MAXIMAL REPRESENTATIONS OF UNIFORM COMPLEX HYPERBOLIC LATTICES

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ABSTRACT. Let ρ be a maximal representation of a uniform lattice $\Gamma \subset \mathrm{SU}(n,1)$, $n \geq 2$, in a classical Lie group of Hermitian type G. We prove that necessarily $G = \mathrm{SU}(p,q)$ with $p \geq qn$ and there exists a holomorphic or antiholomorphic ρ-equivariant map from the complex hyperbolic space to the symmetric space associated to $\mathrm{SU}(p,q)$. This map is moreover a totally geodesic homothetic embedding. In particular, up to a representation in a compact subgroup of $\mathrm{SU}(p,q)$, the representation ρ extends to a representation of $\mathrm{SU}(n,1)$ in $\mathrm{SU}(p,q)$.

CONTENTS

1. Introduction

Lattices in non compact simple Lie groups can be regrouped in two broad classes: those which are superrigid and those which are not. A lattice Γ in a simple noncompact Lie group H is superrigid (over $\mathbb{R}$ or $\mathbb{C}$) if for all simple noncompact Lie group G with trivial center, every homomorphism $\Gamma \to G$ with Zariski-dense image extends to a homomorphism $H \to G$. Lattices in simple Lie groups of real rank at least 2, such as $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$, as well as lattices in the real rank 1 Lie groups $\mathrm{Sp}(n, 1)$ and F_4^{-20}, are superrigid by [Mar91, Cor92, GS92]. This implies that these lattices are all arithmetic. On the other hand, lattices in the remaining simple Lie groups of real rank 1, $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$, are not superrigid in general. In particular, the study of their representations does not reduce to the study of the representations of the Lie group they live in. There are however important differences between real hyperbolic lattices, i.e. lattices in $\mathrm{SO}(n, 1)$, and complex hyperbolic lattices, i.e. lattices in $\mathrm{SU}(n, 1)$. Real hyperbolic objects are softer and more flexible than their complex counterparts. From the perspective of representations of lattices, for example, it is sometimes possible to deform non trivially lattices of $\mathrm{SO}(n, 1)$ in $\mathrm{SO}(m, 1)$, $m > n \geq 3$, see e.g. [JM87]. The analogous statement does not hold for lattices in $\mathrm{SU}(n, 1)$, $n \geq 2$: W. Goldman and J. Millson [GM87] proved that if $\Gamma \in \mathrm{SU}(n, 1)$, $n \geq 2$, is a uniform lattice and if $\rho : \Gamma \to \mathrm{SU}(m, 1)$, $m \geq n$, is the composition of the inclusion $\Gamma \hookrightarrow \mathrm{SU}(n, 1)$ with the natural embedding $\mathrm{SU}(n, 1) \hookrightarrow \mathrm{SU}(m, 1)$, then ρ, although not necessarily infinitesimally rigid, is locally rigid. From a maybe more subjective point of view, non arithmetic lattices in $\mathrm{SO}(n, 1)$ can be constructed for all n [GPS88] but there are no similar constructions in the complex case and examples of non arithmetic lattices in $\mathrm{SU}(n, 1)$ are very difficult to come by (and none are known for $n \geq 4$).

We will be interested here in global rigidity results for representations of complex hyperbolic lattices in semisimple Lie groups of Hermitian type with no compact factors which generalize the local rigidity we just mentioned. Recall that a Lie group G is of Hermitian type if its associated symmetric space is a Hermitian symmetric space. The classical noncompact groups of Hermitian type are $\mathrm{SU}(p, q)$ with $p \geq q \geq 1$, $\mathrm{SO}_0(p, 2)$ with $p \geq 3$, $\mathrm{Sp}(m, \mathbb{R})$ with $m \geq 2$ and $\mathrm{SO}^\star(2m)$ with $m \geq 4$.

Let Γ be a lattice in $\mathrm{SU}(n, 1)$. The group Γ acts on complex hyperbolic n-space $\mathbb{H}_\mathbb{C}^n = \mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$. The space $\mathbb{H}_\mathbb{C}^n$ is the rank 1 Hermitian symmetric space of non compact type and of complex dimension n. From a Riemannian point of view, it is up to isometry the unique complete simply connected Kähler manifold of constant negative holomorphic sectional curvature. The $\mathrm{SU}(n, 1)$-invariant metric on $\mathbb{H}_\mathbb{C}^n$ will be normalized so that its holomorphic sectional curvature is -1. As a bounded symmetric domain, $\mathbb{H}_\mathbb{C}^n$ is biholomorphic to the unit ball in $\mathbb{C}^n$.

For simplicity in this introduction, and because this is needed in our main result, the lattice Γ is assumed to be uniform (and torsion free) unless otherwise specified, so that the quotient $X := \Gamma \backslash \mathbb{H}_\mathbb{C}^n$ is a compact Kähler manifold.

Let also G be a semisimple Lie group of Hermitian type without compact factors, $\mathcal{Y}$ the symmetric space associated to G and ρ a representation of Γ in G, i.e. a group homomorphism $\rho : \Gamma \to G$. There is a natural way to measure the "complex size" of the representation ρ by using the invariant Kähler forms of the involved symmetric spaces. The *Toledo invariant* of ρ is defined as follows:

$$\tau(\rho) = \frac{1}{n!} \int_X f^\star \omega_\mathcal{Y} \wedge \omega^{n-1},$$

where $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}$ is any ρ-equivariant map, ω is the Kähler form of X coming from the invariant Kähler form of $\mathbb{H}^n_{\mathbb{C}}$, $\omega_{\mathcal{Y}}$ is the G-invariant Kähler form of $\mathcal{Y}$ normalized so that its holomorphic sectional curvatures are in $[-1, -1/\mathrm{rk}\mathcal{Y}]$, and $f^\star\omega_{\mathcal{Y}}$ is understood as a 2-form on X.

It should be noted that ρ-equivariant maps $\mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}$ always exist, because $\mathcal{Y}$ is contractible, and that any two such maps are equivariantly homotopic, so that the Toledo invariant depends only on ρ, not on the choice of f. In fact, it depends only on the connected component of $\mathrm{Hom}(\Gamma, G)$ containing ρ, because it can be seen as a characteristic class of the flat bundle on X associated to ρ. The definition of the Toledo invariant can be extended to non uniform lattices with a bit more work.

A fundamental fact about the Toledo invariant that was established in full generality by M. Burger and A. Iozzi in [BI07] is that it satisfies the following *Milnor-Wood type inequality*:

$$|\tau(\rho)| \leq \mathrm{rk}(\mathcal{Y}) \, \mathrm{vol}(X).$$

This allows to single out a special class of representations, namely those for which this inequality is an equality. These are the *maximal representations* we are interested in.

The Toledo invariant was first considered for representations of surface groups, i.e. when Γ is the fundamental group of a closed Riemann surface, which can be seen as a uniform lattice in $\mathrm{SU}(1, 1)$. It appeared for the first time in D. Toledo's 1979 paper [Tol79] and more explicitly in [Tol89], where the Milnor-Wood inequality was proved for $n = 1$ and $\mathrm{rk}\,\mathcal{Y} = 1$, namely when $G = \mathrm{SU}(m, 1)$ for some $m \geq 1$. Toledo proved that maximal representations are faithful with discrete image, and stabilize a complex line in complex hyperbolic m-space, thus generalizing a theorem of Goldman for $G = \mathrm{SL}(2, \mathbb{R})$ [Gol80, Gol88]. Analogous results in the non uniform case were proved in [BI07, KM08a]. L. Hernandez showed in [Her91] that maximal representations of surface groups in $G = \mathrm{SU}(p, 2)$, $p \geq 2$, are also discrete and faithful and stabilize a symmetric subspace associated to the subgroup $\mathrm{SU}(2, 2)$ in $\mathcal{Y}$. Maximal representations of surface groups are now known to be reductive, discrete and faithful, to stabilize a maximal tube type subdomain in $\mathcal{Y}$, and in general to carry interesting geometric structures, see e.g. [BIW10, GW12]. They are nevertheless quite flexible. They can for example always be deformed to representations that are Zariski-dense in the subgroup corresponding to the tube type subdomain they stabilize [BIW10].

On the other hand, as indicated by the local rigidity result of [GM87], maximal representations of higher dimensional complex hyperbolic lattices, that is, lattices in $\mathrm{SU}(n, 1)$ for n greater than 1, are expected to be much more rigid.

This was confirmed for rank 1 targets by K. Corlette in [Cor88] (the statement was given for representations maximizing the so-called volume instead of the Toledo invariant but the proof for the Toledo invariant is essentially the same). Corlette proved that if ρ is a volume-maximal representation of a uniform lattice $\Gamma \subset \mathrm{SU}(n, 1)$, $n \geq 2$, in $G = \mathrm{SU}(m, 1)$, then there exists a ρ-equivariant holomorphic totally geodesic embedding $\mathbb{H}^n_{\mathbb{C}} \to \mathbb{H}^m_{\mathbb{C}}$. This answered a conjecture of Goldman and Millson and implies the local rigidity of [GM87]. This was later shown to hold also in the case of non uniform lattices [BI08, KM08a].

For $n \geq 2$ and higher rank targets, the situation was until now far from being well understood. The case of real rank 2 target Lie groups has been treated in [KM08b] (for uniform lattices), but the proof did not go through to higher ranks. In [BIW09], M. Burger, A. Iozzi and A. Wienhard proved that maximal representations are necessarily reductive (this holds also for $n = 1$ and without assuming the lattice to be uniform). Very recently, M. B. Pozzetti [Poz15] succeeded in generalizing the approach of [BI08] and proved that for $n \geq 2$ there are no Zariski dense maximal representations of a lattice $\Gamma \subset \mathrm{SU}(n, 1)$ in $\mathrm{SU}(p, q)$

if $p > q > 1$. There is no rank restriction in her result, and it is also valid for non uniform lattices, but as of now it seems to depend strongly on having a non tube type target (this is the meaning of the assumption $p \neq q$).

In this paper, we prove the expected global rigidity for maximal representations of uniform lattices of $\mathrm{SU}(n,1)$, $n \geq 2$, in all classical Lie groups of Hermitian type:

Theorem 1.1. *Let Γ be a uniform (torsion free) lattice in $\mathrm{SU}(n,1)$, $n \geq 2$. Let ρ be a group homomorphism of Γ in a classical noncompact Lie group of Hermitian type G, i.e. G is either $\mathrm{SU}(p,q)$ with $p \geq q \geq 1$, $\mathrm{SO}_0(p,2)$ with $p \geq 3$, $\mathrm{Sp}(m,\mathbb{R})$ with $m \geq 2$, or $\mathrm{SO}^\star(2m)$ with $m \geq 4$.*

If ρ is maximal, then $G = \mathrm{SU}(p,q)$ with $p \geq qn$, ρ is reductive and there exists a holomorphic or antiholomorphic ρ-equivariant map from $\mathbb{H}^n_{\mathbb{C}}$ to the symmetric space $\mathcal{Y}_{p,q}$ associated to $\mathrm{SU}(p,q)$.

As a consequence, maximal representations can be described completely:

Corollary 1.2. *Let $n \geq 2$ and $p \geq qn$. Let $\rho : \Gamma \to \mathrm{SU}(p,q)$ be a maximal representation of a uniform torsion free lattice $\Gamma \subset \mathrm{SU}(n,1)$. Then:*

– the ρ-equivariant holomorphic or antiholomorphic map $\mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}_{p,q}$ whose existence is guaranteed by Theorem 1.1 is a totally geodesic homothetic embedding; it is unique up to composition by an element of $\mathrm{SU}(p,q)$;

– the representation ρ is faithful, discrete, and $\rho(\Gamma)$ stabilizes (and acts cocompactly on) a totally geodesic image of $\mathbb{H}^n_{\mathbb{C}}$ in $\mathcal{Y}_{p,q}$, of induced holomorphic sectional curvature $-\frac{1}{q}$;

– up to conjugacy, the representation ρ is a product $\rho_{\mathrm{diag}} \times \rho_{\mathrm{cpt}}$, where ρ_{diag} is the standard diagonal embedding $\mathrm{SU}(n,1) \hookrightarrow \mathrm{SU}(n,1)^q \hookrightarrow \mathrm{SU}(nq,q) \hookrightarrow \mathrm{SU}(p,q)$, and ρ_{cpt} is a representation of Γ in the centralizer of $\rho_{\mathrm{diag}}(\mathrm{SU}(n,1))$ in $\mathrm{SU}(p,q)$, which is compact.

Because as we said the Toledo invariant is constant on connected components of $\mathrm{Hom}(\Gamma, G)$, this also implies the local rigidity of maximal representations and in particular we have:

Corollary 1.3. *Let $n \geq 2$ and $p \geq qn$. Then the restriction to a uniform lattice $\Gamma \subset \mathrm{SU}(n,1)$ of the standard diagonal embedding $\rho_{\mathrm{diag}} : \mathrm{SU}(n,1) \hookrightarrow \mathrm{SU}(n,1)^q \hookrightarrow \mathrm{SU}(p,q)$ is locally rigid (up to a representation in the compact centralizer of $\rho_{\mathrm{diag}}(\mathrm{SU}(n,1))$ in $\mathrm{SU}(p,q)$).*

This last corollary is in fact true without assuming the lattice Γ to be uniform [Poz15, Corollary 1.5]. It is also a special case of the main result of [Kli11], where B. Klingler gave a general algebraic condition for representations of uniform lattices in $\mathrm{SU}(n,1)$ induced by representations of $\mathrm{SU}(n,1)$ to be locally rigid.

To prove Theorem 1.1, we work with a reductive representation $\rho : \Gamma \to G$ (non reductive representations can be ruled out a priori by [BIW09], or later, see §4.5) and we consider the *harmonic Higgs bundle* (E, θ) on the closed complex hyperbolic manifold $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ associated to ρ by the work of K. Corlette [Cor88] and C. Simpson [Sim92]. This Higgs bundle is *polystable* and has a *real structure* which comes from the fact that it is constructed out of a representation in a Lie group of Hermitian type (and not merely in the general linear group). The Toledo invariant is interpreted in this setting as the degree of a vector bundle on X. See §2.1, §4.2 and §4.3.1. These facts can be used in some situations to (re)prove the Milnor-Wood inequality and study maximal representations. This has been widely done for representations of surface groups, see e.g. [Xia00, MX02, BGPG03, BGPG06], and also, with limited success, for higher dimensional lattices [KM08b].

The main novelty here is the study of the interplay between the Higgs bundle point of view and the geometry and dynamics of the *tautological foliation* $\mathcal{T}$ on the projectivized tangent bundle $\mathbb{P}T_X$ of the complex hyperbolic manifold X.

When the base (Kähler) manifold Y of a harmonic Higgs bundle $(E, \theta) \to Y$ comes with a smooth holomorphic foliation $\mathcal{T}$ by complex curves, and this foliation admits an invariant transverse measure, one can investigate the behaviour of the Higgs bundle *along the leaves* of $\mathcal{T}$. This is the content of §2.2. The transverse measure indeed yields a closed current of integration and we define the *foliated degree* of a coherent sheaf on Y by integrating its first Chern class against it. We call a subsheaf of $\mathcal{O}_Y(E)$ a *leafwise Higgs subsheaf* of E if it is invariant by the Higgs field θ in the directions tangent to the leaves. With these definitions we introduce notions of *leafwise semistability* and *leafwise polystability* and we prove (Proposition 2.2) that they are satisfied by the Higgs bundle (E, θ) when the invariant transverse measure is induced by an invariant transverse volume form.

Now, there is a well-defined notion of *complex geodesics* in complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. This implies that the projectivized tangent bundle $\mathbb{P}T_X$ of the complex hyperbolic manifold $X = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ carries a smooth holomorphic 1-dimensional foliation $\mathcal{T}$ by lifts of tangent spaces of (local) complex geodesics, see §3.1. The tangential line subbundle L of the tangent bundle of $\mathbb{P}T_X$, i.e. the subbundle of tangent vectors tangent to the leaves of the foliation, identifies naturally with the tautological line bundle $\mathcal{O}_{\mathbb{P}T_X}(-1)$ on $\mathbb{P}T_X$. The tautological foliation is endowed with a homogeneous transverse structure, where the $\mathrm{SU}(n, 1)$-homogeneous space in question is the space $\mathcal{G}$ of complex geodesics of $\mathbb{H}^n_{\mathbb{C}}$. This space supports an invariant indefinite but non degenerate Kähler metric $\omega_{\mathcal{G}}$, hence an invariant volume form which defines a transverse measure $\mu_{\mathcal{G}}$ for the foliation $\mathcal{T}$, cf. §3.2. The fundamental feature of the induced current of integration is that it enables to compute the Toledo invariant of the representation ρ and degrees of vector bundles on X as foliated degrees of vector bundles on $\mathbb{P}T_X$ (Proposition 3.1).

The idea is then to pull-back the Higgs bundle $(E, \theta) \to X$ associated to the representation ρ to obtain a harmonic Higgs bundle $(\tilde{E}, \tilde{\theta})$ over the projectivized tangent bundle $\mathbb{P}T_X$ and to take advantage of the leafwise stability properties of this new Higgs bundle with respect to the tautological foliation $\mathcal{T}$ and its invariant transverse measure $\mu_{\mathcal{G}}$. This allows to give a new proof of the Milnor-Wood inequality for reductive representations of uniform lattices and to gain a lot of information in the maximal case, see e.g. §4.3 for representations in $\mathrm{SU}(p, q)$. To conclude the proof one needs a dynamical argument to understand closures of projections to X of subsets of $\mathbb{P}T_X$ which are saturated under the tautological foliation. This is done using results of M. Ratner on unipotent flows, see §3.3.

The interpretation of the Toledo invariant as a "foliated Toledo number" is sketched by M. Burger and A. Iozzi in [BI08, p. 183], where it is attributed to F. Labourie. This point of view is indeed strongly related with their approach, and the one of M. B. Pozzetti, where one wants to prove that when a representation is maximal, there exists an equivariant measurable map between the Shilov boundaries that preserves a special incidence geometry. In the complex hyperbolic case, this incidence geometry is the geometry of *chains*, i.e. of boundaries at infinity of complex geodesics. Tautological foliations on the projectivized tangent bundle of manifolds carrying a holomorphic projective structure (in particular complex hyperbolic manifolds) are also discussed and used by N. Mok in [Mok05]. Some time ago, without at first grasping the foliated side of the story, the authors of the present paper made some quickly unsuccessful attempts at working with Higgs bundles on the projectivized tangent bundle. Reading F. Labourie's suggestion in [BI08] and N. Mok's article [Mok05] encouraged them to try again.

Combining foliations and Higgs bundle theory to prove rigidity properties of lattices is of course reminiscent of the work of M. Gromov on foliated harmonic maps in [Gro91a, Gro91b]. M. Gromov considered foliations by (lifts of) totally geodesic subspaces in (bundles over) locally symmetric spaces of which the tautological foliation discussed here is a particular

case. There is however a difference. In his application to quaternionic rigidity [Gro91b, §7.E] (see also [Cor92] for a different proof), M. Gromov uses his existence theorem for foliated harmonic maps to produce maps on quaternionic hyperbolic space which are harmonic along totally geodesic complex subspaces but not (a priori) harmonic on the whole space, because the harmonic map on the whole space is not (a priori) harmonic when restricted to these subspaces. In this paper, since we work on Kähler manifolds, and our leaves are complex curves, harmonic maps are pluriharmonic (see §2.1) and their restrictions to the leaves are automatically harmonic. Therefore foliated harmonic maps are not needed and neither is a fully fledged theory of foliated Higgs bundles (e.g. on real manifolds foliated by Kähler submanifolds), although such a theory would probably be interesting to develop.

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2. Higgs bundles on foliated Kähler manifolds

In this section, we first give a brief account on harmonic Higgs bundles on a compact Kähler manifold Y. When the manifold Y admits a holomorphic foliation by complex curves and the foliation admits an invariant transverse measure, we define a notion of foliated degree for $\mathcal{O}_Y$-coherent sheaves. If moreover the transverse measure is induced by an invariant transverse volume form, we exhibit some stability properties of the Higgs bundle with respect to the foliated degree.

2.1. **Harmonic Higgs bundles.**

Let Y be a compact manifold, Γ its fundamental group, and $\rho : \Gamma \to G$ a group homomorphism in a real algebraic semisimple Lie group without compact factors $G \subset \mathrm{SL}(N, \mathbb{C})$.

We assume in this section that ρ is *reductive*, i.e. the Zariski closure of $\rho(\Gamma)$ in G is a reductive group. By a fundamental result of K. Corlette [Cor88], this is equivalent to the existence of a ρ-equivariant *harmonic map f* from the universal cover $\tilde{Y}$ of Y to the symmetric space $\mathcal{Y}$ associated to G.

When the manifold Y is moreover Kähler, it follows from a Bochner formula due to J. H. Sampson [Sam86] and Y.-T. Siu [Siu80], that the harmonic map f is pluriharmonic, and that the image of the $(1,0)$-part $d^{1,0}f : T^{1,0}\tilde{Y} \to T^{\mathbb{C}}\mathcal{Y}$ of its complexified differential is Abelian (as a subspace of the complexification of the Lie algebra of G). This has been shown by C. Simpson [Sim88, Sim92] to give a *harmonic Higgs bundle (E, θ)* on Y.

The fact that the representation ρ takes its values in the real group G endows the Higgs bundle with a *real structure*. This real structure is important and will be discussed later in different particular cases. However, at this point it is not relevant and in this section we see ρ as a homomorphism in $\mathrm{SL}(N, \mathbb{C})$.

