Tropical methods for ergodic control and zero-sum games

Minilecture, Part II

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Dynamical Optimization in PDE and Geometry
Applications to Hamilton-Jacobi
Ergodic Optimization, Weak KAM
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Today

From tropical algebra to nonexpansive mappings and back

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

- $[n] := \{1, \dots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$

Shapley operators

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T is order preserving, additively homogeneous $\Rightarrow$ sup-norm nonexpansive:

$$x \leq y \implies T(x) \leq T(y)$$
$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$
$$\|T(x) - T(y)\| \leq \|x - y\|$$

Shapley operators

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Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic) Gunawardena, Sparrow; Singer, Rubinov,

$$T_i(x) = \sup_{y \in \mathbb{R}} \Big(T_i(y) + \min_{1 \le i \le n} (x_i - y_i) \Big)$$

Repeated games

The value of the game in horizon k starting from state i is $(T^k(0))_i$.

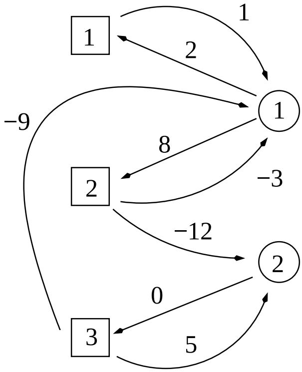
We are interested in the long term payment per time unit

$$\chi(T) := \lim_{k\to\infty} T^k(0)/k$$

Example: mean payoff games

$G = (V, E)$ bipartite graph. r_{ij} price of the arc $(i, j) \in E$. "Max" and "Min" move a token. The player receives the amount of the arc.

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



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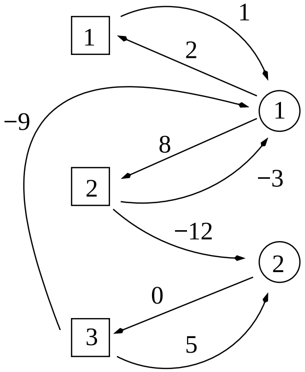


$$\chi(T) = \lim_k v^k/k = (-1,5)$$

$\chi(T) = \lim_k T^k(0)/k$ may not exist if the action spaces are infinite (Kohlberg, Neyman). Counter example in dimension 3.

However. Let v_α denote the discounted value

$$v_\alpha = T(\alpha v_\alpha), \qquad 0 < \alpha < 1 \ .$$

Theorem (Neyman 04 - book edited with Sorin)

If $\alpha \mapsto (1-\alpha)v_\alpha$ has bounded variation as $\alpha \to 1$, then

$$\lim_k T^k(0)/k = \lim_{\alpha \to 1^-} (1-\alpha)v_\alpha \qquad \textit{does exist}$$

Corollary (Neyman 04, Bewley and Kohlberg 76)
If the graph of T is semi-algebraic, then $\chi(T)$ does exists.

This is the case in particular if the action spaces are finite.

Original motivation. Von Neumann's value of a matrix game with imperfect information (rock-sissors-stone), given a $n \times p$ matrix M,

$$\operatorname{val} M = \min_{x \in \Sigma_n} \max_{y \in \Sigma_p} x^\top M y$$

where $\Sigma_n := \{x \in \mathbb{R}^n_+ \mid \sum_i x_i = 1\}$. The graph of $M \mapsto \operatorname{val} M$ is semi-algebraic.

Ingredients of the proof

A real Puiseux series in a parameter t is of the form

$$\sum_{k \geq K} a_k t^{k/r}, \qquad a_k \in \mathbb{R},\ r \geq 1,\ K \in \mathbb{Z}\ .$$

Eg., $-3/\sqrt{t} + 7 + 8\sqrt{t} + t + t^{3/2} + \dots$.

Can consider formal series, or converging series in $0 < |t| < D$ for D small enough.

Puiseux series constitute a real closed field (every square is nonnegative, every equation of odd degree has at least one root).

