Autosimilarité et anisotropie : applications en imagerie médicale

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journées MAS, Bordeaux, 03/09/2010

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Outlines

1 Fractal analysis of medical images

ROI medical images = textures

Goal: use texture analysis to extract diagnostically meaningful information

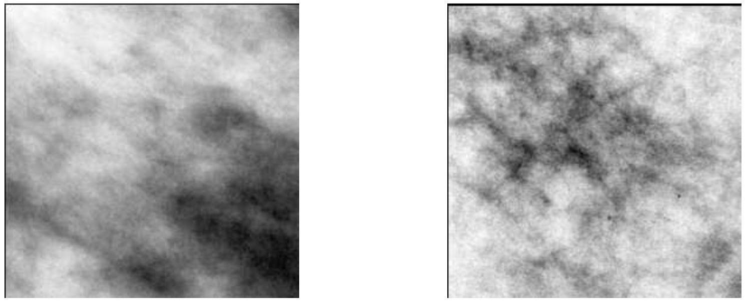
One tool: fractal analysis to characterize texture via statistical self-similarity or scale invariance through a fractal index $H \in (0, 1)$

Numerous methods and studies!

[Lopes and Betrouni, 2009]

1.1 Examples

Mammography

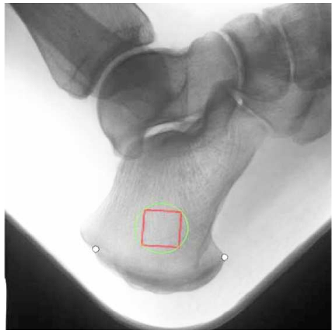
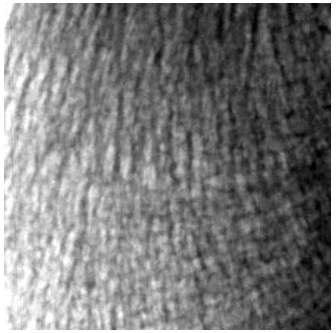
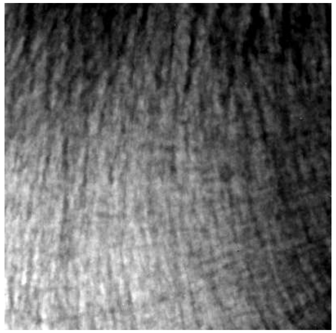


dense breast tissue fatty breast tissue

- Validation of self-similarity using power spectrum method $H \in [0.33, 0.42]$ [Heine et al, 2002].
- Discrimination of dense $H \in [0.55, 0.75]$ and fatty $H \in [0.2, 0.35]$ breast tissue using WTMM method [Kestener et al, 2001]

Trabecular bone microarchitecture

Dataset of 211 high-resolution digital X-ray images of calcaneum (a heel bone) with standardized acquisition procedure [Lespessailles et al., 2007]:

ROI location control case osteoporotic case

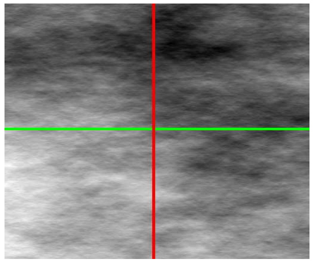
- Validation of self-similarity using variogram and power spectrum methods on calcaneous bone [Benhamou et al, 94], on cancellous bone [Caldwell et al, 94]
- Discrimination of osteoporotic cases $H_{mean} = 0.679 \pm 0.053$ (osteoporotic cases)/ $H_{mean} = 0.696 \pm 0.030$ (control cases) [Benhamou et al, 2001]

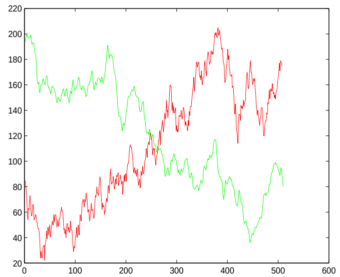
1.2 Variogram method

Let us consider the discrete image

$$I(k,l), 0 \leq k \leq n-1,\ 0 \leq l \leq n-1.$$

Extract a line of the image $I_\theta(k)$, $0 \leq k \leq n_\theta - 1$ for θ a direction.



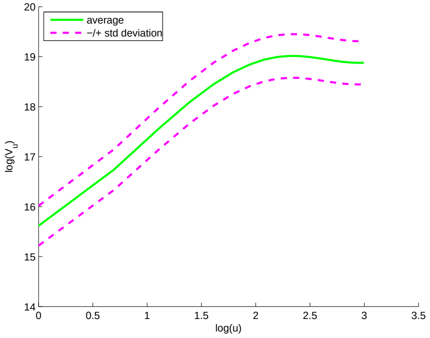


Compute $v_\theta(u) = \dfrac{1}{n_\theta - u} \displaystyle\sum_{k=0}^{n_\theta - 1 - u} \left(I_\theta(k+u) - I_\theta(k)\right)^2$, $1 \leq u \leq n_\theta - 1$.