The bundle E, as a C^∞-bundle, is the flat complex vector bundle of rank N with holonomy ρ. The Higgs field θ is a holomorphic $(1,0)$-form with values in $\mathrm{End}(E)$, which can be seen as the $(1,0)$-part $d^{1,0}f$ of the complexified differential of the harmonic map f. It satisfies the *integrability condition* $[\theta, \theta] = 0$. The ρ-equivariant harmonic map can also be thought of as a reduction of the structure group of E to the maximal compact subgroup $\mathrm{SU}(N)$ of $\mathrm{SL}(N, \mathbb{C})$, see e.g. [Cor88]. Therefore choosing a $\mathrm{SU}(N)$-invariant Hermitian metric on a fiber

of E defines a Hermitian metric on E, called the *harmonic metric*, which has the following properties. If D is the flat connection on E and ∇ the component of D which preserves this metric, then $(\nabla'')^2 = 0$ and $\nabla''\theta = 0$, so that ∇'' defines a holomorphic structure on E for which θ is holomorphic. Moreover D, ∇ and θ are related by

$$D = \nabla + \theta + \theta^\star,$$

where $\theta^\star$ is the adjoint of θ w.r.t. the harmonic metric. This, together with the Chern-Weil formula, implies that (E, θ) is a *polystable* Higgs bundle of degree 0 on Y, see [Sim88] and the proof of Proposition 2.2 below. This means first that (E, θ) is a *semistable* Higgs bundle, namely that if $\mathcal{F} \subset \mathcal{O}_Y(E)$ is a Higgs subsheaf of E, i.e. a subsheaf such that $\theta(\mathcal{F} \otimes T_Y) \subset \mathcal{F}$, then

$$\deg \mathcal{F} := \frac{1}{m!} \int_Y c_1(\mathcal{F}) \wedge \omega_Y^{m-1} \leq 0 = \deg E$$

where m is the dimension and ω_Y the Kähler form of Y (the last equality holds because E is flat). Second, whenever $\mathcal{F}$ is a Higgs subsheaf of E of degree equal to 0, its saturation (see below) is the sheaf of sections of a holomorphic vector subbundle F of E stable by θ and the orthogonal complement $F^\perp$ of F w.r.t the harmonic metric is also a holomorphic subbundle of E stable by θ, so that we have a Higgs bundle orthogonal decomposition

$$(E, \theta) = (F, \theta_{|F}) \oplus (F^\perp, \theta_{|F^\perp}).$$

Remark 2.1. In general, when dealing with notions of stability of vector bundles, one uses slopes rather than degrees. However, in our case E is flat and there is no need to consider slopes.

2.2. **Higgs bundles and foliations.**

Assume now that the compact Kähler manifold Y, with its harmonic Higgs bundle (E, θ), also admits a smooth holomorphic foliation $\mathcal{T}$ by complex curves, and that this foliation has an invariant transverse invariant (positive) measure μ. Our goal in this section is to understand the behaviour of the Higgs bundle (E, θ) with respect to the foliation $\mathcal{T}$ and its transverse measure μ.

We begin by defining adapted notions of Higgs subsheaves and degree.

Let $L \subset T_Y$ be the tangential line field of the foliation $\mathcal{T}$, i.e. the holomorphic line subbundle of vectors which are tangent to the leaves of $\mathcal{T}$, and let $L^\vee$ be its dual. We restrict the Higgs field θ to L, i.e. we see it as a holomorphic section of $\mathrm{End}(E) \otimes L^\vee$.

A subsheaf $\mathcal{F} \subset \mathcal{O}_Y(E)$ is *invariant along the leaves* or is a *leafwise Higgs subsheaf* if the Higgs field θ maps $\mathcal{F} \otimes L$ to $\mathcal{F}$.

We make the observation that a Higgs subsheaf of (E, θ) is a leafwise Higgs subsheaf, but that the converse does not hold, so that there is no reason why the degree (computed w.r.t. the Kähler form ω_Y) of a leafwise Higgs subsheaf should be nonpositive.

The invariant transverse measure μ defines a closed current $\int_{\mathcal{T}, \mu}$ of bidegree $(m-1, m-1)$ on Y which is $\mathcal{T}$-invariant and positive since μ is (see [God91, V.3.5] and [Sul76]). Let indeed α be a 2-form on Y. Take a covering $(U_i)_{i \in I}$ of Y by regular open sets for the foliation $\mathcal{T}$, and a partition of unity $(\chi_i)_{i \in I}$ subordinated to it. Let T_i be the space of plaques of U_i and call again μ the measure on T_i given by the transverse measure. The forms $\chi_i \alpha$ are compactly supported in the open sets U_i and by integrating $\chi_i \alpha$ on the plaques of U_i, we obtain a compactly supported function on the space T_i which we can then integrate against the measure μ to get

$$\int_{\mathcal{T}, \mu} \alpha := \sum_{i \in I} \int_{T_i} \left(\int_t \chi_i \alpha \right) d\mu(t).$$

Invariant transverse measures to $\mathcal{T}$ can be atomic, for example when $\mathcal{T}$ admits a closed leaf C in which case the current is given by integration on C. Or they can be diffuse, for example when the foliation admits an invariant transverse volume form (cf. [God91, V.3.7(i)] for the definition) in which case there exists a smooth closed *basic* $(m-1, m-1)$-form Ω on Y such that $\int_{\mathcal{T}, \mu} \alpha = \int_Y \alpha \wedge \Omega$ for any smooth 2-form α on Y. A form Ω is basic w.r.t the foliation $\mathcal{T}$ if $\iota_\xi \Omega = \iota_\xi d\Omega = 0$ for all $\xi \in L$.

The *foliated degree* $\deg_{\mathcal{T}, \mu} \mathcal{F}$ of an $\mathcal{O}_Y$-coherent sheaf $\mathcal{F}$ on Y is defined by

$$\deg_{\mathcal{T}, \mu} \mathcal{F} = \int_{\mathcal{T}, \mu} c_1(\mathcal{F})$$

where $c_1(\mathcal{F})$ is any smooth representative of the first Chern class of $\mathcal{F}$.

Our main technical tool will be a *weak polystability* property of the harmonic Higgs bundle *along the leaves*, in the case when the current $\int_{\mathcal{T}, \mu}$ is sufficiently regular, namely when the transverse measure comes from a transverse volume form. Before giving the statement, we need some definitions (unfortunately, because this seems to be the admitted terminology in the literature, we have to use the word "saturated" with two different meanings, but no confusion should arise).

A subset $S \subset Y$ is $\mathcal{T}$-*saturated* if it is a union of leaves of the foliation $\mathcal{T}$, i.e. for all $x \in S$, the leaf $\mathcal{L}_x$ of the foliation $\mathcal{T}$ through x is included in S. If S is $\mathcal{T}$-saturated then so is $Y \backslash S$.

A coherent subsheaf $\mathcal{F}$ of the sheaf $\mathcal{O}_Y(E)$ is *saturated* if $\mathcal{O}_Y(E)/\mathcal{F}$ is torsion free. A saturated subsheaf of $\mathcal{O}_Y(E)$ is reflexive and therefore normal. If $\mathcal{F}$ is a coherent subsheaf of $\mathcal{O}_Y(E)$, its *saturation* is the kernel of $\mathcal{O}_Y(E) \to (\mathcal{O}_Y(E)/\mathcal{F})/\mathrm{Tor}(\mathcal{O}_Y(E)/\mathcal{F})$. It is a saturated subsheaf of $\mathcal{O}_Y(E)$.

In this paper, the *singular locus* $\mathcal{S}(\mathcal{F})$ of a coherent subsheaf $\mathcal{F}$ of $\mathcal{O}_Y(E)$ is the analytic subset of Y where the quotient $\mathcal{O}_Y(E)/\mathcal{F}$ is not locally free. This is not the usual definition. The complement $Y \backslash \mathcal{S}(\mathcal{F})$ of $\mathcal{S}(\mathcal{F})$ is the biggest subset of Y where $\mathcal{F}$ is the sheaf of sections of a subbundle F of E. If $\mathcal{F}$ is saturated then $\mathcal{S}(\mathcal{F})$ has codimension at least 2 in Y.

Proposition 2.2. *Let Y be a compact Kähler manifold and (E, θ) be a harmonic Higgs bundle on Y. Let $\mathcal{T}$ be a smooth holomorphic foliation of Y by complex curves. Assume that $\mathcal{T}$ admits an invariant transverse measure μ given by an invariant transverse volume form.*

   (1) (Semistability along the leaves) *For any leafwise Higgs subsheaf $\mathcal{F} \subset \mathcal{O}_Y(E)$ of (E, θ),* $\deg_{\mathcal{T}, \mu} \mathcal{F} \leq 0$.
   (2) (Weak polystability along the leaves) *If $\mathcal{F} \subset \mathcal{O}_Y(E)$ is a saturated leafwise Higgs subsheaf of (E, θ) such that $\deg_{\mathcal{T}, \mu} \mathcal{F} = 0$, then*
     (a) *the singular locus $\mathcal{S}(\mathcal{F})$ of $\mathcal{F}$ is $\mathcal{T}$-saturated;*
     (b) *on $Y \backslash \mathcal{S}(\mathcal{F})$, if F is the subbundle of E such that $\mathcal{F} = \mathcal{O}_Y(F)$ and $F^\perp$ is its orthogonal complement w.r.t. the harmonic metric on E, we have $\theta(F^\perp \otimes L) \subset F^\perp$ and the C^∞-decomposition $E = F \oplus F^\perp$ is holomorphic along the leaves of the foliation $\mathcal{T}$, i.e. for any leaf $\mathcal{L}$ of $\mathcal{T}$ such that $\mathcal{L} \subset Y \backslash \mathcal{S}(\mathcal{F})$, $E_{|\mathcal{L}} = F_{|\mathcal{L}} \oplus F^\perp_{|\mathcal{L}}$ is a holomorphic orthogonal direct sum on $\mathcal{L}$.*

Proof. We first prove the semistability along the leaves, using the Chern-Weil formula. Let Ω be the closed basic $(m-1, m-1)$-form on Y given by the invariant transverse volume form to the foliation $\mathcal{T}$, so that for all 2-form α on Y, $\int_{\mathcal{T}, \mu} \alpha = \int_Y \alpha \wedge \Omega$.

Let $\mathcal{F}$ be a leafwise Higgs subsheaf of (E, θ). The foliated degree of the saturation $\overline{\mathcal{F}}$ of $\mathcal{F}$ is greater than or equal to the foliated degree of $\mathcal{F}$. This is because there exists an effective divisor D such that $\det \overline{\mathcal{F}} = (\det \mathcal{F}) \otimes [D]$ (see e.g. [Kob87, Chap. V (8.5) p. 180]), so that $\deg_{\mathcal{T}, \mu} \overline{\mathcal{F}} = \int_{\mathcal{T}, \mu} c_1(\overline{\mathcal{F}}) = \int_Y c_1(\det \overline{F}) \wedge \Omega = \deg_{\mathcal{T}, \mu} \mathcal{F} + \int_D \Omega \geq \deg_{\mathcal{T}, \mu} \mathcal{F}$. Therefore it is

enough to prove (1) for saturated leafwise Higgs subsheaves and we assume for now on that $\mathcal{F}$ is saturated, so that the codimension of $\mathcal{S}(\mathcal{F})$ is at least 2.

There exists a holomorphic subbundle F of E defined outside of the singular locus $\mathcal{S}(\mathcal{F})$ of $\mathcal{F}$, such that $\mathcal{F}$ is the sheaf of sections of F on $Y \backslash \mathcal{S}(\mathcal{F})$. On $Y \backslash \mathcal{S}(\mathcal{F})$, we can decompose the flat connection $D = \nabla + \theta + \theta^\star$ with respect to the orthogonal decomposition $E = F \oplus F^\perp$ (for the harmonic metric). Denoting by $\sigma \in C_{1,0}^\infty(Y \backslash \mathcal{S}(\mathcal{F}), \mathrm{Hom}(F, F^\perp))$ the second fundamental form of F, we get:

$$D = \begin{pmatrix} \nabla_F & -\sigma^\star \\ \sigma & \nabla_{F^\perp} \end{pmatrix} + \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix} + \begin{pmatrix} \theta_1^\star & \theta_3^\star \\ \theta_2^\star & \theta_4^\star \end{pmatrix}.$$

On the one hand, the curvature Θ_F of the connection $\nabla_F + \theta_1 + \theta_1^\star$ can be used to compute a representative of the first Chern class of $\mathcal{F}$ on $Y \backslash \mathcal{S}(\mathcal{F})$, namely $c_1(F) = \frac{\sqrt{-1}}{2\pi} \mathrm{tr}\, \Theta_F$, and integrating $\frac{\sqrt{-1}}{2\pi} \mathrm{tr}\, \Theta_F \wedge \Omega$ on $Y \backslash \mathcal{S}(\mathcal{F})$ gives the foliated degree of $\mathcal{F}$. This is sketched in [Sim88, Lemma 3.2], and can be proved as follows. By [Kob87, pp. 180-182] (see also [Sib13, Theorem 2.23 & Lemma 4.6]), if Ξ_F is the curvature of the metric connection ∇_F on $Y \backslash \mathcal{S}(\mathcal{F})$ then integrating $\frac{\sqrt{-1}}{2\pi} \mathrm{tr}\, \Xi_F$ against Ω on $Y \backslash \mathcal{S}(\mathcal{F})$ computes $\int_Y c_1(\mathcal{F}) \wedge \Omega$. Moreover, still on $Y \backslash \mathcal{S}(\mathcal{F})$, we have $\mathrm{tr}\, \Theta_F = \mathrm{tr}\, \Xi_F + d(\mathrm{tr}\,(\theta_1 + \theta_1^\star))$. It is known (see [UY86] and also [Pop05]) that the orthogonal projection $\varpi : E \to F$, which can be seen as an element of $L^\infty(Y, \mathrm{End}(E))$, is also in the Sobolev space $L_1^2(Y, \mathrm{End}(E))$. Therefore $\theta_1 = \varpi \circ \theta \circ \varpi$ and $\theta_1^\star = \varpi \circ \theta^\star \circ \varpi$ are such that $\int_{Y \backslash \mathcal{S}(\mathcal{F})} d(\mathrm{tr}\,(\theta_1 + \theta_1^\star)) \wedge \Omega = \int_Y d(\mathrm{tr}\,(\theta_1 + \theta_1^\star)) \wedge \Omega = 0$ by Stokes formula and density of smooth functions in $L^2(Y)$. To sum up, we have

$$\deg_{\mathcal{T}, \mu} \mathcal{F} = \int_Y c_1(\mathcal{F}) \wedge \Omega = \frac{\sqrt{-1}}{2\pi} \int_{Y \backslash \mathcal{S}(\mathcal{F})} \mathrm{tr}\, \Xi_F \wedge \Omega = \frac{\sqrt{-1}}{2\pi} \int_{Y \backslash \mathcal{S}(\mathcal{F})} \mathrm{tr}\, \Theta_F \wedge \Omega.$$

On the other hand, since $D^2 = 0$, we have $(\nabla_F + \theta_1 + \theta_1^\star)^2 = -(\theta_2 + \theta_3^\star - \sigma^\star) \wedge (\theta_3 + \theta_2^\star + \sigma)$. Therefore

$$\begin{aligned} \Theta_F \wedge \Omega &= (-\theta_2 \wedge \theta_2^\star + \sigma^\star \wedge \sigma - \theta_3^\star \wedge \theta_3 - \theta_3^\star \wedge \sigma + \sigma^\star \wedge \theta_3) \wedge \Omega \\ &= (-\theta_2 \wedge \theta_2^\star + \sigma^\star \wedge \sigma) \wedge \Omega \end{aligned}$$

because $\theta_3 \wedge \Omega = \theta_3^\star \wedge \Omega = 0$, for Ω is a basic $(m-1, m-1)$-form and θ_3 vanishes in the direction of the leaves of $\mathcal{T}$ since $\mathcal{F}$ is a leafwise Higgs subsheaf. Hence

$$\deg_{\mathcal{T}, \mu} \mathcal{F} = \frac{\sqrt{-1}}{2\pi} \int_{Y \backslash \mathcal{S}(\mathcal{F})} \mathrm{tr}\,(-\theta_2 \wedge \theta_2^\star + \sigma^\star \wedge \sigma) \wedge \Omega \leq 0$$

as wanted.

Now let us prove (2). We will follow the proof that Einstein-Hermitian vector bundles are polystable, see [Kob87]. Assume that $\deg_{\mathcal{T}, \mu}(\mathcal{F}) = 0$ for the subsheaf $\mathcal{F}$ of the proof of Assertion (1). Then $\mathrm{tr}\,(-\theta_2 \wedge \theta_2^\star + \sigma^\star \wedge \sigma) \wedge \Omega = 0$ on $Y \backslash \mathcal{S}(\mathcal{F})$ and this implies that for all $\eta \in L_{|Y \backslash \mathcal{S}(\mathcal{F})}$, $\theta_2(\eta) = 0$ and $\sigma(\eta) = 0$. This means on the first hand that $\theta(F^\perp \otimes L) \subset F^\perp$ on $Y \backslash \mathcal{S}(\mathcal{F})$ and on the second hand that if $\mathcal{L}$ is a leaf of $\mathcal{T}$, then on $\mathcal{L} \cap (Y \backslash \mathcal{S}(\mathcal{F}))$, F is a parallel subbundle of E. As in the proof of [Kob87, Theorem 5.8.3], we deduce that the C^∞-decomposition $E = F \oplus F^\perp$ is holomorphic when restricted to $\mathcal{L} \cap (Y \backslash \mathcal{S}(\mathcal{F}))$. We will say that the decomposition, where it is defined, is holomorphic along the leaves of $\mathcal{T}$. This will prove (2b) once (2a) will be established.

Let $S = \{x \in \mathcal{S}(\mathcal{F}) \text{ such that } \mathcal{L}_x \subset \mathcal{S}(\mathcal{F})\}$. This subset of $\mathcal{S}(\mathcal{F})$ is $\mathcal{T}$-saturated, and it is an analytic subset of codimension at least 2 in Y. Indeed, on a regular open set U for the foliation $\mathcal{T}$ identified with an open subset of $\mathbb{C}^m$, we may assume that the leaves of $\mathcal{T}$ are the fibers of a linear projection $p : \mathbb{C}^m \to \mathbb{C}^{m-1}$. Then $S = \{x \in \mathcal{S}(\mathcal{F}) \text{ such that } \dim p^{-1}(p(x)) \geq 1\}$ and hence is analytic by [Fis76, p. 137].

We will prove that the holomorphic subbundle F which is defined outside $\mathcal{S}(\mathcal{F})$ can be extended to a holomorphic subbundle defined on $Y \backslash S$, and that the decomposition $E = F \oplus F^\perp$, which is C^∞ and holomorphic along the leaves outside $\mathcal{S}(\mathcal{F})$, can also be extended to a decomposition on $Y \backslash S$, with the same regularity. This will be a consequence of the following variation on the second Riemann extension theorem:

Lemma 2.3. *Let $\mathcal{O}$ be an open subset of $\mathbb{C}^m$ and V be a 1-dimensional linear subspace in $\mathbb{C}^m$. For $z \in \mathbb{C}^m$, let ℓ_z be the affine line $z + V$. Let A be an analytic subset of $\mathcal{O}$, of codimension at least 2. Let $\varphi : \mathcal{O} \backslash A \to \mathbb{C}$ be a C^∞ map. Assume that φ is holomorphic in the V-direction, meaning that for every $z \in \mathcal{O} \backslash A$, the restriction of φ to a neighborhood of z in ℓ_z is holomorphic. Let a be a point of A which is an isolated point of $A \cap \ell_a$. Then there exist a neighborhood $\mathcal{U}$ of a in $\mathcal{O}$ and a C^∞ map $\Phi : \mathcal{U} \to \mathbb{C}$, holomorphic in the V-direction, such that $\Phi = \varphi$ on $\mathcal{U} \backslash A$.*

(We postpone the proof of the lemma to the end of the present proof.)

Let x be a point of $\mathcal{S}(\mathcal{F}) \backslash S$, i.e. x is an isolated point of $\mathcal{S}(\mathcal{F}) \cap \mathcal{L}_x$. We want to show that F and $F^\perp$ can be extended in a neighborhood of x in Y. Since this is a local problem, we may assume that we are on an open subset $\mathcal{O}$ of $\mathbb{C}^m$, that the leaves of the tautological foliation $\mathcal{T}$ are the affine lines of a given direction $V \subset \mathbb{C}^m$ as in the lemma, and that E is a trivial bundle. Because of the regularity properties of F and $F^\perp$, the section ϕ of $\mathrm{Hom}(E, E)$ defined over $\mathcal{O} \backslash \mathcal{S}(\mathcal{F})$ and corresponding to the orthogonal projection on $F^\perp$ is given by a matrix of functions (ϕ_{ij}) from $\mathcal{O} \backslash \mathcal{S}(\mathcal{F})$ to $\mathbb{C}$ which are C^∞ and holomorphic in the V-direction. By the above lemma, ϕ extends to a section of $\mathrm{Hom}(E, E)$ defined in a neighborhood of x in $\mathcal{O}$. By the lower semi-continuity of the rank, if $x \in \mathcal{S}(\mathcal{F}) \backslash S$, $\mathrm{rk}\,\phi(x) \leq \mathrm{rk}\,E - \mathrm{rk}\,F$. In the same way, $\mathrm{id} - \phi$ can be extended to $\mathcal{S}(\mathcal{F}) \backslash S$ and hence $\mathrm{rk}\,\phi(x) = \mathrm{rk}\,E - \mathrm{rk}\,F$ on $\mathcal{S}(\mathcal{F}) \backslash S$. Hence the subbundles F and $F^\perp$ can be extended to $Y \backslash S$, as C^∞-vector bundles holomorphic along the leaves of $\mathcal{T}$. Since F is holomorphic and orthogonal to $F^\perp$ on $Y \backslash \mathcal{S}(\mathcal{F})$, this is also true on $Y \backslash S$. Finally, because $\mathcal{F}$ is normal, $\mathcal{F}$ coincides with the sheaf of sections of F on $Y \backslash S$.

