THE BOLZA CURVE AND SOME ORBIFOLD BALL QUOTIENT SURFACES

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ABSTRACT. We study Deraux's non-arithmetic orbifold ball quotient surfaces obtained as birational transformations of a quotient X of a particular Abelian surface A. Using the fact that A is the Jacobian of the Bolza genus 2 curve, we identify X as the weighted projective plane $\mathbb{P}(1,3,8)$. We compute the equation of the mirror M of the orbifold ball quotient (X, M) and by taking the quotient by an involution, we obtain an orbifold ball quotient surface with mirror birational to an interesting configuration of plane curves of degrees $1, 2$ and 3. We also exhibit an arrangement of four conics in the plane which provides the above-mentioned ball quotient orbifold surfaces.

1. INTRODUCTION.

Chern numbers of smooth complex surfaces of general type X satisfy the Bogomolov-Miyaoka-Yau inequality $c_1^2(X) \leq 3c_2(X)$. Surfaces for which the equality is reached are ball quotient surfaces: there exists a cocompact torsion-free lattice Γ in the automorphism group $PU(2, 1)$ of the ball B_2 such that $X = B_2/\Gamma$. This description of ball quotient surfaces by uniformisation is of transcendental nature, and in fact among ball-quotient surfaces, very few are constructed geometrically (e.g. by taking cyclic covers of known surfaces or by explicit equations of an embedding in a projective space).

Among lattices in $PU(2, 1)$, only 22 commensurability classes are known to be non-arithmetic. The first examples of such lattices were given by Mostow and Deligne-Mostow (see [22] and [10]), and recently Deraux, Parker and Paupert [12, 13] constructed some more, sometimes related to an earlier work of Couwenberg, Heckman and Looijenga [9].

Being rare and difficult to produce, these examples are particularly interesting and one would like a geometric description of them. To do so, Deraux [14] studies the quotient of the Abelian surface $A = E \times E$, where E is the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$, by an order 48 automorphism group isomorphic to $GL_2(\mathbb{F}_3)$ that we will denote by G_{48}. The ramification locus of the quotient map $A \to A/G_{48}$ is the union of 12 elliptic curves and two orbits of isolated fixed points. The images of these two orbits are singularities of type A_2 and $\frac{1}{8}(1, 3)$, respectively.

Then Deraux proves that (on some birational transforms) the 1-dimensional branch locus M_{48} of the quotient map $A \to A/G_{48}$ and the two singularities are the support of four ball-quotient orbifold structures, three of these corresponding to non-arithmetic lattices in $PU(2, 1)$. Knowing the branch locus M_{48} is therefore important for these ball-quotient orbifolds, since it gives an explicit geometric description of the uniformisation maps from the ball to the surface.

Deraux also remarks in [14] that the invariants of A/G_{48} and its singularities are the same as for the weighted projective plane $\mathbb{P}(1, 3, 8)$ and, in analogy with cases in [11] and [15] where

weighted projective planes appear in the context of ball-quotient surfaces, he asks whether the two surfaces are isomorphic.

In fact, the quotient A/G_{48} can also be seen as a quotient $\mathbb{C}^2/G$ where G is an affine crystallographic complex reflection group. The Chevalley Theorem assert that if G' is a finite reflection group acting on a space V then the quotient V/G' is a weighted projective space. Using theta functions, Bernstein and Schwarzman [2] observed that for many examples of affine crystallographic complex reflection groups G acting on a space V, the quotient V/G is a also weighted projective space. Kaneko, Tokunaga and Yoshida [20] worked out some other cases, and it is believed that this analog of the Chevalley Theorem always happens (see [2], [16, p. 17]), although no general method is known (see also the presentation of the problem given by Deraux in [14], where more details can be found).

In this paper we prove that indeed:

Theorem A. *The surface A/G_{48} is isomorphic to $\mathbb{P}(1, 3, 8)$.*

We obtain this result by exploiting the fact that A is the Jacobian of a smooth genus 2 curve θ, a curve which was first studied by Bolza [5]. The automorphism group of the curve θ induces the action of G_{48} on the Jacobian A. The main idea to obtain Theorem A is to understand the image of the curve θ in A by the quotient map $A \to A/G_{48}$ and to prove that its strict transform in the minimal resolution is a (-1)-curve.

We then construct birational transformations of $\mathbb{P}(1, 3, 8)$ to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and obtain the equations of the images $M_{\mathbb{P}^1 \times \mathbb{P}^1}$, $M_{\mathbb{P}^2}$ of the branch curve M_{48} in these surfaces (and also $M_{48} \subset \mathbb{P}(1, 3, 8)$). In particular:

Theorem B. *In the projective plane, the mirror $M_{\mathbb{P}^2}$ is the quartic curve*
$$(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0.$$
This curve has two smooth flex points and singular set $\mathfrak{a}_1 + 2\mathfrak{a}_2$ (where an $\mathfrak{a}_k$ singularity has local equation $y^2 - x^{k+1} = 0$). The line L_0 through the two residual points of the flex lines F_1, F_2 contains the node (by flex line we mean the tangent line to a flex point).

The curve $M_{\mathbb{P}^2}$ with the two flex lines F_1, F_2 gives rise to the four orbifold ball-quotient surfaces (previously described by Deraux [14]) on suitable birational transformations of the plane. We prove that the configuration of curves described in Theorem B is unique up to projective equivalence.

In [18], Hirzebruch constructed ball quotient surfaces using arrangements of lines and performing Kummer coverings. It is a well-known question whether one can construct other ball quotient surfaces using higher degree curves, the next case being arrangements of conics.

Let φ be the Cremona transformation of the plane centered at the three singularities of $M_{\mathbb{P}^2}$. The image by φ of the curves $M_{\mathbb{P}^2}, F_1, F_2, L_0$ described in Theorem B is a special arrangement of four plane conics. We remark that by performing birational transforms of $\mathbb{P}^2$ and by taking the images of the 4 conics, one can obtain the orbifold ball-quotients of [14]. To our knowledge that gives the first example of orbifold ball quotients obtained from a configuration of conics (ball quotient orbifolds obtained from a configuration of a conic and three tangent lines are studied in [19] and [28]). However we do not know whether one can obtain ball quotient surfaces by performing Kummer coverings branched at these conics.

When preparing this paper, we observed that the mirror $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ and one related orbifold ball quotient surface among the four might be invariant by an order 2 automorphism. Using the equation we have obtained for $M_{\mathbb{P}^2}$, we prove that this is actually the case: there is an involution σ on $\mathbb{P}^1 \times \mathbb{P}^1$ with fixed point set a $(1, 1)$-curve D_i such that the quotient surface is $\mathbb{P}^2$, moreover the image of D_i is a conic C_o and the image of $M_{\mathbb{P}^2}$ is the unique cuspidal cubic curve C_u. In the last section we obtain and describe the following result:

Theorem C. *There is an orbifold ball-quotient structure on a surface W birational to $\mathbb{P}^2$ such that the strict transforms on W of C_o, C_u have weights $2, \infty$ respectively.*

The paper is structured as follows:

In section 2, we recall some results of Deraux on the quotient surface A/G_{48} and introduce some notation. In section 3, we study properties of the surface $\mathbb{P}(1,3,8)$. In section 4, we introduce the Bolza curve θ and prove that A/G_{48} is isomorphic to $\mathbb{P}(1,3,8)$. Section 5 is devoted to the equation of the mirror $M_{\mathbb{P}^2}$. Moreover we describe the four conics configuration. Section 6 deals with Theorem C.

Some of the proofs in sections 5 and 6 use the computational algebra system Magma, version V2.24-5. A text file containing only the Magma code that appear below is available as an auxiliary file on arXiv and at [25].

Along this paper we use intersection theory on normal surfaces as defined by Mumford in [23, Section 2].

2. QUOTIENT OF A BY G_{48} AND IMAGE OF THE MIRRORS

2.1. **Properties of A/G_{48} and image of the mirrors.**
In this section, we collect some facts from [14] about the action of the automorphism subgroup G_{48} on the Abelian surface

$$A := \mathbb{C}^2/(\mathbb{Z}[i\sqrt{2}])^2.$$

There exists a group G_{48} of order 48 acting on A which is isomorphic to $GL_2(\mathbb{F}_3)$ (see [14, Section 3.1] for generators). The action of G_{48} on A has no global fixed points (in particular some elements have a non-trivial translation part).

The group G_{48} contains 12 order 2 reflections, i.e. their linear parts acting on the tangent space $T_A \simeq \mathbb{C}^2$ are complex order 2 reflections. The fix point set of a reflection being usually called a mirror, we similarly call the fixed point set of a reflection τ of G_{48} a *mirror*. The mirror of such a τ is an elliptic curve on A. The group G_{48} acts transitively on the set of the 12 mirrors whose list can be found in [14, Table 1].

We denote by M the union of the mirrors in A and by M_{48} the image of M in the quotient surface A/G_{48}. The curve M_{48} is also called the mirror of A/G_{48}.

Except the points on M, there are two orbits of points in A with non-trivial isotropy, one with isotropy group of order 3 at each point, the other with isotropy group of order 8, see [14, Proposition 4.4]. Correspondingly, the quotient A/G_{48} has two singular points, which are the images of the two special orbits.

