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# Billards Stochastiques et Milieux Aléatoires

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# Introduction

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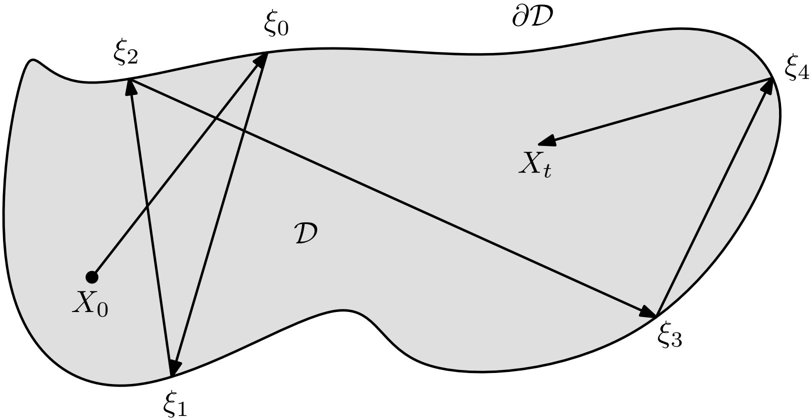

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Kinetic theory of gases: Knudsen [1952]
Diffusion in nanopores: Coppens + Malek [2003], Coppens + Dammers [2006]
Feres [2007]: dynamical systems and ergodic theory
Lalley + Robbins [1987, 1988]: convex $\mathcal{D}$ and cosine law. “princess and monster”
S. Evans [2001]: $C^1$ boundary or polygon, uniform reflection law
Borovkov [1991, 1994], Romeijn [1998]: Monte Carlo Markov chains algorithm (“running shake-and-bake algorithm”)
Goldstein, Kipnis, Ianiro [1985]: a mechanical particle system with stochastic boundary conditions

Our emphasis: we consider general domains

# Outline

# Billiard table

$\mathcal{D} \subset \mathbb{R}^d$ open connected domain, with boundary $\partial\mathcal{D}$ locally Lipschitz and almost everywhere continuously differentiable:

1– $\forall x \in \partial\mathcal{D}$, we can rotate $\partial\mathcal{D}$ so that it is locally the graph of a Lipschitz function.

2– $\exists\, \mathcal{R} \subset \partial\mathcal{D}$ open such that $\partial\mathcal{D}$ is continuously differentiable on $\mathcal{R}$ and the $(d-1)$-dimensional Hausdorff measure of $\partial\mathcal{D} \setminus \mathcal{R}$ is equal to zero.

# Reflection law for stochastic billiard

Outgoing direction is random, with density (in the relative frame) $\gamma$ on the open half sphere $\mathbb{S}_e = \{u \in \mathbb{R}^d : |u| = 1, u \cdot e > 0\}$, with $e =$ the first unit vector, such that

$$\inf_K \gamma > 0 \qquad \forall K \text{ compact } \subset \mathbb{S}_e$$

**Main example** for $\gamma$: cosine density,

$$\gamma(u) = \gamma_d \; e \cdot u \quad \text{on half sphere } \mathbb{S}_e$$

cf Knudsen [1952].

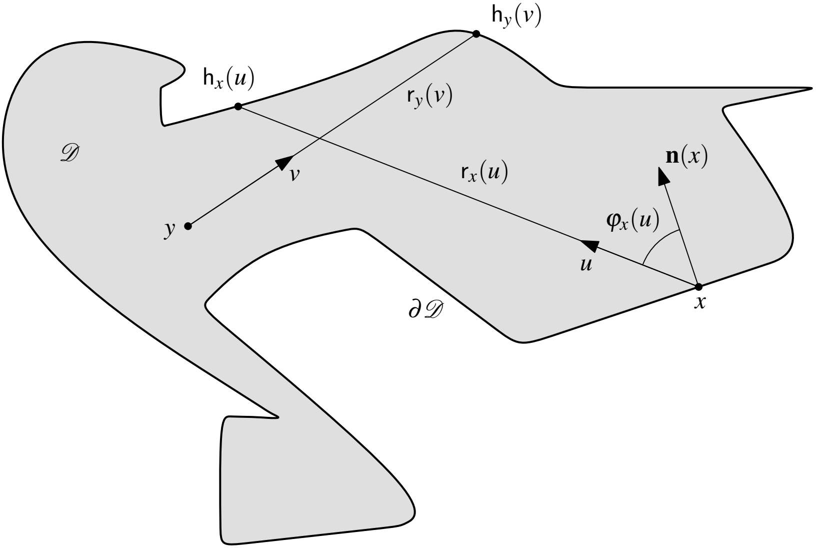



Figure: Bounce at $x \in \partial\mathcal{D}$ in dimension $d = 2$. The outgoing direction $u$ is such that its angle $\varphi_x(u)$ with the normal $n(x)$ has density $\gamma$; independent of the ingoing direction.

