widetilde{\mathscr{H}}$ with domain $\mathscr{G}$ is positive, densely defined and closed. We shall denote by $\widetilde{\Delta}$ the positive self-adjoint operator on $\widetilde{\mathscr{H}}$ associated to it.

We can express $\widetilde{\Delta}$ in more explicit terms as follows. Denote $\widetilde{\mathrm{d}}$ the operator d when viewed as operator acting from $\widetilde{\mathscr{H}}$ to $\widetilde{\mathscr{K}}$. Then $\widetilde{\Delta} = \widetilde{\mathrm{d}}^*\widetilde{\mathrm{d}}$, where $\widetilde{\mathrm{d}}^* : \mathcal{D}(\widetilde{\mathrm{d}}^*) \subset \widetilde{\mathscr{K}} \to \widetilde{\mathscr{H}}$ is the adjoint of $\widetilde{\mathrm{d}} = \mathrm{d}$ with respect to the new Hilbert space structures (the spaces $\widetilde{\mathscr{H}}, \widetilde{\mathscr{K}}$ being also identified with their adjoints). It is easy to check that $\widetilde{\mathrm{d}}^* = \lambda^{-1}\mathrm{d}^*\Lambda$. Thus $\widetilde{\Delta} = \lambda^{-1}\mathrm{d}^*\Lambda\mathrm{d}$.

Now let (X, ρ) be a proper locally compact metric space (see Subsection 2.4) and let us assume that $\mathscr{H}$ and $\mathscr{K}$ are Hilbert X-modules.

Definition 5.1 *A closed densely defined map* $\mathrm{d} : \mathscr{H} \to \mathscr{K}$ *is a* first order operator *if there is* $C \in \mathbb{R}$ *such that for each bounded Lipschitz function* $\varphi : X \to \mathbb{R}$ *the form* $[\mathrm{d}, \varphi(Q)]$ *is a bounded operator and* $\|[\mathrm{d}, \varphi(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq C \operatorname{Lip} \varphi$.

Here

$$\mathrm{Lip}\,\varphi = \inf_{x\neq y} |\varphi(x)-\varphi(y)|\rho(x,y)^{-1}.$$

In more explicit terms, we require

$$|\langle \mathrm{d}^*u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}}| \leq C\,\mathrm{Lip}\,\varphi\,\|u\|_{\mathscr{K}}\|v\|_{\mathscr{H}}$$

for all $u \in \mathcal{D}(\mathrm{d}^*)$ and $v \in \mathcal{D}(d)$. Thus $\langle \mathrm{d}^*u, \varphi(Q)v\rangle - \langle u, \varphi(Q)\mathrm{d}v\rangle$ is a sesquilinear form on the dense subspace $\mathcal{D}(\mathrm{d}^*) \times \mathcal{D}(\mathrm{d})$ of $\mathscr{K} \times \mathscr{H}$ which is continuous for the topology induced by $\mathscr{H} \times \mathscr{K}$. Hence there is a unique continuous operator $[\mathrm{d}, \varphi(Q)] : \mathscr{H} \to \mathscr{K}$ such that

$$\langle \mathrm{d}^*u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}} = \langle u, [\mathrm{d}, \varphi(Q)]v\rangle_{\mathscr{K}}$$

for all $u \in \mathcal{D}(\mathrm{d}^*)$, $v \in \mathcal{D}(\mathrm{d})$ and $\|[\mathrm{d}, \varphi(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq C\,\mathrm{Lip}\,\varphi$.

Lemma 5.2 *The operator* $\mathrm{d}(\Delta+1)^{-1}$ *is quasilocal.*

Proof: We shall prove that $S := \mathrm{d}(\Delta+1)^{-1}$ is a quasilocal operator with the help of Corollary 2.14, more precisely we show that $[S, \varphi(Q)]$ is a bounded operator if φ is a positive Lipschitz function. Let $\varepsilon > 0$ and $\varphi_\epsilon = \varphi(1+\varepsilon\varphi)^{-1}$. Then φ_ε is a bounded function with $|\varphi_\varepsilon| \leq \varepsilon^{-1}$ and

$$|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| = \frac{|\varphi(x)-\varphi(y)|}{(1+\varepsilon\varphi(x))(1+\varepsilon\varphi(y))} \leq |\varphi(x)-\varphi(y)|$$

hence $\mathrm{Lip}\,\varphi_\varepsilon \leq \mathrm{Lip}\,\varphi$. Let $v \in \mathcal{D}(\mathrm{d})$ we have for all $u \in \mathcal{D}(\mathrm{d}^*)$:

$$\begin{aligned} |\langle \mathrm{d}^*u, \varphi_\epsilon(Q)v\rangle_{\mathscr{H}}| &= |\langle u, \varphi_\epsilon(Q)\mathrm{d}v\rangle_{\mathscr{K}} + \langle u, [\mathrm{d}, \varphi_\varepsilon(Q)]v\rangle_{\mathscr{K}}| \\ &\leq \|u\|_{\mathscr{K}}(\varepsilon^{-1}\|\mathrm{d}v\|_{\mathscr{K}} + C\,\mathrm{Lip}\,\varphi_\varepsilon\,\|u\|_{\mathscr{H}}). \end{aligned}$$

Hence $\varphi_\varepsilon(Q)v \in \mathcal{D}(\mathrm{d}^{**}) = \mathcal{D}(\mathrm{d})$ because d is closed. Thus $\varphi_\varepsilon(Q)\mathcal{D}(\mathrm{d}) \subset \mathcal{D}(\mathrm{d})$ and by the closed graph theorem we get $\varphi_\varepsilon(Q) \in \mathcal{B}(\mathscr{G})$, where $\mathscr{G}$ is the domain of d equipped with the graph topology. This also implies that $\varphi_\varepsilon(Q)$ extends to an operator in $\mathcal{B}(\mathscr{G}^*)$ (note that $\varphi_\varepsilon(Q)$ is symmetric in $\mathscr{H}$).

Now, if we think of d as a continuous operator $\mathscr{G} \to \mathscr{K}$, then it has an adjoint $\mathrm{d}^* : \mathscr{K} \to \mathscr{G}^*$ which is the unique continuous extension of the operator $\mathrm{d}^* : \mathcal{D}(\mathrm{d}^*) \subset \mathscr{K} \to \mathscr{H} \subset \mathscr{G}^*$. Thus the canonical extension of Δ to an element of $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ is the product of $\mathrm{d} : \mathscr{G} \to \mathscr{K}$ with $\mathrm{d}^* : \mathscr{K} \to \mathscr{G}^*$ (note $\mathcal{D}(\mathrm{d})$ is the form domain of Δ). Then it is trivial to justify that we have in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$:

$$[\Delta, \varphi_\varepsilon(Q)] = [\mathrm{d}^*, \varphi_\varepsilon(Q)]\mathrm{d} + \mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)].$$

Here $[\mathrm{d}^*, \varphi_\varepsilon(Q)] = [\varphi_\varepsilon(Q), \mathrm{d}]^* \in \mathcal{B}(\mathscr{K}, \mathscr{H})$. Since $\Delta + 1 : \mathscr{G} \to \mathscr{G}^*$ is a linear homeomorphism, we then have in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$:

$$
\begin{aligned}
[\varphi_\varepsilon(Q), (\Delta+1)^{-1}] &= (\Delta+1)^{-1}[\Delta, \varphi_\varepsilon(Q)](\Delta+1)^{-1} \\
&= (\Delta+1)^{-1}[\varphi_\varepsilon(Q), \mathrm{d}]^* \mathrm{d}(\Delta+1)^{-1} \\
&+ (\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)](\Delta+1)^{-1}.
\end{aligned}
$$

Finally, taking once again into account the fact that $\varphi_\varepsilon(Q)$ leaves $\mathscr{G}$ invariant, we have:

$$
\begin{aligned}
[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}] = \; & [\varphi_\varepsilon(Q), \mathrm{d}](\Delta+1)^{-1} \\
& + \mathrm{d}(\Delta+1)^{-1}[\varphi_\varepsilon(Q), \mathrm{d}]^* \mathrm{d}(\Delta+1)^{-1} \\
& + \mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)](\Delta+1)^{-1}.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\|[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq \; & \|[\varphi_\varepsilon(Q), \mathrm{d}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})} \\
& + \|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|[\varphi_\varepsilon(Q), \mathrm{d}]^*\|_{\mathcal{B}(\mathscr{K},\mathscr{H})}\|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\
& + \|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})}\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})}.
\end{aligned}
$$

The most singular factor here is

$$
\|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})} \leq \|\mathrm{d}\|_{\mathcal{B}(\mathscr{G},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})}\|\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{G}^*)}
$$

and this is finite. Thus we get for a finite constant C_1:

$$
\begin{aligned}
\|[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} &\leq C_1\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\
&\leq C_1\, C \,\mathrm{Lip}\, \varphi_\varepsilon \leq C_1\, C \,\mathrm{Lip}\, \varphi
\end{aligned}
$$

Now let $u \in \mathscr{K}_\mathrm{c}$ and $v \in \mathscr{H}_\mathrm{c}$. We get:

$$
\begin{aligned}
|\langle \varphi(Q)u, \mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi(Q)v\rangle| \;&= \\
= \lim_{\varepsilon\to 0} &|\langle \varphi_\varepsilon(Q)u, \mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi_\varepsilon(Q)v\rangle| \\
\leq C_1\, C \,&\mathrm{Lip}\, \varphi
\end{aligned}
$$

Thus $[\varphi(Q), \mathrm{d}(\Delta+1)^{-1}]$ is a bounded operator. ■

Theorem 5.3 *Let (X, ρ) be a proper locally compact metric space. Assume that $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module and that $\mathscr{K}$ is a Hilbert X-module. Let $\mathrm{d}, \lambda, \Lambda$ be operators satisfying the following conditions:*

(i) d *is a closed first order operator from* $\mathscr{H}$ *to* $\mathscr{K}$ *with* $\mathcal{D}(\mathrm{d}) = \mathscr{G}$*;*
(ii) λ *is a bounded self-adjoint operator on* $\mathscr{H}$ *with* $\inf \lambda > 0$ *and such that* $\lambda - 1 \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ *(e.g.* $\lambda - 1 \in \mathcal{B}_0(\mathscr{H})$*);*
(iii) Λ *is a bounded self-adjoint operator on* $\mathscr{K}$ *with* $\inf \Lambda > 0$ *and such that* $\Lambda - 1 \in \mathcal{B}_0(\mathscr{K})$.
Then the self-adjoint operators Δ *and* $\widetilde{\Delta}$ *have the same essential spectrum.*

Proof: In this proof, we shall consider $\widetilde{\Delta}$ as an operator acting on $\mathscr{H}$. Since $\widetilde{\mathscr{H}} = \mathscr{H}$ as topological vector spaces and the notion of spectrum is purely topological, $\widetilde{\Delta}$ is a closed densely defined operator on $\mathscr{H}$ and it has the same spectrum as the self-adjoint $\widetilde{\Delta}$ on $\widetilde{\mathscr{H}}$. Moreover, if we define the essential spectrum $\sigma_{\text{ess}}(A)$ as the set of $z \in \mathbb{C}$ such that either $\ker(A - z)$ is infinite dimensional or the range of $A - z$ is not closed, we see that the essential spectrum is a topological notion, so $\sigma_{\text{ess}}(\widetilde{\Delta})$ is the same, whether we think of $\widetilde{\Delta}$ as operator on $\mathscr{H}$ or on $\widetilde{\mathscr{H}}$. Finally, with this definition of σ_{ess} we have $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ if $(A - z)^{-1} - (B - z)^{-1}$ is compact operator for some $z \in \rho(A) \cap \rho(B)$.