By the definition of $\mathcal{S}(\mathcal{F})$, this implies that $(Y \backslash S) \cap \mathcal{S}(\mathcal{F}) = \emptyset$, so that $\mathcal{S}(\mathcal{F}) = S$ and hence $\mathcal{S}(\mathcal{F})$ is $\mathcal{T}$-saturated. $\square$

Proof of Lemma 2.3. Choose coordinates $(z_1, \ldots, z_m)$ on $\mathbb{C}^m$ such that $a = 0$ and $\ell_0 = V = \{z \mid z_1 = \cdots = z_{m-1} = 0\}$. By assumption 0 is an isolated point of $\ell_0 \cap A$, hence there exists $r > 0$ such that the circle $\{z \mid z_1 = \cdots = z_{m-1} = 0, |z_m| = r\}$ does not meet A. Let $\varepsilon > 0$ be such that the polydisc $\Delta(0, \varepsilon)^{m-1} \times \Delta(0, r + \varepsilon) \subset \mathcal{O}$ and $\{z \mid |z_i| < \varepsilon, \forall 1 \leq i \leq m-1, |z_m| \in (r - \varepsilon, r + \varepsilon)\} \cap A = \emptyset$. Then the function Φ defined by

$$\Phi(z_1, \ldots, z_m) = \frac{1}{2\pi\sqrt{-1}} \int_{|t|=r} \frac{\varphi(z_1, \ldots, z_{m-1}, t)}{t - z_m}\, dt$$

is C^∞ on the polydisc $\mathcal{U} = \Delta(0, \varepsilon)^{m-1} \times \Delta(0, r) \subset \mathcal{O}$. Moreover, for all $(z_1 \ldots, z_{m-1}) \in \Delta(0, \varepsilon)^{m-1}$, the map $z_m \mapsto \Phi(z_1, \ldots, z_{m-1}, z_m)$ is holomorphic on the disc $\Delta(0, r)$.

Let $\mathcal{U}' = \{z \in \mathcal{U} \mid \ell_z \cap A \cap \mathcal{U} = \emptyset\}$. Because φ is holomorphic in the V-direction, for all z in $\mathcal{U}'$, the restriction of Φ to $\ell_z \cap \mathcal{U}$ equals φ by the Cauchy formula. Now $\mathcal{U}'$ is dense in $\mathcal{U}$. Indeed, let p be the projection $\mathbb{C}^m \to \mathbb{C}^m/V$. Since $0 \in A$ is an isolated point of $A \cap p^{-1}(p(0))$, near $0 = p(0)$ the set $p(A)$ is analytic of the same dimension as A ([Fis76, p. 133]), thus it has codimension at least 1. Hence $\Phi = \varphi$ on $\mathcal{U} \backslash A$. $\square$

Remark 2.4. *Closed leaves I: semistability and saturation of sheaves.* If instead of an invariant transverse measure given by an invariant transverse volume form, one considers the measure δ_C given by a closed leaf C of the foliation $\mathcal{T}$ (assuming there is one), a statement like Proposition 2.2 will fail without further assumptions. In fact, it is well known that even

the notion of degree is in general not reasonable in this case. Suppose for example that Y is a compact Kähler surface and that the foliation $\mathcal{T}$ admits a closed leaf of negative self intersection. Then $\deg_{\mathcal{T}, \delta_C} \mathcal{O}_Y(-C) := \int_C \mathcal{O}_Y(-C) = -C^2 > 0$, and the "degree" of $\mathcal{O}_Y(-C)$ is bigger than the "degree" of its saturation $\mathcal{O}_Y$, which of course vanishes. In order to avoid this kind of inconvenience, it is necessary that C enjoys some positivity properties, e.g. the cohomology class of the current $\int_{\mathcal{T}, \delta_C}$ is represented by a smooth semi-positive $(1,1)$-form in the sense of currents. One can then hope to get a leafwise semistability result.

3. The tautological foliation on the projectivized tangent bundle of complex hyperbolic manifolds

In this section we give a detailed description of the tautological foliation $\mathcal{T}$ by complex curves on the projectivized tangent bundle $\mathbb{P}T_X$ of a complex hyperbolic manifold X and of its transverse structure. Together with the results of §2, it will be one of the main tools to (re)prove the Milnor-Wood inequality on the Toledo invariant and to study maximal representations. The section ends with some applications of Ratner's theorem on the closure of orbits under groups generated by unipotent elements to projection to X of subsets of $\mathbb{P}T_X$ saturated under $\mathcal{T}$.

The Klein model of complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ is the set of negative lines in $\mathbb{C}^{n+1}$ for a Hermitian form h of signature $(n,1)$. It is an open set in the projective space $\mathbb{CP}^n$.

The Lie group $\mathrm{SU}(n,1) = \mathrm{SU}(\mathbb{C}^{n+1}, h)$ is the subgroup of $\mathrm{SL}(n+1, \mathbb{C})$ consisting of elements preserving the Hermitian form h. As a group of matrices, in a basis $(e_1, \ldots, e_n, e_{n+1})$ of $\mathbb{C}^{n+1}$ where the matrix of h is the diagonal matrix $I_{n,1} = \mathrm{diag}(1, \ldots, 1, -1)$,

$$\mathrm{SU}(n,1) = \{M \in \mathrm{SL}(n+1, \mathbb{C}) \,|\, M^\star I_{n,1} M = I_{n,1}\},$$

where $M^\star$ denotes the conjugate transpose of M.

The group $\mathrm{SU}(n,1)$ acts transitively on $\mathbb{H}^n_{\mathbb{C}}$. The stabilizer of a point is a maximal compact subgroup of $\mathrm{SU}(n,1)$ and is conjugated to $\mathrm{U}(n) \simeq \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$. This gives a realization of $\mathbb{H}^n_{\mathbb{C}}$ as the Hermitian symmetric space $\mathrm{SU}(n,1)/\mathrm{U}(n)$. As a bounded symmetric domain, complex hyperbolic n-space is biholomorphic to the unit ball in $\mathbb{C}^n$.

We equip the Lie algebra $\mathfrak{su}(n,1)$ of $\mathrm{SU}(n,1)$ with the Killing form $b(A, B) = 2 \mathrm{tr}\,(AB)$, normalized so that the holomorphic sectional curvature of the $\mathrm{SU}(n,1)$-invariant Kähler metric ω it induces on $\mathbb{H}^n_{\mathbb{C}}$ is -1.

An n-dimensional complex hyperbolic manifold X is the quotient of $\mathbb{H}^n_{\mathbb{C}}$ by a discrete torsion free subgroup Γ of $\mathrm{SU}(n,1)$.

3.1. Complex geodesics and the tautological foliation.

The *complex geodesics* of $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{CP}^n$ are the intersections of $\mathbb{H}^n_{\mathbb{C}}$ with the complex lines $\mathbb{CP}^1 \subset \mathbb{CP}^n$. It follows that the space $\mathcal{G}$ of complex geodesics is an open homogeneous set in the Grassmannian of 2-planes in $\mathbb{C}^{n+1}$. More precisely, $\mathrm{SU}(n,1)$ acts transitively on $\mathcal{G}$ and $\mathcal{G} = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n-1) \times \mathrm{U}(1,1))$. Complex geodesics are complex totally geodesic subspaces of $\mathbb{H}^n_{\mathbb{C}}$ isometric (up to a constant) to the Poincaré disc, of induced sectional curvature -1. Given a point in $\mathbb{H}^n_{\mathbb{C}}$ and a complex tangent line at this point, there is a unique complex geodesic through that point tangent to the complex line.

Let $T_{\mathbb{H}^n_{\mathbb{C}}} \to \mathbb{H}^n_{\mathbb{C}}$ be the holomorphic tangent bundle of $\mathbb{H}^n_{\mathbb{C}}$ and consider the *projectivized tangent bundle* $\pi : \mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}} \to \mathbb{H}^n_{\mathbb{C}}$ of $\mathbb{H}^n_{\mathbb{C}}$. It is a holomorphic bundle and the fiber over a point $x \in \mathbb{H}^n_{\mathbb{C}}$ is the projective space of lines in the tangent space $T_{\mathbb{H}^n_{\mathbb{C}}, x}$. A point in the projectivized tangent bundle $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ of $\mathbb{H}^n_{\mathbb{C}}$ is given by two h-orthogonal complex lines in $\mathbb{C}^{n+1}$ spanning a complex geodesic. Hence $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ is the homogeneous space $\mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n-1) \times \mathrm{U}(1) \times \mathrm{U}(1))$.

The central fiber of the holomorphic projection $\pi : \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} \to \mathbb{H}_{\mathbb{C}}^n$ is $\mathrm{U}(n)/(\mathrm{U}(n-1) \times \mathrm{U}(1)) = \mathbb{CP}^{n-1}$ as it should.

The map from $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ to $\mathcal{G}$ associating to a point in the projectivized tangent bundle the complex geodesic it defines is the $\mathrm{SU}(n,1)$-equivariant holomorphic fibration $\pi_{\mathcal{G}} : \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n-1) \times \mathrm{U}(1) \times \mathrm{U}(1)) \to \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n-1) \times \mathrm{U}(1,1))$. The central fiber $\mathrm{U}(1,1)/(\mathrm{U}(1) \times \mathrm{U}(1))$ is isometric to the Poincaré disc so that $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ is a disc bundle over $\mathcal{G}$. This of course defines a foliation on $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ whose leaves are the fibers of $\pi_{\mathcal{G}}$.

If Γ is a discrete torsion free subgroup in $\mathrm{SU}(n,1)$ and $X = \Gamma\backslash\mathbb{H}_{\mathbb{C}}^n$ the corresponding complex hyperbolic manifold, we again call $\pi : \mathbb{P}T_X \to X$ the projectivized tangent bundle of X. The fibration $\pi_{\mathcal{G}}$, by $\mathrm{SU}(n,1)$-equivariance, defines a smooth holomorphic foliation by holomorphic curves on $\mathbb{P}T_X$. This foliation inherits a structure of transversally homogeneous $\mathcal{G}$-foliation, see [God91, § III.3 p. 164], which will be discussed in §3.2.

If $\xi \in \mathbb{P}T_X$, the leaf $\mathcal{L}_\xi$ through ξ is locally given by the holomorphic tangent space of the local complex geodesic tangent to ξ at $\pi(\xi)$ in X. Vectors tangent to the leaves of the foliation form a line subbundle L of $T_{\mathbb{P}T_X}$. Recall that we can pull-back the tangent bundle $T_X \to X$ to $\mathbb{P}T_X$ to obtain a vector bundle $\pi^\star T_X \to \mathbb{P}T_X$ and that the tautological line bundle $\mathcal{O}_{\mathbb{P}T_X}(-1)$ is the subbundle of $\pi^\star T_X$ defined by

$$\mathcal{O}_{\mathbb{P}T_X}(-1) = \{(u,\xi) \in T_X \times \mathbb{P}T_X \mid u \in \xi\}.$$

By construction the differential $\pi_\star$ of π at ξ maps the fiber L_ξ of L to the line $\xi \subset T_{X,\pi(\xi)}$. This means that when considered as a morphism from $T_{\mathbb{P}T_X}$ to $\pi^\star T_X$, $\pi_\star$ realizes an isomorphism between the line subbundle L of $T_{\mathbb{P}T_X}$ and the tautological line subbundle $\mathcal{O}_{\mathbb{P}T_X}(-1)$ of $\pi^\star T_X$.

For these reasons the foliation will be called the *tautological foliation* of $\mathbb{P}T_X$ and will be denoted by $\mathcal{T}$.

3.2. The transverse structure of the tautological foliation.

By construction, the tautological foliation $\mathcal{T}$ has a structure of transversally homogeneous $\mathcal{G}$-foliation, also called a transverse $(\mathrm{SU}(n,1), \mathcal{G})$-structure. In this section, we describe this structure and prove the

Proposition 3.1. *The homogeneous space $\mathcal{G}$ of complex geodesics of $\mathbb{H}_{\mathbb{C}}^n$ admits a $\mathrm{SU}(n,1)$-invariant non-degenerate but indefinite Kähler form $\omega_{\mathcal{G}}$, which is unique up to normalization. This form defines a diffuse invariant transverse measure $\mu_{\mathcal{G}}$ for the tautological foliation $\mathcal{T}$ on the projectivized tangent bundle $\pi : \mathbb{P}T_X \to X$ of a complex hyperbolic manifold X. The associated current of integration on $\mathbb{P}T_X$ satisfies (when suitably normalized)*

$$\int_{\mathcal{T}, \mu_{\mathcal{G}}} \pi^\star \beta = \frac{1}{n!} \int_X \beta \wedge \omega^{m-1}$$

for any compactly supported 2-form β on X.

This result will allow us to compute numerical invariants on the complex hyperbolic manifold X by going up to the projectivized tangent bundle $\mathbb{P}T_X$ and integrating along the leaves of $\mathcal{T}$. For instance, if X is compact and $\mathcal{F}$ is a coherent sheaf on X, then, using the definition of the foliated degree given in §2.2:

$$\deg \mathcal{F} = \frac{1}{n!} \int_X c_1(\mathcal{F}) \wedge \omega^{n-1} = \int_{\mathcal{T}, \mu_{\mathcal{G}}} \pi^\star c_1(\mathcal{F}) = \deg_{\mathcal{T}, \mu_{\mathcal{G}}} (\pi^\star \mathcal{F})$$

In the same way, still assuming that $X = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ is compact, if ρ is a representation of Γ in a Hermitian Lie group G, then the Toledo invariant of ρ is given by

$$\tau(\rho) := \frac{1}{n!} \int_X f^\star \omega_{\mathcal{Y}} \wedge \omega^{n-1} = \int_{\mathcal{T}, \mu_{\mathcal{G}}} \pi^\star f^\star \omega_{\mathcal{Y}}$$

where f is any ρ-equivariant map from $\mathbb{H}^n_{\mathbb{C}}$ to the Hermitian symmetric space $\mathcal{Y}$ associated to G, and $\omega_{\mathcal{Y}}$ is the Kähler form of $\mathcal{Y}$ normalized so that the minimal value of its holomorphic sectional curvature is -1.

The existence of the indefinite Kähler form $\omega_{\mathcal{G}}$ on $\mathcal{G}$ stated in Proposition 3.1 is not new, see [Wol69, Theorem 6.3 and Corollary 6.4]. However we need the correct normalization constants between $\omega_{\mathcal{G}}$, the invariant Kähler form ω on X, and the curvature of the tautological line bundle on $\mathbb{P}T_X$. To work these constants out we now describe the geometry of the double holomorphic fibration between the complex $\mathrm{SU}(n,1)$-homogeneous spaces $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$, $\mathcal{G}$ and $\mathbb{H}^n_{\mathbb{C}}$ and the (pseudo-)Kähler structure of these spaces on the Lie algebra level. The results are summarized in Lemma 3.2 below. The end of the proof of Proposition 3.1 is given in Lemma 3.3.

To lighten the notation, in this section we set $M = \mathrm{S}(\mathrm{U}(n-1) \times \mathrm{U}(1) \times \mathrm{U}(1))$, $H = \mathrm{S}(\mathrm{U}(n-1) \times \mathrm{U}(1,1))$, and we denote their respective Lie algebras by $\mathfrak{m}$ and $\mathfrak{h}$.

The Lie algebra $\mathfrak{su}(n,1)$ of the group $\mathrm{SU}(n,1)$ is:

$$\mathfrak{su}(n,1) = \left\{ \begin{pmatrix} A & \xi \\ \xi^\star & a \end{pmatrix}, A \in M_n(\mathbb{C}), \xi \in \mathbb{C}^n, a \in \mathbb{C} \mid A^\star = -A, a + \mathrm{tr}\, A = 0 \right\}.$$

The Lie algebra of the maximal compact subgroup of $\mathrm{SU}(n,1)$, isomorphic to $\mathrm{U}(n)$, is the subalgebra

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}, A \in M_n(\mathbb{C}), a \in \mathbb{C} \mid A^\star = -A, a + \mathrm{tr}\, A = 0 \right\} \simeq \mathfrak{u}(n),$$

whereas

$$\mathfrak{m} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, A \in \mathfrak{u}(n-1), a,b \in \mathbb{C}, a + b + \mathrm{tr}\, A = 0 \right\}.$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A \in \mathfrak{u}(n-1), B \in \mathfrak{u}(1,1), \mathrm{tr}\, A + \mathrm{tr}\, B = 0 \right\}.$$

The real tangent space of $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ at M is naturally identified with the subspace

$$\mathfrak{s} = \left\{ \xi = \begin{pmatrix} 0 & \xi_3 & \xi_2 \\ -\xi_3^\star & 0 & \xi_1 \\ \xi_2^\star & \xi_1^\star & 0 \end{pmatrix}, \xi_1 \in \mathbb{C}, \xi_2, \xi_3 \in \mathbb{C}^{n-1} \right\} \subset \mathfrak{su}(n,1)$$

and its invariant complex structure J is given at M by

$$J \begin{pmatrix} 0 & \xi_3 & \xi_2 \\ -\xi_3^\star & 0 & \xi_1 \\ \xi_2^\star & \xi_1^\star & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1}\xi_3 & \sqrt{-1}\xi_2 \\ \sqrt{-1}\xi_3^\star & 0 & \sqrt{-1}\xi_1 \\ -\sqrt{-1}\xi_2^\star & -\sqrt{-1}\xi_1^\star & 0 \end{pmatrix}.$$

Define the subspaces

$$\mathfrak{s}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi_1 \\ 0 & \xi_1^\star & 0 \end{pmatrix}, \xi_1 \in \mathbb{C} \right\},$$

$$\mathfrak{s}_2 = \left\{ \begin{pmatrix} 0 & 0 & \xi_2 \\ 0 & 0 & 0 \\ \xi_2^{\star} & 0 & 0 \end{pmatrix}, \xi_2 \in \mathbb{C}^{n-1} \right\},$$

$$\mathfrak{s}_3 = \left\{ \begin{pmatrix} 0 & \xi_3 & 0 \\ -\xi_3^{\star} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \xi_3 \in \mathbb{C}^{n-1} \right\}$$

of $\mathfrak{s}$. It is plain that $\mathfrak{u}(n) \oplus (\mathfrak{s}_1 \oplus \mathfrak{s}_2)$ is a Cartan decomposition of $\mathfrak{su}(n,1)$ so that $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ is invariant under the adjoint action of $U(n)$ and identifies with the tangent space to $\mathbb{H}^n_{\mathbb{C}} = SU(n,1)/U(n)$ at $U(n)$. Similarly, the subspace $\mathfrak{s}_2 + \mathfrak{s}_3$ is H-invariant and identifies with the tangent space of $\mathcal{G} = SU(n,1)/H$ at H. The subspaces $\mathfrak{s}_1$, $\mathfrak{s}_2$ and $\mathfrak{s}_3$ are invariant under the adjoint action of M on $\mathfrak{s}$, and therefore define C^{∞} subbundles of the real tangent bundle of $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$. The subbundle corresponding to $\mathfrak{s}_1$, resp. $\mathfrak{s}_3$, is the tangent bundle of the fibers of $\pi_{\mathcal{G}} : \mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}} \to \mathcal{G}$, resp. $\pi : \mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}} \to \mathbb{H}^n_{\mathbb{C}}$.

Let $\omega_1, \omega_2, \omega_3$ be the skew-symmetric $\mathbb{R}$-bilinear forms on $\mathfrak{s}$ given by

$$\omega_j(\xi, \eta) = 2\sqrt{-1}(\eta_j^{\star}\xi_j - \xi_j^{\star}\eta_j),$$

for $\xi = (\xi_1, \xi_2, \xi_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ in $\mathfrak{s}$. These forms are invariant by M hence they define $SU(n,1)$-invariant 2-forms on $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}} = SU(n,1)/M$ which will be denoted by the same letters.