# Construction of KRW and KSB:

... standard way, with an i.i.d. sequence of law $\gamma$ on $\mathbb{S}_e$.

- Knudsen Random Walk (KRW) $(\xi_n, n \geq 0) =$ sequence of impacts on the boundary.
  Markov chain in $\{\partial\mathcal{D}, \infty, \mathfrak{S}\}$.
  Note:: Start from $\xi_0 \in \mathcal{R}$. Then, with probability 1, $\xi$ does not enter $\mathfrak{S}$.
- Knudsen Stochastic billiard: time-continuous process moving at speed 1. Is defined for all times, a.s..

# Change the variable

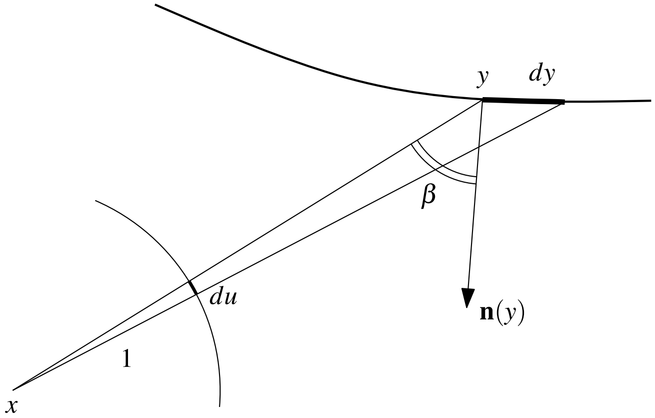



Figure: $du = \|x - y\|^{-(d-1)} \cos\beta \, dy$

# Transition kernel for the random walk

Changing variable from $u \in \mathbb{S}_e$ to $y = h_x(U_x u)$, we get for $x \in \mathcal{R}$,

$$\mathbf{P}[\xi_{n+1} \in A \mid \xi_n = x] = \int_A K(x,y)\, dy \ ,$$

where $dy$ is the surface measure on $\partial\mathcal{D}$ and

$$K(x,y) = \frac{\gamma(U_x^{-1}\frac{y-x}{\|y-x\|})\cos(\widehat{\mathbf{n}(y),y-x})}{\|x-y\|^{d-1}}\mathbf{1}\{x,y \in \mathcal{R}, x \leftrightarrow y\}$$

where we write $x \leftrightarrow y$ *(see each other)* if the open segment $(x,y) \subset \mathcal{D}$.

# Transition kernel for Knudsen random walk

For $\gamma =$ Cosine law, the transition density is then

$$K(x,y) = \gamma_d \frac{\big((y-x)\cdot \mathbf{n}(x)\big)\big((x-y)\cdot \mathbf{n}(y)\big)}{\|x-y\|^{d+1}} \mathbf{1}\{x,y \in \mathcal{R}, x \leftrightarrow y\}$$

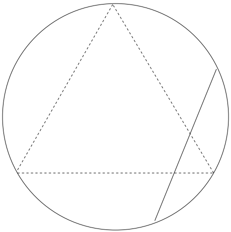
symmetric ! The surface measure $dx$ on $\partial\mathcal{D}$ is reversible,

$$dx\ K(x,dy) = dy\ K(y,dx),$$

and then invariant.

# Le paradoxe de Bertrand, 1889

Comment choisir une corde au hasard sur un domaine $\mathcal{D}$ du plan ?
Probabilité pour que la corde soit plus longue que le coté du triangle équilateral inscrit ?