Average along a set of lines with the same direction $\overline{v_\theta}(u)$ and plot $\log \overline{v_\theta}(u)$ versus $\log(u)$.

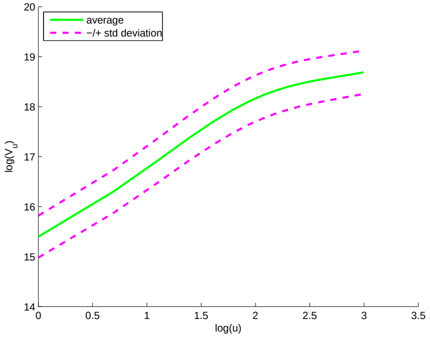
➥ Fractal index $H_\theta = \text{slope}(\theta)/2$

Example on bone radiographs (211 cases):



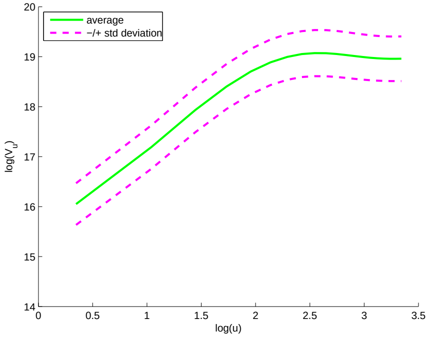
$\theta = (1, 0)$ (horizontal)

$H_\theta = 0.51 \pm 0.08$



$\theta = (0, 1)$ (vertical)

$H_\theta = 0.56 \pm 0.06$



$\theta = (1, 1)/\sqrt{2}$ (diagonal)

$H_\theta = 0.51 \pm 0.08$

1.3 Fractional Brownian motion modeling

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For $H \in (0,1)$, the fractional Brownian motion [Kolmogorov, 1940], [Mandelbrot and Van Ness, 1968] $B_H = \{B_H(t)\,;t \in \mathbb{R}\}$ is a zero mean Gaussian random process such that

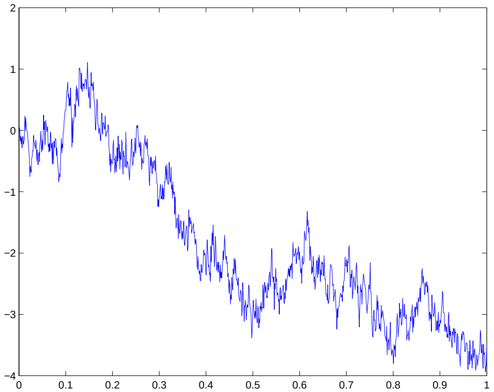
$$\forall t \in \mathbb{R},\ B_H(t) : \Omega \to \mathbb{R} \text{ real random variable } \sim \mathcal{N}\left(0, |t|^{2H}\right).$$

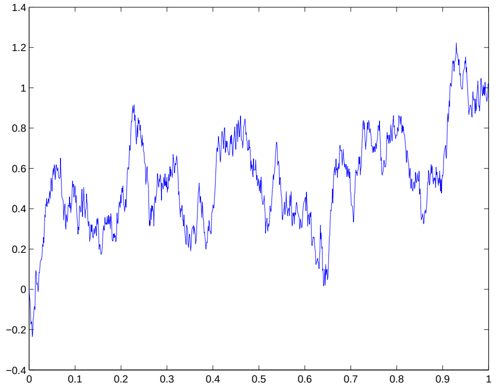
and

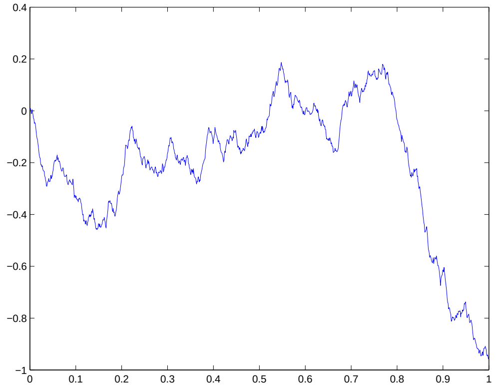
$$\mathrm{Cov}\,(B_H(t), B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Properties :

- $B_H(0) = 0$ a.s.
- Stationary increments: $\forall t_0 \in \mathbb{R},\ B_H(.+t_0) - B_H(t_0) \stackrel{fdd}{=} B_H(.)$
- Self-similarity of order H: $\forall \lambda > 0,\ B_H(\lambda .) \stackrel{fdd}{=} \lambda^H B_H(.)$.