Thus it suffices to prove that $(\Delta + 1)^{-1} - (\widetilde{\Delta} + 1)^{-1} \in \mathcal{K}(\mathscr{H})$. Now we observe that

$$\widetilde{\Delta} + 1 = \lambda^{-1} d^* \Lambda d + 1 = \lambda^{-1}(\mathrm{d}^* \Delta \mathrm{d} + \lambda)$$

and $\Delta_\Lambda = \mathrm{d}^* \Lambda \mathrm{d}$ is the positive self-adjoint operator on $\mathscr{H}$ associated to the closed quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\mathscr{H}$ with domain $\mathscr{G}$. Thus $(\widetilde{\Delta} + 1)^{-1} = (\Delta_\Lambda + \lambda)^{-1}\lambda$ and

$$(\widetilde{\Delta} + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\Delta_\Lambda + \lambda)^{-1}(\lambda - 1) = [(\lambda - 1)(\Delta_\Lambda + \lambda)^{-1}]^*$$

The range of $(\Delta_\Lambda + \lambda)^{-1}$ is included in the form domain of $\Delta_\Lambda + \lambda$, which is $\mathscr{G}$. The map $(\Delta_\Lambda + \lambda)^{-1} : \mathscr{H} \to \mathscr{G}$ is continuous, by closed graph theorem, and $\lambda - 1 : \mathscr{G} \to \mathscr{H}$ is compact. Hence $(\widetilde{\Delta} + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1}$ is compact. Similarly:

$$(\Delta + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\mathrm{d}^*\mathrm{d} + 1)^{-1} - (\mathrm{d}^* \Delta \mathrm{d} + 1)^{-1} \in \mathcal{K}(\mathscr{H})$$

For this we use Theorem 3.6 with: $\mathscr{E} = \mathscr{K}$, $D = \mathrm{d}$, $a = 1$, $b = \Lambda$ and $z = -1$. Since $\mathrm{d}^*\mathrm{d}$ and $\mathrm{d}^*\Lambda d$ are positive self-adjoint operators on $\mathscr{H}$ with the same form domain $\mathscr{G}$, the first condition of Theorem 3.6 is satisfied. Then the second condition holds because $\Lambda - 1 \in \mathcal{B}_0(\mathscr{K})$. Thus it remains to observe that the operator $\mathrm{d}(\Delta + 1)^{-1}$ is quasilocal by Lemma 5.2. ■

Remark: The map $\varphi \mapsto \varphi(Q)$ provides $\widetilde{\mathscr{H}}$ with a Banach X-module structure. $\widetilde{\mathscr{H}}$ is a Hilbert X-module for this structure if and only if λ is $C_0(X)$-linear. Indeed, the adjoint of $\varphi(Q)$ in $\widetilde{\mathscr{H}}$ is $\lambda^{-1}\overline{\varphi}(Q)\lambda$ and $\lambda^{-1}\overline{\varphi}(Q)\lambda = \overline{\varphi}(Q)$ is equivalent to $[\lambda, \overline{\varphi}(Q)] = 0$.

We shall consider now an application of Theorem 5.3 to concrete Riemannian manifolds. It will be clear from what follows that we could treat Lipschitz manifolds with measurable metrics (see [DP, Hi, Te, We] for example), but the case of C^1 manifolds with locally bounded metrics suffices as an example.

Let X be a *non-compact* differentiable manifold of class C^1 and T^*X be its cotangent manifold, a topological vector fiber bundle over X whose fiber over x will be denoted T^*_xX. If $u : X \to \mathbb{R}$ is differentiable then $\mathrm{d}u(x) \in T^*_xX$ is its differential at the point x and its differential $\mathrm{d}u$ is a section of T^*X. Thus for the moment d is a linear map defined on the space of real $C^1(X)$ functions to the space of sections of T^*X.

We now assume that X is equipped with a measurable locally bounded Riemannian structure. To be precise, each T^*_xX is equipped with a scalar product $\langle\cdot|\cdot\rangle_x$ and the associated norm $\|\cdot\|_x$ satisfying the following condition:

$$
(R)\ \begin{cases} \text{if } v \text{ is a continuous section of } T^*X \text{ over a compact set } K \text{ such that} \\ v(x) \neq 0 \text{ for } x \in K, \text{ then } x \mapsto \|v(x)\|_x \text{ is a bounded Borel map on} \\ K \text{ and } \|v(x)\|_x \geq c \text{ for some number } c > 0 \text{ and all } x \in K. \end{cases}
$$

This structure allows one to construct a metric compatible with the topology on X (if the scalar products do not depend continuously on x, this is not a completely trivial matter, see the references above). Since X was assumed to be non-compact, the metric space X is proper in the sense defined in Subsection 2.4 if and only if it is a complete metric space.

It will also be convenient to complexify these structures (i.e. replace T^*_xX by $T^*_xX \otimes \mathbb{C}$ and extend the scalar product to the complexification as usual) but to keep the same notations (we could, of corse, work with real Hilbert spaces, but this would not be coherent with the conventions of the rest of the paper).

Now let μ be a positive measure on X such that:

$$
(M)\ \begin{cases} \mu \text{ is absolutely continuous and its density is locally bounded} \\ \text{and locally bounded from below by strictly positive constants.} \end{cases}
$$

We shall take $\mathscr{H} = L^2(X,\mu)$ and $\mathscr{K}$ equal to the completion of the space of continuous sections of T^*X with compact support under the natural norm

$$
\|v\|^2_{\mathscr{K}} = \int_X \|v(x)\|^2_x d\mu(x).
$$

In fact, $\mathscr{K}$ is the space of (suitably defined) square integrable sections of T^*X.

The operator of exterior differentiation d induces a linear map $C^1_{\mathrm{c}}(X) \to \mathscr{K}$ which is easily seen to be closable as operator from $\mathscr{H}$ to $\mathscr{K}$ (this is a purely local problem and the hypotheses we put on the metric and the measure allow us to reduce ourselves to the Euclidean case). We shall keep the notation d for its closure and we note that its domain $\mathscr{G}$ is the natural first order Sobolev space $\mathscr{H}^1$ defined in this context as the closure of $C^1_\mathrm{c}(X)$ under the norm

$$\|u\|^2_{\mathscr{H}^1} = \int_X \Big(|u(x)|^2 + \|\mathrm{d}u(x)\|^2_x\Big) d\mu(x).$$

Note that the self-adjoint operator $\Delta = \mathrm{d}^*\mathrm{d}$ is a slightly generalized form of the Laplace operator associated to the Riemannian structure of X because μ is not necessarily the Riemannian volume element (but we could choose it so).

We shall now consider perturbations of this structure. We assume that the perturbation preserves the local structure, although Theorem 5.3 allows us to go much further.

Proposition 5.4 *Let X be a non-compact manifold of class C^1 equipped with a Riemannian structure and a measure satisfying the conditions (R) and (M) and such that X is complete for the associated metric. Let λ be a bounded Borel function on X such that $\lambda(x) \geq c$ for some number $c > 0$ and $\lim_{x\to\infty} \lambda(x) = 1$. Assume that a new Riemann structure verifying (R) is given on X such that the associated norms $\|\cdot\|'_x$ verify $\alpha(x)\|\cdot\|_x \leq \|\cdot\|'_x \leq \beta(x)\|\cdot\|_x$ for some functions α, β such that $\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \beta(x) = 1$. Let Δ be as above and Δ' be the analog operator associated to the second Riemann structure and to the measure $\mu' = \lambda\mu$. Then $\sigma_{\mathrm{ess}}(\Delta) = \sigma_{\mathrm{ess}}(\Delta')$.*

Proof: We check that the assumptions of Theorem 5.3 are satisfied. We noted above that X is a proper metric space for the metric associated to the initial Riemann structure. The spaces $\mathscr{H}, \mathscr{K}$ have obvious X-module structures and for each $\varphi \in C_\mathrm{c}(X)$ the operator $\varphi(Q) : \mathscr{H}^1 \to \mathscr{H}$ is compact. Indeed, by using partitions of unity, we may assume that the support of φ is contained in the domain of a local chart and then we are reduced to a known fact in the Euclidean case. Thus $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module. To see that d is a first order operator we observe that if φ is Lipschitz then $[\mathrm{d}, \varphi]$ is the operator of multiplication by the differential $\mathrm{d}\varphi$ of φ and the estimate ess-sup $\|\mathrm{d}\varphi(x)\|_x \leq$ Lip φ is easy to obtain. The conditions on λ in Theorem 5.3 are trivially verified. So it remains to consider the operator Λ. For each $x \in X$ there is a unique operator $\Lambda_0(x)$ on $T^*_x X$ such that $\langle u|v\rangle'_x = \langle u|\Lambda_0(x)v\rangle_x$ for all $u, v \in T^*_x X$ and we have $\alpha(x)^2 \leq \Lambda_0(x) \leq \beta(x)^2$ by hypothesis. Here the inequalities must be interpreted with respect to the initial scalar product on $T^*_x X$. Thus the operator Λ on $\mathscr{K}$ is just the operator of multiplication by the function $x \mapsto \lambda(x)\Lambda(x)$ and the condition (iii) of Theorem 5.3 is clearly satisfied. ■

We note that if μ is the measure canonically associated to the initial Riemann structure then we can choose λ such that μ' be the measure associated to the second Riemann structure. In particular, if we have two locally L^∞ Riemannian metrics on a non-compact C^1 manifold, if the structures are asymptotically equivalent in the sense made precise in Proposition 5.4, and if the manifold is complete for one of the metrics (hence for the other too), then the Laplacians associated to the two metrics have the same essential spectrum. We stress that this is known, and easy to prove if one uses some local regularity estimates for elliptic equations, if one of the metrics is locally Lipschitz or Hölder continuous (in the second case, the required regularity estimate is not so easy, however). On the other hand, it is clear that our arguments, although quite elementary, cover situations when X is not of class C^1 and the metrics are only L^p. In fact, the arguments work without any modification if X is a Lipschitz manifold and a countable atlas has been chosen, because then the tangent space are well defined almost everywhere and the absolute continuity notions that we have used make sense.

6 Weakly vanishing perturbations

6.1 General remarks

The algebraic framework introduced in Section 2 and the abstract Theorems 3.1 and 3.6 should allow one to go beyond the primitive idea of "vanishing at infinity perturbation" that we considered so far. Indeed, we recall that, according to our general definitions, the multiplier algebra of a Banach module should be the prototype of the notion of vanishing at infinity operator. The purpose of this section is to give examples of such extensions.

Let X be a locally compact non-compact topological space and let $\mathscr{H}$ be a Hilbert X-module. Then the C^*-algebra of the operators $\varphi(Q)$ with $\varphi \in C_0(X)$ is the initial multiplier algebra of $\mathscr{H}$ but, due to Lemma 2.12, we can also consider on $\mathscr{H}$ the Hilbert module structure defined by the algebra consisting of the operators $\varphi(Q)$ with φ an arbitrary bounded Borel function on X. These operators cannot be considered as vanishing at infinity, but we could consider some other subalgebras of $B(X)$. It is easy to see that each function $\varphi \in B_0(X)$ can be written as a product $\varphi = \theta\psi$ with $\theta \in C_0(X)$ and $\psi \in B_0(X)$ (this is obvious if one accepts the Cohen-Hewitt Theorem A.3). Thus we get no improvement by going from $C_0(X)$ to $B_0(X)$. Hence we have to point out a class of functions which vanish at infinity in a weaker sense.