Dans le cas du disque, Joseph Bertrand propose plusieurs constructions "raisonables":

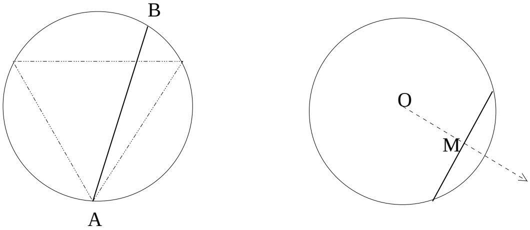



Figure: (1)– Extrémités au hasard: $p = 1/3$.
(2)– Rayon au hasard: ; le milieu $M$ de la corde est tel que direction et longueur de $\overrightarrow{OM}$ sont indépendants uniformes: $p = 1/2$
(3)– $M$ uniformément distribué dans le disque: $p = 1/4$

# Bertrand, la corde et la loi du cosinus

➠ Y a-t-il une unique solution ”correcte” ? Si oui, laquelle ?

Après Bertrand 1889, d’autres se sont penchés sur le paradoxe, dont Borel 1909, Poincaré 1912, Gnedenko 1962, Kendall and Moran 1963, von Mises 1957-64... Seul Borel exprime une préference (sans donner de raison), au contraire de Von Mises qui écrit que ce genre de question (comme celle de l’aiguille de Buffon) n’est pas du resort des probabilités; les autres se contentent de dire que la question n’a pas de réponse précise en l’absence d’une définition du mot ”au hasard”.

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En 1973 Edwin Jaynes a proposé de choisir "la bonne réponse" suivant un principe de minimum d'information (entropie). En demandant que la distribution de la corde au hasard soit invariante par homothétie et translation, le problème devient bien posé, et admet pour solution unique la construction du "rayon aléatoire".

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Nota: l'angle entre la corde et la tangente a densité proportionnelle à son *cosinus*.

⇒ Le billard stochastique avec la loi du cosinus étend cette construction au cas d'un domaine général.

# Outline

# Asymptotics on a bounded table (for a general $\gamma$)

Suppose that

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**Theorem**

(i) *There exists a unique probability measure $\hat{\mu}$ on $\partial\mathcal{D}$ which is invariant for the random walk $\xi_n$. Moreover, $d\hat{\mu} << dx$.*

(ii) $\|\mathbf{P}[\xi_n \in \cdot] - \hat{\mu}\|_v \leq \beta_0 e^{-\beta_1 n}$ *($\|\cdot\|_v$ = total variation distance).*

(iii) *Central Limit Theorem: $\forall A \subset \partial\mathcal{D}$ measurable there exists $\sigma_A$ ($\sigma_A > 0$ if $0 < |A| < |\partial\mathcal{D}|$) such that*

$$n^{-1/2}\left(\sum_{i=1}^{n} \mathbf{1}\{\xi_i \in A\} - n\hat{\mu}(A)\right) \xrightarrow{law} \mathcal{N}(0, \sigma_A^2)$$

For the cosine law, the theorem holds with $d\hat{\mu} = |\partial\mathcal{D}|^{-1} dx$ uniform on $\partial\mathcal{D}$.

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☐ Check Döblin condition: there exist $n$, $\varepsilon > 0$ such that for all $x, y \in \mathcal{R}$

$$K^n(x,y) \ge \varepsilon \qquad (1)$$

☐

# Inheritance by induced chords for KRW (cosine law)

Let $\mathcal{D}' \subset \mathcal{D}$ convex.

The stationary KRW $(\xi_n)_n$ in $\mathcal{D}$ provides a random chord $(\xi_1, \xi_2)$ on $\mathcal{D}$. Conditionally on the chord hitting the domain $\mathcal{D}'$, it intersects $\mathcal{D}'$ according to an "induced" random chord $(\xi_1', \xi_2')$.

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**Theorem**

*The induced chord has the same law (relative to $\mathcal{D}'$) as the original one (relative to $\mathcal{D}$)*

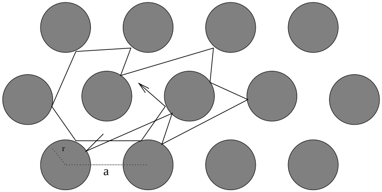
Nice inheritance property!

(A more complicated but similar one holds for non-convex $\mathcal{D}'$.)

# Outline

# Context

Billiard with finite horizon: distance between bounces is uniformly bounded. (Ex: Sinaï billiard on the triangular lattice with $a\sqrt{3}/4 < r < a/2$.)