By Tarski's theorem, if the graph of T is semi-algebraic, the unique fixed point v_α of $x \mapsto T(\alpha x)$ has a converging Puiseux series expansion

$$v_\alpha = \sum_{k \geq K} a_k (1-\alpha)^{k/r}, \qquad r \geq 1 \ .$$

Use nonexpansiveness to deduce that $v_\alpha = O((1-\alpha)^{-1})$, the smallest exponent is ≥ -1.
Indeed,
$\|v_\alpha - T(0)\| = \|T(\alpha v_\alpha) - T(0)\| \leq \alpha \|v_\alpha\| \leq \alpha(\|v_\alpha - T(0)\| + \|T(0)\|)$, and so

$$\|v_\alpha - T(0)\| \leq \frac{\alpha}{1-\alpha}\|T(0)\| \ .$$

So bounded variation of $(1-\alpha)v_\alpha$ holds (converging, sign of derivative constant in a neighborhood of 1^-).

Classical Blackmailer example (Blackwell).

Blackmailer goes to see Victim.

Give me $p \in [0, 1]$

Victim pays, but with probability p^2, goes to see the police.

$$T : \mathbb{R} \to \mathbb{R}, \qquad T(x) = \max_{p \in [0,1]} p + (1 - p^2)x \ .$$

$$p^{\mathrm{opt}} = 1/2x, \qquad T^k(0) \simeq \sqrt{k}, \qquad v_\alpha \simeq 1/\sqrt{1-\alpha} \ .$$

Even if

$$T_i(x) = \max_{a\in A_i} \min_{b\in B_i} \big(r_i^{ab} + \sum_{1\leq j\leq n} P_{ij}^{ab} x_j\big), \qquad i \in [n]$$

with compact action spaces, $(a,b) \mapsto r_i^{ab}$ and $(a,b) \mapsto P_{ij}^{ab}$ continuous;

it is not known whether $\lim_k T^k(0)/k$ exists.

See works by Sorin, Rosenberg, Vigeral,. . .

In general. . .

Escape rate

(X, d) metric space, $T : X \to X$,

$$d(T(x), T(y)) \leq d(x, y) \ .$$

$$\rho(T) := \lim_{k\to\infty} \frac{d(x, T^k(x))}{k} = \inf_{k\geq 1} \frac{d(x, T^k(x))}{k}$$

(independent of $x \in X$ by nonexpansiveness, existence by subadditivity).

Taking $d(x,y) = \|y-x\|$, $\mathrm{t}(y-x)$, or $\mathrm{b}(y-x)$, we get that the following limits do exist and are independent of $x \in \mathbb{R}^n$

$$\lim_{k\to\infty} \frac{\|T^k(x)-x\|_\infty}{k} = \inf_{k\geq 1} \frac{\|T^k(x)-x\|_\infty}{k}$$

$$\overline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{t}(T^k(x)-x)}{k} = \inf_{k\geq 1} \frac{\mathrm{t}(T^k(x)-x)}{k}$$

$$\underline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{b}(T^k(x)-x)}{k} = \sup_{k\geq 1} \frac{\mathrm{b}(T^k(x)-x)}{k}$$

$$\mathrm{t}(z) := \max_i z_i, \qquad \mathrm{b}(z) := \min_i z_i \quad .$$

Theorem (Kohlberg & Neyman, Isr. J. Math., 81)

Assume $\rho(T) > 0$. Then, there exists a linear form $\varphi \in X^*$ of norm one such that for all $x \in X$,

$$\rho(T) = \lim_{k\to\infty} \varphi\big(T^k(x)/k\big) = \inf_{y\in X} \|T(y) - y\|$$

Actually, for all $x \in X$, there exists such a φ such that

$$\varphi(T^k(x)) \geq k\rho(T) + \varphi(x) \enspace .$$

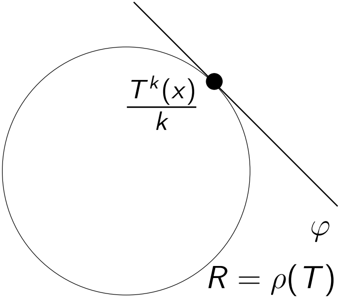
φ can be chosen in the weak-star closure of the set of extreme points of the dual unit ball