$H = 0.3$ $H = 0.5$ $H = 0.7$

- H= Self-similarity/Hölder regularity/Fractal dimension : $2 - H$

  ↪ Numerous estimators of H.

- Law characterized by the variogram $v_H(t) = \mathrm{Var}\left(B_H(t)\right) = |t|^{2H}$.

- Spectral representation $B_H(t) = c_H \int_{\mathbb{R}} \left(e^{-it\xi} - 1\right) |\xi|^{-H-1/2} dW(\xi)$

$$v_H(t) = c_H^2 \int_{\mathbb{R}} \left|e^{-it\xi} - 1\right|^2 |\xi|^{-2H-1} d\xi$$

↪ spectral density $= c_H^2 |\xi|^{-2H-1}$ [$1/f$ process].

Estimation

Let us observe $\left\{I_\theta(k) = c_\theta B_H\left(\frac{k}{n}\right); 0 \leq k \leq n-1\right\}$.

$$V_{1,u}(B_H) := \frac{c_\theta^2}{n-u} \sum_{k=0}^{n-1-u} \left(B_H\left(\frac{k+u}{n}\right) - B_H\left(\frac{k}{n}\right)\right)^2$$

$$\hookrightarrow \quad \mathbb{E}(V_{1,u}(B_H)) = \frac{c_\theta^2}{n-u} \sum_{k=0}^{n-1-u} v_H\left(\frac{u}{n}\right) = c_\theta^2 n^{-2H} u^{2H}.$$

Ergodic Theorem: $V_{1,u}(B_H) / \mathbb{E}(V_{1,u}(B_H)) \underset{n \to +\infty}{\longrightarrow} 1$ a.s.

$$\boxed{\widehat{H} = \frac{1}{2} \log\left(\frac{V_{1,u}(B_H)}{V_{1,v}(B_H)}\right) / \log\left(\frac{u}{v}\right)}$$

Remark:

$$v_H\left(\frac{u}{n}\right) = \text{variogram at small scales} = \begin{cases} -\text{Hölder regularity} \\ -\text{spectral density at high frequencies} \end{cases}$$

Rate of convergence: asymptotic normality?

$$P^K(x) = (1-x)^K = \sum_{l=0}^{K} a_l x^l \text{ filter of order } K \geq 1$$

$$\begin{aligned} \forall t \in \mathbb{R},\ P_u^K(B_H)(t) &= \sum_{l=0}^{K} a_l B_H\left(\frac{t+lu}{n}\right) \\ &= c_H \int_{\mathbb{R}} e^{-i\frac{t\xi}{n}} P\left(e^{-i\frac{u\xi}{n}}\right)^K |\xi|^{-H-\frac{1}{2}} dW(\xi) \end{aligned}$$

➥ stationary ergodic Gaussian process

Generalized quadratic variations [Istas and Lang, 1997]

$$V_{K,u}(B_H) = \frac{1}{n-Ku} \sum_{k=0}^{n-Ku+1} \left(P_u^K(B_H)(k)\right)^2 \text{ with },$$

$$\mathbb{E}\left(V_{K,u}(B_H)\right) = c_{K,H}\, n^{-2H}\, u^{2H}.$$

Ergodic Theorem: $V_{K,u}(B_H)/\mathbb{E}(V_{K,u}(B_H)) \underset{n\to+\infty}{\longrightarrow} 1$ a.s.

$$\sqrt{n-Ku}\left(\frac{V_{K,u}\left(B_H\right)}{\mathbb{E}\left(V_{K,u}\left(B_H\right)\right)}-1\right)\overset{fdd}{=}\frac{1}{\sqrt{n-Ku}}\sum_{k=0}^{n-Ku-1}H_2\left(X_u\right)(k),$$

with $H_2(x)=x^2-1=$ 2th-Hermite polynomial and X_u is a centered stationary Gaussian time series which admits for spectral density

$$F_{X_u}(\xi)=\frac{2\pi}{c_{K,H}u^{2H}}\left|P\left(e^{-iu\xi}\right)\right|^{2K}\sum_{k\in\mathbb{Z}}|\xi+2k\pi|^{-2H-1},$$

ie

$$\mathrm{Cov}(X_u(k+k'),X_u(k'))=\frac{1}{2\pi}\int_{-\pi}^{+\pi}e^{-ik\xi}F_{X_u}(\xi)dx.$$

Theorem:[Breuer-Major, 1983] If $\int_{\mathbb{R}}\left|P\left(e^{-iu\xi}\right)\right|^{4K}|\xi|^{-4H-2}d\xi<+\infty$, then, $F_{X_u}\in L^2([-\pi,\pi])$ and, as $n\to+\infty$,

$$\sqrt{n-Ku}\left(\frac{V_{K,u}\left(B_H\right)}{\mathbb{E}\left(V_{K,u}\left(B_H\right)\right)}-1\right)\to\mathcal{N}(0,\sigma_u^2)$$

with $\sigma_u^2=2F_{X_u}^{*2}(0)$.