A natural idea is to extend the usual notion of neighborhood of infinity. It is usual to define the filter of neighborhoods of infinity as the family of subsets of X with relatively compact complement; we shall call this the *Fréchet filter*. If

$\mathcal{F}$ is a filter on X finer than the Fréchet filter then a function $\varphi : X \to \mathbb{C}$ such that $\lim_{\mathcal{F}} \varphi = 0$ can naturally be thought as convergent to zero at infinity in a generalized sense (we recall that $\lim_{\mathcal{F}} \varphi = 0$ means that for each $\varepsilon > 0$ the set of points x such that $|\varphi(x)| < \varepsilon$ belongs to $\mathcal{F}$). It is clear that

$$B_{\mathcal{F}}(X) := \{\varphi \in B(X) \mid \lim_{\mathcal{F}} \varphi = 0\} \tag{6.20}$$

is a C^*-algebra and that we can consider on $\mathscr{H}$ the Hilbert module structure defined by the multiplier algebra $\mathcal{M}_{\mathcal{F}} := \{\varphi(Q) \mid \varphi \in B_{\mathcal{F}}(X)\}$. We will be interested in the corresponding classes of vanishing at infinity or quasilocal operators. To be precise, we shall speak in this context of (left or right) $\mathcal{F}$*-vanishing at infinity* operators or of (left or right) $\mathcal{F}$*-quasilocal operators*. Below and later on we use the notation $N^{\mathrm{c}} = X \setminus N$.

Lemma 6.1 *Let $\mathscr{H}, \mathscr{K}$ be Hilbert X-modules. Then an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is right $\mathcal{F}$-quasilocal if and only if for each Borel set N with $N^{\mathrm{c}} \in \mathcal{F}$ and for each $\varepsilon > 0$ there is a Borel set $F \in \mathcal{F}$ such that $\|\chi_F(Q) S \chi_N(Q)\| \leq \varepsilon$.*

Proof: We note first that the family of operators χ_N, where N runs over the family of Borel sets with complement in $\mathcal{F}$, is an approximate unit for $B_{\mathcal{F}}(X)$. Indeed, if $\varepsilon > 0$ and $\varphi \in B_{\mathcal{F}}(X)$ then the set $N = \{x \mid |\varphi(x)| > \varepsilon\}$ has the properties required above and $\sup_x |\varphi(x)(1 - \chi_N(x))| \leq \varepsilon$. Thus, according to Proposition 2.9, S is right $\mathcal{F}$-quasilocal if and only if $S\chi_N(Q)$ is left $\mathcal{F}$-vanishing at infinity for each N. Now the result follows from (2.6). ■

The main restriction we have to impose on $\mathcal{F}$ comes from the fact that the Friedrichs couple $(\mathscr{G}, \mathscr{H})$ which is involved in the definition of the class of operators that we study must be such that $\varphi(Q) \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ if $\varphi \in B_{\mathcal{F}}(X)$. That this is an important restriction follows from the following easily proven result:

Lemma 6.2 *Let X be an Euclidean space, $\mathscr{H} = L^2(X)$, and let $\mathscr{G} = \mathscr{H}^s$ be a Sobolev space of order $s > 0$. If $\varphi \in B(X)$ then $\varphi(Q) \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ if and only if*

$$\lim_{a \to \infty} \int_{|x-a| \leq 1} |\varphi(x)| dx = 0. \tag{6.21}$$

The importance of such a condition in questions of stability of the essential spectrum has been noticed in [He, LV, OS, We]. That it is a natural condition follows also from the characterizations that we shall give below in a more general context.

Let X be a locally compact non-compact abelian group. We shall say that a function $\varphi \in B(X)$ is *weakly vanishing (at infinity)* if

$$\lim_{a \to \infty} \int_{a+K} |\varphi(x)| dx = 0 \text{ for each compact set } K. \tag{6.22}$$

We shall denote by $B_{\rm w}(X)$ the set of functions φ satisfying (6.22). This is clearly a C^*-algebra. Note that it suffices that the convergence condition in (6.22) be satisfied for only one compact set K with non-empty interior.

Let us now express the condition (6.22) in terms of convergence to zero along a certain filter. We denote $|K|$ the exterior (Haar) measure of a set $K \subset X$ and we set $K_a = a + K$ if $a \in X$. A subset N is called w-*small (at infinity)* if there is a compact neighborhood K of the origin such that $\lim_{a\to\infty} |N \cap K_a| = 0$. The complement of a w-small set will be called w-*large (at infinity)*. The family $\mathcal{F}_{\rm w}$ of all w-large sets is clearly a filter on X finer than the Fréchet filter.

Observe that a Borel set is w-small if and only if its characteristic function weakly vanishes at infinity. Denote $f * g$ the convolution of two functions on X.

Lemma 6.3 *For a function $\varphi \in B(X)$ the following conditions are equivalent:* (1) *φ is weakly vanishing;* (2) *$\theta * |\varphi| \in C_0(X)$ if $\theta \in C_{\rm c}(X)$;* (3) *$\lim_{\mathcal{F}_{\rm w}} \varphi = 0$;* (4) *$\varphi(Q)\psi(P)$ is a compact operator on $L^2(X)$ for all $\psi \in C_0(X)$.*

Proof: The equivalence of (1) and (2) is clear because $\int_{K_a} |\varphi| dx = (\chi_K * |\varphi|)(a)$. Then (3) means that for each $\varepsilon > 0$ the Borel set N where $|\varphi(x)| > \varepsilon$ is w-small. Since $\chi_N \leq \varphi/\varepsilon$, the implication (2) $\Rightarrow$ (3) is clear, while the reciprocal implication follows from $\chi_K * |\varphi| \leq \sup|\varphi| \chi_K * \chi_N + \varepsilon|K|$. If (4) holds, let us choose ψ such that its Fourier transform $\widehat{\psi}$ be a positive function in $C_{\rm c}(X)$ and let $f \in C_{\rm c}(X)$ be positive and not zero. Since $\psi(P)f$ is essentially the convolution of $\widehat{\psi}$ with f, there is a compact set K with non-empty interior such that $\psi(P)f \geq c\chi_K$ with a number $c > 0$. Let U_a be the unitary operator of translation by a in $L^2(X)$, then $U_a f \to 0$ weakly when $a \to \infty$, hence $\|\varphi(Q)U_a\psi(P)f\| = \|\varphi(Q)\psi(P)U_a f\| \to 0$. Since $U_a^*\varphi(Q)U_a = \varphi(Q-a)$ we get $\|\varphi(Q-a)\chi_K\| \to 0$, hence (1) holds.

Finally, let us prove that (1) $\Rightarrow$ (4). It suffices to prove that $\varphi(Q)\psi(P)$ is compact if $\widehat{\psi} \in C_{\rm c}(X)$ and for this it suffices that $\bar{\psi}(P)|\varphi|^2(Q)\psi(P)$ be compact. Since $\xi := |\varphi|^2 \in B_{\rm w}(X)$ and since $\psi(P)$ is the operator of convolution by a function in $\theta \in C_{\rm c}(X)$, we are reduced to proving that the integral operator S with kernel $S(x,y) = \int \bar{\theta}(z-x)\xi(z)\theta(z-y)dz$ is compact. If $K = \text{supp}\,\theta$ and Λ is the compact set $K - K$, then clearly there is a number C such that

$$|S(x,y)| \leq C \int_{K_x} \xi(z)dz\chi_\Lambda(x-y) \equiv \phi(x)\chi_\Lambda(x-y)$$

where $\phi \in C_0(X)$. The last term here is a kernel which defines a compact operator T. Thus $\eta(Q)S$ is a Hilbert-Schmidt operator for each $\eta \in C_{\rm c}(X)$ and from the preceding estimate we get $\|(S - \eta(Q)S)u\| \leq \|(1-\eta(Q))T|u|\|$ for each

$u \in L^2(X)$. Thus $\|S - \eta(Q)S\| \leq \|(1 - \eta(Q))T\|$ and the right hand side tends to zero if $\eta \equiv \eta_\alpha$ is an approximate unit for $C_0(X)$. ■

We define now a second class of functions which vanishes at infinity in a generalized sense, and this for an arbitrary Borel space X equipped with a positive measure μ such that $\mu(X) = \infty$. Let us say that a set $F \subset X$ is of *cofinite measure* if its complement F^c is of finite (exterior) measure. The family of sets of cofinite measure is clearly a filter $\mathcal{F}_\mu$ and if X is a locally compact space and μ a Radon measure then $\mathcal{F}_\mu$ is finer than the Fréchet filter. Moreover, if X is an abelian locally compact non-compact group then $\mathcal{F}_\mu \subset \mathcal{F}_w$ and the inclusion is strict. If φ is a function on X then $\lim_{\mathcal{F}_\mu} \varphi = 0$ means that for each $\varepsilon > 0$ the set where $|\varphi(x)| \geq \varepsilon$ is of finite measure. We denote $B_\mu(X)$ the C^*-subalgebra of $B(X)$ consisting of functions with this property.

Proposition 6.4 *Let (X, μ) be a positive measure space with $\mu(X) = \infty$ and let us equip $L^2(X)$ with the Hilbert module structure defined by $B_\mu(X)$. If $S \in \mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ for some $p < 2$, then S is right $\mathcal{F}_\mu$-quasilocal.*

Proof: We first show that $\mathcal{M}_\mu := \{\varphi(Q) \mid \varphi \in B_\mu(X)\}$ defines indeed a Hilbert module structure on $\mathscr{H} = L^2(X)$. Let $\mathcal{N}_\mu$ be the set of Borel subsets of finite measure of X. Then $\{\chi_N\}_{N \in \mathcal{N}_\mu}$ is an approximate unit of $B_\mu(X)$ because for each $\varphi \in B_\mu(X)$ and each $\varepsilon > 0$ we have $N = \{x \mid |\varphi(x)| \geq \varepsilon\} \in \mathcal{N}_\mu$ and $\sup |\varphi - \chi_N \varphi| \leq \varepsilon$. That the action of $\mathcal{M}_\mu$ on $\mathscr{H}$ is non-degenerate follows from the density of $L^1 \cap L^\infty$ in L^2 and the fact that each $u \in L^1 \cap L^\infty$ can be written as $u = \varphi v$ with $\varphi = \sqrt{|u|} \in L^2 \cap L^\infty \subset \mathcal{M}_\mu$ and $v = \sqrt{|u|}\,\mathrm{sign}\, u \in L^2$.

Now let $S \in \mathcal{B}(L^2(X))$ such that S induces a continuous operator in $L^p(X)$ for some number p such that $1 < p < 2$. We shall prove that for each $N \in \mathcal{N}_\mu$ the operator $T = S\chi_N(Q)$ has the property: for each $\varepsilon > 0$ there is a Borel set $F \in \mathcal{F}_\mu$ such that $\|\chi_F(Q)T\| \leq \varepsilon$. According to Lemma 6.1, this implies the right $\mathcal{F}_\mu$-quasilocality of S.