Deterministic billiard. Bunimovich, Sinaï, Chernov [1991]: probability of initial conditions (in the unit cell) s.t. $t^{-1/2}X(t) \in \mathcal{O}$ open bounded $\rightarrow$ gaussian measure of $\mathcal{O}$.

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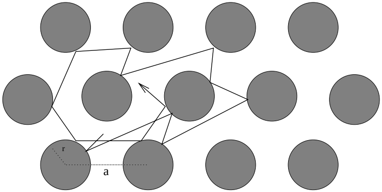


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How can we go beyond periodic domains and still understand diffusivity ?

# Infinite horizontal "tube"

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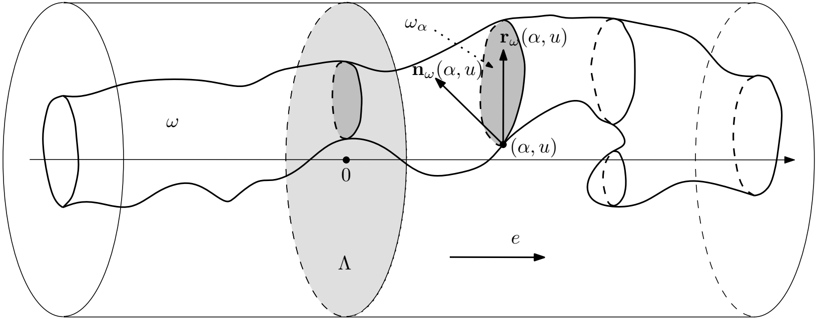


Figure: Random tube. Inwards, normal vectors $\mathbf{n}_\omega(x) = \mathbf{n}_\omega(\alpha, u) \in \mathbb{S}^{d-1}$ and $\mathbf{r}_\omega(x) = \mathbf{r}_\omega(\alpha, u) \in \mathbb{S}^{d-2}$.

# Random tube = random environment

Any tube $\omega = (\omega_\alpha, \alpha \in \mathbb{R})$ is seen as the process of its sections

$$\omega = \{(\alpha, u) \in \mathbb{R}^d : u \in \omega_\alpha\}$$

Let $\mathfrak{E}$ to be the set of all open domains $A \subset \mathbb{R}^{d-1}$ contained in a fixed ball,

$$A \subset \Lambda := \{u \in \mathbb{R}^{d-1} : \|u\| \leq M\}.$$

Let $\Omega = \mathcal{C}(\mathbb{R} \to \mathfrak{E})$ "space of tubes" (equipped with the distance $\rho(A, B) = |(A \setminus B) \cup (B \setminus A)|$ on $\mathfrak{E}$ and cylinder sigma-algebra).

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Assume

$$\omega \sim \mathbb{P},$$

with $\mathbb{P}$ a probability measure on $\Omega$, stationary and ergodic (w.r.t. shifts in $\alpha$).

# Random tube: assumptions, notations

Assumptions: $\mathbb{P}$-a.s., $\omega$ is open, connected, and:
**(L)** $\partial\omega$ is Lipschitz with uniform constants
**(R)** $\{x \in \partial\omega : \partial\omega \text{ is } C^1 \text{ in } x, |\mathbf{n}_\omega(x) \cdot e| \neq 1\}$ has full measure $\mathcal{H}_{d-1}$-measure
**(P)** Points on the boundary which are close, communicate "well" and "quickly": $\exists N, \varepsilon, \delta$: $\mathbb{P}$-a.s., $\forall x, y \in \mathcal{R}$ with $|(x-y)\cdot e| \leq 2, \exists B_1, \ldots, B_n \subset \partial\omega$, $n \leq N$ with $\nu^\omega(B_i) \geq \delta(i = 1, \ldots, n)$, s.t.