New proof by [Nualart, Ortiz-Latorre, 2008], [Nourdin, Peccatti, 2009]
+ vectorial CLT from [Peccatti, Tudor, 2004]

Theorem:[B., Bonami and León, 2010]
Let $Y(t) = \int_{\mathbb{R}} \left(e^{-it\xi} - 1\right) f(\xi)^{1/2} dW(\xi)$, with

$$f(\xi) = \frac{c}{|\xi|^{2H+1}} + \underset{|\xi|\to+\infty}{O}\left(\frac{1}{|\xi|^{2H+1+\gamma}}\right),$$

Then, $\widehat{H} = \frac{1}{2}\log\left(\frac{V_{K,u}(Y)}{V_{K,v}(Y)}\right) / \log\left(\frac{u}{v}\right) \underset{n\to+\infty}{\longrightarrow} H$ a.s.
+ asymptotic normality for $K > H + 1/4$ and $\gamma > 1/2$.

2 Stochastic modeling for images

$\{X(x); x \in \mathbb{R}^2\}$ a zero mean Gaussian random field

Spectral representation: [Bonami and Estrade, 2003]

$$X(x) = \int_{\mathbb{R}^2} \left(e^{-ix\cdot\xi} - 1\right) f(\xi)^{1/2} dW(\xi), x \in \mathbb{R}^2$$

well defined for $f \geq 0$ even and $\int_{\mathbb{R}^2} \min(1, \|\xi\|^2) f(\xi) d\xi < +\infty$.

Properties :

- $X(0) = 0$ a.s.
- Stationary increments: $\forall x_0 \in \mathbb{R}^2$, $X(. + x_0) - X(x_0) \overset{fdd}{=} X(.)$
- $\mathrm{Cov}\,(X(x), X(y)) = \frac{1}{2}\,(v_X(x) + v_X(y) - v_X(x-y))$ with

$$v_X(x) = \int_{\mathbb{R}^2} \left|e^{-ix\cdot\xi} - 1\right|^2 f(\xi) d\xi$$

↪ f spectral density

2.1 Restriction on a line

For all direction $\theta \in S^1$, $\{X(t\theta); t \in \mathbb{R}\}$ is a zero mean Gaussian random process with spectral density given by

$$R_\theta f(\tau) = \int_{\mathbb{R}} f(\tau\theta + \gamma\theta^\perp) d\gamma \text{ a.e. } \tau \in \mathbb{R}$$

Self-similarity: $R_\theta f(\tau) = c_\theta |\tau|^{-2H-1}$

$$\boxed{f(\xi) = c(\arg(\xi)) \|\xi\|^{-2H-2}, \text{ a.e. } \xi \in \mathbb{R}^2}$$

$$\forall \lambda > 0,\ X(\lambda \cdot) \stackrel{fdd}{=} \lambda^H X(\cdot).$$

Isotropy: $R_{\theta'} f = R_\theta f$ for all $\theta' \in S^1$

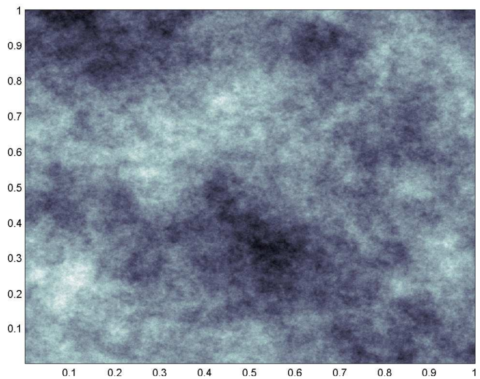
$$\boxed{f \text{ is radial}}$$

$$\forall M \in \mathcal{O}_2(\mathbb{R}),\ X(M \cdot) \stackrel{fdd}{=} X(\cdot).$$

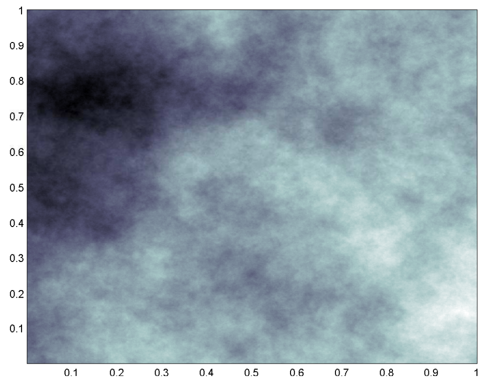
2.2 Fractional Brownian field

$f(\xi) = c_H \|\xi\|^{-2H-2}$ a.e. $\xi \in \mathbb{R}^2$ and $\forall x \in \mathbb{R}^2, v_X(x) = \mathrm{Var}(X(x)) = \|x\|^{2H}$.