Lemma 3.2.
- *The bilinear form $\omega_1 + \omega_2$ defines the $SU(n,1)$-invariant Kähler form ω on $\mathbb{H}^n_{\mathbb{C}}$ normalized so as to have constant holomorphic sectional curvature -1.*
- *The bilinear form $\omega_2 - \omega_3$ defines a $SU(n,1)$-invariant non degenerate but indefinite Kähler form $\omega_{\mathcal{G}}$ on the space of complex geodesics $\mathcal{G}$ of $\mathbb{H}^n_{\mathbb{C}}$.*
- *The bilinear form $\omega_1 + \frac{1}{2}(\omega_2 + \omega_3)$ defines a Kähler form $\pi^{\star}\omega - \frac{1}{2}\pi_{\mathcal{G}}^{\star}\omega_{\mathcal{G}}$ on the projectivized tangent bundle $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ of $\mathbb{H}^n_{\mathbb{C}}$. It is the curvature form of the dual $\mathcal{O}_{\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}}(1)$ of the tautological line bundle over $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ endowed with the natural metric induced by $T_{\mathbb{H}^n_{\mathbb{C}}}$.*

Proof. It is easily checked that the bilinear form $\omega_1 + \omega_2$ on $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ is invariant by $U(n)$, hence that it defines a $SU(n,1)$-invariant 2-form ω on $\mathbb{H}^n_{\mathbb{C}} = SU(n,1)/U(n)$. The form ω is closed (for example because it is a 2-form on a symmetric space and it is invariant by the geodesic symmetries) and it is precisely the invariant Kähler form on $\mathbb{H}^n_{\mathbb{C}}$, normalized so as to have constant holomorphic sectional curvature -1. It is also given by $\omega(\xi, \eta) = b(\zeta, [\xi, \eta]) = b(\mathrm{ad}(\zeta)\xi, \eta)$ for $\xi, \eta \in \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where b is the Killing form on $\mathfrak{su}(n,1)$ and ζ is the element of the 1-dimensional center of $\mathfrak{u}(n)$ such that $\mathrm{ad}(\zeta)$ gives the invariant complex structure of $\mathbb{H}^n_{\mathbb{C}}$:

$$\zeta = \begin{pmatrix} \frac{\sqrt{-1}}{n+1}1_n & 0 \\ 0 & -\frac{n\sqrt{-1}}{n+1} \end{pmatrix}$$

(Here and in the rest of the paper, if k is an integer, 1_k denotes the identity matrix of size k.)

One also checks that the bilinear form $\omega_2 - \omega_3$ on $\mathfrak{s}_2 \oplus \mathfrak{s}_3$ is invariant by H and hence defines a $SU(n,1)$-invariant form $\omega_{\mathcal{G}}$ on $\mathcal{G} = SU(n,1)/H$ that is indeed non degenerate (its signature is $(n-1, n-1)$). Again, this form can be computed as $\omega_{\mathcal{G}}(\xi, \eta) = b(\zeta_{\mathfrak{h}}, [\xi, \eta]) = b(\mathrm{ad}(\zeta_{\mathfrak{h}})\xi, \eta)$ for $\xi, \eta \in \mathfrak{s}_2 \oplus \mathfrak{s}_3$, where

$$\zeta_{\mathfrak{h}} = \begin{pmatrix} \frac{2\sqrt{-1}}{n+1}1_{n-1} & 0 \\ 0 & -\frac{(n-1)\sqrt{-1}}{n+1}1_2 \end{pmatrix}$$

is the element of the 1-dimensional center of $\mathfrak{h}$ such that $\mathrm{ad}(\zeta_{\mathfrak{h}})$ gives the invariant complex structure of $\mathcal{G}$. The $\mathrm{SU}(n,1)$-invariance of the Killing form b and the Jacobi identity imply that $\omega_{\mathcal{G}}$ is closed.

Endow the dual $\mathcal{O}_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}(1)$ of the tautological line bundle over $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ with the natural metric induced from the one of $T_{\mathbb{H}_{\mathbb{C}}^n}$. Its curvature form is a positive (1,1)-form and one can compute (see e.g. [GS69, Part 4] or [CW03, (3.7)]) that:

$$\sqrt{-1}\Theta(\mathcal{O}_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}(1)) = \omega_1 + \frac{1}{2}(\omega_2 + \omega_3) = (\omega_1 + \omega_2) - \frac{1}{2}(\omega_2 - \omega_3) = \pi^\star\omega - \frac{1}{2}\pi_{\mathcal{G}}^\star\omega_{\mathcal{G}}.$$

(Note that we normalized the metric on $T_{\mathbb{H}_{\mathbb{C}}^n}$ in order to have constant holomorphic sectional curvature -1 and that ω_3 restricted to a fiber of π is $2\,\omega_{FS}$ in [CW03].) $\square$

If $X = \Gamma\backslash\mathbb{H}_{\mathbb{C}}^n$ is a complex hyperbolic manifold, the $\mathrm{SU}(n,1)$-invariant (1,1)-forms ω on $\mathbb{H}_{\mathbb{C}}^n$ and $\pi_{\mathcal{G}}^\star\omega_{\mathcal{G}}$ on $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ defined in Lemma 3.2 descend to closed forms on X and $\mathbb{P}T_X$ respectively which will be denoted by the same letters. The form ω is the Kähler form of X. The Kähler form $\pi^\star\omega - \frac{1}{2}\pi_{\mathcal{G}}^\star\omega_{\mathcal{G}}$ is the curvature form of $\mathcal{O}_{\mathbb{P}T_X}(1)$, which is isomorphic to the dual $L^\vee$ of the tangent bundle L to the tautological foliation $\mathcal{T}$ on $\mathbb{P}T_X$.

The non degenerate indefinite Kähler form $\omega_{\mathcal{G}}$ on $\mathcal{G}$ defines an invariant transverse Kähler form for the foliation $\mathcal{T}$ on the projectivized tangent bundle $\mathbb{P}T_X$ of X, hence an invariant transverse volume form and an invariant transverse measure $\mu_{\mathcal{G}}$ for the foliation. We normalize the induced current of integration along the leaves of $\mathcal{T}$ so that for all compactly supported 2-form α on $\mathbb{P}T_X$,

$$\int_{\mathcal{T},\,\mu_{\mathcal{G}}} \alpha = \int_{\mathbb{P}T_X} \alpha \wedge \Omega_{\mathcal{G}},$$

where

$$\Omega_{\mathcal{G}} := \frac{(-1)^{n-1}}{(2n-2)!\,\mathrm{vol}(\mathbb{C}\mathbb{P}^{n-1})}\,\pi_{\mathcal{G}}^\star\omega_{\mathcal{G}}^{2n-2}.$$

The form $\Omega_{\mathcal{G}}$ is a closed basic semi-positive $(2n-2, 2n-2)$-form of rank $4n-4$ on $\mathbb{P}T_X$.

It is now easy to complete the proof of Proposition 3.1:

Lemma 3.3. *Let β be a compactly supported 2-form on X, then*

$$\frac{1}{n!}\int_X \beta \wedge \omega^{n-1} = \int_{\mathbb{P}T_X} \pi^\star\beta \wedge \Omega_{\mathcal{G}}.$$

Proof. By $\mathrm{SU}(n,1)$-invariance the (1,1)-forms ω_1, ω_2 and ω_3 on $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ descend to forms on $\mathbb{P}T_X$ and we again have $\pi_{\mathcal{G}}^\star\omega_{\mathcal{G}} = \omega_2 - \omega_3$. Let α be a compactly supported 2-form on $\mathbb{P}T_X$. Then,

$$\begin{aligned}
\int_{\mathbb{P}T_X} \alpha \wedge \omega_2^{n-1} \wedge \omega_3^{n-1} &= \frac{1}{2}\int_{\mathbb{P}T_X} \langle\alpha,\omega_1\rangle\,\omega_1 \wedge \omega_2^{n-1} \wedge \omega_3^{n-1} \\
&= \frac{1}{2n}\int_{\mathbb{P}T_X} \langle\alpha,\omega_1\rangle\,(\pi^\star\omega)^n \wedge \omega_3^{n-1} \\
&= \frac{1}{2n}\int_X \left(\int_{\pi^{-1}(x)} \langle\alpha,\omega_1\rangle\,\omega_3^{n-1}\right) \omega^n.
\end{aligned}$$

If now β is a compactly supported 2-form on X, one has

$$\int_{\pi^{-1}(x)} \langle\pi^\star\beta,\omega_1\rangle\,\frac{\omega_3^{n-1}}{(n-1)!} = \frac{\mathrm{vol}(\mathbb{C}\mathbb{P}^{n-1})}{n}\langle\beta,\omega\rangle_x.$$

Hence

$$
\begin{aligned}
\int_{\mathbb{P}T_X} \pi^\star\beta \wedge \omega_2^{n-1} \wedge \omega_3^{n-1} &= \frac{(n-1)!\,\mathrm{vol}(\mathbb{CP}^{n-1})}{2n^2} \int_X \langle\beta,\omega\rangle\,\omega^n \\
&= (n-1)!^2\,\mathrm{vol}(\mathbb{CP}^{n-1})\frac{1}{n!}\int_X \beta\wedge\omega^{n-1},
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{1}{n!}\int_X \beta\wedge\omega^{n-1} &= \frac{1}{\mathrm{vol}(\mathbb{CP}^{n-1})}\int_{\mathbb{P}T_X} \pi^\star\beta\wedge\frac{\omega_2^{n-1}}{(n-1)!}\wedge\frac{\omega_3^{n-1}}{(n-1)!} \\
&= \frac{(-1)^{n-1}}{\mathrm{vol}(\mathbb{CP}^{n-1})}\int_{\mathbb{P}T_X}\pi^\star\beta\wedge\frac{(\omega_2-\omega_3)^{2n-2}}{(2n-2)!}.
\end{aligned}
$$

$\square$

Remark 3.4. *Closed leaves II: convergence of currents.* As we said in Remark 2.4, the current of integration given by a closed leaf of a foliation is not in general well-behaved. This is still true in the case of the tautological foliation on the projectivized tangent bundle $\mathbb{P}T_X$ of a complex hyperbolic manifold X. More importantly here, there is no direct relation, such as the one established in Proposition 3.1, between the integration along a single closed leaf of the tautological foliation and integration against ω^{n-1}. It is however possible to exploit the existence of closed totally geodesic curves C_i in $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ to make such a relation, when there are infinitely many such curves. This is true for example if Γ is a so-called arithmetic lattice of type I of $\mathrm{SU}(n,1)$, and moreover in this case the sequence of curves C_i can be chosen so that no subsequence is contained in a proper totally geodesic submanifold of X. As is proved e.g. in [KM14], this implies that the currents $\int_{C_i}$ suitably normalized converge towards ω^{n-1}.

3.3. **Some consequences of Ratner's theorem on orbit closures.**

We just saw that the tautological foliation $\mathcal{T}$ on the projectivized tangent bundle $\mathbb{P}T_X$ of a complex hyperbolic manifold X has a rich *transversal* structure. We consider now the *tangential* structure of the foliation $\mathcal{T}$ and we state fundamental properties of its leaves which follow from the resolution by M. Ratner of Raghunathan's conjecture on orbit closures.

In this section we come back to the setting of the paper so that Γ is a torsion free uniform lattice of $\mathrm{SU}(n,1)$ and $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ is therefore a compact complex hyperbolic manifold.

Let $\mathcal{L}$ be a leaf of the tautological foliation $\mathcal{T}$ on $\mathbb{P}T_X = \Gamma\backslash\mathrm{SU}(n,1)/M$. In this section again, M is short for $\mathrm{S}(\mathrm{U}(n-1)\times\mathrm{U}(1)\times\mathrm{U}(1))$. The leaf $\mathcal{L}$ is of the form $\Gamma\backslash\Gamma\,U_{\mathcal{L}}\,g_{\mathcal{L}}M/M$ for some $g_{\mathcal{L}}\in\mathrm{SU}(n,1)$ and a group $U_{\mathcal{L}}$ locally isomorphic to $\mathrm{SU}(1,1)$. Because $U_{\mathcal{L}}$ is generated by unipotent elements, it follows from the work of Ratner [Rat91a] that the closure of the orbit $\Gamma e\cdot U_{\mathcal{L}}$ in $\Gamma\backslash\mathrm{SU}(n,1)$ is homogeneous, namely that there exists a closed subgroup $S_{\mathcal{L}}$ of $\mathrm{SU}(n,1)$ such that $U_{\mathcal{L}}\subset S_{\mathcal{L}}$ and $\overline{\Gamma e\cdot U_{\mathcal{L}}} = \Gamma e\cdot S_{\mathcal{L}}$. This implies that $\Gamma\cap S_{\mathcal{L}}$ is a lattice in $S_{\mathcal{L}}$ [Rag72, Theorem 1.13] and that $S_{\mathcal{L}}$ is a reductive group with compact center, for example because $\mathrm{rk}_{\mathbb{R}}\mathrm{SU}(n,1) = 1$ [Sha91].

By [Pay99], the fact that $\mathrm{rk}_{\mathbb{R}}U_{\mathcal{L}} = \mathrm{rk}_{\mathbb{R}}\mathrm{SU}(n,1)$ implies that the Lie algebra of $S_{\mathcal{L}}$ is stable by the Cartan involution of $\mathrm{SU}(n,1)$ given by the point $g_{\mathcal{L}}\mathrm{U}(n)$ of $\mathbb{H}^n_{\mathbb{C}} = \mathrm{SU}(n,1)/\mathrm{U}(n)$, so that the orbit $\tilde{Y}_{\mathcal{L}} := S_{\mathcal{L}}\cdot g_{\mathcal{L}}\mathrm{U}(n)$ of $g_{\mathcal{L}}\mathrm{U}(n)$ under $S_{\mathcal{L}}$ in $\mathbb{H}^n_{\mathbb{C}}$ is a totally geodesic submanifold and $Y_{\mathcal{L}} := \Gamma\backslash\Gamma\tilde{Y}_{\mathcal{L}}$ is a closed immersed totally geodesic submanifold of $X = \Gamma\backslash\mathrm{SU}(n,1)/\mathrm{U}(n)$. This submanifold is the closure of the projection $\pi(\mathcal{L})$ of $\mathcal{L}$ in X.

Summing up, we have:

Proposition 3.5. *Let X be a compact complex hyperbolic manifold and let $\mathcal{L}$ be a leaf of the tautological foliation $\mathcal{T}$ on $\mathbb{P}T_X$. The closure $\overline{\pi(\mathcal{L})}$ of the image of $\mathcal{L}$ by the projection $\pi : \mathbb{P}T_X \to X$ is a closed immersed totally geodesic submanifold of X.*

The last proposition has the following consequence on projections to X of $\mathcal{T}$-saturated subsets of $\mathbb{P}T_X$:

Proposition 3.6. *Let X be a compact complex hyperbolic manifold and let S be a closed $\mathcal{T}$-saturated proper subset of $\mathbb{P}T_X$. Then $\pi(S)$ is a proper subset of X.*

Proof. The key point is that there is at most a countable number of closed immersed totally geodesic submanifolds of dimension at least 2 in X. This follows from [Rat91b, Theorem 1.1] but in our case a similar but simpler argument is available.

Let $Y \subset X$ be such a submanifold. This means that $Y = \Gamma\backslash\Gamma\tilde{Y}$ where $\tilde{Y}$ is a symmetric subspace of the noncompact type of $\mathbb{H}^n_{\mathbb{C}}$ whose stabilizer S in $\mathrm{SU}(n,1)$ contains $\Lambda := \Gamma \cap S$ as a lattice. Moreover, there exists $y \in \mathbb{H}^n_{\mathbb{C}}$ and a simple (because $\mathrm{rk}_{\mathbb{R}}\mathrm{SU}(n,1) = 1$) noncompact subgroup H of S such that $\tilde{Y} = S \cdot y = H \cdot y$.

We claim that $\tilde{Y}$, and hence Y, is entirely determined by the intersection $\Lambda = \Gamma \cap S$. Indeed, let Y' be another closed immersed totally geodesic submanifold of dimension at least 2 of X, let $\tilde{Y}'$, S', Λ', H' and y' be defined as above for Y, and assume that $\Lambda' = \Lambda$.

By a strengthening of the Borel density theorem, see e.g. [Dan80, Corollary 4.2], since Λ is a lattice in S and H is a simple noncompact subgroup of S, the Zariski closure $\overline{\Lambda}^{\mathsf{z}}$ of Λ in $\mathrm{SU}(n,1)$ contains H. Therefore $H \subset \overline{\Lambda}^{\mathsf{z}} = \overline{\Lambda'}^{\mathsf{z}} \subset S'$ because S' is Zariski-closed. In the same way, $H' \subset S$.

If d denotes the distance function on $\mathbb{H}^n_{\mathbb{C}}$, the function $x \mapsto d(x, \tilde{Y}')$ is constant on $\tilde{Y}$ because $\tilde{Y}$ is an H-orbit, $\tilde{Y}'$ is an S'-orbit and $H \subset S'$. It must be identically zero, because if not, the convex hull of two distinct points in $\tilde{Y}$ and their (distinct) projections in $\tilde{Y}'$ is Euclidean by the flat quadrilateral theorem [BH99, p. 181], a contradiction since $\mathrm{rk}\,\mathbb{H}^n_{\mathbb{C}} = 1$. Hence $\tilde{Y} \subset \tilde{Y}'$. The same reasoning gives $\tilde{Y}' \subset \tilde{Y}$. This is what we wanted.

Since $\Lambda = \Gamma \cap S$ is finitely generated because it is a lattice in S, and since there are only countably many finite subsets in Γ, this indeed proves that there are at most countably many closed immersed totally geodesic submanifolds of dimension at least 2 in X.

To conclude, let S be a closed $\mathcal{T}$-saturated proper subset of $\mathbb{P}T_X$ and assume that $\pi(S) = X$. Then because S is a union of leaves, X is the union of the projections of the leaves of S, hence of their closures. By Proposition 3.5, these closures are closed immersed totally geodesic submanifolds of X of dimension at least 2. Since there are only countably many such objects, there must be a leaf $\mathcal{L} \subset S$ such that $\overline{\pi(\mathcal{L})} = X$. But $\overline{\pi(\mathcal{L})}$ is the projection to X of the totally geodesic orbit $\tilde{Y}_{\mathcal{L}} = S_{\mathcal{L}} \cdot g_{\mathcal{L}}\mathrm{U}(n)$ in $\mathbb{H}^n_{\mathbb{C}}$, so this orbit must be the whole $\mathbb{H}^n_{\mathbb{C}}$, and $S_{\mathcal{L}}$ being reductive, this implies that $S_{\mathcal{L}} = \mathrm{SU}(n,1)$, so that $\overline{\mathcal{L}} = \mathbb{P}T_X$. Hence $S = \mathbb{P}T_X$, for S is closed. A contradiction. $\square$

4. Representations in $\mathrm{SU}(p,q)$, $p \geq q$

4.1. **Strategy of the proof.**

This section is devoted to the proof of Theorem 1.1 and Corollary 1.2 in the case of representations in the group $\mathrm{SU}(p,q)$. Representations in the other classical Hermitian Lie groups will be treated in §5 using results of this section.

Our primary goal here is to prove:

Theorem 4.1. *Let ρ be a reductive representation of a (torsion free) uniform lattice Γ of $\mathrm{SU}(n,1)$ in $\mathrm{SU}(p,q)$, $p \geq q \geq 1$. If ρ is maximal and $n \geq 2$ then the ρ-equivariant harmonic map f from $\mathbb{H}^n_{\mathbb{C}}$ to the symmetric space $\mathcal{Y}_{p,q}$ of $\mathrm{SU}(p,q)$ is holomorphic or antiholomorphic.*

We will explain in §4.4 why Theorem 1.1 and Corollary 1.2 for reductive representations in $\mathrm{SU}(p,q)$ follow from this result.

As we said in the introduction, it is a theorem of [BIW09] that maximal representations are necessarily reductive, so that non reductive ones could be excluded from the very beginning. We will nevertheless discuss (and indeed exclude, eventually) non reductive representations in §4.5, for two reasons. Firstly, the arguments of [BIW09] are quite different from those of the present paper, and we wish to be as self-contained as possible. Secondly and more importantly, it is interesting to see how the rigidity of reductive maximal representations in turn implies that non reductive ones don't exist.

Our approach to Theorem 4.1 is based on the study of the real Higgs bundle (E, θ) over the compact complex hyperbolic manifold $X = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ constructed from the ρ-equivariant harmonic map $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}_{p,q}$ (which exists since ρ is reductive), see §2.1.

After some preliminaries on the group $\mathrm{SU}(p,q)$ and its symmetric space $\mathcal{Y}_{p,q}$, the *real structure* of the Higgs bundle (E, θ) will be described in §4.2.2. We shall see that E is a direct sum $V \oplus W$ and that the Higgs field θ has two components $\beta : W \otimes T_X \to V$ and $\gamma : V \otimes T_X \to W$ corresponding respectively to the holomorphic and antiholomorphic parts of the ρ-equivariant harmonic map f, so that f is holomorphic, resp. antiholomorphic, if and only if $\gamma = 0$, resp. $\beta = 0$.

The proof of Theorem 4.1 then proceeds in three steps.

The first step is a new proof of the Milnor-Wood inequality obtained by pulling back the Higgs bundle $(V \oplus W, \beta \oplus \gamma)$ over X to a Higgs bundle $(\tilde{V} \oplus \tilde{W}, \tilde{\beta} \oplus \tilde{\gamma})$ over the projectivized tangent bundle $\mathbb{P}T_X$ with its tautological foliation $\mathcal{T}$, so that the ideas concerning foliated Higgs bundles developed in §2.2 and §3.2 can come into play.

Proposition 3.1 is used to show that the Milnor-Wood inequality is equivalent to an inequality between the foliated degrees of certain bundles on $\mathbb{P}T_X$, namely

$$|\deg_{\mathcal{T}, \mu_{\mathcal{G}}} \tilde{W}| \leq q \, \frac{\deg_{\mathcal{T}, \mu_{\mathcal{G}}} L^{\vee}}{2}$$

where $L^{\vee}$ is the dual of the tangent line bundle L to the foliation $\mathcal{T}$.

Thanks to the semistability statement of Proposition 2.2, and because the leaves of $\mathcal{T}$ are complex curves, a more precise statement (Proposition 4.2) is proved, exactly in the same way as for surface groups, i.e. lattices in $\mathrm{SU}(1,1)$, see e.g. [Xia00, MX02, BGPG03].