Proposition 1. *The surface A/G_{48} is rational and its singularities are of type $A_2 + \frac{1}{8}(1,3)$. The minimal resolution $p : X_{48} \to A/G_{48}$ of the surface A/G_{48} has invariants $K^2_{X_{48}} = 5$ and $c_2(X_{48}) = 7$.*

Proof. Let us compute the invariants of X_{48}. Let $\pi : A \to A/G_{48}$ be the quotient map. One has

$$(2.1) \qquad\qquad \mathcal{O}_A = K_A = \pi^* K_{A/G_{48}} + M,$$

moreover, according to [14, §4], each mirror M_i, $i = 1, ..., 12$, satisfies $M_i M = 24$, therefore $M^2 = 288$ and

$$(K_{A/G_{48}})^2 = \frac{1}{48} M^2 = 6.$$

We observe that $M = \pi^* \left(\frac{1}{2} M_{48} \right)$, thus by (2.1), one gets $M_{48} = -2K_{A/G_{48}}$.

The singularities of the quotient surface A/G_{48} are computed in [14, Table 2]. Let C_1, C_2 be the two (-3)-curves above the singularity $\frac{1}{8}(1, 3)$; they are such that $C_1 C_2 = 1$. Since the singularity of type A_2 is an ADE singularity, we obtain:

$$K_{X_{48}} = p^* K_{A/G_{48}} - \frac{1}{2}(C_1 + C_2)$$

and $(K_{X_{48}})^2 = 5$.

Let τ be a reflection in G_{48} and let G be the Klein group of order 4 generated by τ and the involution $[-1]_A \in G_{48}$. One can check that the quotient surface A/G is rational. Being dominated by the rational surface A/G, the surface A/G_{48} is also rational. Thus the second Chern number is $c_2(X_{48}) = 7$ by Noether's formula.   □

The mirror M_{48} (the image of M by the quotient map) does not contain singularities of A/G_{48}, moreover:

Lemma 2. *The pull-back $\tilde{M}_{48}$ of the mirror M_{48} by the resolution map $p : X_{48} \to A/G_{48}$ has self-intersection 24. Its singular set is*

$$2\mathfrak{a}_2 + \mathfrak{a}_3 + \mathfrak{a}_5,$$

where $\mathfrak{a}_k$ denotes a singularity with local equation $y^2 - x^{k+1} = 0$.

Proof. The singularities of $\tilde{M}_{48} = p^* M_{48}$ are the same as the singularities of M_{48} since M_{48} is in the smooth locus of A/G_{48}. For the computation of the singularities of M_{48}, we refer to [14, Table 3], and for the self-intersection of $\tilde{M}_{48}$ (which is the same as the one of M_{48}) to [14, §6.2].

□

3. THE WEIGHTED PROJECTIVE SPACE $\mathbb{P}(1, 3, 8)$.

Since we aim to prove that the quotient surface A/G_{48} is isomorphic to $\mathbb{P}(1, 3, 8)$, one first has to study that weighted projective space: this is the goal of this (technical) section. The reader might at first browse through the main results and notation and proceed to the next section.

3.1. The surface $\mathbb{P}(1, 3, 8)$ and its minimal resolution. The weighted projective space $\mathbb{P}(1, 3, 8)$ is the quotient of $\mathbb{P}^2$ by the group $\mathbb{Z}_3 \times \mathbb{Z}_8$ generated by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & \zeta \end{pmatrix} \in PGL_3(\mathbb{C}),$$

where $j^2 + j + 1 = 0$ and ζ is a primitive 8th root of unity. The fixed point set of the order 24 element σ is

$$p_1 = (1 : 0 : 0), \ p_2 = (0 : 1 : 0), \ p_3 = (0 : 0 : 1).$$

For $i, j \in \{1, 2, 3\}$ with $i \neq j$ let L'_{ij} be the line through p_i and p_j. The fixed point set of an order 3 element (e.g. σ^8) is p_2 and the line L'_{13}. The fixed point set of an order 8 element (e.g. σ^3) and its non-trivial powers is p_3 and the line L'_{12}. Let $\pi : \mathbb{P}^2 \to \mathbb{P}(1, 3, 8)$ be the quotient map: π is ramified with order 3 over L'_{13} and with order 8 over L'_{12}. The surface $\mathbb{P}(1, 3, 8)$ has two singularities, images of p_2 and p_3, which are respectively a cusp A_2 and a

singularity of type $\frac{1}{8}(1,3)$. We denote by $p : Z \to \mathbb{P}(1,3,8)$ the minimal desingularization map. The singularity of type $\frac{1}{8}(1,3)$ is resolved by two rational curves C_1, C_2 with $C_1 C_2 = 1$, $C_1^2 = C_2^2 = -3$, and the singularity A_2 is resolved by two rational curves C_3, C_4 with $C_3 C_4 = 1$, $C_3^2 = C_4^2 = -2$, (see e.g. [1, Chapter III]).

Lemma 3. *The invariants of the resolution Z are*

$$K_Z^2 = 5,\ c_2(Z) = 7,\ p_q = q = 0.$$

Proof. We have:

$$K_{\mathbb{P}^2} \equiv \pi^* K_{\mathbb{P}(1,3,8)} + 2L'_{13} + 7L'_{12},$$

therefore since $K_{\mathbb{P}^2} \equiv -3L$, we obtain $\pi^* K_{\mathbb{P}(1,3,8)} \equiv -12L$ and

$$(K_{\mathbb{P}(1,3,8)})^2 = \frac{(-12L)^2}{24} = 6.$$

We have

$$K_Z \equiv p^* K_{\mathbb{P}(1,3,8)} - \sum_{i=1}^{4} a_i C_i$$

where the a_i are rational numbers. The divisor K_Z must satisfy the adjunction formula i.e. one must have $C_i K_Z = -2 - C_i^2$ for $i \in \{1,2,3,4\}$. That gives:

$$K_Z = p^* K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)$$

and therefore $K_Z^2 = 5$. For the Euler number, one may use the formula in [26, Lemma 3]:

$$e(\mathbb{P}(1,3,8)) = \frac{1}{24}(3 + 2(2-2) + 7(2-2) + 23 \cdot 3) = 3.$$

Thus $e(Z) = e(\mathbb{P}(1,3,8)) - 2 + 3 + 3 = 7$. Since $\mathbb{P}(1,3,8)$ is dominated by $\mathbb{P}^2$, the surface Z is rational, so that $q = p_g = 0$. $\qquad\square$

3.2. **The branch curves in $\mathbb{P}(1,3,8)$ and their pullback in the resolution.** Let L_{ij} be the image of the line L'_{ij} on $\mathbb{P}(1,3,8)$ and let $\bar{L}_{ij}$ be the strict transform of L_{ij} in Z.

Proposition 4. *We have:*

$$\bar{L}_{23}^2 = -1,\ \bar{L}_{23}C_1 = \bar{L}_{23}C_3 = 1,\ \bar{L}_{23}C_2 = \bar{L}_{23}C_4 = 0,$$
$$\bar{L}_{13}^2 = 0,\ \bar{L}_{13}C_2 = 1,\ \bar{L}_{13}C_1 = \bar{L}_{13}C_3 = \bar{L}_{13}C_4 = 0,$$
$$\bar{L}_{12}^2 = 2,\ \bar{L}_{12}C_4 = 1,\ \bar{L}_{12}C_1 = \bar{L}_{12}C_2 = \bar{L}_{12}C_3 = 0.$$

Proof. On $\mathbb{P}(1,3,8)$ one has $L_{23}^2 = \frac{1}{24}L_{23}'^2 = \frac{1}{24}$. Recall that the resolution map is $p : Z \to \mathbb{P}(1,3,8)$. Let $a_1, \ldots, a_4 \in \mathbb{Q}$ such that
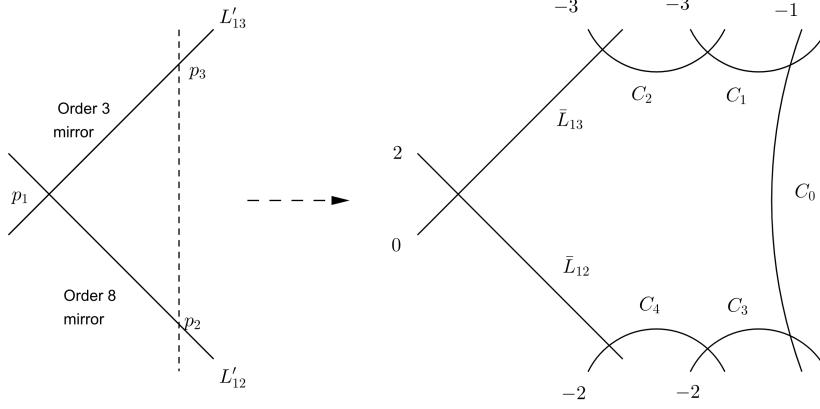
$$\bar{L}_{23} = p^* L_{23} - \sum_{i=1}^{4} a_i C_i,$$

then $C_i p^* L_{23} = 0$ for $i \in \{1,2,3,4\}$. Let $u_i \in \mathbb{N}$ such that $C_i \bar{L}_{23} = u_i$. One gets that