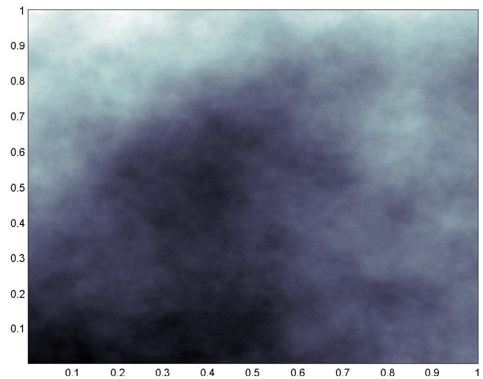
Exact method of simulation [Stein, 2002]



$H = 0.3$

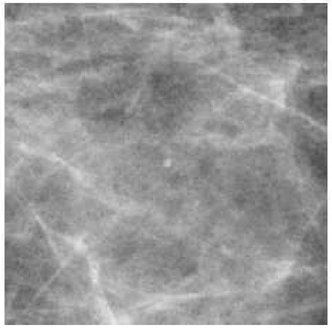


$H = 0.5$

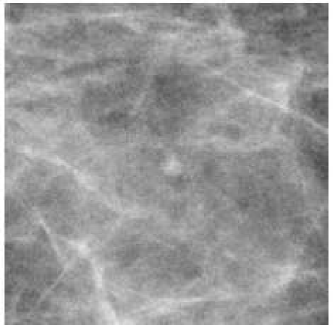


$H = 0.7$

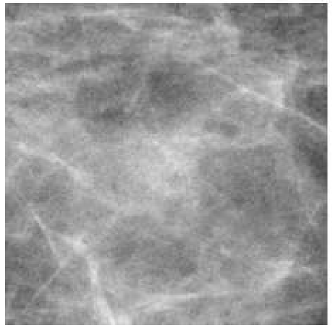
Detectability of spots on mammograms [Grosjean, Moisan, 2009]