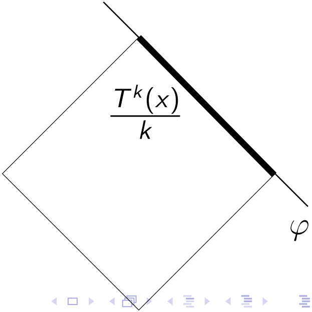
Corollary (Kohlberg & Neyman, Isr. J. Math., 81, extending Reich 73 and Pazy 71)

The limit

$$\lim_{k\to\infty} \frac{T^k(x)}{k}$$

exists in the weak (resp. strong) topology if X is reflexive and strictly convex (resp. if the norm of the dual space $X^\star$ is Frechet differentiable).





The special case of games

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \le j \le n} P_{ij}^{ab} x_j \Big), \qquad 1 \le i \le n$$

$$\rho(T) = \overline{\chi}(T) = \lim_{k \to \infty} \max_{1 \le j \le n} \frac{(T^k(x))_j}{k}$$

Theorem (SG, Gunawardena, TAMS 04)

For all $x \in \mathbb{R}^n$, there exists $1 \le i \le n$ such that

$$\big(T^k(x)\big)_i \ge x_i + k\rho(T), \qquad \forall k \in \mathbb{N} \ .$$

Initial state i guarantees the best reward per time unit.

Consider

$$F = \exp \circ T \circ \log, \qquad \mathbb{R}_+^n \to \mathbb{R}_+^n$$

which is order preserving, positively homogeneous, and continuous.

Theorem (non-linear Collatz-Wielandt, Nussbaum, LAA 86)

$$\begin{aligned}\rho(F) &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad x \in \operatorname{Int} \mathbb{R}_+^n \\ &= \max\{\mu \in \mathbb{R}_+ \mid F(v) = \mu v, v \in \mathbb{R}_+^n, v \neq 0\} \\ &= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in \mathbb{R}_+^n, v \neq 0\} \\ &= \inf\{\mu > 0 \mid F(v) \leq \mu v, v \in \operatorname{int} \mathbb{R}_+^n\}\end{aligned}$$

Proof ingredients

The (reverse) Funk (hemi-)metric on a cone, Hilbert's and Thompson metric

C closed pointed cone, $X = \operatorname{int} C \neq \emptyset$,
$y \geq x \iff y - x \in C$,

$$\delta(x,y) = \operatorname{RFunk}(x,y) := \log \inf\{\lambda > 0 | \lambda x \geq y\}$$

$$\begin{aligned}
\operatorname{RFunk}(x,y) &= \log \max_{\varphi \in C^* \setminus \{0\}} \frac{\varphi(y)}{\varphi(x)} \\
&= \log \max_{\varphi \in \operatorname{Extr} C^*} \frac{\varphi(y)}{\varphi(x)} \\
&= \log \max_{1 \leq i \leq n} \frac{y_i}{x_i} \qquad \text{if} \quad C = \mathbb{R}^n_+ \ ,
\end{aligned}$$

Thompson's metric is the Finsler structure corresponding to the "weighted sup-norm" at point u,

$$\begin{aligned} \|x\|_u &= \inf\{\alpha \mid -\alpha \leq x \leq \alpha u\} \\ &= \max_i \frac{|x_i|}{u_i} \qquad \text{if } C = \mathbb{R}^n_+ \ . \end{aligned}$$

$$d_T(x,y) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} dt, \;\; \gamma(0) = x, \; \gamma(1) = y \ .$$

See Nussbaum.

Nonexpansiveness in the Funk metric

Lemma

$F : C \to C$ is order preserving and homogeneous of degree 1 iff

$$\mathrm{RFunk}(F(x), F(y)) \leq \mathrm{RFunk}(x, y), \qquad \forall x, y \in \mathrm{int}\, C \ .$$

[simple but useful: Gunawardena, Keane, Sparrow, Lemmens, Scheutzow, Walsh.]