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}.$$

We have $\pi^* K_{\mathbb{P}(1,3,8)} = -12L'_{23}$, thus

$$K_{\mathbb{P}(1,3,8)} L_{23} = \frac{1}{24}(-12L'_{23} \cdot L'_{23}) = -\frac{1}{2}.$$

FIGURE 3.1. Image of the lines L'_{ij} in the desingularisation of $\mathbb{P}(1,3,8)$

Since $K_Z = p^* K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)$, we get

$$K_Z \bar{L}_{23} = \left(p^* K_{\mathbb{P}(1,3,8)} - \tfrac{1}{2}(C_1 + C_2)\right)\left(p^* L - \sum_{i=1}^{4} a_i C_i\right)$$
$$= -\tfrac{1}{2} - a_1 - a_2 = -\tfrac{1}{2}(1 + u_1 + u_2),$$

which is in $\mathbb{Z}$, with $u_1, u_2 \in \mathbb{N}$. One computes that

$$\bar{L}_{23}^2 = \frac{1}{24} - \frac{1}{8}(3u_1^2 + 3u_2^2 + 2u_1 u_2) - \frac{2}{3}(u_3^2 + u_3 u_4 + u_4^2) \in \mathbb{Z}_{\leq 0}.$$

Since $K_Z \bar{L}_{23} + \bar{L}_{23}^2 = -2$, the only possibility is

$$\{u_1, u_2\} = \{0, 1\},\ \{u_3, u_4\} = \{0, 1\},$$

which gives the intersection numbers with $\bar{L}_{23}$.

For the curve L_{13}, one has $L_{13} K_{\mathbb{P}(1,3,8)} = -\frac{3}{2}$ and $L_{13}^2 = \frac{3}{8}$. Let $u := \bar{L}_{13} C_1 \in \mathbb{N}$, $v := \bar{L}_{13} C_2 \in \mathbb{N}$. Then one similarly computes that

$$\bar{L}_{13} K_Z = -\frac{1}{2}(3 + u + v) \leq -\frac{3}{2}$$

and

$$\bar{L}_{13}^2 = \frac{1}{8}(3 - 3u^2 - 3v^2 - 2uv) \leq \frac{3}{8}.$$

Therefore $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \leq -\frac{9}{8}$ and since $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \geq -2$, the only solution is $\{u, v\} = \{0, 1\}$, thus $\bar{L}_{13}^2 = 0$ and $\bar{L}_{13} K_Z = -2$.

For the curve L_{12}, which does not go through the $\frac{1}{8}(1,3)$ singularity, one has

$$\bar{L}_{12} K_Z = L_{12} K_{\mathbb{P}(1,3,8)} = -4$$

and $L_{12}^2 = \frac{8}{3}$. Let $w := \bar{L}_{12} C_3$, $t := \bar{L}_{12} C_4$. Then

$$\bar{L}_{12}^2 = \frac{1}{3}(8 - 2w^2 - 2t^2 - 2wt) \leq \frac{8}{3}.$$

Therefore $\bar{L}_{12}^2 + K_Z \bar{L}_{12} \leq -\frac{4}{3}$ and the only solution is $\{w, t\} = \{0, 1\}$, thus $\bar{L}_{12}^2 = 2$. $\qquad\square$

3.3. **From $\mathbb{P}(1,3,8)$ to the Hirzebruch surface $\mathbb{F}_3$ and back.** By contracting the (-1)-curve $C_0 := \bar{L}_{23}$ and then the other (-1)-curves appearing from the configuration $C_1, \ldots, C_4, \bar{L}$, one gets a rational surface with

$$K^2 = 2c_2 = 8$$

containing (depending on the choice of the (-1)-curves we contract) a curve which either is a (-2)-curve or a (-3)-curve. Thus that surface is one of the Hirzebruch surfaces $\mathbb{F}_2$ or $\mathbb{F}_3$. Conversely one can reverse the process and obtain the surface $\mathbb{P}(1,3,8)$ by performing a sequence of blow-ups and blow-downs. This process is unique: this follows from the fact that the automorphism group of a Hirzebruch surface $\mathbb{F}_n$, $n \geq 1$ has two orbits, which are the unique $(-n)$-curve and its open complement (see e.g. [4]). In the sequel, only the connection between $\mathbb{P}(1,3,8)$ and $\mathbb{F}_3$ will be used.

4. THE BOLZA GENUS 2 CURVE IN A AND ITS IMAGE BY THE QUOTIENT MAP

In this section we prove that A/G_{48} is isomorphic to $\mathbb{P}(1,3,8)$.

Let us consider the genus 2 curve θ whose affine model is

$$(4.1) \qquad\qquad y^2 = x^5 - x.$$

It was proved by Bolza [5] that the automorphism group of θ is $GL_2(\mathbb{F}_3) \simeq G_{48}$ and θ is the unique genus 2 curve with such an automorphism group.

The automorphisms of θ are generated by the hyperelliptic involution λ and the lift of the automorphism group G of $\mathbb{P}^1$ that preserves the set of 6 branch points $0, \infty, \pm 1, \pm i$ of the canonical map $\theta \to \mathbb{P}^1$ (i.e. the set of points which are fixed by λ). Note that actually, any map of degree 2 from θ to $\mathbb{P}^1$ is the composition of this map with an automorphism of $\mathbb{P}^1$. This is a consequence of the two following facts: on the one hand the 6 ramification points (by the Riemann-Hurwitz formula) of such a map are Weierstrass points, and on the other hand the genus 2 curve θ has exactly 6 Weierstrass points.

By the universal property of the Abel-Jacobi map, the group $GL_2(\mathbb{F}_3)$ acts naturally on the Jacobian variety $J(\theta)$ of θ, the action on θ and $J(\theta)$ being equivariant.

There is only one Abelian surface with an action of $GL_2(\mathbb{F}_3)$, which is $A = E \times E$, where $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$ as above (see Fujiki [17] or [3]). We identify $J(\theta)$ with A. There are up to conjugation only two possible actions of $GL_2(\mathbb{F}_3)$ on A (see [24]):
a) The action of $G_{48} \simeq GL_2(\mathbb{F}_3)$ which is described in sub-section 2.1; it has no global fixed points;
b) The one obtained by forgetting the translation part of that action. That second action globally fixes the 0 point in A.

Let $\alpha : \theta \hookrightarrow J(\theta) = A$ be the embedding of θ sending the point at infinity of the affine model (4.1) to 0; we identify θ with its image.

Note that the morphism $\theta \times \theta \to A$, $(x,y) \mapsto [y] - [x] \in \mathrm{Div}_0(\theta) \simeq A$ is onto since $\theta \times \theta$ and A are both two-dimensional. Actually, this map has generic degree 2 and contracts the diagonal. Indeed, assume that $[y] - [x] = [y'] - [x']$ i.e. $[y] + [x'] - [x] - [y'] = 0 \in \mathrm{Div}_0(\theta)$. If $y' = y$ then $x' = x$ (and conversely) because there is no degree 1 map from θ to $\mathbb{P}^1$. In the same way, $y = x$ iff $y' = x'$. In the remaining cases, there exists a function of degree 2 from θ to $\mathbb{P}^1$ whose zeroes are y and x' and poles are x and y'. But by the remark above, we must have $x' = \lambda(y)$ and $y' = \lambda(x)$. Conversely, by the same argument, it is clear that for all x and y in θ, $[\lambda(y)] - [\lambda(x)] = [x] - [y]$.

This also implies that the points of the type $[y] - [x]$ with x and y being distinct Weierstrass points are exactly the 2-torsion points of A. Indeed, since there are 6 Weierstrass points on θ, we have 15 points of that type in A satisfying $[y] - [x] = [\lambda(x)] - [\lambda(y)] = [x] - [y]$ i.e. they are 2-torsion points.

The induced linear action b) is given by $g([y] - [x]) = [g(y)] - [g(x)]$ for which $0 \in \mathrm{Div}_0(\theta)$ is a fixed point.

If we fix the base point $\infty \in \theta$ then for each $y \in \theta$, $\alpha(x) = [x] - [\infty]$. The induced action of $g \in \mathrm{Aut}(\theta)$ on A is then given by $g([y] - [x]) = [g(y)] - [g(x)] + [g(\infty)] - [\infty]$. This is indeed the only action of $\mathrm{Aut}(\theta)$ on A commuting with α.

Lemma 5. *The action of $GL_2(\mathbb{F}_3)$ on A inducing the action of $\mathrm{Aut}(\theta)$ on the curve $\theta \hookrightarrow A$ has no global fixed points.*

Proof. The fixed points on A for the action of the hyperelliptic involution λ are its points of 2-torsion (and 0). Indeed, $\lambda([y] - [x]) = [\lambda(y)] - [\lambda(x)] \in \mathrm{Div}_0(\theta)$ since $\infty \in \theta$ is fixed by λ and, as a consequence of the discussion above, if $[y] - [x] = [\lambda(y)] - [\lambda(x)]$ then either $y = x$ or $y = \lambda(x)$ i.e. $[y] - [x] = [x] - [y]$ and we saw that this implies that x and y are Weierstrass points.