Since N is of finite measure, $\chi_N(Q)$ is a bounded operator $L^2 \to L^p$, hence $T \in \mathcal{B}(L^2, L^p)$. The rest of the proof is a straightforward application of the following factorization theorem, due to Bernard Maurey [Ma]:
Let $1 < p < 2$ and let T be an arbitrary continuous linear map from a Hilbert space $\mathscr{H}$ into L^p. Then there is $R \in \mathcal{B}(\mathscr{H}, L^2)$ and there is a function $g \in L^q$, where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, such that $T = g(Q)R$.
In our case $\mathscr{H} = L^2$. Let $a > 0$ real and let F be the set of points x such that $|g(x)| \leq a$. Since $g \in L^q$ with $q < \infty$, we have $F \in \mathcal{F}_\mu$ and

$$\|\chi_F(Q)T\|_{\mathcal{B}(L^2)} = \|\chi_F(Q)g(Q)R\|_{\mathcal{B}(L^2)} \leq a\|R\|_{\mathcal{B}(L^2)}.$$

Thus it suffices to choose a such that $a\|R\|_{\mathcal{B}(L^2)} = \varepsilon$. ■

We introduce now classes of vanishing at infinity functions of a more topological nature. We consider only the case of an Euclidean space X, the extension to the case of locally compact groups or metric spaces being obvious. We set $B_a(r) = \{x \in X \mid |x-a| < r\}$, $B_a = B_a(1)$ and $B(r) = B_0(r)$.

Let us fix a *uniformly discrete set* $L \subset X$, i.e. a set such that $\inf |a-b| > 0$ where the infimum is taken over couples of distinct points $a, b \in L$. Let $L_\varepsilon = L + B(\varepsilon)$ be the set of points at distance $< \varepsilon$ from L. We say that a subset $N \subset X$ is *L-thin* if for each $\varepsilon > 0$ there is $r < \infty$ such that $N \setminus B(r) \subset L_\varepsilon$. In other terms, N is L-thin if there is a family $\{\delta_a\}_{a \in L}$ of positive real numbers with $\delta_a \to 0$ as $a \to \infty$ such that $N \subset \bigcup B_a(\delta_a)$. The complement of such a set will be called *L-fat*. We denote $\mathcal{F}_L$ the family of L-fat sets, we note that $\mathcal{F}_L$ is a filter on X contained in $\mathcal{F}_{\mathrm{w}}$ and finer than the Fréchet filter, and we denote $B_L(X)$ the set of bounded Borel functions such that $\lim_{\mathcal{F}_L} \varphi = 0$. So $\varphi \in B(X)$ belongs to $B_L(X)$ if and only if the set $\{|\varphi| \geq \lambda\}$ is L-thin for each $\lambda > 0$. The advantage of this filter is that we have a simple criterion of $\mathcal{F}_L$-quasilocality.

Proposition 6.5 *Let $X = \mathbb{R}^n$ and let S be a bounded operator on $L^2(X)$ such that on the region $x \neq y$ its distribution kernel is a function satisfying the estimate $|S(x,y)| \leq c|x-y|^{-m}$ for some $m > n$. Then S is $\mathcal{F}_L$-quasilocal.*

Proof: Let $\theta \in C_{\mathrm{b}}(X)$ such that $\theta(x) = 0$ on a neighborhood of the origin and $S_\theta(x,y) = \theta(x-y)S(x,y)$. If $\xi(x) = \theta(x)|x|^{-m}$ then for the operator S_θ of kernel $S_\theta(x,y)$ we have $\|S_\theta u\| \leq c\|\xi * |u|\|$ hence $\|S_\theta\| \leq c\|\xi\|_{L^1}$ By choosing a convenient sequence of functions θ we see that S is the norm limit of a sequence of operators which besides the properties from the statement of the proposition are such that $S(x,y) = 0$ if $|x-y| > R(S)$. Since the set of $\mathcal{F}_L$-quasilocal operators is closed in norm (see Subsection 2.3), we may assume in the rest of the proof that the kernel of S has this property. In fact, in order to simplify the notations and without loss of generality, we shall assume $S(x,y) = 0$ if $|x-y| > 1$.

Let N be an L-thin Borel set and let $\varepsilon > 0$. We shall construct an L-fat Borel set with $F \subset N^{\mathrm{c}}$ such that $\|\chi_N(Q) S \chi_F(Q)\| \leq \varepsilon$. Since the adjoint operator S^* has the same properties as S, this suffices to prove quasilocality.

We shall only need two simple estimates. First, if $\rho_x(G)$ is the distance from a Borel set G to a point x, then

$$\int_G \frac{dy}{|x-y|^{2m}} \leq C(m,n)\rho_x(G)^{n-2m}. \tag{6.23}$$

Then, if B_0, B are two balls with the same center and radiuses δ and $\delta + \varepsilon$, then

$$\int_{B_0} \rho_x(B^{\mathrm{c}})^{n-2m} dx \leq C(m,n)\varepsilon^{n-2m}\delta^n. \tag{6.24}$$

We shall choose $\varepsilon = \delta^{n/2m}$. Then $\chi_{B_0}(Q)S\chi_{B^c}(Q)$ is an operator with integral kernel and we can estimate its Hilbert-Schmidt norm as follows:

$$\begin{aligned}
\|\chi_{B_0}(Q)S\chi_{B^c}(Q)\|_{HS}^2 &= \int_{X\times X} \chi_{B_0}(x)|S(x,y)|^2\chi_{B^c}(y)dxdy \\
&\leq c\int_{B_0} dx \int_{B^c} \frac{dy}{|x-y|^{2m}} \leq C\int_{B_0} \rho_x(B^c)^{n-2m}dx \\
&\leq C'\varepsilon^{n-2m}\delta^n = C'\delta^\lambda \qquad (6.25)
\end{aligned}$$

where $\lambda = n^2/2m > 0$.

We can assume that $N = \bigcup_a B_a(\delta_a)$, where the sequence of numbers δ_a satisfies $\delta_a \to 0$ as $a \to \infty$. Denote $N_a = B_a(\delta_a)$ and $M_a = B_a(\delta_a + \varepsilon_a)$, where we choose $\varepsilon_a = \delta_a^{n/2m}$ as above. Choose r such that the balls N_a are pairwise disjoint and $\delta_a + \varepsilon_a < 1$ if $|a| > r$ and let R such that $\chi_{N_a}(Q)S\chi_{B(R)^c}(Q) = 0$ if $|a| \leq r$. Let $M = \bigcup M_a$ and $F = M^c \setminus B(R)$, so that F is a closed L-fat set. Then for any $u \in L^2(X)$ we have:

$$\|\chi_N(Q)S\chi_F(Q)u\|^2 = \sum_{|a|>r} \|\chi_{N_a}(Q)S\chi_F(Q)u\|^2.$$

Since S is of range 1 we have $\chi_{N_a}(Q)S\chi_{B_a(2)^c}(Q) = 0$ if $\delta_a < 1$. Thus

$$\|\chi_N(Q)S\chi_F(Q)u\|^2 \leq \sum_{|a|>r} \|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\|^2 \, \|\chi_{B_a(2)}(Q)u\|^2$$

The number of $b \in L$ such that $B_b(2)$ meets $B_a(2)$ is a bounded function of a, hence there is a constant C depending only on L such that

$$\|\chi_N(Q)S\chi_F(Q)u\| \leq C \sup_{|a|>r} \|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\| \, \|u\|.$$

We have $F \subset M^c \subset M_a^c$ hence

$$\|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\| \leq \|\chi_{N_a}(Q)S\chi_{M_a^c}(Q)\|_{HS} \leq C'\delta_a^{\lambda/2}$$

because of (6.25). So the norm $\|\chi_N(Q)S\chi_F(Q)\|$ can be made as small as we wish by choosing r large enough. ■

Corollary 6.6 *Let $X = \mathbb{R}^n$, μ the Lebesgue measure, and L a uniformly discrete subset of $\mathbb{R}^n$. Then a pseudo-differential operator of class S^0 on $L^2(X)$ is both $\mathcal{F}_\mu$-quasilocal and $\mathcal{F}_L$-quasilocal.*

Proof: In the first case we use Proposition 6.4 by taking into account that a pseudo-differential operator of class S^0 belongs to $\mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and that the adjoint of such an operator is also pseudo-differential of class S^0. For the second case, note that the distribution kernel of such an operator verifies the estimates $|S(x,y)| \leq C_k |x-y|^{-n}(1+|x-y|)^{-k}$ for any $k > 0$, see [Ho]. ■

We shall consider now a general class of filters defined in terms of the metric and measure space structure of the euclidean X. To each function $\nu : X \to]0, \infty[$ such that $\liminf_{a\to\infty} \nu(a) = 0$ we associate a set of subsets of X as follows:

$$\mathscr{N}_\nu = \{N \subset X \mid \limsup_{a\to\infty} \nu(a)^{-1}|N \cap B_a| < \infty\}. \tag{6.26}$$

We recall that B_a is the unit ball centered at a. Clearly $\mathscr{F}_\nu = \{F \subset X \mid F^{\mathrm{c}} \in \mathscr{N}_\nu\}$ is a filter on X finner than the Fréchet filter. Our purpose is to give a criterion of $\mathscr{F}_\nu$-quasilocality. For this we make a preliminary remark concerning the class of $C^{\mathrm{u}}(Q)$. We shall say that an operator $S \in \mathcal{B}(L^2(X))$ is of *finite range* if there is $r < \infty$ such that its distribution kernel satisfies $S(x,y) = 0$ for $|x-y| > r$.

Proposition 6.7 *The set of linear continuous finite range operators on $L^2(X)$ is a dense $*$-subalgebra of $C^{\mathrm{u}}(Q)$.*

Proof: The fact that the set of finite range operators in $\mathcal{B}(L^2(X))$ is a $*$-algebra is easy to check. We prove now that a finite range operator $S \in \mathcal{B}(L^2(X))$ is of class $C^{\mathrm{u}}(Q)$. Let us denote $Z = \mathbb{Z}^n$ and for each $a \in Z$ let $K_a = a + K$, where $K =]-1/2, 1/2]^n$, so that K_a is a unit cube centered at a and we have $X = \bigcup_{a\in Z} K_a$ disjoint union. Let χ_a be the characteristic function of K_a and let us abbreviate $\chi_a = \chi(Q)$. If r is as above, we similarly define $L =]-r-1, r+1]^n$, $L_a = a + L$ and denote φ_a the characteristic function of L_a. Note that there is a number N such that any cube L_a intersects at most N other cubes L_b.