So F is nonexpansive in the RFunk metric, in Thompson metric $d_T(x, y) = \mathrm{RFunk}(x, y) \vee \mathrm{RFunk}(y, x)$, and Hilbert's projective metric $d_H(x, y) = \mathrm{RFunk}(y, x) - \mathrm{RFunk}(x, y)$.

Take q denote a positively homogeneous order preserving map from $\mathrm{Int}\, C$ to $(0,\infty)$, and let $u \in \mathrm{int}\, C$,

$$T_\epsilon(x) = T(x) + \epsilon q(x) u \enspace .$$

If C is normal, meaning that $0 \leq x \leq y \implies \|x\| \leq M\|y\|$ for some constant M, then $(\{x \in \mathrm{int}\, C, q(x) = 1\}, d_H)$ is complete, and T_ϵ is a strict contraction in d_H.
Then,

$$T_\epsilon(v_\epsilon) = \rho(T_\epsilon) v_\epsilon \enspace, v_\epsilon \in \mathrm{int}\, C \enspace .$$

Letting $\epsilon \to 0$, and taking an accumulation point v of v_ϵ, we get

$$T(v) = \mu v$$

so $\mu \leq \rho(T)$, but $\mu \geq \liminf_\epsilon \rho(T_\epsilon) \geq \rho(T)$.
This shows the Collatz-Wielandt theorem.

$$
\begin{aligned}
\rho(F) &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad x \in \operatorname{Int} C \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) = \mu v, v \in C, v \neq 0\} \\
&= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in C, v \neq 0\} \\
&= \inf\{\mu > 0 \mid F(v) \leq \mu v, v \in \operatorname{int} C\}
\end{aligned}
$$

Work of Akian, SG, Nussbaum: extension to the case of normal cones in Banach spaces, under compactness conditions (essential spectral radius).

Compare now Collatz-Wielandt

$$\begin{aligned}\rho(F) &= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in \mathbb{R}_+^n, v \neq 0\}\\ &= \inf\{\mu > 0 \mid F(w) \leq \mu w, w \in \operatorname{int} \mathbb{R}_+^n\}\\ &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad \forall x \in \operatorname{int} \mathbb{R}_+^n\end{aligned}$$

and so

$$\inf_{w\in \operatorname{int} \mathbb{R}_+^n} \max_{1\leq i\leq n} \frac{(F(w))_i}{w_i} = \rho(F) = \max_{\substack{v\in\mathbb{R}_+^n\\ v\neq 0}} \min_{\substack{1\leq i\leq n\\ v_i\neq 0}} \frac{(F(v))_i}{v_i} \quad .$$

with Kohlberg and Neyman

$$\rho(T) := \lim_{k\to\infty} \left\| \frac{T^k(x)}{k} \right\| = \inf_{y\in X} \|T(y) - y\| = \lim_{k\to\infty} \varphi\big(T^k(x)/k\big) \quad .$$

Is there an explanation of this analogy ?

Collatz-Wielandt and Kohlberg-Neyman are special cases of a general result.

Theorem (SG and Vigeral, Math. Proc. Phil. Soc. 11)

Let T be a nonexpansive self-map of a complete hemi-metric space (X, d) of non-positive curvature in the sense of Busemann. Let

$$\rho(T) := \lim_{k\to\infty} \frac{d(x, T^k(x))}{k}$$

Then, there exists a Martin function h such that

$$h(T(x)) \geq \rho(T) + h(x), \qquad \forall x$$

Moreover,

$$\rho(T) = \inf_{y\in X} d(y, T(y)) \ .$$

If in addition X is a metric space and $\rho(T) > 0$, then h is an horofunction.

It follows that

$$h(T^k(x)) \geq k\rho(T) + h(x), \qquad \forall x \in X \ .$$

Karlsson (Ergodic. Th. and Dyn. S., 01) proved an analogous result without nonpositive curvature, but with h depending on x (uses only subadditivity).

Indeed, $\rho(T) < \inf_{y \in X} d(x, T(y))$ may happen without nonpositive curvature (no strong duality).

See also works of Beardon, and Lins.