But for any pair (x, y) of distinct Weierstrass points, it is easy to find $g \in \mathrm{Aut}(\theta)$ (lifting an automorphism of $\mathbb{P}^1$) such that $g(\infty) = \infty$ but $[g(y)] - [g(x)] \neq [y] - [x]$. $\square$

For $t \in A$, let θ_t be the curve $\theta_t = t + \theta$. The previous result does not depend on the choice of the embedding $\theta \hookrightarrow A$: indeed the group of automorphisms acting on A and preserving θ_t is conjugated by the translation $x \mapsto x + t$ to the group of automorphisms acting on A and preserving θ.

We denote by H_{48} the order 48 group acting on A and inducing the automorphism group of the curve $\theta \hookrightarrow A$ by restriction. As a consequence of Lemma 5, we get:

Corollary 6. *There exists an isomorphism between H_{48} and G_{48}. That isomorphism is induced by an automorphism g of the surface A such that $H_{48} = gG_{48}g^{-1}$.*

By [6, Theorem (0.3)], the embedding $\alpha : \theta \hookrightarrow A$ is such that the torsion points of A contained in θ are 16 torsion points of order 6, 5 torsion points of order 2 and the origin, moreover the x-coordinates of the 22 torsion points on θ satisfy

$$x^4 - 4ix^2 - 1 = 0, \ \ x^4 + 4ix^2 - 1 = 0$$
$$x^5 - x = 0, \ \ x = \infty.$$

Proposition 7. *(a) These 22 torsion points of θ are not in the mirror of any of the 12 complex reflections of H_{48};*
(b) Each of these 22 points has a non-trivial stabilizer.

Proof. Let us prove part (a).

The hyperelliptic involution is given by $(x, y) \to (x, -y)$. By [7], the rational map

$$v : (x, y) \mapsto \left(-\frac{x + i}{ix + 1}, \sqrt{2}\frac{i - 1}{(ix + 1)^3}y \right)$$

defines a non-hyperelliptic involution v on θ. The x-coordinates of the fixed point set of v are $x_\pm = i(1 \pm \sqrt{2})$. These coordinates $x_\pm$ are not among the x-coordinates of the 22 torsion points in θ. Let $\mathbf{v}$ be the automorphism of A induced by v. The fixed point set of $\mathbf{v}$ is a smooth genus 1 curve E_v (a mirror) and we have just proved that E_v contains no torsion points of θ. By transitivity of the group H_{48} on its set of 12 non-hyperelliptic involutions, one gets that no mirror contains any of the 22 torsion points.

Let us prove part (b).

The six 2-torsion points are the Weierstrass points of the curve θ, they are fixed by the hyperelliptic involution (whose action on A has only 16 fixed points).

The transformation

$$w : (x, y) \mapsto \left(\frac{(1+i)x - (1+i)}{(1-i)x + (1-i)}, -\frac{1}{((1-i)x + (1-i))^3} y \right)$$

defines an order 3 automorphism of θ, which acts symplectically on A and one computes that it fixes a torsion point $p_0 = (x_0, y_0)$ on θ with x_0 such that $x_0^4 + 4ix_0^2 - 1 = 0$, i.e. it is an order 6 torsion point. This torsion point is an isolated fixed point for each non-trivial element of its stabilizer (since by part (a), it is not on a mirror).

Recall that by [14, Table 2], there are exactly two orbits of points of respective orders 6 and 16 with non-trivial stabilizer under G_{48} which are isolated fixed points of the non-trivial elements of their stabilizer (by a direct computation one can check that these two orbits are 16 points of order 6 and 6 points of order 2). Since H_{48} is conjugate to G_{48}, the 15 other 6-torsion points on θ are also isolated fixed points for each non-trivial element of their stabilizer. $\qquad\square$

Since one can change the embedding $\theta \hookrightarrow A$ by composing with the automorphism g such that $H_{48} = gG_{48}g^{-1}$, let us identify H_{48} with G_{48}.

By sub-section 2.1 (or [14]), the images of the 22 torsion points of θ on the quotient surface A/G_{48} give the singularities A_2 and $\frac{1}{8}(1, 3)$.

Let m be the mirror of one of the 12 complex reflections in G_{48}.

Lemma 8. *One has $\theta \cdot m = 2$.*

Proof. The intersection number $\theta \cdot m$ is the number of fixed points of the involution ι_m with mirror m restricted to θ. Since ι_m fixes exactly one holomorphic form, the quotient of θ by ι_m is an elliptic curve, thus by the Hurwitz formula $\theta \cdot m = 2$. $\qquad\square$

Let θ_{48} be the image of θ in A/G_{48}. One has:

Proposition 9. *The strict transform C_0 of θ_{48} by the resolution $X_{48} \to A/G_{48}$ is a (-1)-curve and we have $\tilde{M}_{48} C_0 = 1$.*

Proof. One has

$$\theta_{48}^2 = \frac{1}{48} \theta^2 = \frac{1}{24}.$$

Let $\pi : A \to A/G_{48}$ be the quotient map; it is ramified with order 2 on the union M of the 12 mirrors. One has $\pi^*(K_{A/G_{48}} + \frac{1}{2} M_{48}) = K_A = 0$, thus

$$K_{A/G_{48}} \theta_{48} = -\frac{1}{48}(M\theta) = -\frac{1}{48} 12 \cdot 2 = -\frac{1}{2}.$$

The curve θ_{48} contains the singularities $\frac{1}{8}(1, 3)$ and A_2 (image respectively of the 2-torsion points and the 6-torsion points of θ). We are then left with the same combinatorial situation as in the computation of $\bar{L}_{23}^2$ in Proposition 4, thus we conclude that $C_0^2 = -1$.

The two intersection points of m and θ in Lemma 8 are permuted by the hyperelliptic involution of θ thus $M_{48}\theta_{48} = 1$, which implies $\tilde{M}_{48} C_0 = 1$. $\qquad\square$

We obtain:

Theorem 10. *The surface A/G_{48} is isomorphic to $\mathbb{P}(1, 3, 8)$.*

Proof. Let us denote the resolution map by $p : X_{48} \to A/G_{48}$. Let C_1, C_2 be the resolution curves of the singularity $\frac{1}{8}(1, 3)$, and C_3, C_4 be the resolution of A_2. Let $a \in A$ be an isolated fixed point of an automorphism τ of order 3 or 8. The tangent space $T_{\theta,a} \subset T_{A,a}$ is stable by the action of τ. Since the local setup is the same, we can reason as in Proposition 4 and we obtain that the curve C_0 is such that

$$C_0 C_1 = C_0 C_3 = 1, \; C_0 C_2 = C_0 C_4 = 0.$$

Contracting the curves C_0, C_1, C_2, one gets a rational surface with a (-3)-curve and with invariants $K^2 = 2c_2 = 8$. This is therefore the Hirzebruch surface $\mathbb{F}_3$. From section 3, we know that reversing the contraction process one gets the weighted projective plane $\mathbb{P}(1,3,8)$ (contracting the curves C_0, C_1, C_3, one would have obtained the Hirzebruch surface $\mathbb{F}_2$). $\square$

Remark 11. Now we identify $\mathbb{P}(1,3,8)$ with A/G_{48} and we use the notation in section 3. In particular $Z = X_{48}$ is the minimal resolution of $\mathbb{P}(1,3,8)$, the curves $C_1, \ldots, C_4$ are exceptional divisors of the resolution map $Z \to \mathbb{P}(1,3,8)$ and $C_0 = \bar{L}_{23}$ is a (-1)-curve in Z.

Let us observe that the divisor $\tilde{F} = C_1 + 3C_0 + 2C_3 + C_4$ satisfies

$$\tilde{F}C_1 = \tilde{F}C_0 = \tilde{F}C_3 = \tilde{F}C_4 = 0,$$

thus $\tilde{F}^2 = 0$, moreover $\tilde{F}C_2 = \bar{L}_{13}\tilde{F} = 1$, $\tilde{F}\bar{L}_{13} = 0$ and $\bar{L}_{13}^2 = 0$. This implies that the curves $\tilde{F}$ and $\bar{L}_{13}$ are fibers of the same fibration onto $\mathbb{P}^1$ and C_2 is a section of that fibration.

The curves $C_0, \ldots, C_4$ are exceptional divisors or strict transform of generators of the Néron-Severi group of a minimal rational surface. Thus the Néron-Severi group of the rational surface X_{48} is generated by these curves. Knowing the intersection of curves $\bar{L}_{12}$, $\bar{L}_{13}$, $\tilde{M}_{48}$ with these curves (see Propositions 4 and 9) it is easy to obtain their classes in the Néron-Severi group, in particular one gets that $\bar{L}_{12}\tilde{M}_{48} = 8$, $\bar{L}_{13}\tilde{M}_{48} = 3$.