It suffices to prove that for each linear function $\xi : X \to \mathbb{R}$ the commutator $[\xi(Q), S]$ is bounded, because this is equivalent to the fact that the map $k \mapsto V_k^* S V_k$ is Lipschitz. We have $\sum_u \chi_a = 1$ strongly on L^2 and $[\xi(Q), S]\chi_a = \varphi_a[\xi(Q), S]\chi_a$ due to the assumption concerning the range of S. Thus there is a constant C depending only on N such that for $u \in L^2$ with compact support:

$$\begin{aligned} \|[\xi(Q), S]u\|^2 &\leq C \sum \|\varphi_a[\xi(Q), S]\chi_a u\|^2 \\ &= C \sum \|\varphi_a[\xi(Q) - \xi(a), S]\chi_a \cdot \chi_a u\|^2 \\ &\leq C \sum C' \|\chi_a u\|^2 = CC'\|u\|^2 \end{aligned}$$

Now we shall prove that any operator of class $C^{\mathrm{u}}(Q)$ is a norm limit of finite range operators and this in the more general setting of Hilbert X-modules. Let X

be an abelian locally compact group and let $\mathscr{H}, \mathscr{K}$ be Hilbert X-modules. We fix a Haar measure dk on X^* and if $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $\Theta \in L^1(X^*)$ we define

$$S_\Theta = \int_{X^*} V_k^* S V_k \Theta(k) dk. \tag{6.27}$$

The integral is well defined because $k \mapsto V_k^* S V_k \in \mathcal{B}(\mathscr{H})$ is a bounded strongly continuous map. In order to explain the main idea of the proof we shall make a formal computation involving the spectral measure $E(A) = \chi_A(Q)$, see Lemmas 2.12 and 2.18 (we shall use the same notation for the spectral measures in $\mathscr{H}$ and $\mathscr{K}$). We have for $k \in X^*$ and $\varphi(Q) \in B(X)$

$$\varphi(Q) V_k^* = \varphi(Q) k(Q)^* = (\varphi \overline{k})(Q) = \int \varphi(x) \overline{k}(x) E(dx).$$

Note also that for $x, y \in X$ we have $\overline{k}(x) k(y) = k(-x) k(y) = k(y - x)$. Let $\widehat{\Theta}(x) = \int \overline{k(x)} \Theta(k) dk$ be the Fourier transform of Θ. Then we have for all $\varphi, \psi \in B(X)$:

$$\begin{aligned} \varphi(Q) S_\Theta \psi(Q) &= \int_{X^*} \Theta(k) dk \int_X \int_X \varphi(x) \overline{k}(x) k(y) \psi(y) E(dx) S E(dy) \\ &= \int_X \int_X \widehat{\Theta}(x - y) \varphi(x) \psi(y) E(dx) S E(dy). \end{aligned} \tag{6.28}$$

We can rigorously justify this computation and give a meaning to the last integral by taking into account that $E(A) S E(B)$ induces a finitely additive measure on the algebra generated by rectangles $A \times B$ in $X \times X$ (note that $\widehat{\Theta} \in C_0(X)$). If S is Hilbert-Schmidt then the measure is in fact σ-additive and the result becomes obvious. We shall, however, avoid these questions and we shall directly prove only what we need. Namely, we show the following:

$$\textbf{(}*\textbf{)} \quad \begin{cases} \text{If the support of } \widehat{\Theta} \text{ is a compact set } \Lambda \text{ and if supp } \varphi \cap (\Lambda + \text{supp } \psi) = \varnothing \\ \text{then } \varphi(Q) S_\Theta \psi(Q) = 0. \end{cases}$$

Observe that if $(*)$ holds for a certain set of operators S then it also holds for the strongly closed linear subspace of $\mathcal{B}(\mathscr{H}, \mathscr{K})$ generated by it. So it suffices to prove $(*)$ for S an operator of rank one $Sf = v\langle u, f\rangle$ with some fixed $u \in \mathscr{H}$ and $v \in \mathscr{K}$. Now the computation giving (6.28) obviously makes sense in the weak topology and gives for $f \in \mathscr{H}$ and $g \in \mathscr{K}$:

$$\langle g, \varphi(Q) S_\Theta \psi(Q) f\rangle = \int_X \int_X \widehat{\Theta}(x - y) \varphi(x) \psi(y) \langle g, E(dx) u\rangle \langle u, E(dy) f\rangle,$$

hence $(*)$ holds for such S.

Finally, note that if $S \in C^{\mathrm{u}}(Q)$ then S is norm limit of operators of the form S_Θ. For this it suffices to take $\Theta = |K|^{-1}\chi_K$ where K runs over the set of open relatively compact neighbourhoods of the neutral element of X^*, $|K|$ being the Haar measure of K. Then, by approximating conveniently Θ in L^1 norm, one shows that S is norm limit of operators S_Θ such that $\widehat{\Theta}$ has compact support. ■

Remark: This proposition gives a new proof of Proposition 2.23 for the case of Hilbert X-modules. Indeed, it is obvious that a finite range operator is quasilocal.

Theorem 6.8 *Let $X = \mathbb{R}^n$ and let $\nu : X \to]0,\infty[$ such that $\liminf_{a\to\infty} \nu(a) = 0$ and $\sup_{|b-a|\le r} \nu(b)/\nu(a) < \infty$ for each real r. If $S \in \mathcal{B}(L^2(X))$ is of class $C^{\mathrm{u}}(Q)$ and if $S \in \mathcal{B}(L^p(X))$ for some $p < 2$, then S is right $\mathscr{F}_\nu$-quasilocal.*

Proof: We can approximate in norm in $\mathcal{B}(L^2(X))$ the operator S by operators which are in $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ and have finite range. Indeed, the approximation procedure (6.27) used in the proof of Proposition 6.7 is such that it leaves $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ invariant (because V_k are isometries in L^p too). Since the set of right $\mathscr{F}_\nu$-quasilocal operators is norm closed in $\mathcal{B}(L^2(X))$, we may assume in the rest of the proof that S is of finite range. According to Lemma 6.1, it suffices to show that, for a given Borel set $N \in \mathscr{N}_\nu$ and for any number $\varepsilon > 0$, there is a Borel set $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q)S\chi_N(Q)\| < \varepsilon$.

In the rest of the proof we shall freely use the notations introduced in the in the second part of the Appendix (see also the proof of Proposition 6.7). In particular, q is defined by $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. If $f \in L^2(X)$ we have

$$\|\chi_N f\|_{L^p(K_a)} \le \|\chi_N\|_{L^q(K_a)}\|f\|_{L^2(K_a)} \le |N \cap K_a|^{1/q}\|f\|_{L^2(K_a)}.$$

Since $N \in \mathscr{N}_\nu$ we can find a constant c such that $|N \cap K_a| \le c\nu(a)$ (note that the definition (6.26) does not involve the restriction of ν to bounded sets). Thus, if we take $\lambda_a = \nu(a)^{-1/q}$ for $a \in Z \equiv \mathbb{Z}^n$, we get $\chi_N f \in \mathscr{L}$ with the notations of the Appendix. In other terms, we see that we have $\chi_N(Q) \in \mathcal{B}(L^2(X), \mathscr{L})$. Let $T = S\chi_N(Q)$ and let us assume that we also have $S \in \mathcal{B}(\mathscr{L})$. Then $T \in \mathcal{B}(L^2(X), \mathscr{L})$ and we can apply the Maurey type factorization theorem Theorem A.8, where $\mathscr{H} = L^2(X)$. Thus we can write $T = g(Q)R$ for some $R \in \mathcal{B}(L^2(X))$ and some function $g \in \mathscr{M}$, which means that $G := \sup_{a\in Z} \nu(a)^{-1/q}\|g\|_{L^q(K_a)}$ is a finite number. If $t > 0$ and $M = \{x \mid g(x) > t\}$ then we get for all $a \in Z$:

$$|M \cap K_a| = \|\chi_M\|^q_{L^q(K_a)} \le \|g/t\|^q_{L^q(K_a)} \le (G/t)^q\nu(a).$$

Note that the second condition imposed on ν in Theorem 6.8 ca be stated as follows: there is an increasing strictly positive function δ on $[0,\infty[$ such that $\nu(b) \le$

$\delta(|b-a|)\nu(a)$ for all a, b. Indeed, we may take $\delta(r) = \sup_{|b-a|\le r} \nu(b)/\nu(a)$. Now let $a \in X$ and let $D(a)$ be the set of $b \in Z$ such that K_b intersects B_a. Clearly $D(a)$ contains at most 2^n points b all of them satisfying $|b-a| \le \sqrt{n}+1$. Hence:

$$|M \cap K_a| \le \sum_{b \in D(a)} |M \cap K_b| \le 2^n \sup_{b \in D(a)} (G/t)^q \nu(b) \le 2^n (G/t)^q \delta(\sqrt{n}+1)\nu(a),$$

which proves that M belongs to $\mathscr{N}_\nu$. On the other hand, we have:

$$\|\chi_{M^c}(Q)T\| = \|\chi_{M^c}(Q)g(Q)R\| \le \|\chi_{M^c} g\|_{L^\infty} \|R\| \le t\|R\|.$$

To finish the proof of the theorem it suffices to take $t = \varepsilon/\|R\|$.

We still have to prove that $S \in \mathcal{B}(\mathscr{L})$. Since S is of finite range, there is a number r such that $\chi_a(Q)\chi_b(Q) = 0$ if $|a-b| \ge r$. Then for any $f \in \mathscr{L}$:

$$\sum_a \lambda_a^2 \|\chi_a S f\|_{L^p}^2 = \sum_a \lambda_a^2 \|\sum_{|b-a|<r} \chi_a S \chi_b f\|_{L^p}^2 \le C \sum_{|b-a|<r} \lambda_a^2 \|\chi_a S \chi_b f\|_{L^p}^2$$

where C is a number depending only on r and n. Since S is bounded in L^p the last term is less than $CC' \sum_{|b-a|<r} \lambda_a^2 \|\chi_b f\|_{L^p}^2$ for some constant C'. Finally, from $\nu(b) \le \delta(|b-a|)\nu(a) \le \delta(r)\nu(a)$ we get

$$\sum_{|a-b|<r} \lambda_a^2 = \sum_{|a-b|<r} \nu(a)^{-2/q} \le L(r)\delta(r)^{2/q}\lambda_b^2$$

where $L(r)$ is the maximum number of points from Z inside a ball of radius r. Thus we have $\|S\|_{\mathcal{B}(\mathscr{L})}^2 \le CC'L(r)\delta(r)^{2/q}$. ■

Corollary 6.9 *Let $X = \mathbb{R}^n$ and let S be a pseudo-differential operator of class S^0. Then S is $\mathcal{F}_{\mathrm{w}}$-quasilocal, i.e. for each $\varphi \in B_{\mathrm{w}}(X)$ there are $\psi_1, \psi_2 \in B_{\mathrm{w}}(X)$ and $T_1, T_2 \in \mathcal{B}(L^2(X))$ such that $\varphi(Q)S = T_1\psi_1(Q)$ and $S\varphi(Q) = \psi_2(Q)T_2$.*

Proof: Since the adjoint of S is also a pseudo-differential operator of class S^0, it suffices to show that S is right $\mathcal{F}_{\mathrm{w}}$-quasilocal. We have $S \in \mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and S is of class $C^{\mathrm{u}}(Q)$ because the commutators $[Q_j, S]$ are bounded operators for all $1 \le j \le n$. Thus we can apply Theorem 6.8 and deduce that for any function ν as in the statement of the theorem, for any $\varepsilon > 0$, and for any $N \in \mathscr{N}_\nu$ there is $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q)S\chi_N(Q)\| \le \varepsilon$. Now let N be a Borel w-small set, i.e. such that $|N \cap B_a| \to 0$ if $a \to \infty$. We shall prove that there is a function ν with the properties required in Theorem 6.8 and with $\lim_{a\to\infty} \nu(a) = 0$ such that $N \in \mathscr{N}_\nu$. This finishes the proof of the corollary because the relation $M \in \mathscr{N}_\nu$ implies now that M is w-small.