radius 5



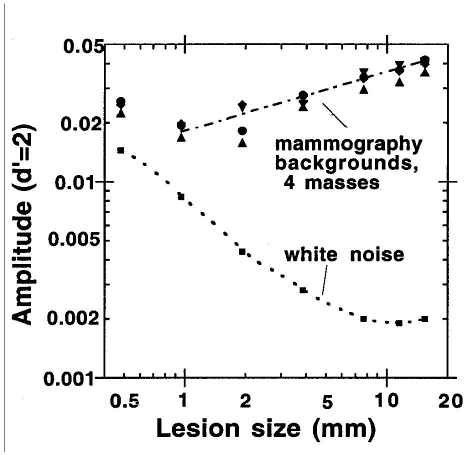
radius 10



radius 50

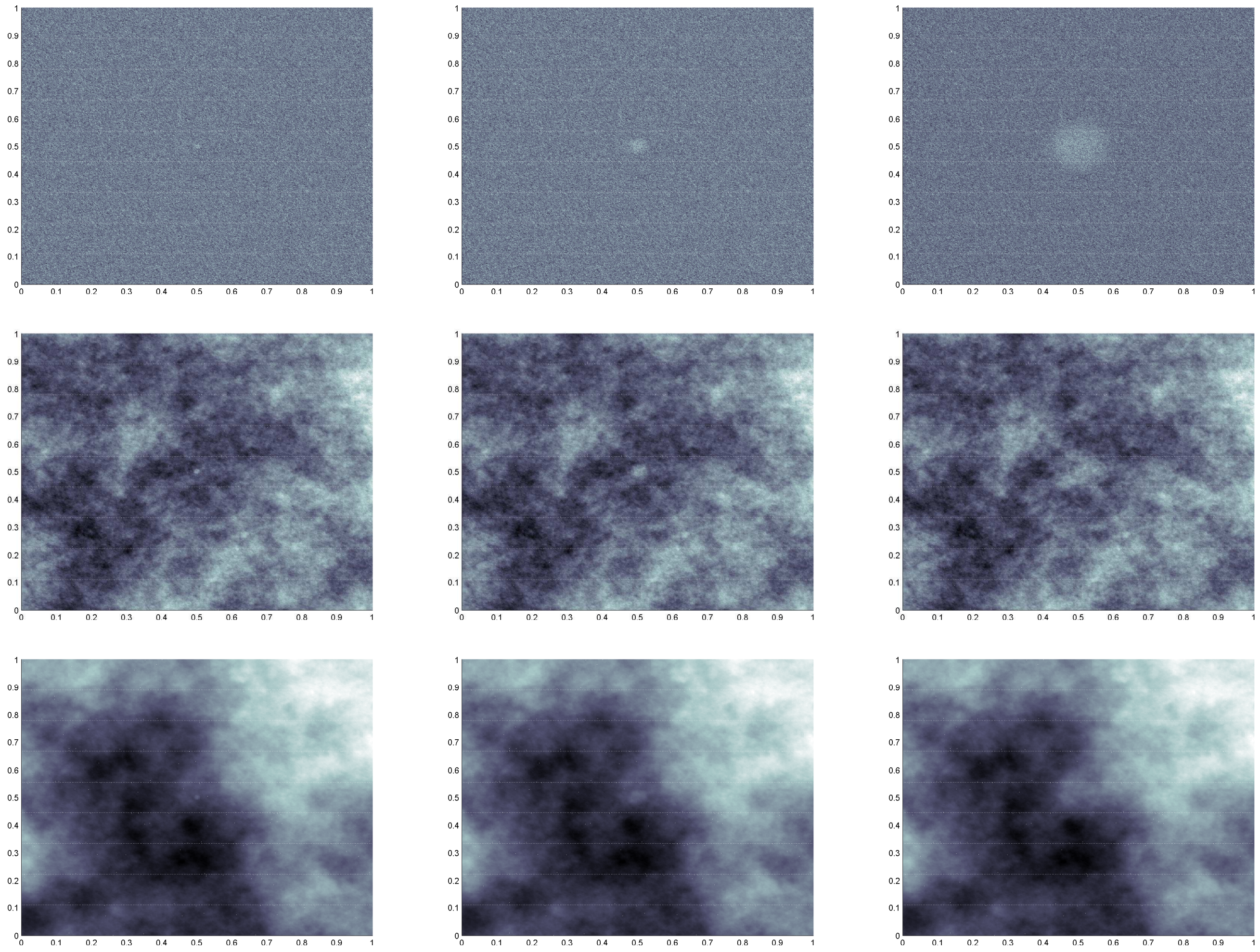
Simulated spots with identical contrast on a real mammogram

Burgess' Law [Burgess et al, 2001]



Link between size and contrast for human perception of opacities and mammograms

Simulated spots with identical contrast



WN

$H = 0.3$

$H = 0.7$

radius 5 radius 10 radius 50

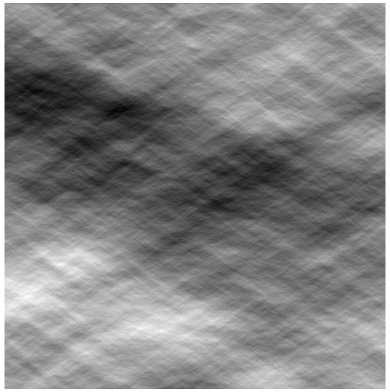
2.3 Anisotropic Gaussian field

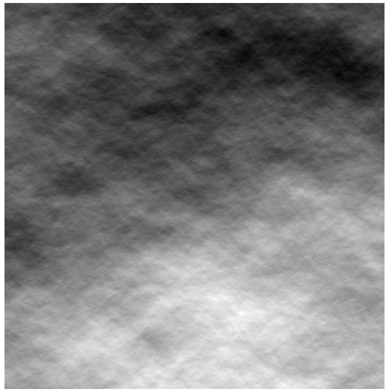
$$\boxed{f(\xi) = c(arg(\xi))\|\xi\|^{-2H-2} \text{ a.e. } \xi \in \mathbb{R}^2}$$

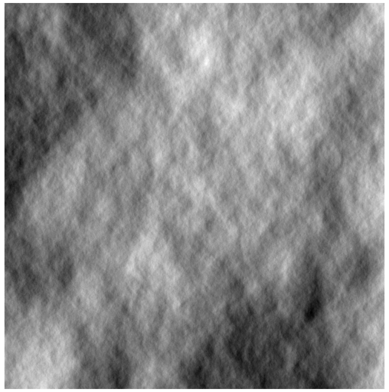
$$\forall x \in \mathbb{R}^2, v_X(x) = \mathrm{Var}(X(x)) = C(arg(x))\|x\|^{2H}.$$

C = topothesy function [Davies and Hall, 1999] with $\forall \theta, C(\theta) > 0$

Simulations for $H = 0.4$ [Coll: Moisan, Richard (MAP5)]







$c(\theta) = \mathbf{1}_{[\pi/4, 3\pi/4]}(|\theta|)$ $\quad$ $c(\theta) = |\sin(\theta)|$ $\quad$ $c(\theta) = |\cos(\theta)|$

- Identification of c [Istas, 2007]
- Anisotropy test for C [Coll: Bonami, León, Richard]

Anisotropic self-similarity?