- $K(x,z) \geq \varepsilon$ for all $z \in B_1$, $K(y,z) \geq \varepsilon$ for all $z \in B_n$,
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Notations:
$\mu_\alpha^\omega$ = restriction of $(d-2)$-dimensional Hausdorff measure on $\partial\omega_\alpha$
$\nu^\omega$ = restriction of $(d-1)$-dimensional Hausdorff measure on $\partial\omega$

Disintegration formula: with $x = (\alpha, u)$,

$$d\nu^\omega(x) = \kappa_{\alpha,u}^{-1}\, d\mu_\alpha^\omega(u)\, d\alpha \,, \qquad \kappa_{\alpha,u} = \mathbf{n}_\omega(x) \cdot \mathbf{r}_\omega(x).$$

## What we prove: from now on, $\gamma$= cosine density

Denote by $P_\omega$ the quenched law, i.e. the law of KRW in the environment $\omega$

**Theorem**

*Let $d \geq 3$. Then, $\mathbb{P}$-a.s., we have under the quenched law $P_\omega$,*

$$n^{-1/2}\xi_n \cdot e \xrightarrow{law} \mathcal{N}(0, \sigma^2)$$

*and a similar result for continuous time stochastic billiard.*

Paradigm: KRW is “alike” RW in RE, $d = 1$, with random conductances.

$\mathcal{U}$ = projection on $\mathbb{R}^{d-1}$. Define the environment as seen from the walker:

$$\big((\theta_{\xi_n \cdot e}\omega, \mathcal{U}\xi_n), n \geq 0\big)$$

with state space

$$\mathfrak{S} = \{(\omega, u) : \omega \in \Omega, u \in \partial\omega_0\}$$

*Note: environment consists not only in tube with an appropriate horizontal shift, but also in the transverse component of the walk.*

Markov chain with transition operator $G$,

$$\begin{aligned} Gf(\omega, u) &= \mathsf{E}_\omega\big(f(\theta_{\xi_1 \cdot e}\omega, \mathcal{U}\xi_1) \mid \xi_0 = (0, u)\big) \\ &= \int_{-\infty}^{+\infty} d\alpha \int_\Lambda d\mu_\alpha^\omega(v)\, \kappa_{\alpha,v}^{-1} f(\theta_\alpha\omega, v) K(0, u; \alpha, v). \end{aligned}$$

# Environment viewed from the particle:

## Proposition

*The probability measure $\mathbb{Q}$ on $\mathfrak{S}$*

$$d\mathbb{Q}(\omega,u) = \frac{1}{\mathcal{Z}}\kappa_{0,u}^{-1}\, d\mu_0^{\omega}(u)\, d\mathbb{P}(\omega)$$

*is reversible for G (hence invariant), i.e. $\langle g, Gf\rangle_{\mathbb{Q}} = \langle f, Gg\rangle_{\mathbb{Q}}$. Moreover, the (stationary) Markov process of environment viewed from the particle is ergodic.*

Define the local drift and the second moment of the jump (projected on the horizontal direction):

$$\begin{aligned} \Delta_\beta(\omega,u) &= \mathrm{E}_\omega\big((\xi_1-\xi_0)\cdot e \mid \xi_0=(\beta,u)\big) \\ b_\beta(\omega,u) &= \mathrm{E}_\omega\big(((\xi_1-\xi_0)\cdot e)^2 \mid \xi_0=(\beta,u)\big) \end{aligned}$$

Then, $\langle \Delta_\beta\rangle_{\mathbb{Q}} = 0$.

# Our Results

By invariance, $\langle b_\beta \rangle_{\mathbb{Q}}$ does not depend on $\beta$.

**Theorem (Quenched Invariance Principle)**

*Assume*

$$\langle b_\beta \rangle_{\mathbb{Q}} < \infty. \tag{2}$$

*Then, $\exists \sigma > 0$ such that for $\mathbb{P}$-almost all $\omega$ and any starting point, the polygonal interpolation of $\frac{n}{m} \mapsto m^{-1/2}\xi_n \cdot e$ converges as $m \to \infty$ in law, under $\mathrm{P}_\omega$, to the Brownian motion with diffusion coefficient $\sigma$.*

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**Proposition**

*If $d \geq 3$, then condition (2) holds.*
*If $d = 2$ and if $\exists S \subset \Lambda$ interval s.t. $\mathbb{R} \times S \subset \omega$ for $\mathbb{P}$-a.a. $\omega$, then (2) does not hold.*

*Remark:* if $d = 2$ it may happen that (2) holds. For ex., in case of uniformly bounded jumps.

# Ideas of proof of Diffusivity

Condition (2) is known to imply the Invariance Principle under the *annealed law*:
Kipnis, Varadhan (1986), De Masi, Ferrari, Goldstein, Wick (1989).
But we want more. . .