We construct ν as follows. The relation $\theta(r) = \sup_{|a|\geq r} |N \cap B_a|$ defines a positive decreasing function on $[0, \infty[$ which tends to zero at infinity and such that $|N \cap B_a| \leq \theta(|a|)$ for all $a \in X$. We set $\xi(t) = \theta(0)$ if $0 \leq t < 1$ and for $k \geq 0$ integer and $2^k \leq t < 2^{k+1}$ we define $\xi(t) = \max\{\xi(2^{k-1})/2, \theta(2^k)\}$. So ξ is a strictly positive decreasing function on $[0, \infty[$ which tends to zero at infinity and such that $\theta \leq \xi$. Moreover, if $2^k \leq s < 2^{k+1}$ and $2^{k+p} \leq t < 2^{k+p+1}$ then

$$\xi(t) = \xi(2^{k+p}) \geq \xi(2^{k+p-1})/2 \geq \ldots \geq 2^{-p}\xi(2^k) = 2^{-p}\xi(s)$$

hence $\xi(s) \geq \xi(t) \geq \frac{s}{2t}\xi(s)$ if $1 \leq s \leq t$. We take $\nu(a) = \xi(|a|)$, so ν is a bounded strictly positive function on X with $\lim_{a\to\infty} \nu(a) = 0$ and $|N \cap B_a| \leq \nu(a)$ for all a. If a, b are points with $|a|, |b| \geq 1$ and $|a - b| \leq r$ then $\nu(b)/\nu(a) \leq 1$ if $|a| \leq |b|$ and if $|a| > |b|$ then

$$\frac{\nu(b)}{\nu(a)} = \frac{\xi(|b|)}{\xi(|a|)} \leq \frac{2|a|}{|b|} \leq 2(1 + r).$$

Thus the second condition imposed on ν in Theorem 6.8 is also satisfied. ■

Remark 6.10 We stress that we shall need this corollary for a very simple class of operators, namely $S = \psi(P)$ with $\psi(k) = k^\alpha(\sum_{|\beta|\leq m} k^{2\beta})^{-1/2}$ and $|\alpha| \leq m$.

6.2 Applications

We shall give an application of the formalism presented in Subsection 6.1 in the framework of Subsection 4.1. We consider on $\mathscr{H}$ the class of "vanishing at infinity" functions corresponding to the C^*-algebra of multipliers $B_{\mathrm{w}}(X)$. The conditions of decay at infinity (4.18) imposed in Proposition 4.1 come from the consideration of $\mathscr{H}$ equipped with the Hilbert module structure defined by the algebra $B_0(X)$. Note that if we equip $\mathscr{H}$ with the Hilbert module structure defined by the algebra $B_{\mathrm{w}}(X)$ the property of compactness of the Friedrichs module $(\mathscr{G}, \mathscr{H})$ remains valid, cf. Lemma 6.2 and the space $\mathscr{K}$ inherits a natural direct sum Hilbert module structure.

Our purpose is to apply Theorem 3.6 in this setting. The only thing which remains to be checked is the left $\mathcal{F}_{\mathrm{w}}$-quasilocality of the operator $D(\Delta_a^* - \bar{z})^{-1}$. We shall establish such a result below assuming that the lower order coefficients are also bounded operators, but it is clear that this assumption can be replaced by much more general ones. Note also that in this subsection we are less precise and identify the operators D^*aD and Δ_a, although they act in different spaces.

In the next lemma we consider only the filter $\mathcal{F}_{\mathrm{w}}$. Of course, the result remains true if $\mathcal{F}_{\mathrm{w}}$ is replaced by $\mathcal{F}_\mu$ or $\mathcal{F}_L$.

Lemma 6.11 *Let* $\Delta_a = \sum_{|\alpha|,|\beta|\leq m} P^\alpha a_{\alpha\beta} P^\beta$ *with* $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$ $\mathcal{F}_{\rm w}$*-quasilocal (e.g.* $a_{\alpha\beta} \in B(X)$*) and let as assume that the operator* $\Delta_a : \mathscr{H}^m \to \mathscr{H}^{-m}$ *is coercive, i.e. there are numbers* $\mu, \nu > 0$ *such that for all* $u \in \mathscr{H}^m$*:*

$$\text{Re}\,\langle u, \Delta_a u\rangle \geq \mu \|u\|^2_{\mathscr{H}^m} - \nu \|u\|^2_{\mathscr{H}}. \tag{6.29}$$

Then $P^\alpha(\Delta_a + z)^{-1}$ *is* $\mathcal{F}_{\rm w}$*-quasilocal if* $|\alpha| \leq m$ *and if* $\text{Re}\, z > 0$ *is large.*

Proof: We shall denote by Δ the operator Δ_a corresponding to the case when a is the identity matrix, so $\Delta = D^*D = \sum_{|\alpha|\leq m} P^{2\alpha}$ (of course, this is not the Laplace operator). In fact, Δ is the canonical (Riesz) positive isomorphism of $\mathscr{G}$ onto $\mathscr{G}^*$ and (6.29) means $\text{Re}\,\Delta_a \geq \mu\Delta - \nu$. Note that we can include ν in the term of order zero of Δ_a, hence there is no loss of generality if we assume $\nu = 0$. Later computations look simpler if $\mu = 1$ and we can reduce ourselves to this situations by replacing a by a/μ. Thus we may assume that we have the estimate $\text{Re}\,\Delta_a \geq \Delta$. Now let us decompose $\Delta_a = \Delta + D^*(a-1)D \equiv \Delta + V$ and, if θ is a positive number, let us set $A_\theta = \Delta + \theta V$. Then $A_\theta \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and we have $\text{Re}\, A_\theta \geq \Delta$, so that if $\text{Re}\, z \geq 0$ then $A_\theta + z : \mathscr{G} \to \mathscr{G}^*$ is bijective and $\|(A_\theta + z)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})} \leq 1$ (see the Appendix). It follows easily that the function $\theta \mapsto (A_\theta + z)^{-1} \in \mathcal{B}(\mathscr{G}^*, \mathscr{G})$ is real analytic on $]0, \infty[$ which implies that the function $\theta \mapsto P^\alpha(A_\theta + z)^{-1} \in \mathcal{B}(\mathscr{H})$ is real analytic too. The set of $\mathcal{F}_{\rm w}$-quasilocal operators is a closed subspace of the Banach space $\mathcal{B}(\mathscr{H})$ and an analytic function which on an open set takes values in a closed subspace remains in that subspace for ever. Thus it suffices to how that the operator $P^\alpha(A_\theta + z)^{-1}$ is $\mathcal{F}_{\rm w}$-quasilocal for small values of θ. The operator $P^\alpha(A_\theta + z)^{-1}$ is also a holomorphic function of z in the region $\text{Re}\, z > 0$, so by a similar argument we see that it suffices to consider $z \geq 0$. Below we shall take $z = 0$, the argument in general is identical.

For reasons of simplicity, we change again the notations: we set $b = \theta(a-1)$, we assume $\|b\|_{\mathcal{B}(\mathscr{K})} < 1$, and denote $V = D^*bD$ and $A = \Delta + V$. Let $S = \Delta^{-1/2}$, where Δ is considered as self-adjoint operator on $\mathscr{H}$. Note that S is an isometry of $\mathscr{G}^*$ onto $\mathscr{H}$ and of $\mathscr{H}$ onto $\mathscr{G}$. Then we have:

$$A^{-1} = S(1 + SVS)^{-1}S = \sum_{k\geq 0} (-1)^k S(SVS)^k S$$

the series being norm convergent in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$. Indeed, $\|DS\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} = 1$, hence

$$\|S(SVS)^k S\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})} = \|(SD^*bDS)^k\|_{\mathcal{B}(\mathscr{H})} \leq \|b\|^k_{\mathcal{B}(\mathscr{K})}$$

and $\|b\|_{\mathcal{B}(\mathscr{K})} < 1$. Thus $P^\alpha A^{-1}$ is a sum of terms $P^\alpha(-1)^k S(SVS)^k S$ which converges in norm, so it suffices that each of them be $\mathcal{F}_{\rm w}$-quasilocal. But

$$P^\alpha S(SVS)^k S = (P^\alpha S)(SD^*)b(DS)\dots(SD^*)b(DS)S$$

and each factor in the product is $\mathcal{F}_{\mathrm{w}}$-quasilocal: for b this is an hypothesis (or trivial if the $a_{\alpha\beta}$ are functions), and for $P^\alpha S$, DS and SD^* because of Corollary 6.9. ■

Below we give just an example of application of Theorem 3.6. The conditions on the lower order coefficients can be improved without difficulty.

Theorem 6.12 *Let Δ_a be as in Lemma 6.11 and let $b = (b_{\alpha\beta})_{|\alpha|,|\beta|\leq m}$ with $b_{\alpha\beta}$ bounded operators $\mathscr{H}^{m-|\beta|} \to \mathscr{H}^{|\alpha|-m}$ such that Δ_b is coercive. For $|\alpha|+|\beta| = m$ assume that $b_{\alpha\beta}-a_{\alpha\beta}$ is left $\mathcal{F}_{\mathrm{w}}$-vanishing at infinity (which holds if $b_{\alpha\beta}-a_{\alpha\beta} \in B_{\mathrm{w}}(X)$). If $|\alpha|+|\beta| < m$ we assume $b_{\alpha\beta} - a_{\alpha\beta} \in \mathcal{K}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$. Then the operator Δ is a compact perturbation of Δ_a, in particular Δ_a and Δ_b have the same essential spectrum.*

Proof: We check the conditions of Theorem 3.6. Because of the coercivity assumptions, condition (1) is fulfilled, and (2) is satisfied by Lemma 6.11. The part of condition (3) involving the coefficients such that $|\alpha|+|\beta| = m$) is satisfied by definition, for the lower order coefficients it suffices to use (2.10). ■

Remark 6.13 If $b_{\alpha\beta} \in B(X)$ and $b_{\alpha\beta} - a_{\alpha\beta} \in B_{\mathrm{w}}(X)$ for all α, β, then the compactness conditions on the lower order coefficients are satisfied. Indeed, if $\varphi \in B_{\mathrm{w}}(X)$ then $\varphi(Q) : \mathscr{H}^s \to \mathscr{H}^{-t}$ is compact if $s, t \geq 0$ and one of them is not zero, see Lemma 6.2.

A Appendix

1. This Appendix consists of two parts: in the first one we discuss some elementary abstract facts which are used without comment in the main text and in the second one we present a Maurey type factorization theorem adapted to our needs.

Let $(\mathscr{G}, \mathscr{H})$ be a Friedrichs couple and $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ the Gelfand triplet associated to it. To an operator $S \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ (which is the same as a continuous sesquilinear form on $\mathscr{G}$) we associate an operator $\widehat{S}$ acting in $\mathscr{H}$ according to the rules: $\mathcal{D}(\widehat{S}) = S^{-1}(\mathscr{H})$, $\widehat{S} = S|\mathcal{D}(\widehat{S})$. Due to the identification $\mathscr{G}^{**} = \mathscr{G}$, the operator S^* is an element of $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$, so $\widehat{S^*}$ makes sense. On the other hand, if $\widehat{S}$ is densely defined in $\mathscr{H}$ then the adjoint $\widehat{S}^*$ of $\widehat{S}$ with respect to $\mathscr{H}$ is also well defined and we clearly have $\widehat{S^*} \subset \widehat{S}^*$.