FIGURE 4.1.   Configuration of curves $\tilde{M}_{48}$, $\bar{L}_{12}, \bar{L}_{13}$ etc... in X_{48} and their intersection numbers
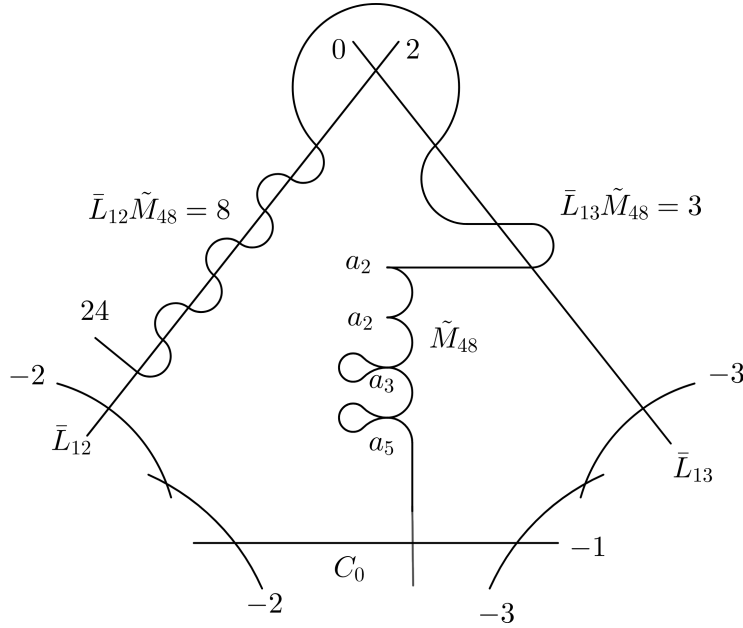
5. A MODEL OF THE MIRROR

5.1. **A birational map from $\mathbb{P}(1,3,8)$ to $\mathbb{P}^1 \times \mathbb{P}^1$; images of the mirror.**

5.1.1. *A rational map* $\mathbb{P}(1,3,8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. As above, we identify $\mathbb{P}(1,3,8)$ with A/G_{48}; we use the notation of sections 3 and 4.

Take a point p in the Hirzebruch surface $\mathbb{F}_n$ that is not in the negative section. By blowing-up at p, and then by blowing-down the strict transform of the fiber through p, we get the Hirzebruch surface $\mathbb{F}_{n-1}$. This process is called an *elementary transformation*.

Recall from sections 3 and 4 that there is a map $\psi : \mathbb{P}(1,3,8) \dashrightarrow \mathbb{F}_3$ that contracts the curves C_0, C_3, C_4 to a smooth point.

Performing any sequence of three elementary transformations as above, we get a map $\rho : \mathbb{F}_3 \dashrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. This can be seen as a birational transform that, by blowing-up three times at a point q not contained in the negative section, takes the fibre F_q through q to a chain of curves with self intersections $(-1), (-2), (-2), (-1)$, then followed by the contraction of the $(-1), (-2), (-2)$ chain (which contains the strict transform of F_q). For our purpose, we choose the three points to blow-up in a specific way, see subsection 5.1.2.

Consider
$$\phi := \rho \circ \psi : \mathbb{P}(1,3,8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

We observe that given any two points $t, t' \in \mathbb{P}^1 \times \mathbb{P}^1$ not in a common fiber, the map ϕ can be chosen such that the inverse ϕ^{-1} is not defined at t, t' and $\phi^{-1}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}(1,3,8)$.

FIGURE 5.1. From X_{48} to $\mathbb{P}^1 \times \mathbb{P}^1$ and back
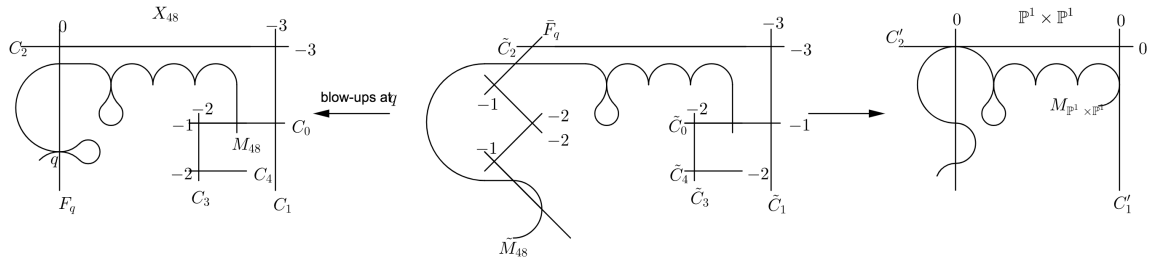
5.1.2. *Image of the mirror M_{48} in $\mathbb{P}^1 \times \mathbb{P}^1$.* Let us describe how to choose ϕ such that the image $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ of the mirror curve M_{48} is a $(3,3)$-curve with singularities $\mathfrak{a}_3 + 2\mathfrak{a}_2$ and two special fibers tangent to it with multiplicity 3.

The map $\mathbb{P}(1,3,8) \dashrightarrow \mathbb{F}_3$ factors through a morphism $\varphi : X_{48} \to \mathbb{F}_3$. Consider the point $t_0 := \varphi(C_0)$. Since $M_{48}C_0 = 1$, then $\varphi(M_{48})$ is a curve which is smooth at t_0 and its intersection number with the curve $\varphi(C_1)$ at t_0 is 3. The curve $C_1' := \rho \circ \varphi(C_1)$ is a fiber of $\mathbb{P}^1 \times \mathbb{P}^1$.

Then we choose q to be the $\mathfrak{a}_5$-singularity of M_{48}. The fiber F_q through q cuts M_{48} at q with multiplicity 2 or 3. Suppose that the multiplicity is 3. Then by taking the blow-up at that point and computing the strict transform of the curves F_q and M_{48}, one can check that $F_q M_{48} \geq 4$. But $F_q M_{48} = \bar{L}_{13} M_{48} = 3$ by Remark 11. Therefore the fiber F_q through q cuts M_{48} at q with multiplicity 2, and at another point.

Remark 12. An analogous reasoning gives that the fiber through the $\mathfrak{a}_3$-singularity has the same property: it is transverse to the tangent of the $\mathfrak{a}_3$-singularity.

The three successive blow-ups above q are chosen such that they resolve the singularity $\mathfrak{a}_5$. The three blow-downs we described create a multiplicity 3 tangent point between $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ (the image of M_{48} in $\mathbb{P}^1 \times \mathbb{P}^1$) and the curve C_2' (the image of C_2), thus $C_2' M_{\mathbb{P}^1 \times \mathbb{P}^1} = 3$. Moreover $C_2'^2 = 0, C_1' C_2' = 1$ (see figure 5.1).

The mirror M_{48} does not cut the curves C_1 and C_2. The transforms of these curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are fibers C_1', C_2' such that C_i' cuts $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ at one point only, with multiplicity 3.

In particular, the class of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ in the Néron-Severi group of $\mathbb{P}^1 \times \mathbb{P}^1$ is $3C_1' + 3C_2'$. The singularities of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ are $\mathfrak{a}_3 + 2\mathfrak{a}_2$.

5.1.3. *From $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ and back.* Let us recall that the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point, followed by the blow-down of the strict transform of the two fibers through that point, gives a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$.

  We choose to blow-up the point at the $\mathfrak{a}_3$-singularity s_0, so that the strict transform of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ has a node above s_0. The two fibers F_1, F_2 of $\mathbb{P}^1 \times \mathbb{P}^1$ passing through s_0 cut $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ in two other points respectively s_1, s_2 (see Remark 12; the result is preserved through the birational process). The fibers F_1, F_2 are contracted into points in $\mathbb{P}^2$ by the rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$, the images of s_1, s_2 by that map are on the image of the exceptional divisor, which is a line L_0 through the node. This implies that the strict transform of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ is a plane quartic curve $M_{\mathbb{P}^2}$. The process in illustrated in Figure 5.2.

FIGURE 5.2.   From $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$
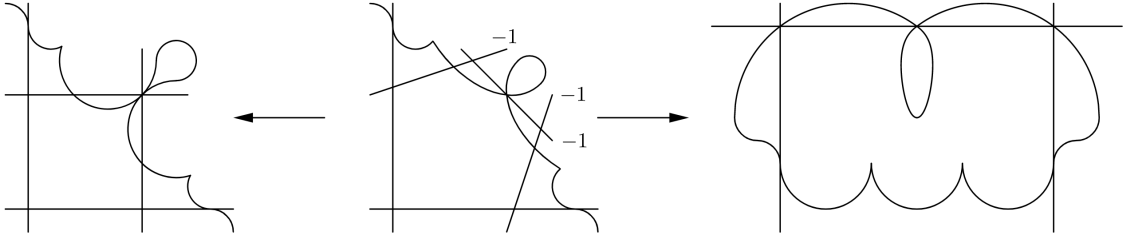
  The total transform of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ in $\mathbb{P}^2$ is the union of $2L_0$ with $M_{\mathbb{P}^2}$. This quartic $M_{\mathbb{P}^2}$ has the following properties which follow from its description and the choice of the transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$:

Proposition 13. *The singular set of the quartic curve $M_{\mathbb{P}^2}$ is $\mathfrak{a}_1 + 2\mathfrak{a}_2$, and the nodal point is contained in the line L_0. The curve $M_{\mathbb{P}^2}$ contains two flex points such that each corresponding tangent line meets the quartic at a second point that is contained in the line L_0.*

5.2. **The yoga between the mirrors $M_{\mathbb{P}^2}$ and M_{48}.** Using the previous description the reader can follow the transformations between the surfaces $\mathbb{P}(1, 3, 8)$ and the plane. The link between Deraux's ball quotient orbifolds described in [14, Theorem 5] and the quartic $M_{\mathbb{P}^2}$ is as follows:

  The singularities $\mathfrak{a}_1 + 2\mathfrak{a}_2$ of $M_{\mathbb{P}^2}$ correspond respectively to singularities $\mathfrak{a}_3 + 2\mathfrak{a}_2$ of M_{48}, so that in order to get the curves F, G, H in [14, Figure 1] one has to blow-up and contract at these 3 points as it is done in [14]. In order to obtain the curve E in [14, Figure 1], one has to blow-up the two flexes three times in order to separate $M_{\mathbb{P}^2}$ and the flex lines. One obtain two chains of $(-1), (-2), (-2)$ curves. Contracting one of the two $(-2), (-2)$ chains one gets an A_2-singularity. The curve E is the image by the contraction map of the remaining (-1)-curve of the chain. The resolution of the singularity A_2 on $\mathbb{P}(1, 3, 8)$ corresponds to the two (-2)-curves on the other chain of $(-1), (-2), (-2)$ curves. After taking the blow-up at the residual intersection of the quartic and the flex lines and after separating the flex lines and the mirror $M_{\mathbb{P}^2}$, one gets two (-3)-curves intersecting transversally at one point. In that way the resolution of the singularity $\frac{1}{8}(1, 3)$ on $\mathbb{P}(1, 3, 8)$ by two (-3)-curves corresponds to the two flex lines.