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(i) Construct a corrector: $\exists \psi : \mathfrak{M} = \{(\omega, u, y) : \omega \in \Omega, u \in \partial\omega_0, y \in \partial\omega\} \to \mathbb{R}$ such that

$$\xi_n \cdot e + \psi(\omega, u, \xi_n)$$

is a martingale under the quenched law $\mathrm{P}_\omega$ (for all $u \in \partial\omega_0$).
Using condition (2), $\psi$ can be obtained as an orthogonal projection on the space of gradients (Kozlov'85, Landim-Olla'08, Mathieu-Piatniski'07, Biskup-Prescott'07).

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(ii) Apply CLT to this martingale.
(iii) Show that the corrector can be neglected using the ergodic theorem □

# Easy proofs of finite/infinite norm

(i) Elementary proof of (2) for $d \geq 4$:

$$|\{s \in \mathbb{S}^{d-1} : x + hs \in \mathbb{R} \times \Lambda\}| = O(h^{-(d-1)}) \qquad \text{as } h \to \infty,$$

uniformly in $x \in \mathbb{R} \times \Lambda$. This integrate $h$, so $\langle b_\beta \rangle_{\mathbb{Q}} < \infty$.

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uniformly in $x \in \mathbb{R} \times \Lambda$. This integrate $h$, so $\langle b_\beta \rangle_{\mathbb{Q}} < \infty$.

Want to use what we know for straight cylinders...

(ii) Case $d = 2$ and $\omega$ contains an infinite straight strip of height $r > 0$, $\mathbb{P}$-a.s. Simple calculation for the strip: the expected squared length of a chord is infinite. We use the property of induced chords to deduce that $\langle b_\beta \rangle_{\mathbb{Q}} = \infty$.

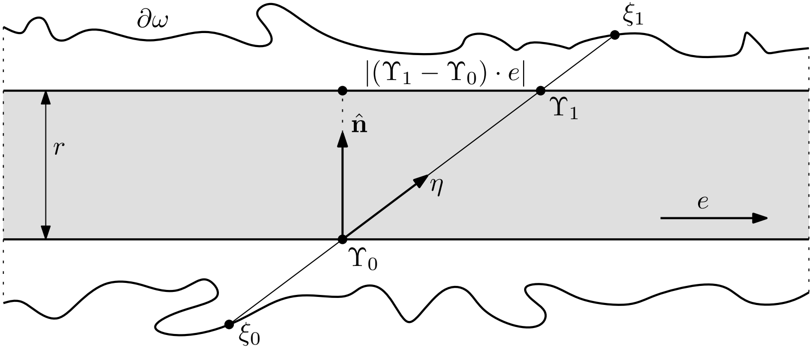


Figure: $d = 2$, $\omega$ containing an infinite straight strip

## Sketch of proof, $d = 3$

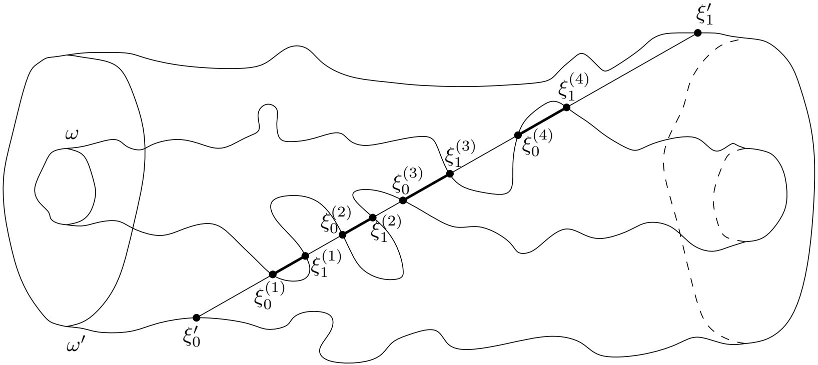


Figure: $d = 3$. If the outer domain $\omega'$ is a cylinder, the expected squared length of a chord is finite

We use the property of induced chords with non-convex interior domain to prove $\langle b_\beta \rangle_{\mathbb{Q}} = \infty$.

# Outline

# Stochastic billiard with drift in a random tube

Same assumptions as above on the random tube.