Lemma A.1 *If $S - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some $z \in \mathbb{C}$, then $\widehat{S}$ is a closed densely defined operator, we have $\widehat{S}^* = \widehat{S^*}$ and $z \in \rho(\widehat{S})$. Moreover, the domains $\mathcal{D}(\widehat{S})$ and $\mathcal{D}(\widehat{S}^*)$ are dense subspaces of $\mathscr{G}$.*

Proof: Clearly we can assume $z = 0$. From the bijectivity of $S : \mathscr{G} \to \mathscr{G}^*$ and the inverse mapping theorem it follows that S and S^* are homeomorphisms of $\mathscr{G}$ onto $\mathscr{G}^*$. Since $\mathscr{H}$ is dense in $\mathscr{G}^*$, we see that $\mathcal{D}(\widehat{S})$ and $\mathcal{D}(\widehat{S^*})$ are dense in $\mathscr{G}$, hence in $\mathscr{H}$. Since $\widehat{S^*} \subset \widehat{S}^*$, the operator $\widehat{S}^*$ is also densely defined in $\mathscr{H}$. Thus $\widehat{S}$ is densely defined and closable. We now show that it is closed. Consider a sequence of elements $u_n \in \mathcal{D}(\widehat{S})$ such that $u_n \to u$ and $\widehat{S}u_n \to v$ in $\mathscr{H}$. Then $Su_n \to v$ in $\mathscr{G}^*$ hence, S^{-1} being continuous, $u_n \to S^{-1}v$ in $\mathscr{G}$, so in $\mathscr{H}$. Hence $u = S^{-1}v \in \mathcal{D}(\widehat{S})$ and $\widehat{S}u = v$.

We have proved that $\widehat{S}$ is densely defined and closed and clearly $0 \in \rho(\widehat{S})$. Then we also have $0 \in \rho(\widehat{S}^*)$, so $\widehat{S}^* : \mathcal{D}(\widehat{S}^*) \to \mathscr{H}$ is bijective. Since $\widehat{S^*} : \mathcal{D}(\widehat{S^*}) \to \mathscr{H}$ is also bijective and $\widehat{S}^*$ is an extension of $\widehat{S^*}$, we get $\widehat{S^*} = \widehat{S}^*$. ■

A standard example of operator satisfying the condition required above is a *coercive* operator, i.e. such that Re $\langle u, Su\rangle \geq \mu\|u\|^2_{\mathscr{G}} - \nu\|u\|^2_{\mathscr{H}}$ for some strictly positive constants μ, ν and all $u \in \mathscr{G}$. Indeed, replacing S by $S + \nu$, we may assume Re $\langle u, Su\rangle \geq \mu\|u\|^2_{\mathscr{G}}$. Since S^* verifies the same estimate, this clearly gives $\|Su\|_{\mathscr{G}^*} \geq \mu\|u\|_{\mathscr{G}}$ and $\|S^*u\|_{\mathscr{G}^*} \geq \mu\|u\|_{\mathscr{G}}$ for all $u \in \mathscr{G}$. Thus S and S^* are injective operators with closed range, which implies that they are bijective.

If A is a self-adjoint operator on $\mathscr{H}$ then there is a natural Gelfand triplet associated to it, namely $\mathcal{D}(|A|^{1/2}) \subset \mathscr{H} \subset \mathcal{D}(|A|^{1/2})^*$. Then A extends to a continuous operator $A_0 : \mathcal{D}(|A|^{1/2}) \to \mathcal{D}(|A|^{1/2})^*$ which fulfills the conditions of Lemma A.1 and one has $\widehat{A}_0 = A$. In our applications it is interesting to know whether there are other Gelfand triplets $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ with $\mathcal{D}(A) \subset \mathscr{G}$ and such that A extends to a continuous operator $\mathscr{G} \to \mathscr{G}^*$. For not semibounded operators, e.g. for Dirac operators, many other possibilities exist such that $\mathscr{G}$ is not comparable to $\mathcal{D}(|A|^{1/2})$. But if A is semibounded, then the class of spaces $\mathscr{G}$ is rather restricted, as the next lemma shows.

Lemma A.2 *Assume that A is a bounded from below self-adjoint operator on $\mathscr{H}$ and such that $\mathcal{D}(A) \subset \mathscr{G}$ densely. Then A extends to a continuous operator $\widetilde{A} : \mathscr{G} \to \mathscr{G}^*$ if and only if $\mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ and in this case $\widetilde{A} = A_0|_{\mathscr{G}}$.*

Proof: We prove only the nontrivial implication of the lemma. So let us assume that A extends to some $\widetilde{A} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$. Replacing A by $A + \lambda$ with λ a large enough number, we can assume that $A \geq 1$. For $u \in \mathcal{D}(A)$ we have

$$\|A^{1/2}u\|_{\mathscr{H}} = \sqrt{\langle u, Au\rangle} = \sqrt{\langle u, \widetilde{A}u\rangle} \leq C\|u\|_{\mathscr{G}},$$

where $C^2 = \|\widetilde{A}\|_{\mathscr{G}\to\mathscr{G}^*}$. Since $\mathcal{D}(A)$ is dense in $\mathscr{G}$, it follows that the inclusion map $\mathcal{D}(A) \to \mathcal{D}(A^{1/2})$ extends to a continuous linear map $J : \mathscr{G} \to \mathcal{D}(A^{1/2})$.

If $u \in \mathscr{G}$ then there is a sequence $\{u_n\}$ in $\mathcal{D}(A)$ such that $u_n \to u$ in $\mathscr{G}$. Then $J(u_n) \to J(u)$ in $\mathcal{D}(A^{1/2})$. Since $\mathscr{G}$ and $\mathcal{D}(A^{1/2})$ are continuously embedded in $\mathscr{H}$ we shall have $u_n \to u$ in $\mathscr{H}$ and $u_n = J(u_n) \to J(u)$ in $\mathscr{H}$, hence $J(u) = u$ for all $u \in \mathscr{G}$. In other terms, $\mathscr{G} \subset \mathcal{D}(A^{1/2})$. ■

We note that, under the conditions of the lemma, the inclusions $\mathcal{D}(A) \subset \mathscr{G}$ and $\mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ are continuous (by the closed graph theorem), so we have a scale

$$\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2}) \subset \mathscr{H} \subset \mathcal{D}(|A|^{1/2})^* \subset \mathscr{G}^* \subset \mathcal{D}(A)^*$$

with continuous and dense embeddings (because $\mathcal{D}(A)$ is dense in $\mathcal{D}(|A|^{1/2})$).

In view of its importance in this paper, we state below the Cohen-Hewitt factorization theorem [FD, Ch. V–9.2].

Theorem A.3 *Let $\mathscr{C}$ be a Banach algebra with an approximate unit, let $\mathscr{H}$ be a Banach space, and let $Q : \mathscr{C} \to \mathcal{B}(\mathscr{H})$ be a continuous morphism. Denote $\mathscr{H}_0$ the closed linear subspace of $\mathscr{H}$ generated by the elements of the form $Q(\varphi)v$ with $\varphi \in \mathscr{C}$ and $v \in \mathscr{H}$. Then for each $u \in \mathscr{H}_0$ there are $\varphi \in \mathscr{C}$ and $v \in \mathscr{H}$ such that $u = Q(\varphi)v$.*

2. In this second part of the appendix we shall prove a version of the factorization theorem due to Bernard Maurey (see the proof of Proposition 6.4 here and Theorem 8 in [Ma]). Our proof follows closely that of Maurey; we shall, however, give all the details, since the Banach space techniques involved in it are not very usual in the context of spectral theory. We first recall the Ky Fan's Lemma, see [DJT, 9.10].

Proposition A.4 *Let $\mathcal{K}$ be a compact convex subset of a Hausdorff topological vector space and let $\mathscr{F}$ be a convex set of functions $F : \mathcal{K} \to]-\infty, +\infty]$ such that each $F \in \mathscr{F}$ is convex and lower semicontinuous. If for each $F \in \mathscr{F}$ there is $g \in \mathcal{K}$ such that $F(g) \leq 0$, then there is $g \in \mathcal{K}$ such that $F(g) \leq 0$ for all $F \in \mathscr{F}$.*

We need a second general fact that we state below. Let (X, μ) be a σ-finite positive measure space and let $L^0(X)$ be the space of μ-equivalence classes of complex valued measurable functions on X with the topology of convergence in measure. Let $\mathscr{L}$ be a Banach space with $\mathscr{L} \subset L^0(X)$ linearly and continuously and such that if $f \in L^0(X)$, $g \in \mathscr{L}$ and $|f| \leq |g|$ (μ-a.e.) then $f \in \mathscr{L}$ and $\|f\|_{\mathscr{L}} \leq \|g\|_{\mathscr{L}}$.

Proposition A.5 *There is a number C, independent of $\mathscr{L}$, such that for any Hilbert space $\mathscr{H}$ and any $T \in \mathcal{B}(\mathscr{H}, \mathscr{L})$ the following inequality holds*

$$\|(\textstyle\sum_j |Tu_j|^2)^{1/2}\|_{\mathscr{L}} \leq C\|T\|_{\mathcal{B}(\mathscr{H},\mathscr{L})}(\textstyle\sum_j \|u_j\|^2)^{1/2} \tag{1.30}$$

for all finite families $\{u_j\}$ of vectors in $\mathscr{H}$.

This a rather standard consequence of Khinchin's inequality [DJT, 1.10]. The result is stated in [Pi] with an explicit value for C.

From now on we work in a setting adapted to our needs in Section 6, although it is clear that we could treat by the same methods a general abstract situation. Let $X = \mathbb{R}^n$ equipped with the Lebesgue measure, denote $Z = \mathbb{Z}^n$, and for each $a \in Z$ let $K_a = a + K$, where $K =]-1/2, 1/2]^n$, so that K_a is a unit cube centered at a and we have $X = \bigcup_{a \in Z} K_a$ disjoint union. Let χ_a be the characteristic function of K_a and if $f : X \to \mathbb{C}$ let $f_a = f|K_a$. We fix a number $1 < p < 2$ and a family $\{\lambda_a\}_{a \in Z}$ of strictly positive numbers $\lambda_a > 0$ and we define $\mathscr{L} \equiv \ell^2_\lambda(L^p)$ as the Banach space of all (equivalence classes) of complex functions f on X such that

$$\|f\|_{\mathscr{L}} := \Big(\sum_{a \in Z} \|\lambda_a \chi_a f\|^2_{L^p} \Big)^{1/2} < \infty. \tag{1.31}$$

Here $L^p = L^p(X)$ but note that, by identifying $\chi_a f \equiv f_a$, we can also interpret $\mathscr{L}$ as a conveniently normed direct sum of the spaces $L^p(K_a)$, see [DJT, page XIV]. If $\lambda_a = 1$ for all a we set $\ell^2_\lambda(L^p) = \ell^2(L^p)$. Observe that $\ell^2(L^2) = L^2(X)$.

Let q be given by $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, so that $1 < p < 2 < q < \infty$. We also need the space $\mathscr{M} \equiv \ell^\infty_\lambda(L^q)$ defined by the condition

$$\|g\|_{\mathscr{M}} := \sup_{a \in Z} \|\lambda_a \chi_a g\|_{L^q} < \infty. \tag{1.32}$$

The definitions are chosen such that $\|gu\|_{\mathscr{L}} \le \|g\|_{\mathscr{M}} \|u\|_{L^2}$ where $L^2 = L^2(X)$. As explained at [DJT, page XV], the space $\mathscr{M}$ is naturally identified with the dual space of the Banach space $\mathscr{M}_* \equiv \ell^1_{\lambda^{-1}}(L^{q'})$, where $\frac{1}{q} + \frac{1}{q'} = 1$, defined by the norm

$$\|h\|_{\mathscr{M}_*} := \sum_{a \in Z} \|\lambda_a^{-1} \chi_a h\|_{L^{q'}}.$$

Below, when we speak about w^*-topology on $\mathscr{M}$ we mean the $\sigma(\mathscr{M}, \mathscr{M}_*)$-topology. Clearly

$$\mathscr{M}_1^+ = \{g \in \mathscr{M} \mid g \ge 0, \|g\|_{\mathscr{M}} \le 1\}$$

is a convex compact subset of $\mathscr{M}$ for the w^*-topology.