The second step is an analysis of the singular loci of the components β and γ of the Higgs field. Consider β for example. Define the singular locus $\mathcal{S}_{\tilde{\beta}}$ of $\tilde{\beta}$ as the following subset of $\mathbb{P}T_X$:

$$\mathcal{S}_{\tilde{\beta}} = \{\xi \in \mathbb{P}T_X \mid \mathrm{rk}\, \tilde{\beta}_{\xi} < \mathrm{rk}\, \tilde{\beta}\}$$

where $\mathrm{rk}\, \tilde{\beta}$ is the generic rank of $\tilde{\beta} : \tilde{W} \otimes L \to \tilde{V}$, and the singular locus $\mathcal{S}_{\beta}$ of β as the projection $\pi(\mathcal{S}_{\tilde{\beta}})$ of $\mathcal{S}_{\tilde{\beta}}$ to X.

The set $\mathcal{S}_{\tilde{\beta}}$ is a proper analytic subset of $\mathbb{P}T_X$. We want to prove that when the inequality of Proposition 4.2 is an equality, and say $\deg_{\mathcal{T}, \mu_{\mathcal{G}}} \tilde{W} > 0$, then $\mathcal{S}_{\beta}$ is also a proper analytic subset in X. This is achieved by first proving that $\mathcal{S}_{\tilde{\beta}}$ is saturated under the tautological foliation $\mathcal{T}$. This follows from our proof of the inequality, and from the weak polystability of $(\tilde{E}, \tilde{\theta})$ along the leaves of $\mathcal{T}$, see Proposition 2.2. We may then apply Proposition 3.6 which indeed implies that $\mathcal{S}_{\beta}$ is a proper analytic subset of X.

The third step is the conclusion where we prove that in the maximal case, and say if $\tau(\rho) > 0$, the injectivity of $\beta_x(\xi)$ for all $x \in X \backslash \mathcal{S}_\beta$ and $\xi \in T_{X,x} \backslash \{0\}$ forces γ to vanish, hence the ρ-equivariant harmonic map to be holomorphic. Here we use the integrability condition $[\theta, \theta] = 0$ and our standing assumption that $n \geq 2$. If $\tau(\rho) < 0$, the harmonic map is proved to be antiholomorphic by considering γ instead of β.

4.2. Preliminaries.

4.2.1. *The symmetric space $\mathcal{Y}_{p,q}$.*

We recall here some necessary facts on the symmetric space $\mathcal{Y}_{p,q}$ associated to the group $\mathrm{SU}(p,q)$. We refer to [KM08b, §3.1] for details.

Let $\mathbb{E}$ be the vector space $\mathbb{C}^{p+q}$ equipped with a Hermitian form $h_{p,q}$ of signature (p,q), $p \geq q$. The group $\mathrm{SU}(p,q) = \mathrm{SU}(\mathbb{E}, h_{p,q})$ acts transitively on $\mathcal{Y}_{p,q}$, the open subset of the Grassmann manifold of q-dimensional subspaces of $\mathbb{E}$ consisting of q-subspaces on which $h_{p,q}$ restricts to a negative definite Hermitian form. Let $\mathbb{W} \subset \mathbb{E}$ be a point in $\mathcal{Y}_{p,q}$, and $\mathbb{V} \subset \mathbb{E}$ be its orthogonal complement w.r.t. $h_{p,q}$. The stabilizer of $\mathbb{W}$ is a maximal compact subgroup of $\mathrm{SU}(p,q)$ and is isomorphic to $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$. Hence $\mathcal{Y}_{p,q} = \mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$. As a bounded symmetric domain, it is naturally identified with $\{Z \in M_{p,q}(\mathbb{C}) \,|\, 1_q - Z^\star Z > 0\} \subset \mathbb{C}^{pq}$. The rank of the symmetric space $\mathcal{Y}_{p,q}$ is $\min\{p,q\} = q$.

We have the tangent spaces identifications $T_{\mathcal{Y}_{p,q},\mathbb{W}} \simeq T_{\mathbb{W}}^{1,0}\mathcal{Y}_{p,q} \simeq \mathbb{W}^\star \otimes \mathbb{V}$ and $T_{\mathbb{W}}^{0,1}\mathcal{Y}_{p,q} \simeq \mathbb{V}^\star \otimes \mathbb{W}$. We normalize the $\mathrm{SU}(p,q)$-invariant metric $\omega_{\mathcal{Y}_{p,q}}$ on $\mathcal{Y}_{p,q}$ so that, representing an element of $T_{\mathcal{Y}_{p,q},\mathbb{W}}$ by a matrix in $M_{p,q}(\mathbb{C})$, the holomorphic sectional curvature for the complex line $\langle A \rangle$ generated by a nonzero $A \in M_{p,q}(\mathbb{C})$ is given by

$$\kappa(\langle A \rangle) = -\frac{\mathrm{tr}\left((A^\star A)^2\right)}{(\mathrm{tr}\,(A^\star A))^2}.$$

This formula shows that $\kappa(\langle A \rangle)$ is pinched between -1 and $-1/q$ and that $\kappa(\langle A \rangle) = -1/q$ if and only if the column vectors of A are pairwise orthogonal and have the same norm (for the standard Hermitian scalar product in $\mathbb{C}^p$).

The symmetric space $\mathcal{Y}_{p,q}$ is a Kähler-Einstein manifold, and with our curvature normalization, the first Chern form of its holomorphic tangent bundle $T_{\mathcal{Y}_{p,q}}$ is $c_1(T_{\mathcal{Y}_{p,q}}) = -\frac{1}{2\pi} \frac{p+q}{2} \omega_{\mathcal{Y}_{p,q}}$.

4.2.2. *The real structure of a $\mathrm{SU}(p,q)$-Higgs bundle.*

Harmonic Higgs bundles arising from reductive representations into real reductive subgroups G of $\mathrm{SL}(N, \mathbb{C})$ have an additional *real structure*, compared to those arising from representations in $\mathrm{SL}(N, \mathbb{C})$ without further restriction, see e.g. [Sim92, p. 90-91] or [Mau15, §3.6].

We describe this real structure in our case, namely for $G = \mathrm{SU}(p,q) \subset \mathrm{SL}(p+q, \mathbb{C})$. More details can be found for example in [KM08b, §2.4 & §3.2] or [Mau15, §3.6.2 & §3.6.3].

The first observation is that a Higgs bundle $(E, \theta) \to Y$ associated to a reductive representation in $\mathrm{SU}(p,q)$ of the fundamental group of a compact Kähler manifold Y splits holomorphically as a sum $E = V \oplus W$, where V has rank p and W has rank q. Indeed, as a smooth bundle, E is the flat bundle associated to the standard representation of $\mathrm{SL}(p+q, \mathbb{C})$ on $\mathbb{E} = \mathbb{C}^{p+q}$. But the ρ-equivariant harmonic map $f : \tilde{Y} \to \mathcal{Y}_{p,q}$ defines a reduction of its structure group to the maximal compact subgroup $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ of $\mathrm{SU}(p,q)$, hence to its complexification $\mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$. Since this group preserves the decomposition $\mathbb{E} = \mathbb{V} \oplus \mathbb{W}$, we indeed get the holomorphic splitting $E = V \oplus W$.

Note that $\deg V + \deg W = \deg E = 0$ because E is flat, and that since the harmonic metric on E is defined by the reduction of the structure group of E to $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$, the direct sum $E = V \oplus W$ is orthogonal for the harmonic metric.

The Higgs field is by construction the $(1,0)$-part $d^{1,0}f : T^{1,0}\tilde{Y} \to T^{\mathbb{C}}\mathcal{Y}_{p,q}$ of the complexified differential of the harmonic map f, seen as an endomorphism of E in the following way. The complexified tangent bundle $T^{\mathbb{C}}\mathcal{Y}_{p,q}$ is the bundle on $\mathcal{Y}_{p,q} = \mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ associated to the adjoint action of $\mathrm{S}(\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}))$ on $T^{\mathbb{C}}_{\mathbb{W}}\mathcal{Y}_{p,q} = T^{1,0}_{\mathbb{W}}\mathcal{Y}_{p,q} \oplus T^{0,1}_{\mathbb{W}}\mathcal{Y}_{p,q} = (\mathbb{W}^{\star} \otimes \mathbb{V}) \oplus (\mathbb{V}^{\star} \otimes \mathbb{W})$. Therefore the pull-back bundle $f^{\star}T^{\mathbb{C}}\mathcal{Y}_{p,q}$ over Y is the subbundle $(W^{\star} \otimes V) \oplus (V^{\star} \otimes W)$ of $\mathrm{End}(E)$ and the Higgs field θ is $d^{1,0}f$ seen as a holomorphic 1-form with values in this bundle.

This means that the Higgs field seen as a sheaf morphism $\theta : E \otimes T_Y \to E$ is off-diagonal w.r.t. the decomposition $E = V \oplus W$: it has two components $\beta : W \otimes T_Y \to V$ and $\gamma : V \otimes T_Y \to W$. Moreover the vanishing of γ, resp. β, exactly means that $d^{1,0}f : T^{1,0}\tilde{Y} \to T^{\mathbb{C}}\mathcal{Y}_{p,q}$ maps $T^{1,0}\tilde{Y}$ to $T^{1,0}\mathcal{Y}_{p,q}$, resp. $T^{0,1}\mathcal{Y}_{p,q}$, i.e. that the harmonic map $f : \tilde{Y} \to \mathcal{Y}_{p,q}$ is holomorphic, resp. antiholomorphic.

4.3. **Proof of Theorem 4.1.**

In this section Γ is a torsion free uniform complex hyperbolic lattice Γ in $\mathrm{SU}(n,1)$, $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ the corresponding compact complex hyperbolic manifold, $\pi : \mathbb{P}T_X \to X$ the projectivized tangent bundle of X, $\mathcal{T}$ the tautological foliation on $\mathbb{P}T_X$, ρ a reductive representation of Γ in $\mathrm{SU}(p,q)$ and $(E,\theta) = (V \oplus W, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})$ the $\mathrm{SU}(p,q)$-Higgs bunde over X associated to ρ.

We consider the pull-back $(\pi^{\star}E, \pi^{\star}\theta)$ of the Higgs bundle $(E,\theta) \to X$ to the projectivized tangent bundle $\mathbb{P}T_X$. This is the harmonic Higgs bundle over $\mathbb{P}T_X$ associated to the representation of $\pi_1(\mathbb{P}T_X) \simeq \pi_1(X)$ induced by ρ. To lighten the notation, pulled-back objects will be denoted with a "$\sim$". In particular, $\tilde{W}$ is the rank q holomorphic bundle $\pi^{\star}W$ on $\mathbb{P}T_X$, $\tilde{V}$ is the rank p holomorphic bundle $\pi^{\star}V$, and

$$\begin{cases} \tilde{\beta} : \tilde{W} \otimes L \longrightarrow \tilde{V} \\ \tilde{\gamma} : \tilde{V} \otimes L \longrightarrow \tilde{W} \end{cases}$$

are the two components of the lifted Higgs field $\tilde{\theta}$ restricted to the tangent line bundle L of the tautological foliation $\mathcal{T}$ on $\mathbb{P}T_X$. (From now on, we shall denote by the same letter a vector bundle defined on $\mathbb{P}T_X$, or X, and the sheaf of its sections.)

Summing up, on the projectivized tangent bundle $\mathbb{P}T_X$, we have the harmonic $\mathrm{SU}(p,q)$-Higgs bundle $(\tilde{E},\tilde{\theta})$ and the tautological foliation $\mathcal{T}$ with its invariant transverse measure $\mu_{\mathcal{G}}$ given by the transverse indefinite Kähler form $\omega_{\mathcal{G}}$, see Proposition 3.1. By Proposition 2.2, $(\tilde{E},\tilde{\theta})$ is weakly polystable along the leaves of $\mathcal{T}$ with respect to $\mu_{\mathcal{G}}$ and we will exploit this fact to prove Theorem 4.1.

From now on, foliated degrees of sheaves on $\mathbb{P}T_X$ will always be computed with the transverse measure $\mu_{\mathcal{G}}$. Hence we will abbreviate the notation $\deg_{\mathcal{T},\mu_{\mathcal{G}}}$ to $\deg_{\mathcal{T}}$.

4.3.1. *Milnor-Wood inequality.*

We begin by reformulating the Milnor-Wood inequality on the Toledo invariant of ρ in terms of foliated degrees of vector bundles on $\mathbb{P}T_X$.

If $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}_{p,q}$ is the ρ-equivariant harmonic map, the Toledo invariant of ρ is given by

$$\tau(\rho) = \frac{1}{n!} \int_X f^{\star}\omega_{\mathcal{Y}_{p,q}} \wedge \omega^{n-1}.$$

We saw in §4.2.1 and §4.2.2 that $c_1(T\mathcal{Y}_{p,q}) = -\frac{1}{2\pi}\frac{p+q}{2}\,\omega_{\mathcal{Y}_{p,q}}$ and $f^\star T\mathcal{Y}_{p,q} \simeq W^\star \otimes V$, where $f^\star T\mathcal{Y}_{p,q}$ is here seen as a bundle on $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ by ρ-equivariance of f. Remembering that $\deg V = -\deg W$, we get

$$\tau(\rho) = -\frac{4\pi}{p+q}\deg f^\star T\mathcal{Y}_{p,q} = -\frac{4\pi}{p+q}\left(-p\deg W + q\deg V\right) = 4\pi\deg W = 4\pi\deg_{\mathcal{T}}\tilde{W},$$

where the last equality is given by Proposition 3.1.

On the other hand, by the last item of Lemma 3.2 we have $c_1(L^\vee) = \frac{1}{2\pi}(\pi^\star\omega - \frac{1}{2}\pi^\star_{\mathcal{G}}\omega_{\mathcal{G}})$, so that again by Proposition 3.1 and the definition of the invariant transverse measure $\mu_{\mathcal{G}}$ just before Lemma 3.3:

$$\deg_{\mathcal{T}}L^\vee = \frac{1}{2\pi}\int_{\mathbb{P}T_X}(\pi^\star\omega - \frac{1}{2}\pi^\star_{\mathcal{G}}\omega_{\mathcal{G}})\wedge\Omega_{\mathcal{G}} = \frac{1}{2\pi}\int_{\mathbb{P}T_X}\pi^\star\omega\wedge\Omega_{\mathcal{G}} = \frac{1}{2\pi\,n!}\int_X\omega^n = \frac{1}{2\pi}\operatorname{vol}(X).$$

Therefore the Milnor-Wood inequality $|\tau(\rho)| \le q\operatorname{vol}(X)$ for reductive representations is equivalent to

$$|\deg_{\mathcal{T}}\tilde{W}| \le q\,\frac{\deg_{\mathcal{T}}L^\vee}{2}$$

and reductive maximal representations are those for which this inequality is an equality. The proof of this inequality will mimic the "Higgs bundles proof" of the Milnor-Wood inequality in the one dimensional case, i.e. for representations of surface groups (see e.g. [Xia00]). It is based on the semistability of the Higgs bundle $(\tilde{E}, \theta)$ along the leaves of $\mathcal{T}$.

To be more precise, let $\operatorname{rk}\tilde{\beta}$ and $\operatorname{rk}\tilde{\gamma}$ be the generic ranks of $\tilde{\beta} : \tilde{W}\otimes L \to \tilde{V}$ and $\tilde{\gamma} : \tilde{V}\otimes L \to \tilde{W}$. Since $\operatorname{rk}\tilde{\beta}$ and $\operatorname{rk}\tilde{\gamma}$ are bounded above by q, the Milnor-Wood inequality follows from

Proposition 4.2. *We have* $-\operatorname{rk}\tilde{\gamma}\,\dfrac{\deg_{\mathcal{T}}L^\vee}{2} \le \deg_{\mathcal{T}}\tilde{W} \le \operatorname{rk}\tilde{\beta}\,\dfrac{\deg_{\mathcal{T}}L^\vee}{2}.$

Proof. We first prove that $\deg_{\mathcal{T}}\tilde{W} \le \operatorname{rk}\tilde{\beta}\,\frac{\deg_{\mathcal{T}}L^\vee}{2}$. Consider $\tilde{\beta} : \tilde{W}\otimes L \to \tilde{V}$.

If $\tilde{\beta} = 0$ then $\tilde{W}$ is a leafwise Higgs subsheaf of $(\tilde{E}, \tilde{\theta})$ hence by semistability along the leaves of $\mathcal{T}$, see Proposition 2.2, $\deg_{\mathcal{T}}\tilde{W} \le 0$ and we are done.

Assume therefore that $\tilde{\beta} \ne 0$. Let $\mathcal{N} = \operatorname{Ker}\tilde{\beta} \subset \tilde{W}\otimes L$ and $\mathcal{I}$ be the saturation (as a sheaf) of $\operatorname{Im}\tilde{\beta} \subset \tilde{V}$. By construction, $\mathcal{N}\otimes L^\vee$ and $\tilde{W}\oplus\mathcal{I}$ are leafwise Higgs subsheaves of $(\tilde{E}, \tilde{\theta})$ and, again by leafwise semistability,

$$\deg_{\mathcal{T}}(\mathcal{N}\otimes L^\vee) \le 0 \ \text{ and } \ \deg_{\mathcal{T}}\tilde{W} + \deg_{\mathcal{T}}\mathcal{I} \le 0.$$

Moreover, we have $\deg_{\mathcal{T}}(W\otimes L) = \deg_{\mathcal{T}}\mathcal{N} + \deg_{\mathcal{T}}\operatorname{Im}\tilde{\beta} \le \deg_{\mathcal{T}}\mathcal{N} + \deg_{\mathcal{T}}\mathcal{I}$ since $\deg_{\mathcal{T}}\mathcal{I} \ge \deg_{\mathcal{T}}\operatorname{Im}\tilde{\beta}$, see the proof of Proposition 2.2 (1). Because the generic rank of $\tilde{\beta}$ is the rank of $\mathcal{I}$ and of $(\tilde{W}\otimes L)/\mathcal{N}$, we thus have

$$\deg_{\mathcal{T}}\tilde{W} + q\deg_{\mathcal{T}}L \le \deg_{\mathcal{T}}\mathcal{N} + \deg_{\mathcal{T}}\mathcal{I} \le (q - \operatorname{rk}\tilde{\beta})\deg_{\mathcal{T}}L - \deg_{\mathcal{T}}\tilde{W}$$

and hence $\deg_{\mathcal{T}}\tilde{W} \le \operatorname{rk}\tilde{\beta}\,\dfrac{\deg_{\mathcal{T}}L^\vee}{2}$.

In the same way, using $\tilde{\gamma} : \tilde{V}\otimes L \to \tilde{W}$, we obtain $-\deg_{\mathcal{T}}\tilde{W} = \deg_{\mathcal{T}}\tilde{V} \le \operatorname{rk}\tilde{\gamma}\,\dfrac{\deg_{\mathcal{T}}L^\vee}{2}$. $\quad\square$

Remark 4.3. This proposition holds more generally in the setting of §2.2, i.e. for a harmonic $\operatorname{SU}(p,q)$-Higgs bundle over a compact Kähler manifold Y with a smooth foliation $\mathcal{T}$ by holomorphic curves and an invariant transverse volume form, so that Proposition 2.2 applies. Note that in this case $\tilde{\beta}$ and $\tilde{\gamma}$ have to be understood as the restriction of the components β and γ of the Higgs field to the leaves.

We remark that by the proof of the proposition, if the foliation $\mathcal{T}$ is such that $\deg_{\mathcal{T},\mu} L^{\vee} \leq 0$, then necessarily $\tilde{\beta}$ and $\tilde{\gamma}$ vanish identically (i.e the harmonic map $f : \tilde{Y} \to \mathcal{Y}_{p,q}$ is constant along the lifted leaves on $\tilde{Y}$) and $\deg_{\mathcal{T},\mu} \tilde{W} = 0$. Observe however that if $\deg_{\mathcal{T},\mu} L^{\vee} < 0$, it follows from a result of Bogomolov and McQuillan [BM01] and from the stability for holomorphic foliations, see [Per01] for instance, that the foliation is a fibration whose fibers are rational curves (at least if M is projective). But then, the harmonic map is constant in the fibers of the fibration (by pluriharmonicity and the fact that a harmonic map on $\mathbb{P}^1_{\mathbb{C}}$ with values in a non positively curved manifold is constant) so that the result was known a priori.

Remark 4.4. *Closed leaves III: Milnor-Wood inequality.* In a similar spirit, the convergence of currents alluded to in Remark 3.4 allows to give another proof of the Milnor-Wood inequality on compact complex hyperbolic manifolds given by arithmetic lattices of type I by deducing it from the inequality on the (infinitely many) totally geodesic curves they contain. It seems however difficult to build on this idea to infer the rigidity of maximal representations in this special case, while the more general approach presented here will prove more fruitful.

4.3.2. *The singular locus of the Higgs field.*

We now study the equality case in Proposition 4.2. Using the weak polystability of the Higgs bundle along the leaves of the tautological foliation $\mathcal{T}$ and the results of §3.3 on the dynamics of $\mathcal{T}$, we show that if equality holds, then a component of the Higgs field, β or γ, is regular (in a sense to be defined below) on an everywhere dense subset of X.

Say that a point $\xi \in \mathbb{P}T_X$ is a $\tilde{\beta}$-*regular* point, or that $\tilde{\beta}$ is regular at ξ, if the rank of $\tilde{\beta}_\xi : \tilde{W}_\xi \otimes L_\xi \to \tilde{V}_\xi$ is the generic rank of $\tilde{\beta} : \tilde{W} \otimes L \to \tilde{V}$. Say that a point $x \in X$ is a β-*regular* point, or that β is regular at x, if the fiber of $\mathbb{P}T_X$ above x consists only of $\tilde{\beta}$-regular points. Points in X, resp. $\mathbb{P}T_X$, which are not β-regular, resp. $\tilde{\beta}$-regular, are β-*singular*, resp. $\tilde{\beta}$-*singular*.