Dynamics for KRW with drift: acceptance/rejection. Fix $\lambda > 0$. If $\xi_n = x$,

- First select $y \in \partial\omega$, $y \sim K(x, y)$
- Then,
  - if $(y - x) \cdot \mathbf{e} > 0$, set $\xi_{n+1} = y$ ,
  - if $(y - x) \cdot \mathbf{e} < 0$,
    set $\xi_{n+1} = y$ with probability $\exp\{-\lambda |(y - x) \cdot \mathbf{e}|\}$, and $\xi_{n+1} = x$ otherwise.

Then, the measure $\nu_\lambda^\omega$ with

$$\frac{d\nu_\lambda^\omega}{d\nu^\omega}(x) = \exp\{\lambda x \cdot \mathbf{e}\}$$

is invariant and reversible for $\xi_n$.

# Law of large numbers

**Theorem**

*Assume $d \geq 3$. There exists $\hat{v} > 0$ deterministic such that, a.s.,*

$$\frac{\xi_n \cdot \mathbf{e}}{n} \to \hat{v} \qquad \text{as } n \to \infty$$

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Idea of proof: using condition (P), we make a coupling of $\xi$ in a fixed $\omega$, with a Random Walk in Random Environment (RWRE) on $\mathbb{Z}$, with unbounded jumps and stationary ergodic environment. We use a (new) Law of Large Numbers for the latter:

# Outline

# Random walk in random environment with unbounded jumps on $\mathbb{Z}$

Attention: now,

$$\omega = (\omega_{x,y}; x, y \in \mathbb{Z}), \quad \omega_{x,y} \geq 0, \sum_y \omega_{x,y} = 1.$$

Let $S_n$ be the RWRE in $\mathbb{Z}$ with $P_\omega(S_{n+1} = x + y | S_n = x) = \omega_{x,y}$.

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Assume $(\omega_{x,\cdot})_x$ is stationary and ergodic under some $\mathbb{P}$.

Consider also the RW in the truncated environment $\omega^\varrho$:

$$\omega^\varrho_{xy} = \begin{cases} \omega_{xy}, & \text{if } 0 < |y| < \varrho, \\ 0, & \text{if } |y| \geq \varrho, \\ \omega_{x0} + \sum_{y:|y|\geq\varrho} \omega_{xy}, & \text{if } y = 0, \end{cases}$$

# RWRE: assumptions

Assume uniform ellipticity, uniform tails, strong transience (no traps):

**Condition E.** There exists $\tilde{\varepsilon}$ such that $\mathbb{P}[\omega_{01} \geq \tilde{\varepsilon}] = 1$.

**Condition C.** $\exists \alpha > 1, \gamma_1 > 0$ s.t. for all $s \geq 1$,

$$\sum_{y:|y|\geq s} \omega_{0y} \leq \gamma_1 s^{-\alpha}, \qquad \mathbb{P}\text{-a.s.}$$

**Condition D.** $\exists g_1 \geq 0$ with $\sum_{k=1}^{\infty} k g_1(k) < \infty, \exists \varrho_0 < \infty$, such that $\forall x \leq 0, \varrho \geq \varrho_0$,

$$E_\omega^0 N_\infty^\varrho(x) \leq g_1(|x|), \qquad \mathbb{P} - \text{a.s.}$$

with $N_n^\varrho(x) = \sum_{k \leq n} \mathbf{1}\{S_k^\varrho = x\}$.

# Law of Large Numbers for ballistic RWRE with unbounded jumps

**Theorem**

*Then,* $\forall \varrho \in [\varrho_0, \infty], \exists v_\varrho > 0$ *s.t. we have*

$$\frac{S_n^\varrho}{n} \to v_\varrho, \quad n \to \infty, \quad \text{a.s.}$$

# Law of Large Numbers for ballistic RWRE with unbounded jumps

**Theorem**

*Then,* $\forall \varrho \in [\varrho_0, \infty], \exists v_\varrho > 0$ *s.t. we have*

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- No reversibility is assumed.
- RWRE on $\mathbb{Z}$ with bounded jumps: long-time behavior determined by middle Lyapunov exponents of random matrices. Transience/recurrence by Key [1984], LLN by Goldsheid [2003, 2008], Brémont [2009]; lingering "à la Sinai" by Bolthausen and Goldsheid [2008].
- Only reference for unbounded jumps: 0-1 law by Andjel [1988].

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Au nom de tous les participants,

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