Lemma A.6 *For each $f \in \mathscr{L}$ there is $g \in \mathscr{M}_1^+$ such that $\|f\|_{\mathscr{L}} = \|g^{-1} f\|_{L^2}$.*

Proof: We can assume $f \ge 0$. Since $1 = \frac{p}{2} + \frac{p}{q}$, we have:

$$\|f_a\|_{L^p} = \|f_a\|^{p/2}_{L^p} \|f_a\|^{p/q}_{L^p} = \|f_a^{p/2}\|_{L^2} \|f_a^{p/q}\|_{L^q} = \|f_a^{-p/q} f\|_{L^2} \|f_a^{p/q}\|_{L^q}$$

with the usual convention $0/0 = 0$. Now we define g_a on K_a as follows. If $f_a = 0$ then we take any $g_a \geq 0$ satisfying $\lambda_a \|g_a\|_{L^q} = 1$. If $f_a \neq 0$ let

$$g_a = \lambda_a^{-1} \big(f_a / \|f_a\|_{L^p} \big)^{p/q} = \lambda_a^{-1} \|f_a^{p/q}\|_{L^q}^{-1} f_a^{p/q}.$$

Thus we have $\lambda_a \|g_a\|_{L^q} = 1$ for all a, in particular $\|g\|_{\mathscr{M}} = 1$. By the preceding computations we also have $\|f_a\|_{L^p} = \|g_a^{-1} f_a\|_{L^2} \|g_a\|_{L^q}$ and so

$$\|f\|_{\mathscr{L}}^2 = \sum \lambda_a^2 \|f_a\|_{L^p}^2 = \sum \lambda_a^2 \|g_a\|_{L^q}^2 \|g_a^{-1} f_a\|_{L^2}^2 = \sum \|g_a^{-1} f_a\|_{L^2}^2$$

which is just $\|g^{-1} f\|_{L^2}^2$. ∎

The main technical result follows.

Proposition A.7 *Let $(f^u)_{u \in U}$ be a family of functions in $\mathscr{L}$ such that, for each $\alpha = (\alpha_u)_{u \in U}$ with $\alpha_u \in \mathbb{R}$, $\alpha_u \geq 0$ and $\alpha_u \neq 0$ for at most a finite number of u, the function $f^\alpha := (\sum_u |\alpha_u f^u|^2)^{1/2}$ satisfies $\|f^\alpha\|_{\mathscr{L}} \leq \|\alpha\|_{\ell^2(U)}$. Then there is $g \in \mathscr{M}_1^+$ such that $\|g^{-1} f^u\|_{L^2} \leq 1$ for all $u \in U$.*

Proof: For each α as in the statement of the proposition we define a function $F_\alpha : \mathscr{M}_1^+ \to]-\infty, +\infty]$ as follows:

$$F_\alpha(g) = \|g^{-1} f^\alpha\|_{L^2}^2 - \|\alpha\|_{\ell^2(U)}^2 = \sum_u \alpha_u^2 \big(\|g^{-1} f^u\|_{L^2}^2 - 1 \big).$$

Our purpose is to apply Proposition A.4 with $\mathscr{K} = \mathscr{M}_1^+$ equipped with the w^*-topology and $\mathscr{F}$ equal to the set of all functions F_α defined above. We saw before that $\mathscr{K}$ is a convex compact set. From the second representation of F_α given above it follows that $\mathscr{F}$ is a convex set. Each F_α is a convex function because $\|g^{-1} f^\alpha\|_{L^2}^2 = \int g^{-2} (f^\alpha)^2 \mathrm{d}x$ and the map $t \mapsto t^{-2}$ is convex on $[0, \infty[$. We shall prove in a moment that F_α is lower semicontinuous. From Lemma A.6 it follws that there is $g_\alpha \in \mathscr{K}$ such that $\|f^\alpha\|_{\mathscr{L}} = \|g_\alpha^{-1} f^\alpha\|_{L^2}$. Then by our assumptions we have

$$F_\alpha(g_\alpha) = \|f^\alpha\|_{\mathscr{L}}^2 - \|\alpha\|_{\ell^2(U)}^2 \leq 0.$$

From Ky Fan's Lemma it follows that one can choose $g \in \mathscr{K}$ such that $F_\alpha(g) \leq 0$ for all α, which finishes the proof of the proposition.

It remains to show the lower semicontinuity of F_α. For this it suffices to prove that $g \mapsto \|g^{-1} f\|_{L^2}^2 \in [0, \infty]$ is lower semicontinuous on $\mathscr{K}$ if $f \in \mathscr{L}, f \geq 0$. But

$$\|g^{-1} f\|_{L^2}^2 = \sum_a \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$$

and the set of lower semicontinuous functions $\mathscr{K} \to [0, \infty]$ is stable under sums and upper bounds of arbitrary families. Hence it suffices to prove that each map $g \mapsto \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$ is lower semicontinuous. This map can be written as a composition $\phi \circ J_a$ where $J_a : \mathscr{M} \to L^q(K_a)$ is the restriction map $J_a g = g_a$ and $\phi : L^q(K_a) \to [0, \infty]$ is defined by $\phi(\theta) = \int_{K_a} \theta^{-2} f_a^2 \mathrm{d}x$. The map J_a is continuous if we equip $L^q(K_a)$ with the weak topology and $\mathscr{M}$ with the w^*-topology because it is the adjoint of the norm continuous map $L^{q'}(K_a) \to \mathscr{M}_*$ which sends u into the function equal to u on K_a and 0 elsewhere. Thus it suffices to show that ϕ is lower semicontinuous on the positive part of $L^q(K_a)$ equipped with the weak topology and for this we can use exactly the same argument as Maurey. We must prove that the set $\{\theta \in L^q(K_a) \mid \theta \geq 0, \phi(\theta) \leq r\}$ is weakly closed for each real r. Since ϕ is convex, this set is convex, so it suffices to show that it is norm closed. But this is clear by the Fatou Lemma. ■

Theorem A.8 *Let $\mathscr{H}$ be a Hilbert space and $T : \mathscr{H} \to \mathscr{L}$ a linear continuous map. Then there exist a linear continuous map $R : \mathscr{H} \to L^2(X)$ and a positive function $g \in \mathscr{M}$ such that $T = g(Q)R$.*

Proof: Let U be the unit ball of $\mathscr{H}$ and for each $u \in U$ let $f^u = Tu$. From Proposition A.5 we get

$$\|f^\alpha\|_{\mathscr{L}} = \|(\textstyle\sum_u |T(\alpha_u u)|^2)^{1/2}\|_{\mathscr{L}} \leq A(\sum_u \|\alpha_u u\|^2)^{1/2} \leq A(\sum_u |\alpha_u|^2)^{1/2}$$

where $A = C\|T\|_{\mathcal{B}(\mathscr{H},\mathscr{L})}$. Since there is no loss of generality in assuming $A \leq 1$, we see that the assumptions of Proposition A.7 are satisfied. So there is $g \in \mathscr{M}_1^+$ such that $\|g^{-1}Tu\|_{L^2(X)} \leq 1$ for all $u \in U$. Thus it suffices to define R by the rule $Ru = g^{-1}Tu$ for all $u \in \mathscr{H}$. ■

References

[ABG] W. Amrein, A. Boutet de Monvel, V. Georgescu, *C_0-Groups, commutator methods and spectral theory of N-body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.

[Ar] M. Arai, *On essential self-adjointness, distinguished selfadjoint extensions and essential spectrum of Dirac operators with matrix valued potentials*, Publ. RIMS, Kyoto University, **19** (1983), 33–57.

[AY] M. Arai, O. Yamada, *Essential self-adjointness and invariance of the essential spectrum for Dirac operators*, Publ. RIMS Kyoto University, **18** (1982), 973–985.

[DP] G. De Cecco, G. Palmieri, *p-Energy of a curve on LIP-manifolds and on general metric spaces,* preprint.

[Di] J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Paris, Gauthier-Villars, 1969.

[DJT] J. Diestel, H. Jarchow, A. Tonge, *Absolutely summing operators,* Cambridge studies in advanced mathematics, Cambridge University Press, 1995.

[FD] J.M.G. Fell and R.S. Doran, *Representations of $*$-algebras, locally compact groups, and Banach $*$-algebraic bundles, Vol. 1,* Academic Press, 1988.

[GM] V. Georgescu, M. Măntoiu, *On the spectral theory of singular Dirac type hamiltonians,* J. Operator Theory **46** (2001), 289–321.

[GW] K. Gustafson, J. Weidmann, *On the essential spectrum*, J. Math. Anal. Appl. **25** (1969) 121–127.

[He] R. Hempel, *Perturbation by Quadratic Forms and Invariance of Essential Spectra,* Math. Z. **185** (1984), 281–289.

[Hi] M. Hilsum, *Structures riemmanniennes L^p et K-homologie*, Annals of Math. **149**, 1007–1022.

[Ho] L. Hörmander, *The analysis of linear partial differential operators vol. I-IV*, Springer, 1983-1987.

[Kl] M. Klaus, *Dirac Operators with Several Coulomb Singularities,* Helv. Phys. Acta **l53** (1980), 463–482.

[LV] V. Liskevich, H. Vogt, *On L^p-spectra and essential spectra of second-order elliptic operators*, Proc. London Math. Soc. (3) **80** (2000), 590–610.

[Lo] L.H. Loomis, *An introduction to abstract harmonic analysis,* Van Nostrand, 1953.

[Ma] B. Maurey, *Théoremes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*, Astérisque **11**, Société Math. de France.

[N1] G. Nenciu, *Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms,* Comm. Math. Phys. **48** (1976), 235–247.

[N2] G. Nenciu, *Distinguished self-adjoint extensions for Dirac operators with potentials dominated by multicenter Coulomb potentials*, Helv. Phys. Acta **50** (1977), 1–3.

[OS] E.M. Ouhabaz, P. Stollmann, *Stability of the ensential spectrum of second-order complex elliptic operators,* J. Reine Angew. Math. **500** (1998), 113–126.

[Pi] G. Pisier, *Factorization of operators and geometry of Banach spaces*, Amer. Math. Soc., Providence R.I., CBMS **60**, 1985.

[Sa] L. Saloff-Coste, *Opérateurs pseudo-différentiels sur un corps local*, thèse de troisième cycle, Université Pierre et Marie Curie, 1983.

[Ta] M. H. Taibleson, *Fourier analysis on local fields*, Princeton University Press, 1975.

[Te] N. Teleman, *The index of signature operators on Lipschitz manifolds*, Publ. Math. I.H.E.S. **58** (1983), 39–78.

[Wa] N. Weaver, *Lipschitz algebras and derivations II: exterior differentiation*, preprint 1998.

[We] J. Weidmann, *Linear operators in Hilbert space,* Springer, 1980.