Define accordingly the *singular locus* $\mathcal{S}_{\tilde{\beta}}$ of $\tilde{\beta} : \tilde{W} \otimes L \to \tilde{V}$ as the subset of $\tilde{\beta}$-singular points in $\mathbb{P}T_X$:

$$\mathcal{S}_{\tilde{\beta}} = \{\xi \in \mathbb{P}T_X \mid \operatorname{rk}(\tilde{\beta}_\xi) < \operatorname{rk}\tilde{\beta}\}$$

and the *singular locus* $\mathcal{S}_\beta$ of $\beta : W \otimes T_X \to V$ as the subset of β-singular points in X. Note that $\mathcal{S}_\beta$ is by definition the projection of $\mathcal{S}_{\tilde{\beta}}$ to X:

$$\mathcal{S}_\beta = \pi(\mathcal{S}_{\tilde{\beta}}) = \{x \in X \mid \exists \xi \in T_{X,x}, \xi \neq 0, \text{ such that } \operatorname{rk}\beta_x(\xi) < \operatorname{rk}\tilde{\beta}\}.$$

One defines similarly $\tilde{\gamma}$- and γ- regular and singular points as well as the singular loci $\mathcal{S}_{\tilde{\gamma}}$ and $\mathcal{S}_\gamma$.

Observe that while $\mathcal{S}_{\tilde{\beta}}$ and $\mathcal{S}_{\tilde{\gamma}}$ are proper analytic subsets of $\mathbb{P}T_X$, $\mathcal{S}_\beta$ and $\mathcal{S}_\gamma$ might well be the whole X.

Lemma 4.5. *If* $\deg_{\mathcal{T}} \tilde{W} = \operatorname{rk}\tilde{\beta} \frac{\deg_{\mathcal{T}} L^{\vee}}{2}$, *the singular locus* $\mathcal{S}_{\tilde{\beta}}$ *of* $\tilde{\beta} : \tilde{W} \otimes L \to \tilde{V}$ *is a proper* $\mathcal{T}$-*saturated subset of* $\mathbb{P}T_X$. *If* $\deg_{\mathcal{T}} \tilde{W} = -\operatorname{rk}\tilde{\gamma} \frac{\deg_{\mathcal{T}} L^{\vee}}{2}$, *then the singular locus* $\mathcal{S}_{\tilde{\gamma}}$ *of* $\tilde{\gamma} : \tilde{V} \otimes L \to \tilde{W}$ *is a proper* $\mathcal{T}$-*saturated subset of* $\mathbb{P}T_X$.

Proof. We prove the assertion on $\tilde{\beta}$. Call r the generic rank of $\tilde{\beta}$. Let $\mathcal{N}$ and $\mathcal{I}$ be respectively the kernel sheaf and the saturation of the image sheaf of $\tilde{\beta} : \tilde{W} \otimes L \to \tilde{V}$ and let $\mathcal{S}(\mathcal{N})$ and $\mathcal{S}(\mathcal{I})$ be their singular loci (as defined just before Proposition 2.2). Observe that by definition, outside of $\mathcal{S}_{\tilde{\beta}}$, the rank of $\tilde{\beta}_\xi$ is constant equal to r, so that $\mathcal{N}$, resp. $\mathcal{I}$, is the sheaf of sections of a subbundle of $\tilde{W} \otimes L$, resp. $\tilde{V}$. This implies that $\mathcal{S}(\mathcal{N})$ and $\mathcal{S}(\mathcal{I})$ are included in $\mathcal{S}_{\tilde{\beta}}$.

By the proof of Proposition 4.2, if $\deg_{\mathcal{T}}\tilde{W} = r\frac{\deg_{\mathcal{T}}L^\vee}{2}$ then the foliated degrees of $\mathcal{N}\otimes L^\vee$ and $\mathcal{I}\oplus\tilde{W}$, which are leafwise Higgs subsheaves of $\tilde{E}$, vanish. Since $\tilde{W}/(\mathcal{N}\otimes L^\vee)$ and $\tilde{V}/\mathcal{I}$ are torsion free, by the weak polystability property of $(\tilde{E},\tilde{\theta})$ proved in Proposition 2.2 (2a), $\mathcal{S}(\mathcal{N})$ and $\mathcal{S}(\mathcal{I})$ are both $\mathcal{T}$-saturated. Moreover there exist a rank $q-r$ holomorphic subbundle N of $\tilde{W}$, a rank r holomorphic subbundle I of $\tilde{V}$, both defined outside of the codimension at least 2 subset $S := \mathcal{S}(\mathcal{N})\cup\mathcal{S}(\mathcal{I})$ of $\mathbb{P}T_X$, such that on $\mathbb{P}T_X\backslash S$, $\mathcal{N}\otimes L^\vee$ and $\mathcal{I}$ are the sheaves of sections of N and I.

Since, outside of S, $\tilde{\beta}$ maps $(\tilde{W}/N)\otimes L$ to I and $\operatorname{rk} I = r = \operatorname{rk}(\tilde{W}/N)$, the set of points $\xi\in\mathbb{P}T_X\backslash S$ where $\tilde{\beta}_\xi$ is not of rank r is locally given by the vanishing of a single holomorphic function and hence has codimension 1 if not empty. This means that the components of $\mathcal{S}_{\tilde{\beta}}$ of higher codimension are included in S and hence that $\tilde{\beta}:\tilde{W}\otimes L\to\tilde{V}$ has rank r, as a vector bundle map, outside $S\cup|\Delta|$, where $|\Delta|$ is the (possibly empty) divisorial part of $\mathcal{S}_{\tilde{\beta}}$, i.e. the union of its irreducible components Δ_j of codimension 1. Thus $\mathcal{S}_{\tilde{\beta}}$ is included in $S\cup|\Delta|$, so that in fact, by our first observation, $\mathcal{S}_{\tilde{\beta}} = S\cup|\Delta| = \mathcal{S}(\mathcal{N})\cup\mathcal{S}(\mathcal{I})\cup|\Delta|$.

By an argument similar to [Kob87, Chap. V (8.5) p. 180], there is a line bundle $[\Delta]$ on $\mathbb{P}T_X$ corresponding to a divisor $\Delta = \sum_j a_j\Delta_j$ whose support is $|\Delta|$ (i.e. $a_j\geq 1$ for all j) such that $\det\mathcal{I}\simeq\det(\operatorname{Im}\tilde{\beta})\otimes[\Delta]$ on $\mathbb{P}T_X$. Again by the proof of Proposition 4.2, $\deg_{\mathcal{T}}\tilde{W} = r\frac{\deg_{\mathcal{T}}L^\vee}{2}$ implies $\deg_{\mathcal{T}}\operatorname{Im}\tilde{\beta} = \deg_{\mathcal{T}}\mathcal{I}$, thus $\deg_{\mathcal{T}}[\Delta] = \sum_j a_j\int_{\Delta_j}\Omega_{\mathcal{G}} = 0$. This means that for all j, and at each smooth point x of Δ_j, the leaf $\mathcal{L}_x$ of $\mathcal{T}$ through x is tangent to Δ_j. As the foliation is smooth, $\mathcal{L}_x$ must be contained in Δ_j. Now in Δ_j, the smooth points are dense and the set of points whose leaves stay in Δ_j is closed, for it is analytic as explained in the proof of (2) in Proposition 2.2. Thus Δ_j is $\mathcal{T}$-saturated for all j. $\qquad\square$

Remark 4.6. As it is clear from its proof, Lemma 4.5 also holds more generally in the setting of §2.2 if Proposition 2.2 applies.

If we now consider the singular locus of β or γ in X, Proposition 3.6 implies immediately:

Corollary 4.7. *If $\deg_{\mathcal{T}}\tilde{W} = \operatorname{rk}\tilde{\beta}\frac{\deg_{\mathcal{T}}L^\vee}{2}$, the singularity set $\mathcal{S}_\beta$ of $\beta:W\otimes T_X\to V$ is a proper analytic subset of X. If $\deg_{\mathcal{T}}\tilde{W} = -\operatorname{rk}\tilde{\gamma}\frac{\deg_{\mathcal{T}}L^\vee}{2}$ then the singular locus $\mathcal{S}_\gamma$ of $\gamma:V\otimes T_X\to W$ is a proper analytic subset of X.*

Proof. We prove the assertion on β. Since $\mathcal{S}_{\tilde{\beta}}$ is a proper closed subset of $\mathbb{P}T_X$ and is $\mathcal{T}$-saturated by Lemma 4.5, Proposition 3.6 implies that $\mathcal{S}_\beta = \pi(\mathcal{S}_{\tilde{\beta}})$ is a proper subset of X. Now $\mathcal{S}_{\tilde{\beta}}$ is an analytic subset and π a proper map, so $\mathcal{S}_\beta$ is also an analytic subset of X. $\quad\square$

4.3.3. *Conclusion.*

We are now in position to conclude the proof of Theorem 4.1. So we assume that the reductive representation ρ is maximal. We want to prove that the ρ-equivariant harmonic map f is holomorphic or antiholomorphic, i.e. that one of the components of the Higgs field it defines vanishes. By the previous paragraph, we already know that one component is regular outside a proper analytic subset of X. The idea is that if $n\geq 2$, the integrability property $[\theta,\theta] = 0$ of the Higgs field forces the other component to vanish outside of this subset, hence everywhere.

Suppose that $\tau(\rho) > 0$, so that $\deg_{\mathcal{T}}\tilde{W} = q\frac{\deg_{\mathcal{T}}L^\vee}{2}$. We know from Proposition 4.2 that $\operatorname{rk}\tilde{\beta} = q$, hence from §4.3.2 that the set of β-regular points

$$X\backslash\mathcal{S}_\beta = \{x\in X\mid\beta_x(\xi):W_x\longrightarrow V_x\text{ is injective for all }\xi\neq 0\text{ in }T_{X,x}\}$$

is everywhere dense in X.

Let us fix a β-regular point $x \in X$, i.e. $x \notin \mathcal{S}_\beta$. For $\xi \neq 0$ in $T_{X,x}$, call $I_\xi \subset V_x$ the image of $\beta_x(\xi) : W_x \to V_x$ (which is injective), and $I_\xi^\perp$ its orthogonal complement in V_x w.r.t. the harmonic metric. Observe that $I_\xi^\perp$ is also the orthogonal complement of $I_\xi \oplus W_x$ in $E_x = V_x \oplus W_x$, because V_x and W_x are orthogonal for the harmonic metric.

Using the integrability property of the Higgs field and again the weak polystability along the leaves, we first prove

Lemma 4.8. *For all η and all $\xi \neq 0$ in $T_{X,x}$, $\gamma_x(\eta)$ vanishes on $I_\xi^\perp$.*

Proof. Since $\deg_{\mathcal{T}} \tilde{W} = q \frac{\deg_{\mathcal{T}} L^\vee}{2}$, we know from the proof of Proposition 4.2 that the kernel sheaf $\mathcal{N}$ of $\tilde{\beta} : \tilde{W} \otimes L \to \tilde{V}$ is zero and that its image sheaf $\mathcal{I}$ satisfies $\deg_{\mathcal{T}}(\tilde{W} \oplus \mathcal{I}) = 0$. By weak polystability along the leaves, see Proposition 2.2, outside the singular locus $\mathcal{S}_{\tilde{\beta}}$ of $\tilde{\beta}$, there is a subbundle I of $\tilde{V}$ such that $\mathcal{I}$ is the sheaf of sections of I and $(\tilde{E}, \theta) = (\tilde{W} \oplus I, \theta_{|W \oplus I}) \oplus (I^\perp, \theta_{|I^\perp})$ is a Higgs bundle decomposition along the leaves of $\mathcal{T}$, where $I^\perp$ is the orthogonal complement in $\tilde{E}$ of $\tilde{W} \oplus I$ w.r.t. the lifted harmonic metric on $\tilde{E}$. Since $x \notin \mathcal{S}_\beta$, this means that for all $\xi \neq 0$ in $T_{X,x}$, $W_x \oplus I_\xi$ and $I_\xi^\perp$ are invariant by $\theta_x(\xi)$.

But $I_\xi^\perp \subset V_x$ and $\theta_x(\xi)_{|V_x} = \gamma_x(\xi)$ maps V_x to W_x. Hence $I_\xi^\perp \subset \operatorname{Ker} \gamma_x(\xi)$.

On the other hand, the integrability property $[\theta, \theta] = 0$ of the Higgs field means that $\beta_x(\xi) \circ \gamma_x(\eta) = \beta_x(\eta) \circ \gamma_x(\xi)$ for all $\xi, \eta \in T_{X,x}$. Therefore, if $v \in \operatorname{Ker} \gamma_x(\xi)$ for some $\xi \neq 0$, then for all η we have $\beta_x(\xi)(\gamma_x(\eta)v) = \beta_x(\eta)(\gamma_x(\xi)v) = 0$ and hence $\gamma_x(\eta)v = 0$ since $\beta_x(\xi)$ is injective. Hence for all $\xi \neq 0$ and all η, $\operatorname{Ker} \gamma_x(\xi) \subset \operatorname{Ker} \gamma_x(\eta)$. $\square$

The next lemma shows that the subspaces $I_\xi^\perp$ for $\xi \neq 0$ generate V_x. This is the only point in the proof for which the assumption that $n \geq 2$ is required.

Lemma 4.9. *Assume that $n \geq 2$. Then $\cap_{\xi \neq 0} I_\xi = \{0\}$.*

Proof. Let indeed v be in $\cap_{\xi \neq 0} I_\xi$. For all $\xi \neq 0$, there exists $\varphi(\xi) \in W_x$ such that $\beta_x(\xi)\varphi(\xi) = v$. By the injectivity of $\beta_x(\xi)$, $\varphi(\xi)$ is unique and φ is a well-defined map from $T_{X,x} \backslash \{0\}$ to W_x. Since φ is locally given by inverting a q-by-q submatrix of $\beta_x(\xi)$, it is holomorphic on $T_{X,x} \backslash \{0\}$. Because $n \geq 2$, the map φ can be extended holomorphically to $0 \in T_{X,x}$ and necessarily $\beta_x(0)\varphi(0) = v$ so that $v = 0$ since $\beta_x : T_{X,x} \to \operatorname{Hom}(W_x, V_x)$ is linear. $\square$

Together these lemmas imply that for $n \geq 2$, γ vanishes outside a proper analytic subset of X, hence everywhere and the ρ-equivariant harmonic map f is holomorphic.

In the same manner, if $\tau(\rho) < 0$, then $\deg_{\mathcal{T}} \tilde{W} = -\frac{q}{2} \deg_{\mathcal{T}} L^\vee$, and if $n \geq 2$, β vanishes outside the singular locus $\mathcal{S}_\gamma$ of γ, hence identically, and the ρ-equivariant harmonic map f is antiholomorphic.

4.4. **Proof of Theorem 1.1 and Corollary 1.2 for reductive representations.**

Recall from §4.2.1 that the maximal value of the holomorphic sectional curvature of the $\operatorname{SU}(p,q)$-invariant metric $\omega_{\mathcal{Y}_{p,q}}$ of $\mathcal{Y}_{p,q}$ is $-1/q$. The Ahlfors-Schwarz-Pick lemma [Roy80] therefore implies that if $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}_{p,q}$ is holomorphic, then $f^\star \omega_{\mathcal{Y}_{p,q}} \leq q\,\omega$. Moreover, this inequality is an equality only if the induced holomorphic sectional curvature on the image of f is everywhere maximal, i.e. equal to $-1/q$, and we proved the following result in [KM08b, §3.1]:

Proposition 4.10. *Let $f : \mathbb{H}_{\mathbb{C}}^n \to \mathcal{Y}_{p,q}$ be a holomorphic map such that $f^\star \omega_{\mathcal{Y}_{p,q}} = q\,\omega$. Then $p \geq qn$ and up to the composition of f by an isometry of $\mathcal{Y}_{p,q}$, f is equal to the following holomorphic totally geodesic embedding:*

$$f_{\mathrm{diag}} : \mathbb{H}_{\mathbb{C}}^n \ni z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \longmapsto Z = \begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & z & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{Y}_{p,q}.$$

The totally geodesic map f_{diag} is equivariant with respect to the standard diagonal embedding $\rho_{\mathrm{diag}} : \mathrm{SU}(n,1) \hookrightarrow \mathrm{SU}(n,1)^q \hookrightarrow \mathrm{SU}(nq,q) \hookrightarrow \mathrm{SU}(p,q)$. The stabilizer of its image in $\mathcal{Y}_{p,q}$ is an almost-direct product of $\rho_{\mathrm{diag}}(\mathrm{SU}(n,1))$ with its centralizer K in $\mathrm{SU}(p,q)$, which is compact and acts trivially on $f_{\mathrm{diag}}(\mathbb{H}_{\mathbb{C}}^n)$.

This proposition shows that Theorem 1.1 and Corollary 1.2 for reductive representations in $\mathrm{SU}(p,q)$ are direct consequences of Theorem 4.1. Indeed, this theorem says that if ρ is a reductive maximal representation in $\mathrm{SU}(p,q)$ of a torsion free uniform lattice Γ of $\mathrm{SU}(n,1)$, $n \geq 2$, then the ρ-equivariant harmonic map $f : \mathbb{H}_{\mathbb{C}}^n \to \mathcal{Y}_{p,q}$ is holomorphic or antiholomorphic.

If f is holomorphic then the Ahlfors-Schwarz-Pick lemma gives the pointwise inequality $f^\star \omega_{\mathcal{Y}_{p,q}} \leq q\,\omega$ whereas the maximality of ρ means that $\int_X f^\star \omega_{\mathcal{Y}_{p,q}} \wedge \omega^{n-1} = q \int_X \omega^n$, so that necessarily $f^\star \omega_{\mathcal{Y}_{p,q}} = q\,\omega$. By Proposition 4.10 we have $p \geq qn$ (which proves Theorem 1.1) and up to composition by an element of $\mathrm{SU}(p,q)$, $f = f_{\mathrm{diag}}$, which is the first assertion of Corollary 1.2. The second assertion follows easily (to prove that ρ is faithful, note that Γ being torsion free, it is isomorphic to its projection to $\mathrm{PU}(n,1)$ which acts effectively on $\mathbb{H}_{\mathbb{C}}^n$). The third assertion follows from our description of the stabilizer of $f_{\mathrm{diag}}(\mathbb{H}_{\mathbb{C}}^n)$ and the fact that up to conjugacy we may assume that f_{diag} is ρ-equivariant so that for $\gamma \in \Gamma$, $\rho(\gamma)$ acts on $f_{\mathrm{diag}}(\mathbb{H}_{\mathbb{C}}^n)$ as $\rho_{\mathrm{diag}}(\gamma)$.

If f is antiholomorphic then the maximality of ρ implies $f^\star \omega_{\mathcal{Y}_{p,q}} = -q\,\omega$, so that again $p \geq qn$ and essentially $f = \bar{f}_{\mathrm{diag}}$.

4.5. **Non reductive representations.**

A very general result of M. Burger, A. Iozzi and A. Wienhard asserts that so-called *tight* representations of lattices, uniform or not, of $\mathrm{SU}(n,1)$, $n \geq 1$, in Hermitian Lie groups are always reductive, see [BIW09, Corollary 4]. Maximal representations are tight, so that the results of §4.3 and §4.4 for reductive representations imply Theorem 1.1 and Corollary 1.2 in the general case.

It is however interesting to see that one can deduce the inexistence of non reductive maximal representations in $\mathrm{SU}(p,q)$ from the rigidity just established for reductive maximal ones. We explain here how to do this by deforming non reductive representations to reductive ones, a known operation sometimes called *semi-simplification*:

Lemma 4.11. *A non reductive group homomorphism ρ of a group Γ in a semisimple Lie group G without compact factors can be deformed to a reductive homomorphism $\rho_{ss} : \Gamma \to P$, where P is a proper parabolic subgroup of G.*

Together with our results for reductive representations, this implies:

Corollary 4.12. *Let ρ be a non reductive representation of a torsion free uniform lattice Γ of $\mathrm{SU}(n,1)$ in $\mathrm{SU}(p,q)$, $p \geq q \geq 1$. Then ρ satisfies the Milnor-Wood inequality $|\tau(\rho)| \leq q\,\mathrm{vol}(X)$. Moreover, if $n \geq 2$, ρ is not maximal.*

Therefore the proofs of Theorem 1.1 and Corollary 1.2 for representations in $\mathrm{SU}(p,q)$ are complete.

Proof of Lemma 4.11. This follows e.g. from [Ric88]. Let $\rho : \Gamma \to G$ a non reductive homomorphism: the Zariski closure $\overline{\rho(\Gamma)}^{\mathbf{z}}$ of $\rho(\Gamma)$ in G is not a reductive group, so that its unipotent radical U is not trivial. Let L be a Levi factor of $\overline{\rho(\Gamma)}^{\mathbf{z}}$. By [Ric88, Proposition 2.6], there exists a 1-parameter subgroup λ of G, such that $\overline{\rho(\Gamma)}^{\mathbf{z}}$ is contained in the parabolic subgroup $P(\lambda) := \{g \in G \,|\, \lim_{t\to+\infty} \lambda(-t)\, g\, \lambda(t) \text{ exists}\}$, U is contained in the unipotent radical $N(\lambda) := \{g \in G \,|\, \lim_{t\to+\infty} \lambda(-t)\, g\, \lambda(t) = 1\}$ of $P(\lambda)$, and L is contained in $L(\lambda) := \{g \in G \,|\, \lim_{t\to+\infty} \lambda(-t)\, g\, \lambda(t) = g\}$ which is a Levi subgroup of $P(\lambda)$.