Corollary (SG and Vigeral)

Let T be a Shapley operator, $\mathcal{C}(K) \to \mathcal{C}(K)$, K compact. Then, for all $u \in \mathcal{C}(K)$, there exists a point $z \in K$ such that

$$[T^k(u)](z) \geq k\rho(T) + u(z)$$

where

$$\rho(T) = \lim_k \max_{y \in K} [T^k(u)](y)/k$$

E.g., Semigroup of Isaacs equation on a compact domain K, $v(s, y)$ value function at time s and point $y \in K$, there exists a state $z \in K$ such that

$$v(s, z) \geq s\rho(T) + v(0, z), \qquad \forall s \ .$$

No reachability assumption, only preserving $\mathcal{C}(K)$.

The formal analogue of the upper bound in Collatz-Wielandt for the Isaacs equation

$$v_t - H(x, Dv, D^2 v) = 0, \qquad H(x, p, \cdot) \text{ order preserving}$$

is

$$\lambda = \lim_{s\to\infty} \sup v(s,\cdot)/s = \inf_{\phi\in\mathcal{C}^2} \sup_x H(x, D\phi(x), D^2\phi(x))$$

Let us explain the different notions appearing in this theorem . . .

Hemi-metric

δ is an hemi-metric on X if

- $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$
- $\delta(x,y) = \delta(y,x) = 0$ if and only if $x = y$.

Variant: weak metric of Papadoupoulos, Troyanov.

(X,δ) is complete if X is complete for the metric $d(x,y) := \max(\delta(x,y), \delta(y,x))$.

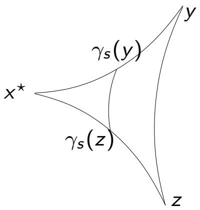
Example: the (reverse) Funk (hemi-)metric

$$\delta(x,y) = \mathrm{RFunk}(x,y) := \log\inf\{\lambda > 0 | \lambda x \geq y\}$$

Busemann convexity / nonpositive curvature condition

We say that (X, δ) is metrically star-shaped with center $x^\star$ if there exists a family of geodesics $\{\gamma_y\}_{y\in X}$, such that γ_y joins the center $x^\star$ to the point y, and

$$\delta\left(\gamma_y(s), \gamma_z(s)\right) \leq s\delta(y,z), \qquad \forall (y,z) \in X^2, \quad \forall s \in [0,1]$$



Examples of nonpositively curved spaces. . .

X Banach space, with the choice of straight lines as goedesics

Busemann nonpositive curvature is weaker than CAT(0)

$X = \operatorname{int} S_n^+$, where S_n^+ is the cone of positive definite matrices, equipped with the hemi-metric

$$\delta(A, B) = \nu(\log \operatorname{Spec}(A^{-1}B)) \enspace ,$$

where $\nu : \mathbb{R}^n \to \mathbb{R}$ is symmetric, continuous, convex, positively homogeneous of degree 1 (so $\nu(\lambda x) = \lambda\nu(x)$, for all $\lambda \geq 0$), and such that $\nu(x) = \nu(-x) = 0 \implies x = 0$.

This coincides with the invariant Finsler structure

$$\delta(A, B) = \inf_{\gamma(0)=A,\, \gamma(1)=B} \int_0^1 \nu(\operatorname{Spec}(\gamma(s)^{-1}\dot{\gamma}(s))ds =$$

$$\Gamma_X(A) := XAX^*, \qquad \delta(\Gamma_X(A), \Gamma_X(B)) = \delta(A, B)$$

$\gamma_B(s) = P^{\frac{1}{2}} \left(P^{-\frac{1}{2}} B P^{-\frac{1}{2}}\right)^s P^{\frac{1}{2}}$ is a choice of geodesics with center P.

Riemannian metric

Cartan/Mostow, $\nu := \|\cdot\|_2$,

$$\delta(A,B) = \sqrt{\sum_i (\log \lambda_i)^2}, \quad \lambda_i \text{ eigen. of. } A^{-1}B$$

Thompson's part metric / Funk metric, (Nussbaum), $\nu := \|\cdot\|_\infty$, $\nu(x) = \max_i x_i$,

$$\begin{aligned} d_T(A,B) &= \max(\mathrm{RFunk}(A,B), \mathrm{RFunk}(A,B)) \\ &= \max_i |\log \lambda_i| \end{aligned}$$

ν symmetric jauge function: Bhatia.