Problem: Let $\theta_1 = (1,0)$ and $\theta_2 = (0,1)$. Find X such that

- $\{X(t\theta_1); t \in \mathbb{R}\}$ =FBM of order H_1
- $\{X(t\theta_2); t \in \mathbb{R}\}$ =FBM of order H_2 with $H_1 < H_2$

Remark: It is not possible to get more than 2 different directions due to S.I.

Toy example solution:

$X(x) = X(x_1, x_2) = B_{H_1}(x_1) + B_{H_2}(x_2)$ with B_{H_1} and B_{H_2} ind. FBMs.

Then

$$v_X(x) = |x_1|^{2H_1} + |x_2|^{2H_2} \longrightarrow \text{ no spectral density.}$$

Note that

$$\forall \lambda > 0, \quad v_X(\lambda x_1, \lambda^a x_2) = \lambda^{2H_1} v_X(x_1, x_2) \text{ with } a = H_1/H_2.$$

Operator scaling property

For $a = H_1/H_2$, $\forall \lambda > 0$,

$$\{X(\lambda x_1, \lambda^a x_2); (x_1, x_2) \in \mathbb{R}^2\} \overset{fdd}{=} \lambda^{H_1} \{X(x_1, x_2); (x_1, x_2) \in \mathbb{R}^2\}$$

A solution with spectral density f:

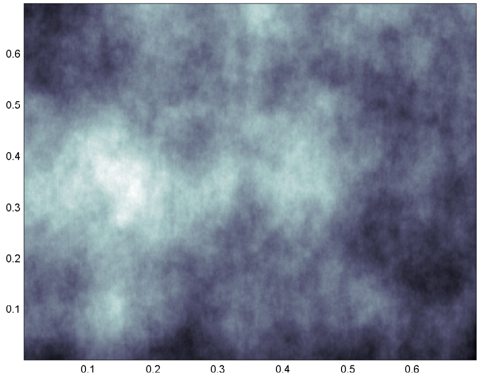
$$f(\lambda \xi_1, \lambda^a \xi_2) = \lambda^{-\beta} f(\xi_1, \xi_2), \quad \beta = 2H_1 + 1 + a$$

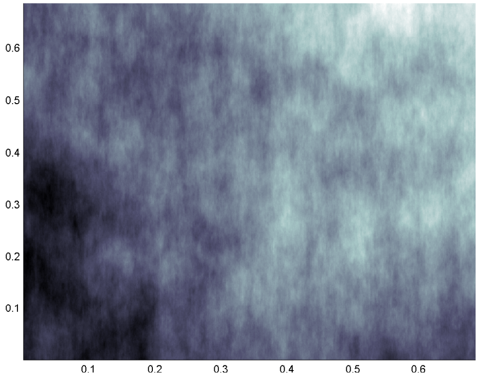
One example: $f(\xi_1, \xi_2) = \left(\xi_1^2 + \xi_2^{2a}\right)^{-\beta/2}$

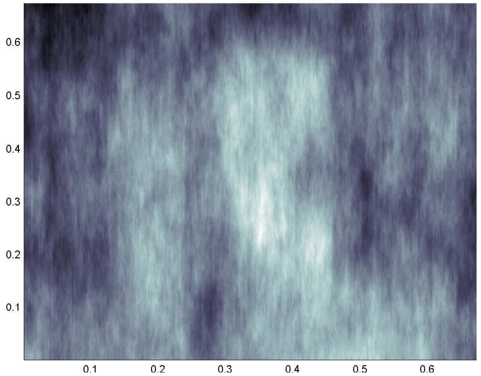
Theorem:[B., Meerschaert, Scheffler, 2007]: X is well defined and satisfies the operator scaling property

$$\forall \lambda > 0, \; X(\lambda^E \cdot) \overset{fdd}{=} \lambda^{H_1} X(\cdot), \text{ with } E = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

Simulations for $H_2 = 0.6$







$H_1 = 0.55$ $H_1 = 0.5$ $H_1 = 0.45$

What for other directions?

If $\theta = (\theta_1, \theta_2) \in S^1$ with $\theta_1 \neq 0$ and $\theta_2 \neq 0$,