5.3. **A particular quartic curve in $\mathbb{P}^2$.** The aim of this sub-section is to prove the following result:

Theorem 14. *Up to projective equivalence, there is a unique quartic curve Q in $\mathbb{P}^2$ with distinct points $p_1, \ldots, p_7$ such that:*

   (1) *Q has a node at p_1 and ordinary cusps at p_2, p_3;*
   (2) *the points p_4, p_5 are flex points of Q;*
   (3) *the tangent lines to Q at p_4, p_5 contain p_6, p_7, respectively;*
   (4) *the line through p_6, p_7 contains p_1.*

   We can assume that

$$p_1 = [0:0:1], \ p_2 = [0:1:1], \ p_3 = [1:0:1].$$

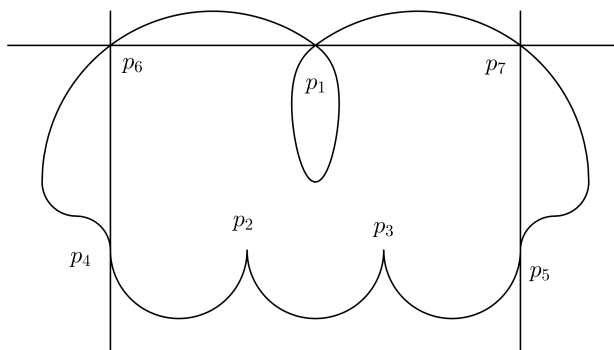*Then the equation of Q is*

$$(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0,$$

and the points p_4, p_5 and p_6, p_7 are, respectively,

$$\left[\pm 2\sqrt{-2} + 8 : \mp 2\sqrt{-2} + 8 : 25\right], \ \left[\pm 2\sqrt{-2} : \mp 2\sqrt{-2} : 1\right].$$

Corollary 15. *The mirror $M_{\mathbb{P}^2}$ described on sub-section 5.1.3 satisfies the hypothesis of Theorem 14, thus $M_{\mathbb{P}^2}$ is projectively equivalent to the quartic Q.*

FIGURE 5.3. The quartic Q

In order to prove 14, let us first give a criterion for the existence of roots of multiplicity at least 3 on homogeneous quartic polynomials on two variables. We use the computational algebra system Magma; see [25] for a copy-paste ready version of the Magma code.

Lemma 16. *The polynomial*

$$P(x, z) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4$$

has a root of multiplicity at least 3 if and only if

$$12ae - 3bd + c^2 = 27ad^2 + 27b^2e - 27bcd + 8c^3 = 0.$$

Proof. The computation below is self-explanatory.

```
R<u,v,m,n,a,b,c,d,e>:=PolynomialRing(Rationals(),9);
P<x,z>:=PolynomialRing(R,2);
f:=(u*x+v*z)^3*(m*x+n*z);
s:=Coefficients(f);
I:=ideal<R|a-s[5],b-s[4],c-s[3],d-s[2],e-s[1]>;
EliminationIdeal(I,4);
```

$\square$

Let us now prove Theorem 14:

Proof. We have already chosen 3 points p_1, p_2, p_3 in $\mathbb{P}^2$. Instead of choosing a fourth point for having a projective base, one can fix two infinitely near points over p_2 and p_3. Indeed the projective transformations that fix points p_1, p_2, p_3 are of the form

$$\phi : [x : y : z] \longmapsto [ax : by : (a-1)x + (b-1)y + z]$$

and these transformations act transitively on the lines through p_2 and p_3. Thus up to projective equivalence, we can fix the tangent cones (which are double lines) of the curve Q at the cusps p_2, p_3. Let us choose for these cones the lines with equations $y = z$ and $x = z$, respectively.

The linear system of quartic curves in $\mathbb{P}^2$ is 14 dimensional. The imposition of a node and two ordinary cusps (with given tangent cones) corresponds to 13 conditions, thus we get a pencil of curves. We compute that this pencil is generated by the following quartics:

$$(x^2 + xy + y^2 - xz - yz)^2 = 0, \quad xy(x + y - z)^2 = 0.$$

Notice that, at the points p_2, p_3, the first generator is of multiplicity 2 and the second generator is of multiplicity 3, thus a generic element in the pencil has a cusp singularity at p_2, p_3.

Let us compute the quartic curves Q satisfying condition (1) to (4) of Theorem 14. The method is to define a scheme by imposing the vanishing of certain polynomials $P_i = 0$, and the non-vanishing of another ones $D_i \neq 0$, which is achieved by using an auxiliary parameter n and imposing $1 + nD_i = 0$.

```
K:=Rationals();
R<a,q1,q2,m,d1,d2,n>:=PolynomialRing(K,7);
P<x,y,z>:=ProjectiveSpace(R,2);
```

The defining polynomial of Q, depending on one parameter:

```
F:=(x^2 + x*y + y^2 - x*z - y*z)^2 + a*x*y*(x + y - z)^2;
```

The points p_6, p_7 are in a line $y = mx$, hence they are of the form

```
p6:=[q1,m*q1,1];
p7:=[q2,m*q2,1];
```

and we must have the vanishing of

```
P1:=Evaluate(F,[q1,m*q1,1]);
P2:=Evaluate(F,[q2,m*q2,1]);
```

The defining polynomials of lines through that points are:

```
L1:=-y+d1*x+(m*q1-d1*q1)*z;
L2:=-y+d2*x+(m*q2-d2*q2)*z;
```

We need to impose that these lines are not tangent to Q at p_6, p_7, thus the following matrices must be of rank 2.

```
M1:=Matrix([JacobianSequence(F),JacobianSequence(L1)]);
M1:=Evaluate(M1,[q1,m*q1,1]);
M2:=Matrix([JacobianSequence(F),JacobianSequence(L2)]);
M2:=Evaluate(M2,[q2,m*q2,1]);
```

The matrix M_i is of rank 2 if one of its minors is non-zero. Here we make a choice for these minors, but in order to cover all cases the computations must be repeated for all other choices.

```
D1:=Minors(M1,2)[1];
D2:=Minors(M2,2)[1];
```

Now we intersect the quartic Q with the lines L_1, L_2 :

```
R1:=Evaluate(F,y,d1*x+(m*q1-d1*q1)*z);
R2:=Evaluate(F,y,d2*x+(m*q2-d2*q2)*z);
```

and we use Lemma 16 to impose that these lines are tangent to Q at flex points of Q:

```
c:=Coefficients(R1);
P3:=c[1]*c[5]-1/4*c[2]*c[4]+1/12*c[3]^2;
P4:=c[1]*c[4]^2+c[2]^2*c[5]-c[2]*c[3]*c[4]+8/27*c[3]^3;
c:=Coefficients(R2);
P5:=c[1]*c[5]-1/4*c[2]*c[4]+1/12*c[3]^2;
P6:=c[1]*c[4]^2+c[2]^2*c[5]-c[2]*c[3]*c[4]+8/27*c[3]^3;
```

We note that the lines L_1, L_2 cannot contain the points p_2, p_3 :

```
D3:=Evaluate(L1,[0,1,1]);
D4:=Evaluate(L1,[1,0,1]);
D5:=Evaluate(L2,[0,1,1]);
D6:=Evaluate(L2,[1,0,1]);
```

Also the line L_i cannot contain the point p_1, $i = 1, 2$:

```
D7:=(m-d1)*(m-d2);
```

And it is clear that the following must be non-zero:

```
D8:=a*q1*q2*(q1-q2);
```

Finally we define a scheme with all these conditions.

```
A:=AffineSpace(R);
S:=Scheme(A,[P1,P2,P3,P4,P5,P6,1+n*D1*D2*D3*D4*D5*D6*D7*D8]);
```

We compute (that takes a few hours):

```
PrimeComponents(S);
```

and get the unique solution $a = -8$. □

From the equation of the quartic $Q = M_{\mathbb{P}^2}$, one can compute a degree 24 equation for the mirror M_{48}, which is:

```
(31072410*r+44060139)*x^24+(599304420*r-4660302600)*x^21*y+(-106415505000*r+18054913500)*x^18
*y^2+(796474485000*r+3638808225000)*x^15*y^3+(-27123660*r-18697014)*x^16*z+(34521715125000
*r-31210968093750)*x^12*y^4+(107726220*r+2948918400)*x^13*y*z+(-257483985484500*r-
516632817969000)*x^9*y^5+(42798843000*r-32351244300)*x^10*y^2*z+(-1747212737190000*r
+3228789525752500)*x^6*y^6+(-407331396000*r-935091495000)*x^7*y^3*z+(-655139025450000*r+
10855982580975000)*x^3*y^7+(7724970*r-2222037)*x^8*z^2+(-3383703150000*r+9052448883750)
*x^4*y^4*z+(1544666220033750*r+11942493993804375)*y^8+(-102498120*r-465161400)*x^5*y*z^2+
(-319463676000*r+12613760073000)*x*y^5*z+(-2705586000*r+7086771600)*x^2*y^2*z^2+(-712080*r
+1186268)*z^3=0
```

where $r = \sqrt{-2}$.