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ν symmetric jauge function: Bhatia.
Nonpositive curvature corresponds to the log-majorization inequality: for all $U, V \in \operatorname{int} V$, and for all $0 < s < 1$,

$$\log(\mathrm{Spec}(U^s V^s)) \prec s \log(\mathrm{Spec}(UV)) .$$

The discrete Riccati equation:

$$T(X) = A + M(B + X^{-1})^{-1}M^*, \qquad A, B \in S_n^+$$

is nonexpansive in any of these metrics!

Wojtowksi: Thompson metric; Bougerol: Riemanian metric; Lee and Lim: invariant Finsler metrics.

The horoboundary of a metric space

Defined by Gromov (81), see also Rieffel (Doc. Math. 02).
Fix a basepoint $\bar{x} \in X$.
$i : X \to \mathscr{C}(X)$,

$$i(x) : y \to [i(x)](y) := \delta(\bar{x}, x) - \delta(y, x).$$

so that

$$i(x)(\bar{x}) = 0, \quad \forall x \in X$$

Martin space: $\mathcal{M} := \overline{i(X)}$ (eg: product topology)
Boundary: $\mathcal{H} := \mathcal{M} \setminus i(X)$. An element of $\mathcal{H}$ is an horofunction.
A Busemann point is the limit $\lim_t i(x_t)$, where $(x_t)_{t\geq 0}$ is an infinite (almost) geodesic.
Busemann points $\subseteq$ boundary points, with equality for a polyhedral norm. See Walsh (boundary of normed space).

$i : X \to \mathscr{C}(X)$

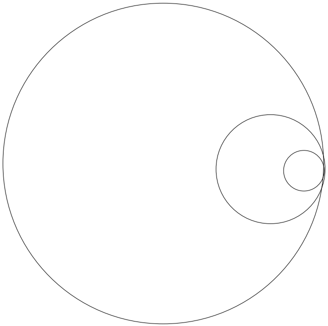
$$i(x) : y \to [i(x)](y) := \delta(\bar{x}, x) - \delta(y, x).$$

The boundary is usually defined as the closure of $i(X)$ in the topology of uniform convergence on bounded sets (Gromov, Ballman).
Indeed, i is continuous, and it is known to be an embedding if X is a complete geodesic space.
By Ascoli's theorem, on the equicontinuous set $i(X)$, the topology of uniform convergence on bounded sets and the pointwise convergence topology coincide if every closed ball is compact.

In general (infinite dimension), we need to take the pointwise convergence topology, then $i(X)$ is relatively compact, but i is no longer an embedding (the topology on $i(X)$ is too weak).

Exercise. Compute the horoboundary of $(\mathbb{R}, |\cdot|)$ and of $(\mathbb{R}^2, \|\cdot\|_1)$.

In the Poincare disk model, the level lines of horofunctions are horocircles



The Wolff-Denjoy theorem (1926) says that the orbits of a fixed point free analytic function leaving invariant the open disk converge to a boundary point (and that horodisks are invariants).

Theorem (SG and Vigeral, ibid.)

Let T be a nonexpansive self-map of a complete hemi-metric space (X, d) of non-positive curvature in the sense of Busemann. Then, there exists a Martin function h such that

$$h(T(x)) \geq \rho(T) + h(x), \qquad \forall x$$

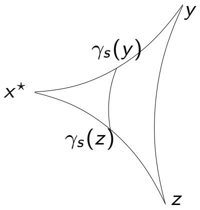
If in addition X is a metric space and $\rho(T) > 0$, then h is an horofunction.