The homomorphism ρ_{ss} is then defined by $\rho_{ss}(\gamma) = \lim_{t\to+\infty} \lambda(-t)\, \rho(\gamma)\, \lambda(t) \in L(\lambda)$ for all $\gamma \in \Gamma$. It is reductive and maps Γ to $L \subset L(\lambda) \subset P(\lambda)$. The parabolic subgroup $P(\lambda)$ is indeed a proper subgroup of G since its unipotent radical contains U. $\qquad\square$

Proof of Corollary 4.12. Deform ρ to the reductive representation ρ_{ss} as in Lemma 4.11. The representation ρ_{ss} belongs to the connected component of ρ in the space $\mathrm{Hom}(\Gamma, \mathrm{SU}(p,q))$, and therefore $\tau(\rho_{ss}) = \tau(\rho)$. Proposition 4.2 gives the Milnor-Wood inequality on $\tau(\rho_{ss})$, hence on $\tau(\rho)$.

Assume moreover that $n \geq 2$ and that ρ is maximal. Hence so is ρ_{ss}, and by §4.4 we know that $p \geq nq$ and that, up to conjugacy by an element of $\mathrm{SU}(p,q)$, ρ_{ss} is a product $\rho_{\mathrm{cpt}} \times \rho_{\mathrm{diag}}$. Moreover $\rho_{ss}(\Gamma)$ is a lattice in the stabilizer of $\rho_{\mathrm{diag}}(\mathbb{H}_{\mathbb{C}}^n)$, which by Proposition 4.10 is an almost direct product of the simple noncompact group $\rho_{\mathrm{diag}}(\mathrm{SU}(n,1))$ with its compact centralizer K. Therefore by [Dan80, Corollary 4.2], the Zariski closure $\overline{\rho_{ss}(\Gamma)}^{\mathbf{z}}$ of $\rho_{ss}(\Gamma)$ must contain $\rho_{\mathrm{diag}}(\mathrm{SU}(n,1))$, so that the centralizer of $\overline{\rho_{ss}(\Gamma)}^{\mathbf{z}}$ in $\mathrm{SU}(p,q)$ is included in K. This is a contradiction since by Lemma 4.11, $\overline{\rho_{ss}(\Gamma)}^{\mathbf{z}}$ sits in a Levi subgroup of a proper parabolic subgroup of G, hence its centralizer is not compact. $\qquad\square$

5. Representations in classical Hermitian Lie groups other than $\mathrm{SU}(p,q)$

In order to conclude the proof of Theorem 1.1, one needs to rule out possible maximal representations in the remaining classical Hermitian Lie groups, namely $\mathrm{SO}_0(p,2)$ with $p \geq 3$, $\mathrm{Sp}(m,\mathbb{R})$ with $m \geq 2$, and $\mathrm{SO}^\star(2m)$ with $m \geq 4$. We recall that their associated symmetric spaces' ranks are 2, m and $\lfloor m/2 \rfloor$ respectively.

This section is therefore devoted to the proof of the following:

Theorem 5.1. *There are no maximal representations of a uniform (torsion free) lattice of $\mathrm{SU}(n,1)$, $n \geq 2$, in the classical Hermitian Lie groups $\mathrm{SO}_0(p,2)$ with $p \geq 3$, $\mathrm{Sp}(m,\mathbb{R})$ with $m \geq 2$, and $\mathrm{SO}^\star(2m)$ with $m \geq 4$.*

Representations in $\mathrm{SO}_0(p,2)$ can be dealt with using the results of [KM08b], see §5.1. Representations in $\mathrm{Sp}(m,\mathbb{R})$ and $\mathrm{SO}^\star(2m)$ will be treated in §5.2 and §5.3, respectively. In these two latter cases the proof relies on the results of §4. Indeed, representations in these groups can be seen as representations in $\mathrm{SU}(m,m)$. As we shall see, in the case of a representation into $\mathrm{Sp}(m,\mathbb{R})$ or into $\mathrm{SO}^\star(2m)$ for m even, easy curvature computations show that the Milnor-Wood inequality the representation should satisfy is the same as the Milnor-Wood inequality it satisfies (by §4) when seen as a representation in $\mathrm{SU}(m,m)$. A maximal representation in these groups would therefore be a particular maximal representation in $\mathrm{SU}(m,m)$, and this is impossible, again by §4. For representation in $\mathrm{SO}^\star(2m)$ with m odd however, and although the general idea is the same as in the $\mathrm{SU}(p,q)$ case, more work is needed, including rather painful verifications.

Remark 5.2. We assume $p \geq 3$ for $\mathrm{SO}_0(p, 2)$ because $\mathrm{SO}_0(2, 2)$ is not simple (it is locally isomorphic to $\mathrm{SU}(1, 1) \times \mathrm{SU}(1, 1)$), $m \geq 2$ for $\mathrm{Sp}(m, \mathbb{R})$ because $\mathrm{Sp}(1, \mathbb{R})$ is isomorphic to $\mathrm{SU}(1, 1)$, and $m \geq 4$ for $\mathrm{SO}^\star(2m)$ because for $m = 2$, $\mathrm{SO}^\star(4)$ is not simple (it is locally isomorphic to $\mathrm{SU}(1, 1) \times \mathrm{SU}(1, 1)$), whereas for $m = 3$, $\mathrm{SO}^\star(6)$ is locally isomorphic to $\mathrm{SU}(3, 1)$ (and therefore in this last case there are maximal representations of lattices of $\mathrm{SU}(2, 1)$ and $\mathrm{SU}(3, 1)$ in this group). Note that $\mathrm{SO}^\star(8)$ is locally isomorphic to $\mathrm{SO}_0(6, 2)$.

Remark 5.3. Since there are no maximal (in our sense) representations of a uniform lattice Γ of $\mathrm{SU}(n, 1)$, $n \geq 2$, in any of the groups $\mathrm{SO}_0(p, 2)$ with $p \geq 3$, $\mathrm{Sp}(m, \mathbb{R})$ with $m \geq 2$, $\mathrm{SO}^\star(2m)$ with $m \geq 4$, or $\mathrm{SU}(p, q)$ with $p \geq q \geq 1$ but $p < qn$, it is natural to wonder what is the maximal possible value of the Toledo invariant of a representation $\rho : \Gamma \to G$, for G a specific group in this list, and whether a representation realizing this maximal value has some nice geometric properties. This seems to be a difficult question, whose answer probably depends heavily on the specific target Lie group G.

Observe however that the Milnor-Wood inequality is satisfied by representations of surface groups, i.e. uniform lattices in $\mathrm{SU}(1, 1)$, and that in this case maximal representations (as defined in this paper) exist in any Hermitian Lie group.

5.1. **Representations in** $\mathrm{SO}_0(p, 2)$**,** $p \geq 3$**.**

The case of representations in the groups $\mathrm{SO}_0(p, 2)$ $(p \geq 3)$ has already been treated in [KM08b] where it was shown that such representations satisfy the inequality $|\tau(\rho)| \leq \frac{n+1}{n} \mathrm{vol}(X)$. This is stronger than the Milnor-Wood inequality since the rank of the symmetric space associated to $\mathrm{SO}_0(p, 2)$ is 2 and $n \geq 2$. Hence there are no maximal representations in this case.

5.2. **Representations in** $\mathrm{Sp}(m, \mathbb{R})$**,** $m \geq 2$**.**

This group may be described as the following subgroup of $\mathrm{SU}(m, m)$:

$$\mathrm{Sp}(m, \mathbb{R}) = \{g \in \mathrm{SU}(m, m) \,|\, g^\top J_{m,m}\, g = J_{m,m}\}$$

where $g^\top$ is the transpose of the matrix g and $J_{m,m}$ is the $2m$-by-$2m$ matrix

$$J_{m,m} = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}.$$

The associated symmetric space $\mathcal{Y}$ is totally geodesically, holomorphically, and $\mathrm{Sp}(m, \mathbb{R})$-equivariantly, embedded in the symmetric space $\mathcal{Y}_{m,m}$ associated to $\mathrm{SU}(m, m)$ as

$$\mathcal{Y} = \{Z \in M_m(\mathbb{C}) \,|\, 1_m - Z^\star Z > 0 \text{ and } Z^\top = Z\} \subset \{Z \in M_m(\mathbb{C}) \,|\, 1_m - Z^\star Z > 0\} = \mathcal{Y}_{m,m}$$

Let us call $\iota : \mathcal{Y} \to \mathcal{Y}_{m,m}$ this embedding, $\omega_{\mathcal{Y}}$ the $\mathrm{Sp}(m, \mathbb{R})$-invariant metric on $\mathcal{Y}$ and $\omega_{\mathcal{Y}_{m,m}}$ the $\mathrm{SU}(m, m)$-invariant metric of $\mathcal{Y}_{m,m}$, both normalized to that the minimum of their holomorphic sectional curvature is -1.

Lemma 5.4. *We have* $\iota^\star \omega_{\mathcal{Y}_{m,m}} = \omega_{\mathcal{Y}}$.

Proof. Both metrics are $\mathrm{Sp}(m, \mathbb{R})$-invariant metrics on $\mathcal{Y}$, it is therefore enough to show that their normalizations agree, namely that the minimum of the holomorphic sectional curvature of $\iota^\star \omega_{\mathcal{Y}_{m,m}}$ is -1. Because ι is totally geodesic, the holomorphic sectional curvature of $\iota^\star \omega_{\mathcal{Y}_{m,m}}$ is the restriction of the holomorphic sectional curvature of the metric $\omega_{\mathcal{Y}_{m,m}}$ to complex lines in $T_{\mathcal{Y}}$. Now, at a point $o \in \mathcal{Y} \subset \mathcal{Y}_{m,m}$, the holomorphic tangent space $T_{\mathcal{Y}, o}$ to $\mathcal{Y}$ identifies with the subspace $S_m(\mathbb{C})$ of symmetric matrices in $M_m(\mathbb{C}) \simeq T_{\mathcal{Y}_{m,m}, o}$. Therefore, by the formula of §4.2, the holomorphic sectional curvature of $\iota^\star \omega_{\mathcal{Y}_{m,m}}$ on the complex line $\langle A \rangle$ generated by a nonzero symmetric $A \in M_m(\mathbb{C})$ is $-\mathrm{tr}\,((A^\star A)^2)/(\mathrm{tr}\,(A^\star A))^2$ so that its minimum value is

indeed -1 (attained for example by a diagonal matrix with only one non zero entry equal to 1). $\hfill\square$

Let ρ be a representation of a lattice Γ of $\mathrm{SU}(n,1)$ in $\mathrm{Sp}(m,\mathbb{R})$, and let $\rho' : \Gamma \to \mathrm{SU}(m,m)$ be ρ composed with the inclusion $\mathrm{Sp}(m,\mathbb{R}) \subset \mathrm{SU}(m,m)$. By the very definition of the Toledo invariant, we have

$$\tau(\rho) = \frac{1}{n!}\int_X f^\star \omega_{\mathcal{Y}} \wedge \omega^{n-1} = \frac{1}{n!}\int_X f^\star \iota^\star \omega_{\mathcal{Y}_{m,m}} \wedge \omega^{n-1} = \frac{1}{n!}\int_X (\iota \circ f)^\star \omega_{\mathcal{Y}_{m,m}} \wedge \omega^{n-1} = \tau(\rho')$$

which means that the Toledo invariant of $\rho : \Gamma \to \mathrm{Sp}(m,\mathbb{R})$ is the same as the Toledo invariant of $\rho' : \Gamma \to \mathrm{SU}(m,m)$. Since the ranks of $\mathcal{Y}$ and $\mathcal{Y}_{m,m}$ are both equal to m, the results of §4 give the Milnor-Wood inequality $|\tau(\rho)| \leq m \,\mathrm{vol}(X)$. Moreover, if the representation ρ in $\mathrm{Sp}(m,\mathbb{R})$ is maximal, then the representation ρ' in $\mathrm{SU}(m,m)$ is also maximal. But there are no such representations since by §4, maximal representations in $\mathrm{SU}(p,q)$ exist only if $p \geq nq$ (as always, we assume that $n \geq 2$).

5.3. **Representations in** $\mathrm{SO}^\star(2m)$**,** $m \geq 4$**.**

We proceed as in the previous paragraph by considering representations with values in $\mathrm{SO}^\star(2m)$ as special representations with values in $\mathrm{SU}(m,m)$. For m even, this allows to conclude as quickly as in the $\mathrm{Sp}(m,\mathbb{R})$ case.

The group $\mathrm{SO}^\star(2m)$ may be described as the following subgroup of $\mathrm{SU}(m,m)$:

$$\mathrm{SO}^\star(2m) = \{g \in \mathrm{SU}(m,m) \,|\, g^\top J'_{m,m}\, g = J'_{m,m}\}$$

where $J'_{m,m}$ is the $2m$-by-$2m$ matrix

$$J'_{m,m} = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}.$$

Observe that if $q'_{m,m}$ is the quadratic form on $\mathbb{C}^{2m}$ whose matrix in the canonical basis is $J'_{m,m}$ then $\mathrm{SO}^\star(2m)$ is a subgroup of $\mathrm{SO}(2m,\mathbb{C}) = \mathrm{SO}(\mathbb{C}^{2m}, q'_{m,m})$. In fact it is a real form of this complex group.

The associated symmetric space $\mathcal{Y}$ is totally geodesically, holomorphically, and $\mathrm{SO}^\star(2m)$-equivariantly, embedded in $\mathcal{Y}_{m,m}$ as

$$\mathcal{Y} = \{Z \in M_m(\mathbb{C}) \,|\, 1_m - Z^\star Z > 0 \text{ and } Z^\top = -Z\} \subset \{Z \in M_m(\mathbb{C}) \,|\, 1_m - Z^\star Z > 0\} = \mathcal{Y}_{m,m}$$

Again, call $\iota : \mathcal{Y} \to \mathcal{Y}_{m,m}$ this embedding, $\omega_{\mathcal{Y}}$ the $\mathrm{SO}^\star(2m)$-invariant metric on $\mathcal{Y}$ and $\omega_{\mathcal{Y}_{m,m}}$ the $\mathrm{SU}(m,m)$-invariant metric of $\mathcal{Y}_{m,m}$, both normalized to that the minimum of their holomorphic sectional curvature is -1.

Lemma 5.5. *We have* $\iota^\star \omega_{\mathcal{Y}_{m,m}} = 2\,\omega_{\mathcal{Y}}$.

Proof. The proof is entirely similar to the proof of Lemma 5.4: these two metrics on $\mathcal{Y}$ are $\mathrm{SO}^\star(2m)$-invariant and all we need to prove is that the minimum of the holomorphic sectional curvature of $\iota^\star \omega_{\mathcal{Y}_{m,m}}$ is $-\frac{1}{2}$.

At a point $o \in \mathcal{Y} \subset \mathcal{Y}_{m,m}$, the holomorphic tangent space $T_{\mathcal{Y},o}$ to $\mathcal{Y}$ identifies with the subspace of skew-symmetric matrices in $M_m(\mathbb{C}) \simeq T_{\mathcal{Y}_{m,m},o}$. Therefore, as in the proof of Lemma 5.4, the holomorphic sectional curvature of $\iota^\star \omega_{\mathcal{Y}_{m,m}}$ on the complex line $\langle A \rangle$ generated by a nonzero skew-symmetric $A \in M_m(\mathbb{C})$ is $-\mathrm{tr}\,((A^\star A)^2)/(\mathrm{tr}\,(A^\star A))^2$.

By Youla's decomposition [You61], there exists a unitary matrix $U \in \mathrm{U}(m)$ such that $U^\top A U$ is a block diagonal matrix with $\lfloor m/2 \rfloor$ skew-symmetric 2-by-2 blocks $\begin{pmatrix} 0 & -\alpha_i \\ \alpha_i & 0 \end{pmatrix}$, where

$\alpha_i \in \mathbb{R}$, and one additional zero on the diagonal if m is odd. Then, as $(U^\top AU)^\star(U^\top AU) = U^{-1}A^\star AU$, we get

$$\frac{\operatorname{tr}\left((A^\star A)^2\right)}{\left(\operatorname{tr}(A^\star A)\right)^2} = \frac{2\sum \alpha_i^4}{(2\sum \alpha_i^2)^2}$$

which clearly implies the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let ρ be a representation of a lattice Γ of $\mathrm{SU}(n,1)$ in $\mathrm{SO}^\star(2m)$, and $\rho' : \Gamma \to \mathrm{SU}(m,m)$ be ρ composed with the inclusion $\mathrm{SO}^\star(2m) \subset \mathrm{SU}(m,m)$. By definition, we have

$$\tau(\rho) = \frac{1}{n!}\int_X f^\star\omega_{\mathcal{Y}} \wedge \omega^{n-1} = \frac{1}{2}\frac{1}{n!}\int_X (\iota \circ f)^\star\omega_{\mathcal{Y}_{m,m}} \wedge \omega^{n-1} = \frac{1}{2}\,\tau(\rho').$$

As a consequence, the Milnor-Wood inequality $|\tau(\rho)| \leq \lfloor m/2\rfloor\operatorname{vol}(X)$ is equivalent to the inequality $|\tau(\rho')| \leq 2\lfloor m/2\rfloor\operatorname{vol}(X)$.

If m is even, the Milnor-Wood inequality for $\rho : \Gamma \to \mathrm{SO}^\star(2m)$ is therefore the usual Milnor-Wood inequality for $\rho' : \Gamma \to \mathrm{SU}(m,m)$ and ρ is maximal if and only if ρ' is maximal. As in the previous paragraph we may apply the results of §4 to obtain the inexistence of maximal representations in $\mathrm{SO}^\star(2m)$, m even.

We assume from now on that m is odd. Theorem 5.1 in this case is a consequence of the following two results:

Proposition 5.6. *Let ρ be a reductive maximal representation of a uniform lattice $\Gamma \subset \mathrm{SU}(n,1)$ in $\mathrm{SO}^\star(2m)$. Assume that $n \geq 2$ and that $m \geq 3$ is odd. Then if $\tau(\rho) > 0$, resp. $\tau(\rho) < 0$, there exists a ρ-equivariant holomorphic, resp. antiholomorphic, map $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}$.*

Proposition 5.7. *If $n \geq 2$ and $m \geq 4$, there are no holomorphic map $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}$ such that $f^\star\omega_{\mathcal{Y}} = \lfloor m/2\rfloor\,\omega$.*

Indeed, to prove Theorem 5.1, assume that there is a maximal representation of a lattice Γ of $\mathrm{SU}(n,1)$, $n \geq 2$, in $\mathrm{SO}^\star(2m)$, with m odd and $m \geq 5$. Then we may either apply [BIW09, Corollary 4] to get that this representation is reductive, or semisimplify this representation as in §4.5 to obtain a reductive representation ρ with the same Toledo invariant, so that ρ is again maximal. By Proposition 5.6, if $\tau(\rho) > 0$, there exists a ρ-invariant holomorphic map $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}$ and by the Ahlfors-Schwarz-Pick lemma [Roy80], $f^\star\omega_{\mathcal{Y}} \leq \lfloor m/2\rfloor\,\omega$. Since ρ is maximal, necessarily $f^\star\omega_{\mathcal{Y}} = \lfloor m/2\rfloor\,\omega$ and this is a contradiction by Proposition 5.7. If $\tau(\rho) < 0$ then there is a antiholomorphic map $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}$. But in this case the conjugate $\bar{f}$ is holomorphic and satisfies $\bar{f}^\star\omega_{\mathcal{Y}} = \lfloor m/2\rfloor\,\omega$, again a contradiction.

Proof of Proposition 5.6. We work with the Higgs bundle (E, θ) associated to the reductive representation ρ in $\mathrm{SO}^\star(2m)$. As in the $\mathrm{SU}(p,q)$-case, this Higgs bundle has a real structure. Since as we saw $\mathrm{SO}^\star(2m)$ is a subgroup of $\mathrm{SU}(m,m)$, the Higgs bundle (E, θ) is in particular a $\mathrm{SU}(m,m)$-Higgs bundle, so that we have $(E, \theta) = (V \oplus W, \beta \oplus \gamma)$, with $\operatorname{rk} V = \operatorname{rk} W = m$, $\beta : W \otimes T_X \to V$ and $\gamma : V \otimes T_X \to W$. Because ρ takes its values in $\mathrm{SO}^\star(2m)$, which is a real form of $\mathrm{SO}(2m, \mathbb{C}) = \mathrm{SO}(\mathbb{C}^{2m}, q'_{m,m})$, we have moreover an identification of V with $W^\star$, and for all $\xi \in T_X$, $\beta(\xi) \in \operatorname{Hom}(W, W^\star)$ and $\gamma(\xi) \in \operatorname{Hom}(W^\star, W)$ are skew-symmetric, namely for all $w_1, w_2 \in W$, $(\beta(\xi)w_1)(w_2) = -(\beta(\xi)w_2)(w_1)$ and for all $v_1, v_2 \in V = W^\star$, $v_1(\gamma(\xi)v_2) = -v_2(\gamma(\xi)v_1)$.