5.4. A configuration of four plane conics related to the orbifold ball quotient.
In this subsection we describe the configuration of conics which we announced in the introduction.

Let us consider a conic tangent to two lines of a triangle in $\mathbb{P}^2$, and going through two points of the remaining line. Performing a Cremona transformation at the three vertices of the triangle one obtains a quartic curve in $\mathbb{P}^2$ with singularities $\mathfrak{a}_1 + 2\mathfrak{a}_2$. Conversely, starting with such a quartic, its image by the Cremona transform at the three singularities is a conic with three lines having the above configuration.

Thus we consider the Cremona transform φ at the three singularities of the quartic $M_{\mathbb{P}^2}$. Let $D_1, \ldots, D_4$ be respectively the images of $M_{\mathbb{P}^2}$, the line L_0 through the node and the two residual points of the flex lines, and the two flex lines. Using Magma, we see that these are 4 conics meeting in 10 points, as follows:

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}
D_1	1+	1+	0	0	0	0	1	1	1	1
D_2	1	1	1	1	1	0	0	0	1	1
D_3	0	1+	1	1	1	1	1	0	0	0
D_4	1+	0	1	1	1	1	0	1	0	0

Here two + in the column of q_j mean that the two curves meet with multiplicity 3 at point q_i. The other intersections are transverse. We see that the various ball-quotient orbifolds that Deraux described in [14] may be obtained from a configuration of conics by performing birational transformations.

6. One further quotient by an involution

6.1. The quotient morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, image of the mirror as the cuspidal cubic.
Consider the plane quartic curve Q from Theorem 14. Here we show the existence of a birational map

$$\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$$

and an involution σ on $\mathbb{P}^1 \times \mathbb{P}^1$ that preserves $\rho(Q)$ and fixes the diagonal D of $\mathbb{P}^1 \times \mathbb{P}^1$ pointwise. Moreover, we have $(\mathbb{P}^1 \times \mathbb{P}^1) /\sigma = \mathbb{P}^2$, and the images C_u, C_o of $\rho(Q)$, D are curves of degrees 3, 2, respectively. The curve C_u has a cusp singularity and intersects C_o at three points, with intersection multiplicities $4, 1, 1$. The map ρ is the inverse of the birational transform $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ described in sub-section 5.1.3, whose indeterminacy is at the singularity $\mathfrak{a}_3$ of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$.

```
K:=Rationals();
R<r>:=PolynomialRing(K);
K<r>:=ext<K|r^2+2>;
P2<x,y,z>:=ProjectiveSpace(K,2);
Q:=Curve(P2,(x^2+x*y+y^2-x*z-y*z)^2-8*x*y*(x+y-z)^2);
p6:=P2![2*r,-2*r,1];
```

```
p7:=P2![-2*r,2*r,1];
```

We compute the linear system of conics through the cuspidal points p_2, p_3 and take the corresponding map to $\mathbb{P}^3$.

```
L:=LinearSystem(LinearSystem(P2,2),[p6,p7]);
P3<a,b,c,d>:=ProjectiveSpace(K,3);
rho:=map<P2->P3|Sections(L)>;
```

The image of $\mathbb{P}^2$ is a quadric surface Q_2 ($\cong \mathbb{P}^1 \times \mathbb{P}^1$).

```
Q2:=rho(P2);Q2;
C:=rho(Q);C;
```

There is an involution preserving both Q_2 and the curve $C := \rho(Q)$.

```
sigma:=map<P3->P3|[d,b,c,a]>;
C:=rho(Q);C;
sigma(Q2) eq Q2;
sigma(C) eq C;
```

We compute the corresponding map to the quotient. The image of C is a cubic curve, and the image of the diagonal is a conic.

```
psi:=map<P3->P2|[a+d,b,c]>;
Cu:=psi(C);
Co:=psi(Scheme(rho(P2),[a-d]));
Co:=Curve(P2,DefiningEquations(Co));
```

The curve C_u has a cusp singularity:

```
pts:=SingularPoints(Cu);
ResolutionGraph(Cu,pts[1]);
```

The intersections of C_o and C_u :

```
Degree(ReducedSubscheme(Co meet Cu)) eq 3;
pt:=Points(Co meet Cu)[1];
IntersectionNumber(Co,Cu,pt) eq 4;
```

Let C_1', C_2' be the fibers that intersect $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ each at a unique point with multiplicity 3.These fibers are exchanged by the involution σ and are sent to a line F_l which cuts the cubic curve C_u at a unique point: this is a flex line. That line F_l also cuts the conic C_o at a unique point.

Conversely, let us start from the data of a conic C_o and a cuspidal cubic C_u intersecting as above, with the flex line (at the smooth flex point) of the cubic tangent to the conic. One can take the double cover of the plane branched over C_o, which is $\mathbb{P}^1 \times \mathbb{P}^1$. The pull-back of C_u is then a curve satisfying the properties of Theorem 14, thus the configuration (C_o, C_u) we described is unique in $\mathbb{P}^2$, up to projective automorphisms.

6.2. **An orbifold ball-quotient structure from** $(\mathbb{P}^2, (C_o, C_u))$**.** Let $C_u \hookrightarrow \mathbb{P}^2$ be the unique plane cuspidal curve and let c_1 be its cuspidal point. Let F_l be the flex line through the unique smooth flex point c_2 of C_u. By the previous subsection, one has the following result:

Proposition 17. *There exists a unique conic $C_o \hookrightarrow \mathbb{P}^2$ such that the following holds:*
i) F_l is tangent to C_o;
ii) C_o cuts C_u at points c_3, c_4, c_5 ($\neq c_1, c_2$) with intersection multiplicities $4, 1, 1$, respectively.

In this subsection we prove that there is a natural birational transformation $W \dashrightarrow \mathbb{P}^2$ such that together with the strict transform of the curves C_o and C_u one gets an orbifold ball quotient surface. For definitions and results on orbifold theory, we use [8, 11] and [29].

Let us blow-up over points c_1, c_2, c_3 and then contract some divisors as follows (for a pictural description see figure 6.1):

We blow up over c_1 three times, the first blow-up resolves the cusp of C_u and the exceptional divisor intersects the strict transform of C_u tangentially, the second blow-up is at that point of tangency and the third blow-up separates the strict transforms of the first exceptional divisor and the curve C_u. One obtains in that way a chain of (-3), (-1) and (-2)-curves. We then contract the (-2) and (-3)-curves obtaining in that way singularities A_1 and $\frac{1}{3}(1,1)$. The image of the (-1)-curve by that contraction map is denoted by H. As an orbifold we put multiplicity 2 on H.

We blow up over c_2 (the flex point) three times in order that the strict transform of the curves F_l and C_u get separated over c_2. We obtain in that way a chain of (-1), (-2), (-2)-curves. We then contract the two (-2)-curves and obtain an A_2-singularity. The strict transform of the line F_l is a (-2)-curve, which we also contract, obtaining in that way an A_1-singularity. The contracted curve being tangent to $\tilde{C}_0$, the image $\bar{C}_0$ has a cusp $\mathfrak{a}_2$ at the singularity A_1.

We moreover blow up over c_3 four times, in order that the strict transform of the curves C_o and C_u get separated over c_3. We obtain in that way a chain of (-1), (-2), (-2), (-2)-curves. We then contract the three (-2)-curves and obtain an A_3-singularity. The image of the (-1)-curve by the contraction map is a curve denoted by F_d, we give the weight 2 to that curve.