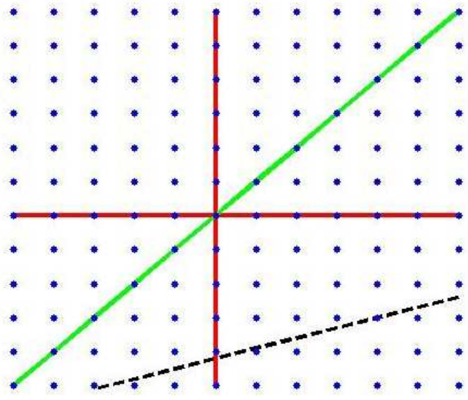
$$R_\theta f(p) \underset{+\infty}{\asymp} c_\theta |p|^{-2H_1 - 1} \text{ with } c_\theta = \frac{1}{|\theta_1|} \int_{\mathbb{R}} (s^{2a} + \theta_1^{-2})^{-\beta/2} ds$$

$$\hookrightarrow \quad v_\theta(t) = \mathrm{Var}\,(X(t\theta)) \underset{0}{\asymp} d_\theta |t|^{2H_1} \text{ with } d_\theta > 0.$$

2.4 Validation for bone radiographs

[Benhamou, B., Richard, 2009]

Implementation issues



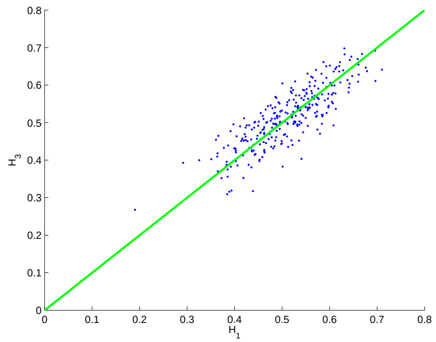
Black = out of lattice.

Precision of

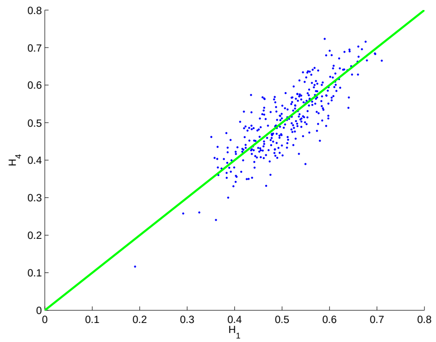
red = 1, green = $\sqrt{2}$

- Estimation on oriented lines **without interpolation**.
- Precision is not the same in all directions.
- Accuracy of orientation analysis $\leftrightarrow$ Precision of the image.

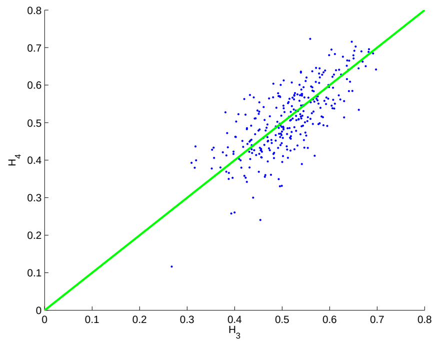
Comparison of the index in different directions



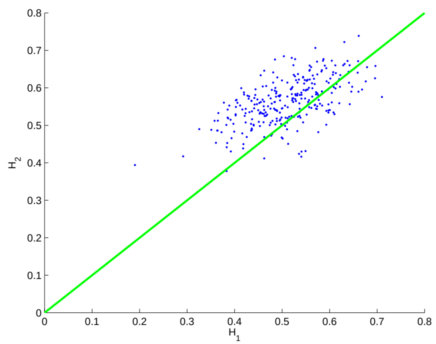
H_{θ_3} vs H_{θ_1}



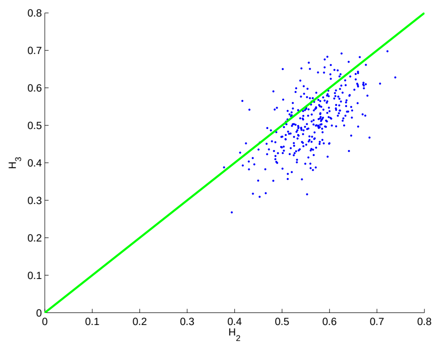
H_{θ_4} vs H_{θ_1}



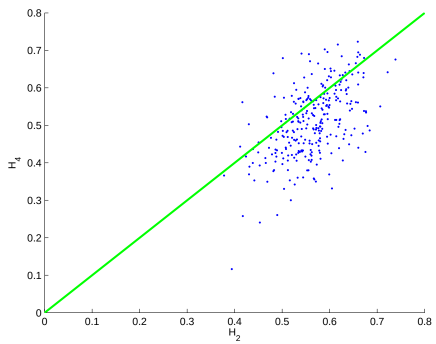
H_{θ_4} vs H_{θ_3}



H_{θ_2} vs H_{θ_1}



H_{θ_3} vs H_{θ_2}



H_{θ_4} vs H_{θ_2}

Directions:

1: $\theta_1 = (1, 0)$ (horizontal), 2: $\theta_2 = (0, 1)$ (vertical),

3: $\theta_3 = (1, 1)/\sqrt{2}$ (diagonal), 4: $\theta_4 = (-1, 1)/\sqrt{2}$ (diagonal).