The harmonic metric on $E = V \oplus W$ comes from a reduction of the structure group of E to the maximal compact subgroup $\mathrm{U}(n)$ of $\mathrm{SO}^\star(2n)$. Therefore it is also compatible with the real structure in the sense that if $(w_1, \ldots, w_m)$ is an orthonormal basis of the fiber W_x above some $x \in X$ then the dual basis $(w_1^\star, \ldots, w_m^\star)$ of $V_x = W_x^\star$ is also orthonormal. Equivalently,

for all subspace F of W_x or of V_x, we have $(F^\perp)^\circ = (F^\circ)^\perp$, where if F is a subspace of W_x, resp. V_x, $F^\perp$ is the orthogonal complement of F in W_x, resp. V_x, w.r.t. the harmonic metric, and $F^\circ = \{v \in V_x = W_x^\star \mid v_{|F} = 0\} \subset V_x$, resp. $F^\circ = \{w \in W_x \mid v(w) = 0, \forall v \in F\} \subset W_x$.

As in §4, we lift the Higgs bundle over X to the projectivized tangent bundle $\mathbb{P}T_X$ and the Milnor-Wood inequality $|\tau(\rho)| \leq \lfloor \frac{m}{2} \rfloor \operatorname{vol}(X)$ is equivalent to $|\deg_\mathcal{T} \tilde{W}| \leq 2 \lfloor \frac{m}{2} \rfloor \frac{\deg_\mathcal{T} L^\vee}{2}$ (the factor 2 comes from Lemma 5.5).

Since m is odd, we therefore need to prove that $|\deg_\mathcal{T} \tilde{W}| \leq (m-1) \frac{\deg_\mathcal{T} L^\vee}{2}$. Now, $\tilde{\beta}$ and $\tilde{\gamma}$ being skew-symmetric, their generic ranks are bounded above by $m-1$ (again because m is odd). Thus Proposition 4.2 proves the Milnor-Wood inequality.

If the representation is maximal, say with $\deg_\mathcal{T} \tilde{W} > 0$, then $\deg_\mathcal{T} \tilde{W} = (m-1) \frac{\deg_\mathcal{T} L^\vee}{2}$ and the generic rank of $\tilde{\beta} : \tilde{W} \otimes L \to \tilde{V}$ on $\mathbb{P}T_X$ is $m-1$. Moreover, by Corollary 4.7, the singular locus $\mathcal{S}_\beta$ of β is a proper analytic subset of X.

We again work above a single point $x \in X$, $x \notin \mathcal{S}_\beta$. If $\xi \in T_{X,x}$, we will write $\beta(\xi)$, resp. $\gamma(\xi)$, for $\beta_x(\xi)$, resp. $\gamma_x(\xi)$.

If $\xi \neq 0$, we know since $x \notin \mathcal{S}_\beta$ that $\beta(\xi)$ has rank $m-1$. We write N_ξ for the 1-dimensional kernel of $\beta(\xi) : W_x \to V_x$, and I_ξ for its $(m-1)$-dimensional image. We denote by $N_\xi^\perp \subset W_x$ and $I_\xi^\perp \subset V_x$ their orthogonal complements w.r.t. the harmonic metric. We remark that by skew-symmetry of $\beta(\xi)$, $I_\xi \subset N_\xi^\circ$ and that since $\operatorname{rk} \beta(\xi) = m-1$, in fact $I_\xi = N_\xi^\circ$.

We want to proceed as for the $\mathrm{SU}(p,q)$ case in §4.3.3. We have the exact same statement as Lemma 4.8, although the proof is slightly different:

Lemma 5.8. *For all η and all $\xi \neq 0$ in $T_{X,x}$, $\gamma(\eta)$ vanishes on $I_\xi^\perp$ and hence maps V_x to $\cap_{\xi \neq 0} N_\xi^\perp$.*

Proof. Let $\xi \neq 0$. Exactly as in Lemma 4.8, $\gamma(\xi)$ vanishes on $I_\xi^\perp \subset V_x$ by weak polystability along the leaves, because $I_\xi^\perp$ must be stable by the Higgs field.

Hence, for all η, by integrability of the Higgs field, $\beta(\xi) \circ \gamma(\eta) = \beta(\eta) \circ \gamma(\xi)$ vanishes on $I_\xi^\perp$ so that $\gamma(\eta)$ maps $I_\xi^\perp$ to N_ξ. But, again by weak polystability, $\gamma(\eta)$ also maps V_x to $N_\eta^\perp$, because $N_\eta^\perp \oplus I_\eta$ is stable by the Higgs field. Therefore $\gamma(\eta)(I_\xi^\perp) \subset N_\xi \cap N_\eta^\perp$, so that for η close to ξ, and hence for all η, $\gamma(\eta)(I_\xi^\perp) = 0$.

Now, $\gamma(\eta)$ being skew-symmetric, $\operatorname{Im} \gamma(\eta) \subset (\operatorname{Ker} \gamma(\eta))^\circ$, so that $\operatorname{Im} \gamma(\eta) \subset (I_\xi^\perp)^\circ = N_\xi^\perp$, since as we saw $I_\xi = N_\xi^\circ$. Hence our claim. $\square$

The fact that the $\beta(\xi)$'s are not injective here makes the situation a little more complicated than in §4.3.3, and for example Lemma 4.9 does not hold. It is however possible to exploit the fact that the $\beta(\xi)$'s all have the same rank.

Since $n \geq 2$ we may choose two linearly independent tangent vectors ξ and η. The letter ζ will denote a tangent vector in $\langle \xi, \eta \rangle$. (In this proof, whenever $(v_i)_{i \in I}$ is a family of vectors or subspaces in a vector space, $\langle v_i, i \in I \rangle$ denotes the subspace generated by the v_i's.)

Lemma 5.9. *There exist decompositions $W_x = W_1 \oplus W_2$ and $V_x = V_1 \oplus V_2$ such that*
- $\beta(\zeta)(W_i) \subset V_i$, for all ζ and all $1 \leq i \leq 2$;
- $\dim V_1 = \dim W_1 + 1$ and $\dim V_2 = \dim W_2 - 1$;
- $\beta(\zeta)_{|W_1} : W_1 \to V_1$ is one-to-one for all $\zeta \neq 0$;
- $\beta(\zeta)_{|W_2} : W_2 \to V_2$ is onto for all $\zeta \neq 0$.

Proof. The set $\{\beta(\zeta), \zeta \in \langle \xi, \eta \rangle\}$ is a 2-dimensional linear subspace of $\operatorname{Hom}(W_x, V_x)$, whose non zero elements are all of rank $(m-1)$ (it is 2-dimensional because $\zeta \mapsto \beta(\zeta)$ is linear).

Therefore by [Wes72, Theorem 3.1], there exist $r \geq 1$, and decompositions $W_x = W_0 \oplus W_1 \oplus \cdots \oplus W_r$ and $V_x = V_0 \oplus V_1 \oplus \cdots \oplus V_r$ such that

- $\beta(\zeta)(W_0) = \{0\}$, for all ζ;
- $\beta(\zeta)(W_i) \subset V_i$, for all ζ and all $1 \leq i \leq r$;
- $\dim V_i = \dim W_i \pm 1$ for all $1 \leq i \leq r$;
- $\mathrm{rk}\,\beta(\zeta)_{|W_i} = \min\{\dim W_i, \dim V_i\}$, for all $\zeta \neq 0$ and all $1 \leq i \leq r$.

We remark that since ξ and η are linearly independent, the kernels N_ξ and N_η of $\beta(\xi)$ and $\beta(\eta)$ are distinct, and so are their images I_ξ and I_η. Indeed, the equality of the images is equivalent to the equality of the kernels by skew-symmetry. Therefore if they were equal, we would get that $\beta(\zeta)$ defines an isomorphism $N_\xi^\perp \to I_\xi$ for all $\zeta \neq 0$. This is impossible for example because since $\langle \xi, \eta \rangle$ is 2-dimensional, $\zeta \mapsto \det \beta(\zeta)$ cannot vanish only for $\zeta = 0$. Therefore $W_0 = \{0\}$. Also, $V_0 = \{0\}$ because if not then necessarily $\dim V_0 = 1$ and $I_\zeta = V_1 \oplus \cdots \oplus V_r$ for all $\zeta \neq 0$.

Moreover, since $\dim W_x = \dim V_x$, there must be at least one i such that $\dim V_i = \dim W_i + 1$. Say that $\dim V_i = \dim W_i + 1$ for $1 \leq i \leq s$ and $\dim V_i = \dim W_i - 1$ for $s + 1 \leq i \leq r$. Then for $\zeta \neq 0$, $\mathrm{rk}\beta(\zeta) = m - (r - s) = m - s$, so that $s = 1$ and $r = 2$. $\qquad \square$

This allows to give the analog of Lemma 4.9 in the present situation:

Lemma 5.10. *We have that $\cap_{\zeta \neq 0} \beta(\zeta) W_1 = \{0\}$ and $W_2 = \langle N_\zeta, \zeta \neq 0 \rangle$.*

Proof. For all $\zeta \neq 0$, $\beta(\zeta)_{|W_1}$ is injective. Hence the proof of the first statement is the same as that of Lemma 4.9. By duality, the same reasoning implies the second statement. $\qquad \square$

The first point of Lemma 5.10 implies that $\gamma(\eta)$ vanishes on $V_2^\perp$, since it vanishes on each $I_\zeta^\perp$ by Lemma 5.8 and

$$\langle I_\zeta^\perp, \zeta \neq 0 \rangle = \left(\bigcap_{\zeta \neq 0} I_\zeta \right)^\perp = \left(\bigcap_{\zeta \neq 0} (V_2 \oplus \beta(\zeta) W_1) \right)^\perp = \left(V_2 \oplus \left(\bigcap_{\zeta \neq 0} \beta(\zeta) W_1 \right) \right)^\perp = V_2^\perp.$$

The second point implies that $\gamma(\eta)(V_2) \subset W_2$, and hence that $\gamma(\eta)$ vanishes on V_2. Indeed, since $\beta(\xi)_{|W_2} : W_2 \to V_2$ is surjective, V_2 is generated by vectors of the form $\beta(\xi)w$ with $w \in W_2$ such that $\beta(\zeta)w = 0$ for some $\zeta \neq 0$. Hence, using the integrability condition $[\theta, \theta] = 0$ of the Higgs field, we get $\beta(\zeta) \circ \gamma(\eta) \circ \beta(\xi)w = \beta(\eta) \circ \gamma(\xi) \circ \beta(\zeta)w = 0$, so that $\gamma(\eta) \circ \beta(\xi)w \in N_\zeta \subset W_2$. Now we saw in Lemma 5.8 that $\gamma(\eta)(V_x) \subset \cap_{\zeta \neq 0} N_\zeta^\perp = W_2^\perp$. Hence $\gamma(\eta)(V_2) = \{0\}$.

We conclude that $\gamma = 0$ outside of $\mathcal{S}_\beta$, hence everywhere, so that the ρ-equivariant harmonic map f is holomorphic.

In the same way, if the representation is maximal and $\deg_{\mathcal{T}} \tilde{W} < 0$, we get that $\beta = 0$ so that the ρ-equivariant harmonic map f is antiholomorphic. $\qquad \square$

Proof of Proposition 5.7. Assume that there exists a holomorphic map $f : \mathbb{H}^n_{\mathbb{C}} \to \mathcal{Y}$ such that $f^\star \omega_{\mathcal{Y}} = \lfloor \frac{m}{2} \rfloor \omega$. By the equality case of the Ahlfors-Schwarz-Pick lemma, see [Roy80], for all $\xi \neq 0$ in the image of df, the holomorphic sectional curvature of $\omega_{\mathcal{Y}}$ in the direction of ξ is maximal, i.e. equal to $-\frac{1}{\lfloor m/2 \rfloor}$. Moreover, the map f is an immersion, so that the image of df in $T_{\mathcal{Y}}$ has dimension n at each point.

The lemma will follow if we prove that for $m \geq 4$, and for o a point in $\mathcal{Y}$, the maximal dimension of a subspace of $T_{\mathcal{Y},o}$ on which the holomorphic sectional curvatures of $\omega_{\mathcal{Y}}$ equal $-\frac{1}{\lfloor m/2 \rfloor}$ is 1. Lemma 5.5 and its proof show that this is equivalent to proving that for $m \geq 4$,

the dimension of a maximal linear subspace of skew-symmetric matrices in $M_m(\mathbb{C})$ such that $\frac{\operatorname{tr}((A^\star A)^2)}{(\operatorname{tr}(A^\star A))^2} = \frac{1}{2\lfloor m/2 \rfloor}$ is 1.

The Youla decomposition of a skew-symmetric matrix A (see the proof of Lemma 5.5) shows that $\frac{\operatorname{tr}((A^\star A)^2)}{(\operatorname{tr}(A^\star A))^2} = \frac{1}{2\lfloor m/2 \rfloor}$ if and only if $A^\star A$ is unitary conjugate to $\alpha^2 1_m$ if m is even (and hence equal to $\alpha^2 1_m$), or to $\alpha^2 \operatorname{diag}(1, \ldots, 1, 0)$ if m is odd.

We will prove our claim by contradiction. So let now A and B be two linearly independent skew-symmetric matrix in $M_m(\mathbb{C})$, such that each non zero matrix C in the two-dimensional vector space they generate satisfies $\frac{\operatorname{tr}((C^\star C)^2)}{(\operatorname{tr}(C^\star C))^2} = \frac{1}{2\lfloor m/2 \rfloor}$. We normalize A and B such that $\operatorname{tr}(A^\star A) = \operatorname{tr}(B^\star B) = 2\lfloor m/2 \rfloor$. We can also suppose that A and B are orthogonal (i.e. $\operatorname{tr}(A^\star B) = 0$).

If m is even then clearly we have a contradiction. Indeed, for all $\lambda, \mu \in \mathbb{C}$, $(\lambda A + \mu B)^\star(\lambda A + \mu B)$ is a multiple of 1_m, hence $\bar{\lambda}\mu A^\star B + \lambda\bar{\mu} B^\star A$ is also a multiple of 1_m, but it is trace free so $A^\star B$ must be equal to zero, which is not possible as the column vectors of A (and B) generate $\mathbb{C}^m$.

From now on, we assume that m is odd. Then for any $\lambda, \mu \in \mathbb{C}$, $(\lambda, \mu) \neq (0,0)$, the matrix

$$(\lambda A + \mu B)^\star(\lambda A + \mu B) - \frac{1}{m-1}\operatorname{tr}\left[(\lambda A + \mu B)^\star(\lambda A + \mu B)\right]1_m$$

has rank 1 and since $\operatorname{tr}(A^\star B) = 0$, $\operatorname{tr}\left[(\lambda A + \mu B)^\star(\lambda A + \mu B)\right] = (m-1)(|\lambda|^2 + |\mu|^2)$. In other words $N_{\lambda,\mu} := M_{\lambda,\mu} - (|\lambda|^2 + |\mu|^2)1_m$ has rank 1, where we denoted $(\lambda A + \mu B)^\star(\lambda A + \mu B)$ by $M_{\lambda,\mu}$.

There exists a hyperplane E (resp. F) in $\mathbb{C}^m$ such that the endomorphism of $\mathbb{C}^m$ whose matrix is $A^\star A$ (resp. $B^\star B$) is the identity when restricted to E (resp. F).

If $E = F$ then upon replacing A and B by $U^\top AU$ and $U^\top BU$ for some well chosen $U \in \mathrm{U}(m)$, we may assume that the m-th column vectors of A and B are trivial so that the m-th column and the m-th line of $M_{\lambda,\mu}$ both are trivial. As $N_{\lambda,\mu}$ has rank 1, this implies that for any λ, μ in $\mathbb{C}$, the upper left $(m-1)$-by-$(m-1)$ block of $M_{\lambda,\mu}$ is equal to $(|\lambda|^2 + |\mu|^2)1_{m-1}$. As in the case when m is even, the upper left $(m-1)$-by-$(m-1)$ block of $A^\star B$ should be equal to zero and this is impossible because the column vectors of A and B generate hyperplanes which must intersect non trivially.

Assume now that $E \cap F$ has codimension 2. We will use the notation $\langle x, y \rangle = x^\star y$ for the standard Hermitian product on $\mathbb{C}^m$ and write $|x|^2 = x^\star x$. Let us denote by $v_1, \ldots, v_m$, resp. $w_1, \ldots, w_m$, the column vectors of A, resp. B. Again upon replacing A and B by $U^\top AU$ and $U^\top BU$ for some well chosen $U \in \mathrm{U}(m)$, one can assume that $v_m = 0$ and that $(v_1, \ldots, v_{m-1})$ and $(w_1, \ldots, w_{m-2})$ are orthonormal families. Moreover, w_{m-1} and w_m are linearly dependent because the bottom right 2-by-2 block of $B^\star B$ must have determinant 0. Finally we also have $|w_{m-1}|^2 + |w_m|^2 = 1$.

The bottom right 2-by-2 block of $N_{\lambda,\mu}$ is

$$\begin{pmatrix} |\mu|^2(|w_{m-1}|^2 - 1) + \lambda\bar{\mu}\langle w_{m-1}, v_{m-1}\rangle + \bar{\lambda}\mu\langle v_{m-1}, w_{m-1}\rangle & |\mu|^2\langle w_{m-1}, w_m\rangle + \bar{\lambda}\mu\langle v_{m-1}, w_m\rangle \\ |\mu|^2\langle w_m, w_{m-1}\rangle + \lambda\bar{\mu}\langle w_m, v_{m-1}\rangle & -|\lambda|^2 + |\mu|^2(|w_m|^2 - 1) \end{pmatrix}$$

In the determinant of this block, the coefficient of $|\mu|^4$ is equal to 0 since

$$(1 - |w_{m-1}|^2)(1 - |w_m|^2) - |\langle w_m, w_{m-1}\rangle|^2 = |w_{m-1}|^2|w_m|^2 - |\langle w_m, w_{m-1}\rangle|^2$$

and w_m and w_{m-1} are linearly dependent.

The coefficient of $|\lambda|^2|\mu|^2$ is

$$1 - |w_{m-1}|^2 - |\langle w_m, v_{m-1}\rangle|^2$$

and by Schwarz inequality, it vanishes if and only if w_m and v_{m-1} are proportional since $|w_{m-1}|^2 + |\langle w_m, v_{m-1}\rangle|^2 \le |w_{m-1}|^2 + |w_m|^2 = 1$. In this case, there exist complex numbers a and b such that $w_{m-1} = a\,v_{m-1}$, $w_m = b\,v_{m-1}$ and $|a|^2 + |b|^2 = 1$. The above determinant is then equal to $-(|\lambda|^2 + |\mu|^2)(\lambda\bar{\mu}\bar{a} + \bar{\lambda}\mu a)$ and vanishes identically if and only if $a = 0$.

So $a = 0$ and $|b| = 1$. This immediately implies that $N_{\lambda,\mu}$ is block diagonal with a $(m-2)$-by-$(m-2)$ upper left block and a 2-by-2 bottom right block because $A^\star A$ et $B^\star B$ have the same block decomposition, hence $A^\star B$ and $B^\star A$ too, since $w_{m-1} = v_m = 0$ and $w_m = b\,v_{m-1}$. The 2-by-2 bottom right block of $N_{\lambda,\mu}$ is equal to

$$\begin{pmatrix} -|\mu|^2 & b\bar{\lambda}\mu \\ \bar{b}\lambda\bar{\mu} & -|\lambda|^2 \end{pmatrix}$$

hence has rank 1 for each $(\lambda, \mu) \ne (0, 0)$. As the matrix $N_{\lambda,\mu}$ has rank 1 for all $(\lambda, \mu) \ne (0, 0)$, we must have $\langle v_i, w_j \rangle = 0$ for any $1 \le i, j \le m - 2$. If $m \ge 5$, this is impossible since the two subspaces of codimension 2 generated respectively by the families $\{v_i\}_{1 \le i \le m-2}$ and $\{w_i\}_{1 \le i \le m-2}$ have a non trivial intersection. $\square$

5.4. **Representations in Hermitian groups without exceptional factors.**

It is easy to generalize the statement of Theorem 1.1 to the case where the lattice Γ is assumed uniform but not torsion free, and the target Lie group G is assumed to be a semisimple Lie group of Hermitian type without compact or exceptional factors. By this we mean that G is an almost-direct product of simple noncompact Lie groups of Hermitian type which are each isogenous to one of the classical groups we have been considering.

In this case, by Selberg's lemma, there is a normal subgroup Γ' of finite index d in Γ such that Γ' is torsion free and the representation $\rho' = \rho_{|\Gamma'}$ is a product of k representations $\rho'_i : \Gamma' \to G_i$, where the G_i's are classical Hermitian noncompact Lie groups. One defines the Toledo invariant of ρ to be $\frac{1}{d}\,\tau(\rho')$. Since $\mathrm{vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}) = \frac{1}{d}\,\mathrm{vol}(\Gamma'\backslash\mathbb{H}^n_{\mathbb{C}})$, the representation ρ is maximal if and only if the representation ρ' is. Since $\tau(\rho') = \sum_{i=1}^k \tau(\rho'_i)$ and $\mathrm{rk}_{\mathbb{R}}\,G = \sum_{i=1}^k \mathrm{rk}_{\mathbb{R}}\,G_i$, ρ' is maximal if and only if each ρ'_i is. Therefore in this case $G_i = \mathrm{SU}(p_i, q_i)$ with $p_i \ge q_i n$ for all i and there is a ρ-equivariant holomorphic or antiholomorphic map from $\mathbb{H}^n_{\mathbb{C}}$ to the symmetric space $\mathcal{Y} = \Pi_{i=1}^k \mathcal{Y}_{p_i,q_i}$ associated to G.

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