FIGURE 6.1. The plane, the surfaces Z and W
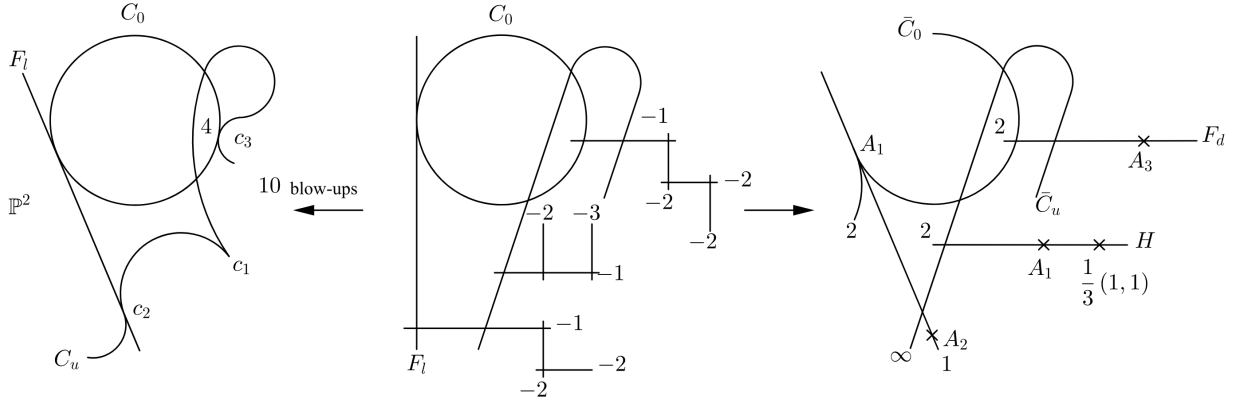
Let us denote by W the resulting surface. For a curve D on $\mathbb{P}^2$, we denote by $\tilde{D}$ its strict transform on W. Let $\mathcal{W}$ be the orbifold with same subjacent topological space, with divisorial part:

$$\Delta = \left(1 - \frac{1}{\infty}\right)\bar{C}_u + \left(1 - \frac{1}{2}\right)\left(\bar{C}_o + F_d + H\right).$$

The singular points of W are

$$A_1 + A_1 + A_2 + A_3 + \frac{1}{3}(1,1),$$

and they have an isotropy β of order $16, 4, 3, 8, 6$ respectively, for $\mathcal{W}$. The computation of the isotropy is immediate, except for the first point (that we shall denote by r_1), which is also a cusp on the curve $\bar{C}_0$ (which has weight 2). Let SD_{16} be the the semidihedral group of order 16, generated by the matrices

$$g_1 = \begin{pmatrix} 0 & -\zeta \\ -\zeta^3 & 0 \end{pmatrix}, \ g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ζ is a primitive 8th root of unity. The order 2 elements g_2, $g_1^{-1}g_2g_1$ generate an order 8 reflection group D_4. The quotient of $\mathbb{C}^2$ by SD_{16} has a A_1 singularity and one computes that the image of the 4 mirrors of D_4 is a curve with a cusp $\mathfrak{a}_2$ at the A_1 singularity of $\mathbb{C}^2/SD_{16}$. The isotropy group of the point r_1 in the orbifold is therefore the semidihedral group SD_{16} of order 16. The following proposition is an application of the main result of [21]:

Proposition 18. *The Chern numbers of the orbifold* $\mathcal{W} = (W, \Delta)$ *satisfy*

$$c_1^2(\mathcal{W}) = 3c_2(\mathcal{W}) = \frac{9}{16},$$

in particular $\mathcal{W}$ *is an orbifold ball quotient.*

Proof. Let us compute the orbifold second Chern number of $\mathcal{W}$. We have (see e.g. [27]):

$$c_2(\mathcal{W}) = e(W) - \left((1 - \tfrac{1}{\infty})e(\bar{C}_u \setminus S) + (1 - \tfrac{1}{2})e(\bar{C}_o \setminus S) + \right.$$
$$\left. + (1 - \tfrac{1}{2})e(F_d \setminus S) + (1 - \tfrac{1}{2})e(H \setminus S)\right) - \sum_{p \in S}\left(1 - \tfrac{1}{\beta(p)}\right),$$

where S is the union of the singular points of W with the singular points of the round-up divisor $\lceil \Delta \rceil$, and where moreover $\beta(p)$ is the isotropy order of the point p, so that for example for p on $\bar{C}_u$, $\beta(p) = \infty$ and the unique point p in F_d and $\bar{C}_o$ has $\beta(p) = 4$. Since we have blown-up $\mathbb{P}^2$ over 10 points and we have contracted 8 rational curves, we get

$$e(W) = 3 + 10 - 8 = 5.$$

We obtain

$$c_2(\mathcal{W}) = 5 - \left((2 - 4) + \frac{1}{2}(2 - 4) + \frac{1}{2}(2 - 3) + \frac{1}{2}(2 - 3)\right)$$
$$- \left(10 - \frac{1}{16} - \frac{1}{4} - \frac{1}{3} - \frac{1}{8} - \frac{1}{6} - \frac{1}{4} - 4 \cdot \frac{1}{\infty}\right),$$

thus $c_2(\mathcal{W}) = \frac{3}{16}$.

Let us compute $c_1^2(\mathcal{W})$. One has

$$c_1^2(\mathcal{W}) = (K_W + \Delta)^2,$$

so that

$$c_1^2(\mathcal{W}) = K_W^2 + 2K_W\bar{C}_u + K_W(\bar{C}_o + F_d + H) + \tfrac{1}{4}(\bar{C}_o^2 + F_d^2 + H^2) + \bar{C}_u^2$$
$$+ \bar{C}_u(\bar{C}_o + F_d + H) + \tfrac{1}{2}(\bar{C}_oF_d + \bar{C}_oH + F_dH).$$

Let $p : Z \to W$ be the surface above W which resolves W and is a blow-up of $\mathbb{P}^2$. Since Z is obtained by 10 blow-ups of $\mathbb{P}^2$ one has $K_Z^2 = 9 - 10 = -1$. Moreover, since all singularities but one are ADE, one has $K_Z = p^*K_W - \frac{1}{3}D_1$ where D_1 is the (-3)-curve on Z which is contracted to the $\frac{1}{3}(1,1)$ singularity on W. Since $p^*K_W \cdot D_1 = 0$, we obtain

$$K_W^2 = -\frac{2}{3}.$$

The curve $\bar{C}_u$ is a smooth curve of genus 0 on the smooth locus of W. The blow-up at the $\mathfrak{a}_2$-singularity of the cuspidal cubic decreases the self-intersection by 4, the remaining blow-ups decrease the self-intersection by 1. Since one has $4 + 2 + 3 = 9$ such blow-ups, one gets

$$\bar{C}_u^2 = 3^2 - 4 - 9 = -4,$$

and therefore $K_W\bar{C}_u = 2$. Let $\tilde{D}$ be the strict transform on Z of a curve D on W or $\mathbb{P}^2$. We have

$$\tilde{C}_o = p^*\bar{C}_o - aF_l.$$

Since $\tilde{C}_o F_l = 2$, then a is equal to 1. Since moreover $\tilde{C}_o^{\,2} = 0$, we get $0 = (\tilde{C}_o)^2 = \bar{C}_o^2 - 2$, thus $\bar{C}_o^2 = 2$. We have

$$K_W \bar{C}_o = (\tilde{C}_o + F_l)\left(K_W + \frac{1}{3}D_1\right) = -2.$$

Let $F_1, F_2, F_3 \subset Z$ be the chain of three (-2)-curves above the A_3 singularity in W, so that $\tilde{F}_d F_1 = 1$. One computes that

$$\tilde{F}_d = p^* F_d - \frac{1}{4}(3F_1 + 2F_2 + F_3)$$

(it is easy to check that $\tilde{F}_d F_1 = 1$, $\tilde{F}_d F_2 = \tilde{F}_d F_3 = 0$). Then

$$-1 = \tilde{F}_d^2 = F_d^2 - \frac{3}{4}$$

gives $F_d^2 = -\frac{1}{4}$. One has

$$K_W F_d = \left(K_Z + \frac{1}{3}D_1\right)\left(\tilde{F}_d + \frac{1}{4}(3F_1 + 2F_2 + F_3)\right) = -1.$$

Let D_1, D_2 be respectively the (-3) and (-2) curves intersecting $\tilde{H}$. Since $\tilde{H}D_1 = \tilde{H}D_2 = 1$, one has

$$\tilde{H} = p^* H - \frac{1}{3}D_1 - \frac{1}{2}D_2,$$

thus

$$-1 = \tilde{H}^2 = H^2 - \frac{1}{3} - \frac{1}{2}$$

and $H^2 = -\frac{1}{6}$. Moreover

$$K_W H = \left(K_Z + \frac{1}{3}D_1\right)\left(\tilde{H} + \frac{1}{3}D_1 + \frac{1}{2}D_2\right) = -1 + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = -\frac{2}{3}.$$

We compute therefore

$$c_1^2(\mathcal{W}) = -\frac{2}{3} + 2 \cdot 2 + \left(-2 - 1 - \frac{2}{3}\right) + \frac{1}{4}\left(2 - \frac{1}{4} - \frac{1}{6}\right) - 4$$
$$+ (2 + 1 + 1) + \frac{1}{2}(1 + 0 + 0) = \frac{9}{16},$$

thus $c_1^2(\mathcal{W}) = 3c_2(\mathcal{W}) = \frac{9}{16}$. $\square$

Remark 19. In [14], Deraux obtains 4 different orbifold ball-quotient structures on surfaces birational to A/G_{48}. Among these, only the fourth one, W', is invariant by the involution σ, the obstruction being the divisor E in [14] which creates an asymmetry, unless it has weight 1. The orbifold $\mathcal{W}$ we just described can be seen as the quotient of W' by the involution σ.

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