Kohlberg-Neyman is a direct corollary. Since $h = \lim_\alpha -\| \cdot - x_\alpha \|$ modulo constants, h is concave. Take any $\varphi \in \partial h(x)$. Then,

$$\varphi(T^k(x) - x) \geq h(T^k(x)) - h(x) \geq k\rho(T) \ .$$

Proof idea. Fix a center $x^\star$, with geodesics $\gamma_y(0) = x^\star$, $\gamma_y(1) = y$. Busemann nonpositive curvature

$$\delta\left(\gamma_y(s), \gamma_z(s)\right) \leq s\delta(y, z), \qquad \forall (y, z) \in X^2, \quad \forall s \in [0, 1]$$



says that

$$r_\alpha(y) := \gamma_y(\alpha)$$

is a contraction of rate α.
The Martin function h is constructed as an accumulation point of $i(y_\alpha)$ as $\alpha \to 1^-$, where y_α is the fixed point of $T \circ r_\alpha$.

Collatz-Wielandt revisited

Let $F : C \to C$, where C is a symmetric cone (self-dual cone with a group of automorphisms acting transitively on it), say $C = \mathbb{R}^n_+$ or $C = S_n^+$.

Recall F is nonexpansive in RFunk iff it is order preserving and homogeneous of degree one.

Walsh (Adv. Geom. 08): the horoboundary of C in the (reverse) Funk metric is the Euclidean boundary: any Martin function h corresponds to some $u \in C \setminus \{0\}$:

$$h(x) = -\operatorname{RFunk}(x, u) + \operatorname{RFunk}(x^*, u) \quad , \forall x \in \operatorname{int} C,$$

h is a horofunction iff $u \in \partial C \setminus \{0\}$.

Corollary (Collatz-Wielandt recovered, and more)

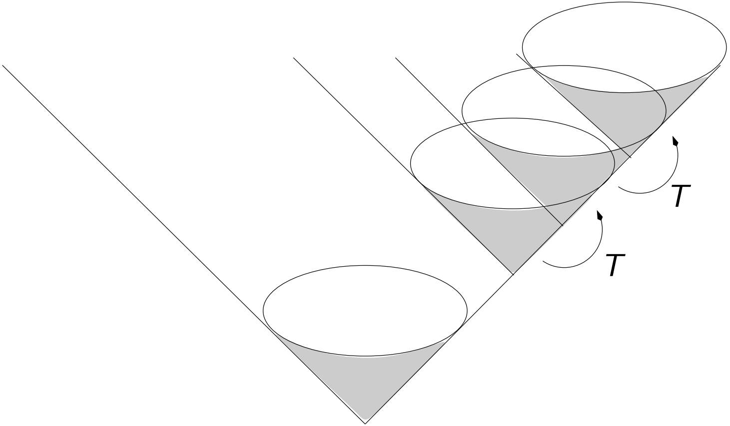
Let $T : \operatorname{int} C \to \operatorname{int} C$, *order-preserving and positively homogeneous,* C *symmetric cone. Then,*

$$\begin{aligned}
\rho(T) &:= \lim_{k\to\infty} \frac{\mathrm{RFunk}(x, T^k(x))}{k}, \qquad \forall x \in \operatorname{int} C \\
&= \inf_{y\in \operatorname{int} C} \mathrm{RFunk}(y, T(y)) \\
&= \log \inf\{\lambda > 0 \mid \exists y \in \operatorname{int} C,\ T(y) \leq \lambda y\} \\
&= \max_{u\in C\setminus\{0\}} -\mathrm{RFunk}(T(u), u) \\
&= \log \max\{\mu \geq 0 \mid \exists u \in C \setminus \{0\},\ T(u) \geq \mu u\}
\end{aligned}$$

and there is a generator w *of an extreme ray of* C *such that*

$$\log \left(w, T^k(x)\right) \geq \log \left(w, x\right) + k\rho(T), \qquad \forall k \in \mathbb{N}$$

Refines Gunawardena and Walsh, Kibernetica, 03.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Back to combinatorial games and tropical convexity.

Recall that $C \subset \mathbb{R}_{\max}^n$ is a tropical convex cone if

$$\lambda \in \mathbb{R}_{\max},\ u, v \in C \implies \sup\{u, v\} \in C,\ \lambda + u \in C \quad .$$

Correspondence between tropical convexity and zero-sum games, part II

Theorem (Akian, SG, Guterman, arXiv:0912.2462 → IJAC)

TFAE:

- *C closed tropical convex cone*
- *$C = \{u \in (\mathbb{R} \cup \{-\infty\})^n \mid u \leq T(u)\}$ for some Shapley operator T*

and MAX has at least one winning state ($\overline{\chi}(T) \geq 0$) if and only if

$$C \neq \{(-\infty, \ldots, -\infty)\} \enspace .$$

Proof of last statement. Think of T as a Perron-Frobenius operator in log-glasses:

$$F = \exp \circ T \circ \log, \qquad \mathbb{R}_+^n \to \mathbb{R}_+^n$$

$\overline{\chi}(T) \geq 0 \iff C \neq \{-\infty\}$ follows from Nussbaum's Collatz-Wielandt theorem, $F := \exp \circ T \circ \log$,

$$\begin{aligned} \overline{\chi}(T) &\geq 0 \\ \rho(F) &\geq 1 \\ \exists v \in \mathbb{R}_+^n, v \not\equiv 0, F(v) &\geq v \\ \exists u \not\equiv -\infty, T(u) &\geq u \end{aligned}$$

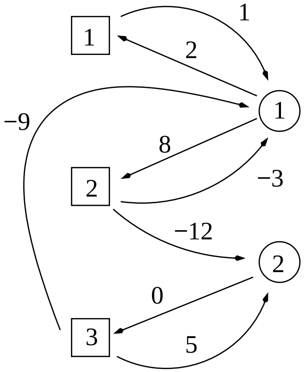
Polyhedral case

Theorem (Akian, SG, Guterman arXiv:0912.2462 → IJAC)

If the game is deterministic with finite action spaces (i.e. C is a tropical polyhedron), then the set of winning states is the support of C:

$$\{i \mid \exists u \in C,\ u_i \neq -\infty\} = \{i \mid \chi_i(T) \geq 0\}$$

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



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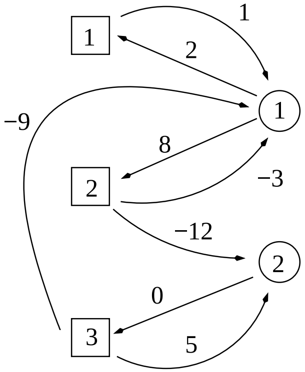

$\chi(T) = (-1, 5), x = (-\infty, 0)$ sol.

Relies on Kohlberg's theorem 1980.

A nonexpansive piecewise affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ admits an invariant half-line

$$\exists v \in \mathbb{R}^n,\ \vec{\eta} \in \mathbb{R}^n, \quad T(v + s\vec{\eta}) = v + (s+1)\vec{\eta} \ .$$

The vector u such that $T(u) \geq u$ is obtained from v, η ($u_i = -\infty$ if $\eta_i < 0$, $u_i = v_i + s\vec{\eta}_i$ for large s otherwise).

Kohlberg's theorem uses vanishing discount

Proposition

If T is nonexpansive and piecewise affine $\mathbb{R}^n \to \mathbb{R}^n$, the discounted value $v_\alpha = T(\alpha v_\alpha)$ has a Laurent series expansion

$$v_\alpha = \frac{a_{-1}}{1-\alpha} + a_0 + (1-\alpha)a_1 + \ldots, a_i \in \mathbb{R}^n$$

Nonexpansiveness $\implies$ 1 is necessarily a semisimple eigenvalue of $DT(x)$ at any point $x \in \mathbb{R}^n \implies$ pole of order ≤ 1.

$$T(v_\alpha - (1-\alpha)v_\alpha) = v_\alpha$$

$$T(sa_{-1} - a_0) = sa_{-1}, \qquad s \text{ large}$$

because T is piecewise affine.

Menu of tomorrow

- extreme points of tropical convex sets
- a bit more tropical geometry (tropical polynomials)
- some max-plus spectral theory
- max-plus Martin representation theorem
- deformation of Perron-Frobenius theory

Thank you !